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Uniform Boundedness Principle

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Summary. In this article at first, we proved the lemma of the inferior limit and the superior limit. Next, we proved the Baire category theorem (Banach space version) [20], [9], [3], quoted it and proved the uniform boundedness principle. Moreover, the proof of the Banach-Steinhaus theorem is added.

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The articles [17], [18], [15], [12], [19], [1], [21], [5], [8], [7], [16], [10], [6], [13], [4], [2], [14], and [11] provide the terminology and notation for this paper.

1. Uniform Boundedness Principle

The following two propositions are true:

(1) For every sequence s_1 of real numbers and for every real number r such that s_1 is bounded and $0 \leq r$ holds $\liminf(r\, s_1) = r \cdot \liminf s_1$.

(2) For every sequence s_1 of real numbers and for every real number r such that s_1 is bounded and $0 \leq r$ holds $\limsup(r\, s_1) = r \cdot \limsup s_1$.

Let X be a real Banach space. One can verify that MetricSpaceNorm X is complete.

Let X be a real Banach space, let x_0 be a point of X, and let r be a real number. The functor $\mathrm{Ball}(x_0, r)$ yielding a subset of X is defined as follows:

(Def. 1) $\mathrm{Ball}(x_0, r) = \{x; x$ ranges over points of X: $\|x_0 - x\| < r\}$.

The following propositions are true:

(3) Let X be a real Banach space and Y be a sequence of subsets of X. Suppose $\bigcup \mathrm{rng}\, Y =$ the carrier of X and for every element n of $\mathbb{N}$ holds $Y(n)$ is closed. Then there exists an element n_0 of $\mathbb{N}$ and there exists

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a real number r and there exists a point x_0 of X such that $0 < r$ and $\mathrm{Ball}(x_0, r) \subseteq Y(n_0)$.

(4) Let X, Y be real normed spaces and f be a bounded linear operator from X into Y. Then

(i) f is Lipschitzian on the carrier of X and continuous on the carrier of X, and

(ii) for every point x of X holds f is continuous in x.

(5) Let X be a real Banach space, Y be a real normed space, and T be a subset of the real norm space of bounded linear operators from X into Y. Suppose that for every point x of X there exists a real number K such that $0 \leq K$ and for every point f of the real norm space of bounded linear operators from X into Y such that $f \in T$ holds $\|f(x)\| \leq K$. Then there exists a real number L such that

(i) $0 \leq L$, and

(ii) for every point f of the real norm space of bounded linear operators from X into Y such that $f \in T$ holds $\|f\| \leq L$.

Let X, Y be real normed spaces, let H be a function from $\mathbb{N}$ into the carrier of the real norm space of bounded linear operators from X into Y, and let x be a point of X. The functor $H\#x$ yields a sequence of Y and is defined by:

(Def. 2) For every element n of $\mathbb{N}$ holds $(H\#x)(n) = H(n)(x)$.

The following proposition is true

(6) Let X be a real Banach space, Y be a real normed space, v_1 be a sequence of the real norm space of bounded linear operators from X into Y, and t_1 be a function from X into Y. Suppose that for every point x of X holds $v_1\#x$ is convergent and $t_1(x) = \lim(v_1\#x)$. Then

(i) t_1 is a bounded linear operator from X into Y,

(ii) for every point x of X holds $\|t_1(x)\| \leq \liminf\|v_1\| \cdot \|x\|$, and

(iii) for every point t_2 of the real norm space of bounded linear operators from X into Y such that $t_2 = t_1$ holds $\|t_2\| \leq \liminf\|v_1\|$.

2. Banach-Steinhaus Theorem

We now state two propositions:

(7) Let X be a real Banach space, X_0 be a subset of LinearTopSpaceNorm X, Y be a real Banach space, and v_1 be a sequence of the real norm space of bounded linear operators from X into Y. Suppose that

(i) X_0 is dense,

(ii) for every point x of X such that $x \in X_0$ holds $v_1\#x$ is convergent, and

(iii) for every point x of X there exists a real number K such that $0 \leq K$ and for every element n of $\mathbb{N}$ holds $\|(v_1\#x)(n)\| \leq K$.

Let x be a point of X. Then $v_1\#x$ is convergent.

(8) Let X, Y be real Banach spaces, X_0 be a subset of LinearTopSpaceNorm X, and v_1 be a sequence of the real norm space of bounded linear operators from X into Y. Suppose that (i) X_0 is dense,
(ii) for every point x of X such that $x \in X_0$ holds $v_1\#x$ is convergent, and
(iii) for every point x of X there exists a real number K such that $0 \leq K$ and for every element n of $\mathbb{N}$ holds $\|(v_1\#x)(n)\| \leq K$.

Then there exists a point t_1 of the real norm space of bounded linear operators from X into Y such that for every point x of X holds $v_1\#x$ is convergent and $t_1(x) = \lim(v_1\#x)$ and $\|t_1(x)\| \leq \liminf\|v_1\| \cdot \|x\|$ and $\|t_1\| \leq \liminf\|v_1\|$.

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