FORMALIZED MATHEMATICS
Vol. 32, No. 1, pp. 155–162, 2024
DOI: 10.2478/forma-2024-0013
e-ISSN: 1898–9934

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sciendo.com/journal/forma

Differentiability Properties of Lipschitzian Bilinear Operators in Real Normed Spaces[1]

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Summary. In this article we formalize in Mizar [2], [3] various differentiability properties of Lipschitzian bilinear operators in real normed spaces. It covers topics such as partial differentiability, continuity, and total differentiability of these operators. The work extends results for linear operators to the bilinear case and provides theorems on the behavior of differential operators up to arbitrary order. Key results include the Lipschitz continuity of partial derivatives, the representation of the total derivative in terms of partial derivatives, and the continuous differentiability of Lipschitzian bilinear operators on open subsets of the product space. We referred to [9], [20], [15], [16] in this formalization.

MSC: 46G05 47A07 68V20

Keywords: functional analysis; bilinear operators; Lipschitz continuity; differentiability; normed spaces

MML identifier: NDIFF12, version: 8.1.14 5.85.1476

1. Fundamental Properties and Partial Differentiability

From now on E, F, G, S, T, W, Y denote real normed spaces, f, f_1, f_2 denote partial functions from S to T, Z denotes a subset of S, and i, n denote natural numbers.

Now we state the propositions:

(1) Let us consider a bilinear operator f from $E \times F$ into G, and a point z of $E \times F$. Then

[1]This work was supported by JSPS KAKENHI Grant Number 24K14897.

(i) $f \cdot (\mathrm{reproj1}(z))$ is a linear operator from E into G, and

(ii) $f \cdot (\mathrm{reproj2}(z))$ is a linear operator from F into G.

Proof: Reconsider $L_1 = f \cdot (\mathrm{reproj1}(z))$ as a function from E into G. For every elements x, y of E, $L_1(x+y) = L_1(x) + L_1(y)$ by [5, (15)], [11, (12)]. For every vector x of E and for every real number a, $L_1(a \cdot x) = a \cdot L_1(x)$ by [5, (15)], [11, (12)]. Reconsider $L_2 = f \cdot (\mathrm{reproj2}(z))$ as a function from F into G. For every elements x, y of F, $L_2(x+y) = L_2(x) + L_2(y)$ by [5, (15)], [11, (12)]. For every vector x of F and for every real number a, $L_2(a \cdot x) = a \cdot L_2(x)$ by [5, (15)], [11, (12)]. □

(2) Let us consider a Lipschitzian bilinear operator f from $E \times F$ into G, and a point z of $E \times F$. Then

(i) $f \cdot (\mathrm{reproj1}(z))$ is a Lipschitzian linear operator from E into G, and

(ii) $f \cdot (\mathrm{reproj2}(z))$ is a Lipschitzian linear operator from F into G, and

(iii) there exists a point g of $\mathrm{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(E, F, G)$ such that $f = g$ and for every vector x of E, $\|(f \cdot (\mathrm{reproj1}(z)))(x)\| \leqslant \|g\| \cdot \|(z)_{\mathbf{2}}\| \cdot \|x\|$ and for every vector y of F, $\|(f \cdot (\mathrm{reproj2}(z)))(y)\| \leqslant \|g\| \cdot \|(z)_{\mathbf{1}}\| \cdot \|y\|$.

Proof: Reconsider $g = f$ as a point of $\mathrm{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(E, F, G)$. Set $K = \|g\|$. Reconsider $L_1 = f \cdot (\mathrm{reproj1}(z))$ as a linear operator from E into G. Reconsider $L_2 = f \cdot (\mathrm{reproj2}(z))$ as a linear operator from F into G. Set $K_1 = K \cdot \|(z)_{\mathbf{2}}\|$. Set $K_2 = K \cdot \|(z)_{\mathbf{1}}\|$. For every vector x of E, $\|L_1(x)\| \leqslant K_1 \cdot \|x\|$ by [5, (15)], [10, (16)]. For every vector y of F, $\|L_2(y)\| \leqslant K_2 \cdot \|y\|$ by [5, (15)], [10, (16)]. □

(3) Let us consider a Lipschitzian bilinear operator f from $E \times F$ into G. Then there exists a real number K such that

(i) $0 \leqslant K$, and

(ii) for every point z of $E \times F$, for every point x of E, $\|(f \cdot (\mathrm{reproj1}(z)))(x)\| \leqslant K \cdot \|(z)_{\mathbf{2}}\| \cdot \|x\|$ and for every point y of F, $\|(f \cdot (\mathrm{reproj2}(z)))(y)\| \leqslant K \cdot \|(z)_{\mathbf{1}}\| \cdot \|y\|$.

Proof: Consider K being a real number such that $0 \leqslant K$ and for every vector x of E and for every vector y of F, $\|f(x, y)\| \leqslant K \cdot \|x\| \cdot \|y\|$. Set $L_1 = f \cdot (\mathrm{reproj1}(z))$. Set $K_1 = K \cdot \|(z)_{\mathbf{2}}\|$. For every vector x of E, $\|L_1(x)\| \leqslant K_1 \cdot \|x\|$ by [5, (15)]. □

(4) Let us consider a Lipschitzian bilinear operator f from $E \times F$ into G, and a point z of $E \times F$. Then

(i) f is partially differentiable in z w.r.t. 1, and

(ii) $\mathrm{partdiff}(f, z)$ w.r.t. $1 = f \cdot (\mathrm{reproj1}(z))$, and

(iii) f is partially differentiable in z w.r.t. 2, and

(iv) $\text{partdiff}(f, z)\,\text{w.r.t.}\,2 = f \cdot (\text{reproj2}(z))$.

The theorem is a consequence of (2).

(5) Let us consider points s, t of $E \times F$, and a real number a. Then

(i) $s = \langle (s)_{\mathbf{1}}, (s)_{\mathbf{2}} \rangle$, and

(ii) $((s+t))_{\mathbf{1}} = (s)_{\mathbf{1}} + (t)_{\mathbf{1}}$, and

(iii) $((s+t))_{\mathbf{2}} = (s)_{\mathbf{2}} + (t)_{\mathbf{2}}$, and

(iv) $((s-t))_{\mathbf{1}} = (s)_{\mathbf{1}} - (t)_{\mathbf{1}}$, and

(v) $((s-t))_{\mathbf{2}} = (s)_{\mathbf{2}} - (t)_{\mathbf{2}}$, and

(vi) $(a \cdot s)_{\mathbf{1}} = a \cdot ((s)_{\mathbf{1}})$, and

(vii) $(a \cdot s)_{\mathbf{2}} = a \cdot ((s)_{\mathbf{2}})$.

(6) Let us consider a Lipschitzian bilinear operator f from $E \times F$ into G. Then there exists a real number K such that

(i) $0 \leqslant K$, and

(ii) for every point z of $E \times F$, $\|\text{partdiff}(f, z)\,\text{w.r.t.}\,1\| \leqslant K \cdot \|z\|$ and $\|\text{partdiff}(f, z)\,\text{w.r.t.}\,2\| \leqslant K \cdot \|z\|$.

The theorem is a consequence of (3), (2), (4), and (5).

2. Total Differentiability and Continuity

Let us consider a Lipschitzian bilinear operator f from $E \times F$ into G. Now we state the propositions:

(7) (i) for every points z_1, z_2 of $E \times F$, $\text{partdiff}(f, z_1 + z_2)\,\text{w.r.t.}\,1 = \text{partdiff}(f, z_1)\,\text{w.r.t.}\,1 + \text{partdiff}(f, z_2)\,\text{w.r.t.}\,1$, and

(ii) for every point z of $E \times F$ and for every real number a, $\text{partdiff}(f, a \cdot z)\,\text{w.r.t.}\,1 = a \cdot (\text{partdiff}(f, z)\,\text{w.r.t.}\,1)$, and

(iii) for every points z_1, z_2 of $E \times F$, $\text{partdiff}(f, z_1 - z_2)\,\text{w.r.t.}\,1 = \text{partdiff}(f, z_1)\,\text{w.r.t.}$ $\text{partdiff}(f, z_2)\,\text{w.r.t.}\,1$.

The theorem is a consequence of (4) and (5).

(8) (i) for every points z_1, z_2 of $E \times F$, $\text{partdiff}(f, z_1 + z_2)\,\text{w.r.t.}\,2 = \text{partdiff}(f, z_1)\,\text{w.r.t.}\,2 + \text{partdiff}(f, z_2)\,\text{w.r.t.}\,2$, and

(ii) for every point z of $E \times F$ and for every real number a, $\text{partdiff}(f, a \cdot z)\,\text{w.r.t.}\,2 = a \cdot (\text{partdiff}(f, z)\,\text{w.r.t.}\,2)$, and

(iii) for every points z_1, z_2 of $E \times F$, $\mathrm{partdiff}(f, z_1 - z_2)$ w.r.t. $2 = \mathrm{partdiff}(f, z_1)$ w.r.t. $2 - \mathrm{partdiff}(f, z_2)$ w.r.t. 2.

The theorem is a consequence of (4) and (5).

Now we state the proposition:

(9) Let us consider a Lipschitzian bilinear operator f from $E \times F$ into G, and a subset Z of $E \times F$. Suppose Z is open. Then

(i) f is partially differentiable on Z w.r.t. 1, and

(ii) f is partially differentiable on Z w.r.t. 2, and

(iii) $f \restriction^1 Z$ is continuous on Z, and

(iv) $f \restriction^2 Z$ is continuous on Z.

PROOF: For every point x of $E \times F$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. 1. For every point x of $E \times F$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. 2. Set $g_1 = f \restriction^1 Z$. Set $g_2 = f \restriction^2 Z$. Consider K being a real number such that $0 \leqslant K$ and for every point z of $E \times F$, $\|\mathrm{partdiff}(f, z)\,\text{w.r.t.}\,1\| \leqslant K \cdot \|z\|$ and $\|\mathrm{partdiff}(f, z)\,\text{w.r.t.}\,2\| \leqslant K \cdot \|z\|$. For every point t_0 of $E \times F$ and for every real number r such that $t_0 \in Z$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point t_1 of $E \times F$ such that $t_1 \in Z$ and $\|t_1 - t_0\| < s$ holds $\|g_{1/t_1} - g_{1/t_0}\| < r$ by (7), [13, (4)]. For every point t_0 of $E \times F$ and for every real number r such that $t_0 \in Z$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point t_1 of $E \times F$ such that $t_1 \in Z$ and $\|t_1 - t_0\| < s$ holds $\|g_{2/t_1} - g_{2/t_0}\| < r$ by (8), [13, (4)]. $\square$

Let us consider a Lipschitzian bilinear operator f from $E \times F$ into G. Now we state the propositions:

(10) There exists a real number K such that

(i) $0 \leqslant K$, and

(ii) for every point z of $E \times F$, there exists a Lipschitzian linear operator L from $E \times F$ into G such that for every point d_1 of E and for every point d_2 of F, $L(d_1, d_2) = f(d_1, (z)_{\mathbf{2}}) + f((z)_{\mathbf{1}}, d_2)$ and for every point s of $E \times F$, $\|L(s)\| \leqslant K \cdot \|z\| \cdot \|s\|$.

PROOF: Consider K being a real number such that $0 \leqslant K$ and for every vector x of E and for every vector y of F, $\|f(x, y)\| \leqslant K \cdot \|x\| \cdot \|y\|$. Define $\mathcal{Q}(\text{element of } E, \text{element of } F) = f(\$_1, (z)_{\mathbf{2}}) + f((z)_{\mathbf{1}}, \$_2)$. Consider L_0 being a function from (the carrier of E) $\times$ (the carrier of F) into the carrier of G such that for every element x of the carrier of E and for every element y of the carrier of F, $L_0(x, y) = \mathcal{Q}(x, y)$ from [4, Sch. 4]. Reconsider $L = L_0$ as a function from $E \times F$ into G. For every elements x, y of $E \times$

F, $L(x+y) = L(x) + L(y)$ by [12, (18)], [11, (12)]. For every vector x of $E \times F$ and for every real number a, $L(a \cdot x) = a \cdot L(x)$ by [12, (18)], [11, (12)]. Set $K_1 = 2 \cdot K \cdot \|z\|$. For every vector w of $E \times F$, $\|L(w)\| \leqslant K_1 \cdot \|w\|$ by [12, (18)], [14, (15)], [13, (4)]. □

(11) There exists a real number K such that

(i) $0 \leqslant K$, and

(ii) for every point z of $E \times F$, f is differentiable in z and for every point d_1 of E and for every point d_2 of F, $f'(z)(d_1, d_2) = f(d_1, (z)_{\mathbf{2}}) + f((z)_{\mathbf{1}}, d_2)$ and $\|f'(z)\| \leqslant K \cdot \|z\|$.

Proof: Consider K_0 being a real number such that $0 \leqslant K_0$ and for every vector x of E and for every vector y of F, $\|f(x, y)\| \leqslant K_0 \cdot \|x\| \cdot \|y\|$. Consider K being a real number such that $0 \leqslant K$ and for every point z of $E \times F$, there exists a Lipschitzian linear operator L from $E \times F$ into G such that for every point d_1 of E for every point d_2 of F, $L(d_1, d_2) = f(d_1, (z)_{\mathbf{2}}) + f((z)_{\mathbf{1}}, d_2)$ and for every point s of $E \times F$, $\|L(s)\| \leqslant K \cdot \|z\| \cdot \|s\|$. Consider L being a Lipschitzian linear operator from $E \times F$ into G such that for every point d_1 of E and for every point d_2 of F, $L(d_1, d_2) = f(d_1, (z)_{\mathbf{2}}) + f((z)_{\mathbf{1}}, d_2)$ and for every point s of $E \times F$, $\|L(s)\| \leqslant K \cdot \|z\| \cdot \|s\|$. Reconsider $L_0 = L$ as a point of the real norm space of bounded linear operators from $E \times F$ into G. Define $\mathcal{Q}(\text{element of } E, \text{element of } F) = f(\$_1, \$_2)$. Consider R being a function from (the carrier of E)$\times$(the carrier of F) into the carrier of G such that for every element d_1 of the carrier of E and for every element d_2 of the carrier of F, $R(d_1, d_2) = \mathcal{Q}(d_1, d_2)$ from [4, Sch. 4]. For every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every point w of $E \times F$ such that $w \neq 0_{E \times F}$ and $\|w\| < d$ holds $\|w\|^{-1} \cdot \|R_{/w}\| < r$ by [13, (4)], [12, (18)], [14, (15)]. For every point w of $E \times F$ such that $w \in$ the neighbourhood of z holds $f_{/w} - f_{/z} = L_0(w - z) + R_{/w-z}$ by [12, (18)], [19, (1)], [11, (12)]. □

Now we state the propositions:

(12) Let us consider a Lipschitzian bilinear operator f from $E \times F$ into G, a point z of $E \times F$, a point d_1 of E, and a point d_2 of F. Then $f'(z)(d_1, d_2) = (\text{partdiff}(f, z) \text{ w.r.t. } 1)(d_1) + (\text{partdiff}(f, z) \text{ w.r.t. } 2)(d_2)$. The theorem is a consequence of (11) and (4).

(13) Let us consider a Lipschitzian bilinear operator f from $E \times F$ into G. Then

(i) for every points z_1, z_2 of $E \times F$, $f'(z_1 + z_2) = f'(z_1) + f'(z_2)$, and

(ii) for every point z of $E \times F$ and for every real number a, $f'(a \cdot z) = a \cdot f'(z)$, and

(iii) for every points z_1, z_2 of $E \times F$, $f'(z_1 - z_2) = f'(z_1) - f'(z_2)$.

The theorem is a consequence of (12), (7), and (8).

(14) Let us consider a Lipschitzian bilinear operator f from $E \times F$ into G, and a subset Z of $E \times F$. Suppose Z is open. Then

(i) f is differentiable on Z, and

(ii) $f'_{\restriction Z}$ is continuous on Z.

PROOF: Consider K being a real number such that $0 \leqslant K$ and for every point z of $E \times F$, f is differentiable in z and for every point d_1 of E and for every point d_2 of F, $f'(z)(d_1, d_2) = f(d_1, (z)_{\mathbf{2}}) + f((z)_{\mathbf{1}}, d_2)$ and $\|f'(z)\| \leqslant K \cdot \|z\|$. Set $g_1 = f'_{\restriction Z}$. For every point t_0 of $E \times F$ and for every real number r such that $t_0 \in Z$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point t_1 of $E \times F$ such that $t_1 \in Z$ and $\|t_1 - t_0\| < s$ holds $\|g_{1/t_1} - g_{1/t_0}\| < r$ by (13), [13, (4)]. $\square$

3. Higher-Order Derivatives and Special Properties

Now we state the propositions:

(15) Let us consider a Lipschitzian bilinear operator f from $E \times F$ into G. Then

(i) $f'_{\restriction \Omega_{E \times F}}$ is Lipschitzian linear operator from $E \times F$ into the real norm space of bounded linear operators from $E \times F$ into G, differentiable on $\Omega_{E \times F}$, and continuous on the carrier of $E \times F$, and

(ii) for every point z of $E \times F$, $(f'_{\restriction \Omega_{E \times F}})'(z) = f'_{\restriction \Omega_{E \times F}}$.

The theorem is a consequence of (14), (13), and (11).

(16) Let us consider a Lipschitzian linear operator L from E into F. Then

(i) $L'(\Omega_E)(0) = L$, and

(ii) $L'(\Omega_E)(1) = \Omega_E \longmapsto L$, and

(iii) $L'(\Omega_E)(2) = \Omega_E \longmapsto (\Omega_E \longmapsto (\Omega_E \longmapsto 0_F))$, and

(iv) $L'(\Omega_E)(3) = \Omega_E \longmapsto (\Omega_E \longmapsto (\Omega_E \longmapsto (\Omega_E \longmapsto 0_F)))$.

PROOF: For every object z such that $z \in \operatorname{dom} L'_{\restriction \Omega_E}$ holds $L'_{\restriction \Omega_E}(z) = L$ by [7, (26)]. For every object z such that $z \in \operatorname{dom}(L'_{\restriction \Omega_E})'_{\restriction \Omega_E}$ holds $(L'_{\restriction \Omega_E})'_{\restriction \Omega_E}(z) = \Omega_E \longmapsto (\Omega_E \longmapsto 0_F)$ by [8, (33)]. Reconsider $L_1 = L'(\Omega_E)(2)$ as a partial function from E to the real norm space of bounded linear operators from E into the real norm space of bounded linear operators from E into F. For every object z such that $z \in \operatorname{dom} L_1{'}{\restriction}\Omega_E$ holds $L_1{'}{\restriction}\Omega_E(z) = \Omega_E \longmapsto (\Omega_E \longmapsto (\Omega_E \longmapsto 0_F))$ by [8, (33)]. □

(17) Let us consider a natural number i. Then $0_{\mathrm{diff}_{\mathrm{SP}}(E^{(i+1)},F)} = \Omega_E \longmapsto 0_{\mathrm{diff}_{\mathrm{SP}}(E^{i},F)}$.

Let us consider a Lipschitzian linear operator L from E into F and a natural number i. Now we state the propositions:

(18) $\mathrm{diff}_{\Omega_E}(L, i+2) = \Omega_E \longmapsto 0_{\mathrm{diff}_{\mathrm{SP}}(E^{(i+2)},F)}$.
Proof: Define $\mathcal{P}$[natural number] $\equiv \mathrm{diff}_{\Omega_E}(L, \$_1{+}2) = \Omega_E \longmapsto 0_{\mathrm{diff}_{\mathrm{SP}}(E^{(\$_1+2)},F)}$. $\mathcal{P}[0]$ by [6, (7), (10)], [17, (31)], (16). For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [18, (13), (8)], [8, (33)], [6, (10), (13)]. For every natural number i, $\mathcal{P}[i]$ from [1, Sch. 2]. □

(19) (i) $\mathrm{diff}_{\Omega_E}(L, i+1)$ is differentiable on Ω_E, and

(ii) $\mathrm{diff}_{\Omega_E}(L, i+1)'{\restriction}\Omega_E = \Omega_E \longmapsto 0_{\mathrm{diff}_{\mathrm{SP}}(E^{(i+2)},F)}$, and

(iii) $\mathrm{diff}_{\Omega_E}(L, i+1)'{\restriction}\Omega_E$ is continuous on Ω_E.

Proof: Define $\mathcal{P}$[natural number] $\equiv \mathrm{diff}_{\Omega_E}(L, \$_1+1)$ is differentiable on Ω_E and $\mathrm{diff}_{\Omega_E}(L, \$_1+1)'{\restriction}\Omega_E = \Omega_E \longmapsto 0_{\mathrm{diff}_{\mathrm{SP}}(E^{(\$_1+2)},F)}$ and $\mathrm{diff}_{\Omega_E}(L, \$_1+1)'{\restriction}\Omega_E$ is continuous on Ω_E. $\mathcal{P}[0]$ by [6, (7), (10)], [17, (31)], (16). For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [6, (13)], [18, (13), (8)], [8, (33)]. For every natural number i, $\mathcal{P}[i]$ from [1, Sch. 2]. □

(20) (i) $\mathrm{diff}_{\Omega_E}(L, i)$ is differentiable on Ω_E, and

(ii) $\mathrm{diff}_{\Omega_E}(L, i)'{\restriction}\Omega_E$ is continuous on Ω_E.

The theorem is a consequence of (16) and (19).

Now we state the proposition:

(21) Let us consider a Lipschitzian bilinear operator B from $E \times F$ into G, and a natural number i. Then

(i) $\mathrm{diff}_{\Omega_{E\times F}}(B, i)$ is differentiable on $\Omega_{E\times F}$, and

(ii) $\mathrm{diff}_{\Omega_{E\times F}}(B, i)'{\restriction}\Omega_{E\times F}$ is continuous on $\Omega_{E\times F}$.

Proof: Reconsider $L = B'{\restriction}\Omega_{E\times F}$ as a Lipschitzian linear operator from $E\times F$ into the real norm space of bounded linear operators from $E\times F$ into G. Set G_1 = the real norm space of bounded linear operators from $E\times F$ into G. Define $\mathcal{P}$[natural number] $\equiv \mathrm{diff}_{\Omega_{E\times F}}(B, \$_1+1) = \mathrm{diff}_{\Omega_{E\times F}}(L, \$_1)$ and $\mathrm{diff}_{\mathrm{SP}}((E\times F)^{(\$_1+1)}, G) = \mathrm{diff}_{\mathrm{SP}}((E\times F)^{\$_1}, G_1)$. $\mathcal{P}[0]$ by [6, (11), (7)]. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [6, (10), (13)]. For every natural number n, $\mathcal{P}[n]$ from [1, Sch. 2]. □

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Accepted December 9, 2024