The Friendship Theorem

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Summary. In this article we prove the friendship theorem according to
the article [1], which states that if a group of people has the property that any
pair of persons have exactly one common friend, then there is a universal friend,
i.e. a person who is a friend of every other person in the group.

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The papers [3], [2], [6], [7], [11], [8], [9], [15], [14], [4], [13], [5], [17], [18], [12],
[16], and [10] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we adopt the following rules: $x$, $y$, $z$ are sets, $i$, $k$, $n$ are
natural numbers, $R$ is a binary relation, $P$ is a finite binary relation, and $p$, $q$
are finite sequences.

Let us consider $P$, $x$. Observe that $P^0x$ is finite.

We now state several propositions:
1. $\overline{R} = R^\sim$.
2. If $R$ is symmetric, then $R^0x = R^{-1}(x)$.
3. If $(p\downharpoonright k) \cap (p\downharpoonright k) = (q\downharpoonright n) \cap (q\downharpoonright n)$ and $k \leq n \leq \text{len} p$, then $p = (q\downharpoonright n - k) \cap (q\downharpoonright n - k)$.
4. If $n \in \text{dom} q$ and $p = (q\downharpoonright n) \cap (q\downharpoonright n)$, then $q = (p\downharpoonright \text{len} p - n) \cap (p\downharpoonright \text{len} p - n)$.

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(5) If \((p|k) \sim (p|k) = (q|n) \sim (q|i)\), then there exists \(i\) such that \(p = (q|i) \sim (q|i)\).

The scheme \(Sch\) deals with a non empty set \(A\), a non zero natural number \(B\), and a unary predicate \(P\), and states that:

\[
\{F \in A^B : P[F]\}
\]

provided the following requirements are met:

- For all finite sequences \(p, q\) of elements of \(A\) such that \(p \sim q\) is \(B\)-element and \(P[p \sim q]\) holds \(P[q \sim p]\), and
- For every element \(p\) of \(A^B\) such that \(P[p]\) and for every natural number \(i\) such that \(i < B\) and \(p = (p|i) \sim (p|i)\) holds \(i = 0\).

One can prove the following propositions:

(6) Let \(X\) be a non empty set, \(A\) be a non empty finite subset of \(X\), and \(P\) be a function from \(X\) into \(2^X\). Suppose that for every \(x\) such that \(x \in X\) holds \(P(x) = n\). Then

\[
\{F \in X^{k+1} : F(1) \in A \land \bigwedge_i (i \in \text{Seg} k \Rightarrow F(i + 1) \in P(F(i)))\} = \overline{A} \cdot n^k.
\]

(7) If \(\text{len}\; p\) is prime and there exists \(i\) such that \(0 < i < \text{len}\; p\) and \(p = (p|i) \sim (p|i)\), then \(\text{rng}\; p \subseteq \{p(1)\}\).

2. The Friendship Graph

Let us consider \(R\) and let \(x\) be an element of field \(R\). We say that \(x\) is universal friend if and only if:

(Def. 1) For every \(y\) such that \(y \in \text{field}\; R \setminus \{x\}\) holds \(\langle x, y \rangle \in R\).

Let \(R\) be a binary relation. We say that \(R\) has universal friend if and only if:

(Def. 2) There exists an element of field \(R\) which is universal friend.

Let \(R\) be a binary relation. We introduce \(R\) is without universal friend as an antonym of \(R\) has universal friend.

Let \(R\) be a binary relation. We say that \(R\) is friendship graph like if and only if:

(Def. 3) For all \(x, y\) such that \(x, y \in \text{field}\; R\) and \(x \neq y\) there exists \(z\) such that

\[
R^o x \cap \text{Coim}(R, y) = \{z\}.
\]

Let us observe that there exists a binary relation which is finite, symmetric, irreflexive, and friendship graph like.

A friendship graph is a finite symmetric irreflexive friendship graph like binary relation.

In the sequel \(F_1\) is a friendship graph.

The following propositions are true:
The friendship theorem

(8) $2 \mid \overline{F_1^\circ x}$.

(9) If $x, y \in \text{field } F_1$ and $\langle x, y \rangle \notin F_1$, then $\overline{F_1^\circ x} = \overline{F_1^\circ y}$.

(10) If $F_1$ is without universal friend and $x \in \text{field } F_1$, then $\overline{F_1^\circ x} > 2$.

(11) If $F_1$ is without universal friend and $x, y \in \text{field } F_1$, then $\overline{F_1^\circ x} = \overline{F_1^\circ y}$.

(12) If $F_1$ is without universal friend and $x \in \text{field } F_1$, then $\text{field } F_1 = 1 + \overline{F_1^\circ x} \cdot (\overline{F_1^\circ x} - 1)$.

(13) For all elements $x, y$ of field $F_1$ such that $x$ is universal friend and $x \neq y$ there exists $z$ such that $F_1^\circ y = \{x, z\}$ and $F_1^\circ z = \{x, y\}$.

3. The Friendship Theorem

Next we state the proposition

(14) If $F_1$ is non empty, then $F_1$ has universal friend.

References


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