Constructing Binary Huffman Tree

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Summary. Huffman coding is one of a most famous entropy encoding methods for lossless data compression [16]. JPEG and ZIP formats employ variants of Huffman encoding as lossless compression algorithms. Huffman coding is a bijective map from source letters into leaves of the Huffman tree constructed by the algorithm. In this article we formalize an algorithm constructing a binary code tree, Huffman tree.

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The notation and terminology used in this paper have been introduced in the following articles: [9], [11], [20], [21], [14], [11], [12], [24], [22], [2], [3], [13], [19], [17], [25], [26], [24], [1], [5], [6], 7, and [13].

1. Constructing Binary Decoded Trees

Let $D$ be a non empty set and $x$ be an element of $D$. Observe that the root tree of $x$ is binary as a decorated tree.

The functor $\mathbb{R}_N$ yielding a set is defined by the term

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(Def. 1) \( \mathbb{N} \times \mathbb{R} \).

Note that \( \mathbb{R}_\mathbb{N} \) is non empty.

Let \( D \) be a non empty set. The binary finite trees of \( D \) yielding a set of trees decorated with elements of \( D \) is defined by

(Def. 2) Let us consider a tree \( T \) decorated with elements of \( D \). Then \( \text{dom}\ T \) is finite and \( T \) is binary if and only if \( T \in \text{it} \).

The Boolean binary finite trees of \( D \) yielding a non empty subset of \( \text{FinTrees}(\mathbb{R}_\mathbb{N}) \) is defined by the term

(Def. 3) \( \{ x, \text{where } x \text{ is an element of } 2^\alpha : x \text{ is finite and } x \neq \emptyset \} \), where \( \alpha \) is the binary finite trees of \( D \).

In this paper \( \mathcal{S} \) denotes a non empty finite set, \( p \) denotes a probability on the trivial \( \sigma \)-field of \( \mathcal{S} \), \( T_1 \) denotes a finite sequence of elements of the Boolean binary finite trees of \( \mathbb{R}_\mathbb{N} \), and \( q \) denotes a finite sequence of elements of \( \mathbb{N} \).

Let us consider \( \mathcal{S} \) and \( p \). The functor \( \text{InitTrees} \) yielding a non empty finite subset of the binary finite trees of \( \mathbb{R}_\mathbb{N} \) is defined by the term

(Def. 4) \( \{ T, \text{where } T \text{ is an element of } \text{FinTrees}(\mathbb{R}_\mathbb{N}) : T \text{ is a finite binary tree decorated with elements of } \mathbb{R}_\mathbb{N} \text{ and there exists an element } x \text{ of } \mathcal{S} \text{ such that } T = \text{the root tree of } ((\text{CFS}(\mathcal{S}))^{-1}(x), p(\{x\})) \} \).

Let \( p \) be a tree decorated with elements of \( \mathbb{R}_\mathbb{N} \). The value of root from right of \( p \) yielding a real number is defined by the term

(Def. 5) \( p(\emptyset)_2 \).

The value of root from left of \( p \) yielding a natural number is defined by the term

(Def. 6) \( p(\emptyset)_1 \).

Let \( T \) be a finite binary tree decorated with elements of \( \mathbb{R}_\mathbb{N} \) and \( p \) be an element of \( \text{dom}\ T \). The value of tree of \( p \) yielding a real number is defined by the term

(Def. 7) \( T(p)_2 \).

Let \( p, q \) be finite binary trees decorated with elements of \( \mathbb{R}_\mathbb{N} \) and \( k \) be a natural number. The functor \( \text{MakeTree}(p, q, k) \) yielding a finite binary tree decorated with elements of \( \mathbb{R}_\mathbb{N} \) is defined by the term

(Def. 8) \( \langle k, (\text{the value of root from right of } p) + (\text{the value of root from right of } q) \rangle \text{-tree}(p, q) \).

Let \( X \) be a non empty finite subset of the binary finite trees of \( \mathbb{R}_\mathbb{N} \). The maximal value of \( X \) yielding a natural number is defined by

(Def. 9) There exists a non empty finite subset \( L \) of \( \mathbb{N} \) such that

(i) \( L = \{ \text{the value of root from left of } p, \text{ where } p \text{ is an element of the binary finite trees of } \mathbb{R}_\mathbb{N} : p \in X \} \), and

(ii) \( \text{it} = \max L \).

Now we state the propositions:
(1) Let us consider a non empty finite subset $X$ of the binary finite trees of $\mathbb{R}_N$ and a finite binary tree $w$ decorated with elements of $\mathbb{R}_N$. Suppose $X = \{w\}$. Then the maximal value of $X = \text{the value of root from left of } w$. \textbf{Proof:} Consider $L$ being a non empty finite subset of $\mathbb{N}$ such that $L = \{\text{the value of root from left of } p, \text{ where } p \in X\}$ and the maximal value of $X = \max L$. For every element $n$ such that $n \in L$ holds $n = \text{the value of root from left of } w$. For every element $n$ such that $n = \text{the value of root from left of } w$ holds $n \in L$. \hfill $\square$

(2) Let us consider non empty finite subsets $X$, $Y$, $Z$ of the binary finite trees of $\mathbb{R}_N$. Suppose $Z = X \cup Y$. Then the maximal value of $Z = \max(\text{the maximal value of } X, \text{the maximal value of } Y)$.

(3) Let us consider non empty finite subsets $X$, $Z$ of the binary finite trees of $\mathbb{R}_N$ and a set $Y$. Suppose $Z = X \setminus Y$. Then the maximal value of $Z \leq \text{the maximal value of } X$. The theorem is a consequence of (2).

(4) Let us consider a non empty finite subset $X$ of the binary finite trees of $\mathbb{R}_N$ and an element $p$ of the binary finite trees of $\mathbb{R}_N$. Suppose $p \in X$. Then the value of root from left of $p \leq \text{the maximal value of } X$.

Let $X$ be a non empty finite subset of the binary finite trees of $\mathbb{R}_N$. A minimal value tree of $X$ is a finite binary tree decorated with elements of $\mathbb{R}_N$ and is defined by

(Def. 10) (i) $it \in X$, and

(ii) for every finite binary tree $q$ decorated with elements of $\mathbb{R}_N$ such that $q \in X$ holds the value of root from right of $it \leq \text{the value of root from right of } q$.

Now we state the propositions:

(5) $\text{InitTrees}_p = \mathcal{S}$. \textbf{Proof:} Reconsider $f_1 = (\text{CFS}(\mathcal{S}))^{-1}$ as a function from $\mathcal{S}$ into Seg $\mathcal{S}$. Define $\mathcal{P}[\text{element, element}] \equiv \mathcal{S}_2 = \text{the root tree of } (f_1(\mathcal{S}_1), p(\{\mathcal{S}_1\})).$ For every element $x$ such that $x \in \mathcal{S}$ there exists an element $y$ such that $y \in \text{InitTrees}_p$ and $\mathcal{P}[x, y]$ by [12, (5)], [13, (87)], [17, (3)]. Consider $f$ being a function from $\mathcal{S}$ into $\text{InitTrees}_p$ such that for every element $x$ such that $x \in \mathcal{S}$ holds $\mathcal{P}[x, f(x)]$ from [12, Sch. 1]. \hfill $\square$

(6) Let us consider a non empty finite subset $X$ of the binary finite trees of $\mathbb{R}_N$ and finite binary trees $s$, $t$ decorated with elements of $\mathbb{R}_N$. Then $\text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1)) \notin X$.

Let $X$ be a set. The set of leaves of $X$ yielding a subset of $2^{\mathbb{R}_N}$ is defined by the term

(Def. 11) \{Leaves($p$), where $p$ is an element of the binary finite trees of $\mathbb{R}_N : p \in X$\}.

Now we state the propositions:
(7) Let us consider a finite binary tree \(X\) decorated with elements of \(\mathbb{R}_N\). Then the set of leaves of \(\{X\} = \{\text{Leaves}(X)\}\). \textbf{Proof:} For every element \(x, x \in \text{the set of leaves of } \{X\} \text{ iff } x \in \{\text{Leaves}(X)\}\). \(\Box\)

(8) Let us consider sets \(X, Y\). Then the set of leaves of \(X \cup Y = (\text{the set of leaves of } X) \cup (\text{the set of leaves of } Y)\). \textbf{Proof:} For every element \(x, x \in \text{the set of leaves of } X \cup Y \text{ iff } x \in (\text{the set of leaves of } X) \cup (\text{the set of leaves of } Y)\). \(\Box\)

(9) Let us consider trees \(s, t\). Then \(\emptyset \notin \text{Leaves}(\widehat{s,t})\). \textbf{Proof:} For every element \(p, p \in \widehat{s,t} \text{ iff } p \in \text{the elementary tree of } 0\) by [4, (19), (29)], [10 (130)]. \(\Box\)

(10) Let us consider trees \(s, t\). Then \(\text{Leaves}(\widehat{s,t}) = (\{0\} \smallsetminus p, \text{ where } p \text{ is an element of } t \bigcap p \in \text{Leaves}(t)) \cup (\{1\} \smallsetminus p, \text{ where } p \text{ is an element of } s : p \in \text{Leaves}(s)\). The theorem is a consequence of (9). \textbf{Proof:} Set \(L = \{0\} \smallsetminus p, \text{ where } p \text{ is an element of } t : p \in \text{Leaves}(t)\}. Set \(R = \{1\} \smallsetminus p, \text{ where } p \text{ is an element of } s : p \in \text{Leaves}(s)\}. Set \(H = \text{Leaves}(\widehat{s,t})\). For every element \(x, x \in H \text{ iff } x \in L \cup R\) by [21 (23)], [9 (6)]. \(\Box\)

Let us consider decorated trees \(s, t\), an element \(x\), and a finite sequence \(q\) of elements of \(\mathbb{N}\). Now we state the propositions:

(11) If \(q \in \text{dom } t\), then \((x\text{-tree}(t,s))(\{0\} \smallsetminus q) = t(q)\).

(12) If \(q \in \text{dom } s\), then \((x\text{-tree}(t,s))(\{1\} \smallsetminus q) = s(q)\).

Now we state the propositions:

(13) Let us consider decorated trees \(s, t\) and an element \(x\).

Then \(\text{Leaves}(x\text{-tree}(t,s)) = \text{Leaves}(t) \cup \text{Leaves}(s)\). The theorem is a consequence of (10), (11), and (12). \textbf{Proof:} Set \(L = \{0\} \smallsetminus p, \text{ where } p \text{ is an element of } \text{dom } t : p \in \text{Leaves}(\text{dom } t)\}. Set \(R = \{1\} \smallsetminus p, \text{ where } p \text{ is an element of } \text{dom } s : p \in \text{Leaves}(\text{dom } s)\}. \text{For every element } z, z \in (x\text{-tree}(t,s))L \text{ iff } z \in t^\circ(\text{Leaves}(\text{dom } t)). \text{For every element } z, z \in (x\text{-tree}(t,s))R \text{ iff } z \in s^\circ(\text{Leaves}(\text{dom } s))\). \(\Box\)

(14) Let us consider a natural number \(k\) and finite binary trees \(s, t\) decorated with elements of \(\mathbb{R}_N\). Then \(\bigcup \text{the set of leaves of } \{s,t\} = \bigcup \text{the set of leaves of } \{\text{MakeTree}(t,s,k)\}\). The theorem is a consequence of (8), (7), and (13).

(15) \(\text{Leaves}(\text{the elementary tree of } 0) = \text{the elementary tree of } 0\). \textbf{Proof:} For every element \(x, x \in \text{Leaves}(\text{the elementary tree of } 0) \text{ iff } x \in \text{the elementary tree of } 0\) by [4 (29), (54)]. \(\Box\)

(16) Let us consider an element \(x\), a non empty set \(D\), and a finite binary tree \(T\) decorated with elements of \(D\). Suppose \(T = \text{the root tree of } x\). Then \(\text{Leaves}(T) = \{x\}\). The theorem is a consequence of (15).
2. Binary Huffman Tree

Let us consider $S$, $p$, $T_1$, and $q$. We say that $T_1$, $q$, and $p$ are constructing binary Huffman tree if and only if

(Def. 12) (i) $T_1(1) = \text{InitTrees} p$, and

(ii) $\text{len } T_1 = \overline{S}$, and

(iii) for every natural number $i$ such that $1 \leq i < \text{len } T_1$ there exist non empty finite subsets $X$, $Y$ of the binary finite trees of $\mathbb{R}_N$ and there exists a minimal value tree $s$ of $X$ and there exists a minimal value tree $t$ of $Y$ and there exists a finite binary tree $v$ decorated with elements of $\mathbb{R}_N$ such that $T_1(i) = X$ and $Y = X \setminus \{s\}$ and $v \in \{\text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1)), \text{MakeTree}(s, t, ((\text{the maximal value of } X) + 1))\}$ and $T_1(i + 1) = (X \setminus \{t, s\}) \cup \{v\}$, and

(iv) there exists a finite binary tree $T$ decorated with elements of $\mathbb{R}_N$ such that $\{T\} = T_1(\text{len } T_1)$, and

(v) $\text{dom } q = \overline{S}$, and

(vi) for every natural number $k$ such that $k \in \overline{S}$ holds $q(k) = \overline{T_1(k)}$ and $q(k) \neq 0$, and

(vii) for every natural number $k$ such that $k < \overline{S}$ holds $q(k + 1) = q(1) - k$, and

(viii) for every natural number $k$ such that $1 \leq k < \overline{S}$ holds $2 \leq q(k)$.

Now we state the proposition:

(17) There exists $T_1$ and there exists $q$ such that $T_1$, $q$, and $p$ are constructing binary Huffman tree. The theorem is a consequence of (5) and (6). PROOF: Define $\mathcal{A}[\text{natural number, set, set}] \equiv$ if there exist elements $u, v$ such that $u \neq v$ and $u, v \in \mathbb{S}_2$, then there exist non empty finite subsets $X$, $Y$ of the binary finite trees of $\mathbb{R}_N$ and there exists a minimal value tree $s$ of $X$ and there exists a minimal value tree $t$ of $Y$ and there exists a finite binary tree $w$ decorated with elements of $\mathbb{R}_N$ such that $\mathbb{S}_2 = X$ and $\mathbb{S}_1 = Y = X \setminus \{s\}$ and $w \in \{\text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1)), \text{MakeTree}(s, t, ((\text{the maximal value of } X) + 1))\}$ and $\mathbb{S}_3 = (X \setminus \{t, s\}) \cup \{w\}$. For every natural number $n$ such that $1 \leq n < \overline{S}$ for every element $x$ of the Boolean binary finite trees of $\mathbb{R}_N$, there exists an element $y$ of the Boolean binary finite trees of $\mathbb{R}_N$ such that $\mathcal{A}[n, x, y]$. Reconsider $I = \text{InitTrees} p$ as an element of the Boolean binary finite trees of $\mathbb{R}_N$. Consider $T_1$ being a finite sequence of elements of the Boolean binary finite trees of $\mathbb{R}_N$ such that $\text{len } T_1 = \overline{S}$ and $T_1(1) = I$ or $\overline{S} = 0$ and for every natural number $n$ such that $1 \leq n < \overline{S}$ holds $\mathcal{A}[n, T_1(n), T_1(n + 1)]$ from [15, Sch. 4]. Define $\mathcal{B}[\text{element, element}] \equiv$ there exists a finite set $X$ such that
$T_1(\mathcal{S}_1) = X$ and $\mathcal{S}_2 = \overline{X}$ and $\mathcal{S}_2 \neq 0$. For every natural number $k$ such that $k \in \text{Seg} \overline{\mathcal{S}}$ there exists an element $x$ of $\mathbb{N}$ such that $\mathcal{B}[k, x]$ by [11] (3)). Consider $q$ being a finite sequence of elements of $\mathbb{N}$ such that $\text{dom} q = \text{Seg} \overline{\mathcal{S}}$ and for every natural number $k$ such that $k \in \text{Seg} \overline{\mathcal{S}}$ holds $\mathcal{B}[k, q(k)]$ from $\mathcal{S}$, Sch. 5]. For every natural number $k$ such that $k \in \text{Seg} \overline{\mathcal{S}}$ holds $q(k) = T_1(k)$ and $q(k) \neq 0$. For every natural number $k$ such that $1 \leq k < \overline{\mathcal{S}}$ holds if $2 \leq q(k)$, then $q(k+1) = q(k) - 1$ by $\mathcal{S}$, (1)], [2] (10), (11), (13)]. Define $\mathcal{C}[\text{natural number}] \equiv$ if $\mathcal{S}_1 < \overline{\mathcal{S}}$, then $q(\mathcal{S}_1 + 1) = q(1) - \mathcal{S}_1$. For every natural number $n$ such that $\mathcal{C}[n]$ holds $\mathcal{C}[n + 1]$ by [2] (10), \mathcal{S}, (1)], [2] (14), (13)]. For every natural number $n$, $\mathcal{C}[n]$ from [2, Sch. 2]. For every natural number $n$ such that $1 \leq n < \overline{\mathcal{S}}$ holds $2 \leq q(n)$ by [2] (21), (13)]. For every natural number $k$ such that $1 \leq k < \text{len} T_1$ there exist non empty finite subsets $X$, $Y$ of the binary finite trees of $\mathbb{R}_\mathbb{N}$ and there exists a minimal value tree $s$ of $X$ and there exists a minimal value tree $t$ of $Y$ and there exists a finite binary tree $w$ decorated with elements of $\mathbb{R}_\mathbb{N}$ such that $T_1(k) = X$ and $Y = X \setminus \{s\}$ and $w \in \{\text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1)), \text{MakeTree}(s, t, ((\text{the maximal value of } X) + 1))\}$ and $T_1(k + 1) = (X \setminus \{t, s\}) \cup \{w\}$ by \mathcal{S}, (1)]. Consider $T_2$ being a finite set such that $T_1(\mathcal{S}) = T_2$ and $q(\mathcal{S}) = T_2$ and $q(\mathcal{S}) \neq 0$. Consider $u$ being an element such that $T_2 = \{u\}$.

Let us consider $\mathcal{S}$ and $p$. A binary Huffman tree of $p$ is a finite binary tree decorated with elements of $\mathbb{R}_\mathbb{N}$ and is defined by

(Def. 13) There exists a finite sequence $T_1$ of elements of the Boolean binary finite trees of $\mathbb{R}_\mathbb{N}$ and there exists a finite sequence $q$ of elements of $\mathbb{N}$ such that $T_1$, $q$, and $p$ are constructing binary Huffman tree and $\{it\} = T_1(\text{len} T_1)$.

In this paper $T$ denotes a binary Huffman tree of $p$.

Now we state the propositions:

(18) $\cup$ the set of leaves of InitTrees $p = \{z\}$, where $z$ is an element of $\mathbb{N} \times \mathbb{R}$ : there exists an element $x$ of $\mathcal{S}$ such that $z = ((\text{CFS}(\mathcal{S}))^{-1}(x), \text{p}(\{x\}))$. The theorem is a consequence of (16). PROOF: Set $L = \cup$ the set of leaves of InitTrees $p$. Set $R = \{z\}$, where $z$ is an element of $\mathbb{N} \times \mathbb{R}$ : there exists an element $x$ of $\mathcal{S}$ such that $z = ((\text{CFS}(\mathcal{S}))^{-1}(x), \text{p}(\{x\}))$. For every element $x, x \in L$ iff $x \in R$ by [13] (87)], [7] (3)]. □

(19) Suppose $T_1$, $q$, and $p$ are constructing binary Huffman tree. Let us consider a natural number $i$. Suppose $1 \leq i \leq \text{len} T_1$. Then $\cup$ the set of leaves of $T_1(i) = \{z\}$, where $z$ is an element of $\mathbb{N} \times \mathbb{R}$ : there exists an element $x$ of $\mathcal{S}$ such that $z = ((\text{CFS}(\mathcal{S}))^{-1}(x), \text{p}(\{x\}))$. The theorem is a consequence of (18), (8), and (14). PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\mathcal{S}_1 < \text{len} T_1$, then $\cup$ the set of leaves of $T_1(\mathcal{S}_1 + 1) = \{z\}$, where $z$ is an element of $\mathbb{N} \times \mathbb{R}$ : there exists an element $x$ of $\mathcal{S}$ such that $z = ((\text{CFS}(\mathcal{S}))^{-1}(x)$,
Leaves(T) = \{ z, \text{where } z \text{ is an element of } \mathbb{N} \times \mathbb{R} : \text{there exists an element } x \text{ of } \mathbb{S} \text{ such that } z = \langle (\text{CFS}(\mathbb{S}))^{-1}(x), p(\{x\}) \rangle \}. The theorem is a consequence of (19) and (7).

Let us consider elements t, s, r of dom T. Suppose

(i) t ∈ dom T \ Leaves(dom T), and
(ii) s = t \langle 0 \rangle, and
(iii) r = t \langle 1 \rangle.

Then the value of tree of t = (the value of tree of s) + (the value of tree of r). The theorem is a consequence of (21).

Let us consider a non empty finite subset X of the binary finite trees of \( \mathbb{R}_N \). Suppose a finite binary tree T decorated with elements of \( \mathbb{R}_N \) and for every natural number i, \( \mathbb{P}[i] \) from [2 Sch. 2].

Proof: For every element a such that
Suppose $T_1$, $q$, and $p$ are constructing binary Huffman tree. Let us consider a natural number $i$. Suppose $1 \leq i < \text{len} T_1$. Let us consider non empty finite subsets $X$, $Y$ of the binary finite trees of $\mathbb{R}_N$. Suppose

(i) $X = T_1(i)$, and

(ii) $Y = T_1(i+1)$.

Then the maximal value of $Y = (\text{the maximal value of } X) + 1$. **Proof:**

Consider $X$, $Y$ being non empty finite subsets of the binary finite trees of $\mathbb{R}_N$, $s$ being a minimal value tree of $X$, $t$ being a minimal value tree of $Y$, $v$ being a finite binary tree decorated with elements of $\mathbb{R}_N$ such that $T_1(i) = X$ and $Y = X \setminus \{s\}$ and $v \in \{\text{MakeTree}(t,s,((\text{the maximal value of } X) + 1)), \text{MakeTree}(s,t,((\text{the maximal value of } X) + 1))\}$ and $T_1(i + 1) = (X \setminus \{t,s\}) \cup \{v\}$. Consider $L_1$ being a non empty finite subset of $\mathbb{N}$ such that $L_1 = \{\text{the value of root from left of } p, \text{ where } p \text{ is an element of the binary finite trees of } \mathbb{R}_N : p \in X0\}$ and the maximal value of $X0 = \max L_1$. Consider $L_4$ being a non empty finite subset of $\mathbb{N}$ such that $L_4 = \{\text{the value of root from left of } p, \text{ where } p \text{ is an element of the binary finite trees of } \mathbb{R}_N : p \in Y0\}$ and the maximal value of $Y0 = \max L_4$. Reconsider $p_1 = v$ as an element of the binary finite trees of $\mathbb{R}_N$. For every extended real $x$ such that $x \in L_4$ holds $x \leq \text{the value of root from left of } p_1$ by [2, (16)]. □

Let us consider a natural number $i$, a non empty finite subset $X$ of the binary finite trees of $\mathbb{R}_N$, a finite binary tree $T$ decorated with elements of $\mathbb{R}_N$, an element $p$ of $\text{dom } T$, and an element $r$ of $\mathbb{N}$. Now we state the propositions:

(25) Suppose $T_1$, $q$, and $p$ are constructing binary Huffman tree. Then if $X = T_1(i)$, then if $T \in X$, then if $r = T(p)_{1}$, then $r \leq \text{the maximal value of } X$.

(26) Suppose $T_1$, $q$, and $p$ are constructing binary Huffman tree. Then if $X = T_1(i)$, then if $T \in X$, then if $r = T(p)_{1}$, then $r \leq \text{the maximal value of } X$.

Now we state the proposition:

(27) Suppose $T_1$, $q$, and $p$ are constructing binary Huffman tree. Let us consider a natural number $i$, finite binary trees $s$, $t$ decorated with elements of $\mathbb{R}_N$, and a non empty finite subset $X$ of the binary finite trees of $\mathbb{R}_N$. Suppose

(i) $X = T_1(i)$, and

(ii) $s$, $t \in X$. 

Let us consider a finite binary tree $z$ decorated with elements of $\mathbb{R}_N$. Suppose $z \in X$. Then $\{((\text{the maximal value of } X) + 1, (\text{the value of root from right of } t) + (\text{the value of root from right of } s)) \notin \text{rng } z\}$. The theorem is a consequence of (26).

Let $x$ be an element. Note that the root tree of $x$ is one-to-one.

Now we state the propositions:

(28) Let us consider a non empty finite subset $X$ of the binary finite trees of $\mathbb{R}_N$ and finite binary trees $s, t, w$ decorated with elements of $\mathbb{R}_N$. Suppose

(i) for every finite binary tree $T$ decorated with elements of $\mathbb{R}_N$ such that $T \in X$ for every element $p$ of $\text{dom } T$ for every element $r$ of $\mathbb{N}$ such that $r = T(p)_1$ holds $r \leq \text{the maximal value of } X$, and

(ii) for every finite binary trees $p, q$ decorated with elements of $\mathbb{R}_N$ such that $p, q \in X$ and $p \neq q$ holds $\text{rng } p \cap \text{rng } q = \emptyset$, and

(iii) $s, t \in X$, and

(iv) $s \neq t$, and

(v) $w \in X \setminus \{s, t\}$.

Then $\text{rng } \text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1)) \cap \text{rng } w = \emptyset$. The theorem is a consequence of (11) and (12). Proof: Set $d = \text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1))$. For every element $a$ such that $a \in \text{dom } d$ holds $a = \emptyset$ or there exists an element $f$ of $\text{dom } t$ such that $a = (0) \cap f$ or there exists an element $f$ of $\text{dom } s$ such that $a = (1) \cap f$ by [2] (23)]. Consider $n_2$ being an element such that $n_2 \in \text{rng } d \cap \text{rng } w$. Consider $a_1$ being an element such that $a_1 \in \text{dom } d$ and $n_2 = d(a_1)$. Consider $b_1$ being an element such that $b_1 \in \text{dom } w$ and $n_2 = w(b_1)$. $w \in X$ and $w \neq s$ and $w \neq t$. □

(29) Suppose $T_1, q, p$ are constructing binary Huffman tree. Let us consider a natural number $i$ and finite binary trees $T, S$ decorated with elements of $\mathbb{R}_N$. Suppose

(i) $T, S \in T_1(i)$, and

(ii) $T \neq S$.

Then $\text{rng } T \cap \text{rng } S = \emptyset$. The theorem is a consequence of (26) and (28). Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ if $1 \leq s_1 \leq \text{len } T_1$, then for every finite binary trees $T, S$ decorated with elements of $\mathbb{R}_N$ such that $T, S \in T_1(s_1)$ and $T \neq S$ holds $\text{rng } T \cap \text{rng } S = \emptyset$. For every natural number $i$ such that $\mathcal{P}[i]$ holds $\mathcal{P}[i + 1]$ by [21] (8)], [2] (16), (14)]. For every natural number $i$, $\mathcal{P}[i]$ from [2] Sch. 2]. □

(30) Let us consider a non empty finite subset $X$ of the binary finite trees of $\mathbb{R}_N$ and finite binary trees $s, t$ decorated with elements of $\mathbb{R}_N$. Suppose

(i) $s$ is one-to-one, and
(ii) $t$ is one-to-one, and
(iii) $t, s \in X$, and
(iv) $\text{rng } s \cap \text{rng } t = \emptyset$, and
(v) for every finite binary tree $z$ decorated with elements of $\mathbb{R}_N$ such that $z \in X$ holds $\langle (\text{the maximal value of } X) + 1, (\text{the value of root from right of } t) + (\text{the value of root from right of } s) \rangle \notin \text{rng } z$.

Then $\text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1))$ is one-to-one. The theorem is a consequence of (11) and (12). \textbf{Proof:} Set $d = \text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1))$. For every element $a$ such that $a \in \text{dom } d$ holds

$a = \emptyset$ or there exists an element $f$ of $\text{dom } t$ such that $a = \langle 0 \rangle \uparrow f$ or there exists an element $f$ of $\text{dom } s$ such that $a = \langle 1 \rangle \uparrow f$ by [2, (23)].

For every element $x$ such that $x \in \text{dom } d$ and $x \neq \emptyset$ holds $d(x) \neq d(\emptyset)$ by [11, (3)]. For every elements $x_1, x_2$ such that $x_1, x_2 \in \text{dom } d$ and $d(x_1) = d(x_2)$ holds it is not true that there exists an element $f$ of $\text{dom } s$ such that $x_1 = \langle 1 \rangle \uparrow f$ and there exists an element $f$ of $\text{dom } t$ such that $x_2 = \langle 0 \rangle \uparrow f$ by [11, (3)]. For every elements $x_1, x_2$ such that $x_1, x_2 \in \text{dom } d$ and $d(x_1) = d(x_2)$ holds $x_1 = x_2$. \hfill $\square$

(31) Suppose $T_1, q$, and $p$ are constructing binary Huffman tree. Let us consider a natural number $i$ and a finite binary tree $T$ decorated with elements of $\mathbb{R}_N$. If $T \in T_1(i)$, then $T$ is one-to-one. The theorem is a consequence of (27), (29), and (30). \textbf{Proof:} Define $\mathcal{P}[\text{natural number}] \equiv$ if $1 \leq i_1 \leq \text{len } T_1$, then for every finite binary tree $T$ decorated with elements of $\mathbb{R}_N$ such that $T \in T_1(i_1)$ holds $T$ is one-to-one. For every natural number $i$ such that $\mathcal{P}[i]$ holds $\mathcal{P}[i + 1]$ by [2] (16), (14)]. For every natural number $i$, $\mathcal{P}[i]$ from [2] Sch. 2. \hfill $\square$

Let us consider $p$.

Now we are at the position where we can present the Main Theorem of the paper: Every binary Huffman tree of $p$ is one-to-one.

\textbf{References}

Constructing binary Huffman tree


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