FORMALIZED MATHEMATICS
Vol. 32, No. 1, pp. 111–120, 2024
DOI: 10.2478/forma-2024-0008
e-ISSN: 1898–9934

sciendo.com/journal/forma

Formal Proof of Transcendence of the Number e. Part I

Yasushige Watase
Suginami-ku Matsunoki 6, 3-21 Tokyo
Japan

Summary. In this article, we prove the transcendence of the number e using the Mizar formalism [4], along with Hurwitz's proof [11]. This subject has been implemented over the past decade in other theorem provers, such as HOL Light [6] and Coq [5]. This article prepares the necessary definitions and lemmas. The main body of the proof will be presented separately.
At the beginning, we formalize a lemma about algebraic numbers, namely for a polynomial over $\mathbb{Z}$ which has a root equal to e, if suppose e is algebraic, in another word e is not transcendental number as theorem. (see `E_TRANS2:41`). It corresponds to the equation (3) of [11].

Then, we define a polynomial transformation F. For a polynomial f over $\mathbb{Q}$ with degree r, we introduce a functor: (see `E_TRANS1:def 11`)

$$F : f(x) \mapsto f(x) + f'(x) + f''(x) + \cdots + f^{(r)}(x)$$

In Hurwitz's proof he defines $F(x) = f(x) + f'(x) + f''(x) + \cdots + f^{(r)}(x)$ as equation (1) in [11]. In the actual formalization for constructing F we generate a finite sequence of polynomials defined by $G = \{f^{(i)}(x)\}$, then F is formalized as the summation of it, namely $F = SumG$. Since higher order derivations for a ring have been implemented in [30], we are able to formalize i^{th} component of G. Then we apply the mean value theorem to $e^x F(x)$ on an interval and formalize the following equation quoted as equation (2) in [11]: (see `E_TRANS1:34`)

$$F(x) - e^x F(0) = -xe^{(1-\vartheta)x} f(\vartheta x).$$

The rest of the section is devoted to preparing lemmas to define the particular polynomial $f(x) = \frac{1}{(p-1)!} x^{p-1}(1-x)^p(2-x)^p \cdots (n-x)^p$ which play an important role of the main proof.

MSC: 11J81 68V20

Keywords: transcendental number; Hurwitz

MML identifier: `E_TRANS1`, version: `8.1.14 5.83.1471`

1. Preliminaries

From now on n, k denote natural numbers, L denotes a commutative ring, R denotes an integral domain, and x_0 denotes a positive real number.

The functor $\frac{1}{\exp_R}$ yielding a function from $\mathbb{R}$ into $\mathbb{R}$ is defined by the term

(Def. 1) $\frac{1}{\text{the function exp}}$.

One can verify that $\frac{1}{\exp_R}$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$ and the function exp is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$.

Now we state the propositions:

(1) Let us consider natural numbers n, m, and an element b of R. Then $(n \cdot m) \cdot b = n \cdot (m \cdot b)$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ $(\$_1 \cdot m) \cdot b = \$_1 \cdot (m \cdot b)$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [22, (15), (13)]. For every natural number n, $\mathcal{P}[n]$ from [2, Sch. 2]. □

(2) Let us consider finite sequences F, G of elements of $\mathbb{R}_{\rm F}$. Suppose $\operatorname{len} F = \operatorname{len} G$ and for every natural number i such that $i \in \operatorname{dom} F$ holds $F(i) \leqslant G(i)$. Then $\sum F \leqslant \sum G$.

Proof: $\sum F \leqslant \sum G$ by [3, (4), (59)], [2, (11)], [3, (5)]. □

Now we state the propositions:

(3) Generalization of ZMATRLIN:42:
Let us consider an ideal I of L, and a finite sequence F of elements of L. Suppose for every natural number i such that $i \in \operatorname{dom} F$ holds $F(i) \in I$. Then $\sum F \in I$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequence F of elements of L such that $\operatorname{len} F = \$_1$ and for every natural number i such that $i \in \operatorname{dom} F$ holds $F(i) \in I$ holds $\sum F \in I$. $\mathcal{P}[0]$ by [29, (43)], [1, (2)]. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [3, (4), (59)], [2, (11)], [3, (5)]. For every natural number n, $\mathcal{P}[n]$ from [2, Sch. 2]. □

(4) Let us consider an element a of L, and a non empty finite sequence p of elements of the carrier of L. Suppose for every natural number j such that $j \in \operatorname{dom} p$ holds $a \mid p_{/j}$. Then $a \mid \sum p$.

Proof: For every natural number i such that $i \in \operatorname{dom} p$ holds $p(i) \in \{a\}$–ideal by [13, (18)]. □

Let k, j be natural numbers. The functor $\eta_{k,j}$ yielding an element of $\mathbb{N}$ is defined by the term

(Def. 2) $\frac{k!}{(k -' j)!}$.

Now we state the proposition:

(5) Let us consider natural numbers k, j. If $j \leqslant k$, then $j! \cdot \binom{k}{j} = \eta_{k,j}$.

Let R be a ($\mathbb{Z}^{\mathrm{R}}$)-extending commutative ring and i be an integer. One can verify that $i(\in R)$ reduces to i.

Now we state the propositions:

(6) Let us consider a natural number n, and an element f of the carrier of Polynom-Ring $\mathbb{F}_{\mathbb{Q}}$. Then $n \cdot f = n(\in \mathbb{F}_{\mathbb{Q}}) \cdot f$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv \$_1 \cdot f = \$_1(\in \mathbb{F}_{\mathbb{Q}}) \cdot f$. $\mathcal{P}[0]$ by [17, (26)], [22, (12)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [10, (7)], [22, (13), (15)]. For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 2]. $\square$

(7) Let us consider a natural number n, and elements f, g of L. If $f \mid g$, then $f \mid n \cdot g$.

Let R be an add-associative, right zeroed, right complementable, distributive, non empty double loop structure and f be an element of the carrier of Polynom-Ring R. The functor $\curvearrowleft f$ yielding a polynomial over R is defined by the term

(Def. 3) f.

Let p be a polynomial over R. The functor $\hat{p}$ yielding an element of the carrier of Polynom-Ring R is defined by the term

(Def. 4) p.

Observe that there exists a finite sequence of elements of $\mathbb{F}_{\mathbb{Q}}$ which is $\mathbb{Z}$-valued and $\mathbf{0}.\mathbb{F}_{\mathbb{Q}}$ is $\mathbb{Z}$-valued and $\mathbf{1}.\mathbb{F}_{\mathbb{Q}}$ is $\mathbb{Z}$-valued and there exists a polynomial over $\mathbb{F}_{\mathbb{Q}}$ which is monic and $\mathbb{Z}$-valued.

Now we state the proposition:

(8) Let us consider an element f of the carrier of Polynom-Ring R. Then $\operatorname{rng} f = f^{\circ}(\operatorname{Support} f) \cup \{0_R\}$.
Proof: For every object y such that $y \in f^{\circ}(\mathbb{N} \setminus (\operatorname{Support} f))$ holds $y \in \{0_R\}$. For every object y such that $y \in \{0_R\}$ holds $y \in f^{\circ}(\mathbb{N} \setminus (\operatorname{Support} f))$ by [18, (8)]. $\square$

Let f be an element of the carrier of Polynom-Ring $\mathbb{F}_{\mathbb{Q}}$. The functor denomi-set(f) yielding a non empty, finite subset of $\mathbb{N}$ is defined by the term

(Def. 5) $(\mathrm{TRANQN})^{\circ}(\operatorname{rng} f)$.

The functor denomi-seq(f) yielding a non empty finite sequence of elements of $\mathbb{N}$ is defined by the term

(Def. 6) CFS(denomi-set(f)).

Now we state the propositions:

(9) Let us consider an element f of the carrier of Polynom-Ring $\mathbb{F}_{\mathbb{Q}}$. Then $\prod$ denomi-seq(f) is not zero.

(10) Let us consider an element f of the carrier of Polynom-Ring $\mathbb{F}_{\mathbb{Q}}$, and a natural number i. Then

(i) $\operatorname{den} f(i) \in$ denomi-set(f), and

(ii) there exists an integer z such that $z \cdot (\operatorname{den} f(i)) = \prod$ denomi-seq(f).

(11) Let us consider fields K, L, and an element w of L. Suppose K is a subring of L and w is integral over K. Then AnnPoly(w, K) is maximal.

(12) Let us consider an element f of Polynom-Ring $\mathbb{F}_{\mathbb{Q}}$, and a non zero natural number n. If f is irreducible, then $n \cdot f$ is irreducible. The theorem is a consequence of (7) and (6).

(13) Let us consider an element x of $\mathbb{R}_{\mathrm{F}}$. Suppose x is irrational. Let us consider a non zero polynomial g over $\mathbb{F}_{\mathbb{Q}}$. If ExtEval$(g, x) = 0$, then $\deg(g) \geqslant 2$.

2. Some Properties of Algebraic Numbers

Now we state the proposition:

(14) Let us consider a polynomial g over $\mathbb{F}_{\mathbb{Q}}$. Suppose $\deg(g) \geqslant 2$ and $\hat{g}$ is irreducible. Then $g(0) \neq 0_{\mathbb{F}_{\mathbb{Q}}}$.

Proof: Reconsider $g_1 =$ NormPoly $\hat{g}$ as a polynomial over $\mathbb{F}_{\mathbb{Q}}$. $g_1(0) \neq 0_{\mathbb{F}_{\mathbb{Q}}}$ by [17, (31)], [21, (50)], [17, (40)], [25, (30), (37)]. □

3. Constructing Polynomial Transformation $'F'$

Now we state the propositions:

(15) Let us consider a non degenerated integral domain L, a non zero natural number n, and a non zero element a of L. If char$(L) = 0$, then $n \cdot a \neq 0_L$.

(16) Let us consider a commutative ring R, an element f of the carrier of Polynom-Ring R, and a natural number i. Suppose $i \geqslant 1$ and the length of f is at most i and $f(i-1) \neq 0_R$. Then $\operatorname{len} f = i$.

Proof: For every natural number i such that $i \geqslant 1$ and the length of f is at most i and $f(i-1) \neq 0_R$ holds $\operatorname{len} f = i$ by [2, (13)], [18, (8)]. □

(17) Let us consider an integral domain R, and an element f of the carrier of Polynom-Ring R. Suppose $\operatorname{len} f > 1$ and char$(R) = 0$. Then $\operatorname{len}(\mathrm{Der1}(R))(f) = \operatorname{len} f - 1$.

Proof: Reconsider $l_1 = \operatorname{len} f - 1$ as a natural number. For every natural number i such that $i \geqslant l_1$ holds $(\mathrm{Der1}(R))(f)(i) = 0_R$ by [18, (8)]. □

(18) Let us consider an integral domain L, a derivation D of L, an element f of the carrier of L, and natural numbers j, n. Then $D^n(j \cdot f) = j \cdot D^n(f)$.
PROOF: For every element f of the carrier of L and for every natural numbers j, n, $D^n(j \cdot f) = j \cdot D^n(f)$ by [19, (18)], [30, (9), (6)]. □

(19) Let us consider a natural number k, and an element f of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Suppose $(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^1(f^1) = 1_{\text{Polynom-Ring}\,\mathbb{Z}^{\mathrm{R}}}$. Let us consider a natural number j. Suppose $1 \leqslant j \leqslant k$. Then $(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^j(f^k) = \eta_{k,j} \cdot f^{k-'j}$.
PROOF: Set $D = \mathrm{Der1}(\mathbb{Z}^{\mathrm{R}})$. Define $\mathcal{P}$[natural number] $\equiv$ for every natural number j such that $1 \leqslant j \leqslant \$_1$ holds $D^j(f^{\$_1}) = \eta_{\$_1,j} \cdot f^{\$_1-'j}$. For every natural number k such that for every natural number n such that $n < k$ holds $\mathcal{P}[n]$ holds $\mathcal{P}[k]$ by [19, (19)], [30, (7)], [15, (15)], [19, (20)]. For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 4]. □

(20) Let us consider a natural number k, and an element f of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Suppose $(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^1(f^1) = 1_{\text{Polynom-Ring}\,\mathbb{Z}^{\mathrm{R}}}$. Then $(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^k(f^k) = k! \cdot (1_{\text{Polynom-Ring}\,\mathbb{Z}^{\mathrm{R}}})$. The theorem is a consequence of (19).

(21) Let us consider a natural number j. Suppose $j > k$. Let us consider an element f of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Suppose $(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^1(f^1) = 1_{\text{Polynom-Ring}\,\mathbb{Z}^{\mathrm{R}}}$. Then $(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^j(f^k) = 0_{\text{Polynom-Ring}\,\mathbb{Z}^{\mathrm{R}}}$.
PROOF: Set $L =$ Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Set $D = \mathrm{Der1}(\mathbb{Z}^{\mathrm{R}})$. For every element f of the carrier of L such that $D^1(f^1) = 1_{\text{Polynom-Ring}\,\mathbb{Z}^{\mathrm{R}}}$ holds $D^j(f^k) = 0_L$ by [26, (3)], [2, (14)], [19, (20)], [9, (15)]. □

(22) Let us consider an integral domain R, an element f of the carrier of Polynom-Ring R, a natural number k, and a natural number i. Then $(\mathrm{Der1}(R))^k(f)(i) = \eta_{i+k,k} \cdot f(i+k)$.
PROOF: Set $D = \mathrm{Der1}(R)$. Define $\mathcal{P}$[natural number] $\equiv$ for every natural number i for every natural number i, $D^{\$_1}(f)(i) = \eta_{i+\$_1,\$_1} \cdot f(i+\$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [30, (9)], (1). For every natural number i, $D^0(f)(i) = \eta_{i+0,0} \cdot f(i+0)$ by [19, (18)], [22, (13)]. For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 2]. □

(23) Let us consider a function h from R into R, and a finite sequence s of elements of the carrier of R. If h is additive, then $h(\sum s) = \sum h \cdot s$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every function h from R into R for every finite sequence s of elements of R such that $\operatorname{len} s = \$_1$ and h is additive holds $h(\sum s) = \sum h \cdot s$. $\mathcal{P}[0]$ by [29, (75)], [13, (6)], [31, (27)], [29, (43)]. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [8, (3)], [3, (4), (59)], [2, (11)]. For every natural number n, $\mathcal{P}[n]$ from [2,

Sch. 2]. □

(24) Let us consider an integral domain R, an element f of the carrier of Polynom-Ring R, and a natural number j. Suppose $\operatorname{len} f > j$ and $\operatorname{char}(R) = 0$. Then $\operatorname{len} (\mathrm{Der1}(R))^j(f) = \operatorname{len} f - j$.

PROOF: Reconsider $l_1 = \operatorname{len} f - 1$ as a natural number. Reconsider $l_3 = \operatorname{len} f - j$ as a natural number. Reconsider $l_4 = l_3 - 1$ as a natural number. Reconsider $l_5 = \binom{l_4+j}{l_4} \cdot (j!)$ as a natural number. $\eta_{l_4+j,j} = \binom{l_4+j}{j} \cdot (j!)$. $(\mathrm{Der1}(R))^j(f)(l_4) = l_5 \cdot f(l_1)$. For every natural number i such that $i \geqslant l_3$ holds $(\mathrm{Der1}(R))^j(f)(i) = 0_R$ by [18, (8)], (22). □

Let p be an element of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. The functor ${}^{@}p$ yielding an element of the carrier of Polynom-Ring $\mathbb{R}_{\mathrm{F}}$ is defined by the term

(Def. 7) p.

Let F be a finite sequence of elements of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. The functor ${}^{@}F$ yielding a finite sequence of elements of the carrier of Polynom-Ring $\mathbb{R}_{\mathrm{F}}$ is defined by

(Def. 8) $\operatorname{dom} it = \operatorname{dom} F$ and for every natural number i such that $i \in \operatorname{dom} F$ holds $it(i) = {}^{@}F_{/i}$.

Let L be a commutative ring, F be a finite sequence of elements of the carrier of Polynom-Ring L, and x be an element of L. The functor $\mathrm{eval}(F, x)$ yielding a finite sequence of elements of the carrier of L is defined by

(Def. 9) $\operatorname{dom} it = \operatorname{dom} F$ and for every natural number i such that $i \in \operatorname{dom} F$ holds $it(i) = \mathrm{eval}(\curvearrowleft F_{/i}, x)$.

Now we state the propositions:

(25) Let us consider a natural number N_0, a commutative ring L, a finite sequence F of elements of the carrier of Polynom-Ring L, and an element x of L. Suppose $\operatorname{len} F = N_0 + 1$. Then $\mathrm{eval}(F, x) = \mathrm{eval}(F \upharpoonright N_0, x) \frown \langle \mathrm{eval}(\curvearrowleft F_{/\operatorname{len} F}, x) \rangle$.

PROOF: For every natural number k such that $1 \leqslant k \leqslant \operatorname{len} \mathrm{eval}(F, x)$ holds $(\mathrm{eval}(F, x))(k) = (\mathrm{eval}(F \upharpoonright N_0, x) \frown \langle \mathrm{eval}(\curvearrowleft F_{/\operatorname{len} F}, x) \rangle)(k)$ by [3, (9)], [27, (18)], [8, (47)], [3, (6), (4)]. □

(26) Let us consider a commutative ring L, a finite sequence F of elements of the carrier of Polynom-Ring L, and an element x of L. Then $\mathrm{eval}(\curvearrowleft \sum F, x) = \sum \mathrm{eval}(F, x)$. The theorem is a consequence of (25).

(27) Let us consider elements p, q of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Then

(i) ${}^{@}(p + q) = {}^{@}p + {}^{@}q$, and

(ii) ${}^{@}p \cdot q = ({}^{@}p) \cdot ({}^{@}q)$.

Let f be an element of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. The functor $\mathcal{G}\,f$ yielding a finite sequence of elements of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$ is defined by

(Def. 10) len it = len f and for every natural number i such that $i \in \mathrm{dom}\, it$ holds $it(i) = (\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^{i-'1}(f)$.

Now we state the propositions:

(28) Let us consider a finite sequence F of elements of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$, an element x of $\mathbb{Z}^{\mathrm{R}}$, and an element x_1 of $\mathbb{R}_{\mathrm{F}}$. If $x = x_1$, then $\mathrm{eval}({}^{@}F, x_1) = \mathrm{eval}(F, x)$.
PROOF: For every natural number i such that $i \in \mathrm{dom}(\mathrm{eval}({}^{@}F, x_1))$ holds $(\mathrm{eval}({}^{@}F, x_1))(i) = (\mathrm{eval}(F, x))(i)$ by [8, (3)], [23, (27)]. $\square$

(29) Let us consider a finite sequence F of elements of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Then $\sum {}^{@}F = {}^{@}\sum F$. The theorem is a consequence of (27).

(30) Let us consider an element x_0 of $\mathbb{Z}^{\mathrm{R}}$, an element x of $\mathbb{R}_{\mathrm{F}}$, and a finite sequence F of elements of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Suppose $x = x_0$. Then $(\mathrm{Eval}(\curvearrowleft{}^{@}\sum F))(x) = \sum \mathrm{eval}(F, x_0)$. The theorem is a consequence of (28), (29), and (26).

The Definition below corresponds to the Transformation (1) in [**?**]

Let f be an element of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. The functor $\mathcal{F}\,f$ yielding a function from $\mathbb{R}$ into $\mathbb{R}$ is defined by the term

(Def. 11) $\mathrm{Eval}(\curvearrowleft{}^{@}\sum \mathcal{G}\,f)$.

4. CONSTRUCT THE EQUATION (2) IN [11]

Now we state the proposition:

(31) Let us consider an element p of the carrier of Polynom-Ring $\mathbb{R}_{\mathrm{F}}$. Then $\mathrm{Eval}(\curvearrowleft p)\,`| = \mathrm{Eval}(\curvearrowleft(\mathrm{Der1}(\mathbb{R}_{\mathrm{F}}))(p))$.
PROOF: Set $D_1 = \mathrm{Der1}(\mathbb{R}_{\mathrm{F}})$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every element p of the carrier of Polynom-Ring $\mathbb{R}_{\mathrm{F}}$ such that $\mathrm{len}\curvearrowleft p \leqslant \$_1$ holds $\mathrm{Eval}(\curvearrowleft p)\,`| = \mathrm{Eval}(\curvearrowleft D_1(p))$. $\mathcal{P}[0]$ by [16, (5)], [24, (58)], [12, (52), (54)]. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$ by [12, (36)], [16, (3)], [12, (37), (55), (14)]. $\mathcal{P}[n]$ from [2, Sch. 2]. $\square$

Let f be an element of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. The functor $\Phi(f)$ yielding a function from $\mathbb{R}$ into $\mathbb{R}$ is defined by the term

(Def. 12) $\frac{1}{\exp_R} \cdot \mathcal{F}\,f$.

Note that $\mathcal{F}\,f$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$.

Let us consider an element f of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Now we state the propositions:

(32) $(\frac{1}{\exp_R} \cdot \mathcal{F} f){\restriction}[0, x_0]$ is continuous.

PROOF: Set $f_1 = \frac{1}{\text{the function exp}}$. Set $f_2 = \mathcal{F} f$. For every real number r such that $r \in \operatorname{dom}((f_1 \cdot f_2){\restriction}[0, x_0])$ holds $(f_1 \cdot f_2){\restriction}[0, x_0]$ is continuous in r by [14, (45)], [20, (7)]. □

(33) $\frac{1}{\exp_R} \cdot \mathcal{F} f$ is differentiable on $]0, x_0[$.

Now we state the proposition:

(34) THE FOLLOWING THEOREM CORRESPONDS TO THE EQUATION (2) IN [**?**].:

Let us consider an element f of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$, and a positive real number x_0. Suppose $\operatorname{len} f > 0$. Then there exists a real number s such that

(i) $0 < s < 1$, and

(ii) $(\mathcal{F} f)(x_0) - (\text{the function exp})(x_0) \cdot (\mathcal{F} f)(0) = -x_0 \cdot (\text{the function exp})(x_0 \cdot (1 -$

Now we state the proposition:

(35) RING EXTENDED VERSION OF FIELD_13:13.:

Let us consider an integral domain F, a ring extension E of F, a polynomial p over F, a polynomial q over E, an element a of F, and an element b of E. If $p = q$ and $a = b$, then $a \cdot p = b \cdot q$.

Now we state the propositions:

(36) RING EXTENSION VERSION OF REALALG3:16.:

Let us consider an integral domain F, a domain ring extension E of F, a polynomial p over F, an element a of F, and elements x, b of E. If $b = a$, then $\operatorname{ExtEval}(a \cdot p, x) = b \cdot (\operatorname{ExtEval}(p, x))$. The theorem is a consequence of (35).

(37) Let us consider a non degenerated commutative ring L, a non empty finite sequence F of elements of the carrier of Polynom-Ring L, and an element x of L. Then $\operatorname{eval}(\curvearrowleft\prod F, x) = \prod \operatorname{eval}(F, x)$.

PROOF: For every non zero natural number k such that $\operatorname{len} F = k$ holds $\operatorname{eval}(\curvearrowleft\prod F, x) = \prod \operatorname{eval}(F, x)$ by [8, (3)], [3, (40)], [28, (9)], [7, (19)]. □

(38) Let us consider a non empty finite sequence F of elements of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$, and an element x of $\mathbb{R}_{\mathrm{F}}$. Then $\operatorname{eval}(\curvearrowleft{}^{@}\prod F, x) = \prod \operatorname{eval}({}^{@}F, x)$. The theorem is a consequence of (37).

REFERENCES

[1] Jonathan Backer, Piotr Rudnicki, and Christoph Schwarzweller. Ring ideals. *Formalized Mathematics*, 9(**3**):565–582, 2001.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[4] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[5] Sophie Bernard, Yves Bertot, Laurence Rideau, and Pierre-Yves Strub. Formal proofs of transcendence for e and π as an application of multivariate and symmetric polynomials. In Jeremy Avigad and Adam Chlipala, editors, *Proceedings of the 5th ACM SIGPLAN Conference on Certified Programs and Proofs*, pages 76–87. ACM, 2016. doi:10.1145/2854065.2854072.

[6] Jesse Bingham. Formalizing a proof that e is transcendental. *Journal of Formalized Reasoning*, 4:71–84, 2011.

[7] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.

[9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[10] Ewa Grądzka. The algebra of polynomials. *Formalized Mathematics*, 9(**3**):637–643, 2001.

[11] Adolf Hurwitz. Beweis der Transcendenz der Zahl e. *Mathematische Annalen*, 43:220–221, 1893.

[12] Artur Korniłowicz. Differentiability of polynomials over reals. *Formalized Mathematics*, 25(**1**):31–37, 2017. doi:10.1515/forma-2017-0002.

[13] Artur Korniłowicz and Christoph Schwarzweller. The first isomorphism theorem and other properties of rings. *Formalized Mathematics*, 22(**4**):291–301, 2014. doi:10.2478/forma-2014-0029.

[14] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Formalized Mathematics*, 1(**4**):703–709, 1990.

[15] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(**5**):887–890, 1990.

[16] Robert Milewski. The evaluation of polynomials. *Formalized Mathematics*, 9(**2**):391–395, 2001.

[17] Robert Milewski. Fundamental theorem of algebra. *Formalized Mathematics*, 9(**3**):461–470, 2001.

[18] Michał Muzalewski and Lesław W. Szczerba. Construction of finite sequences over ring and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):97–104, 1991.

[19] Karol Pąk. Eigenvalues of a linear transformation. *Formalized Mathematics*, 16(**4**):289–295, 2008. doi:10.2478/v10037-008-0035-x.

[20] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(**4**):787–791, 1990.

[21] Piotr Rudnicki. Little Bezout theorem (factor theorem). *Formalized Mathematics*, 12(**1**): 49–58, 2004.

[22] Christoph Schwarzweller. The binomial theorem for algebraic structures. *Formalized Mathematics*, 9(**3**):559–564, 2001.

[23] Christoph Schwarzweller. Field extensions and Kronecker's construction. *Formalized Mathematics*, 27(**3**):229–235, 2019. doi:10.2478/forma-2019-0022.

[24] Christoph Schwarzweller and Agnieszka Rowińska-Schwarzweller. Simple extensions. *Formalized Mathematics*, 31(1):287–298, 2023. doi:10.2478/forma-2023-0023.

[25] Christoph Schwarzweller, Artur Korniłowicz, and Agnieszka Rowińska-Schwarzweller. Some algebraic properties of polynomial rings. *Formalized Mathematics*, 24(**3**):227–237, 2016. doi:10.1515/forma-2016-0019.

[26] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.

[27] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *Formalized Mathematics*, 1(**3**):569–573, 1990.

[28] Wojciech A. Trybulec. Lattice of subgroups of a group. Frattini subgroup. *Formalized Mathematics*, 2(**1**):41–47, 1991.

[29] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[30] Yasushige Watase. Derivation of commutative rings and the Leibniz formula for power of derivation. *Formalized Mathematics*, 29(**1**):1–8, 2021. doi:10.2478/forma-2021-0001.

[31] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1 (**1**):73–83, 1990.

Accepted November 17, 2024