FORMALIZED MATHEMATICS
Vol. 34, No. 1, pp. 89–111, 2026
DOI: 10.2478/forma-2026-0007
e-ISSN: 1898–9934

Paradigm
reference-global.com

Surreal Numbers and Conway Names: A Formal Introduction to Their Interrelations

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Summary. This paper analyzes the relationship between the classical recursive definition of arithmetic operations on surreal numbers and their representation in Conway Normal Form. While the traditional approach uses transfinite induction and recursion to define numbers as cuts between left and right sets, Conway Normal Form allows them to be treated as a vector-like representation within a transfinite power basis.

This study shows that addition in this representation has a structure similar to vector summation, requiring only the exponent supports to be aligned. Furthermore, the multiplication of surreal numbers in normal form is interpreted as a transfinite generalization of the Cauchy product for power series.

MSC: 03H05 12J15 68V20

Keywords: surreal number; normal form

MML identifier: SURREAL1, version: 8.1.15 5.105.1515

Introduction

In our previous work [22], we focused on formalizing the Conway Normal Form (CNF), as set out in Theorem 21 of Conway's seminal handbook [12], by adapting the framework proposed by Ehrlich [14, 13]. Under this approach, every surreal number can be uniquely represented as a sum $\sum_{\alpha<\beta} \boldsymbol{\omega}^{\mathtt{y}_\alpha}\mathtt{r}_\alpha$ where $\{\mathtt{y}_\alpha\}_{\alpha<\beta}$ is a strictly decreasing sequence of surreal numbers and $\{\mathtt{r}_\alpha\}_{\alpha<\beta}$ is a sequence of non-zero real numbers. Furthermore, both Conway and Ehrlich

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extend this definition somewhat casually by allowing the coefficients $\mathtt{r}_\alpha$ to be zero where necessary. Although they provide minimal justification for this step, inserting a potentially transfinite number of zeros into an infinite summation is far from trivial. In this extended context, the formal expression

$$\sum_{y\in\mathbf{No}} \boldsymbol{\omega}^y a_y \tag{I.1}$$

is well-defined, provided that the support $\{y \in \mathbf{No} \mid a_y \neq 0\}$ can be well-ordered into a single descending sequence. The existence of such an ordering uniquely determines the descending sequence of exponents, $\{\mathtt{y}_\alpha\}_{\alpha<\beta}$, and this sequence in turn fixes the sequence of coefficients $\{\mathtt{r}_\alpha\}_{\alpha<\beta}$ via $\mathtt{r}_\alpha := a_{\mathtt{y}_\alpha}$ for all $\alpha < \beta$. While this approach allows for an intuitive formulation of fundamental properties, such as the addition law:

$$\sum_{y\in\mathbf{No}} \boldsymbol{\omega}^y(a_y + b_y) = \sum_{y\in\mathbf{No}} \boldsymbol{\omega}^y a_y + \sum_{y\in\mathbf{No}} \omega^y b_y \tag{I.2}$$

it significantly complicates formal verification by obscuring foundational subtleties that are usually left implicit or taken for granted in informal reasoning. In particular, both summations of (I.2) implicitly assume the existence of two descending sequences associated with the coefficients a_y and b_y, respectively. Furthermore, it is assumed that the union of the supports of these sequences can be well-ordered into a single descending sequence. Certain explanations regarding the formal correctness of this equation were exclusive to Gonshor [15]; however, his proposed CNF diverges slightly from the approach previously considered in [18]. For this reason, our primary focus is first on well-ordered sets, followed by an examination of the consistency of the CNF across these different approaches, with the ultimate goal of establishing the fundamental arithmetic operations.

We begin Section 1 by defining the operation of the transfinite shifting of a sequence. Subsequently, we establish fundamental relationships between the existence of monotone sequences within structures and the property of well-foundedness, building upon the framework formalized in [23]. We demonstrate that a relation well-orders a set if and only if its restriction to that set is reflexive and antisymmetric, and the set itself constitutes the range of an increasing sequence (see Theorem

In Section 6, we establish several properties intended to bridge the approaches of Gonshor and Conway. To recall Gonshor's framework, the CNF is defined inductively; the simpler case of a successor ordinal (i.e., $\beta = \gamma + 1$) is given as a natural sum:

$$\sum_{\alpha<\beta} \boldsymbol{\omega}^{\mathtt{y}_\alpha}\mathtt{r}_\alpha := \sum_{\alpha<\gamma} \boldsymbol{\omega}^{\mathtt{y}_\alpha}\mathtt{r}_\alpha + \boldsymbol{\omega}^{\mathtt{y}_\gamma}\mathtt{r}_\gamma, \tag{I.3}$$

while the more complex case of a limit ordinal β yields the expression $\sum_{\alpha<\beta} \boldsymbol{\omega}^{\mathtt{y}_\alpha}\mathtt{r}_\alpha$, defined as the surreal number:

$$\left\{\sum_{\alpha<\kappa} \boldsymbol{\omega}^{\mathtt{y}_\alpha}\mathtt{r}_\alpha + \boldsymbol{\omega}^{\mathtt{y}_\kappa}(\mathtt{r}_\kappa - \epsilon) \mid \sum_{\alpha<\kappa} \boldsymbol{\omega}^{\mathtt{y}_\alpha}\mathtt{r}_\alpha + \boldsymbol{\omega}^{\mathtt{y}_\kappa}(\mathtt{r}_\kappa + \epsilon)\right\}_{\kappa<\beta}^{\epsilon\in\mathbb{R}^+} \tag{I.4}$$

where $\mathbb{R}^+$ is identified with the subset of positive real surreal numbers. It is worth noting that the proofs of the fundamental properties of GonshorŹs CNF established in [15] have been adapted here to reconcile this framework with ConwayŹs original definition. Indeed, our point of departure is the formalization of Lemma 5.2 from [15] (cf. Theorem

Based on this connection, we establish the fundamental properties of the CNF noted by Ehrlich [14]; although the author emphasizes the importance of these relations and refers to Gonshor's work, he does so without providing explicit proofs. In particular, we formalize the *Basic Observation* [14], also named *the associative law* [15], generalizing the result to the case where the sequence $\{\mathtt{r}_\alpha\}$ allows for zero values (cf. Theorem

The final two sections are devoted exclusively to the multiplication of surreal numbers in normal form. In his monograph, Gonshor establishes that the surreal product agrees with the formal product (see Theorem 5.8 in [15]). A similar perspective is shared by Ehrlich [14], who formulates the multiplication of normal forms as follows:

$$\sum_{y\in\mathbf{No}} \boldsymbol{\omega}^y a_y \cdot \sum_{y\in\mathbf{No}} \boldsymbol{\omega}^y b_y = \sum_{y\in\mathbf{No}} \boldsymbol{\omega}^y \cdot \left[\sum_{\substack{(\mu,\nu)\in\mathbf{No}\times\mathbf{No}\\ \mu+\nu=y}} a_\mu b_\nu\right] \tag{I.5}$$

The key to understanding the inner sum "where $(\mu,\nu)\in\mathbf{No}\times\mathbf{No}$ and $\mu+\nu=y$" is the observation that the vast majority of its terms vanish. Furthermore, since the non-zero coefficients a_μ and b_ν can be arranged in strictly decreasing transfinite sequences $\{\mathtt{y}_\alpha\}_{\alpha<\beta}$ and $\{\mathtt{z}_\gamma\}_{\gamma<\delta}$ respectively, the number of terms where both factors a_μ and b_ν are simultaneously non-zero is guaranteed to be finite. We prove this rigorously in Definition

1. Transfinite Shifting of a Sequence

From now on A, B, C denote ordinal numbers and f, g denote transfinite sequences.

Now we state the propositions:

(1) $f \frown \emptyset = f$.

(2) If $A \subseteq \operatorname{dom} f$, then $(f \frown g)\restriction A = f\restriction A$.

PROOF: For every object x such that $x \in A$ holds $((f \frown g)\restriction A)(x) = (f\restriction A)(x)$ by [8, (49)]. □

(3) If $A \subseteq B$, then $B = A + (B - A)$.

(4) If $\operatorname{dom} f \subseteq A$, then $(f \frown g)\restriction A = f \frown (g\restriction(A - \operatorname{dom} f))$. The theorem is a consequence of (3).

(5) Let us consider transfinite sequences g_1, g_2. If $f \frown g_1 = f \frown g_2$, then $g_1 = g_2$.

Let f be a transfinite sequence and A be an ordinal number. The functor $f_{\downarrow A}$ yielding a transfinite sequence is defined by

(Def. 1) $\operatorname{dom} it = \operatorname{dom} f - A$ and $f = (f\restriction A) \frown it$.

Now we state the propositions:

(6) $\operatorname{dom}(f_{\downarrow A}) \subseteq \operatorname{dom} f$.

(7) If $A \subseteq B$, then $f(B) = (f_{\downarrow A})(B - A)$ and ($B \in \operatorname{dom} f$ iff $B - A \in \operatorname{dom}(f_{\downarrow A})$).

PROOF: $B \in \operatorname{dom} f$ iff $B - A \in \operatorname{dom}(f_{\downarrow A})$ by [5, (53)], [3, (16)], [5, (59)]. $f(B) = (f_{\downarrow A})(B - A)$ by [3, (12)], [26, (62)], (3). □

(8) (i) $f(A + B) = (f_{\downarrow A})(B)$, and

(ii) $A + B \in \operatorname{dom} f$ iff $B \in \operatorname{dom}(f_{\downarrow A})$.

The theorem is a consequence of (7).

Let D be a set, f be a transfinite sequence of elements of D, and A be an ordinal number. Note that $f_{\downarrow A}$ is D-valued.

Now we state the proposition:

(9) If $\operatorname{succ} A \subseteq \operatorname{dom} f$, then $(f\restriction \operatorname{succ} A)_{\downarrow A} = \langle f(A)\rangle$. The theorem is a consequence of (8).

2. Preliminaries on Monotone Sequences in Well-Founded Structures

Now we state the propositions:

(10) Let us consider a relational structure R, and a subset X of R. Then X is well founded if and only if $\operatorname{sub}(X)$ is well founded.

PROOF: Set $I =$ the internal relation of R. For every set Y such that $Y \subseteq X$ and $Y \neq \emptyset$ there exists an object a such that $a \in Y$ and I-Seg(a) misses Y by [6, (1)], [11, (87)]. □

(11) Let us consider a relational structure R. Then $\emptyset_R$ is well founded.

(12) Let us consider a non empty relational structure R, and a subset X of R. Then X is well founded if and only if there exists no sequence f of R such that $\operatorname{rng} f \subseteq X$ and f is descending. The theorem is a consequence of (11) and (10).

(13) Let us consider a non empty, antisymmetric, transitive relational structure R, and a sequence f of R. If f is descending, then every subsequence of f is descending.

PROOF: For every natural number n, $s(n+1) \neq s(n)$ and $\langle s(n+1), s(n)\rangle \in$ the internal relation of R by [9, (15)], [2, (13)], [17, (8)]. □

Let R be a non empty, antisymmetric, transitive relational structure. Note that every sequence of R which is descending is also one-to-one.

Now we state the propositions:

(14) Let us consider a non empty relational structure R, a sequence f of R, and a subset X of R. Suppose $X \cap \operatorname{rng} f$ is infinite. Then there exists a subsequence s of f such that $\operatorname{rng} s \subseteq X$.

(15) Let us consider a relational structure R, and subsets X, Y of R. If $X \subseteq Y$ and Y is well founded, then X is well founded. The theorem is a consequence of (11) and (12).

(16) Let us consider an antisymmetric, transitive relational structure R, and a subset X of R. If X is finite, then X is well founded. The theorem is a consequence of (11) and (12).

(17) Let us consider an antisymmetric, transitive relational structure R, and well founded subsets X, Y of R. Then $X \cup Y$ is well founded. The theorem is a consequence of (12), (14), and (13).

Let R be a non empty relational structure and s be a sequence of R. We say that s is non descending if and only if

(Def. 2) for every natural number n, $\langle s(n), s(n+1)\rangle \in$ the internal relation of R.

Now we state the propositions:

(18) Let us consider a non empty, transitive relational structure R, and a sequence f of R. Suppose f is non descending. Let us consider natural numbers i, j. Suppose $i < j$. Then $\langle f(i), f(j)\rangle \in$ the internal relation of R.

PROOF: Set $I =$ the internal relation of R. Define $\mathcal{P}$[natural number] $\equiv$ for every natural number i such that $i < \$_1$ holds $\langle f(i), f(\$_1)\rangle \in I$. For every natural number j such that $\mathcal{P}[j]$ holds $\mathcal{P}[j+1]$ by [2, (13)]. For every natural number j, $\mathcal{P}[j]$ from [2, Sch. 2]. □

(19) Let us consider a non empty, reflexive relational structure R, and a se-

quence f of R. Suppose R is connected. Let us consider a subset X of R. Suppose $\operatorname{rng} f \subseteq X$ and X is well founded. Then there exists a subsequence s of f such that s is non descending.

PROOF: Set $I =$ the internal relation of R. Consider m being an object such that $m \in \operatorname{rng} f$ and I-Seg(m) misses $\operatorname{rng} f$. Consider a being an object such that $a \in \operatorname{dom} f$ and $f(a) = m$. Define $\mathcal{P}[\text{natural number}, \text{set}, \text{set}] \equiv$ for every natural number i such that $i = \$_2$ holds $\$_3$ is a natural number and for every natural number j such that $j = \$_3$ holds $i < j$ and $f(j) \in \operatorname{rng}(f \upharpoonright (i+1))$ and I-Seg$(f(j))$ misses $\operatorname{rng}(f \upharpoonright (i+1))$. For every natural number n and for every set x, there exists a set y such that $\mathcal{P}[n, x, y]$ by [2, (55), (11), (13)]. Consider N being a function such that $\operatorname{dom} N = \mathbb{N}$ and $N(0) = a$ and for every natural number n, $\mathcal{P}[n, N(n), N(n+1)]$ from [16, Sch. 1]. Define $\mathcal{P}[\text{natural number}] \equiv N(\$_1)$ is a natural number. For every natural number j, $\mathcal{P}[j]$ from [2, Sch. 2]. $\operatorname{rng} N \subseteq \mathbb{N}$. Reconsider $f_1 = f \cdot N$ as a subsequence of f. Define $\mathcal{R}[\text{natural number}] \equiv \langle f_1(\$_1),\ f_1(\$_1+1)\rangle \in I$ and I-Seg$(f_1(\$_1+1))$ misses $\operatorname{rng}(f \upharpoonright (N(\$_1)+1))$. $\mathcal{R}[0]$ by [9, (15)], [6, (1)]. For every natural number n such that $\mathcal{R}[n]$ holds $\mathcal{R}[n+1]$ by [9, (15)], [2, (21), (48), (55)]. For every natural number n, $\mathcal{R}[n]$ from [2, Sch. 2]. □

(20) Let us consider a non empty, transitive relational structure R, and a sequence f of R. If f is non descending, then every subsequence of f is non descending.

PROOF: For every natural number n, $\langle s(n),\ s(n+1)\rangle \in$ the internal relation of R by [9, (15)], [2, (13)], (18). □

Let R be a binary relation and s be a transfinite sequence of elements of field R. We say that s is increasing if and only if

(Def. 3) for every ordinal numbers A, B such that $A \in B \in \operatorname{dom} s$ holds $s(A) \neq s(B)$ and $\langle s(A),\ s(B)\rangle \in R$.

Let us note that every transfinite sequence of elements of field R which is increasing is also one-to-one.

Let us consider a binary relation R and a set X. Now we state the propositions:

(21) Suppose $X = \operatorname{field}(R \mid^2 X)$. Then

(i) if $R \mid^2 X$ is reflexive, then R is reflexive in X, and

(ii) if $R \mid^2 X$ is connected, then R is connected in X, and

(iii) if $R \mid^2 X$ is well founded, then R is well founded in X.

PROOF: For every set Y such that $Y \subseteq X$ and $Y \neq \emptyset$ there exists an object a such that $a \in Y$ and R-Seg(a) misses Y by [6, (3), (1)], [11, (87)]. □

(22) (i) if $R \mid^2 X$ is transitive, then R is transitive in X, and

(ii) if $R\,|^2\,X$ is antisymmetric, then R is antisymmetric in X.

Now we state the propositions:

(23) Let us consider a binary relation R, and a subset X of field R. Suppose $R|^2 X$ is reflexive in X and antisymmetric in X. Let us consider a transfinite sequence s of elements of field R. Suppose $\operatorname{rng} s = X$ and s is increasing. Then s is an isomorphism between $\subseteq_{\operatorname{dom} s}$ and $R\,|^2\,X$.

PROOF: Set $R_2 = R\,|^2\,X$. $X \subseteq \operatorname{field} R_2$ by [26, (15)]. Set $O = \operatorname{dom} s$. Set $R_5 = \subseteq_O$. For every objects a, b, $\langle a, b\rangle \in R_5$ iff a, $b \in \operatorname{field} R_5$ and $\langle s(a), s(b)\rangle \in R_2$ by [26, (15)], [3, (11)], [11, (87)], [3, (16)]. □

(24) Let us consider a binary relation R, and a subset X of field R. Then R well orders X if and only if $R\,|^2\,X$ is reflexive in X and antisymmetric in X and there exists a transfinite sequence s of elements of field R such that $\operatorname{rng} s = X$ and s is increasing.

PROOF: Set $R_2 = R\,|^2\,X$. If R well orders X, then R_2 is reflexive in X and antisymmetric in X and there exists a transfinite sequence s of elements of field R such that $\operatorname{rng} s = X$ and s is increasing by [7, (1)], [6, (4), (40)], [3, (10)]. $X \subseteq \operatorname{field} R_2$ by [26, (15)]. Set $O = \operatorname{dom} s$. Set $R_5 = \subseteq_O$. s is an isomorphism between R_5 and R_2. R is reflexive in X, transitive in X, antisymmetric in X, connected in X, and well founded in X. □

(25) Let us consider a binary relation R, and a subset X of field R. Suppose $R|^2 X$ is reflexive in X and antisymmetric in X. Let us consider transfinite sequences s, t of elements of field R. Suppose $\operatorname{rng} s = X = \operatorname{rng} t$ and s is increasing and t is increasing. Then $s = t$. The theorem is a consequence of (23).

Let R be a relational structure and A be a subset of R. We say that A is well-ordering if and only if

(Def. 4) the internal relation of R well orders A.

Observe that $\emptyset_R$ is well-ordering.

Now we state the propositions:

(26) Let us consider a relational structure R, and subsets X, Y of R. If $X \subseteq Y$ and Y is well-ordering, then X is well-ordering. The theorem is a consequence of (15).

(27) Let us consider a subset X of A. Then there exists a sequence s of ordinal numbers such that

(i) s is increasing, and

(ii) $\operatorname{rng} s = X$.

3. Selected Properties of Operators on Sequences of Surreal Numbers

From now on s, x, x_1, x_2, y, y_1, y_2, z denote surreal numbers, r, r_1, r_2, e denote real numbers, and S, T denote surreal-membered sets.

Now we state the propositions:

(28) If $x_1 \approx x_2$ and $y_1 \approx y_2$, then $x_1 - y_1 \approx x_2 - y_2$.

(29) If $x_1 \approx x_2$ and $y_1 \approx y_2$, then $x_1 \cdot y_1 \approx x_2 \cdot y_2$.

(30) (i) $s + \mathrm{s}_{\mathbb{R}}(r + e) \cdot z \approx s + \mathrm{s}_{\mathbb{R}}(r) \cdot z + \mathrm{s}_{\mathbb{R}}(e) \cdot z$, and

(ii) $s + \mathrm{s}_{\mathbb{R}}(r - e) \cdot z \approx s + \mathrm{s}_{\mathbb{R}}(r) \cdot z - \mathrm{s}_{\mathbb{R}}(e) \cdot z$.

(31) If $\mathbf{0_{No}} \leqslant x <^{\infty} y$, then $|\mathrm{s}_{\mathbb{R}}(r) \cdot x| <^{\infty} y$.

(32) $\mathrm{s}_{\mathbb{R}}(r_1) - \mathrm{s}_{\mathbb{R}}(r_2) \approx \mathrm{s}_{\mathbb{R}}(r_1 - r_2)$.

(33) $x \cdot \mathrm{s}_{\mathbb{R}}(r) \approx -x \cdot \mathrm{s}_{\mathbb{R}}(-r)$.

(34) Let us consider sets X_1, X_2, Y_1, Y_2. Suppose $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$. Then $X_1 \oplus Y_1 \subseteq X_2 \oplus Y_2$.

Let o be an object and f be a surreal valued function. Assume $o \in \operatorname{dom} f$. The functor $f \upharpoonleft o$ yielding a surreal number is defined by the term

(Def. 5) $f(o)$.

Let S be a surreal sequence. Observe that S is positive if and only if the condition (Def. 6) is satisfied.

(Def. 6) for every A such that $A \in \operatorname{dom} S$ holds $S \upharpoonleft A$ is positive.

Let us note that S is decreasing infinitely if and only if the condition (Def. 7) is satisfied.

(Def. 7) for every A and B such that $A \in B \in \operatorname{dom} S$ holds $S \upharpoonleft B <^{\infty} S \upharpoonleft A$.

One can check that every surreal sequence which is decreasing infinitely is also one-to-one.

Let f be a surreal sequence and A be an ordinal number. Observe that $f_{\restriction A}$ is surreal valued.

Let us consider a surreal sequence f. Now we state the propositions:

(35) If $A \subseteq B \in \operatorname{dom} f$, then $f \upharpoonleft B = f_{\restriction A} \upharpoonleft (B - A)$. The theorem is a consequence of (7).

(36) If $A + B \in \operatorname{dom} f$ or $B \in \operatorname{dom}(f_{\restriction A})$, then $f \upharpoonleft (A + B) = f_{\restriction A} \upharpoonleft B$. The theorem is a consequence of (8).

Let f be a positive surreal sequence and A be an ordinal number. One can check that $f_{\restriction A}$ is positive.

Let f be a decreasing infinitely surreal sequence. One can verify that $f_{\restriction A}$ is decreasing infinitely.

Let S be a surreal-membered set. The functor $\mathrm{Unique}_{\mathbf{No}}(S)$ yielding a unique surreal-membered set is defined by

(Def. 8) for every x, $x \in it$ iff there exists y such that $y \in S$ and $x = \mathrm{Unique}_{\mathbf{No}}(y)$.

Let s be a surreal sequence. The functor $\mathrm{Unique}_{\mathbf{No}}(s)$ yielding a uniq-surreal sequence is defined by

(Def. 9) $\mathrm{dom}\, it = \mathrm{dom}\, s$ and for every A such that $A \in \mathrm{dom}\, s$ holds $it(A) = \mathrm{Unique}_{\mathbf{No}}(s \hookrightarrow A)$.

Now we state the propositions:

(37) Let us consider a surreal sequence s. Then $\mathrm{Unique}_{\mathbf{No}}(\mathrm{rng}\, s) = \mathrm{rng}\, \mathrm{Unique}_{\mathbf{No}}(s)$.

(38) $\mathrm{Unique}_{\mathbf{No}}(S \oplus T) = \mathrm{Unique}_{\mathbf{No}}(\mathrm{Unique}_{\mathbf{No}}(S) \oplus \mathrm{Unique}_{\mathbf{No}}(T))$.
PROOF: Set $U_1 = \mathrm{Unique}_{\mathbf{No}}(S)$. Set $U_2 = \mathrm{Unique}_{\mathbf{No}}(T)$. $\mathrm{Unique}_{\mathbf{No}}(S \oplus T) \subseteq \mathrm{Unique}_{\mathbf{No}}(U_1 \oplus U_2)$ by [21, (43)], [20, (4), (50)]. $\mathrm{Unique}_{\mathbf{No}}(U_1 \oplus U_2) \subseteq \mathrm{Unique}_{\mathbf{No}}(S \oplus T)$ by [21, (43)], [20, (4), (50)]. □

(39) $\mathrm{Unique}_{\mathbf{No}}(S) = \emptyset$ if and only if $S = \emptyset$.

4. Well-Ordered Substructures of Surreal Numbers under the Reverse Order

Let X be a surreal-membered set. The functor $\mathrm{Rev}_{\mathbf{No}}(X)$ yielding a strict relational structure is defined by

(Def. 10) the carrier of $it = X$ and for every surreal numbers a, b such that a, $b \in X$ holds $a \leqslant b$ iff $\langle b, a \rangle \in$ the internal relation of it.

Let us observe that $\mathrm{Rev}_{\mathbf{No}}(X)$ is reflexive and transitive.

Let X be a non empty, surreal-membered set. One can check that $\mathrm{Rev}_{\mathbf{No}}(X)$ is non empty and $\mathrm{Rev}_{\mathbf{No}}(X)$ is connected.

Let X be a surreal-membered set. We say that X is $\approx$distinct if and only if

(Def. 11) if x_1, $x_2 \in X$ and $x_1 \approx x_2$, then $x_1 = x_2$.

One can check that every unique surreal-membered set is $\approx$distinct and there exists a surreal-membered set which is non empty and $\approx$distinct.

Let S be a $\approx$distinct, surreal-membered set. Observe that $\mathrm{Rev}_{\mathbf{No}}(S)$ is antisymmetric.

Let S be a surreal-membered set. We say that S is desc-ordering if and only if

(Def. 12) there exists a strictly decreasing surreal sequence y such that $\mathrm{rng}\, y = S$.

Let us note that every surreal-membered set which is desc-ordering is also $\approx$distinct.

Let S be a $\approx$distinct, surreal-membered set. Observe that every subset of S is $\approx$distinct.

Now we state the proposition:

(40) Let us consider a subset A of S. Then field((the internal relation of $\mathrm{Rev}_{\mathbf{No}}(S)) \restriction^2 A) = A$.
PROOF: Set $R = \mathrm{Rev}_{\mathbf{No}}(S)$. Set $I =$ the internal relation of R. $A \subseteq \mathrm{field}(I \restriction^2 A)$ by [11, (87)], [26, (15)]. □

Let S be a surreal-membered set. Note that every subset of $\mathrm{Rev}_{\mathbf{No}}(S)$ is surreal-membered.

Now we state the propositions:

(41) field(the internal relation of $\mathrm{Rev}_{\mathbf{No}}(S)) = S$.
PROOF: Set $I =$ the internal relation of $\mathrm{Rev}_{\mathbf{No}}(S)$. Set $R = \mathrm{Rev}_{\mathbf{No}}(S)$. Set $I =$ the internal relation of R. $S \subseteq \mathrm{field}\, I$ by [26, (15)]. □

(42) Let us consider a surreal sequence s, and a $\approx$distinct, surreal-membered set X. Suppose $\mathrm{rng}\, s \subseteq X$. Then

(i) s is a transfinite sequence of elements of field(the internal relation of $\mathrm{Rev}_{\mathbf{No}}(X)$), and

(ii) for every transfinite sequence t of elements of field(the internal relation of $\mathrm{Rev}_{\mathbf{No}}(X)$) such that $t = s$ holds s is strictly decreasing iff t is increasing.

PROOF: Set $R = \mathrm{Rev}_{\mathbf{No}}(X)$. Set $I =$ the internal relation of R. $\mathrm{field}\, I = X$. If s is strictly decreasing, then t is increasing by [3, (10)], [20, (3)]. □

Now we state the proposition:

(43) CHARACTERIZATION OF WELL-ORDERING VIA STRICTLY DESCENDING SEQUENCES:
Let us consider a $\approx$distinct, surreal-membered set S, and a subset A of $\mathrm{Rev}_{\mathbf{No}}(S)$. Then A is desc-ordering if and only if A is well-ordering.
PROOF: Set $R_6 = \mathrm{Rev}_{\mathbf{No}}(S)$. Set $I =$ the internal relation of R_6. $\mathrm{field}\, I = S$. $\mathrm{field}(I \restriction^2 A) = A$. If A is desc-ordering, then A is well-ordering by (42), [6, (15), (18)], (24). Consider s being a transfinite sequence of elements of $\mathrm{field}\, I$ such that $\mathrm{rng}\, s = A$ and s is increasing. Reconsider $t = s$ as a surreal sequence. t is strictly decreasing. □

Let s be a strictly decreasing surreal sequence. Let us observe that $\mathrm{rng}\, s$ is desc-ordering.

5. DESCENDING ORDERING SUBSETS OF SURREAL NUMBERS

Let us observe that there exists a unique surreal-membered set which is non empty and desc-ordering and $\emptyset$ is desc-ordering.

Now we state the proposition:

(44) Let us consider a $\approx$distinct, surreal-membered set S, and a subset A of $\mathrm{Rev}_{\mathbf{No}}(S)$. If A is well founded, then A is well-ordering. The theorem is a consequence of (41).

Let S be a desc-ordering, surreal-membered set. Let us observe that every subset of S is desc-ordering.

Let S be a $\approx$distinct, surreal-membered set. One can check that every subset of S which is finite is also desc-ordering and there exists a subset of S which is finite and desc-ordering.

Let S be a $\approx$distinct, surreal-membered set and A, B be desc-ordering subsets of S. Let us note that $A \cup B$ is desc-ordering.

Now we state the proposition:

(45) Let us consider a surreal sequence s. Then s is strictly decreasing if and only if $\mathrm{Unique}_{\mathbf{No}}(s)$ is strictly decreasing.
PROOF: Set $U = \mathrm{Unique}_{\mathbf{No}}(s)$. If s is strictly decreasing, then U is strictly decreasing by [3, (10)], [20, (9)]. □

Let z be a strictly decreasing surreal sequence. Note that $\mathrm{Unique}_{\mathbf{No}}(z)$ is strictly decreasing.

Now we state the proposition:

(46) Let us consider a desc-ordering, surreal-membered set S. Then $\mathrm{Unique}_{\mathbf{No}}(S)$ is desc-ordering. The theorem is a consequence of (37).

Now we state the propositions:

(47) CLOSURE OF DESCENDING ORDERINGS UNDER ADDITION:
Let us consider desc-ordering, surreal-membered sets S, T. Then $\mathrm{Unique}_{\mathbf{No}}(S \oplus T)$ is desc-ordering. The theorem is a consequence of (39), (41), (38), (43), (12), (46), (19), (13), and (20).

(48) If $S \subseteq T$, then $\mathrm{Unique}_{\mathbf{No}}(S) \subseteq \mathrm{Unique}_{\mathbf{No}}(T)$.

Let S be a surreal-membered set. Assume S is desc-ordering. The functor $\mathrm{ord}(S)$ yielding a strictly decreasing surreal sequence is defined by

(Def. 13) $\mathrm{rng}\, it = S$.

Let us note that every surreal sequence which is strictly decreasing is also one-to-one.

6. REPRESENTATIONS FOR PARTICULAR SURREAL NUMBERS

In the sequel e, e_1, e_2 denote positive real numbers and a, b, c denote surreal numbers.

Let us consider r. Now we state the propositions:

(49) $\langle$the set of all $\mathrm{s}_{\mathbb{R}}(r - e_1)$, the set of all $\mathrm{s}_{\mathbb{R}}(r + e_2)\rangle$ is a surreal number.
PROOF: Set S_l = the set of all $\mathrm{s}_{\mathbb{R}}(r-e_1)$. Set S_r = the set of all $\mathrm{s}_{\mathbb{R}}(r+e_2)$. Reconsider $S_L = S_l$ as a surreal-membered set. Reconsider $S_R = S_r$ as a surreal-membered set. Consider M being an ordinal number such that for every object o such that $o \in S_L \cup S_R$ there exists an ordinal number A such that $A \in M$ and $o \in \mathrm{Day}A$. $S_l \ll S_r$ by [19, (51)]. □

(50) $\langle\{\mathrm{s}_{\mathbb{R}}(r - e_1) : r - e_1 > 0\}$, the set of all $\mathrm{s}_{\mathbb{R}}(r + e_2)\rangle$ is a surreal number.
PROOF: Set $S_l = \{\mathrm{s}_{\mathbb{R}}(r-e_1) : r-e_1 > 0\}$. Set S_r = the set of all $\mathrm{s}_{\mathbb{R}}(r+e_2)$. Reconsider $S_L = S_l$ as a surreal-membered set. Reconsider $S_R = S_r$ as a surreal-membered set. Consider M being an ordinal number such that for every object o such that $o \in S_L \cup S_R$ there exists an ordinal number A such that $A \in M$ and $o \in \mathrm{Day}A$. $S_l \ll S_r$ by [19, (51)]. □

Now we state the propositions:

(51) Suppose $z > \mathbf{0_{No}}$. Then $\langle$the set of all $\mathrm{s}_{\mathbb{R}}(r-e_1)\cdot z$, the set of all $\mathrm{s}_{\mathbb{R}}(r+e_2)\cdot z\rangle$ is a surreal number.
PROOF: Set S_l = the set of all $\mathrm{s}_{\mathbb{R}}(r - e) \cdot a$ where e is a positive real number. Set S_r = the set of all $\mathrm{s}_{\mathbb{R}}(r + e) \cdot a$ where e is a positive real number. Consider M being an ordinal number such that for every object o such that $o \in S_l \cup S_r$ there exists an ordinal number A such that $A \in M$ and $o \in \mathrm{Day}A$. $S_l \ll S_r$ by (30), [19, (56), (51)], [20, (9)]. □

(52) Suppose $z > \mathbf{0_{No}}$. Then $\langle$the set of all $s+\mathrm{s}_{\mathbb{R}}(r-e_1)\cdot z$, the set of all $s+\mathrm{s}_{\mathbb{R}}(r + e_2) \cdot z\rangle$ is a surreal number.
PROOF: Set S_l = the set of all $s + \mathrm{s}_{\mathbb{R}}(r - e) \cdot N$ where e is a positive real number. Set S_r = the set of all $s + \mathrm{s}_{\mathbb{R}}(r + e) \cdot N$ where e is a positive real number. Consider M being an ordinal number such that for every object o such that $o \in S_l \cup S_r$ there exists an ordinal number A such that $A \in M$ and $o \in \mathrm{Day}A$. $S_l \ll S_r$ by (30), [19, (56), (51)], [20, (9)]. □

(53) Suppose $\mathrm{L}x \ll \{y\} \ll \mathrm{R}x$. Then
(i) $\mathfrak{born}\,\mathrm{Unique}_{\mathbf{No}}(x) \subseteq \mathfrak{born}\,y$, and
(ii) if $x \not\approx y$, then $\mathfrak{born}\,\mathrm{Unique}_{\mathbf{No}}(x) \in \mathfrak{born}\,y$.
PROOF: Define $\mathcal{P}$[surreal number] $\equiv \mathrm{L}x \ll \{\$_1\} \ll \mathrm{R}x$. Consider s being a unique surreal number such that $\mathcal{P}[s]$ and for every unique surreal number a such that $\mathcal{P}[a]$ and $a \neq s$ holds $\mathfrak{born}\,s \in \mathfrak{born}\,a$ from [**?** , Sch. 1]. For every surreal number z such that $\mathrm{L}x \ll \{z\} \ll \mathrm{R}x$ holds $\mathfrak{born}\,s \subseteq \mathfrak{born}\,z$ by [20, (17), (18), (48)]. □

(54) Suppose $x = \langle$the set of all $s+\mathrm{s}_{\mathbb{R}}(r-e_1)\cdot z$, the set of all $s+\mathrm{s}_{\mathbb{R}}(r+e_2)\cdot z\rangle$. Then

(i) x is (s,z,r)-Term, and

(ii) for every unique surreal numbers u_1, u_2 such that $u_1 \approx x$ and $u_1 \neq u_2$ and u_2 is (s,z,r)-Term holds $\mathfrak{born}\, u_1 \in \mathfrak{born}\, u_2$.

PROOF: $|x - (s + \mathrm{s}_{\mathbb{R}}(r) \cdot N)| <^{\infty} N$ by (30), [20, (11), (9)], [21, (41), (42)]. $\mathrm{L}_x \ll \{u_2\} \ll \mathrm{R}_x$ by [**?** , (13), (9), (52)], [21, (42)]. □

(55) Let us consider a positive real number r, and x. Suppose $x = \langle \{\mathrm{s}_{\mathbb{R}}(r - e_1) : r - e_1 > 0\}$, the set of all $\mathrm{s}_{\mathbb{R}}(r + e_2)\rangle$. Then $x \approx \mathrm{s}_{\mathbb{R}}(r)$.

PROOF: $\mathrm{L}_a \ll \{\mathrm{s}_{\mathbb{R}}(r)\}$ by [19, (51)]. $\{\mathrm{s}_{\mathbb{R}}(r)\} \ll \mathrm{R}_a$ by [19, (51)]. $\mathrm{L}_{\mathrm{s}'_{\mathbb{R}}(r)} \ll \{a\}$ by [19, (41), (46)], [20, (11)], [19, (24), (51)]. $\{a\} \ll \mathrm{R}_{\mathrm{s}'_{\mathbb{R}}(r)}$ by [19, (41), (46)], [20, (11)]. □

(56) Suppose $x = \langle$the set of all $\mathrm{s}_{\mathbb{R}}(r - e_1)$, the set of all $\mathrm{s}_{\mathbb{R}}(r + e_2)\rangle$. Then $x \approx \mathrm{s}_{\mathbb{R}}(r)$.

PROOF: $\mathrm{L}_a \ll \{\mathrm{s}_{\mathbb{R}}(r)\}$ by [19, (51)]. $\{\mathrm{s}_{\mathbb{R}}(r)\} \ll \mathrm{R}_a$ by [19, (51)]. $\mathrm{L}_{\mathrm{s}'_{\mathbb{R}}(r)} \ll \{a\}$ by [19, (41), (46)], [20, (11)]. $\{a\} \ll \mathrm{R}_{\mathrm{s}'_{\mathbb{R}}(r)}$ by [19, (41), (46)], [20, (11)]. □

(57) Suppose $x = \langle$the set of all $\mathrm{s}_{\mathbb{R}}(r - e_1) \cdot (\boldsymbol{\omega}^z)$, the set of all $\mathrm{s}_{\mathbb{R}}(r + e_2) \cdot (\boldsymbol{\omega}^z)\rangle$. Then $\mathrm{s}_{\mathbb{R}}(r) \cdot (\boldsymbol{\omega}^z) \approx x$.

PROOF: Reconsider $R = \langle$the set of all $\mathrm{s}_{\mathbb{R}}(r - e_1)$, the set of all $\mathrm{s}_{\mathbb{R}}(r + e_2)\rangle$ as a surreal number. $R \approx \mathrm{s}_{\mathbb{R}}(r)$. Set $N = \boldsymbol{\omega}^a$. Set $R_4 = R \cdot N$. $\mathrm{L}_{R_4} \ll \{X\}$ by [**?** , (22)], [20, (11)], [21, (37)], [19, (56)]. $\{R_4\} \ll \mathrm{R}_X$ by [20, (9)], [19, (51)], [21, (70)]. $\mathrm{L}_X \ll \{R_4\}$ by [20, (9)], [19, (51)], [21, (70)]. $\{X\} \ll \mathrm{R}_{R_4}$ by [**?** , (22)], [20, (11)], [21, (37)], [19, (56)]. $R_4 \approx \mathrm{s}_{\mathbb{R}}(r) \cdot N$. □

7. Additive Properties and Laws of Surreal Representations

In the sequel r, r_1, r_2, r_{12}, s denote transfinite sequences of elements of $\mathbb{R}$ and y, y_1, y_2, y_{12}, z, y_9 denote strictly decreasing surreal sequences.

Let us consider a positive, decreasing infinitely surreal sequence y. Now we state the propositions:

(58) If $A \subseteq \operatorname{dom} r \cap \operatorname{dom} y$, then $(\mathrm{PartialSums}(r, y))(A) = \sum_{\kappa=0}^{y\restriction A} (r\restriction A)(\kappa)$.

(59) Suppose $\operatorname{succ} A \subseteq \operatorname{dom} r \cap \operatorname{dom} y$. Then $\langle$the set of all $\sum_{\kappa=0}^{y\restriction A} (r\restriction A)(\kappa) + \mathrm{s}_{\mathbb{R}}(r(A) - e_1) \cdot (y \upharpoonleft A)$, the set of all $\sum_{\kappa=0}^{y\restriction A} (r\restriction A)(\kappa) + \mathrm{s}_{\mathbb{R}}(r(A) + e_2) \cdot (y \upharpoonleft A)\rangle$ is a surreal number. The theorem is a consequence of (52).

(60) Suppose $\operatorname{succ} A \subseteq \operatorname{dom} r \cap \operatorname{dom} y$ and $x = \langle$the set of all $\sum_{\kappa=0}^{y\restriction A} (r\restriction A)(\kappa) + \mathrm{s}_{\mathbb{R}}(r(A) - e_1) \cdot (y \upharpoonleft A)$, the set of all $\sum_{\kappa=0}^{y\restriction A} (r\restriction A)(\kappa) + \mathrm{s}_{\mathbb{R}}(r(A) + e_2) \cdot (y \upharpoonleft A)\rangle$. Then $\sum_{\kappa=0}^{y\restriction \operatorname{succ} A} (r\restriction \operatorname{succ} A)(\kappa) = \mathrm{Unique}_{\mathbf{No}}(x)$.

PROOF: Set $N = y \upharpoonleft A$. Set $S_3 = \sum_{\kappa=0}^{y\restriction A} (r\restriction A)(\kappa)$. Set $s_1 = \operatorname{succ} A$. $(\mathrm{PartialSums}(r, y))(\operatorname{succ} A) = \sum_{\kappa=0}^{y\restriction s_1} (r\restriction s_1)(\kappa)$. $S_3 = (\mathrm{PartialSums}(r, y))(A)$.

Set $U = \mathrm{Unique}_{\mathbf{No}}(x)$. x is $(S_3,N,(r(A)))$-Term. $U \in \bigcap \mathrm{PartialSums}(r,y),y,r,s_1$ by [3, (8)], [**?** , (74)], [8, (49)], [**?** , (87)]. $\sum_{\kappa=0}^{y\restriction s_1}(r\restriction s_1)(\kappa) = \mathrm{Unique}_{\mathbf{No}}(x)$. □

(61) Suppose $A \subseteq \mathrm{dom}\, r \cap \mathrm{dom}\, y$. Then $\langle\{\sum_{\kappa=0}^{y\restriction B}(r\restriction B)(\kappa)+\mathrm{s}_{\mathbb{R}}(r(B)-e)\cdot(y \hookleftarrow B)$, where B is an element of $A : B \in A\}$, $\{\sum_{\kappa=0}^{y\restriction B}(r\restriction B)(\kappa)+\mathrm{s}_{\mathbb{R}}(r(B)+e)\cdot(y \hookleftarrow B)$, where B is an element of $A : B \in A\}\rangle$ is a surreal number.
Proof: Set $S_l = \{\sum_{\kappa=0}^{y\restriction B}(r\restriction B)(\kappa) + \mathrm{s}_{\mathbb{R}}(r(B) - e_1)\cdot(y \hookleftarrow B)$, where B is an element of $A : B \in A\}$. Set $S_r = \{\sum_{\kappa=0}^{y\restriction B}(r\restriction B)(\kappa)+\mathrm{s}_{\mathbb{R}}(r(B)+e_2)\cdot(y \hookleftarrow B)$, where B is an element of $A : B \in A\}$. Consider M being an ordinal number such that for every object o such that $o \in S_l \cup S_r$ there exists an ordinal number A such that $A \in M$ and $o \in \mathrm{Day}A$. $S_l \ll S_r$ by [20, (9)]. □

(62) Suppose $A \subseteq \mathrm{dom}\, r \cap \mathrm{dom}\, y$ and $x = \langle\{\sum_{\kappa=0}^{y\restriction B}(r\restriction B)(\kappa)+\mathrm{s}_{\mathbb{R}}(r(B)-e)\cdot(y \hookleftarrow B)$, where B is an element of $A : B \in A\}$, $\{\sum_{\kappa=0}^{y\restriction B}(r\restriction B)(\kappa)+\mathrm{s}_{\mathbb{R}}(r(B)+e)\cdot(y \hookleftarrow B)$, where B is an element of $A : B \in A\}\rangle$. Then $\sum_{\kappa=0}^{y\restriction A}(r\restriction A)(\kappa) = \mathrm{Unique}_{\mathbf{No}}(x)$. The theorem is a consequence of (58), (53), and (30).

(63) Suppose $\mathrm{succ}\, A \subseteq \mathrm{dom}\, r \cap \mathrm{dom}\, y$ and for every surreal number r_9 such that $r_9 = \langle$the set of all $\mathrm{s}_{\mathbb{R}}(r(A) - e_1)\cdot(y \hookleftarrow A)$, the set of all $\mathrm{s}_{\mathbb{R}}(r(A) + e_2)\cdot(y \hookleftarrow A)\rangle$ holds $r_9 \approx \mathrm{s}_{\mathbb{R}}(r(A))\cdot(y \hookleftarrow A)$. Then $\sum_{\kappa=0}^{y\restriction \mathrm{succ}\, A}(r\restriction \mathrm{succ}\, A)(\kappa) \approx \sum_{\kappa=0}^{y\restriction A}(r\restriction A)(\kappa) + \mathrm{s}_{\mathbb{R}}(r(A))\cdot(y \hookleftarrow A)$.
Proof: Set $r_4 = r(A)$. Set $y_4 = y \hookleftarrow A$. Reconsider $r_9 = \langle$the set of all $\mathrm{s}_{\mathbb{R}}(r_4 - e_1)\cdot y_4$, the set of all $\mathrm{s}_{\mathbb{R}}(r_4+e_2)\cdot y_4\rangle$ as a surreal number. Reconsider $S_8 = \langle$the set of all $\mathbf{0}_{\mathbf{No}} + \sum_{\kappa=0}^{y\restriction A}(r\restriction A)(\kappa) + \mathrm{s}_{\mathbb{R}}(r_4 - e_1)\cdot y_4$, the set of all $\mathbf{0}_{\mathbf{No}} + \sum_{\kappa=0}^{y\restriction A}(r\restriction A)(\kappa) + \mathrm{s}_{\mathbb{R}}(r_4 + e_2)\cdot y_4\rangle$ as a surreal number. Reconsider $S_9 = \langle\{\sum_{\kappa=0}^{y\restriction B}(r\restriction B)(\kappa) + \mathrm{s}_{\mathbb{R}}(r(B) - e)\cdot(y \hookleftarrow B)$, where B is an element of $\mathrm{succ}\, A : B \in \mathrm{succ}\, A\}$, $\{\sum_{\kappa=0}^{y\restriction B}(r\restriction B)(\kappa)+\mathrm{s}_{\mathbb{R}}(r(B)+e)\cdot(y \hookleftarrow B)$, where B is an element of $\mathrm{succ}\, A : B \in \mathrm{succ}\, A\}\rangle$ as a surreal number. $\sum_{\kappa=0}^{y\restriction \mathrm{succ}\, A}(r\restriction \mathrm{succ}\, A)(\kappa) = \mathrm{Unique}_{\mathbf{No}}(S_9) \approx S_9$. $\sum_{\kappa=0}^{y\restriction \mathrm{succ}\, A}(r\restriction \mathrm{succ}\, A)(\kappa) = \mathrm{Unique}_{\mathbf{No}}(S_8) \approx S_8$. Reconsider $S_{10} = \langle\{\sum_{\kappa=0}^{y\restriction B}(r\restriction B)(\kappa) + \mathrm{s}_{\mathbb{R}}(r(B) - e)\cdot(y \hookleftarrow B)$, where B is an element of $A : B \in A\}$, $\{\sum_{\kappa=0}^{y\restriction B}(r\restriction B)(\kappa)+\mathrm{s}_{\mathbb{R}}(r(B)+e)\cdot(y \hookleftarrow B)$, where B is an element of $A : B \in A\}\rangle$ as a surreal number. $\sum_{\kappa=0}^{y\restriction A}(r\restriction A)(\kappa) = \mathrm{Unique}_{\mathbf{No}}(S_{10}) \approx S_{10}$. $\mathrm{L}_{S_8} \ll \{S_{10} + r_9\}$ by [20, (11)], [21, (44)]. $\{S_{10} + r_9\} \ll \mathrm{R}_{S_8}$ by [20, (11)], [21, (44)]. $\mathrm{L}_{(S_{10}+r_9)} \ll \{S_8\}$ by [19, (55)], [21, (54), (67)], [20, (9)]. $\{S_8\} \ll \mathrm{R}_{(S_{10}+r_9)}$ by [21, (54), (67)], [19, (55)], [20, (9)]. □

Now we state the proposition:

(64) Let us consider a strictly decreasing surreal sequence y. Suppose $\mathrm{succ}\, A \subseteq \mathrm{dom}\, r \cap \mathrm{dom}\, y$. Then $\sum_{\kappa=0}^{\boldsymbol{\omega}^{(y\restriction \mathrm{succ}\, A)}}(r\restriction \mathrm{succ}\, A)(\kappa) \approx \sum_{\kappa=0}^{\boldsymbol{\omega}^{(y\restriction A)}}(r\restriction A)(\kappa)+\mathrm{s}_{\mathbb{R}}(r(A))\cdot(\boldsymbol{\omega}^{(y\hookleftarrow A)})$. The theorem is a consequence of (57) and (63).

Let us consider a positive, decreasing infinitely surreal sequence y. Now we state the propositions:

(65) Suppose $A \in \operatorname{dom} r \cap \operatorname{dom} y$ and $r(A) = 0$. Then $\sum_{\kappa=0}^{y\restriction A}(r\restriction A)(\kappa) = \sum_{\kappa=0}^{y\restriction \operatorname{succ} A}(r\restriction \operatorname{succ} A)(\kappa)$.
PROOF: Set $P = \operatorname{PartialSums}(r, y)$. Set $S_{11} = \sum_{\kappa=0}^{y\restriction \operatorname{succ} A}(r\restriction \operatorname{succ} A)(\kappa)$. Set $S_3 = \sum_{\kappa=0}^{y\restriction A}(r\restriction A)(\kappa)$. $P(\operatorname{succ} A) = S_{11}$. $P(A) = S_3$. $S_3 = S_{11}$ by [3, (8)], [**?** , (48)], [19, (47)], [21, (39)]. □

(66) If $\operatorname{rng} r \subseteq \{0\}$, then $\sum_{\kappa=0}^{y} r(\kappa) = \mathbf{0_{No}}$.
PROOF: Define $\mathcal{P}$[ordinal number] $\equiv$ if $\$_1 \subseteq \operatorname{dom} r \cap \operatorname{dom} y$, then $\sum_{\kappa=0}^{y\restriction \$_1}(r\restriction \$_1)(\kappa) = \mathbf{0_{No}}$. For every ordinal number A such that for every ordinal number C such that $C \in A$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[A]$ by (61), [19, (47), (51)], [21, (74), (72)]. For every ordinal number A, $\mathcal{P}[A]$ from [3, Sch. 2]. Set $A = \operatorname{dom} r \cap \operatorname{dom} y$. $(\operatorname{PartialSums}(r, y))(A) = \sum_{\kappa=0}^{y\restriction A}(r\restriction A)(\kappa)$. □

(67) $\sum_{\kappa=0}^{y\restriction 0}(r\restriction 0)(\kappa) = \mathbf{0_{No}}$. The theorem is a consequence of (58).

(68) Suppose $B \in A \subseteq \operatorname{dom} r \cap \operatorname{dom} y$. Then $\sum_{\kappa=0}^{y\restriction A}(r\restriction A)(\kappa)$ is $((\sum_{\kappa=0}^{y\restriction B}(r\restriction B)(\kappa)),(y \hookleftarrow B),(r(B)))$-Term. The theorem is a consequence of (58).

Now we state the propositions:

(69) Let us consider a positive, decreasing infinitely surreal sequence y. Suppose $B \in A \subseteq \operatorname{dom} r \cap \operatorname{dom} y$. Then $\sum_{\kappa=0}^{y\restriction B}(r\restriction B)(\kappa) + \mathrm{s}_{\mathbb{R}}(r(B) - e) \cdot (y \hookleftarrow B) < \sum_{\kappa=0}^{y\restriction A}(r\restriction A)(\kappa) < \sum_{\kappa=0}^{y\restriction B}(r\restriction B)(\kappa) + \mathrm{s}_{\mathbb{R}}(r(B) + e) \cdot (y \hookleftarrow B)$. The theorem is a consequence of (61) and (62).

(70) Let us consider a positive, decreasing infinitely surreal sequence y. Suppose $0 \in \operatorname{dom} y \cap \operatorname{dom} r$ and $y \hookleftarrow 0 <^{\infty} a$. Then $|\sum_{\kappa=0}^{y} r(\kappa)| <^{\infty} a$. The theorem is a consequence of (67), (68), (31), and (58).

Let us consider a positive, decreasing infinitely surreal sequence y. Now we state the propositions:

(71) DECOMPOSITION OF SELECTED SURREAL REPRESENTATIONS INTO HEAD AND TAIL:
Suppose $A \subseteq \operatorname{dom} r \cap \operatorname{dom} y$. Then $\sum_{\kappa=0}^{y\restriction A}(r\restriction A)(\kappa) + \sum_{\kappa=0}^{y_{\downarrow A}} r_{\downarrow A}(\kappa) \approx \sum_{\kappa=0}^{y} r(\kappa)$.
PROOF: Set $r_4 = r\restriction A$. Set $y_4 = y\restriction A$. Set $r_3 = r_{\downarrow A}$. Set $y_3 = y_{\downarrow A}$. Reconsider $X_3 = \langle\{\sum_{\kappa=0}^{y\restriction B}(r\restriction B)(\kappa) + \mathrm{s}_{\mathbb{R}}(r(B) - e) \cdot (y \hookleftarrow B)$, where B is an element of $A : B \in A\}$, $\{\sum_{\kappa=0}^{y\restriction B}(r\restriction B)(\kappa) + \mathrm{s}_{\mathbb{R}}(r(B) + e) \cdot (y \hookleftarrow B)$, where B is an element of $A : B \in A\}\rangle$ as a surreal number. Define $\mathcal{P}$[ordinal number] $\equiv$ if $\$_1 \subseteq \operatorname{dom} r_3 \cap \operatorname{dom} y_3$, then $\sum_{\kappa=0}^{y\restriction A}(r\restriction A)(\kappa) + \sum_{\kappa=0}^{y_3\restriction \$_1}(r_3\restriction \$_1)(\kappa) \approx \sum_{\kappa=0}^{y\restriction (A+\$_1)}(r\restriction (A + \$_1))(\kappa)$. $\sum_{\kappa=0}^{y_4} r_4(\kappa) = \operatorname{Unique}_{\mathbf{No}}(X_3) \approx X_3$. For every ordinal number D such that for every ordinal number C such that $C \in D$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[D]$ by (3), [5, (18)], [4, (27)], (67). Set $D = \operatorname{dom} r_3 \cap \operatorname{dom} y_3$. For every ordinal number A, $\mathcal{P}[A]$ from [3, Sch. 2].

$A + D = \operatorname{dom} r \cap \operatorname{dom} y$ by [5, (13), (23)]. □

(72) Suppose $A \subseteq B \subseteq \operatorname{dom} r \cap \operatorname{dom} y$ and for every C such that $A \subseteq C \in B$ holds $r(C) = 0$. Then $\sum_{\kappa=0}^{y\restriction B}(r\restriction B)(\kappa) = \sum_{\kappa=0}^{y\restriction A}(r\restriction A)(\kappa)$.

PROOF: Reconsider $r_5 = r\restriction B$ as a transfinite sequence of elements of $\mathbb{R}$. Set $y_5 = y\restriction B$. Set $r_6 = r_{5\restriction A}$. Set $y_6 = y_{5\restriction A}$. Set $D = \operatorname{dom} r_6 \cap \operatorname{dom} y_6$. $\sum_{\kappa=0}^{y_5\restriction A}(r_5\restriction A)(\kappa) + \sum_{\kappa=0}^{y_{5\restriction A}} r_{5\restriction A}(\kappa) \approx \sum_{\kappa=0}^{y_5} r_5(\kappa)$. $\sum_{\kappa=0}^{y_6} r_6(\kappa) = (\operatorname{PartialSums}(r_6, y_6))(D) = \sum_{\kappa=0}^{y_6\restriction D}(r_6\restriction D)(\kappa)$. $\operatorname{rng}(r_6\restriction D) \subseteq \{0\}$ by [8, (47)], [26, (57)], (8), [5, (24)]. $\sum_{\kappa=0}^{y_6\restriction D}(r_6\restriction D)(\kappa) = \mathbf{0_{No}}$. □

Now we state the proposition:

(73) INVARIANCE OF SURREAL SUMS UNDER ZERO COEFFICIENT EXTENSION:

Let us consider transfinite sequences r, r_7 of elements of $\mathbb{R}$, and positive, decreasing infinitely surreal sequences y, y_E. Suppose $\operatorname{dom} r = \operatorname{dom} y$ and $\operatorname{dom} r_7 = \operatorname{dom} y_E$ and $\operatorname{rng} y \subseteq \operatorname{rng} y_E$ and for every A and B such that $A \in \operatorname{dom} y$ and $B \in \operatorname{dom} y_E$ and $y(A) = y_E(B)$ holds $r(A) = r_7(B)$ and for every A such that $A \in \operatorname{dom} y_E$ and $y_E(A) \notin \operatorname{rng} y$ holds $r_7(A) = 0$. Then $\sum_{\kappa=0}^{y} r(\kappa) = \sum_{\kappa=0}^{y_E} r_7(\kappa)$. The theorem is a consequence of (58), (67), (66), (72), (61), (62), (68), (30), and (33).

Now we state the propositions:

(74) THE ADDITIVE LAW FOR COMPATIBLE SELECTED SURREAL SERIES:

Let us consider a positive, decreasing infinitely surreal sequence y, and transfinite sequences r, s of elements of $\mathbb{R}$. Suppose $\operatorname{dom} r = \operatorname{dom} s$ and $A \subseteq \operatorname{dom} r \cap \operatorname{dom} y$. Then $\sum_{\kappa=0}^{y\restriction A}((r+s)\restriction A)(\kappa) \approx \sum_{\kappa=0}^{y\restriction A}(r\restriction A)(\kappa) + \sum_{\kappa=0}^{y\restriction A}(s\restriction A)(\kappa)$.

PROOF: Define $\mathcal{P}$[ordinal number] $\equiv$ for every transfinite sequences r, s of elements of $\mathbb{R}$ such that $\operatorname{dom} r = \operatorname{dom} s$ and $\$_1 \subseteq \operatorname{dom} r \cap \operatorname{dom} y$ holds $\sum_{\kappa=0}^{y\restriction \$_1}((r+s)\restriction \$_1)(\kappa) \approx \sum_{\kappa=0}^{y\restriction \$_1}(r\restriction \$_1)(\kappa) + \sum_{\kappa=0}^{y\restriction \$_1}(s\restriction \$_1)(\kappa)$. For every ordinal number A such that for every ordinal number C such that $C \in A$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[A]$ by (61), (62), [21, (44)], (69). For every ordinal number A, $\mathcal{P}[A]$ from [3, Sch. 2]. □

(75) Let us consider a positive, decreasing infinitely surreal sequence y, and a transfinite sequence r of elements of $\mathbb{R}$. Suppose $A \subseteq \operatorname{dom} r \cap \operatorname{dom} y$. Then $\sum_{\kappa=0}^{y\restriction A}((-r)\restriction A)(\kappa) \approx -\sum_{\kappa=0}^{y\restriction A}(r\restriction A)(\kappa)$.

PROOF: Set $m_2 = -r$. Define $\mathcal{P}$[ordinal number] $\equiv$ if $\$_1 \subseteq \operatorname{dom} r \cap \operatorname{dom} y$, then $\sum_{\kappa=0}^{y\restriction \$_1}(m_2\restriction \$_1)(\kappa) \approx -\sum_{\kappa=0}^{y\restriction \$_1}(r\restriction \$_1)(\kappa)$. For every ordinal number A such that for every ordinal number C such that $C \in A$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[A]$ by (61), (62), [21, (7)], [19, (56)]. For every ordinal number A, $\mathcal{P}[A]$ from [3, Sch. 2]. □

Let r be a transfinite sequence of elements of $\mathbb{R}$, y be a one-to-one transfinite sequence, and y_E be a transfinite sequence. The functor r-extended by (y, y_E)

yielding a transfinite sequence of elements of $\mathbb{R}$ is defined by

(Def. 14) $\operatorname{dom} it = \operatorname{dom} y_E$ and for every A and B such that $A \in \operatorname{dom} y$ and $B \in \operatorname{dom} y_E$ and $y(A) = y_E(B)$ holds $r(A) = it(B)$ and for every A such that $A \in \operatorname{dom} y_E$ and $y_E(A) \notin \operatorname{rng} y$ holds $it(A) = 0$.

Now we state the proposition:

(76) Let us consider strictly decreasing surreal sequences y, y_E. Suppose $\operatorname{rng}(y\restriction \operatorname{dom} r) \subseteq \operatorname{rng} y_E$ and $\operatorname{rng} y_E$ is $\approx$distinct. Then $\sum_{\kappa=0}^{\boldsymbol{\omega}^y} r(\kappa) = \sum_{\kappa=0}^{\boldsymbol{\omega}^{y_E}}$($r$-extended PROOF: Set $D = \operatorname{dom} y \cap \operatorname{dom} r$. Set $N_7 = \boldsymbol{\omega}^y$. Set $N_8 = \boldsymbol{\omega}^{y_E}$. Set $r_7 = r$-extended by $(y\restriction \operatorname{dom} r, y_E)$. $\operatorname{rng}(N_7\restriction D) \subseteq \operatorname{rng} N_8$ by [**?** , (88)]. For every ordinal numbers A, B such that $A \in \operatorname{dom}(N_7\restriction D)$ and $B \in \operatorname{dom} N_8$ and $(N_7\restriction D)(A) = N_8(B)$ holds $(r\restriction D)(A) = r_7(B)$ by [26, (57)], [**?** , (88), (9), (25)]. For every ordinal number A such that $A \in \operatorname{dom} N_8$ and $N_8(A) \notin \operatorname{rng}(N_7\restriction D)$ holds $r_7(A) = 0$ by [**?** , (88)]. $\sum_{\kappa=0}^{N_7\restriction D}(r\restriction D)(\kappa) = \sum_{\kappa=0}^{N_8} r_7(\kappa)$. $(\operatorname{PartialSums}(r, N_7))(D) = \sum_{\kappa=0}^{N_7\restriction D}(r\restriction D)(\kappa)$. □

8. PRODUCT CONSTRUCTION FOR SELECTED SURREAL SERIES

Let A, B be ordinal numbers and r, s be transfinite sequences of elements of $\mathbb{R}$. The functor $\operatorname{Product}(r, s, A, B)$ yielding a function from $A \times B$ into $\mathbb{R}$ is defined by

(Def. 15) for every ordinal numbers a, b such that $a \in A$ and $b \in B$ holds $it(a, b) = r(a) \cdot s(b)$.

Let y, z be strictly decreasing surreal sequences. Assume $\operatorname{dom} y \subseteq A$ and $\operatorname{dom} z \subseteq B$. Let s be a surreal number. The functor $[y + z = s]_{A\times B}$ yielding an element of $\operatorname{Fin}(A \times B)$ is defined by

(Def. 16) for every ordinal numbers a, b, $\langle a, b\rangle \in it$ iff $a \in \operatorname{dom} y$ and $b \in \operatorname{dom} z$ and $y \hookleftarrow a + z \hookleftarrow b \approx s$.

Now we state the propositions:

(77) If $\operatorname{dom} y \subseteq A$ and $\operatorname{dom} z \subseteq B$ and $a \approx b$, then $[y + z = a]_{A\times B} \subseteq [y + z = b]_{A\times B}$.

(78) Suppose $\operatorname{dom} y \subseteq A$ and $\operatorname{dom} z \subseteq B$ and for every a such that $a \approx c$ holds $a \notin \operatorname{rng} y \oplus \operatorname{rng} z$. Then $[y + z = c]_{A\times B} = \emptyset$.

(79) Let us consider a positive, decreasing infinitely surreal sequence y, and a real number x. Suppose $A \subseteq \operatorname{dom} r \cap \operatorname{dom} y$. Then $\sum_{\kappa=0}^{y\restriction A}((x \cdot r)\restriction A)(\kappa) \approx \mathrm{s}_{\mathbb{R}}(x) \cdot (\sum_{\kappa=0}^{y\restriction A}(r\restriction A)(\kappa))$. The theorem is a consequence of (66) and (75).

Let s be a surreal sequence and a be a surreal number. The functor $s + a$ yielding a surreal sequence is defined by

(Def. 17) $\operatorname{dom} it = \operatorname{dom} s$ and for every A such that $A \in \operatorname{dom} s$ holds $it(A) = s \upharpoonleft A + a$.

Let s be a strictly decreasing surreal sequence. Observe that $s + a$ is strictly decreasing.

Now we state the propositions:

(80) Let us consider a surreal sequence s. Then $s\restriction A + x = (s + x)\restriction A$.
PROOF: For every object z such that $z \in \operatorname{dom} s \cap A$ holds $(s\restriction A + x)(z) = ((s + x)\restriction A)(z)$ by [8, (47)]. □

(81) $\operatorname{rng}(y + a) = \operatorname{rng} y \oplus \{a\}$.

(82) Let us consider a surreal sequence s. Suppose $A \in \operatorname{dom} s$. Then $(\boldsymbol{\omega}^a) \cdot ((\boldsymbol{\omega}^s) \upharpoonleft A) \approx (\boldsymbol{\omega}^{(s+a)}) \upharpoonleft A$.

(83) Let us consider a strictly decreasing surreal sequence y. Suppose $A \subseteq \operatorname{dom} r \cap \operatorname{dom} y$. Then $\sum_{\kappa=0}^{\boldsymbol{\omega}^{(y+a)}\restriction A}(r\restriction A)(\kappa) \approx (\boldsymbol{\omega}^a) \cdot (\sum_{\kappa=0}^{\boldsymbol{\omega}^y\restriction A}(r\restriction A)(\kappa))$.
PROOF: Set $N_7 = \boldsymbol{\omega}^y$. Set $y_8 = y + a$. Set $N_9 = \boldsymbol{\omega}^{y_8}$. Set $N_6 = \boldsymbol{\omega}^a$. Define $\mathcal{P}[\text{ordinal number}] \equiv$ if $\$_1 \subseteq \operatorname{dom} r \cap \operatorname{dom} y$, then $\sum_{\kappa=0}^{N_9\restriction \$_1}(r\restriction \$_1)(\kappa) \approx N_6 \cdot (\sum_{\kappa=0}^{N_7\restriction \$_1}(r\restriction \$_1)(\kappa))$. For every ordinal number A such that for every ordinal number C such that $C \in A$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[A]$ by (61), (62), [21, (50)], (82). For every ordinal number A, $\mathcal{P}[A]$ from [3, Sch. 2]. □

In the sequel S, T denote non empty ordinal numbers.

Let S, T be non empty ordinal numbers, y, z be strictly decreasing surreal sequences, r, s be transfinite sequences of elements of $\mathbb{R}$, and a be a surreal number. The functor Partial-Sums(y, z, r, s, a, S, T) yielding a real number is defined by the term

(Def. 18) $+_{\mathbb{R}}\text{-}\sum_{[y+z=a]_{S\times T}} \operatorname{Product}(r, s, S, T)$.

Now we state the propositions:

(84) If $\operatorname{dom} y \subseteq S$ and $\operatorname{dom} z \subseteq T$, then Partial-Sums$(y, z, r, s, a, S, T)$ = Partial-Sums(z, y, s, r, a, T, S).
PROOF: Set $S_1 = [y + z = a]_{S\times T}$. Set $S_2 = [z + y = a]_{T\times S}$. Set $A_1 = \operatorname{Product}(r, s, S, T)$. Set $A_2 = \operatorname{Product}(s, r, T, S)$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ for every objects i, j such that $\$_1 = \langle i, j\rangle$ holds $\$_2 = \langle j, i\rangle$. For every object e such that $e \in S_1$ there exists an object u such that $\mathcal{P}[e, u]$. Consider T_1 being a function such that $\operatorname{dom} T_1 = S_1$ and for every object e such that $e \in S_1$ holds $\mathcal{P}[e, T_1(e)]$ from [1, Sch. 1]. $\operatorname{rng} T_1 \subseteq S_2$. $S_2 \subseteq \operatorname{rng} T_1$. T_1 is one-to-one. For every object e such that $e \in S_1$ holds $(A_1\restriction S_1)(e) = (A_2 \cdot T_1)(e)$ by [8, (12), (49)]. □

(85) Let us consider a transfinite sequence r_{12} of elements of $\mathbb{R}$. Suppose $r_1 \frown r_2 = r_{12}$ and $\operatorname{dom} y \subseteq \operatorname{dom} r_1$ and $\operatorname{dom} y \subseteq S$ and $\operatorname{dom} z \subseteq T$. Then Partial-Sums$(y, z, r_1, s, a, S, T)$ = Partial-Sums$(y, z, r_{12}, s, a, S, T)$.

PROOF: Set $S_1 = [y + z = a]_{S\times T}$. Set $A_1 = \mathrm{Product}(r_1, s, S, T)$. Set $A_4 = \mathrm{Product}(r_{12}, s, S, T)$. For every object x such that $x \in S_1$ holds $(A_4 \restriction S_1)(x) = (A_1 \restriction S_1)(x)$ by [8, (49)]. □

(86) Suppose $y_1 \frown y_2 = y_{12}$ and $r_1 \frown r_2 = r_{12}$ and $\mathrm{dom}\, y_{12} \subseteq S$ and $\mathrm{dom}\, z \subseteq T$ and $\mathrm{dom}\, y_1 = \mathrm{dom}\, r_1$ and $\mathrm{dom}\, y_2 \subseteq \mathrm{dom}\, r_2$. Then $\mathrm{Partial\text{-}Sums}(y_{12}, z, r_{12}, s, a, S, T) = \mathrm{Partial\text{-}Sums}(y_1, z, r_1, s, a, S, T) + \mathrm{Partial\text{-}Sums}(y_2, z, r_2, s, a, S, T)$.
PROOF: Set $S_1 = [y_1 + z = a]_{S\times T}$. Set $S_2 = [y_2 + z = a]_{S\times T}$. Set $S_{12} = [y_{12} + z = a]_{S\times T}$. Set $A_2 = \mathrm{Product}(r_2, s, S, T)$. Set $A_4 = \mathrm{Product}(r_{12}, s, S, T)$. $\mathrm{Partial\text{-}Sums}(y_1, z, r_1, s, a, S, T) = \mathrm{Partial\text{-}Sums}(y_1, z, r_{12}, s, a, S, T)$. Set $S_5 = S_{12} \setminus S_1$. $S_1 \subseteq S_{12}$. Define $\mathcal{P}[\mathrm{object}, \mathrm{object}] \equiv$ for every ordinal numbers i, j such that $\$_1 = \langle i, j\rangle$ holds $\$_2 = \langle i - \mathrm{dom}\, y_1, j\rangle$. For every object e such that $e \in S_5$ there exists an object u such that $\mathcal{P}[e, u]$. Consider T_1 being a function such that $\mathrm{dom}\, T_1 = S_5$ and for every object e such that $e \in S_5$ holds $\mathcal{P}[e, T_1(e)]$ from [1, Sch. 1]. For every ordinal numbers i, j, $\langle i, j\rangle \in S_5$ iff $\mathrm{dom}\, y_1 \subseteq i$ and $\langle i - \mathrm{dom}\, y_1, j\rangle \in S_2$ by [3, (16)], (3), [4, (33), (32)]. $\mathrm{rng}\, T_1 \subseteq S_2$. $S_2 \subseteq \mathrm{rng}\, T_1$ by [5, (24), (52)]. T_1 is one-to-one. For every object e such that $e \in S_5$ holds $(A_4 \restriction S_5)(e) = (A_2 \cdot T_1)(e)$ by [11, (87)], (3), [8, (12), (49)]. □

Let s be a surreal number. Let us observe that $\langle s\rangle$ is strictly decreasing.

Now we state the propositions:

(87) Suppose $\mathrm{dom}\, y \subseteq S$ and $\mathrm{dom}\, z \subseteq T$ and $a \approx b$. Then $\mathrm{Partial\text{-}Sums}(y, z, r, s, a, S, T) = \mathrm{Partial\text{-}Sums}(y, z, r, s, b, S, T)$. The theorem is a consequence of (77).

(88) Suppose $\mathrm{dom}\, y \subseteq S$ and $\mathrm{dom}\, z \subseteq T$ and for every a such that $a \approx c$ holds $a \notin \mathrm{rng}\, y \oplus \mathrm{rng}\, z$. Then $\mathrm{Partial\text{-}Sums}(y, z, r, s, c, S, T) = 0$. The theorem is a consequence of (78).

(89) Let us consider a real number r. Suppose $\mathrm{dom}\, z \subseteq T$ and $A \in \mathrm{dom}\, z$ and $a \approx c + z \restriction\!\!\upharpoonright A$. Then $\mathrm{Partial\text{-}Sums}(\langle c\rangle, z, \langle r\rangle, s, a, S, T) = r \cdot s(A)$.
PROOF: $[\langle c\rangle + z = a]_{S\times T} \subseteq \{\langle\langle 0, A\rangle\rangle\}$ by [3, (8)], [20, (4)], [21, (44)]. □

9. COMPATIBILITY OF SERIES PRODUCTS WITH CONWAY NORMAL FORM

Let S, T be non empty ordinal numbers, y, z, y_9 be strictly decreasing surreal sequences, and r, s be transfinite sequences of elements of $\mathbb{R}$. The functor $\mathrm{Sum}(y, z, y_9, r, s, S, T)$ yielding a transfinite sequence of elements of $\mathbb{R}$ is defined by

(Def. 19) $\mathrm{dom}\, it = \mathrm{dom}\, y_9$ and for every A such that $A \in \mathrm{dom}\, y_9$ holds $it(A) = \mathrm{Partial\text{-}Sums}(y, z, r, s, y_9 \restriction\!\!\upharpoonright A, S, T)$.

Let f be a strictly decreasing surreal sequence and A be an ordinal number. Let us observe that $f_{\restriction A}$ is strictly decreasing.

Now we state the propositions:

(90) If $\operatorname{dom} y \subseteq S$ and $\operatorname{dom} z \subseteq T$, then $\operatorname{Sum}(y, z, y_9, r, s, S, T) = \operatorname{Sum}(z, y, y_9, s, r, T, S)$. The theorem is a consequence of (84).

(91) If $y = \emptyset$ and $\operatorname{dom} z \subseteq T$, then $\operatorname{rng} \operatorname{Sum}(y, z, y_9, r, s, S, T) \subseteq \{0\}$. The theorem is a consequence of (88).

(92) Let us consider ordinal numbers A, Y, Z. Suppose $\operatorname{dom} y \subseteq S$ and $\operatorname{dom} z \subseteq T$ and $A \in \operatorname{dom} y_9$ and $y_9 \hookleftarrow A > y \hookleftarrow Y + z \hookleftarrow Z$. Then $(\operatorname{Sum}(y_{\restriction Y}, z_{\restriction Z}, y_9, r, s, S, T))(A) = 0$.
PROOF: Set $y_7 = y_{\restriction Y}$. Set $z_1 = z_{\restriction Z}$. $[y_7 + z_1 = y_9 \hookleftarrow A]_{S \times T} = \emptyset_{S \times T}$ by (6), (8), [4, (27)], [20, (3)]. □

(93) Suppose $A \subseteq \operatorname{dom} r \cap \operatorname{dom} y_1$ and $\operatorname{dom} y_1 = \operatorname{dom} y_2$ and for every B such that $B \in A$ holds $y_1 \hookleftarrow B \approx y_2 \hookleftarrow B$. Then $\sum_{\kappa=0}^{\boldsymbol{\omega}^{y_1}\restriction A}(r\restriction A)(\kappa) = \sum_{\kappa=0}^{\boldsymbol{\omega}^{y_2}\restriction A}(r\restriction A)(\kappa)$.
PROOF: Set $N_1 = \boldsymbol{\omega}^{y_1}$. Set $N_2 = \boldsymbol{\omega}^{y_2}$. Define $\mathcal{P}[\text{ordinal number}] \equiv$ if $\$_1 \subseteq O$, then $\sum_{\kappa=0}^{N_1\restriction \$_1}(r\restriction \$_1)(\kappa) = \sum_{\kappa=0}^{N_2\restriction \$_1}(r\restriction \$_1)(\kappa)$. For every ordinal number B such that $B \in O$ holds $N_1 \hookleftarrow B \approx N_2 \hookleftarrow B$ by [**?** , (24)]. For every ordinal number A such that for every ordinal number C such that $C \in A$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[A]$ by (61), [3, (10)], [21, (54), (66)]. For every ordinal number A, $\mathcal{P}[A]$ from [3, Sch. 2]. □

(94) Suppose $A \subseteq \operatorname{dom} r \cap \operatorname{dom} y$. Then $\sum_{\kappa=0}^{\boldsymbol{\omega}^{y}\restriction A}(r\restriction A)(\kappa) = \sum_{\kappa=0}^{\boldsymbol{\omega}^{\operatorname{Unique}_{\mathbf{No}}(y)}\restriction A}(r\restriction A)(\kappa)$. The theorem is a consequence of (93).

(95) Let us consider a strictly decreasing uniq-surreal sequence y_9, and a real number r. Suppose $\operatorname{dom} s = \operatorname{dom} z \subseteq T$ and $\operatorname{Unique}_{\mathbf{No}}(\{a\} \oplus \operatorname{rng} z) \subseteq \operatorname{rng} y_9$. Then $\sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}} \operatorname{Sum}(\langle a\rangle, z, y_9, \langle r\rangle, s, S, T)(\kappa) \approx \mathrm{s}_{\mathbb{R}}(r)\cdot(\boldsymbol{\omega}^{a})\cdot(\sum_{\kappa=0}^{\boldsymbol{\omega}^{z}} s(\kappa))$.
PROOF: Set $D = \operatorname{dom} s$. $\sum_{\kappa=0}^{\boldsymbol{\omega}^{(z+a)}\restriction D}(s\restriction D)(\kappa) \approx (\boldsymbol{\omega}^{a})\cdot(\sum_{\kappa=0}^{\boldsymbol{\omega}^{z}\restriction D}(s\restriction D)(\kappa))$. Set $U = \operatorname{Unique}_{\mathbf{No}}(z+a)$. $\operatorname{rng}(z+a) = \operatorname{rng} z \oplus \{a\}$. $\operatorname{rng} U = \operatorname{Unique}_{\mathbf{No}}(\operatorname{rng} z \oplus \{a\})$. Set $e_3 = s$-extended by (U, y_9). Set $r_8 = r \cdot e_3$. Set $Y = \langle a\rangle$. Set $R = \langle r\rangle$. Set $P = \operatorname{Sum}(Y, z, y_9, R, s, S, T)$. For every object o such that $o \in \operatorname{dom} y_9$ holds $r_8(o) = P(o)$ by (89), [5, (8)], [3, (21)], [25, (33)]. $\sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}} e_3(\kappa) = \sum_{\kappa=0}^{\boldsymbol{\omega}^{U}\restriction D}(s\restriction D)(\kappa)$. $\sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}\restriction \operatorname{dom} e_3}(r_8\restriction \operatorname{dom} e_3)(\kappa) \approx \mathrm{s}_{\mathbb{R}}(r)\cdot(\sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}\restriction \operatorname{dom} e_3}(e_3\restriction \operatorname{dom} e_3)(\kappa))$. □

(96) Let us consider a strictly decreasing uniq-surreal sequence y_9, and a real number s. Suppose $\operatorname{dom} r = \operatorname{dom} y \subseteq S$ and $\operatorname{Unique}_{\mathbf{No}}(\operatorname{rng} y \oplus \{a\}) \subseteq \operatorname{rng} y_9$. Then $\sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}} \operatorname{Sum}(y, \langle a\rangle, y_9, r, \langle s\rangle, S, T)(\kappa) \approx \mathrm{s}_{\mathbb{R}}(s)\cdot(\boldsymbol{\omega}^{a})\cdot(\sum_{\kappa=0}^{\boldsymbol{\omega}^{y}} r(\kappa))$. The theorem is a consequence of (95) and (90).

(97) If $y = \emptyset$ and $\operatorname{dom} z \subseteq T$, then $\sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}} \operatorname{Sum}(y, z, y_9, r, s, S, T)(\kappa) = \mathbf{0}_{\mathbf{No}}$. The theorem is a consequence of (91) and (66).

(98) Suppose $\operatorname{dom} y \subseteq S$ and $\operatorname{dom} z \subseteq T$ and $\operatorname{dom} y \subseteq \operatorname{dom} r$ and $A \subseteq$

$\operatorname{dom} y$. Then $\operatorname{Sum}(y{\restriction}A, z, y_9, r{\restriction}A, s, S, T) + \operatorname{Sum}(y_{\lfloor A}, z, y_9, r_{\lfloor A}, s, S, T) = \operatorname{Sum}(y, z, y_9, r, s, S, T)$.

PROOF: Set $P_1 = \operatorname{Sum}(y{\restriction}A, z, y_9, r{\restriction}A, s, S, T)$. Set $P_2 = \operatorname{Sum}(y_{\lfloor A}, z, y_9, r_{\lfloor A}, s, S, T)$. Set $P_{12} = \operatorname{Sum}(y, z, y_9, r, s, S, T)$. For every object a such that $a \in \operatorname{dom} y_9$ holds $P_{12}(a) = (P_1 + P_2)(a)$ by [26, (62)], [5, (59)], (86). $\square$

(99) Suppose $\operatorname{dom} y \subseteq S$ and $\operatorname{dom} z \subseteq T$ and $\operatorname{dom} y \subseteq \operatorname{dom} r$ and $A \subseteq \operatorname{dom} y$. Then $\sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}} \operatorname{Sum}(y, z, y_9, r, s, S, T)(\kappa) \approx \sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}} \operatorname{Sum}(y{\restriction}A, z, y_9, r{\restriction}A, s, S, T)(\kappa) + \sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}} \operatorname{Sum}(y_{\lfloor A}, z, y_9, r_{\lfloor A}, s, S, T)(\kappa)$. The theorem is a consequence of (98) and (74).

(100) Suppose $\operatorname{dom} y \subseteq S$ and $\operatorname{dom} z \subseteq T$ and $\operatorname{dom} z \subseteq \operatorname{dom} s$ and $A \subseteq \operatorname{dom} z$. Then $\sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}} \operatorname{Sum}(y, z, y_9, r, s, S, T)(\kappa) \approx \sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}} \operatorname{Sum}(y, z{\restriction}A, y_9, r, s{\restriction}A, S, T)(\kappa) + \sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}} \operatorname{Sum}(y, z_{\lfloor A}, y_9, r, s_{\lfloor A}, S, T)(\kappa)$. The theorem is a consequence of (99) and (90).

(101) Let us consider a real number r. Then $\sum_{\kappa=0}^{\boldsymbol{\omega}^{\langle a \rangle}} \langle r \rangle(\kappa) \approx \mathrm{s}_{\mathbb{R}}(r) \cdot (\boldsymbol{\omega}^a)$. The theorem is a consequence of (67) and (64).

(102) Suppose $\operatorname{succ} A \subseteq \operatorname{dom} r \cap \operatorname{dom} y$. Then

(i) $\sum_{\kappa=0}^{\boldsymbol{\omega}^y {\restriction} A} (r{\restriction}A)(\kappa) + \mathrm{s}_{\mathbb{R}}(r(A) - e) \cdot ((\boldsymbol{\omega}^y) \leftharpoonup A) \approx \sum_{\kappa=0}^{\boldsymbol{\omega}^y {\restriction} \operatorname{succ} A} (r{\restriction}\operatorname{succ} A)(\kappa) + \mathrm{s}_{\mathbb{R}}(-e) \cdot (\boldsymbol{\omega}^{(y \leftharpoonup A)})$, and

(ii) $\sum_{\kappa=0}^{\boldsymbol{\omega}^y {\restriction} A} (r{\restriction}A)(\kappa) + \mathrm{s}_{\mathbb{R}}(r(A) + e) \cdot ((\boldsymbol{\omega}^y) \leftharpoonup A) \approx \sum_{\kappa=0}^{\boldsymbol{\omega}^y {\restriction} \operatorname{succ} A} (r{\restriction}\operatorname{succ} A)(\kappa) + \mathrm{s}_{\mathbb{R}}(e) \cdot (\boldsymbol{\omega}^{(y \leftharpoonup A)})$.

The theorem is a consequence of (64), (33), and (30).

(103) Let us consider surreal numbers x, x_0, y, y_0, s, t. Then $x \cdot (y_0 + s) + (x_0 + t) \cdot y - (x_0 + t) \cdot (y_0 + s) \approx x \cdot y_0 + x_0 \cdot y - x_0 \cdot y_0 + ((x - x_0) \cdot s + (y - y_0) \cdot t) - t \cdot s$.

(104) Let us consider a real number e. Suppose $\operatorname{succ} B \subseteq A \subseteq \operatorname{dom} r \cap \operatorname{dom} y$. Then $|(\sum_{\kappa=0}^{\boldsymbol{\omega}^{(y {\restriction} A)}} (r{\restriction}A)(\kappa) - \sum_{\kappa=0}^{\boldsymbol{\omega}^{(y {\restriction} \operatorname{succ} B)}} (r{\restriction}\operatorname{succ} B)(\kappa)) \cdot (\mathrm{s}_{\mathbb{R}}(e) \cdot (\boldsymbol{\omega}^a))| <^{\infty} \boldsymbol{\omega}^{(y \leftharpoonup B + a)}$.

(105) $\text{Partial-Sums}(y, z, -r, s, a, S, T) = -\text{Partial-Sums}(y, z, r, s, a, S, T)$.

PROOF: Set $A_3 = \operatorname{Product}(r, s, S, T)$. Set $m_1 = \operatorname{Product}(-r, s, S, T)$. Set $S_6 = S \times T$. Define $\mathcal{P}[\text{set}] \equiv$ for every element X of $\operatorname{Fin} S_6$ such that $X = \$_1$ holds $+_{\mathbb{R}}\text{-}\sum_X m_1 = -(+_{\mathbb{R}}\text{-}\sum_X A_3)$. $\mathcal{P}[\emptyset_{S_6}]$ by [24, (31)]. For every element B' of $\operatorname{Fin} S_6$ and for every element b of S_6 such that $\mathcal{P}[B']$ and $b \notin B'$ holds $\mathcal{P}[B' \cup \{b\}]$ by [10, (2)]. For every element B of $\operatorname{Fin} S_6$, $\mathcal{P}[B]$ from [24, Sch. 2]. $\square$

(106) Suppose $\operatorname{dom} y \subseteq S$ and $\operatorname{dom} z \subseteq T$. Then

(i) $\operatorname{Sum}(y, z, y_9, -r, s, S, T) = -\operatorname{Sum}(y, z, y_9, r, s, S, T)$, and

(ii) $\operatorname{Sum}(y, z, y_9, r, -s, S, T) = -\operatorname{Sum}(y, z, y_9, r, s, S, T)$.

The theorem is a consequence of (90).

(107) Suppose $\operatorname{dom} y \subseteq S$ and $\operatorname{dom} z \subseteq T$ and $\operatorname{dom} y \subseteq \operatorname{dom} r$ and $\operatorname{dom} z \subseteq \operatorname{dom} s$ and $A \subseteq \operatorname{dom} y$ and $B \subseteq \operatorname{dom} z$. Then $\sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}} \operatorname{Sum}(y, z, y_9, r, s, S, T)(\kappa) - \sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}} \operatorname{Sum}(y_{\restriction A}, z_{\restriction B}, y_9, r_{\restriction A}, s_{\restriction B}, S, T)(\kappa) \approx \sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}} \operatorname{Sum}(y{\restriction}A, z, y_9, r{\restriction}A, s, S, T)(\kappa) + \sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}} \operatorname{Sum}(y, z{\restriction}B, y_9, r, s{\restriction}B, S, T)(\kappa) - \sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}} \operatorname{Sum}(y{\restriction}A, z{\restriction}B, y_9, r{\restriction}A, s{\restriction}B, S, T)(\kappa)$. The theorem is a consequence of (99), (6), and (100).

(108) Suppose $\operatorname{dom} y \subseteq S$ and $\operatorname{dom} z \subseteq T$ and $\operatorname{Unique}_{\mathbf{No}}(\operatorname{rng} y \oplus \operatorname{rng} z) \subseteq \operatorname{rng} y_9$ and $A \in \operatorname{dom} y$ and $B \in \operatorname{dom} z$ and $y \hookrightarrow A + z \hookrightarrow B < a$. Then $|\sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}} \operatorname{Sum}(y_{\restriction A}, z_{\restriction B}, y_9, r, s, S, T)(\kappa)| <^{\infty} \boldsymbol{\omega}^a$.

PROOF: Set $y_7 = y_{\restriction A}$. Set $z_1 = z_{\restriction B}$. Set $P = \operatorname{Sum}(y_7, z_1, y_9, r, s, S, T)$. Set $N_{10} = \boldsymbol{\omega}^{y_9}$. Consider O being an object such that $O \in \operatorname{dom} y_9$ and $y_9(O) = \operatorname{Unique}_{\mathbf{No}}(y \hookrightarrow A + z \hookrightarrow B)$. $\sum_{\kappa=0}^{N_{10}} P(\kappa) \approx \sum_{\kappa=0}^{N_{10}{\restriction}O} (P{\restriction}O)(\kappa) + \sum_{\kappa=0}^{N_{10}{\restriction}O} P_{\restriction O}(\kappa)$. $\operatorname{rng}(P{\restriction}O) \subseteq \{0\}$ by [26, (57)], [8, (47)], [20, (4)], (92). $\sum_{\kappa=0}^{N_{10}{\restriction}O} (P{\restriction}O)(\kappa) = \mathbf{0}_{\mathbf{No}}$. □

(109) Let us consider strictly decreasing uniq-surreal sequences y_9, y_{10}. Suppose $\operatorname{dom} y \subseteq S$ and $\operatorname{dom} z \subseteq T$ and $\operatorname{Unique}_{\mathbf{No}}(\operatorname{rng} y \oplus \operatorname{rng} z) \subseteq \operatorname{rng} y_9 \subseteq \operatorname{rng} y_{10}$. Then $\sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}} \operatorname{Sum}(y, z, y_9, r, s, S, T)(\kappa) = \sum_{\kappa=0}^{\boldsymbol{\omega}^{y_{10}}} \operatorname{Sum}(y, z, y_{10}, r, s, S, T)(\kappa)$.

PROOF: Set $P = \operatorname{Sum}(y, z, y_9, r, s, S, T)$. Set $P_3 = \operatorname{Sum}(y, z, y_{10}, r, s, S, T)$. Set $N = \boldsymbol{\omega}^{y_9}$. Set $N_5 = \boldsymbol{\omega}^{y_{10}}$. $\operatorname{rng} N \subseteq \operatorname{rng} N_5$. For every ordinal numbers C, D such that $C \in \operatorname{dom} N$ and $D \in \operatorname{dom} N_5$ and $N(C) = N_5(D)$ holds $P(C) = P_3(D)$ by [**?** , (25)], [20, (3)], [**?** , (9)], (87). For every ordinal number C such that $C \in \operatorname{dom} N_5$ and $N_5(C) \notin \operatorname{rng} N$ holds $P_3(C) = 0$ by [20, (4), (50)], (88). □

Now we state the proposition:

(110) CORRECTNESS OF THE PRODUCT FOR SELECTED REPRESENTATIONS: Let us consider a strictly decreasing uniq-surreal sequence y_9. Suppose $\operatorname{dom} y \subseteq S$ and $\operatorname{dom} z \subseteq T$ and $\operatorname{Unique}_{\mathbf{No}}(\operatorname{rng} y \oplus \operatorname{rng} z) \subseteq \operatorname{rng} y_9$ and $\operatorname{dom} y = \operatorname{dom} r$ and $\operatorname{dom} z = \operatorname{dom} s$. Then $(\sum_{\kappa=0}^{\boldsymbol{\omega}^y} r(\kappa)) \cdot (\sum_{\kappa=0}^{\boldsymbol{\omega}^z} s(\kappa)) \approx \sum_{\kappa=0}^{\boldsymbol{\omega}^{y_9}} \operatorname{Sum}(y, z, y_9, r, s, S, T)(\kappa)$.

PROOF: Set $N_7 = \boldsymbol{\omega}^y$. Set $N_{11} = \boldsymbol{\omega}^z$. Set $N_{10} = \boldsymbol{\omega}^{y_9}$. Define $\mathcal{P}$[ordinal number] $\equiv$ for every ordinal numbers Y, Z such that $Y \subseteq \operatorname{dom} y$ and $Z \subseteq \operatorname{dom} z$ and $Y \oplus Z = \$_1$ holds $(\sum_{\kappa=0}^{N_7{\restriction}Y} (r{\restriction}Y)(\kappa)) \cdot (\sum_{\kappa=0}^{N_{11}{\restriction}Z} (s{\restriction}Z)(\kappa)) \approx \sum_{\kappa=0}^{N_{10}} \operatorname{Sum}(y{\restriction}Y, z{\restriction}Z, y_9, r{\restriction}Y, s{\restriction}Z, S, T)(\kappa)$. For every ordinal number O such that for every ordinal number C such that $C \in O$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[O]$ by [26, (62)], (47), (61), (62). For every ordinal number A, $\mathcal{P}[A]$ from [3, Sch. 2]. □

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Received September 6, 2026