Extended Real-Valued Double Sequence and Its Convergence

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Summary. In this article we introduce the convergence of extended real-valued double sequences [16], [17]. It is similar to our previous articles [15], [10]. In addition, we also prove Fatou’s lemma and the monotone convergence theorem for double sequences.

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The notation and terminology used in this paper have been introduced in the following articles: [5], [21], [15], [10], [12], [6], [7], [22], [13], [11], [14], [1], [2], [8], [18], [24], [25], [26], [20], [23], [3], [4], and [9].

1. Preliminaries

Let $X$ be a non empty set. One can verify that there exists a function from $X$ into $\mathbb{R}$ which is non-negative and non-positive and there exists a function from $X$ into $\mathbb{R}$ which is without $-\infty$, without $+\infty$, non-negative, and non-positive and every function from $X$ into $\mathbb{R}$ which is non-negative is also without $-\infty$ and every function from $X$ into $\mathbb{R}$ which is non-positive is also without $+\infty$ and there exists a without $+\infty$ function from $X$ into $\mathbb{R}$ which is without $-\infty$.

Let $f$ be a function from $X$ into $\mathbb{R}$. Let us observe that the functor $-f$ yields a function from $X$ into $\mathbb{R}$. Let $f$ be a without $-\infty$ function from $X$ into $\mathbb{R}$. Note that $-f$ is without $+\infty$.

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Let $f$ be a without $+\infty$ function from $X$ into $\mathbb{R}$. Let us observe that $-f$ is without $-\infty$.

Let $f$ be a non-negative function from $X$ into $\mathbb{R}$. Note that $-f$ is non-positive.

Let $f$ be a non-positive function from $X$ into $\mathbb{R}$. Let us observe that $-f$ is non-negative.

Let $A, B$ be non-empty sets and $f$ be a without $-\infty$ function from $A \times B$ into $\mathbb{R}$. Let us observe that $f^T$ is without $-\infty$.

Let $f$ be a without $+\infty$ function from $A \times B$ into $\mathbb{R}$. One can verify that $f^T$ is without $+\infty$.

Let $f$ be a non-negative function from $A \times B$ into $\mathbb{R}$. One can check that $f^T$ is non-negative.

Let $f$ be a non-positive function from $A \times B$ into $\mathbb{R}$. Note that $f^T$ is non-positive.

Now we state the propositions:

(1) Let us consider a sequence $s$ of extended reals. Then $(\sum_{\alpha=0}^{\kappa}(-s)(\alpha))_{\kappa \in \mathbb{N}} = - (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$.

Proof: Define $Q[\text{natural number}] \equiv (- (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}})(S_1) = - (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(S_1)$. For every natural number $n$, $Q[n]$ from [1] Sch. 2. Define $P[\text{natural number}] \equiv (\sum_{\alpha=0}^{\kappa}(-s)(\alpha))_{\kappa \in \mathbb{N}}(S_1) = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(S_1)$. For every natural number $n$ such that $P[n]$ holds $P[n+1]$. For every natural number $n$, $P[n]$ from [1] Sch. 2. □

(2) Let us consider a non-empty set $X$, and a partial function $f$ from $X$ to $\mathbb{R}$. Then $-f = f$.

(3) Let us consider non-empty sets $X, Y$, and a function $f$ from $X \times Y$ into $\mathbb{R}$. Then $(-f)^T = -f^T$.

Let $s$ be a non-negative sequence of extended reals. One can verify that $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$ is non-negative.

Let $s$ be a non-positive sequence of extended reals. Let us observe that $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$ is non-positive.

Now we state the propositions:

(4) Let us consider a non-negative sequence $s$ of extended reals, and a natural number $m$. Then $s(m) \leq (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m)$.

Proof: Define $P[\text{natural number}] \equiv s(S_1) \leq (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(S_1)$. For every natural number $k$ such that $P[k]$ holds $P[k+1]$ by [4] (51)]. For every natural number $k$, $P[k]$ from [1] Sch. 2. □

(5) Let us consider a non-positive sequence $s$ of extended reals, and a natural number $m$. Then $s(m) \geq (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m)$. The theorem is a consequence of (4), (1), and (2).
Let us consider a non empty set $X$. Then every without $-\infty$, without $+\infty$ function from $X$ into $\mathbb{R}$ is a function from $X$ into $\mathbb{R}$.

Let $X$ be a non empty set and $f_1, f_2$ be without $-\infty$ functions from $X$ into $\mathbb{R}$. One can verify that the functor $f_1 + f_2$ yields a without $-\infty$ function from $X$ into $\mathbb{R}$. Let $f_1, f_2$ be without $+\infty$ functions from $X$ into $\mathbb{R}$. One can verify that the functor $f_1 + f_2$ yields a without $+\infty$ function from $X$ into $\mathbb{R}$. Let $f_1$ be a without $-\infty$ function from $X$ into $\mathbb{R}$ and $f_2$ be a without $+\infty$ function from $X$ into $\mathbb{R}$. Let us observe that the functor $f_1 - f_2$ yields a without $-\infty$ function from $X$ into $\mathbb{R}$. Let $f_1$ be a without $+\infty$ function from $X$ into $\mathbb{R}$ and $f_2$ be a without $-\infty$ function from $X$ into $\mathbb{R}$. Observe that the functor $f_1 - f_2$ yields a without $+\infty$ function from $X$ into $\mathbb{R}$. Now we state the propositions:

Let us consider a non empty set $X$, an element $x$ of $X$, and functions $f_1, f_2$ from $X$ into $\mathbb{R}$. Then

(i) if $f_1$ is without $-\infty$ and $f_2$ is without $-\infty$, then $(f_1 + f_2)(x) = f_1(x) + f_2(x)$, and

(ii) if $f_1$ is without $+\infty$ and $f_2$ is without $+\infty$, then $(f_1 + f_2)(x) = f_1(x) + f_2(x)$, and

(iii) if $f_1$ is without $-\infty$ and $f_2$ is without $+\infty$, then $(f_1 - f_2)(x) = f_1(x) - f_2(x)$, and

(iv) if $f_1$ is without $+\infty$ and $f_2$ is without $-\infty$, then $(f_1 - f_2)(x) = f_1(x) - f_2(x)$.

Let us consider a non empty set $X$, and without $-\infty$ functions $f_1, f_2$ from $X$ into $\mathbb{R}$. Then

(i) $f_1 + f_2 = f_1 - -f_2$, and

(ii) $-(f_1 + f_2) = -f_1 - f_2$.

The theorem is a consequence of (7).

Let us consider a non empty set $X$, and without $+\infty$ functions $f_1, f_2$ from $X$ into $\mathbb{R}$. Then

(i) $f_1 + f_2 = f_1 - -f_2$, and

(ii) $-(f_1 + f_2) = -f_1 - f_2$.

The theorem is a consequence of (7).

Let us consider a non empty set $X$, a without $-\infty$ function $f_1$ from $X$ into $\mathbb{R}$, and a without $+\infty$ function $f_2$ from $X$ into $\mathbb{R}$. Then

(i) $f_1 - f_2 = f_1 + -f_2$, and

(ii) $f_2 - f_1 = f_2 + -f_1$, and

(iii) $-(f_1 - f_2) = -f_1 + f_2$, and
(iv) \(-(f_2 - f_1) = -f_2 + f_1.\)

The theorem is a consequence of (8), (2), and (9).

Let \(f\) be a function from \(\mathbb{N} \times \mathbb{N}\) into \(\mathbb{R}\) and \(n, m\) be natural numbers. One can check that the functor \(f(n, m)\) yields an element of \(\mathbb{R}\). Now we state the propositions:

(11) Let us consider without \(-\infty\) functions \(f_1, f_2\) from \(\mathbb{N} \times \mathbb{N}\) into \(\mathbb{R}\), and natural numbers \(n, m\). Then \((f_1 + f_2)(n, m) = f_1(n, m) + f_2(n, m)\). The theorem is a consequence of (7).

(12) Let us consider without \(+\infty\) functions \(f_1, f_2\) from \(\mathbb{N} \times \mathbb{N}\) into \(\mathbb{R}\), and natural numbers \(n, m\). Then \((f_1 + f_2)(n, m) = f_1(n, m) + f_2(n, m)\). The theorem is a consequence of (7).

(13) Let us consider a without \(-\infty\) function \(f_1\) from \(\mathbb{N} \times \mathbb{N}\) into \(\mathbb{R}\), a without \(+\infty\) function \(f_2\) from \(\mathbb{N} \times \mathbb{N}\) into \(\mathbb{R}\), and natural numbers \(n, m\). Then

\[
\begin{align*}
(i) \quad (f_1 - f_2)(n, m) &= f_1(n, m) - f_2(n, m), \\
(ii) \quad (f_2 - f_1)(n, m) &= f_2(n, m) - f_1(n, m).
\end{align*}
\]

The theorem is a consequence of (7).

(14) Let us consider non empty sets \(X, Y\), and without \(-\infty\) functions \(f_1, f_2\) from \(X \times Y\) into \(\mathbb{R}\). Then \((f_1 + f_2)^T = f_1^T + f_2^T\). The theorem is a consequence of (7).

(15) Let us consider non empty sets \(X, Y\), and without \(+\infty\) functions \(f_1, f_2\) from \(X \times Y\) into \(\mathbb{R}\). Then \((f_1 + f_2)^T = f_1^T + f_2^T\). The theorem is a consequence of (7).

(16) Let us consider non empty sets \(X, Y\), a without \(-\infty\) function \(f_1\) from \(X \times Y\) into \(\mathbb{R}\), and a without \(+\infty\) function \(f_2\) from \(X \times Y\) into \(\mathbb{R}\). Then

\[
\begin{align*}
(i) \quad (f_1 - f_2)^T &= f_1^T - f_2^T, \\
(ii) \quad (f_2 - f_1)^T &= f_2^T - f_1^T.
\end{align*}
\]

The theorem is a consequence of (7).

One can verify that every sequence of extended reals which is convergent to \(+\infty\) is also convergent and every sequence of extended reals which is convergent to \(-\infty\) is also convergent and every sequence of extended reals which is convergent to a finite limit is also convergent and there exists a sequence of extended reals which is convergent and there exists a without \(-\infty\) sequence of extended reals which is convergent and there exists a without \(+\infty\) sequence of extended reals which is convergent.

Now we state the proposition:

(17) Let us consider a convergent sequence \(s\) of extended reals. Then
(i) \( s \) is convergent to a finite limit iff \( -s \) is convergent to a finite limit, and
(ii) \( s \) is convergent to \(+\infty\) iff \( -s \) is convergent to \(-\infty\), and
(iii) \( s \) is convergent to \(-\infty\) iff \( -s \) is convergent to \(+\infty\), and
(iv) \(-s\) is convergent, and
(v) \( \lim(-s) = -\lim s \).

The theorem is a consequence of (2).

Let us consider without \(-\infty\) sequences \(s_1, s_2\) of extended reals. Now we state the propositions:

(18) Suppose \( s_1 \) is convergent to \(+\infty\) and \( s_2 \) is convergent to \(+\infty\). Then
(i) \( s_1 + s_2 \) is convergent to \(+\infty\) and convergent, and
(ii) \( \lim(s_1 + s_2) = +\infty \).

The theorem is a consequence of (7).

(19) Suppose \( s_1 \) is convergent to \(+\infty\) and \( s_2 \) is convergent to a finite limit. Then
(i) \( s_1 + s_2 \) is convergent to \(+\infty\) and convergent, and
(ii) \( \lim(s_1 + s_2) = +\infty \).

The theorem is a consequence of (7).

Now we state the proposition:

(20) Let us consider without \(+\infty\) sequences \(s_1, s_2\) of extended reals. Suppose \( s_1 \) is convergent to \(+\infty\) and \( s_2 \) is convergent to a finite limit. Then
(i) \( s_1 + s_2 \) is convergent to \(+\infty\) and convergent, and
(ii) \( \lim(s_1 + s_2) = +\infty \).

The theorem is a consequence of (7).

Let us consider without \(-\infty\) sequences \(s_1, s_2\) of extended reals. Now we state the propositions:

(21) Suppose \( s_1 \) is convergent to \(-\infty\) and \( s_2 \) is convergent to \(-\infty\). Then
(i) \( s_1 + s_2 \) is convergent to \(-\infty\) and convergent, and
(ii) \( \lim(s_1 + s_2) = -\infty \).

The theorem is a consequence of (7).

(22) Suppose \( s_1 \) is convergent to \(-\infty\) and \( s_2 \) is convergent to a finite limit. Then
(i) \( s_1 + s_2 \) is convergent to \(-\infty\) and convergent, and
(ii) \( \lim(s_1 + s_2) = -\infty \).
The theorem is a consequence of (7).

(23) Suppose \( s_1 \) is convergent to a finite limit and \( s_2 \) is convergent to a finite limit. Then

(i) \( s_1 + s_2 \) is convergent to a finite limit and convergent, and
(ii) \( \lim(s_1 + s_2) = \lim s_1 + \lim s_2 \).

The theorem is a consequence of (7).

Now we state the propositions:

(24) Let us consider without \(+\infty\) sequences \( s_1, s_2 \) of extended reals. Then

(i) if \( s_1 \) is convergent to \(+\infty\) and \( s_2 \) is convergent to \(+\infty\), then \( s_1 + s_2 \) is convergent to \(+\infty\) and convergent and \( \lim(s_1 + s_2) = +\infty \), and

(ii) if \( s_1 \) is convergent to \(+\infty\) and \( s_2 \) is convergent to a finite limit, then \( s_1 + s_2 \) is convergent to \(+\infty\) and convergent and \( \lim(s_1 + s_2) = +\infty \), and

(iii) if \( s_1 \) is convergent to \(-\infty\) and \( s_2 \) is convergent to \(-\infty\), then \( s_1 + s_2 \) is convergent to \(-\infty\) and convergent and \( \lim(s_1 + s_2) = -\infty \), and

(iv) if \( s_1 \) is convergent to \(-\infty\) and \( s_2 \) is convergent to a finite limit, then \( s_1 + s_2 \) is convergent to \(-\infty\) and convergent and \( \lim(s_1 + s_2) = -\infty \), and

(v) if \( s_1 \) is convergent to a finite limit and \( s_2 \) is convergent to a finite limit, then \( s_1 + s_2 \) is convergent to a finite limit and convergent and \( \lim(s_1 + s_2) = \lim s_1 + \lim s_2 \).

The theorem is a consequence of (17), (21), (10), (9), (2), (22), (18), (19), and (23).

(25) Let us consider a without \(-\infty\) sequence \( s_1 \) of extended reals, and a without \(+\infty\) sequence \( s_2 \) of extended reals. Then

(i) if \( s_1 \) is convergent to \(+\infty\) and \( s_2 \) is convergent to \(-\infty\), then \( s_1 - s_2 \) is convergent to \(+\infty\) and convergent and \( \lim(s_1 - s_2) = +\infty \) and \( \lim(s_2 - s_1) = -\infty \), and

(ii) if \( s_1 \) is convergent to \(+\infty\) and \( s_2 \) is convergent to a finite limit, then \( s_1 - s_2 \) is convergent to \(+\infty\) and convergent and \( \lim(s_1 - s_2) = +\infty \) and \( \lim(s_2 - s_1) = -\infty \), and

(iii) if \( s_1 \) is convergent to \(-\infty\) and \( s_2 \) is convergent to a finite limit, then \( s_1 - s_2 \) is convergent to \(-\infty\) and convergent and \( \lim(s_1 - s_2) = -\infty \) and \( \lim(s_2 - s_1) = +\infty \), and
(iv) if \( s_1 \) is convergent to a finite limit and \( s_2 \) is convergent to a finite limit, then \( s_1 - s_2 \) is convergent to a finite limit and convergent and \( s_2 - s_1 \) is convergent to a finite limit and convergent and \( \lim(s_1 - s_2) = \lim s_1 - \lim s_2 \) and \( \lim(s_2 - s_1) = \lim s_2 - \lim s_1 \).

The theorem is a consequence of (17), (24), (18), (10), (19), (22), (23), and (2).

2. Subsequences of Convergent Extended Real-Valued Sequences

Let us consider sequences \( s_1, s_2 \) of extended reals. Now we state the propositions:

(26) Suppose \( s_2 \) is a subsequence of \( s_1 \) and \( s_1 \) is convergent to a finite limit. Then

(i) \( s_2 \) is convergent to a finite limit, and
(ii) \( \lim s_1 = \lim s_2 \).

Proof: Consider \( g \) being a real number such that \( \lim s_1 = g \) and for every real number \( p \) such that \( 0 < p \) there exists a natural number \( n \) such that for every natural number \( m \) such that \( n \leq m \) holds \( |s_1(m) - \lim s_1| < p \) and \( s_1 \) is convergent to a finite limit. Reconsider \( L = \lim s_1 \) as an extended real number. There exists a real number \( g \) such that for every real number \( p \) such that \( 0 < p \) there exists a natural number \( n \) such that for every natural number \( m \) such that \( n \leq m \) holds \( |(s_2(m) - g) \text{ qua extended real)}| < p \) by [19] (14), [7] (15). For every real number \( p \) such that \( 0 < p \) there exists a natural number \( n \) such that for every natural number \( m \) such that \( n \leq m \) holds \( |s_2(m) - L| < p \) by [19] (14), [7] (15).

(27) Suppose \( s_2 \) is a subsequence of \( s_1 \) and \( s_1 \) is convergent to \( +\infty \). Then

(i) \( s_2 \) is convergent to \( +\infty \), and
(ii) \( \lim s_2 = +\infty \).

(28) Suppose \( s_2 \) is a subsequence of \( s_1 \) and \( s_1 \) is convergent to \( -\infty \). Then

(i) \( s_2 \) is convergent to \( -\infty \), and
(ii) \( \lim s_2 = -\infty \).
3. Convergency for Extended Real-Valued Double Sequences

Let us consider a function $R$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Now we state the propositions:

(29) Suppose the lim in the first coordinate of $R$ is convergent. Then the first coordinate major iterated lim of $R = \lim \text{ (the lim in the first coordinate of } R)$. 

(30) Suppose the lim in the second coordinate of $R$ is convergent. Then the second coordinate major iterated lim of $R = \lim \text{ (the lim in the second coordinate of } R)$. 

Let $E$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. We say that $E$ is P-convergent to a finite limit if and only if 

(Def. 1) there exists a real number $p$ such that for every real number $e$ such that $0 < e$ there exists a natural number $N$ such that for every natural numbers $n, m$ such that $n \geq N$ and $m \geq N$ holds $|E(n, m) - (p \text{ qua extended real})| < e$. 

We say that $E$ is P-convergent to $+\infty$ if and only if 

(Def. 2) for every real number $g$ such that $0 < g$ there exists a natural number $N$ such that for every natural numbers $n, m$ such that $n \geq N$ and $m \geq N$ holds $g \leq E(n, m)$. 

We say that $E$ is P-convergent to $-\infty$ if and only if 

(Def. 3) for every real number $g$ such that $g < 0$ there exists a natural number $N$ such that for every natural numbers $n, m$ such that $n \geq N$ and $m \geq N$ holds $E(n, m) \leq g$. 

Let $f$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. We say that $f$ is convergent in the first coordinate to $+\infty$ if and only if 

(Def. 4) for every element $m$ of $\mathbb{N}$, curry'(f, m) is convergent to $+\infty$. 

We say that $f$ is convergent in the first coordinate to $-\infty$ if and only if 

(Def. 5) for every element $m$ of $\mathbb{N}$, curry'(f, m) is convergent to $-\infty$. 

We say that $f$ is convergent in the first coordinate to a finite limit if and only if 

(Def. 6) for every element $m$ of $\mathbb{N}$, curry'(f, m) is convergent to a finite limit. 

We say that $f$ is convergent in the first coordinate if and only if 

(Def. 7) for every element $m$ of $\mathbb{N}$, curry'(f, m) is convergent. 

We say that $f$ is convergent in the second coordinate to $+\infty$ if and only if 

(Def. 8) for every element $m$ of $\mathbb{N}$, curry(f, m) is convergent to $+\infty$. 

We say that $f$ is convergent in the second coordinate to $-\infty$ if and only if 

(Def. 9) for every element $m$ of $\mathbb{N}$, curry(f, m) is convergent to $-\infty$. 
We say that \( f \) is convergent in the second coordinate to a finite limit if and only if
(Def. 10) for every element \( m \) of \( \mathbb{N} \), \( \text{curry}(f, m) \) is convergent to a finite limit.

We say that \( f \) is convergent in the second coordinate if and only if
(Def. 11) for every element \( m \) of \( \mathbb{N} \), \( \text{curry}(f, m) \) is convergent.

Now we state the propositions:

(31) Let us consider a function \( f \) from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \). Then
(i) if \( f \) is convergent in the first coordinate to \( +\infty \) or convergent in the first coordinate to \( -\infty \) or convergent in the first coordinate to a finite limit, then \( f \) is convergent in the first coordinate, and
(ii) if \( f \) is convergent in the second coordinate to \( +\infty \) or convergent in the second coordinate to \( -\infty \) or convergent in the second coordinate to a finite limit, then \( f \) is convergent in the second coordinate.

(32) Let us consider non empty sets \( X, Y, Z \), a function \( F \) from \( X \times Y \) into \( Z \), and an element \( x \) of \( X \). Then \( \text{curry}(F, x) = \text{curry'}(F^T, x) \).

(33) Let us consider non empty sets \( X, Y, Z \), a function \( F \) from \( X \times Y \) into \( Z \), and an element \( y \) of \( Y \). Then \( \text{curry'}(F, y) = \text{curry}(F^T, y) \).

(34) Let us consider non empty sets \( X, Y \), a function \( F \) from \( X \times Y \) into \( \mathbb{R} \), and an element \( x \) of \( X \). Then \( \text{curry}(-F, x) = -\text{curry}(F, x) \).

(35) Let us consider non empty sets \( X, Y \), a function \( F \) from \( X \times Y \) into \( \mathbb{R} \), and an element \( y \) of \( Y \). Then \( \text{curry'}(-F, y) = -\text{curry'}(F, y) \).

Let us consider a function \( f \) from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \). Now we state the propositions:

(36) (i) \( f \) is convergent in the first coordinate to \( +\infty \) iff \( f^T \) is convergent in the second coordinate to \( +\infty \), and
(ii) \( f \) is convergent in the second coordinate to \( +\infty \) iff \( f^T \) is convergent in the first coordinate to \( +\infty \), and
(iii) \( f \) is convergent in the first coordinate to \( -\infty \) iff \( f^T \) is convergent in the second coordinate to \( -\infty \), and
(iv) \( f \) is convergent in the second coordinate to \( -\infty \) iff \( f^T \) is convergent in the first coordinate to \( -\infty \), and
(v) \( f \) is convergent in the first coordinate to a finite limit iff \( f^T \) is convergent in the second coordinate to a finite limit, and
(vi) \( f \) is convergent in the second coordinate to a finite limit iff \( f^T \) is convergent in the first coordinate to a finite limit.

The theorem is a consequence of (33) and (32).

(37) (i) \( f \) is convergent in the first coordinate to \( +\infty \) iff \( -f \) is convergent in the first coordinate to \( -\infty \), and
(ii) $f$ is convergent in the first coordinate to $-\infty$ iff $-f$ is convergent in the first coordinate to $+\infty$, and

(iii) $f$ is convergent in the first coordinate to a finite limit iff $-f$ is convergent in the first coordinate to a finite limit, and

(iv) $f$ is convergent in the second coordinate to $+\infty$ iff $-f$ is convergent in the second coordinate to $-\infty$, and

(v) $f$ is convergent in the second coordinate to $-\infty$ iff $-f$ is convergent in the second coordinate to $+\infty$, and

(vi) $f$ is convergent in the second coordinate to a finite limit iff $-f$ is convergent in the second coordinate to a finite limit.

The theorem is a consequence of (35), (17), (2), and (34).

Let $f$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. The functors: the lim in the first coordinate of $f$ and the lim in the second coordinate of $f$ yielding sequences of extended reals are defined by conditions

(Def. 12) for every element $m$ of $\mathbb{N}$, the lim in the first coordinate of $f(m) = \lim \text{curry}'(f, m),$

(Def. 13) for every element $n$ of $\mathbb{N}$, the lim in the second coordinate of $f(n) = \lim \text{curry}(f, n),$

respectively. Now we state the proposition:

(38) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Then

(i) the lim in the first coordinate of $f = $ the lim in the second coordinate of $f^T$, and

(ii) the lim in the second coordinate of $f = $ the lim in the first coordinate of $f^T$.

The theorem is a consequence of (33) and (32).

Let $X$, $Y$ be non empty sets, $F$ be a without $+\infty$ function from $X \times Y$ into $\mathbb{R}$, and $x$ be an element of $X$. Let us observe that $\text{curry}(F, x)$ is without $+\infty$.

Let $y$ be an element of $Y$. One can verify that $\text{curry}'(F, y)$ is without $+\infty$.

Let $F$ be a without $-\infty$ function from $X \times Y$ into $\mathbb{R}$ and $x$ be an element of $X$. Let us note that $\text{curry}(F, x)$ is without $-\infty$.

Let $y$ be an element of $Y$. Observe that $\text{curry}'(F, y)$ is without $-\infty$.

Let $f$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. The partial sums in the second coordinate of $f$ yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ is defined by

(Def. 14) for every natural numbers $n$, $m$, $it(n, 0) = f(n, 0)$ and $it(n, m + 1) = it(n, m) + f(n, m + 1)$.

The partial sums in the first coordinate of $f$ yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ is defined by
for every natural numbers \( n, m \), \( it(0, m) = f(0, m) \) and \( it(n + 1, m) = it(n, m) + f(n + 1, m) \).

Let \( f \) be a without \(-\infty\) function from \( \mathbb{N} \times \mathbb{N} \) into \( \overline{\mathbb{R}} \). Let us note that the partial sums in the second coordinate of \( f \) is without \(-\infty\).

Let \( f \) be a without \(+\infty\) function from \( \mathbb{N} \times \mathbb{N} \) into \( \overline{\mathbb{R}} \). Observe that the partial sums in the second coordinate of \( f \) is without \(+\infty\).

Let \( f \) be a non-negative function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \). Let us observe that the partial sums in the second coordinate of \( f \) is non-negative as a function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \).

Let \( f \) be a non-positive function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \). One can check that the partial sums in the second coordinate of \( f \) is non-positive as a function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \).

Let \( f \) be a without \(-\infty\) function from \( \mathbb{N} \times \mathbb{N} \) into \( \overline{\mathbb{R}} \). Let us note that the partial sums in the first coordinate of \( f \) is without \(-\infty\).

Let \( f \) be a without \(+\infty\) function from \( \mathbb{N} \times \mathbb{N} \) into \( \overline{\mathbb{R}} \). Observe that the partial sums in the first coordinate of \( f \) is without \(+\infty\).

Let \( f \) be a non-negative function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \). Let us observe that the partial sums in the first coordinate of \( f \) is non-negative as a function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \).

Let \( f \) be a non-positive function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \). One can check that the partial sums in the first coordinate of \( f \) is non-positive as a function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \).

Let \( f \) be a function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \). The functor \((\sum_{\kappa=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}\) yielding a function from \( \mathbb{N} \times \mathbb{N} \) into \( \overline{\mathbb{R}} \) is defined by the term

\[(\text{Def. 16}) \quad \text{the partial sums in the second coordinate of the partial sums in the first coordinate of } f.\]

Now we state the propositions:

(39) Let us consider a function \( f \) from \( \mathbb{N} \times \mathbb{N} \) into \( \overline{\mathbb{R}} \), and natural numbers \( n, m \). Then

(i) \( (\text{the partial sums in the first coordinate of } f)(n, m) = (\text{the partial sums in the second coordinate of } f^T)(m, n) \), and

(ii) \( (\text{the partial sums in the second coordinate of } f)(n, m) = (\text{the partial sums in the first coordinate of } f^T)(m, n) \).

**Proof:** Define \( \mathcal{P}[\text{natural number}] \equiv (\text{the partial sums in the first coordinate of } f)(\$1, m) = (\text{the partial sums in the second coordinate of } f^T)(m, \$1) \). For every natural number \( k \) such that \( \mathcal{P}[k] \) holds \( \mathcal{P}[k+1] \). For every natural number \( k \), \( \mathcal{P}[k] \) from [1 Sch. 2]. Define \( \mathcal{Q}[\text{natural number}] \equiv (\text{the partial sums in the second coordinate of } f)(n, \$1) = (\text{the partial sums in the first coordinate of } f^T)(m, n) \).
coordinate of $f^T(\$1, n)$. For every natural number $k$ such that $Q[k]$ holds $Q[k + 1]$. For every natural number $k$, $Q[k]$ from [1] Sch. 2. □

(40) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Then

(i) (the partial sums in the first coordinate of $f^T = $ the partial sums in the second coordinate of $f^T$, and
(ii) (the partial sums in the second coordinate of $f^T = $ the partial sums in the first coordinate of $f^T$.

The theorem is a consequence of (39).

(41) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$, an extended real-valued function $g$, and a natural number $n$. Suppose for every natural number $k$,

(i) for every natural number $k$, $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n, k) = (\sum_{\alpha=0}^{\kappa} g(\alpha))_{\kappa \in \mathbb{N}}(k)$, and
(ii) (the lim in the second coordinate of $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}})(n) = \sum g$.

(42) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Then

(i) the partial sums in the second coordinate of $-f = -(\text{the partial sums in the second coordinate of } f)$, and
(ii) the partial sums in the first coordinate of $-f = -(\text{the partial sums in the first coordinate of } f)$.

**Proof:** For every element $z$ of $\mathbb{N} \times \mathbb{N}$, $-\text{(the partial sums in the second coordinate of } f)(z) = \text{(the partial sums in the second coordinate of } -f)(z)$ by [9, (87)]. For every element $z$ of $\mathbb{N} \times \mathbb{N}$,

$-\text{(the partial sums in the first coordinate of } f)(z) = \text{(the partial sums in the first coordinate of } -f)(z)$ by [9, (87)]. □

(43) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$, and elements $m$, $n$ of $\mathbb{N}$. Then

(i) (the partial sums in the first coordinate of $f)(m, n) = (\sum_{\alpha=0}^{\kappa} (\text{curry'}(f, n))(\alpha))_{\kappa \in \mathbb{N}}(m)$, and
(ii) (the partial sums in the second coordinate of $f)(m, n) = (\sum_{\alpha=0}^{\kappa} (\text{curry}(f, m))(\alpha))_{\kappa \in \mathbb{N}}(n)$.

**Proof:** Define $P[\text{natural number}] \equiv (\text{the partial sums in the first coordinate of } f)(\$1, n) = (\sum_{\alpha=0}^{\kappa} (\text{curry'}(f, n))(\alpha))_{\kappa \in \mathbb{N}}(\$1)$. For every natural number $k$ such that $P[k]$ holds $P[k + 1]$. For every natural number $k$, $P[k]$ from [1] Sch. 2. Define $Q[\text{natural number}] \equiv (\text{the partial sums in the second coordinate of } f)(m, \$1) = (\sum_{\alpha=0}^{\kappa} (\text{curry}(f, m))(\alpha))_{\kappa \in \mathbb{N}}(\$1)$. For
every natural number $k$ such that $Q[k]$ holds $Q[k + 1]$. For every natural number $k$, $Q[k]$ from [Sch. 2]. □

(44) Let us consider without $-\infty$ functions $f_1, f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Then

(i) the partial sums in the second coordinate of $f_1 + f_2 = (\text{the partial sums in the second coordinate of } f_1) + (\text{the partial sums in the second coordinate of } f_2)$, and

(ii) the partial sums in the first coordinate of $f_1 + f_2 = (\text{the partial sums in the first coordinate of } f_1) + (\text{the partial sums in the first coordinate of } f_2)$.

The theorem is a consequence of (11).

(45) Let us consider without $+\infty$ functions $f_1, f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Then

(i) the partial sums in the second coordinate of $f_1 + f_2 = (\text{the partial sums in the second coordinate of } f_1) + (\text{the partial sums in the second coordinate of } f_2)$, and

(ii) the partial sums in the first coordinate of $f_1 + f_2 = (\text{the partial sums in the first coordinate of } f_1) + (\text{the partial sums in the first coordinate of } f_2)$.

The theorem is a consequence of (10), (9), (2), (42), (44), and (8).

(46) Let us consider a without $-\infty$ function $f_1$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$, and a without $+\infty$ function $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Then

(i) the partial sums in the second coordinate of $f_1 - f_2 = (\text{the partial sums in the second coordinate of } f_1) - (\text{the partial sums in the second coordinate of } f_2)$, and

(ii) the partial sums in the first coordinate of $f_1 - f_2 = (\text{the partial sums in the first coordinate of } f_1) - (\text{the partial sums in the first coordinate of } f_2)$, and

(iii) the partial sums in the second coordinate of $f_2 - f_1 = (\text{the partial sums in the second coordinate of } f_2) - (\text{the partial sums in the second coordinate of } f_1)$, and

(iv) the partial sums in the first coordinate of $f_2 - f_1 = (\text{the partial sums in the first coordinate of } f_2) - (\text{the partial sums in the first coordinate of } f_1)$.

The theorem is a consequence of (10), (44), (42), and (45).

(47) Let us consider a without $-\infty$ function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$, and natural numbers $n, m$. Then

(i) $(\sum_{\alpha=0}^\kappa f(\alpha))_{\kappa \in \mathbb{N}}(n+1, m) = (\text{the partial sums in the second coordinate of } f)(n+1, m) + (\sum_{\alpha=0}^\kappa f(\alpha))_{\kappa \in \mathbb{N}}(n, m)$, and
(ii) (the partial sums in the first coordinate of the partial sums in the second coordinate of \(f\))(n, m + 1) = (the partial sums in the first coordinate of \(f\))(n, m + 1) + (the partial sums in the first coordinate of the partial sums in the second coordinate of \(f\))(n, m).

**Proof:** Set \(R_1 = (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}\). Set \(C_1 = \) the partial sums in the first coordinate of the partial sums in the second coordinate of \(f\). Set \(R_2 = \) the partial sums in the first coordinate of \(f\). Set \(C_2 = \) the partial sums in the second coordinate of \(f\). Define \(P[\text{natural number}] \equiv R_1(n + 1, \$_1) = C_2(n + 1, \$_1) + R_1(n, \$_1)\). For every natural number \(k\) such that \(P[k]\) holds \(P[k + 1]\). For every natural number \(k, P[k]\) from \([\Pi\text{ Sch. 2}]. Define \(Q[\text{natural number}] \equiv C_1(\$_1, m + 1) = R_2(\$_1, m + 1) + C_1(\$_1, m)\). For every natural number \(k\) such that \(Q[k]\) holds \(Q[k + 1]\). For every natural number \(k, Q[k]\) from \([\Pi\text{ Sch. 2}]. □

(48) Let us consider a without \(+\infty\) function \(f\) from \(\mathbb{N} \times \mathbb{N}\) into \(\mathbb{R}\), and natural numbers \(n, m\). Then

(i) \((\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n+1, m) = (\text{the partial sums in the second coordinate of } f)(n+1, m) + (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n, m), \) and

(ii) (the partial sums in the first coordinate of the partial sums in the second coordinate of \(f\))(n, m + 1) = (the partial sums in the first coordinate of \(f\))(n, m + 1) + (the partial sums in the first coordinate of the partial sums in the second coordinate of \(f\))(n, m).

The theorem is a consequence of (2), (42), and (47).

(49) Let us consider a function \(f\) from \(\mathbb{N} \times \mathbb{N}\) into \(\mathbb{R}\). Suppose \(f\) is without \(-\infty\) or without \(+\infty\). Then \((\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}} = \) the partial sums in the first coordinate of the partial sums in the second coordinate of \(f\).

(50) Let us consider without \(-\infty\) functions \(f_1, f_2\) from \(\mathbb{N} \times \mathbb{N}\) into \(\mathbb{R}\). Then \((\sum_{\alpha=0}^{\kappa} (f_1 + f_2)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}} + (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}}\). The theorem is a consequence of (44).

(51) Let us consider without \(+\infty\) functions \(f_1, f_2\) from \(\mathbb{N} \times \mathbb{N}\) into \(\mathbb{R}\). Then \((\sum_{\alpha=0}^{\kappa} (f_1 + f_2)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}} + (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}}\). The theorem is a consequence of (45).

(52) Let us consider a without \(-\infty\) function \(f_1\) from \(\mathbb{N} \times \mathbb{N}\) into \(\mathbb{R}\), and a without \(+\infty\) function \(f_2\) from \(\mathbb{N} \times \mathbb{N}\) into \(\mathbb{R}\). Then

(i) \((\sum_{\alpha=0}^{\kappa} (f_1 - f_2)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}} - (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}}, \) and

(ii) \((\sum_{\alpha=0}^{\kappa} (f_2 - f_1)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}} - (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}}\).

The theorem is a consequence of (46).

(53) Let us consider a function \(f\) from \(\mathbb{N} \times \mathbb{N}\) into \(\mathbb{R}\), and an element \(k\) of \(\mathbb{N}\). Then
(i) \( \text{curry}'(\text{the partial sums in the first coordinate of } f, k) = (\sum_{\alpha=0}^{\kappa}(\text{curry}'(f, k))(\alpha))_{\kappa \in \mathbb{N}} \), and

(ii) \( \text{curry}(\text{the partial sums in the second coordinate of } f, k) = (\sum_{\alpha=0}^{\kappa}(\text{curry}(f, k))(\alpha))_{\kappa \in \mathbb{N}}. \)

The theorem is a consequence of (43).

(54) Let us consider a function \( f \) from \( \mathbb{N} \times \mathbb{N} \) into \( \overline{\mathbb{R}} \). Suppose \( f \) is without \(-\infty\) or without \(+\infty\). Then

(i) \( \text{curry}(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}, 0) = \text{curry}(\text{the partial sums in the second coordinate of } f, 0), \) and

(ii) \( \text{curry}'(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}, 0) = \text{curry}'(\text{the partial sums in the first coordinate of } f, 0). \)

(55) Let us consider non empty sets \( C, D \), without \(-\infty\) functions \( F_1, F_2 \) from \( C \times D \) into \( \overline{\mathbb{R}} \), and an element \( c \) of \( C \). Then \( \text{curry}(F_1 + F_2, c) = \text{curry}(F_1, c) + \text{curry}(F_2, c) \). The theorem is a consequence of (7).

(56) Let us consider non empty sets \( C, D \), without \(-\infty\) functions \( F_1, F_2 \) from \( C \times D \) into \( \overline{\mathbb{R}} \), and an element \( d \) of \( D \). Then \( \text{curry}'(F_1 + F_2, d) = \text{curry}'(F_1, d) + \text{curry}'(F_2, d) \). The theorem is a consequence of (7).

(57) Let us consider non empty sets \( C, D \), without \(+\infty\) functions \( F_1, F_2 \) from \( C \times D \) into \( \overline{\mathbb{R}} \), and an element \( c \) of \( C \). Then \( \text{curry}(F_1 + F_2, c) = \text{curry}(F_1, c) + \text{curry}(F_2, c) \). The theorem is a consequence of (7).

(58) Let us consider non empty sets \( C, D \), without \(+\infty\) functions \( F_1, F_2 \) from \( C \times D \) into \( \overline{\mathbb{R}} \), and an element \( d \) of \( D \). Then \( \text{curry}'(F_1 + F_2, d) = \text{curry}'(F_1, d) + \text{curry}'(F_2, d) \). The theorem is a consequence of (7).

(59) Let us consider non empty sets \( C, D \), a without \(-\infty\) function \( F_1 \) from \( C \times D \) into \( \overline{\mathbb{R}} \), a without \(+\infty\) function \( F_2 \) from \( C \times D \) into \( \overline{\mathbb{R}} \), and an element \( c \) of \( C \). Then

(i) \( \text{curry}(F_1 - F_2, c) = \text{curry}(F_1, c) - \text{curry}(F_2, c) \), and

(ii) \( \text{curry}'(F_2 - F_1, c) = \text{curry}'(F_2, c) - \text{curry}'(F_1, c) \).

The theorem is a consequence of (7).

(60) Let us consider non empty sets \( C, D \), a without \(-\infty\) function \( F_1 \) from \( C \times D \) into \( \overline{\mathbb{R}} \), a without \(+\infty\) function \( F_2 \) from \( C \times D \) into \( \overline{\mathbb{R}} \), and an element \( d \) of \( D \). Then

(i) \( \text{curry}'(F_1 - F_2, d) = \text{curry}'(F_1, d) - \text{curry}'(F_2, d) \), and

(ii) \( \text{curry}'(F_2 - F_1, d) = \text{curry}'(F_2, d) - \text{curry}'(F_1, d) \).

The theorem is a consequence of (7).
Now we state the propositions:

(61) Let us consider a non-negative sequence $s$ of extended reals. Suppose $\left(\sum_{\alpha=0}^{\kappa} s(\alpha)\right)_{\kappa \in \mathbb{N}}$ is not convergent to $+\infty$. Let us consider a natural number $n$. Then $s(n)$ is a real number.

(62) Let us consider a non-negative sequence $s$ of extended reals. Suppose $s$ is non-decreasing. Then $s$ is convergent to $+\infty$ or convergent to a finite limit.

Let $f$ be a non-negative function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ and $n$ be an element of $\mathbb{N}$. Let us observe that curry$(f, n)$ is non-negative and curry$'$$(f, n)$ is non-negative.

Now we state the propositions:

(63) Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$, and an element $m$ of $\mathbb{N}$. Then curry$(the \ partial \ sums \ in \ the \ second \ coordinate \ of \ f, m)$ is non-decreasing.

**Proof:** Set $P = curry$(the partial sums in the second coordinate of $f, m$). For every natural numbers $n, j$ such that $j \leq n$ holds $P(j) \leq P(n)$ by [4, (51)], [1, (13), (20)]. □

(64) Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$, and an element $n$ of $\mathbb{N}$. Then curry$'$$(the \ partial \ sums \ in \ the \ first \ coordinate \ of \ f, n)$ is non-decreasing. The theorem is a consequence of (63), (40), and (33).

Let $f$ be a non-negative function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ and $m$ be an element of $\mathbb{N}$. One can check that curry$(the \ partial \ sums \ in \ the \ second \ coordinate \ of \ f, m)$ is non-decreasing and curry$'$$(the \ partial \ sums \ in \ the \ first \ coordinate \ of \ f, m)$ is non-decreasing.

Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Now we state the propositions:

(65) (i) if $f$ is convergent in the first coordinate, then the lim in the first coordinate of $f$ is non-negative, and

(ii) if $f$ is convergent in the second coordinate, then the lim in the second coordinate of $f$ is non-negative.

(66) (i) the partial sums in the first coordinate of $f$ is convergent in the first coordinate, and

(ii) the partial sums in the second coordinate of $f$ is convergent in the second coordinate.

Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$, an element $m$ of $\mathbb{N}$, and a natural number $n$.

Let us assume that curry$'$$(the \ partial \ sums \ in \ the \ first \ coordinate \ of \ f, m)$ is not convergent to $+\infty$. Now we state the propositions:
(67) $f(n, m)$ is a real number.

(68) $f(m, n)$ is a real number.

Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ and natural numbers $n, m$. Now we state the propositions:

(69) Suppose for every natural number $i$ such that $i \leq n$ holds $f(i, m)$ is a real number. Then (the partial sums in the first coordinate of $f)(n, m) < +\infty$. 

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\frac{1}{i} \leq n$, then (the partial sums in the first coordinate of $f)(\frac{1}{i}, m) < +\infty$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$ by [4, (51)], [1, (13)]. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. □

(70) Suppose for every natural number $i$ such that $i \leq m$ holds $f(n, i)$ is a real number. Then (the partial sums in the second coordinate of $f)(n, m) < +\infty$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\frac{1}{i} \leq m$, then (the partial sums in the second coordinate of $f)(n, \frac{1}{i}) < +\infty$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$ by [4, (51)], [1, (13)]. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. □

Now we state the proposition:

(71) Let us consider a without $-\infty$ function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Suppose $\left(\sum_{\alpha=0}^{\kappa} f(\alpha)\right)_{\kappa \in \mathbb{N}}$ is convergent in the first coordinate to $-\infty$. Then there exists an element $m$ of $\mathbb{N}$ such that $\text{curry}'$ (the partial sums in the first coordinate of $f, m$) is convergent to $-\infty$. The theorem is a consequence of (54).

Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ and a natural number $m$. Now we state the propositions:

(72) for every element $k$ of $\mathbb{N}$ such that $k \leq m$ holds $\text{curry}'$ (the partial sums in the second coordinate of $f, k$) is not convergent to $+\infty$ if and only if for every element $k$ of $\mathbb{N}$ such that $k \leq m$ holds $\lim \text{curry}'$ (the partial sums in the second coordinate of $f, k) < +\infty$. The theorem is a consequence of (62).

(73) for every element $k$ of $\mathbb{N}$ such that $k \leq m$ holds $\text{curry}'$ (the partial sums in the first coordinate of $f, k$) is not convergent to $+\infty$ if and only if for every element $k$ of $\mathbb{N}$ such that $k \leq m$ holds $\lim \text{curry}'$ (the partial sums in the first coordinate of $f, k) < +\infty$. The theorem is a consequence of (62).

(74) $\left(\sum_{\alpha=0}^{\kappa} (\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)(\alpha)\right)_{\kappa \in \mathbb{N}}(m) = +\infty$ if and only if there exists an element $k$ of $\mathbb{N}$ such that $k \leq m$ and $\text{curry}'$ (the partial sums in the second coordinate
of $f, k$ is convergent to $+\infty$. The theorem is a consequence of (72), (65), and (4).

(75) \( \sum_{\alpha=0}^{\kappa} (\text{the lim in the first coordinate of the partial sums in the first coordinate of } f)(\alpha) \in \mathbb{N}(m) = +\infty \) if and only if there exists an element $k$ of $\mathbb{N}$ such that $k \leq m$ and curry'(the partial sums in the first coordinate of $f, k$) is convergent to $+\infty$. The theorem is a consequence of (38), (40), (74), and (32).

Now we state the proposition:

(76) Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$, and natural numbers $n, m$. Then

(i) (the partial sums in the first coordinate of $f)(n, m) \geq f(n, m)$, and
(ii) (the partial sums in the second coordinate of $f)(n, m) \geq f(n, m)$.

PROOF: Define $P[\text{natural number}] \equiv \text{if } \$1 \leq n, then (the partial sums in the first coordinate of } f)(\$1, m) \geq f(\$1, m). \text{ For every natural number } k \text{ such that } P[k] \text{ holds } P[k+1] \text{ by } [4, (51)]. \text{ For every natural number } k, P[k] \text{ from } \Pi \text{ Sch. 2}. \text{ Define } Q[\text{natural number}] \equiv \text{if } \$1 \leq m, then (the partial sums in the second coordinate of } f)(n, \$1) \geq f(n, \$1). \text{ For every natural number } k \text{ such that } Q[k] \text{ holds } Q[k+1] \text{ by } [4, (51)]. \text{ For every natural number } k, Q[k] \text{ from } \Pi \text{ Sch. 2}. \square

Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ and an element $m$ of $\mathbb{N}$. Now we state the propositions:

(77) Suppose there exists an element $k$ of $\mathbb{N}$ such that $k \leq m$ and curry(the partial sums in the second coordinate of $f, k$) is convergent to $+\infty$. Then

(i) curry(the partial sums in the second coordinate of the partial sums in the first coordinate of $f, m$) is convergent to $+\infty$, and
(ii) $\lim \text{curry'(the partial sums in the second coordinate of the partial sums in the first coordinate of } f, m) = +\infty$.

PROOF: For every real number $g$ such that $0 < g$ there exists a natural number $N$ such that for every natural number $n$ such that $N \leq n$ holds $g \leq (\text{curry'(the partial sums in the second coordinate of the partial sums in the first coordinate of } f, m))(n)$ by $[26(7)], (76). \square$

(78) Suppose there exists an element $k$ of $\mathbb{N}$ such that $k \leq m$ and curry'(the partial sums in the first coordinate of $f, k$) is convergent to $+\infty$. Then

(i) curry'(the partial sums in the first coordinate of the partial sums in the second coordinate of $f, m$) is convergent to $+\infty$, and
(ii) $\lim \text{curry'(the partial sums in the first coordinate of the partial sums in the second coordinate of } f, m) = +\infty$. 

The theorem is a consequence of (40), (32), and (77).

Now we state the propositions:

(79) Let us consider a without $-\infty$ function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the first coordinate to a finite limit if and only if the partial sums in the first coordinate of $f$ is convergent in the first coordinate to a finite limit. The theorem is a consequence of (54), (47), (7), and (23).

(80) Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the first coordinate to a finite limit. Let us consider an element $m$ of $\mathbb{N}$. Then $(\sum_{\alpha=0}^{\kappa}(\text{the lim in the first coordinate of the partial sums in the first coordinate of } f(\alpha))_{\kappa \in \mathbb{N}}(m) = \lim \text{curry}'(\text{the partial sums in the first coordinate of the partial sums in the second coordinate of } f, m)$.

Proof: The partial sums in the first coordinate of $f$ is convergent in the first coordinate to a finite limit. Define $\mathcal{P}[\text{natural number}] \equiv$ for every element $k$ of $\mathbb{N}$ such that $k \leq S_1$ holds $(\sum_{\alpha=0}^{\kappa}(\text{the lim in the first coordinate of the partial sums in the first coordinate of } f(\alpha))_{\kappa \in \mathbb{N}}(k) = \lim \text{curry}'(\text{the partial sums in the first coordinate of the partial sums in the second coordinate of } f, k)$.


(81) Let us consider a without $-\infty$ function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the second coordinate to a finite limit if and only if the partial sums in the second coordinate of $f$ is convergent in the second coordinate to a finite limit. The theorem is a consequence of (36), (40), and (79).

(82) Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the second coordinate to a finite limit. Let us consider an element $m$ of $\mathbb{N}$. Then $(\sum_{\alpha=0}^{\kappa}(\text{the lim in the second coordinate of the partial sums in the second coordinate of } f(\alpha))_{\kappa \in \mathbb{N}}(m) = \lim \text{curry}(\text{the partial sums in the second coordinate of the partial sums in the first coordinate of } f, m)$. The theorem is a consequence of (36), (40), (38), (80), and (32).

Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and a sequence $s$ of extended reals. Now we state the propositions:

(83) Suppose for every element $m$ of $\mathbb{N}$, $s(m) = \lim \inf \text{curry}'(f, m)$. Then $\sum s \leq \lim \inf(\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)$.

Proof: For every element $m$ of $\mathbb{N}$ and for every elements $N, n$ of $\mathbb{N}$
such that \( n \geq N \) holds (the inferior real sequence curry'\((f, m)\))(N) \( \leq f(n, m) \) by \( [26] \) (7), (8). Define \( F(\text{element of } \mathbb{N}) = \) the inferior real sequence curry'\((f, s_1)\). Define \( G(\text{element of } \mathbb{N}, \text{element of } \mathbb{N}) = \) (the inferior real sequence curry'\((f, s_2)\))(\(s_1\)). Consider \( g \) being a function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \) such that for every element \( n \) of \( \mathbb{N} \) and for every element \( m \) of \( \mathbb{N} \), \( g(n, m) = G(n, m) \) from \([5\) Sch. 4\]. For every element \( m \) of \( \mathbb{N} \) and for every elements \( N, n \) of \( \mathbb{N} \) such that \( n \geq N \) holds (the partial sums in the second coordinate of \( g)\)(\(N, m) \leq (\text{the partial sums in the second coordinate of } f)(n, m)\). For every element \( m \) of \( \mathbb{N} \) and for every elements \( N, n \) of \( \mathbb{N} \) such that \( n \geq N \) holds (the partial sums in the second coordinate of \( g)\)(\(N, m) \leq (\text{the inferior real sequence curry'}\((f, m)\))(N) \leq f(n, m) \) by \([26] \) (37), (23)]. Define \( Q[\text{natural number}] \equiv \) for every element \( m \) of \( \mathbb{N} \) such that \( m = s_1 \) holds \( \left( \sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m) = \text{lim inf (the partial sums in the second coordinate of } g, m)\). For every element \( m \) of \( \mathbb{N} \), \( \text{curry'}\((\text{the partial sums in the second coordinate of } g, m)\) is convergent by \([26] \) (7), (37)]. For every natural number \( k \) such that \( Q[k] \) holds \( Q[k+1] \) by \([26] \) (37)], \([4] \) (51), (52)], \([14] \) (11)]. For every natural number \( k, Q[k] \) from \([1\) Sch. 2\]. For every natural number \( m, (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m) \leq \text{lim inf (the lim in the second coordinate of the partial sums in the second coordinate of } f)\) by \([26] \) (37), (38)]. For every object \( m \) such that \( m \in \text{dom } s \) holds \( 0 \leq s(m) \) by \([4] \) (51), (52)], \([26] \) (23)]. \( \square \)

(84) Suppose for every element \( m \) of \( \mathbb{N} \), \( s(m) = \text{lim inf curry'\((f, m)\)}\). Then \( \sum s \leq \text{lim inf (the lim in the first coordinate of the partial sums in the first coordinate of } f)\). The theorem is a consequence of \((32), (83), (38), \) and \((40)\).

Now we state the proposition:

(85) Let us consider a function \( f \) from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \), a sequence \( s \) of extended reals, and natural numbers \( n, m \). Then

(i) if for every natural numbers \( i, j, f(i, j) \leq s(i) \), then (the partial sums in the first coordinate of \( f)\)(\(n, m) \leq (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n)\), and

(ii) if for every natural numbers \( i, j, f(i, j) \leq s(j) \), then (the partial sums in the second coordinate of \( f)\)(\(n, m) \leq (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m)\).

PROOF: Define \( \mathcal{P}[\text{natural number}] \equiv \) (the partial sums in the second coordinate of \( f)\)(\(n, s_1) \leq (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(s_1)\). For every natural number \( k \) such that \( \mathcal{P}[k] \) holds \( \mathcal{P}[k+1] \). For every natural number \( k, \mathcal{P}[k] \) from \([1\) Sch. 2\]. \( \square \)

Let us consider a sequence \( s \) of extended reals and an extended real number \( r. \) Now we state the propositions:
(86) If for every natural number \( n \), \( s(n) \leq r \), then \( \limsup s \leq r \).

**Proof:** Define \( F(\text{element of } \mathbb{N}) = r \). Consider \( f \) being a function from \( \mathbb{N} \) into \( \mathbb{R} \) such that for every element \( n \) of \( \mathbb{N} \), \( f(n) = F(n) \) from \([7, \text{Sch. 4}]\). For every natural number \( n \), \( f(n) = r \). For every natural number \( n \), \( s(n) \leq r \). □

(87) If for every natural number \( n \), \( r \leq s(n) \), then \( r \leq \liminf s \).

**Proof:** Define \( F(\text{element of } \mathbb{N}) = r \). Consider \( f \) being a function from \( \mathbb{N} \) into \( \mathbb{R} \) such that for every element \( n \) of \( \mathbb{N} \), \( f(n) = F(n) \) from \([7, \text{Sch. 4}]\). For every natural number \( n \), \( f(n) = r \). For every natural number \( n \), \( f(n) \leq s(n) \). □

Now we state the proposition:

(88) Let us consider a non-negative function \( f \) from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \). Then

(i) for every natural numbers \( i_1, i_2, j \) such that \( i_1 \leq i_2 \) holds (the partial sums in the first coordinate of \( f \))(\( i_1, j \)) \leq (the partial sums in the first coordinate of \( f \))(\( i_2, j \)), and

(ii) for every natural numbers \( i, j_1, j_2 \) such that \( j_1 \leq j_2 \) holds (the partial sums in the second coordinate of \( f \))(\( i, j_1 \)) \leq (the partial sums in the second coordinate of \( f \))(\( i, j_2 \)).

Let us consider a function \( f \) from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \) and natural numbers \( i, j, k \). Now we state the propositions:

(89) Suppose for every element \( m \) of \( \mathbb{N} \), \( \text{curry}'(f, m) \) is non-decreasing and \( i \leq j \). Then (the partial sums in the second coordinate of \( f \))(\( i, k \)) \leq (the partial sums in the second coordinate of \( f \))(\( j, k \)).

**Proof:** Define \( P[\text{natural number}] \equiv (\text{the partial sums in the second coordinate of } f)(i, \$1) \leq (\text{the partial sums in the second coordinate of } f)(j, \$1) \). For every natural number \( n \) such that \( P[n] \) holds \( P[n + 1] \) by \([20, (7)]\). For every natural number \( n \), \( P[n] \) from \([1, \text{Sch. 2}]\). □

(90) Suppose for every element \( n \) of \( \mathbb{N} \), \( \text{curry}(f, n) \) is non-decreasing and \( i \leq j \). Then (the partial sums in the first coordinate of \( f \))(\( k, i \)) \leq (the partial sums in the first coordinate of \( f \))(\( k, j \)). The theorem is a consequence of (32), (89), and (39).

Let us consider a non-negative function \( f \) from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \) and a sequence \( s \) of extended reals. Now we state the propositions:

(91) Suppose for every element \( m \) of \( \mathbb{N} \), \( \text{curry}'(f, m) \) is non-decreasing and \( s(m) = \lim \text{curry}'(f, m) \). Then

(i) the lim in the second coordinate of the partial sums in the second coordinate of \( f \) is non-decreasing, and
(ii) \[ \sum s = \lim \text{(the lim in the second coordinate of the partial sums in the second coordinate of } f). \]

**Proof:** \[ \sum s \leq \lim \inf \text{(the lim in the second coordinate of the partial sums in the second coordinate of } f). \] For every natural numbers \( n, m, f(n, m) \leq s(m) \) by [26 (37)], [8] (3)]. For every natural numbers \( n, m \) such that \( m \leq n \) holds (the lim in the second coordinate of the partial sums in the second coordinate of \( f)(m) \leq (\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)(n) \) by [26 (37)], (89), [26 (38)]. For every natural number \( n \), (the lim in the second coordinate of the partial sums in the second coordinate of \( f)(n) \leq \sum s \) by [26 (37)], [4] (39)], (87), [26 (41)]. \( \lim \sup \text{(the lim in the second coordinate of the partial sums in the second coordinate of } f) \leq \sum s. \square \)

(92) Suppose for every element \( m \) of \( \mathbb{N} \), \( \text{curry}(f, m) \) is non-decreasing and \( s(m) = \lim \text{curry}(f, m) \). Then

(i) the lim in the first coordinate of the partial sums in the first coordinate of \( f \) is non-decreasing, and

(ii) \[ \sum s = \lim \text{(the lim in the first coordinate of the partial sums in the first coordinate of } f). \]

The theorem is a consequence of (32), (91), (33), and (40).

5. **Pringsheim Sense Convergence for Extended Real-Valued Double Sequences**

Let us consider a function \( f \) from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \). Now we state the propositions:

(93) If \( f \) is P-convergent to \( +\infty \), then \( f \) is not P-convergent to \( -\infty \) and \( f \) is not P-convergent to a finite limit.

(94) If \( f \) is P-convergent to \( -\infty \), then \( f \) is not P-convergent to \( +\infty \) and \( f \) is not P-convergent to a finite limit.

Let \( f \) be a function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \). We say that \( f \) is P-convergent if and only if

(Def. 17) \( f \) is P-convergent to a finite limit or P-convergent to \( +\infty \) or P-convergent to \( -\infty \).

Assume \( f \) is P-convergent. The functor \( P\lim f \) yielding an extended real is defined by

(Def. 18) there exists a real number \( p \) such that \( it = p \) and for every real number \( e \) such that \( 0 < e \) there exists a natural number \( N \) such that for every natural numbers \( n, m \) such that \( n \geq N \) and \( m \geq N \) holds \(|f(n, m) - it| < e\).
and \( f \) is P-convergent to a finite limit or \( it = +\infty \) and \( f \) is P-convergent to \( +\infty \) or \( it = -\infty \) and \( f \) is P-convergent to \( -\infty \).

Now we state the propositions:

(95) Let us consider a function \( f \) from \( \mathbb{N} \times \mathbb{N} \) into \( \overline{\mathbb{R}} \), and a real number \( r \). Suppose for every natural numbers \( n, m, f(n, m) = r \). Then

(i) \( f \) is P-convergent to a finite limit, and
(ii) \( \text{P-lim } f = r \).

(96) Let us consider a function \( f \) from \( \mathbb{N} \times \mathbb{N} \) into \( \overline{\mathbb{R}} \). Suppose for every natural numbers \( n_1, m_1, n_2, m_2 \) such that \( n_1 \leq n_2 \) and \( m_1 \leq m_2 \) holds \( f(n_1, m_1) \leq f(n_2, m_2) \). Then

(i) \( f \) is P-convergent, and
(ii) \( \text{P-lim } f = \sup \text{rng } f \).

(97) Let us consider functions \( f_1, f_2 \) from \( \mathbb{N} \times \mathbb{N} \) into \( \overline{\mathbb{R}} \). Suppose for every natural numbers \( n, m, f_1(n, m) \leq f_2(n, m) \). Then \( \sup \text{rng } f_1 \leq \sup \text{rng } f_2 \).

(98) Let us consider a function \( f \) from \( \mathbb{N} \times \mathbb{N} \) into \( \overline{\mathbb{R}} \), and natural numbers \( n, m \). Then \( f(n, m) \leq \sup \text{rng } f \).

Let us consider a function \( f \) from \( \mathbb{N} \times \mathbb{N} \) into \( \overline{\mathbb{R}} \) and an extended real number \( K \). Now we state the propositions:

(99) If for every natural numbers \( n, m, f(n, m) \leq K \), then \( \sup \text{rng } f \leq K \).

(100) If \( K \neq +\infty \) and for every natural numbers \( n, m, f(n, m) \leq K \), then \( \sup \text{rng } f < +\infty \).

Now we state the propositions:

(101) Let us consider a without \( -\infty \) function \( f \) from \( \mathbb{N} \times \mathbb{N} \) into \( \overline{\mathbb{R}} \). Then \( \sup \text{rng } f \neq +\infty \) if and only if there exists a real number \( K \) such that \( 0 < K \) and for every natural numbers \( n, m, f(n, m) \leq K \).

(102) Let us consider a function \( f \) from \( \mathbb{N} \times \mathbb{N} \) into \( \overline{\mathbb{R}} \), and an extended real \( c \). Suppose for every natural numbers \( n, m, f(n, m) = c \). Then

(i) \( f \) is P-convergent, and
(ii) \( \text{P-lim } f = c \), and
(iii) \( \text{P-lim } f = \sup \text{rng } f \).

(103) Let us consider a function \( f \) from \( \mathbb{N} \times \mathbb{N} \) into \( \overline{\mathbb{R}} \), and without \( -\infty \) functions \( f_1, f_2 \) from \( \mathbb{N} \times \mathbb{N} \) into \( \overline{\mathbb{R}} \). Suppose for every natural numbers \( n_1, m_1, n_2, m_2 \) such that \( n_1 \leq n_2 \) and \( m_1 \leq m_2 \) holds \( f_1(n_1, m_1) \leq f_1(n_2, m_2) \) and for every natural numbers \( n_1, m_1, n_2, m_2 \) such that \( n_1 \leq n_2 \) and \( m_1 \leq m_2 \) holds \( f_2(n_1, m_1) \leq f_2(n_2, m_2) \) and for every natural numbers \( n, m, f_1(n, m) + f_2(n, m) = f(n, m) \). Then
(i) \( f \) is P-convergent, and
(ii) \( \text{P-lim } f = \sup \text{rng } f \), and
(iii) \( \text{P-lim } f = \text{P-lim } f_1 + \text{P-lim } f_2 \), and
(iv) \( \sup \text{rng } f = \sup \text{rng } f_1 + \sup \text{rng } f_2 \).

The theorem is a consequence of (96) and (101).

Let us consider a without \(-\infty\) function \( f_1 \) from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \), a function \( f_2 \) from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \), and a real number \( c \). Now we state the propositions:

(104) Suppose \( 0 \leq c \) and for every natural numbers \( n, m \), \( f_2(n, m) = c \cdot f_1(n, m) \). Then

(i) \( \sup \text{rng } f_2 = c \cdot \sup \text{rng } f_1 \), and
(ii) \( f_2 \) is without \(-\infty\).

The theorem is a consequence of (102) and (101).

(105) Suppose \( 0 \leq c \) and for every natural numbers \( n_1, m_1, n_2, m_2 \) such that \( n_1 \leq n_2 \) and \( m_1 \leq m_2 \) holds \( f_1(n_1, m_1) \leq f_1(n_2, m_2) \) and for every natural numbers \( n, m \), \( f_2(n, m) = c \cdot f_1(n, m) \). Then

(i) for every natural numbers \( n_1, m_1, n_2, m_2 \) such that \( n_1 \leq n_2 \) and \( m_1 \leq m_2 \) holds \( f_2(n_1, m_1) \leq f_2(n_2, m_2) \), and
(ii) \( f_2 \) is without \(-\infty\) and P-convergent, and
(iii) \( \text{P-lim } f_2 = \sup \text{rng } f_2 \), and
(iv) \( \text{P-lim } f_2 = c \cdot \text{P-lim } f_1 \).

The theorem is a consequence of (96) and (104).

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