Borel-Cantelli Lemma

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Summary. This article is about the Borel-Cantelli Lemma in probability theory. Necessary definitions and theorems are given in [10] and [7].

The notation and terminology used here have been introduced in the following papers: [17], [3], [4], [8], [13], [1], [2], [5], [15], [14], [21], [9], [12], [11], [16], [6], [20], [19], and [18].

For simplicity, we adopt the following rules: \( O_1 \) is a non empty set, \( S_1 \) is a \( \sigma \)-field of subsets of \( O_1 \), \( P_1 \) is a probability on \( S_1 \), \( A \) is a sequence of subsets of \( S_1 \), and \( n \) is an element of \( \mathbb{N} \).

Let \( D \) be a set, let \( x, y \) be extended real numbers, and let \( a, b \) be elements of \( D \). Then \((x > y \rightarrow a, b)\) is an element of \( D \).

We now state two propositions:

1. For every element \( k \) of \( \mathbb{N} \) and for every element \( x \) of \( \mathbb{R} \) such that \( k \) is odd and \( x > 0 \) and \( x \leq 1 \) holds \((-x \exp\text{Seq}_{\mathbb{R}})(k+1)+(-x \exp\text{Seq}_{\mathbb{R}})(k+2) \geq 0\).

2. For every element \( x \) of \( \mathbb{R} \) holds \(1 + x \leq (\text{the function } \exp)(x)\).

Let \( s \) be a sequence of real numbers. The functor \( \text{ExpFuncWithElementOf} \) yielding a sequence of real numbers is defined as follows:

\[
\text{Def. 1} \quad \text{For every natural number } d \text{ holds } \langle \text{ExpFuncWithElementOf}(P_1 \cdot A) \rangle(d) = \sum_{\kappa = 0}^{d} -s(d) \exp\text{Seq}_{\mathbb{R}}.
\]

Next we state two propositions:

3. \( \langle \text{The partial product of } \text{ExpFuncWithElementOf}(P_1 \cdot A) \rangle(n) = \langle \text{the function } \exp \rangle(-\langle \sum_{\kappa = 0}^{\kappa}(P_1 \cdot A)(\kappa) \rangle_{\kappa \in \mathbb{N}}(n)). \)

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(4) \((\text{the partial product of } P \cdot A^c)\)(n) \(\leq\) \((\text{the partial product of } \text{ExpFuncWithElementOf}(P \cdot A))\)(n).

Let \(n_1, n_2\) be elements of \(\mathbb{N}\). The functor \(\text{SeqOfIFGT1}(n_1, n_2)\) yielding a sequence of \(\mathbb{N}\) is defined by:

(Def. 2) For every element \(n\) of \(\mathbb{N}\) holds \((\text{SeqOfIFGT1}(n_1, n_2))(n) = (n > n_1 \rightarrow n + n_2, n)\).

Let \(k\) be an element of \(\mathbb{N}\). The \(\text{SeqOfIFGT2} k\) yields a sequence of \(\mathbb{N}\) and is defined by:

(Def. 3) For every element \(n\) of \(\mathbb{N}\) holds \((\text{SeqOfIFGT2} k)(n) = n + k\).

Let \(k\) be an element of \(\mathbb{N}\). The \(\text{SeqOfIFGT3} k\) yields a sequence of \(\mathbb{N}\) and is defined as follows:

(Def. 4) For every element \(n\) of \(\mathbb{N}\) holds \((\text{SeqOfIFGT3} k)(n) = (n > k \rightarrow 0, 1)\).

Let \(n_1, n_2\) be elements of \(\mathbb{N}\). One can verify that \(\text{SeqOfIFGT1}(n_1, n_2)\) is one-to-one and \(\text{SeqOfIFGT4}(n_1, n_2)\) is one-to-one.

(5)(i) For all sequences \(A, B\) of subsets of \(S_1\) such that \(n > n_1\) and \(B = A \cdot \text{SeqOfIFGT1}(n_1, n_2)\) holds \((\text{the partial product of } P \cdot B)(n) = (\text{the partial product of } P \cdot A)(n_1) \cdot (\text{the partial product of } P \cdot \text{SeqOfIFGT3}(A, n_1 + n_2 + 1))(n - n_1 - 1)\), and

(ii) for all sequences \(A, B, C\) of subsets of \(S_1\) and for every sequence \(e\) of \(\mathbb{N}\) such that \(n > n_1\) and \(C = A \cdot e\) and \(B = C \cdot \text{SeqOfIFGT1}(n_1, n_2)\) holds \((\text{the partial Intersection of } B)(n) = (\text{the partial Intersection of } C)(n_1) \cap (\text{the partial Intersection of } \text{SeqOfIFGT3}(C, n_1 + n_2 + 1))(n - n_1 - 1)\).

Let \(O_1\) be a non empty set, let \(S_1\) be a \(\sigma\)-field of subsets of \(O_1\), let \(P_1\) be a probability on \(S_1\), and let \(A\) be a sequence of subsets of \(S_1\). We say that \(A\) is all independent w.r.t. \(P_1\) if and only if the condition (Def. 8) is satisfied.
(Def. 8) Let $B$ be a sequence of subsets of $S_1$. Given a sequence $e$ of $\mathbb{N}$ such that $e$ is one-to-one and for every element $n$ of $\mathbb{N}$ holds $A(e(n)) = B(n)$. Let $n$ be an element of $\mathbb{N}$. Then (the partial product of $P_1 \cdot B)(n) = P_1((\text{the partial Intersection of } B)(n))$.

The following propositions are true:

(6) Suppose $n > n_1$ and $A$ is all independent w.r.t. $P_1$. Then $P_1((\text{the partial Intersection of } A^c)(n_1)) \cap (\text{the partial Intersection of } @\text{ShiftSeq}(A, n_1 + n_2 + 1))(n - n_1 - 1)) = (\text{the partial product of } P_1 \cdot A^c)(n_1) \cdot (\text{the partial product of } P_1 \cdot @\text{ShiftSeq}(A, n_1 + n_2 + 1))(n - n_1 - 1)$.

(7) $(\text{The partial Intersection of } A^c)(n) = (\text{the partial Union of } A)(n)^c$.

(8) $P_1((\text{the partial Intersection of } A^c)(n)) = 1 - P_1((\text{the partial Union of } A)(n))$.

Let $X$ be a set and let $A$ be a sequence of subsets of $X$. The UnionShiftSeq $A$ yielding a sequence of subsets of $X$ is defined as follows:

(Def. 9) For every element $n$ of $\mathbb{N}$ holds $(\text{the UnionShiftSeq } A)(n) = \bigcup \text{ShiftSeq}(A, n)$.

Let $O_1$ be a non empty set, let $S_1$ be a $\sigma$-field of subsets of $O_1$, and let $A$ be a sequence of subsets of $S_1$. The $\text{UnionShiftSeq } A$ yields a sequence of subsets of $S_1$ and is defined as follows:

(Def. 10) The $@\text{UnionShiftSeq } A = \text{the UnionShiftSeq } A$.

Let $O_1$ be a non empty set, let $S_1$ be a $\sigma$-field of subsets of $O_1$, and let $A$ be a sequence of subsets of $S_1$. The $@\text{lim sup } A$ yielding an event of $S_1$ is defined as follows:

(Def. 11) The $@\text{lim sup } A = \bigcap (\text{the UnionShiftSeq } A)$.

Let $X$ be a set and let $A$ be a sequence of subsets of $X$. The IntersectShiftSeq $A$ yielding a sequence of subsets of $X$ is defined as follows:

(Def. 12) For every element $n$ of $\mathbb{N}$ holds $(\text{the IntersectShiftSeq } A)(n) = \text{Intersection ShiftSeq}(A, n)$.

Let $O_1$ be a non empty set, let $S_1$ be a $\sigma$-field of subsets of $O_1$, and let $A$ be a sequence of subsets of $S_1$. The $@\text{IntersectShiftSeq } A$ yielding a sequence of subsets of $S_1$ is defined as follows:

(Def. 13) The $@\text{IntersectShiftSeq } A = \text{the IntersectShiftSeq } A$.

Let $O_1$ be a non empty set, let $S_1$ be a $\sigma$-field of subsets of $O_1$, and let $A$ be a sequence of subsets of $S_1$. The $@\text{lim inf } A$ yielding an event of $S_1$ is defined by:

(Def. 14) The $@\text{lim inf } A = \bigcup (\text{the IntersectShiftSeq } A)$.

The following propositions are true:

(9) $(\text{The IntersectShiftSeq } A^c)(n) = (\text{the UnionShiftSeq } A)(n)^c$. 

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(10) Suppose \( A \) is all independent w.r.t. \( P_1 \). Then \( P_1((\text{the partial Intersection of } A^c(n)) = (\text{the partial product of } P_1 \cdot A^c)(n) \).

(11) Let \( X \) be a set and \( A \) be a sequence of subsets of \( X \). Then

(i) the superior setsequence \( A = \text{the UnionShiftSeq} A \), and

(ii) the inferior setsequence \( A = \text{the IntersectShiftSeq} A \).

(12)(i) The superior setsequence \( A = \text{the @UnionShiftSeq} A \), and

(ii) the inferior setsequence \( A = \text{the @IntersectShiftSeq} A \).

Let \( O_1 \) be a non empty set, let \( S_1 \) be a \( \sigma \)-field of subsets of \( O_1 \), let \( P_1 \) be a probability on \( S_1 \), and let \( A \) be a sequence of subsets of \( S_1 \). The functor \( \text{SumShiftSeq}(P_1, A) \) yields a sequence of real numbers and is defined by:

(Def. 15) For every element \( n \) of \( \mathbb{N} \) holds \( (\text{SumShiftSeq}(P_1, A))(n) = \sum(P_1 \cdot \text{@ShiftSeq}(A, n)) \).

We now state several propositions:

(13) If \( (\sum_{\alpha=0}^n (P_1 \cdot A)((\alpha))_{\alpha \in \mathbb{N}} \) is convergent, then \( P_1((\text{the @lim sup} A) = 0 \) and \( \lim \text{SumShiftSeq}(P_1, A) = 0 \) and \( \text{SumShiftSeq}(P_1, A) \) is convergent.

(14)(i) For every set \( X \) and for every sequence \( A \) of subsets of \( X \) and for every element \( n \) of \( \mathbb{N} \) and for every set \( x \) holds there exists an element \( k \) of \( \mathbb{N} \) such that \( x \in (\text{ShiftSeq}(A, n))(k) \) iff there exists an element \( k \) of \( \mathbb{N} \) such that \( k \geq n \) and \( x \in A(k) \),

(ii) for every set \( X \) and for every sequence \( A \) of subsets of \( X \) and for every set \( x \) holds \( x \in \text{Intersection} (\text{the UnionShiftSeq} A) \) iff for every element \( m \) of \( \mathbb{N} \) there exists an element \( n \) of \( \mathbb{N} \) such that \( n \geq m \) and \( x \in A(n) \),

(iii) for every sequence \( A \) of subsets of \( S_1 \) and for every set \( x \) holds \( x \in \bigcap (\text{the @UnionShiftSeq} A) \) iff for every element \( m \) of \( \mathbb{N} \) there exists an element \( n \) of \( \mathbb{N} \) such that \( n \geq m \) and \( x \in A(n) \),

(iv) for every set \( X \) and for every sequence \( A \) of subsets of \( X \) and for every set \( x \) holds \( x \in \bigcup (\text{the IntersectShiftSeq} A) \) iff there exists an element \( n \) of \( \mathbb{N} \) such that \( k \geq n \) holds \( x \in A(k) \),

(v) for every sequence \( A \) of subsets of \( S_1 \) and for every set \( x \) holds \( x \in \bigcup (\text{the @IntersectShiftSeq} A^c) \) iff there exists an element \( n \) of \( \mathbb{N} \) such that for every element \( k \) of \( \mathbb{N} \) such that \( k \geq n \) holds \( x \in A(k) \), and

(vi) for every sequence \( A \) of subsets of \( S_1 \) and for every element \( x \) of \( O_1 \) holds \( x \in \bigcup (\text{the @IntersectShiftSeq} A^c) \) iff there exists an element \( n \) of \( \mathbb{N} \) such that for every element \( k \) of \( \mathbb{N} \) such that \( k \geq n \) holds \( x \notin A(k) \).

(15)(i) \( \lim \sup A = \text{the @lim sup} A, \)

(ii) \( \lim \inf A = \text{the @lim inf} A, \)

(iii) \( \text{the @lim inf} A^c = (\text{the @lim sup} A)^c, \)

(iv) \( P_1(\text{the @lim inf} A^c) + P_1(\text{the @lim sup} A) = 1, \) and

(v) \( P_1(\lim \inf(A^c)) + P_1(\lim \sup A) = 1. \)
(16)(i) If \((\sum_{n=0}^{\infty}(P_{n} \cdot A)(\alpha))_{\kappa \in \mathbb{N}}\) is convergent, then \(P_{1}(\lim \sup A) = 0\) and \(P_{1}(\lim \inf (A^{c})) = 1\), and

(ii) if \(A\) is all independent w.r.t. \(P_{1}\) and \((\sum_{n=0}^{\infty}(P_{n} \cdot A)(\alpha))_{\kappa \in \mathbb{N}}\) is divergent to \(+\infty\), then \(P_{1}(\lim \inf (A^{c})) = 0\) and \(P_{1}(\lim \sup A) = 1\).

(17) If \((\sum_{n=0}^{\infty}(P_{n} \cdot A)(\alpha))_{\kappa \in \mathbb{N}}\) is not convergent and \(A\) is all independent w.r.t. \(P_{1}\), then \(P_{1}(\lim \inf (A^{c})) = 0\) and \(P_{1}(\lim \sup A) = 1\).

(18) If \(A\) is all independent w.r.t. \(P_{1}\), then \(P_{1}(\lim \inf (A^{c})) = 0\) or \(P_{1}(\lim \inf (A^{c})) = 1\) but \(P_{1}(\lim \sup A) = 0\) or \(P_{1}(\lim \sup A) = 1\).

(19) \((\sum_{n=0}^{\infty}(P_{1} \cdot \Delta \text{ShiftSeq}(A, n_{1}+1))(\alpha))_{\kappa \in \mathbb{N}}(n) \leq (\sum_{n=0}^{\infty}(P_{1} \cdot A)(\alpha))_{\kappa \in \mathbb{N}}(n_{1}+1+n) - (\sum_{n=0}^{\infty}(P_{1} \cdot A)(\alpha))_{\kappa \in \mathbb{N}}(n_{1}).

(20) \(P_{1}(\text{the @IntersectShiftSeq } A^{c})(n)) = 1 - P_{1}(\text{the @UnionShiftSeq } A)(n)).

(21)(i) If \(A^{c}\) is all independent w.r.t. \(P_{1}\), then \(P_{1}(\text{the partial Intersection of } A)(n) = (\text{the partial product of } P_{1} \cdot A)(n)\), and

(ii) if \(A\) is all independent w.r.t. \(P_{1}\), then \(1 - P_{1}(\text{the partial Union of } A)(n)) = (\text{the partial product of } P_{1} \cdot A^{c})(n)\).

**References**


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