FORMALIZED MATHEMATICS
Vol. 32, No. 1, pp. 9–31, 2024
DOI: 10.2478/forma-2024-0002
e-ISSN: 1898–9934

sciendo.com/journal/forma

Integral of Continuous Three Variable Functions[1]

Noboru Endou
National Institute of Technology, Gifu College
2236-2 Kamimakuwa, Motosu, Gifu, Japan

Yasunari Shidama
Karuizawa Hotch 244-1
Nagano, Japan

Summary. In this article, following the previous article [11], we continue our proofs on integrals of continuous functions of three variables in Mizar [2], [3]. In the first section, continuity of functions of three variables is shown. These are used in the proofs of the later sections.

The second section summarizes the basic properties of the projection of a continuous function in three variables, a result that is almost as obvious as in two variables, but is used to transform [10] Riemann and Lebesgue integrals for real-valued functions (not extended real-valued functions).

In the last section, we prove integrability and iterated integrals of continuous functions of three variables. Throughout the paper, the basic operations follow [1], [**?**], and [13].

MSC: 68V20

Keywords:

MML identifier: `MESFUN17`, version: `8.1.14 5.79.1465`

1. Preliminaries

Now we state the propositions:

[1]This work was supported by JSPS KAKENHI 23K11242.

(1) Let us consider real normed spaces X, Y, Z, a point u of $X \times Y \times Z$, a point x of X, a point y of Y, and a point z of Z. Suppose $u = \langle x, y, z \rangle$. Then

  (i) $\|u\| \leqslant \|x\| + \|y\| + \|z\|$, and

  (ii) $\|x\| \leqslant \|u\|$, and

  (iii) $\|y\| \leqslant \|u\|$, and

  (iv) $\|z\| \leqslant \|u\|$.

(2) Let us consider closed interval subsets I, J, K of $\mathbb{R}$, and a subset E of ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$). If $E = (I \times J) \times K$, then E is compact.

(3) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a set E. Suppose $f = g$ and $E \subseteq \operatorname{dom} f$. Then f is uniformly continuous on E if and only if for every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every real numbers x_1, x_2, y_1, y_2, z_1, z_2 such that $\langle x_1, y_1, z_1 \rangle$, $\langle x_2, y_2, z_2 \rangle \in E$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $|z_2 - z_1| < r$ holds $|g(\langle x_2, y_2, z_2 \rangle) - g(\langle x_1, y_1, z_1 \rangle)| < e$.
PROOF: For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every points p_1, p_2 of ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) such that $p_1, p_2 \in E$ and $\|p_1 - p_2\| < r$ holds $\|f_{/p_1} - f_{/p_2}\| < e$ by [17, (9)], [14, (4)], [4, (60)], (1). □

(4) Let us consider intervals I, J, K. Then

  (i) $(I \times J) \times K$ is a subset of ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$), and

  (ii) $(I \times J) \times K \in \sigma(\mathrm{MeasRect}(\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$.

(5) Let us consider a point u of (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$), and a real number r. Suppose $0 < r$. Then there exist real numbers s, x, y, z such that

  (i) $0 < s < r$, and

  (ii) $u = \langle x, y, z \rangle$, and

  (iii) $]x - s, x + s[\times]y - s, y + s[\times]z - s, z + s[\subseteq \mathrm{Ball}(u, r)$.

Let us consider a subset A of (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$). Now we state the propositions:

(6) Suppose for every real numbers a, b, c such that $\langle a, b, c\rangle \in A$ there exists a real-membered set R_{12} such that R_{12} is non empty and upper bounded and $R_{12} = \{r$, where r is a real number : $0 < r$ and $]a-r, a+r[\times]b-r, b+r[\times]c-r, c+r[\subseteq A\}$. Then there exists a function F from A into $\mathbb{R}$ such that for every real numbers a, b, c such that $\langle a, b, c\rangle \in A$ there exists a real-membered set R_{12} such that R_{12} is non empty and upper bounded and $R_{12} = \{r$, where r is a real number : $0 < r$ and $]a-r, a+r[\times]b-r, b+r[\times]c-r, c+r[\subseteq A\}$ and $F(\langle a, b, c\rangle) = \frac{\sup R_{12}}{2}$.
PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exist real numbers a, b, c and there exists a real-membered set R_{12} such that $\$_1 = \langle a, b, c\rangle$ and R_{12} is non empty and upper bounded and $R_{12} = \{r$, where r is a real number : $0 < r$ and $]a-r, a+r[\times]b-r, b+r[\times]c-r, c+r[\subseteq A\}$ and $\$_2 = \frac{\sup R_{12}}{2}$. For every object x such that $x \in A$ there exists an object y such that $y \in \mathbb{R}$ and $\mathcal{P}[x, y]$ by [17, (9)]. Consider F being a function from A into $\mathbb{R}$ such that for every object x such that $x \in A$ holds $\mathcal{P}[x, F(x)]$ from [6, Sch. 1]. For every real numbers a, b, c such that $\langle a, b, c\rangle \in A$ there exists a real-membered set R_{12} such that R_{12} is non empty and upper bounded and $R_{12} = \{r$, where r is a real number : $0 < r$ and $]a-r, a+r[\times]b-r, b+r[\times]c-r, c+r[\subseteq A\}$ and $F(\langle a, b, c\rangle) = \frac{\sup R_{12}}{2}$. $\square$

(7) If A is open, then $A \in \sigma(\text{MeasRect}(\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$. The theorem is a consequence of (5), (6), and (1).

Now we state the propositions:

(8) Let us consider closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose f is continuous on $(I \times J) \times K$ and $f = g$. Let us consider a real number e. Suppose $0 < e$. Then there exists a real number r such that

   (i) $0 < r$, and

   (ii) for every real numbers x_1, x_2, y_1, y_2, z_1, z_2 such that $x_1, x_2 \in I$ and $y_1, y_2 \in J$ and $z_1, z_2 \in K$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $|z_2 - z_1| < r$ holds $|g(\langle x_2, y_2, z_2\rangle) - g(\langle x_1, y_1, z_1\rangle)| < e$.

PROOF: Set $E = (I \times J) \times K$. f is uniformly continuous on E. Consider r being a real number such that $0 < r$ and for every real numbers x_1, x_2, y_1, y_2, z_1, z_2 such that $\langle x_1, y_1, z_1\rangle, \langle x_2, y_2, z_2\rangle \in E$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $|z_2 - z_1| < r$ holds $|g(\langle x_2, y_2, z_2\rangle) - g(\langle x_1, y_1, z_1\rangle)| < e$. For every real numbers x_1, x_2, y_1, y_2, z_1, z_2 such that $x_1, x_2 \in I$ and $y_1, y_2 \in J$ and $z_1, z_2 \in K$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $|z_2 - z_1| < r$ holds $|g(\langle x_2, y_2, z_2\rangle) - g(\langle x_1, y_1, z_1\rangle)| < e$ by [7, (87)]. $\square$

(9) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. If $f = g$, then $\|f\| = |g|$.

(10) Let us consider closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose f is continuous on $(I \times J) \times K$ and $f = g$. Let us consider a real number e. Suppose $0 < e$. Then there exists a real number r such that

(i) $0 < r$, and

(ii) for every real numbers x_1, x_2, y_1, y_2, z_1, z_2 such that $x_1, x_2 \in I$ and $y_1, y_2 \in J$ and $z_1, z_2 \in K$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $|z_2 - z_1| < r$ holds $||g|(\langle x_2, y_2, z_2 \rangle) - |g|(\langle x_1, y_1, z_1 \rangle)| < e$.

The theorem is a consequence of (9) and (8).

2. Properties on the Projective Function of a Three-Variable Function

Now we state the propositions:

(11) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and elements x, y of $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$. Then ProjPMap1(g, $\langle x, y \rangle$) is continuous.
Proof: For every real number z_0 such that $z_0 \in$ dom(ProjPMap1(g, $\langle x, y \rangle$)) holds ProjPMap1(g, $\langle x, y \rangle$) is continuous in z_0 by [15, (19)], [14, (4)], [17, (9)], [20, (15)]. □

(12) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function p_2 from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and an element z of $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$ and $p_2 = \text{ProjPMap2}(g, z)$. Then p_2 is continuous on $\operatorname{dom} p_2$.
Proof: For every point x_4 of (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) such that $x_4 \in \operatorname{dom} p_2$ holds $p_2 \restriction \operatorname{dom} p_2$ is continuous in x_4 by [18, (18)], [15, (19)], [17, (9)], [20, (15)]. □

(13) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and elements x, y of $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$. Then ProjPMap1($|g|, \langle x, y \rangle$) is continuous. The theorem is a consequence of (11).

(14) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function p_2 from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and an element z of $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$ and $p_2 =$ ProjPMap2($|g|, z$). Then p_2 is continuous on $\operatorname{dom} p_2$. The theorem is a consequence of (12).

(15) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and elements x, y of $\mathbb{R}$. Suppose f is uniformly continuous on $\operatorname{dom} f$ and $f = g$. Then ProjPMap1($g, \langle x, y \rangle$) is uniformly continuous.

PROOF: For every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every real numbers z_1, z_2 such that z_1, $z_2 \in$ dom(ProjPMap1($g, \langle x, y \rangle$)) and $|z_1 - z_2| < s$ holds $|$(ProjPMap1($g, \langle x, y \rangle$))(z_1) $-$ (ProjPMap1($g, \langle x, y \rangle$))(z_2)$| < r$ by [14, (4)], [17, (9)], [20, (15)], [19, (22)]. □

(16) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function p_2 from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and an element z of $\mathbb{R}$. Suppose f is uniformly continuous on $\operatorname{dom} f$ and $f = g$ and $p_2 =$ ProjPMap2(g, z). Then p_2 is uniformly continuous on $\operatorname{dom} p_2$.

(17) Let us consider elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_8 from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$ and $P_8 =$ ProjPMap1($\overline{\mathbb{R}}(g), \langle x, y \rangle$). Then P_8 is continuous. The theorem is a consequence of (11).

(18) Let us consider an element z of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times$

$\mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_7 from (the real normed space of $\mathbb{R}) \times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. Suppose f is continuous on dom f and $f = g$ and $P_7 = \text{ProjPMap2}(\overline{\mathbb{R}}(g), z)$. Then P_7 is continuous on dom P_7. The theorem is a consequence of (12).

(19) Let us consider elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}) \times$ (the real normed space of $\mathbb{R})) \times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_8 from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is continuous on dom f and $f = g$ and $P_8 = \text{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y \rangle)$. Then P_8 is continuous. The theorem is a consequence of (13).

(20) Let us consider an element z of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}) \times$ (the real normed space of $\mathbb{R})) \times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_7 from (the real normed space of $\mathbb{R}) \times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. Suppose f is continuous on dom f and $f = g$ and $P_7 = \text{ProjPMap2}(|\overline{\mathbb{R}}(g)|, z)$. Then P_7 is continuous on dom P_7. The theorem is a consequence of (14).

3. Integral of Continuous Three-Variable Functions

Let us consider subsets I, J of $\mathbb{R}$, a non empty, closed interval subset K of $\mathbb{R}$, elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}) \times$ (the real normed space of $\mathbb{R})) \times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_8 from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(21) Suppose $x \in I$ and $y \in J$ and dom $f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_8 = \text{ProjPMap1}(\overline{\mathbb{R}}(g), \langle x, y \rangle)$. Then

(i) $P_8 \restriction K$ is bounded, and

(ii) P_8 is integrable on K.

The theorem is a consequence of (17).

(22) Suppose $x \in I$ and $y \in J$ and dom $f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_8 = \text{ProjPMap1}(\overline{\mathbb{R}}(g), \langle x, y \rangle)$. Then

(i) P_8 is integrable on L-Meas, and

(ii) $\int\limits_K P_8(x)dx = \int P_8 \, \mathrm{d}\,\text{L-Meas}$, and

(iii) $\int\limits_K P_8(x)dx = \int \text{ProjPMap1}(\overline{\mathbb{R}}(g), \langle x, y \rangle) \, \mathrm{d}\,\text{L-Meas}$, and

(iv) $\int\limits_{K} P_8(x)dx = (\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)))(\langle x, y\rangle)$.

The theorem is a consequence of (21).

Now we state the propositions:

(23) Let us consider non empty, closed interval subsets I, J of $\mathbb{R}$, a subset K of $\mathbb{R}$, an element z of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_9 from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Suppose $z \in K$ and $\mathrm{dom}\, f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_9 = \mathrm{ProjPMap2}(\overline{\mathbb{R}}(g), z)$. Then

(i) P_9 is integrable on $\mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas})$, and

(ii) $\int P_9 \,\mathrm{d}\, \mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas}) = \int \mathrm{ProjPMap2}(\overline{\mathbb{R}}(g), z) \,\mathrm{d}\, \mathrm{ProdMeas}(\text{L-Mea}$
and

(iii) $\int P_9 \,\mathrm{d}\, \mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas}) = (\mathrm{Integral1}(\mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}$

The theorem is a consequence of (18).

(24) Let us consider subsets I, J of $\mathbb{R}$, a non empty, closed interval subset K of $\mathbb{R}$, elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_8 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $y \in J$ and $\mathrm{dom}\, f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_8 = \mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y\rangle)$. Then

(i) $P_8 \restriction K$ is bounded, and

(ii) P_8 is integrable on K.

The theorem is a consequence of (19).

(25) Let us consider subsets I, J of $\mathbb{R}$, a non empty, closed interval subset K of $\mathbb{R}$, elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function P_8 from $\mathbb{R}$ to $\mathbb{R}$, and an element E of L-Field. Suppose $x \in I$ and $y \in J$ and $\mathrm{dom}\, f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_8 = \mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y\rangle)$ and $E = K$. Then P_8 is E-measurable. The theorem is a consequence of (24).

(26) Let us consider subsets I, J of $\mathbb{R}$, a non empty, closed interval subset K of $\mathbb{R}$, elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of

$\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_8 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $y \in J$ and $\operatorname{dom} f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_8 = \text{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y \rangle)$. Then

(i) P_8 is integrable on L-Meas, and

(ii) $\int\limits_K P_8(x)dx = \int P_8 \, \mathrm{d}\, \text{L-Meas}$, and

(iii) $\int\limits_K P_8(x)dx = \int \text{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y \rangle) \, \mathrm{d}\, \text{L-Meas}$, and

(iv) $\int\limits_K P_8(x)dx = (\text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|))(\langle x, y \rangle)$.

The theorem is a consequence of (24).

(27) Let us consider non empty, closed interval subsets I, J of $\mathbb{R}$, a subset K of $\mathbb{R}$, an element z of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function P_9 from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and an element E of $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$. Suppose $z \in K$ and $\operatorname{dom} f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_9 = \text{ProjPMap2}(|\overline{\mathbb{R}}(g)|, z)$ and $E = I \times J$. Then P_9 is E-measurable. The theorem is a consequence of (20).

(28) Let us consider non empty, closed interval subsets I, J of $\mathbb{R}$, a subset K of $\mathbb{R}$, an element z of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_9 from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Suppose $z \in K$ and $\operatorname{dom} f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_9 = \text{ProjPMap2}(|\overline{\mathbb{R}}(g)|, z)$. Then

(i) P_9 is integrable on ProdMeas(L-Meas, L-Meas), and

(ii) $\int P_9 \, \mathrm{d}\, \text{ProdMeas}(\text{L-Meas}, \text{L-Meas}) = \int \text{ProjPMap2}(|\overline{\mathbb{R}}(g)|, z) \, \mathrm{d}\, \text{ProdMeas}(\text{L-Me}$
and

(iii) $\int P_9 \, \mathrm{d}\, \text{ProdMeas}(\text{L-Meas}, \text{L-Meas}) = (\text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), |\overline{\mathbb{R}}$

The theorem is a consequence of (20).

(29) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element E of

σ(MeasRect(σ(MeasRect(L-Field, L-Field)), L-Field)). Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $E = (I \times J) \times K$. Then g is E-measurable.

PROOF: For every real number r, $E \cap$LE-dom$(g, r) \in \sigma$(MeasRect(σ(MeasRect(L-Fiel
by [11, (17), (24)], (7). $\square$

Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a real number e. Now we state the propositions:

(30) Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then suppose $0 < e$. Then there exists a real number r such that

(i) $0 < r$, and

(ii) for every elements u_1, u_2 of $\mathbb{R} \times \mathbb{R}$ and for every real numbers x_1, y_1, x_2, y_2 such that $u_1 = \langle x_1, y_1 \rangle$ and $u_2 = \langle x_2, y_2 \rangle$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $u_1, u_2 \in I \times J$ for every element z of $\mathbb{R}$ such that $z \in K$ holds $|(\mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, u_2))(z) - (\mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, u_1))(z)| < e$.

PROOF: For every element x of $\mathbb{R} \times \mathbb{R}$ and for every element y of $\mathbb{R}$ such that $x \in I \times J$ and $y \in K$ holds $(\mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, x))(y) = |\overline{\mathbb{R}}(g)|(x, y)$ and $|\overline{\mathbb{R}}(g)|(x, y) = |g|(\langle x, y \rangle)$ by [7, (87)], [12, (12)]. Consider r being a real number such that $0 < r$ and for every real numbers x_1, x_2, y_1, y_2, z_1, z_2 such that $x_1, x_2 \in I$ and $y_1, y_2 \in J$ and $z_1, z_2 \in K$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $|z_2 - z_1| < r$ holds $||g|(\langle x_2, y_2, z_2 \rangle) - |g|(\langle x_1, y_1, z_1 \rangle)| < e$. $\square$

(31) Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then suppose $0 < e$. Then there exists a real number r such that

(i) $0 < r$, and

(ii) for every elements u_1, u_2 of $\mathbb{R} \times \mathbb{R}$ and for every real numbers x_1, y_1, x_2, y_2 such that $u_1 = \langle x_1, y_1 \rangle$ and $u_2 = \langle x_2, y_2 \rangle$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $u_1, u_2 \in I \times J$ for every element z of $\mathbb{R}$ such that $z \in K$ holds $|(\mathrm{ProjPMap1}(\overline{\mathbb{R}}(g), u_2))(z) - (\mathrm{ProjPMap1}(\overline{\mathbb{R}}(g), u_1))(z)| < e$.

The theorem is a consequence of (8).

Now we state the proposition:

(32) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and

a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

(i) Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$) is a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$, and

(ii) Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$)$\restriction(I \times J)$ is a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and

(iii) Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) is a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$, and

(iv) Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J)$ is a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$.

The theorem is a consequence of (26) and (22).

Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function F_4 from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. Now we state the propositions:

(33) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $F_4 =$ Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$)$\restriction(I \times J)$. Then F_4 is uniformly continuous on $I \times J$. The theorem is a consequence of (30), (19), and (24).

(34) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $F_4 =$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J)$. Then F_4 is uniformly continuous on $I \times J$. The theorem is a consequence of (31), (17), (21), and (22).

Now we state the proposition:

(35) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

(i) Integral1(ProdMeas(L-Meas, L-Meas), $|\overline{\mathbb{R}}(g)|$) is a function from $\mathbb{R}$ into $\mathbb{R}$, and

(ii) Integral1(ProdMeas(L-Meas, L-Meas), $|\overline{\mathbb{R}}(g)|$)$\restriction K$ is a partial function from $\mathbb{R}$ to $\mathbb{R}$, and

(iii) Integral1(ProdMeas(L-Meas, L-Meas), $\overline{\mathbb{R}}(g)$) is a function from $\mathbb{R}$ into $\mathbb{R}$, and

(iv) Integral1(ProdMeas(L-Meas, L-Meas), $\overline{\mathbb{R}}(g)$)$\restriction K$ is a partial function from $\mathbb{R}$ to $\mathbb{R}$.

The theorem is a consequence of (20), (28), (18), and (23).

Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function G_3 from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(36) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $G_3 = \text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), |\overline{\mathbb{R}}(g)|) \restriction K$. Then G_3 is continuous.

Proof: Consider a, b being real numbers such that $I = [a, b]$. Consider c, d being real numbers such that $J = [c, d]$. For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every real numbers z_1, z_2 such that $|z_2 - z_1| < r$ and z_1, $z_2 \in K$ for every real numbers x, y such that $x \in I$ and $y \in J$ holds $||g|(\langle x, y, z_2 \rangle) - |g|(\langle x, y, z_1 \rangle)| < e$. Set $R_{11} = \overline{\mathbb{R}}(g)$. For every elements x, y, z of $\mathbb{R}$ such that $x \in I$ and $y \in J$ and $z \in K$ holds $(\text{ProjPMap2}(|R_{11}|, z))(x, y) = |R_{11}|(\langle x, y \rangle, z)$ and $|R_{11}|(\langle x, y \rangle, z) = |g(\langle x, y, z \rangle)|$ and $|R_{11}|(\langle x, y \rangle, z) = |g|(\langle x, y, z \rangle)$ by [7, (87)], [12, (12)]. For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every elements z_1, z_2 of $\mathbb{R}$ such that $|z_2 - z_1| < r$ and z_1, $z_2 \in K$ for every elements x, y of $\mathbb{R}$ such that $x \in I$ and $y \in J$ holds $|(\text{ProjPMap1}(\text{ProjPMap2}(|R_{11}|, z_2), x))(y) - (\text{ProjPMap1}(\text{ProjPMap2}(|R_{11}|, z_1), x))(y)| < e$ by [11, (28)], [7, (87)], [12, (12)]. For every real numbers z_0, r such that $z_0 \in K$ and $0 < r$ there exists a real number s such that $0 < s$ and for every real number z_1 such that $z_1 \in K$ and $|z_1 - z_0| < s$ holds $|G_3(z_1) - G_3(z_0)| < r$ by [11, (30), (28)], (20), [11, (51), (53)]. $\square$

(37) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $G_3 = \text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g)) \restriction K$. Then G_3 is continuous.

Proof: Consider a, b being real numbers such that $I = [a, b]$. Consider c, d being real numbers such that $J = [c, d]$. For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every real numbers z_1, z_2 such that $|z_2 - z_1| < r$ and z_1, $z_2 \in K$ for every real numbers x, y such that $x \in I$ and $y \in J$ holds $|g(\langle x, y, z_2 \rangle) - g(\langle x, y, z_1 \rangle)| < e$. Set $R_{11} = \overline{\mathbb{R}}(g)$. For every elements x, y, z of $\mathbb{R}$ such that $x \in I$ and $y \in J$ and $z \in K$ holds $(\text{ProjPMap2}(R_{11}, z))(x, y) = R_{11}(\langle x, y \rangle, z)$ and $R_{11}(\langle x, y \rangle, z) = g(\langle x, y, z \rangle)$ and $R_{11}(\langle x, y \rangle, z) = g(\langle x, y, z \rangle)$ by [7, (87)]. For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every elements z_1, z_2 of $\mathbb{R}$ such that $|z_2 - z_1| < r$ and z_1, $z_2 \in K$ for every elements x, y of $\mathbb{R}$ such that $x \in I$ and $y \in J$ holds $|(\text{ProjPMap1}(\text{ProjPMap2}(R_{11}, z_2), x))(y) -$

(ProjPMap1(ProjPMap2(R_{11}, z_1), x))(y)| $< e$ by [11, (28)], [7, (87)], [12, (12)]. For every real numbers z_0, r such that $z_0 \in K$ and $0 < r$ there exists a real number s such that $0 < s$ and for every real number z_1 such that $z_1 \in K$ and $|z_1 - z_0| < s$ holds $|G_3(z_1) - G_3(z_0)| < r$ by [11, (30), (28)], (18), [11, (51), (53)]. □

Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(38) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$) is non-negative. The theorem is a consequence of (24) and (25).

(39) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then Integral1(ProdMeas(L-Meas, L-Meas), $|\overline{\mathbb{R}}(g)|$) is non-negative. The theorem is a consequence of (20) and (27).

Now we state the propositions:

(40) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, an element u of $\mathbb{R} \times \mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then (Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$))($u$) $< +\infty$. The theorem is a consequence of (32).

(41) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, an element z of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then (Integral1(ProdMeas(L-Meas, L-Meas), $|\overline{\mathbb{R}}(g)|$))($z$) $< +\infty$. The theorem is a consequence of (35).

(42) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element E of σ(MeasRect(L-Field, L-Field)). Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$) is E-measurable.

PROOF: Set $F =$ Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$). Set $I_1 = I \times J$. Reconsider $G =$ Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$)$\restriction I_1$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $R_4 = \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction I_1$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $G_1 = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. Reconsider $R_6 = R_4$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. G_1 is uniformly continuous on $I \times J$. R_6 is uniformly continuous on $I \times J$. F is non-negative. Reconsider $H = \mathbb{R} \times \mathbb{R}$ as an element of $\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field}))$. For every real number r, $H \cap \text{LE-dom}(F, r) \in \sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field}))$ by [16, (4)], [5, (49)], [16, (3)]. $\square$

Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(43) Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

(i) g is integrable on $\mathrm{ProdMeas}(\mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \text{L-Meas})$, and

(ii) for every element u of $\mathbb{R} \times \mathbb{R}$, $\mathrm{ProjPMap1}(\overline{\mathbb{R}}(g), u)$ is integrable on L-Meas, and

(iii) for every element U of $\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field}))$, $\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ is U-measurable, and

(iv) $\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ is integrable on $\mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas})$, and

(v) $\int g \,\mathrm{d}\, \mathrm{ProdMeas}(\mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \text{L-Meas}) = \int \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}($

PROOF: Set $F = \mathrm{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|)$. Set $I_1 = I \times J$. Reconsider $G = \mathrm{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|) \restriction I_1$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $R_4 = \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction I_1$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $A_1 = I \times J$ as an element of $\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field}))$. Reconsider $G_1 = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. Reconsider $R_6 = R_4$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. G_1 is uniformly continuous on $I \times J$. R_6 is uniformly continuous on $I \times J$. Reconsider $N_1 = (\mathbb{R} \times \mathbb{R}) \setminus A_1$ as an element of $\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field}))$. F is non-negative. Reconsider $H = \mathbb{R} \times \mathbb{R}$ as an element of $\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field}))$. F is H-measurable. Set $F_1 = F \restriction N_1$. For every object x such that $x \in \mathrm{dom}\, F_1$ holds $F_1(x) = 0$ by [5, (49)]. Reconsider $K_1 = (I \times J) \times K$ as an element of $\sigma(\mathrm{MeasRect}(\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$. g is K_1-

measurable. For every element x of $\mathbb{R}\times\mathbb{R}$, (Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$))$(x) < +\infty$. □

(44) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

- (i) for every element z of $\mathbb{R}$, ProjPMap2($\overline{\mathbb{R}}(g), z$) is integrable on ProdMeas(L-Meas and
- (ii) for every element V of L-Field, Integral1(ProdMeas(L-Meas, L-Meas), $\overline{\mathbb{R}}(g)$) is V-measurable, and
- (iii) Integral1(ProdMeas(L-Meas, L-Meas), $\overline{\mathbb{R}}(g)$) is integrable on L-Meas, and
- (iv) $\int g \, \mathrm{d}$ ProdMeas(ProdMeas(L-Meas, L-Meas), L-Meas) $= \int$ Integral1(ProdMeas(

The theorem is a consequence of (43) and (41).

Now we state the propositions:

(45) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, an element x of $\mathbb{R}$, and an element E of L-Field. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $x \in I$. Then ProjPMap1($|$Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$|, x$) is E-measurable.

PROOF: Set $F_4 =$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)$). Reconsider $G_4 =$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) as a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Reconsider $G = G_4 \restriction (I \times J)$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $F = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. F is uniformly continuous on $I \times J$. Set $F_5 =$ ProjPMap1($|F_4|, x$). Set $L_0 = F_5 \restriction J$. For every element t of $\mathbb{R}$ such that $t \in J$ holds $0 \leqslant L_0(t)$ by [5, (49)], [12, (14)]. Reconsider $H = \mathbb{R}$ as an element of L-Field. For every real number r, $H \cap$ LE-dom(F_5, r) $\in$ L-Field by [5, (49)], [16, (4), (3)]. □

(46) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

- (i) for every element x of $\mathbb{R}$, (Integral2(L-Meas, $|$Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$|$))$(x) < +\infty$, and
- (ii) for every element x of $\mathbb{R}$, ProjPMap1(Integral2(L-Meas, $\overline{\mathbb{R}}(g)$), x) is integrable on L-Meas.

PROOF: Reconsider $G_4 = \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ as a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Reconsider $G = G_4 \restriction (I \times J)$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $F = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. F is uniformly continuous on $I \times J$. For every element x of $\mathbb{R}$, $(\mathrm{Integral2}(\text{L-Meas}, |\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))|))(x) < +\infty$ by [11, (25)], [8, (5)], [9, (75)], [16, (5), (6)]. $\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ is integrable on ProdMeas(L-Mea
$\square$

(47) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, an element y of $\mathbb{R}$, and an element E of L-Field. Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $y \in J$. Then $\mathrm{ProjPMap2}(|\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))|, y)$ is E-measurable.

PROOF: Set $F_4 = \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$. Reconsider $G_4 = \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ as a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Reconsider $G = G_4 \restriction (I \times J)$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $F = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. F is uniformly continuous on $I \times J$. Set $F_6 = \mathrm{ProjPMap2}(|F_4|, y)$. Set $L_0 = F_6 \restriction I$. For every element t of $\mathbb{R}$ such that $t \in I$ holds $0 \leqslant L_0(t)$ by [5, (49)], [12, (14)]. For every element r of $\mathbb{R}$, $0_{\overline{\mathbb{R}}} \leqslant F_6(r)$ by [5, (49)]. Reconsider $H = \mathbb{R}$ as an element of L-Field. For every real number r, $H \cap \text{LE-dom}(F_6, r) \in$ L-Field by [16, (4)], [5, (49)], [16, (3)]. $\square$

(48) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

(i) for every element y of $\mathbb{R}$, $(\mathrm{Integral1}(\text{L-Meas}, |\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))|))(y) < +\infty$, and

(ii) for every element y of $\mathbb{R}$, $\mathrm{ProjPMap2}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y)$ is integrable on L-Meas.

PROOF: Reconsider $G_4 = \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ as a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Reconsider $G = G_4 \restriction (I \times J)$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $F = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. F is uniformly continuous on $I \times J$. For every element y of $\mathbb{R}$,

(Integral1(L-Meas, | Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)|))$(y) < +\infty$ by [11, (26)], [8, (5)], [9, (75)], [16, (5), (6)]. Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) is integrable on ProdMeas(L-Mea
□

(49) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element E of σ(MeasRect(L-Field, L-Field)). Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$) is E-measurable.

PROOF: Set $F =$ Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$). Set $F_0 = F \restriction (I \times J)$. Reconsider $G = F_0$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $G_1 = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. G_1 is uniformly continuous on $I \times J$. Reconsider $R_2 = \mathbb{R} \times \mathbb{R}$ as an element of σ(MeasRect(L-Field, L-Field)). F is non-negative. For every real number r, $R_2 \cap$LE-dom$(F, r) \in \sigma$(MeasRect(L-Field, L-F by [16, (4)], [5, (49)], [16, (3)]. □

(50) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element E of L-Field. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then Integral1(ProdMeas(L-Meas, L-Meas), $|\overline{\mathbb{R}}(g)|$) is E-measurable.

PROOF: Set $F =$ Integral1(ProdMeas(L-Meas, L-Meas), $|\overline{\mathbb{R}}(g)|$). Set $F_0 = F \restriction K$. Reconsider $G = F_0$ as a partial function from $\mathbb{R}$ to $\mathbb{R}$. $G \restriction K$ is bounded and G is integrable on K. Reconsider $R = \mathbb{R}$ as an element of L-Field. F is non-negative. For every real number r, $R \cap$ LE-dom$(F, r) \in$ L-Field by [16, (4)], [5, (49)], [16, (3)]. □

(51) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element x of $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

(i) ProjPMap1(Integral2(L-Meas, $\overline{\mathbb{R}}(g)$), x) is a function from $\mathbb{R}$ into $\mathbb{R}$, and

(ii) ProjPMap1(| Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)|, x) is a function from $\mathbb{R}$ into $\mathbb{R}$.

The theorem is a consequence of (32).

(52) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element y of $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

  (i) $\mathrm{ProjPMap2}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y)$ is a function from $\mathbb{R}$ into $\mathbb{R}$, and

  (ii) $\mathrm{ProjPMap2}(|\,\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))|, y)$ is a function from $\mathbb{R}$ into $\mathbb{R}$.

The theorem is a consequence of (32).

(53) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then $|\,\mathrm{Integral1}(\mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g))|$ is a function from $\mathbb{R}$ into $\mathbb{R}$. The theorem is a consequence of (35).

(54) Let us consider an element x of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} g$. Then $\int \mathrm{ProjPMap1}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x) \restriction \mathbb{R} \setminus J \, \mathrm{d}\,\text{L-Meas} = 0$.

(55) Let us consider an element y of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} g$. Then $\int \mathrm{ProjPMap2}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y) \restriction \mathbb{R} \setminus I \, \mathrm{d}\,\text{L-Meas} = 0$.

(56) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} g$. Then $\int \mathrm{Integral1}(\mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g)) \restriction \mathbb{R} \setminus K \, \mathrm{d}\,\text{L-Meas} = 0$.

(57) Let us consider an element x of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_1 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_1 = \mathrm{ProjPMap1}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x) \restriction J$. Then P_1 is continuous. The theorem is a consequence of (32) and (34).

(58) Let us consider an element y of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_2

from $\mathbb{R}$ to $\mathbb{R}$. Suppose $y \in J$ and $(I\times J)\times K = \operatorname{dom} f$ and f is continuous on $(I\times J)\times K$ and $f = g$ and $P_2 = \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y){\restriction}I$. Then P_2 is continuous. The theorem is a consequence of (32) and (34).

(59) Let us consider an element x of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R}\times\mathbb{R})\times\mathbb{R}$ to $\mathbb{R}$, and a partial function P_1 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $(I\times J)\times K = \operatorname{dom} f$ and f is continuous on $(I\times J)\times K$ and $f = g$ and $P_1 = \text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x){\restriction}J$. Then

(i) $P_1 \restriction J$ is bounded, and

(ii) P_1 is integrable on J.

The theorem is a consequence of (32) and (34).

(60) Let us consider an element y of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R}\times\mathbb{R})\times\mathbb{R}$ to $\mathbb{R}$, and a partial function P_2 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $y \in J$ and $(I\times J)\times K = \operatorname{dom} f$ and f is continuous on $(I\times J)\times K$ and $f = g$ and $P_2 = \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y){\restriction}I$. Then

(i) $P_2 \restriction I$ is bounded, and

(ii) P_2 is integrable on I.

The theorem is a consequence of (32) and (34).

(61) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function G_3 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $G_3 = \text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g)){\restriction}K$. Then

(i) $G_3 \restriction K$ is bounded, and

(ii) G_3 is integrable on K.

The theorem is a consequence of (37).

(62) Let us consider an element x of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R}\times\mathbb{R})\times\mathbb{R}$ to $\mathbb{R}$, and a partial function P_1 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $(I\times J)\times K = \operatorname{dom} f$ and f is continuous on

$(I \times J) \times K$ and $f = g$ and $P_1 = \text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x) \restriction J$. Then

(i) $\text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x) \restriction J$ is integrable on L-Meas, and

(ii) $\int\limits_J P_1(x)dx = \int \text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x) \restriction J \, \mathrm{d}\,\text{L-Meas}$, and

(iii) $\int\limits_J P_1(x)dx = \int \text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x) \, \mathrm{d}\,\text{L-Meas}$, and

(iv) $\int\limits_J P_1(x)dx = (\text{Integral2}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))))(x)$.

The theorem is a consequence of (46), (59), and (54).

(63) Let us consider an element y of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_2 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $y \in J$ and $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_2 = \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y) \restriction I$. Then

(i) $\text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y) \restriction I$ is integrable on L-Meas, and

(ii) $\int\limits_I P_2(x)dx = \int \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y) \restriction I \, \mathrm{d}\,\text{L-Meas}$, and

(iii) $\int\limits_I P_2(x)dx = \int \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y) \, \mathrm{d}\,\text{L-Meas}$, and

(iv) $\int\limits_I P_2(x)dx = (\text{Integral1}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))))(y)$.

The theorem is a consequence of (48), (60), and (55).

Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(64) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

(i) for every element U of L-Field, $\text{Integral2}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)))$ is U-measurable, and

(ii) Integral2(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)) is integrable on L-Meas, and

(iii) $\int$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) d ProdMeas(L-Meas, L-Meas) = $\int$ Integral2(L-Meas, and

(iv) $\int g$ d ProdMeas(ProdMeas(L-Meas, L-Meas), L-Meas) = $\int$ Integral2(L-Meas, Int and

(v) Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J)$ is integrable on ProdMeas(L-Meas, L-Meas), and

(vi) $\int$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J)$ d ProdMeas(L-Meas, L-Meas) = $\int$ Integral2(L J)) d L-Meas.

The theorem is a consequence of (32), (43), (46), (40), and (34).

(65) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

(i) for every element V of L-Field, Integral1(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)) is V-measurable, and

(ii) Integral1(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)) is integrable on L-Meas, and

(iii) $\int$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) d ProdMeas(L-Meas, L-Meas) = $\int$ Integral1(L-Meas, and

(iv) $\int g$ d ProdMeas(ProdMeas(L-Meas, L-Meas), L-Meas) = $\int$ Integral1(L-Meas, Int and

(v) $\int$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J)$ d ProdMeas(L-Meas, L-Meas) = $\int$ Integral1(L J)) d L-Meas.

The theorem is a consequence of (32), (43), (48), (40), and (34).

Now we state the propositions:

(66) Let us consider an element x of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_1 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and P_1 = ProjPMap1(Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J), x)$. Then

(i) P_1 is continuous, and

(ii) dom(ProjPMap1(Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J), x)) = J$, and

(iii) $P_1 \restriction J$ is bounded, and

(iv) P_1 is integrable on J, and

(v) $\int\limits_J P_1(x)dx = \int \mathrm{ProjPMap1}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))\restriction(I\times J), x)\, \mathrm{d}\, \text{L-Meas}$, and

(vi) $\int\limits_J P_1(x)dx = (\mathrm{Integral2}(\text{L-Meas}, \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))\restriction(I\times J)))(x)$, and

(vii) $\mathrm{ProjPMap1}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))\restriction(I\times J), x)$ is integrable on L-Meas.

The theorem is a consequence of (32) and (34).

(67) Let us consider an element y of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R}\times\mathbb{R})\times\mathbb{R}$ to $\mathbb{R}$, and a partial function P_2 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $y \in J$ and $(I\times J)\times K = \mathrm{dom}\, f$ and f is continuous on $(I\times J)\times K$ and $f = g$ and $P_2 = \mathrm{ProjPMap2}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))\restriction(I\times J), y)$. Then

(i) P_2 is continuous, and

(ii) $\mathrm{dom}(\mathrm{ProjPMap2}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))\restriction(I\times J), y)) = I$, and

(iii) $P_2\restriction I$ is bounded, and

(iv) P_2 is integrable on I, and

(v) $\int\limits_I P_2(x)dx = \int \mathrm{ProjPMap2}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))\restriction(I\times J), y)\, \mathrm{d}\, \text{L-Meas}$, and

(vi) $\int\limits_I P_2(x)dx = (\mathrm{Integral1}(\text{L-Meas}, \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))\restriction(I\times J)))(y)$, and

(vii) $\mathrm{ProjPMap2}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))\restriction(I\times J), y)$ is integrable on L-Meas.

The theorem is a consequence of (32) and (34).

(68) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R}\times\mathbb{R})\times\mathbb{R}$ to $\mathbb{R}$, and a partial function G_8 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $(I\times J)\times K = \mathrm{dom}\, f$ and f is continuous on $(I\times J)\times K$ and $f = g$ and $G_8 = \mathrm{Integral2}(\text{L-Meas}, \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))\restriction(I\times J))\restriction I$. Then

(i) $\mathrm{dom}\, G_8 = I$, and

(ii) G_8 is continuous, and

(iii) $G_8 \restriction I$ is bounded, and

(iv) G_8 is integrable on I, and

(v) $\mathrm{Integral2}(\text{L-Meas}, \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction (I \times J)) \restriction I$ is integrable on L-Meas, and

(vi) $\int \mathrm{Integral2}(\text{L-Meas}, \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction (I \times J)) \restriction I \, \mathrm{d}\, \text{L-Meas} = \int\limits_I G_8(x)dx$, and

(vii) $\int \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction (I \times J) \, \mathrm{d}\, \mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas}) = \int\limits_I G_8(x)dx$.

The theorem is a consequence of (32) and (34).

(69) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function G_7 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $G_7 = \mathrm{Integral1}(\text{L-Meas}, \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction (I \times J)) \restriction J$. Then

(i) $\mathrm{dom}\, G_7 = J$, and

(ii) G_7 is continuous, and

(iii) $G_7 \restriction J$ is bounded, and

(iv) G_7 is integrable on J, and

(v) $\mathrm{Integral1}(\text{L-Meas}, \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction (I \times J)) \restriction J$ is integrable on L-Meas, and

(vi) $\int \mathrm{Integral1}(\text{L-Meas}, \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction (I \times J)) \restriction J \, \mathrm{d}\, \text{L-Meas} = \int\limits_J G_7(x)dx$, and

(vii) $\int \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction (I \times J) \, \mathrm{d}\, \mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas}) = \int\limits_J G_7(x)dx$.

The theorem is a consequence of (32) and (34).

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Accepted June 18, 2024