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Properties of First and Second Order Cutting of Binary Relations

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Summary. This paper introduces some notions concerning binary relations according to [9]. It is also an attempt to complement the knowledge contained in the Mizar Mathematical Library regarding binary relations. We define here an image and inverse image of element of set A under binary relation of two sets A, B as image and inverse image of singleton of the element under this relation, respectively. Next, we define "The First Order Cutting Relation of two sets A, B under a subset of the set A" as the union of images of elements of this subset under the relation. We also define "The Second Order Cutting Subset of the Cartesian Product of two sets A, B under a subset of the set A" as an intersection of images of elements of this subset under the subset of the Cartesian Product. The paper also defines first and second projection of binary relations. The main goal of the article is to prove properties and collocations of definitions introduced in this paper.

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The articles [10], [6], [11], [7], [12], [13], [5], [3], [4], [2], [8], and [1] provide the notation and terminology for this paper.

1. Preliminaries

We adopt the following rules: x, y, X, Y, A, B, C, M are sets and P, Q, R, R_1, R_2 are binary relations.

Let X be a set. We introduce $\{\{*\} : * \in X\}$ as a synonym of SmallestPartition (X).

The following propositions are true:

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(1) $y \in \{\{*\} : * \in X\}$ iff there exists x such that $y = \{x\}$ and $x \in X$.

(2) $X = \emptyset$ iff $\{\{*\} : * \in X\} = \emptyset$.

(3) $\{\{*\} : * \in X \cup Y\} = \{\{*\} : * \in X\} \cup \{\{*\} : * \in Y\}$.

(4) $\{\{*\} : * \in X \cap Y\} = \{\{*\} : * \in X\} \cap \{\{*\} : * \in Y\}$.

(5) $\{\{*\} : * \in X \setminus Y\} = \{\{*\} : * \in X\} \setminus \{\{*\} : * \in Y\}$.

(6) $X \subseteq Y$ iff $\{\{*\} : * \in X\} \subseteq \{\{*\} : * \in Y\}$.

Let M be a set and let X, Y be families of subsets of M. Then $X \cap Y$ is a family of subsets of M.

We now state two propositions:

(7) For all families B_1, B_2 of subsets of M holds $\mathrm{Intersect}(B_1) \cap \mathrm{Intersect}(B_2) \subseteq \mathrm{Intersect}(B_1 \cap B_2)$.

(8) $(P \cap Q) \cdot R \subseteq (P \cdot R) \cap (Q \cdot R)$.

2. The First Order Cutting of Binary Relation of Two Sets A, B under Subset of the Set A

Let X, Y be sets, let R be a relation between X and Y, and let x be an element of X. The functor $R^\circ x$ yielding a subset of Y is defined as follows:

(Def. 1) $R^\circ x = R^\circ\{x\}$.

The following propositions are true:

(9) $y \in R^\circ\{x\}$ iff $\langle x, y\rangle \in R$.

(10) $(R_1 \cup R_2)^\circ\{x\} = R_1{}^\circ\{x\} \cup R_2{}^\circ\{x\}$.

(11) $(R_1 \cap R_2)^\circ\{x\} = R_1{}^\circ\{x\} \cap R_2{}^\circ\{x\}$.

(12) $(R_1 \setminus R_2)^\circ\{x\} = R_1{}^\circ\{x\} \setminus R_2{}^\circ\{x\}$.

(13) $(R_1 \cap R_2)^\circ\{\{*\} : * \in X\} \subseteq R_1{}^\circ\{\{*\} : * \in X\} \cap R_2{}^\circ\{\{*\} : * \in X\}$.

Let X, Y be sets, let R be a relation between X and Y, and let x be an element of X. The functor $R^{-1}(x)$ yields a subset of X and is defined by:

(Def. 2) $R^{-1}(x) = R^{-1}(\{x\})$.

One can prove the following propositions:

(14) Let A be a set, F be a family of subsets of A, and R be a binary relation. Then $R^\circ \bigcup F = \bigcup\{R^\circ X; X$ ranges over subsets of A: $X \in F\}$.

(15) For every non empty set A and for every subset X of A holds $X = \bigcup\{\{x\}; x$ ranges over elements of A: $x \in X\}$.

(16) For every non empty set A and for every subset X of A holds $\{\{x\}; x$ ranges over elements of A: $x \in X\}$ is a family of subsets of A.

(17) Let A be a non empty set, B be a set, X be a subset of A, and R be a relation between A and B. Then $R^\circ X = \bigcup\{R^\circ x; x$ ranges over elements of A: $x \in X\}$.

(18) Let A be a non empty set, B be a set, X be a subset of A, and R be a relation between A and B. Then $\{R^\circ x; x$ ranges over elements of A: $x \in X\}$ is a family of subsets of B.

Let A, B be sets, let R be a subset of $[\!:A, 2^B:\!]$, and let X be a set. Then $R^\circ X$ is a family of subsets of B.

Let A be a set and let R be a binary relation. The functor R^A yields a function and is defined as follows:

(Def. 3) $\operatorname{dom}(R^A) = 2^A$ and for every set X such that $X \subseteq A$ holds $R^A(X) = R^\circ X$.

Let B, A be sets and let R be a subset of $[\!:A, B:\!]$. We introduce ${}^\circ R$ as a synonym of R^A.

One can prove the following propositions:

(19) For all sets A, B and for every subset R of $[\!:A, B:\!]$ such that $X \in \operatorname{dom} {}^\circ R$ holds $({}^\circ R)(X) = R^\circ X$.

(20) For all sets A, B and for every subset R of $[\!:A, B:\!]$ holds $\operatorname{rng} {}^\circ R \subseteq 2^{\operatorname{rng} R}$.

(21) For all sets A, B and for every subset R of $[\!:A, B:\!]$ holds ${}^\circ R$ is a function from 2^A into $2^{\operatorname{rng} R}$.

Let B, A be sets and let R be a subset of $[\!:A, B:\!]$. Then ${}^\circ R$ is a function from 2^A into 2^B.

Next we state the proposition

(22) For all sets A, B and for every subset R of $[\!:A, B:\!]$ holds $\bigcup(({}^\circ R)^\circ A) \subseteq R^\circ \bigcup A$.

3. The Second Order Cutting of Binary Relation of Two Sets A, B under Subset of the Set A

For simplicity, we adopt the following rules: X, X_1, X_2 are subsets of A, Y is a subset of B, R, R_1, R_2 are subsets of $[\!:A, B:\!]$, F is a family of subsets of A, and F_1 is a family of subsets of $[\!:A, B:\!]$.

Let A, B be sets, let X be a subset of A, and let R be a subset of $[\!:A, B:\!]$. The functor $R[X]$ is defined as follows:

(Def. 4) $R[X] = \operatorname{Intersect}(({}^\circ R)^\circ\{\{*\} : * \in X\})$.

Let A, B be sets, let X be a subset of A, and let R be a subset of $[\!:A, B:\!]$. Then $R[X]$ is a subset of B.

We now state a number of propositions:

(23) $({}^\circ R)^\circ\{\{*\} : * \in X\} = \emptyset$ iff $X = \emptyset$.

(24) If $y \in R[X]$, then for every set x such that $x \in X$ holds $y \in R^\circ\{x\}$.

(25) Let B be a non empty set, A be a set, X be a subset of A, y be an element of B, and R be a subset of $[\!:A, B:\!]$. Then $y \in R[X]$ if and only if for every set x such that $x \in X$ holds $y \in R^\circ\{x\}$.

(26) If $({}^{\circ}R)^{\circ}\{\{*\} : * \in X_1\} = \emptyset$, then $R[X_1 \cup X_2] = R[X_2]$.

(27) $R[X_1 \cup X_2] = R[X_1] \cap R[X_2]$.

(28) Let A be a non empty set, B be a set, F be a family of subsets of A, and R be a relation between A and B. Then $\{R[X]; X$ ranges over subsets of A: $X \in F\}$ is a family of subsets of B.

(29) If $X = \emptyset$, then $R[X] = B$.

(30) $\bigcup F = \emptyset$ iff for every set X such that $X \in F$ holds $X = \emptyset$.

(31) Let A be a set, B be a non empty set, R be a relation between A and B, F be a family of subsets of A, and G be a family of subsets of B. If $G = \{R[Y]; Y$ ranges over subsets of A: $Y \in F\}$, then $R[\bigcup F] = \mathrm{Intersect}(G)$.

(32) If $X_1 \subseteq X_2$, then $R[X_2] \subseteq R[X_1]$.

(33) $R[X_1] \cup R[X_2] \subseteq R[X_1 \cap X_2]$.

(34) $(R_1 \cap R_2)[X] = R_1[X] \cap R_2[X]$.

(35) $(\bigcup F_1)^{\circ}X = \bigcup\{R^{\circ}X; R$ ranges over subsets of $[: A, B :]$: $R \in F_1\}$.

(36) Let F_1 be a family of subsets of $[: A, B :]$, A, B be sets, and X be a subset of A. Then $\{R[X]; R$ ranges over subsets of $[: A, B :]$: $R \in F_1\}$ is a family of subsets of B.

(37) If $R = \emptyset$ and $X \neq \emptyset$, then $R[X] = \emptyset$.

(38) If $R = [: A, B :]$, then $R[X] = B$.

(39) For every family G of subsets of B such that $G = \{R[X]; R$ ranges over subsets of $[: A, B :]$: $R \in F_1\}$ holds $(\mathrm{Intersect}(F_1))[X] = \mathrm{Intersect}(G)$.

(40) If $R_1 \subseteq R_2$, then $R_1[X] \subseteq R_2[X]$.

(41) $R_1[X] \cup R_2[X] \subseteq (R_1 \cup R_2)[X]$.

(42) $y \in (R^{\mathrm{c}})^{\circ}\{x\}$ iff $\langle x, y\rangle \notin R$ and $x \in A$ and $y \in B$.

(43) If $X \neq \emptyset$, then $R[X] \subseteq R^{\circ}X$.

(44) For all sets X, Y holds X meets $(R^{\smile})^{\circ}Y$ iff there exist sets x, y such that $x \in X$ and $y \in Y$ and $x \in (R^{\smile})^{\circ}\{y\}$.

(45) For all sets X, Y holds there exist sets x, y such that $x \in X$ and $y \in Y$ and $x \in (R^{\smile})^{\circ}\{y\}$ iff Y meets $R^{\circ}X$.

(46) X misses $(R^{\smile})^{\circ}Y$ iff Y misses $R^{\circ}X$.

(47) For every set X holds $R^{\circ}X = R^{\circ}(X \cap \pi_1(R))$.

(48) For every set Y holds $(R^{\smile})^{\circ}Y = (R^{\smile})^{\circ}(Y \cap \pi_2(R))$.

(49) $(R[X])^{\mathrm{c}} = (R^{\mathrm{c}})^{\circ}X$.

In the sequel R denotes a relation between A and B and S denotes a relation between B and C.

Let A, B, C be sets, let R be a subset of $[: A, B :]$, and let S be a subset of $[: B, C :]$. Then $R \cdot S$ is a relation between A and C.

One can prove the following propositions:

(50) $(R^\circ X)^c = R^c[X]$.

(51) $\pi_1(R) = (R^\smile)^\circ B$ and $\pi_2(R) = R^\circ A$.

(52) $\pi_1(R \cdot S) = (R^\smile)^\circ \pi_1(S)$ and $\pi_1(R \cdot S) \subseteq \pi_1(R)$.

(53) $\pi_2(R \cdot S) = S^\circ \pi_2(R)$ and $\pi_2(R \cdot S) \subseteq \pi_2(S)$.

(54) $X \subseteq \pi_1(R)$ iff $X \subseteq (R \cdot R^\smile)^\circ X$.

(55) $Y \subseteq \pi_2(R)$ iff $Y \subseteq (R^\smile \cdot R)^\circ Y$.

(56) $\pi_1(R) = (R^\smile)^\circ B$ and $(R^\smile)^\circ R^\circ A = (R^\smile)^\circ \pi_2(R)$.

(57) $(R^\smile)^\circ B = (R \cdot R^\smile)^\circ A$.

(58) $R^\circ A = (R^\smile \cdot R)^\circ B$.

(59) $S[R^\circ X] = (R \cdot S^c)^c[X]$.

(60) $(R^c)^\smile = (R^\smile)^c$.

(61) $X \subseteq R^\smile[Y]$ iff $Y \subseteq R[X]$.

(62) $R^\circ X^c \subseteq Y^c$ iff $(R^\smile)^\circ Y \subseteq X$.

(63) $X \subseteq R^\smile[R[X]]$ and $Y \subseteq R[R^\smile[Y]]$.

(64) $R[X] = R[R^\smile[R[X]]]$ and $R^\smile[Y] = R^\smile[R[R^\smile[Y]]]$.

(65) $\mathrm{id}_A \cdot R = R \cdot \mathrm{id}_B$.

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The Inner Product and Conjugate of Finite Sequences of Complex Numbers

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Summary. The concept of "the inner product and conjugate of finite sequences of complex numbers" is defined here. Addition, subtraction, scalar multiplication and inner product are introduced using correspondent definitions of "conjugate of finite sequences of field". Many equations for such operations consist like a case of "conjugate of finite sequences of field". Some operations on the set of n-tuples of complex numbers are introduced as well. Additionally, difference of such n-tuples, complement of a n-tuple and multiplication of these are defined in terms of complex numbers.

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The terminology and notation used here are introduced in the following articles: [17], [18], [15], [19], [8], [9], [10], [4], [16], [3], [5], [12], [6], [11], [7], [14], [1], [2], and [13].

1. Preliminaries

For simplicity, we adopt the following convention: i, j are natural numbers, x, y, z are finite sequences of elements of $\mathbb{C}$, c is an element of $\mathbb{C}$, and R, R_1, R_2 are elements of $\mathbb{C}^i$.

Let z be a finite sequence of elements of $\mathbb{C}$. The functor $\overline{z}$ yielding a finite sequence of elements of $\mathbb{C}$ is defined by:

(Def. 1) $\operatorname{len}\overline{z} = \operatorname{len} z$ and for every natural number i such that $1 \leq i$ and $i \leq \operatorname{len} z$ holds $\overline{z}(i) = \overline{z(i)}$.

The following propositions are true:

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(1) If $i \in \operatorname{dom}(x+y)$, then $(x+y)(i) = x(i) + y(i)$.

(2) If $i \in \operatorname{dom}(x-y)$, then $(x-y)(i) = x(i) - y(i)$.

Let us consider i, R_1, R_2. Then $R_1 - R_2$ is an element of $\mathbb{C}^i$.

Let us consider i, R_1, R_2. Then $R_1 + R_2$ is an element of $\mathbb{C}^i$.

Let us consider i, let r be a complex number, and let us consider R. Then $r \cdot R$ is an element of $\mathbb{C}^i$.

We now state a number of propositions:

(3) For every complex number a and for every finite sequence x of elements of $\mathbb{C}$ holds $\operatorname{len}(a \cdot x) = \operatorname{len} x$.

(4) For every finite sequence x of elements of $\mathbb{C}$ holds $\operatorname{dom} x = \operatorname{dom}(-x)$.

(5) For every finite sequence x of elements of $\mathbb{C}$ holds $\operatorname{len}(-x) = \operatorname{len} x$.

(6) For all finite sequences x_1, x_2 of elements of $\mathbb{C}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $\operatorname{len}(x_1 + x_2) = \operatorname{len} x_1$.

(7) For all finite sequences x_1, x_2 of elements of $\mathbb{C}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $\operatorname{len}(x_1 - x_2) = \operatorname{len} x_1$.

(8) Every finite sequence f of elements of $\mathbb{C}$ is an element of $\mathbb{C}^{\operatorname{len} f}$.

(9) $R_1 - R_2 = R_1 + -R_2$.

(10) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $x - y = x + -y$.

(11) $(-1) \cdot R = -R$.

(12) For every finite sequence x of elements of $\mathbb{C}$ holds $(-1) \cdot x = -x$.

(13) For every finite sequence x of elements of $\mathbb{C}$ holds $(-x)(i) = -x(i)$.

Let us consider i, R. Then $-R$ is an element of $\mathbb{C}^i$.

The following propositions are true:

(14) If $c = R(j)$, then $(-R)(j) = -c$.

(15) For every complex number a holds $\operatorname{dom}(a \cdot x) = \operatorname{dom} x$.

(16) For every complex number a holds $(a \cdot x)(i) = a \cdot x(i)$.

(17) For every complex number a holds $\overline{a \cdot x} = \overline{a} \cdot \overline{x}$.

(18) $(R_1 + R_2)(j) = R_1(j) + R_2(j)$.

(19) For all finite sequences x_1, x_2 of elements of $\mathbb{C}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $\overline{x_1 + x_2} = \overline{x_1} + \overline{x_2}$.

(20) $(R_1 - R_2)(j) = R_1(j) - R_2(j)$.

(21) For all finite sequences x_1, x_2 of elements of $\mathbb{C}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $\overline{x_1 - x_2} = \overline{x_1} - \overline{x_2}$.

(22) For every finite sequence z of elements of $\mathbb{C}$ holds $\overline{\overline{z}} = z$.

(23) For every finite sequence z of elements of $\mathbb{C}$ holds $\overline{-z} = -\overline{z}$.

(24) For every complex number z holds $z + \overline{z} = 2 \cdot \Re(z)$.

(25) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $(x - y)(i) = x(i) - y(i)$.

(26) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $(x + y)(i) = x(i) + y(i)$.

Let z be a finite sequence of elements of $\mathbb{C}$. The functor $\Re(z)$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def. 2) $\Re(z) = \frac{1}{2} \cdot (z + \overline{z})$.

One can prove the following proposition

(27) For every complex number z holds $z - \overline{z} = 2 \cdot \Im(z) \cdot i$.

Let z be a finite sequence of elements of $\mathbb{C}$. The functor $\Im(z)$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

(Def. 3) $\Im(z) = (-\frac{1}{2} \cdot i) \cdot (z - \overline{z})$.

Let x, y be finite sequences of elements of $\mathbb{C}$. The functor $|(x, y)|$ yields an element of $\mathbb{C}$ and is defined by:

(Def. 4) $|(x, y)| = (|(\Re(x), \Re(y))| - i \cdot |(\Re(x), \Im(y))|) + i \cdot |(\Im(x), \Re(y))| + |(\Im(x), \Im(y))|$.

We now state four propositions:

(28) For all finite sequences x, y, z of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $x + (y + z) = (x + y) + z$.

(29) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $x + y = y + x$.

(30) Let c be a complex number and x, y be finite sequences of elements of $\mathbb{C}$. If $\operatorname{len} x = \operatorname{len} y$, then $c \cdot (x + y) = c \cdot x + c \cdot y$.

(31) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $x - y = x + -y$.

Let us consider i, c. Then $i \mapsto c$ is an element of $\mathbb{C}^i$.

Next we state a number of propositions:

(32) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $x + y = 0_{\mathbb{C}}^{\operatorname{len} x}$ holds $x = -y$ and $y = -x$.

(33) For every finite sequence x of elements of $\mathbb{C}$ holds $x + 0_{\mathbb{C}}^{\operatorname{len} x} = x$.

(34) For every finite sequence x of elements of $\mathbb{C}$ holds $x + -x = 0_{\mathbb{C}}^{\operatorname{len} x}$.

(35) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $-(x + y) = -x + -y$.

(36) For all finite sequences x, y, z of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $x - y - z = x - (y + z)$.

(37) For all finite sequences x, y, z of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $x + (y - z) = (x + y) - z$.

(38) For every finite sequence x of elements of $\mathbb{C}$ holds $--x = x$.

(39) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $-(x - y) = -x + y$.

(40) For all finite sequences x, y, z of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $x - (y - z) = (x - y) + z$.

(41) For every complex number c holds $c \cdot 0_{\mathbb{C}}^{\operatorname{len} x} = 0_{\mathbb{C}}^{\operatorname{len} x}$.

(42) For every complex number c holds $-c \cdot x = c \cdot -x$.

(43) Let c be a complex number and x, y be finite sequences of elements of $\mathbb{C}$. If $\operatorname{len} x = \operatorname{len} y$, then $c \cdot (x - y) = c \cdot x - c \cdot y$.

(44) For all elements x_1, y_1 of $\mathbb{C}$ and for all real numbers x_2, y_2 such that $x_1 = x_2$ and $y_1 = y_2$ holds $+_{\mathbb{C}}(x_1, y_1) = +_{\mathbb{R}}(x_2, y_2)$.

In the sequel C is a function from $[\!:\mathbb{C}, \mathbb{C}:\!]$ into $\mathbb{C}$ and G is a function from $[\!:\mathbb{R}, \mathbb{R}:\!]$ into $\mathbb{R}$.

One can prove the following proposition

(45) Let x_1, y_1 be finite sequences of elements of $\mathbb{C}$ and x_2, y_2 be finite sequences of elements of $\mathbb{R}$. Suppose $x_1 = x_2$ and $y_1 = y_2$ and $\operatorname{len} x_1 = \operatorname{len} y_2$ and for every i such that $i \in \operatorname{dom} x_1$ holds $C(x_1(i), y_1(i)) = G(x_2(i), y_2(i))$. Then $C^\circ(x_1, y_1) = G^\circ(x_2, y_2)$.

Let z be a finite sequence of elements of $\mathbb{R}$ and let i be a set. Then $z(i)$ is an element of $\mathbb{R}$.

We now state several propositions:

(46) Let x_1, y_1 be finite sequences of elements of $\mathbb{C}$ and x_2, y_2 be finite sequences of elements of $\mathbb{R}$. If $x_1 = x_2$ and $y_1 = y_2$ and $\operatorname{len} x_1 = \operatorname{len} y_2$, then $(+_{\mathbb{C}})^\circ(x_1, y_1) = (+_{\mathbb{R}})^\circ(x_2, y_2)$.

(47) Let x_1, y_1 be finite sequences of elements of $\mathbb{C}$ and x_2, y_2 be finite sequences of elements of $\mathbb{R}$. If $x_1 = x_2$ and $y_1 = y_2$ and $\operatorname{len} x_1 = \operatorname{len} y_2$, then $x_1 + y_1 = x_2 + y_2$.

(48) For every finite sequence x of elements of $\mathbb{C}$ holds $\operatorname{len} \Re(x) = \operatorname{len} x$ and $\operatorname{len} \Im(x) = \operatorname{len} x$.

(49) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $\Re(x + y) = \Re(x) + \Re(y)$ and $\Im(x + y) = \Im(x) + \Im(y)$.

(50) Let x_1, y_1 be finite sequences of elements of $\mathbb{C}$ and x_2, y_2 be finite sequences of elements of $\mathbb{R}$. If $x_1 = x_2$ and $y_1 = y_2$ and $\operatorname{len} x_1 = \operatorname{len} y_2$, then $(-_{\mathbb{C}})^\circ(x_1, y_1) = (-_{\mathbb{R}})^\circ(x_2, y_2)$.

(51) Let x_1, y_1 be finite sequences of elements of $\mathbb{C}$ and x_2, y_2 be finite sequences of elements of $\mathbb{R}$. If $x_1 = x_2$ and $y_1 = y_2$ and $\operatorname{len} x_1 = \operatorname{len} y_2$, then $x_1 - y_1 = x_2 - y_2$.

(52) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $\Re(x - y) = \Re(x) - \Re(y)$ and $\Im(x - y) = \Im(x) - \Im(y)$.

(53) For all complex numbers a, b holds $a \cdot (b \cdot z) = (a \cdot b) \cdot z$.

(54) For every complex number c holds $(-c) \cdot x = -c \cdot x$.

In the sequel h is a function from $\mathbb{C}$ into $\mathbb{C}$ and g is a function from $\mathbb{R}$ into $\mathbb{R}$.

One can prove the following propositions:

(55) Let y_1 be a finite sequence of elements of $\mathbb{C}$ and y_2 be a finite sequence of elements of $\mathbb{R}$. If $\operatorname{len} y_1 = \operatorname{len} y_2$ and for every i such that $i \in \operatorname{dom} y_1$ holds $h(y_1(i)) = g(y_2(i))$, then $h \cdot y_1 = g \cdot y_2$.

(56) Let y_1 be a finite sequence of elements of $\mathbb{C}$ and y_2 be a finite sequence of elements of $\mathbb{R}$. If $y_1 = y_2$ and $\operatorname{len} y_1 = \operatorname{len} y_2$, then $-_{\mathbb{C}} \cdot y_1 = -_{\mathbb{R}} \cdot y_2$.

(57) Let y_1 be a finite sequence of elements of $\mathbb{C}$ and y_2 be a finite sequence of elements of $\mathbb{R}$. If $y_1 = y_2$ and $\operatorname{len} y_1 = \operatorname{len} y_2$, then $-y_1 = -y_2$.

(58) For every finite sequence x of elements of $\mathbb{C}$ holds $\Re(i \cdot x) = -\Im(x)$ and $\Im(i \cdot x) = \Re(x)$.

(59) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(i \cdot x, y)| = i \cdot |(x, y)|$.

(60) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(x, i \cdot y)| = -i \cdot |(x, y)|$.

(61) Let a_1 be an element of $\mathbb{C}$, y_1 be a finite sequence of elements of $\mathbb{C}$, a_2 be an element of $\mathbb{R}$, and y_2 be a finite sequence of elements of $\mathbb{R}$. If $a_1 = a_2$ and $y_1 = y_2$ and $\operatorname{len} y_1 = \operatorname{len} y_2$, then $\cdot_{\mathbb{C}}^{(a_1)} \cdot y_1 = \cdot_{\mathbb{R}}^{a_2} \cdot y_2$.

(62) Let a_1 be a complex number, y_1 be a finite sequence of elements of $\mathbb{C}$, a_2 be an element of $\mathbb{R}$, and y_2 be a finite sequence of elements of $\mathbb{R}$. If $a_1 = a_2$ and $y_1 = y_2$ and $\operatorname{len} y_1 = \operatorname{len} y_2$, then $a_1 \cdot y_1 = a_2 \cdot y_2$.

(63) For all complex numbers a, b holds $(a + b) \cdot z = a \cdot z + b \cdot z$.

(64) For all complex numbers a, b holds $(a - b) \cdot z = a \cdot z - b \cdot z$.

(65) Let a be an element of $\mathbb{C}$ and x be a finite sequence of elements of $\mathbb{C}$. Then $\Re(a \cdot x) = \Re(a) \cdot \Re(x) - \Im(a) \cdot \Im(x)$ and $\Im(a \cdot x) = \Im(a) \cdot \Re(x) + \Re(a) \cdot \Im(x)$.

2. The Inner Product and Conjugate of Finite Sequences

The following propositions are true:

(66) For all finite sequences x_1, x_2, y of elements of $\mathbb{C}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_2 = \operatorname{len} y$ holds $|(x_1 + x_2, y)| = |(x_1, y)| + |(x_2, y)|$.

(67) For all finite sequences x_1, x_2 of elements of $\mathbb{C}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $|(-x_1, x_2)| = -|(x_1, x_2)|$.

(68) For all finite sequences x_1, x_2 of elements of $\mathbb{C}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $|(x_1, -x_2)| = -|(x_1, x_2)|$.

(69) For all finite sequences x_1, x_2 of elements of $\mathbb{C}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $|(-x_1, -x_2)| = |(x_1, x_2)|$.

(70) For all finite sequences x_1, x_2, x_3 of elements of $\mathbb{C}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_2 = \operatorname{len} x_3$ holds $|(x_1 - x_2, x_3)| = |(x_1, x_3)| - |(x_2, x_3)|$.

(71) For all finite sequences x, y_1, y_2 of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y_1$ and $\operatorname{len} y_1 = \operatorname{len} y_2$ holds $|(x, y_1 + y_2)| = |(x, y_1)| + |(x, y_2)|$.

(72) For all finite sequences x, y_1, y_2 of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y_1$ and $\operatorname{len} y_1 = \operatorname{len} y_2$ holds $|(x, y_1 - y_2)| = |(x, y_1)| - |(x, y_2)|$.

(73) Let x_1, x_2, y_1, y_2 be finite sequences of elements of $\mathbb{C}$. If $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_2 = \operatorname{len} y_1$ and $\operatorname{len} y_1 = \operatorname{len} y_2$, then $|(x_1+x_2, y_1+y_2)| = |(x_1, y_1)| + |(x_1, y_2)| + |(x_2, y_1)| + |(x_2, y_2)|$.

(74) Let x_1, x_2, y_1, y_2 be finite sequences of elements of $\mathbb{C}$. If $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_2 = \operatorname{len} y_1$ and $\operatorname{len} y_1 = \operatorname{len} y_2$, then $|(x_1 - x_2, y_1 - y_2)| = (|(x_1, y_1)| - |(x_1, y_2)| - |(x_2, y_1)|) + |(x_2, y_2)|$.

(75) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(x, y)| = \overline{|(y, x)|}$.

(76) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(x + y, x + y)| = |(x, x)| + 2 \cdot \Re(|(x, y)|) + |(y, y)|$.

(77) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(x - y, x - y)| = (|(x, x)| - 2 \cdot \Re(|(x, y)|)) + |(y, y)|$.

(78) For every element a of $\mathbb{C}$ and for all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(a \cdot x, y)| = a \cdot |(x, y)|$.

(79) For every element a of $\mathbb{C}$ and for all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(x, a \cdot y)| = \overline{a} \cdot |(x, y)|$.

(80) Let a, b be elements of $\mathbb{C}$ and x, y, z be finite sequences of elements of $\mathbb{C}$. If $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$, then $|(a \cdot x + b \cdot y, z)| = a \cdot |(x, z)| + b \cdot |(y, z)|$.

(81) Let a, b be elements of $\mathbb{C}$ and x, y, z be finite sequences of elements of $\mathbb{C}$. If $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$, then $|(x, a \cdot y + b \cdot z)| = \overline{a} \cdot |(x, y)| + \overline{b} \cdot |(x, z)|$.

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Inferior Limit and Superior Limit of Sequences of Real Numbers

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Summary. The concept of inferior limit and superior limit of sequences of real numbers is defined here. This article contains the following items: definition of the superior sequence and the inferior sequence of real numbers, definition of the superior limit and the inferior limit of real number, and definition of the relation between the limit value and the superior limit, the inferior limit of sequences of real numbers.

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The articles [2], [12], [6], [1], [3], [13], [10], [8], [15], [9], [16], [4], [14], [5], [11], and [7] provide the terminology and notation for this paper.

We adopt the following rules: n, m, k denote natural numbers, r, s, t denote real numbers, and s_1, s_2, s_3 denote sequences of real numbers.

One can prove the following proposition

(1) $s - r < t$ and $s + r > t$ iff $|t - s| < r$.

Let s_1 be a sequence of real numbers. The functor $\sup s_1$ yielding a real number is defined by:

(Def. 1) $\sup s_1 = \sup \operatorname{rng} s_1$.

Let s_1 be a sequence of real numbers. The functor $\inf s_1$ yielding a real number is defined as follows:

(Def. 2) $\inf s_1 = \inf \operatorname{rng} s_1$.

The following propositions are true:

(2) $(s_2 + s_3) - s_3 = s_2$.

(3) $r \in \operatorname{rng} s_1$ iff $-r \in \operatorname{rng}(-s_1)$.

(4) $\operatorname{rng}(-s_1) = -\operatorname{rng} s_1$.

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(5) s_1 is upper bounded iff $\operatorname{rng} s_1$ is upper bounded.

(6) s_1 is lower bounded iff $\operatorname{rng} s_1$ is lower bounded.

(7) Suppose s_1 is upper bounded. Then $r = \sup s_1$ if and only if the following conditions are satisfied:

(i) for every n holds $s_1(n) \leq r$, and

(ii) for every s such that $0 < s$ there exists k such that $r - s < s_1(k)$.

(8) Suppose s_1 is lower bounded. Then $r = \inf s_1$ if and only if the following conditions are satisfied:

(i) for every n holds $r \leq s_1(n)$, and

(ii) for every s such that $0 < s$ there exists k such that $s_1(k) < r + s$.

(9) For every n holds $s_1(n) \leq r$ iff s_1 is upper bounded and $\sup s_1 \leq r$.

(10) For every n holds $r \leq s_1(n)$ iff s_1 is lower bounded and $r \leq \inf s_1$.

(11) s_1 is upper bounded iff $-s_1$ is lower bounded.

(12) s_1 is lower bounded iff $-s_1$ is upper bounded.

(13) If s_1 is upper bounded, then $\sup s_1 = -\inf(-s_1)$.

(14) If s_1 is lower bounded, then $\inf s_1 = -\sup(-s_1)$.

(15) If s_2 is lower bounded and s_3 is lower bounded, then $\inf(s_2 + s_3) \geq \inf s_2 + \inf s_3$.

(16) If s_2 is upper bounded and s_3 is upper bounded, then $\sup(s_2 + s_3) \leq \sup s_2 + \sup s_3$.

Let f be a sequence of real numbers. We introduce f is non-negative as a synonym of f is non-negative yielding.

Let f be a sequence of real numbers. Let us observe that f is non-negative if and only if:

(Def. 3) For every n holds $f(n) \geq 0$.

The following propositions are true:

(17) If s_1 is non-negative, then $s_1 \uparrow k$ is non-negative.

(18) If s_1 is lower bounded and non-negative, then $\inf s_1 \geq 0$.

(19) If s_1 is upper bounded and non-negative, then $\sup s_1 \geq 0$.

(20) Suppose s_2 is lower bounded and non-negative and s_3 is lower bounded and non-negative. Then $s_2\, s_3$ is lower bounded and $\inf(s_2\, s_3) \geq \inf s_2 \cdot \inf s_3$.

(21) Suppose s_2 is upper bounded and non-negative and s_3 is upper bounded and non-negative. Then $s_2\, s_3$ is upper bounded and $\sup(s_2\, s_3) \leq \sup s_2 \cdot \sup s_3$.

(22) If s_1 is non-decreasing and upper bounded, then s_1 is bounded.

(23) If s_1 is non-increasing and lower bounded, then s_1 is bounded.

(24) If s_1 is non-decreasing and upper bounded, then $\lim s_1 = \sup s_1$.

(25) If s_1 is non-increasing and lower bounded, then $\lim s_1 = \inf s_1$.

(26) If s_1 is upper bounded, then $s_1 \uparrow k$ is upper bounded.

(27) If s_1 is lower bounded, then $s_1 \uparrow k$ is lower bounded.

(28) If s_1 is bounded, then $s_1 \uparrow k$ is bounded.

(29) For all s_1, n holds $\{s_1(k) : n \leq k\}$ is a subset of $\mathbb{R}$.

(30) $\mathrm{rng}(s_1 \uparrow k) = \{s_1(n) : k \leq n\}$.

(31) If s_1 is upper bounded, then for every n and for every subset R of $\mathbb{R}$ such that $R = \{s_1(k) : n \leq k\}$ holds R is upper bounded.

(32) If s_1 is lower bounded, then for every n and for every subset R of $\mathbb{R}$ such that $R = \{s_1(k) : n \leq k\}$ holds R is lower bounded.

(33) If s_1 is bounded, then for every n and for every subset R of $\mathbb{R}$ such that $R = \{s_1(k) : n \leq k\}$ holds R is bounded.

(34) If s_1 is non-decreasing, then for every n and for every subset R of $\mathbb{R}$ such that $R = \{s_1(k) : n \leq k\}$ holds $\inf R = s_1(n)$.

(35) If s_1 is non-increasing, then for every n and for every subset R of $\mathbb{R}$ such that $R = \{s_1(k) : n \leq k\}$ holds $\sup R = s_1(n)$.

(36) Let given s_1. Then there exists a function f from $\mathbb{N}$ into $\mathbb{R}$ such that for every n and for every subset Y of $\mathbb{R}$ if $Y = \{s_1(k) : n \leq k\}$, then $f(n) = \sup Y$.

(37) Let given s_1. Then there exists a function f from $\mathbb{N}$ into $\mathbb{R}$ such that for every n and for every subset Y of $\mathbb{R}$ if $Y = \{s_1(k) : n \leq k\}$, then $f(n) = \inf Y$.

Let s_1 be a sequence of real numbers. The inferior realsequence s_1 yields a sequence of real numbers and is defined as follows:

(Def. 4) For every n and for every subset Y of $\mathbb{R}$ such that $Y = \{s_1(k) : n \leq k\}$ holds (the inferior realsequence $s_1)(n) = \inf Y$.

Let s_1 be a sequence of real numbers. The superior realsequence s_1 yields a sequence of real numbers and is defined by:

(Def. 5) For every n and for every subset Y of $\mathbb{R}$ such that $Y = \{s_1(k) : n \leq k\}$ holds (the superior realsequence $s_1)(n) = \sup Y$.

Next we state a number of propositions:

(38) (The inferior realsequence $s_1)(n) = \inf(s_1 \uparrow n)$.

(39) (The superior realsequence $s_1)(n) = \sup(s_1 \uparrow n)$.

(40) If s_1 is lower bounded, then (the inferior realsequence $s_1)(0) = \inf s_1$.

(41) If s_1 is upper bounded, then (the superior realsequence $s_1)(0) = \sup s_1$.

(42) Suppose s_1 is lower bounded. Then $r =$ (the inferior realsequence $s_1)(n)$ if and only if for every k holds $r \leq s_1(n+k)$ and for every s such that $0 < s$ there exists k such that $s_1(n+k) < r + s$.

(43) Suppose s_1 is upper bounded. Then $r =$ (the superior realsequence $s_1)(n)$ if and only if for every k holds $s_1(n+k) \leq r$ and for every s such that $0 < s$ there exists k such that $r - s < s_1(n+k)$.

(44) If s_1 is lower bounded, then for every k holds $r \leq s_1(n+k)$ iff $r \leq$ (the inferior realsequence $s_1)(n)$.

(45) Suppose s_1 is lower bounded. Then for every m such that $n \leq m$ holds $r \leq s_1(m)$ if and only if $r \leq$ (the inferior realsequence $s_1)(n)$.

(46) If s_1 is upper bounded, then for every k holds $s_1(n+k) \leq r$ iff (the superior realsequence $s_1)(n) \leq r$.

(47) Suppose s_1 is upper bounded. Then for every m such that $n \leq m$ holds $s_1(m) \leq r$ if and only if (the superior realsequence $s_1)(n) \leq r$.

(48) If s_1 is lower bounded, then (the inferior realsequence $s_1)(n) = \min(($the inferior realsequence $s_1)(n+1), s_1(n))$.

(49) If s_1 is upper bounded, then (the superior realsequence $s_1)(n) = \max(($the superior realsequence $s_1)(n+1), s_1(n))$.

(50) If s_1 is lower bounded, then (the inferior realsequence $s_1)(n) \leq$ (the inferior realsequence $s_1)(n+1)$.

(51) If s_1 is upper bounded, then (the superior realsequence $s_1)(n+1) \leq$ (the superior realsequence $s_1)(n)$.

(52) If s_1 is lower bounded, then the inferior realsequence s_1 is non-decreasing.

(53) If s_1 is upper bounded, then the superior realsequence s_1 is non-increasing.

(54) If s_1 is bounded, then (the inferior realsequence $s_1)(n) \leq$ (the superior realsequence $s_1)(n)$.

(55) If s_1 is bounded, then (the inferior realsequence $s_1)(n) \leq \inf$ (the superior realsequence s_1).

(56) If s_1 is bounded, then $\sup$ (the inferior realsequence $s_1) \leq$ (the superior realsequence $s_1)(n)$.

(57) If s_1 is bounded, then $\sup$ (the inferior realsequence $s_1) \leq \inf$ (the superior realsequence s_1).

(58) If s_1 is bounded, then the superior realsequence s_1 is bounded and the inferior realsequence s_1 is bounded.

(59) Suppose s_1 is bounded. Then
   (i) the inferior realsequence s_1 is convergent, and
   (ii) $\lim$ (the inferior realsequence $s_1) = \sup$ (the inferior realsequence s_1).

(60) Suppose s_1 is bounded. Then
   (i) the superior realsequence s_1 is convergent, and
   (ii) $\lim$ (the superior realsequence $s_1) = \inf$ (the superior realsequence s_1).

(61) If s_1 is lower bounded, then (the inferior realsequence s_1)(n) $=$ $-$(the superior realsequence $-s_1$)(n).

(62) If s_1 is upper bounded, then (the superior realsequence s_1)(n) $=$ $-$(the inferior realsequence $-s_1$)(n).

(63) If s_1 is lower bounded, then the inferior realsequence s_1 $=$ $-$the superior realsequence $-s_1$.

(64) If s_1 is upper bounded, then the superior realsequence s_1 $=$ $-$the inferior realsequence $-s_1$.

(65) If s_1 is non-decreasing, then $s_1(n) \leq$ (the inferior realsequence s_1)$(n+1)$.

(66) If s_1 is non-decreasing, then the inferior realsequence $s_1 = s_1$.

(67) If s_1 is non-decreasing and upper bounded, then $s_1(n) \leq$ (the superior realsequence s_1)$(n+1)$.

(68) Suppose s_1 is non-decreasing and upper bounded. Then (the superior realsequence s_1)$(n) =$ (the superior realsequence s_1)$(n+1)$.

(69) Suppose s_1 is non-decreasing and upper bounded. Then (the superior realsequence s_1)$(n) = \sup s_1$ and the superior realsequence s_1 is constant.

(70) If s_1 is non-decreasing and upper bounded, then inf (the superior realsequence s_1) $= \sup s_1$.

(71) If s_1 is non-increasing, then (the superior realsequence s_1)$(n+1) \leq s_1(n)$.

(72) If s_1 is non-increasing, then the superior realsequence $s_1 = s_1$.

(73) If s_1 is non-increasing and lower bounded, then (the inferior realsequence s_1)$(n+1) \leq s_1(n)$.

(74) Suppose s_1 is non-increasing and lower bounded. Then (the inferior realsequence s_1)$(n) =$ (the inferior realsequence s_1)$(n+1)$.

(75) Suppose s_1 is non-increasing and lower bounded. Then (the inferior realsequence s_1)$(n) = \inf s_1$ and the inferior realsequence s_1 is constant.

(76) If s_1 is non-increasing and lower bounded, then sup (the inferior realsequence s_1) $= \inf s_1$.

(77) Suppose s_2 is bounded and s_3 is bounded and for every n holds $s_2(n) \leq s_3(n)$. Then
 (i) for every n holds (the superior realsequence s_2)$(n) \leq$ (the superior realsequence s_3)(n), and
 (ii) for every n holds (the inferior realsequence s_2)$(n) \leq$ (the inferior realsequence s_3)(n).

(78) Suppose s_2 is lower bounded and s_3 is lower bounded. Then (the inferior realsequence $s_2 + s_3$)$(n) \geq$ (the inferior realsequence s_2)$(n) +$ (the inferior realsequence s_3)(n).

(79) Suppose s_2 is upper bounded and s_3 is upper bounded. Then (the superior realsequence $s_2 + s_3$)$(n) \leq$ (the superior realsequence s_2)$(n) +$ (the

superior realsequence $s_3)(n)$.

(80) Suppose s_2 is lower bounded and non-negative and s_3 is lower bounded and non-negative. Then (the inferior realsequence $s_2\, s_3)(n) \geq$ (the inferior realsequence $s_2)(n) \cdot$ (the inferior realsequence $s_3)(n)$.

(81) Suppose s_2 is lower bounded and non-negative and s_3 is lower bounded and non-negative. Then (the inferior realsequence $s_2\, s_3)(n) \geq$ (the inferior realsequence $s_2)(n) \cdot$ (the inferior realsequence $s_3)(n)$.

(82) Suppose s_2 is upper bounded and non-negative and s_3 is upper bounded and non-negative. Then (the superior realsequence $s_2\, s_3)(n) \leq$ (the superior realsequence $s_2)(n) \cdot$ (the superior realsequence $s_3)(n)$.

Let s_1 be a sequence of real numbers. The functor $\limsup s_1$ yielding an element of $\mathbb{R}$ is defined as follows:

(Def. 6) $\limsup s_1 = \inf$ (the superior realsequence s_1).

Let s_1 be a sequence of real numbers. The functor $\liminf s_1$ yielding an element of $\mathbb{R}$ is defined by:

(Def. 7) $\liminf s_1 = \sup$ (the inferior realsequence s_1).

Next we state a number of propositions:

(83) If s_1 is bounded, then $\liminf s_1 \leq r$ iff for every s such that $0 < s$ and for every n there exists k such that $s_1(n+k) < r+s$.

(84) If s_1 is bounded, then $r \leq \liminf s_1$ iff for every s such that $0 < s$ there exists n such that for every k holds $r - s < s_1(n+k)$.

(85) Suppose s_1 is bounded. Then $r = \liminf s_1$ if and only if for every s such that $0 < s$ holds for every n there exists k such that $s_1(n+k) < r+s$ and there exists n such that for every k holds $r - s < s_1(n+k)$.

(86) If s_1 is bounded, then $r \leq \limsup s_1$ iff for every s such that $0 < s$ and for every n there exists k such that $s_1(n+k) > r - s$.

(87) If s_1 is bounded, then $\limsup s_1 \leq r$ iff for every s such that $0 < s$ there exists n such that for every k holds $s_1(n+k) < r+s$.

(88) Suppose s_1 is bounded. Then $r = \limsup s_1$ if and only if for every s such that $0 < s$ holds for every n there exists k such that $s_1(n+k) > r-s$ and there exists n such that for every k holds $s_1(n+k) < r+s$.

(89) If s_1 is bounded, then $\liminf s_1 \leq \limsup s_1$.

(90) s_1 is bounded and $\limsup s_1 = \liminf s_1$ iff s_1 is convergent.

(91) If s_1 is convergent, then $\lim s_1 = \limsup s_1$ and $\lim s_1 = \liminf s_1$.

(92) If s_1 is bounded, then $\limsup(-s_1) = -\liminf s_1$ and $\liminf(-s_1) = -\limsup s_1$.

(93) If s_2 is bounded and s_3 is bounded and for every n holds $s_2(n) \leq s_3(n)$, then $\limsup s_2 \leq \limsup s_3$ and $\liminf s_2 \leq \liminf s_3$.

(94) Suppose s_2 is bounded and s_3 is bounded. Then $\liminf s_2 + \liminf s_3 \leq \liminf(s_2+s_3)$ and $\liminf(s_2+s_3) \leq \liminf s_2+\limsup s_3$ and $\liminf(s_2+s_3) \leq \limsup s_2+\liminf s_3$ and $\liminf s_2+\limsup s_3 \leq \limsup(s_2+s_3)$ and $\limsup s_2+\liminf s_3 \leq \limsup(s_2+s_3)$ and $\limsup(s_2+s_3) \leq \limsup s_2+\limsup s_3$ and if s_2 is convergent or s_3 is convergent, then $\liminf(s_2+s_3) = \liminf s_2 + \liminf s_3$ and $\limsup(s_2 + s_3) = \limsup s_2 + \limsup s_3$.

(95) If s_2 is bounded and non-negative and s_3 is bounded and non-negative, then $\liminf s_2 \cdot \liminf s_3 \leq \liminf(s_2\, s_3)$ and $\limsup(s_2\, s_3) \leq \limsup s_2 \cdot \limsup s_3$.

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Formulas and Identities of Inverse Hyperbolic Functions

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Summary. This article describes definitions of inverse hyperbolic functions and their main properties, as well as some addition formulas with hyperbolic functions.

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The papers [1], [8], [4], [2], [9], [3], [6], [5], and [7] provide the terminology and notation for this paper.

1. Preliminaries

In this paper x, y, t denote real numbers.

Next we state a number of propositions:

(1) If $x > 0$, then $\frac{1}{x} = x^{-1}$.

(2) If $x > 1$, then $(\frac{\sqrt{x^{\mathbf{2}}-1}}{x})^{\mathbf{2}} < 1$.

(3) $(\frac{x}{\sqrt{x^{\mathbf{2}}+1}})^{\mathbf{2}} < 1$.

(4) $\sqrt{x^{\mathbf{2}}+1} > 0$.

(5) $\sqrt{x^{\mathbf{2}}+1} + x > 0$.

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(6) If $y \geq 0$ and $x \geq 1$, then $\frac{x+1}{y} \geq 0$.
(7) If $y \geq 0$ and $x \geq 1$, then $\frac{x-1}{y} \geq 0$.
(8) If $x \geq 1$, then $\sqrt{\frac{x+1}{2}} \geq 1$.
(9) If $y \geq 0$ and $x \geq 1$, then $\frac{x^{\mathbf{2}}-1}{y} \geq 0$.
(10) If $x \geq 1$, then $\sqrt{\frac{x+1}{2}} + \sqrt{\frac{x-1}{2}} > 0$.
(11) If $x^{\mathbf{2}} < 1$, then $x+1 > 0$ and $1-x > 0$.
(12) If $x \neq 1$, then $(1-x)^{\mathbf{2}} > 0$.
(13) If $x^{\mathbf{2}} < 1$, then $\frac{x^{\mathbf{2}}+1}{1-x^{\mathbf{2}}} \geq 0$.
(14) If $x^{\mathbf{2}} < 1$, then $(\frac{2 \cdot x}{1+x^{\mathbf{2}}})^{\mathbf{2}} < 1$.
(15) If $0 < x$ and $x < 1$, then $\frac{1+x}{1-x} > 0$.
(16) If $0 < x$ and $x < 1$, then $x^{\mathbf{2}} < 1$.
(17) If $0 < x$ and $x < 1$, then $\frac{1}{\sqrt{1-x^{\mathbf{2}}}} > 1$.
(18) If $0 < x$ and $x < 1$, then $\frac{2 \cdot x}{1-x^{\mathbf{2}}} > 0$.
(19) If $0 < x$ and $x < 1$, then $0 < (1-x^{\mathbf{2}})^{\mathbf{2}}$.
(20) If $0 < x$ and $x < 1$, then $\frac{1+x^{\mathbf{2}}}{1-x^{\mathbf{2}}} > 1$.
(21) If $1 < x^{\mathbf{2}}$, then $(\frac{1}{x})^{\mathbf{2}} < 1$.
(22) If $0 < x$ and $x \leq 1$, then $1 - x^{\mathbf{2}} \geq 0$.
(23) If $1 \leq x$, then $0 < x + \sqrt{x^{\mathbf{2}} - 1}$.
(24) If $1 \leq x$ and $1 \leq y$, then $0 \leq x \cdot \sqrt{y^{\mathbf{2}} - 1} + y \cdot \sqrt{x^{\mathbf{2}} - 1}$.
(25) If $1 \leq x$ and $1 \leq y$ and $|y| \leq |x|$, then $0 < y - \sqrt{y^{\mathbf{2}} - 1}$.
(26) If $1 \leq x$ and $1 \leq y$ and $|y| \leq |x|$, then $0 \leq y \cdot \sqrt{x^{\mathbf{2}} - 1} - x \cdot \sqrt{y^{\mathbf{2}} - 1}$.
(27) If $x^{\mathbf{2}} < 1$ and $y^{\mathbf{2}} < 1$, then $x \cdot y \neq -1$.
(28) If $x^{\mathbf{2}} < 1$ and $y^{\mathbf{2}} < 1$, then $x \cdot y \neq 1$.
(29) If $x \neq 0$, then $\exp x \neq 1$.
(30) If $0 \neq x$, then $(\exp x)^{\mathbf{2}} - 1 \neq 0$.
(31) If $0 < t$, then $\frac{t^{\mathbf{2}}-1}{t^{\mathbf{2}}+1} < 1$.
(32) If $-1 < t$ and $t < 1$, then $0 < \frac{t+1}{1-t}$.

2. Formulas and Identities of Inverse Hyperbolic Functions

Let x be a real number. The functor $\sinh' x$ yields a real number and is defined by:

(Def. 1) $\sinh' x = \log_e(x + \sqrt{x^{\mathbf{2}} + 1})$.

Let x be a real number. The functor $\cosh'_1 x$ yielding a real number is defined by:

(Def. 2) $\cosh_1' x = \log_e(x + \sqrt{x^2 - 1})$.

Let x be a real number. The functor $\cosh_2' x$ yields a real number and is defined by:

(Def. 3) $\cosh_2' x = -\log_e(x + \sqrt{x^2 - 1})$.

Let x be a real number. The functor $\tanh' x$ yields a real number and is defined by:

(Def. 4) $\tanh' x = \frac{1}{2} \cdot \log_e(\frac{1+x}{1-x})$.

Let x be a real number. The functor $\coth' x$ yielding a real number is defined as follows:

(Def. 5) $\coth' x = \frac{1}{2} \cdot \log_e(\frac{x+1}{x-1})$.

Let x be a real number. The functor $\mathrm{sech}_1' x$ yields a real number and is defined by:

(Def. 6) $\mathrm{sech}_1' x = \log_e(\frac{1+\sqrt{1-x^2}}{x})$.

Let x be a real number. The functor $\mathrm{sech}_2' x$ yielding a real number is defined as follows:

(Def. 7) $\mathrm{sech}_2' x = -\log_e(\frac{1+\sqrt{1-x^2}}{x})$.

Let x be a real number. The functor $\mathrm{csch}' x$ yielding a real number is defined by:

(Def. 8)(i) $\mathrm{csch}' x = \log_e(\frac{1+\sqrt{1+x^2}}{x})$ if $0 < x$,

(ii) $\mathrm{csch}' x = \log_e(\frac{1-\sqrt{1+x^2}}{x})$ if $x < 0$,

(iii) $x < 0$, otherwise.

The following propositions are true:

(33) If $0 \leq x$, then $\sinh' x = \cosh_1' \sqrt{x^2 + 1}$.

(34) If $x < 0$, then $\sinh' x = \cosh_2' \sqrt{x^2 + 1}$.

(35) $\sinh' x = \tanh'(\frac{x}{\sqrt{x^2+1}})$.

(36) If $x \geq 1$, then $\cosh_1' x = \sinh' \sqrt{x^2 - 1}$.

(37) If $x > 1$, then $\cosh_1' x = \tanh'(\frac{\sqrt{x^2-1}}{x})$.

(38) If $x \geq 1$, then $\cosh_1' x = 2 \cdot \cosh_1' \sqrt{\frac{x+1}{2}}$.

(39) If $x \geq 1$, then $\cosh_2' x = 2 \cdot \cosh_2' \sqrt{\frac{x+1}{2}}$.

(40) If $x \geq 1$, then $\cosh_1' x = 2 \cdot \sinh' \sqrt{\frac{x-1}{2}}$.

(41) If $x^2 < 1$, then $\tanh' x = \sinh'(\frac{x}{\sqrt{1-x^2}})$.

(42) If $0 < x$ and $x < 1$, then $\tanh' x = \cosh_1'(\frac{1}{\sqrt{1-x^2}})$.

(43) If $x^2 < 1$, then $\tanh' x = \frac{1}{2} \cdot \sinh'(\frac{2 \cdot x}{1-x^2})$.

(44) If $x > 0$ and $x < 1$, then $\tanh' x = \frac{1}{2} \cdot \cosh_1'(\frac{1+x^2}{1-x^2})$.

(45) If $x^2 < 1$, then $\tanh' x = \frac{1}{2} \cdot \tanh'(\frac{2 \cdot x}{1+x^2})$.

(46) If $x^{\mathbf{2}} > 1$, then $\coth' x = \tanh'(\frac{1}{x})$.

(47) If $x > 0$ and $x \leq 1$, then $\mathrm{sech}'_1 x = \cosh'_1(\frac{1}{x})$.

(48) If $x > 0$ and $x \leq 1$, then $\mathrm{sech}'_2 x = \cosh'_2(\frac{1}{x})$.

(49) If $x > 0$, then $\mathrm{csch}' x = \sinh'(\frac{1}{x})$.

(50) If $x \cdot y + \sqrt{x^{\mathbf{2}} + 1} \cdot \sqrt{y^{\mathbf{2}} + 1} \geq 0$, then $\sinh' x + \sinh' y = \sinh'(x \cdot \sqrt{1 + y^{\mathbf{2}}} + y \cdot \sqrt{1 + x^{\mathbf{2}}})$.

(51) $\sinh' x - \sinh' y = \sinh'(x \cdot \sqrt{1 + y^{\mathbf{2}}} - y \cdot \sqrt{1 + x^{\mathbf{2}}})$.

(52) If $1 \leq x$ and $1 \leq y$, then $\cosh'_1 x + \cosh'_1 y = \cosh'_1(x \cdot y + \sqrt{(x^{\mathbf{2}} - 1) \cdot (y^{\mathbf{2}} - 1)})$.

(53) If $1 \leq x$ and $1 \leq y$, then $\cosh'_2 x + \cosh'_2 y = \cosh'_2(x \cdot y + \sqrt{(x^{\mathbf{2}} - 1) \cdot (y^{\mathbf{2}} - 1)})$.

(54) If $1 \leq x$ and $1 \leq y$ and $|y| \leq |x|$, then $\cosh'_1 x - \cosh'_1 y = \cosh'_1(x \cdot y - \sqrt{(x^{\mathbf{2}} - 1) \cdot (y^{\mathbf{2}} - 1)})$.

(55) If $1 \leq x$ and $1 \leq y$ and $|y| \leq |x|$, then $\cosh'_2 x - \cosh'_2 y = \cosh'_2(x \cdot y - \sqrt{(x^{\mathbf{2}} - 1) \cdot (y^{\mathbf{2}} - 1)})$.

(56) If $x^{\mathbf{2}} < 1$ and $y^{\mathbf{2}} < 1$, then $\tanh' x + \tanh' y = \tanh'(\frac{x+y}{1+x \cdot y})$.

(57) If $x^{\mathbf{2}} < 1$ and $y^{\mathbf{2}} < 1$, then $\tanh' x - \tanh' y = \tanh'(\frac{x-y}{1-x \cdot y})$.

(58) If $0 < x$ and $(\frac{x-1}{x+1})^{\mathbf{2}} < 1$, then $\log_e x = 2 \cdot \tanh'(\frac{x-1}{x+1})$.

(59) If $0 < x$ and $(\frac{x^{\mathbf{2}}-1}{x^{\mathbf{2}}+1})^{\mathbf{2}} < 1$, then $\log_e x = \tanh'(\frac{x^{\mathbf{2}}-1}{x^{\mathbf{2}}+1})$.

(60) If $1 < x$ and $1 \leq \frac{x^{\mathbf{2}}+1}{2 \cdot x}$, then $\log_e x = \cosh'_1(\frac{x^{\mathbf{2}}+1}{2 \cdot x})$.

(61) If $0 < x$ and $x < 1$ and $1 \leq \frac{x^{\mathbf{2}}+1}{2 \cdot x}$, then $\log_e x = \cosh'_2(\frac{x^{\mathbf{2}}+1}{2 \cdot x})$.

(62) If $0 < x$, then $\log_e x = \sinh'(\frac{x^{\mathbf{2}}-1}{2 \cdot x})$.

(63) If $y = \frac{1}{2} \cdot (\exp x - \exp(-x))$, then $x = \log_e(y + \sqrt{y^{\mathbf{2}} + 1})$.

(64) If $y = \frac{1}{2} \cdot (\exp x + \exp(-x))$ and $1 \leq y$, then $x = \log_e(y + \sqrt{y^{\mathbf{2}} - 1})$ or $x = -\log_e(y + \sqrt{y^{\mathbf{2}} - 1})$.

(65) If $y = \frac{\exp x - \exp(-x)}{\exp x + \exp(-x)}$, then $x = \frac{1}{2} \cdot \log_e(\frac{1+y}{1-y})$.

(66) If $y = \frac{\exp x + \exp(-x)}{\exp x - \exp(-x)}$ and $x \neq 0$, then $x = \frac{1}{2} \cdot \log_e(\frac{y+1}{y-1})$.

(67) If $y = \frac{1}{\frac{\exp x + \exp(-x)}{2}}$, then $x = \log_e(\frac{1+\sqrt{1-y^{\mathbf{2}}}}{y})$ or $x = -\log_e(\frac{1+\sqrt{1-y^{\mathbf{2}}}}{y})$.

(68) If $y = \frac{1}{\frac{\exp x - \exp(-x)}{2}}$ and $x \neq 0$, then $x = \log_e(\frac{1+\sqrt{1+y^{\mathbf{2}}}}{y})$ or $x = \log_e(\frac{1-\sqrt{1+y^{\mathbf{2}}}}{y})$.

(69) (The function cosh)$(2 \cdot x) = 1 + 2 \cdot$ (the function sinh)$(x)^{\mathbf{2}}$.

(70) (The function cosh)$(x)^{\mathbf{2}} = 1 +$ (the function sinh)$(x)^{\mathbf{2}}$.

(71) (The function sinh)$(x)^{\mathbf{2}} =$ (the function cosh)$(x)^{\mathbf{2}} - 1$.

(72) $\sinh(5 \cdot x) = 5 \cdot \sinh x + 20 \cdot (\sinh x)^3 + 16 \cdot (\sinh x)^5$.

(73) $\cosh(5 \cdot x) = (5 \cdot \cosh x - 20 \cdot (\cosh x)^3) + 16 \cdot (\cosh x)^5$.

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Lines on Planes in n-Dimensional Euclidean Spaces

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Summary. In the paper we introduce basic properties of lines in the plane on this space. Lines and planes are expressed by the vector equation and are the image of $\mathbb{R}$ and $\mathbb{R}^2$. By this, we can say that the properties of the classic Euclid geometry are satisfied also in $\mathcal{R}^n$ as we know them intuitively. Next, we define the metric between the point and the line of this space.

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The notation and terminology used here are introduced in the following papers: [1], [5], [12], [4], [9], [14], [13], [8], [15], [6], [2], [3], [7], [11], and [10].

We follow the rules: a, a_1, a_2, a_3, b, b_1, b_2, b_3, r, s, t, u are real numbers, n is a natural number, and x_0, x, x_1, x_2, x_3, y_0, y, y_1, y_2, y_3 are elements of $\mathcal{R}^n$.

One can prove the following propositions:

(1) $\frac{s}{t} \cdot (u \cdot x) = \frac{s \cdot u}{t} \cdot x$ and $\frac{1}{t} \cdot (u \cdot x) = \frac{u}{t} \cdot x$.

(2) $x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$.

(3) $x - \langle \underbrace{0, \ldots, 0}_{n} \rangle = x$.

(4) $\langle \underbrace{0, \ldots, 0}_{n} \rangle - x = -x$.

(5) $x_1 - (x_2 + x_3) = x_1 - x_2 - x_3$.

(6) $x_1 - x_2 = x_1 + -x_2$.

(7) $x - x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$ and $x + -x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(8) $-a \cdot x = (-a) \cdot x$ and $-a \cdot x = a \cdot -x$.

(9) $x_1 - (x_2 - x_3) = (x_1 - x_2) + x_3$.

(10) $x_1 + (x_2 - x_3) = (x_1 + x_2) - x_3$.

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(11) $x_1 = x_2 + x_3$ iff $x_2 = x_1 - x_3$.

(12) $x = x_1 + x_2 + x_3$ iff $x - x_1 = x_2 + x_3$.

(13) $-(x_1 + x_2 + x_3) = -x_1 + -x_2 + -x_3$.

(14) $x_1 = x_2$ iff $x_1 - x_2 = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(15) If $x_1 - x_0 = t \cdot x$ and $x_1 \neq x_0$, then $t \neq 0$.

(16) $(a - b) \cdot x = a \cdot x + (-b) \cdot x$ and $(a - b) \cdot x = a \cdot x + -b \cdot x$ and $(a - b) \cdot x = a \cdot x - b \cdot x$.

(17) $a \cdot (x - y) = a \cdot x + -a \cdot y$ and $a \cdot (x - y) = a \cdot x + (-a) \cdot y$ and $a \cdot (x - y) = a \cdot x - a \cdot y$.

(18) $(s - t - u) \cdot x = s \cdot x - t \cdot x - u \cdot x$.

(19) $x - (a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3) = x + ((-a_1) \cdot x_1 + (-a_2) \cdot x_2 + (-a_3) \cdot x_3)$.

(20) $x - (s + t + u) \cdot y = x + (-s) \cdot y + (-t) \cdot y + (-u) \cdot y$.

(21) $(x_1 + x_2) + (y_1 + y_2) = x_1 + y_1 + (x_2 + y_2)$.

(22) $(x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) = x_1 + y_1 + (x_2 + y_2) + (x_3 + y_3)$.

(23) $(x_1 + x_2) - (y_1 + y_2) = (x_1 - y_1) + (x_2 - y_2)$.

(24) $(x_1 + x_2 + x_3) - (y_1 + y_2 + y_3) = (x_1 - y_1) + (x_2 - y_2) + (x_3 - y_3)$.

(25) $a \cdot (x_1 + x_2 + x_3) = a \cdot x_1 + a \cdot x_2 + a \cdot x_3$.

(26) $a \cdot (b_1 \cdot x_1 + b_2 \cdot x_2) = a \cdot b_1 \cdot x_1 + a \cdot b_2 \cdot x_2$.

(27) $a \cdot (b_1 \cdot x_1 + b_2 \cdot x_2 + b_3 \cdot x_3) = a \cdot b_1 \cdot x_1 + a \cdot b_2 \cdot x_2 + a \cdot b_3 \cdot x_3$.

(28) $a_1 \cdot x_1 + a_2 \cdot x_2 + (b_1 \cdot x_1 + b_2 \cdot x_2) = (a_1 + b_1) \cdot x_1 + (a_2 + b_2) \cdot x_2$.

(29) $a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3 + (b_1 \cdot x_1 + b_2 \cdot x_2 + b_3 \cdot x_3) = ((a_1 + b_1) \cdot x_1 + (a_2 + b_2) \cdot x_2) + (a_3 + b_3) \cdot x_3$.

(30) $(a_1 \cdot x_1 + a_2 \cdot x_2) - (b_1 \cdot x_1 + b_2 \cdot x_2) = (a_1 - b_1) \cdot x_1 + (a_2 - b_2) \cdot x_2$.

(31) $(a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3) - (b_1 \cdot x_1 + b_2 \cdot x_2 + b_3 \cdot x_3) = (a_1 - b_1) \cdot x_1 + (a_2 - b_2) \cdot x_2 + (a_3 - b_3) \cdot x_3$.

(32) If $a_1 + a_2 + a_3 = 1$, then $a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3 = x_1 + a_2 \cdot (x_2 - x_1) + a_3 \cdot (x_3 - x_1)$.

(33) If $x = x_1 + a_2 \cdot (x_2 - x_1) + a_3 \cdot (x_3 - x_1)$, then there exists a real number a_1 such that $x = a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3$ and $a_1 + a_2 + a_3 = 1$.

(34) For every natural number n such that $n \geq 1$ holds $1 * n \neq \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(35) For every subset A of $\mathcal{R}^n$ and for all x_1, x_2 such that A is a line and $x_1 \in A$ and $x_2 \in A$ and $x_1 \neq x_2$ holds $A = \mathrm{Line}(x_1, x_2)$.

(36) For all elements x_1, x_2 of $\mathcal{R}^n$ such that $y_1 \in \mathrm{Line}(x_1, x_2)$ and $y_2 \in \mathrm{Line}(x_1, x_2)$ there exists a such that $y_2 - y_1 = a \cdot (x_2 - x_1)$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. The predicate $x_1 \parallel x_2$ is defined as follows:

(Def. 1) $x_1 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$ and $x_2 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$ and there exists r such that $x_1 = r \cdot x_2$.

One can prove the following proposition

(37) For all elements x_1, x_2 of $\mathcal{R}^n$ such that $x_1 \parallel x_2$ there exists a such that $a \neq 0$ and $x_1 = a \cdot x_2$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. Let us note that the predicate $x_1 \parallel x_2$ is symmetric.

The following proposition is true

(38) If $x_1 \parallel x_2$ and $x_2 \parallel x_3$, then $x_1 \parallel x_3$.

Let n be a natural number and let x_1, x_2 be elements of $\mathcal{R}^n$. We say that x_1 and x_2 are linearly independent if and only if:

(Def. 2) For all real numbers a_1, a_2 such that $a_1 \cdot x_1 + a_2 \cdot x_2 = \langle\underbrace{0,\ldots,0}_{n}\rangle$ holds $a_1 = 0$ and $a_2 = 0$.

Let us note that the predicate x_1 and x_2 are linearly independent is symmetric.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. We introduce x_1 and x_2 are linearly dependent as an antonym of x_1 and x_2 are linearly independent.

Next we state a number of propositions:

(39) If x_1 and x_2 are linearly independent, then $x_1 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$ and $x_2 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$.

(40) For all x_1, x_2 such that x_1 and x_2 are linearly independent holds if $a_1 \cdot x_1 + a_2 \cdot x_2 = b_1 \cdot x_1 + b_2 \cdot x_2$, then $a_1 = b_1$ and $a_2 = b_2$.

(41) Let given x_1, x_2, y_1, y_1. Suppose y_1 and y_2 are linearly independent. Suppose $y_1 = a_1 \cdot x_1 + a_2 \cdot x_2$ and $y_2 = b_1 \cdot x_1 + b_2 \cdot x_2$. Then there exist real numbers c_1, c_2, d_1, d_2 such that $x_1 = c_1 \cdot y_1 + c_2 \cdot y_2$ and $x_2 = d_1 \cdot y_1 + d_2 \cdot y_2$.

(42) If x_1 and x_2 are linearly independent, then $x_1 \neq x_2$.

(43) If $x_2 - x_1$ and $x_3 - x_1$ are linearly independent, then $x_2 \neq x_3$.

(44) If x_1 and x_2 are linearly independent, then $x_1 + t \cdot x_2$ and x_2 are linearly independent.

(45) Suppose $x_1 - x_0$ and $x_3 - x_2$ are linearly independent and $y_0 \in \mathrm{Line}(x_0, x_1)$ and $y_1 \in \mathrm{Line}(x_0, x_1)$ and $y_0 \neq y_1$ and $y_2 \in \mathrm{Line}(x_2, x_3)$ and $y_3 \in \mathrm{Line}(x_2, x_3)$ and $y_2 \neq y_3$. Then $y_1 - y_0$ and $y_3 - y_2$ are linearly independent.

(46) If $x_1 \parallel x_2$, then x_1 and x_2 are linearly dependent and $x_1 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$ and $x_2 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$.

(47) If x_1 and x_2 are linearly dependent, then $x_1 = \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$ or $x_2 = \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$ or $x_1 \parallel x_2$.

(48) For all elements x_1, x_2, y_1 of $\mathcal{R}^n$ there exists an element y_2 of $\mathcal{R}^n$ such that $y_2 \in \mathrm{Line}(x_1, x_2)$ and $x_1 - x_2$, $y_1 - y_2$ are orthogonal.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. The predicate $x_1 \perp x_2$ is defined by:

(Def. 3) $x_1 \neq \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$ and $x_2 \neq \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$ and x_1, x_2 are orthogonal.

Let us note that the predicate $x_1 \perp x_2$ is symmetric.

The following propositions are true:

(49) If $x \perp y_0$ and $y_0 \parallel y_1$, then $x \perp y_1$.

(50) If $x \perp y$, then x and y are linearly independent.

(51) If $x_1 \parallel x_2$, then $x_1 \not\perp x_2$.

(52) If $x_1 \perp x_2$, then $x_1 \nparallel x_2$.

Let us consider n. The functor $\mathrm{Lines}(\mathcal{R}^n)$ yields a family of subsets of $\mathcal{R}^n$ and is defined by:

(Def. 4) $\mathrm{Lines}(\mathcal{R}^n) = \{\mathrm{Line}(x_1, x_2)\}$.

Let us consider n. Note that $\mathrm{Lines}(\mathcal{R}^n)$ is non empty.

The following proposition is true

(53) $\mathrm{Line}(x_1, x_2) \in \mathrm{Lines}(\mathcal{R}^n)$.

In the sequel L, L_0, L_1, L_2 are elements of $\mathrm{Lines}(\mathcal{R}^n)$.

The following propositions are true:

(54) If $x_1 \in L$ and $x_2 \in L$, then $\mathrm{Line}(x_1, x_2) \subseteq L$.

(55) L_1 meets L_2 iff there exists x such that $x \in L_1$ and $x \in L_2$.

(56) If L_0 misses L_1 and $x \in L_0$, then $x \notin L_1$.

(57) There exist x_1, x_2 such that $L = \mathrm{Line}(x_1, x_2)$.

(58) There exists x such that $x \in L$.

(59) If $x_0 \in L$ and L is a line, then there exists x_1 such that $x_1 \neq x_0$ and $x_1 \in L$.

(60) If $x \notin L$ and L is a line, then there exist x_1, x_2 such that $L = \mathrm{Line}(x_1, x_2)$ and $x - x_1 \perp x_2 - x_1$.

(61) If $x \notin L$ and L is a line, then there exist x_1, x_2 such that $L = \mathrm{Line}(x_1, x_2)$ and $x - x_1$ and $x_2 - x_1$ are linearly independent.

Let n be a natural number, let x be an element of $\mathcal{R}^n$, and let L be an element of $\mathrm{Lines}(\mathcal{R}^n)$. The functor $\rho(x, L)$ yields a real number and is defined by:

(Def. 5) There exists a subset S of $\mathbb{R}$ such that $S = \{|x - x_0|; x_0$ ranges over elements of $\mathcal{R}^n$: $x_0 \in L\}$ and $\rho(x, L) = \inf S$.

Next we state three propositions:

(62) There exists x_0 such that $x_0 \in L$ and $|x - x_0| = \rho(x, L)$.

(63) $\rho(x, L) \geq 0$.

(64) $x \in L$ iff $\rho(x, L) = 0$.

Let us consider n and let us consider L_1, L_2. The predicate $L_1 \parallel L_2$ is defined as follows:

(Def. 6) There exist elements x_1, x_2, y_1, y_2 of $\mathcal{R}^n$ such that $L_1 = \mathrm{Line}(x_1, x_2)$ and $L_2 = \mathrm{Line}(y_1, y_2)$ and $x_2 - x_1 \parallel y_2 - y_1$.

Let us note that the predicate $L_1 \parallel L_2$ is symmetric.

The following proposition is true

(65) If $L_0 \parallel L_1$ and $L_1 \parallel L_2$, then $L_0 \parallel L_2$.

Let us consider n and let us consider L_1, L_2. The predicate $L_1 \perp L_2$ is defined by:

(Def. 7) There exist elements x_1, x_2, y_1, y_2 of $\mathcal{R}^n$ such that $L_1 = \mathrm{Line}(x_1, x_2)$ and $L_2 = \mathrm{Line}(y_1, y_2)$ and $x_2 - x_1 \perp y_2 - y_1$.

Let us note that the predicate $L_1 \perp L_2$ is symmetric.

We now state a number of propositions:

(66) If $L_0 \perp L_1$ and $L_1 \parallel L_2$, then $L_0 \perp L_2$.

(67) If $x \notin L$ and L is a line, then there exists L_0 such that $x \in L_0$ and $L_0 \perp L$ and L_0 meets L.

(68) If L_1 misses L_2, then there exists x such that $x \in L_1$ and $x \notin L_2$.

(69) If $x_1 \in L$ and $x_2 \in L$ and $x_1 \neq x_2$, then $\mathrm{Line}(x_1, x_2) = L$ and L is a line.

(70) If L_1 is a line and L_2 is a line and $L_1 = L_2$, then $L_1 \parallel L_2$.

(71) If $L_1 \parallel L_2$, then L_1 is a line and L_2 is a line.

(72) If $L_1 \perp L_2$, then L_1 is a line and L_2 is a line.

(73) If $x \in L$ and $a \neq 1$ and $a \cdot x \in L$, then $\langle \underbrace{0, \ldots, 0}_{n} \rangle \in L$.

(74) If $x_1 \in L$ and $x_2 \in L$, then there exists x_3 such that $x_3 \in L$ and $x_3 - x_1 = a \cdot (x_2 - x_1)$.

(75) If $x_1 \in L$ and $x_2 \in L$ and $x_3 \in L$ and $x_1 \neq x_2$, then there exists a such that $x_3 - x_1 = a \cdot (x_2 - x_1)$.

(76) If $L_1 \parallel L_2$ and $L_1 \neq L_2$, then L_1 misses L_2.

(77) If $L_1 \parallel L_2$, then $L_1 = L_2$ or L_1 misses L_2.

(78) If $L_1 \parallel L_2$ and L_1 meets L_2, then $L_1 = L_2$.

(79) If L is a line, then there exists L_0 such that $x \in L_0$ and $L_0 \parallel L$.

(80) For all x, L such that $x \notin L$ and L is a line there exists L_0 such that $x \in L_0$ and $L_0 \parallel L$ and $L_0 \neq L$.

(81) For all x_0, x_1, y_0, y_1, L_1, L_2 such that $x_0 \in L_1$ and $x_1 \in L_1$ and $x_0 \neq x_1$ and $y_0 \in L_2$ and $y_1 \in L_2$ and $y_0 \neq y_1$ and $L_1 \perp L_2$ holds $x_1 - x_0 \perp y_1 - y_0$.

(82) For all L_1, L_2 such that $L_1 \perp L_2$ holds $L_1 \neq L_2$.

(83) For all x_1, x_2, L such that L is a line and $L = \mathrm{Line}(x_1, x_2)$ holds $x_1 \neq x_2$.

(84) If $x_0 \in L_1$ and $x_1 \in L_1$ and $x_0 \neq x_1$ and $y_0 \in L_2$ and $y_1 \in L_2$ and $y_0 \neq y_1$ and $L_1 \parallel L_2$, then $x_1 - x_0 \parallel y_1 - y_0$.

(85) Suppose $x_2 - x_1$ and $x_3 - x_1$ are linearly independent and $y_2 \in \mathrm{Line}(x_1, x_2)$ and $y_3 \in \mathrm{Line}(x_1, x_3)$ and $L_1 = \mathrm{Line}(x_2, x_3)$ and $L_2 = \mathrm{Line}(y_2, y_3)$. Then $L_1 \parallel L_2$ if and only if there exists a such that $a \neq 0$ and $y_2 - x_1 = a \cdot (x_2 - x_1)$ and $y_3 - x_1 = a \cdot (x_3 - x_1)$.

(86) For all L_1, L_2 such that L_1 is a line and L_2 is a line and $L_1 \neq L_2$ there exists x such that $x \in L_1$ and $x \notin L_2$.

(87) For all x, L_1, L_2 such that $L_1 \perp L_2$ and $x \in L_2$ there exists L_0 such that $x \in L_0$ and $L_0 \perp L_2$ and $L_0 \parallel L_1$.

(88) For all x, L_1, L_2 such that $x \in L_1$ and $x \in L_2$ and $L_1 \perp L_2$ there exists x_0 such that $x \neq x_0$ and $x_0 \in L_1$ and $x_0 \notin L_2$.

Let n be a natural number and let x_1, x_2, x_3 be elements of $\mathcal{R}^n$. The functor $\mathrm{Plane}(x_1, x_2, x_3)$ yielding a subset of $\mathcal{R}^n$ is defined as follows:

(Def. 8) $\mathrm{Plane}(x_1, x_2, x_3) = \{a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3 : a_1 + a_2 + a_3 = 1\}$.

Let n be a natural number and let x_1, x_2, x_3 be elements of $\mathcal{R}^n$. One can check that $\mathrm{Plane}(x_1, x_2, x_3)$ is non empty.

Let us consider n and let A be a subset of $\mathcal{R}^n$. We say that A is plane if and only if:

(Def. 9) There exist x_1, x_2, x_3 such that $x_2 - x_1$ and $x_3 - x_1$ are linearly independent and $A = \mathrm{Plane}(x_1, x_2, x_3)$.

One can prove the following propositions:

(89) $x_1 \in \mathrm{Plane}(x_1, x_2, x_3)$ and $x_2 \in \mathrm{Plane}(x_1, x_2, x_3)$ and $x_3 \in \mathrm{Plane}(x_1, x_2, x_3)$.

(90) If $x_1 \in \mathrm{Plane}(y_1, y_2, y_3)$ and $x_2 \in \mathrm{Plane}(y_1, y_2, y_3)$ and $x_3 \in \mathrm{Plane}(y_1, y_2, y_3)$, then $\mathrm{Plane}(x_1, x_2, x_3) \subseteq \mathrm{Plane}(y_1, y_2, y_3)$.

(91) Let A be a subset of $\mathcal{R}^n$ and given x, x_1, x_2, x_3. Suppose $x \in \mathrm{Plane}(x_1, x_2, x_3)$ and there exist real numbers c_1, c_2, c_3 such that $c_1 + c_2 + c_3 = 0$ and $x = c_1 \cdot x_1 + c_2 \cdot x_2 + c_3 \cdot x_3$. Then $\langle \underbrace{0, \ldots, 0}_{n} \rangle \in \mathrm{Plane}(x_1, x_2, x_3)$.

(92) If $y_1 \in \mathrm{Plane}(x_1, x_2, x_3)$ and $y_2 \in \mathrm{Plane}(x_1, x_2, x_3)$, then $\mathrm{Line}(y_1, y_2) \subseteq \mathrm{Plane}(x_1, x_2, x_3)$.

(93) For every subset A of $\mathcal{R}^n$ and for every x such that A is plane and $x \in A$ and there exists a such that $a \neq 1$ and $a \cdot x \in A$ holds $\underbrace{\langle 0, \ldots, 0 \rangle}_{n} \in A$.

(94) If $x_1 - x_1$ and $x_3 - x_1$ are linearly independent and $x \in \mathrm{Plane}(x_1, x_2, x_3)$ and $x = a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3$, then $a_1 + a_2 + a_3 = 1$ or $\underbrace{\langle 0, \ldots, 0 \rangle}_{n} \in \mathrm{Plane}(x_1, x_2, x_3)$.

(95) $x \in \mathrm{Plane}(x_1, x_2, x_3)$ iff there exist a_1, a_2, a_3 such that $a_1 + a_2 + a_3 = 1$ and $x = a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3$.

(96) Suppose that

(i) $x_2 - x_1$ and $x_3 - x_1$ are linearly independent,

(ii) $x \in \mathrm{Plane}(x_1, x_2, x_3)$,

(iii) $a_1 + a_2 + a_3 = 1$,

(iv) $x = a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3$,

(v) $b_1 + b_2 + b_3 = 1$, and

(vi) $x = b_1 \cdot x_1 + b_2 \cdot x_2 + b_3 \cdot x_3$.

Then $a_1 = b_1$ and $a_2 = b_2$ and $a_3 = b_3$.

Let us consider n. The functor $\mathrm{Planes}(\mathcal{R}^n)$ yielding a family of subsets of $\mathcal{R}^n$ is defined by:

(Def. 10) $\mathrm{Planes}(\mathcal{R}^n) = \{\mathrm{Plane}(x_1, x_2, x_3)\}$.

Let us consider n. Note that $\mathrm{Planes}(\mathcal{R}^n)$ is non empty.

The following proposition is true

(97) $\mathrm{Plane}(x_1, x_2, x_3) \in \mathrm{Planes}(\mathcal{R}^n)$.

In the sequel P, P_0, P_1, P_2 are elements of $\mathrm{Planes}(\mathcal{R}^n)$.

Next we state several propositions:

(98) If $x_1 \in P$ and $x_2 \in P$ and $x_3 \in P$, then $\mathrm{Plane}(x_1, x_2, x_3) \subseteq P$.

(99) If $x_1 \in P$ and $x_2 \in P$ and $x_3 \in P$ and $x_2 - x_1$ and $x_3 - x_1$ are linearly independent, then $P = \mathrm{Plane}(x_1, x_2, x_3)$.

(100) If P_1 is plane and $P_1 \subseteq P_2$, then $P_1 = P_2$.

(101) $\mathrm{Line}(x_1, x_2) \subseteq \mathrm{Plane}(x_1, x_2, x_3)$ and $\mathrm{Line}(x_2, x_3) \subseteq \mathrm{Plane}(x_1, x_2, x_3)$ and $\mathrm{Line}(x_3, x_1) \subseteq \mathrm{Plane}(x_1, x_2, x_3)$.

(102) If $x_1 \in P$ and $x_2 \in P$, then $\mathrm{Line}(x_1, x_2) \subseteq P$.

Let n be a natural number and let L_1, L_2 be elements of $\mathrm{Lines}(\mathcal{R}^n)$. We say that L_1 and L_2 are coplanar if and only if:

(Def. 11) There exist elements x_1, x_2, x_3 of $\mathcal{R}^n$ such that $L_1 \subseteq \mathrm{Plane}(x_1, x_2, x_3)$ and $L_2 \subseteq \mathrm{Plane}(x_1, x_2, x_3)$.

We now state a number of propositions:

(103) L_1 and L_2 are coplanar iff there exists P such that $L_1 \subseteq P$ and $L_2 \subseteq P$.

(104) If $L_1 \parallel L_2$, then L_1 and L_2 are coplanar.

(105) Suppose L_1 is a line and L_2 is a line and L_1 and L_2 are coplanar and L_1 misses L_2. Then there exists P such that $L_1 \subseteq P$ and $L_2 \subseteq P$ and P is plane.

(106) There exists P such that $x \in P$ and $L \subseteq P$.

(107) If $x \notin L$ and L is a line, then there exists P such that $x \in P$ and $L \subseteq P$ and P is plane.

(108) If $x \in P$ and $L \subseteq P$ and $x \notin L$ and L is a line, then P is plane.

(109) If $x \notin L$ and L is a line and $x \in P_0$ and $L \subseteq P_0$ and $x \in P_1$ and $L \subseteq P_1$, then $P_0 = P_1$.

(110) If L_1 is a line and L_2 is a line and L_1 and L_2 are coplanar and $L_1 \neq L_2$, then there exists P such that $L_1 \subseteq P$ and $L_2 \subseteq P$ and P is plane.

(111) For all L_1, L_2 such that L_1 is a line and L_2 is a line and $L_1 \neq L_2$ and L_1 meets L_2 there exists P such that $L_1 \subseteq P$ and $L_2 \subseteq P$ and P is plane.

(112) If L_1 is a line and L_2 is a line and $L_1 \neq L_2$ and L_1 meets L_2 and $L_1 \subseteq P_1$ and $L_2 \subseteq P_1$ and $L_1 \subseteq P_2$ and $L_2 \subseteq P_2$, then $P_1 = P_2$.

(113) If $L_1 \parallel L_2$ and $L_1 \neq L_2$, then there exists P such that $L_1 \subseteq P$ and $L_2 \subseteq P$ and P is plane.

(114) If $L_1 \perp L_2$ and L_1 meets L_2, then there exists P such that P is plane and $L_1 \subseteq P$ and $L_2 \subseteq P$.

(115) If $L_0 \subseteq P$ and $L_1 \subseteq P$ and $L_2 \subseteq P$ and $x \in L_0$ and $x \in L_1$ and $x \in L_2$ and $L_0 \perp L_2$ and $L_1 \perp L_2$, then $L_0 = L_1$.

(116) If L_1 and L_2 are coplanar and $L_1 \perp L_2$, then L_1 meets L_2.

(117) If $L_1 \subseteq P$ and $L_2 \subseteq P$ and $L_1 \perp L_2$ and $x \in P$ and $L_0 \parallel L_2$ and $x \in L_0$, then $L_0 \subseteq P$ and $L_0 \perp L_1$.

(118) If $L \subseteq P$ and $L_1 \subseteq P$ and $L_2 \subseteq P$ and $L \perp L_1$ and $L \perp L_2$, then $L_1 \parallel L_2$.

(119) Suppose $L_0 \subseteq P$ and $L_1 \subseteq P$ and $L_2 \subseteq P$ and $L_0 \parallel L_1$ and $L_1 \parallel L_2$ and $L_0 \neq L_1$ and $L_1 \neq L_2$ and $L_2 \neq L_0$ and L meets L_0 and L meets L_1. Then L meets L_2.

(120) If L_1 and L_2 are coplanar and L_1 is a line and L_2 is a line and L_1 misses L_2, then $L_1 \parallel L_2$.

(121) If $x_1 \in P$ and $x_2 \in P$ and $y_1 \in P$ and $y_2 \in P$ and $x_2 - x_1$ and $y_2 - y_1$ are linearly independent, then $\mathrm{Line}(x_1, x_2)$ meets $\mathrm{Line}(y_1, y_2)$.

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Cardinal Numbers and Finite Sets[1]

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Summary. In this paper we define class of functions and operators needed for the proof of the principle of inclusions and the disconnections. We also given certain cardinal numbers concerning elementary class of functions (this function mapping finite set in finite set).

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The articles [21], [10], [24], [17], [26], [6], [27], [2], [9], [11], [1], [25], [7], [8], [22], [19], [5], [15], [12], [20], [16], [14], [18], [13], [3], [23], and [4] provide the terminology and notation for this paper.

For simplicity, we use the following convention: x, x_1, x_2, y, z, X' denote sets, X, Y denote finite sets, n, k, m denote natural numbers, and f denotes a function.

Next we state the proposition

(1) If $X \subseteq Y$ and $\operatorname{card} X = \operatorname{card} Y$, then $X = Y$.

In the sequel F is a function from $X \cup \{x\}$ into $Y \cup \{y\}$.

One can prove the following proposition

(2) For all X, Y, x, y such that if $Y = \emptyset$, then $X = \emptyset$ and $x \notin X$ holds $\operatorname{card}(Y^X) = \overline{\overline{\{F : \operatorname{rng}(F{\restriction}X) \subseteq Y \ \wedge \ F(x) = y\}}}$.

In the sequel F is a function from $X \cup \{x\}$ into Y.

One can prove the following two propositions:

(3) For all X, Y, x, y such that $x \notin X$ and $y \in Y$ holds $\operatorname{card}(Y^X) = \overline{\overline{\{F : F(x) = y\}}}$.

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(4) If if $Y = \emptyset$, then $X = \emptyset$, then $\mathrm{card}(Y^X) = (\mathrm{card}\, Y)^{\mathrm{card}\, X}$.

In the sequel F_1 denotes a function from X into Y and F_2 denotes a function from $X \cup \{x\}$ into $Y \cup \{y\}$.

One can prove the following two propositions:

(5) Let given X, Y, x, y. Suppose if Y is empty, then X is empty and $x \notin X$ and $y \notin Y$. Then $\overline{\overline{\{F_1 : F_1 \text{ is one-to-one}\}}} = \overline{\overline{\{F_2 : F_2 \text{ is one-to-one } \wedge\ F_2(x) = y\}}}$.

(6) $\frac{n!}{(n -' k)!}$ is a natural number.

In the sequel F is a function from X into Y.

The following proposition is true

(7) If $\mathrm{card}\, X \leq \mathrm{card}\, Y$, then $\overline{\overline{\{F : F \text{ is one-to-one}\}}} = \frac{(\mathrm{card}\, Y)!}{(\mathrm{card}\, Y -' \mathrm{card}\, X)!}$.

In the sequel F denotes a function from X into X.

The following proposition is true

(8) $\overline{\overline{\{F : F \text{ is a permutation of } X\}}} = (\mathrm{card}\, X)!$.

Let us consider X, k, x_1, x_2. The functor $\mathrm{Choose}(X, k, x_1, x_2)$ yields a subset of $\{x_1, x_2\}^X$ and is defined as follows:

(Def. 1) $x \in \mathrm{Choose}(X, k, x_1, x_2)$ iff there exists a function f from X into $\{x_1, x_2\}$ such that $f = x$ and $\overline{\overline{f^{-1}(\{x_1\})}} = k$.

We now state several propositions:

(9) If $\mathrm{card}\, X \neq k$, then $\mathrm{Choose}(X, k, x_1, x_1)$ is empty.

(10) If $\mathrm{card}\, X < k$, then $\mathrm{Choose}(X, k, x_1, x_2)$ is empty.

(11) If $x_1 \neq x_2$, then $\mathrm{card}\,\mathrm{Choose}(X, 0, x_1, x_2) = 1$.

(12) $\mathrm{card}\,\mathrm{Choose}(X, \mathrm{card}\, X, x_1, x_2) = 1$.

(13) If $f(y) = x$ and $y \in \mathrm{dom}\, f$, then $\{y\} \cup (f{\restriction}(\mathrm{dom}\, f \setminus \{y\}))^{-1}(\{x\}) = f^{-1}(\{x\})$.

In the sequel g denotes a function from $X \cup \{z\}$ into $\{x, y\}$.

The following propositions are true:

(14) If $z \notin X$, then $\mathrm{card}\,\mathrm{Choose}(X, k, x, y) = \overline{\overline{\{g : \overline{\overline{g^{-1}(\{x\})}} = k + 1\ \wedge\ g(z) = x\}}}$.

(15) If $f(y) \neq x$, then $(f{\restriction}(\mathrm{dom}\, f \setminus \{y\}))^{-1}(\{x\}) = f^{-1}(\{x\})$.

(16) If $z \notin X$ and $x \neq y$, then $\mathrm{card}\,\mathrm{Choose}(X, k, x, y) = \overline{\overline{\{g : \overline{\overline{g^{-1}(\{x\})}} = k\ \wedge\ g(z) = y\}}}$.

(17) If $x \neq y$ and $z \notin X$, then $\mathrm{card}\,\mathrm{Choose}(X \cup \{z\}, k + 1, x, y) = \mathrm{card}\,\mathrm{Choose}(X, k + 1, x, y) + \mathrm{card}\,\mathrm{Choose}(X, k, x, y)$.

(18) If $x \neq y$, then $\mathrm{card}\,\mathrm{Choose}(X, k, x, y) = \binom{\mathrm{card}\, X}{k}$.

(19) If $x \neq y$, then $(Y \longmapsto y) {+\cdot} (X \longmapsto x) \in \mathrm{Choose}(X \cup Y, \mathrm{card}\, X, x, y)$.

(20) If $x \neq y$ and X misses Y, then $(X \longmapsto x) {+\cdot} (Y \longmapsto y) \in \mathrm{Choose}(X \cup Y, \mathrm{card}\, X, x, y)$.

Let F, C_1 be functions and let y be a set. The functor $\mathrm{Intersection}(F, C_1, y)$ yielding a subset of $\bigcup \mathrm{rng}\, F$ is defined as follows:

(Def. 2) $z \in \mathrm{Intersection}(F, C_1, y)$ iff $z \in \bigcup \mathrm{rng}\, F$ and for every x such that $x \in \mathrm{dom}\, C_1$ and $C_1(x) = y$ holds $z \in F(x)$.

In the sequel F, C_1 denote functions.

The following propositions are true:

(21) For all F, C_1 such that $\mathrm{dom}\, F \cap {C_1}^{-1}(\{x\})$ is non empty holds $y \in \mathrm{Intersection}(F, C_1, x)$ iff for every z such that $z \in \mathrm{dom}\, C_1$ and $C_1(z) = x$ holds $y \in F(z)$.

(22) If $\mathrm{Intersection}(F, C_1, y)$ is non empty, then ${C_1}^{-1}(\{y\}) \subseteq \mathrm{dom}\, F$.

(23) If $\mathrm{Intersection}(F, C_1, y)$ is non empty, then for all x_1, x_2 such that $x_1 \in {C_1}^{-1}(\{y\})$ and $x_2 \in {C_1}^{-1}(\{y\})$ holds $F(x_1)$ meets $F(x_2)$.

(24) If $z \in \mathrm{Intersection}(F, C_1, y)$ and $y \in \mathrm{rng}\, C_1$, then there exists x such that $x \in \mathrm{dom}\, C_1$ and $C_1(x) = y$ and $z \in F(x)$.

(25) If F is empty or $\bigcup \mathrm{rng}\, F$ is empty, then $\mathrm{Intersection}(F, C_1, y) = \bigcup \mathrm{rng}\, F$.

(26) If $F {\restriction} {C_1}^{-1}(\{y\}) = {C_1}^{-1}(\{y\}) \longmapsto \bigcup \mathrm{rng}\, F$, then $\mathrm{Intersection}(F, C_1, y) = \bigcup \mathrm{rng}\, F$.

(27) If $\bigcup \mathrm{rng}\, F$ is non empty and $\mathrm{Intersection}(F, C_1, y) = \bigcup \mathrm{rng}\, F$, then $F {\restriction} {C_1}^{-1}(\{y\}) = {C_1}^{-1}(\{y\}) \longmapsto \bigcup \mathrm{rng}\, F$.

(28) $\mathrm{Intersection}(F, \emptyset, y) = \bigcup \mathrm{rng}\, F$.

(29) $\mathrm{Intersection}(F, C_1, y) \subseteq \mathrm{Intersection}(F, C_1 {\restriction} X', y)$.

(30) If ${C_1}^{-1}(\{y\}) = (C_1 {\restriction} X')^{-1}(\{y\})$, then $\mathrm{Intersection}(F, C_1, y) = \mathrm{Intersection}(F, C_1 {\restriction} X', y)$.

(31) $\mathrm{Intersection}(F {\restriction} X', C_1, y) \subseteq \mathrm{Intersection}(F, C_1, y)$.

(32) If $y \in \mathrm{rng}\, C_1$ and ${C_1}^{-1}(\{y\}) \subseteq X'$, then $\mathrm{Intersection}(F {\restriction} X', C_1, y) = \mathrm{Intersection}(F, C_1, y)$.

(33) If $x \in {C_1}^{-1}(\{y\})$, then $\mathrm{Intersection}(F, C_1, y) \subseteq F(x)$.

(34) If $x \in {C_1}^{-1}(\{y\})$, then $\mathrm{Intersection}(F, C_1 {\restriction} (\mathrm{dom}\, C_1 \setminus \{x\}), y) \cap F(x) = \mathrm{Intersection}(F, C_1, y)$.

(35) For all functions C_2, C_3 such that ${C_2}^{-1}(\{x_1\}) = {C_3}^{-1}(\{x_2\})$ holds $\mathrm{Intersection}(F, C_2, x_1) = \mathrm{Intersection}(F, C_3, x_2)$.

(36) If ${C_1}^{-1}(\{y\}) = \emptyset$, then $\mathrm{Intersection}(F, C_1, y) = \bigcup \mathrm{rng}\, F$.

(37) If $\{x\} = {C_1}^{-1}(\{y\})$, then $\mathrm{Intersection}(F, C_1, y) = F(x)$.

(38) If $\{x_1, x_2\} = {C_1}^{-1}(\{y\})$, then $\mathrm{Intersection}(F, C_1, y) = F(x_1) \cap F(x_2)$.

(39) For every F such that F is non empty holds $y \in \mathrm{Intersection}(F, \mathrm{dom}\, F \longmapsto x, x)$ iff for every z such that $z \in \mathrm{dom}\, F$ holds $y \in F(z)$.

Let F be a function. We say that F is finite-yielding if and only if:

(Def. 3) For every x holds $F(x)$ is finite.

Let us observe that there exists a function which is non empty and finite-yielding and there exists a function which is empty and finite-yielding.

Let F be a finite-yielding function and let x be a set. Observe that $F(x)$ is finite.

Let F be a finite-yielding function and let X be a set. One can check that $F{\restriction}X$ is finite-yielding.

Let F be a finite-yielding function and let G be a function. Note that $F \cdot G$ is finite-yielding and $\mathrm{Intersect}(F, G)$ is finite-yielding.

In the sequel F_3 is a finite-yielding function.

The following two propositions are true:

(40) If $y \in \operatorname{rng} C_1$, then $\mathrm{Intersection}(F_3, C_1, y)$ is finite.

(41) If $\operatorname{dom} F_3$ is finite, then $\bigcup \operatorname{rng} F_3$ is finite.

Let F be a finite 0-sequence and let us consider n. Then $F{\restriction}n$ is a finite 0-sequence.

Let D be a set, let F be a finite 0-sequence of D, and let us consider n. Then $F{\restriction}n$ is a finite 0-sequence of D.

In the sequel D is a non empty set and b is a binary operation on D.

Next we state several propositions:

(42) For every finite 0-sequence F of D and for all b, n such that $n \in \operatorname{dom} F$ but b has a unity or $n \neq 0$ holds $b(b \odot F{\restriction}n,\ F(n)) = b \odot F{\restriction}(n+1)$.

(43) For every finite 0-sequence F of D and for every n such that $\operatorname{len} F = n+1$ holds $F = (F{\restriction}n) \frown \langle F(n) \rangle$.

(44) For every finite 0-sequence F of $\mathbb{N}$ and for every n such that $n \in \operatorname{dom} F$ holds $\sum(F{\restriction}n) + F(n) = \sum(F{\restriction}(n+1))$.

(45) For every finite 0-sequence F of $\mathbb{N}$ and for every n such that $\operatorname{rng} F \subseteq \{0, n\}$ holds $\sum F = n \cdot \operatorname{card}(F^{-1}(\{n\}))$.

(46) $x \in \mathrm{Choose}(n, k, 1, 0)$ iff there exists a finite 0-sequence F of $\mathbb{N}$ such that $F = x$ and $\operatorname{dom} F = n$ and $\operatorname{rng} F \subseteq \{0, 1\}$ and $\sum F = k$.

(47) For every finite 0-sequence F of D and for every b such that b has a unity or $\operatorname{len} F \geq 1$ holds $b \odot F = b \odot \mathrm{XFS2FS}(F)$.

(48) Let F, G be finite 0-sequences of D and P be a permutation of $\operatorname{dom} F$. Suppose b is commutative and associative but b has a unity or $\operatorname{len} F \geq 1$ but $G = F \cdot P$. Then $b \odot F = b \odot G$.

Let us consider k and let F be a finite-yielding function. Let us assume that $\operatorname{dom} F$ is finite. The card intersection of F wrt k yielding a natural number is defined by the condition (Def. 4).

(Def. 4) Let x, y be sets, X be a finite set, and P be a function from $\operatorname{card} \mathrm{Choose}(X, k, x, y)$ into $\mathrm{Choose}(X, k, x, y)$. Suppose $\operatorname{dom} F = X$ and

P is one-to-one and $x \neq y$. Then there exists a finite 0-sequence X_1 of $\mathbb{N}$ such that $\operatorname{dom} X_1 = \operatorname{dom} P$ and for all z, f such that $z \in \operatorname{dom} X_1$ and $f = P(z)$ holds $X_1(z) = \overline{\overline{\mathrm{Intersection}(F, f, x)}}$ and the card intersection of F wrt $k = \sum X_1$.

One can prove the following propositions:

(49) Let x, y be sets, X be a finite set, and P be a function from $\operatorname{card} \mathrm{Choose}(X, k, x, y)$ into $\mathrm{Choose}(X, k, x, y)$. Suppose $\operatorname{dom} F_3 = X$ and P is one-to-one and $x \neq y$. Let X_1 be a finite 0-sequence of $\mathbb{N}$. Suppose $\operatorname{dom} X_1 = \operatorname{dom} P$ and for all z, f such that $z \in \operatorname{dom} X_1$ and $f = P(z)$ holds $X_1(z) = \overline{\overline{\mathrm{Intersection}(F_3, f, x)}}$. Then the card intersection of F_3 wrt $k = \sum X_1$.

(50) If $\operatorname{dom} F_3$ is finite and $k = 0$, then the card intersection of F_3 wrt $k = \overline{\overline{\bigcup \operatorname{rng} F_3}}$.

(51) If $\operatorname{dom} F_3 = X$ and $k > \operatorname{card} X$, then the card intersection of F_3 wrt $k = 0$.

(52) Let given F_3, X. Suppose $\operatorname{dom} F_3 = X$. Let P be a function from $\operatorname{card} X$ into X. Suppose P is one-to-one. Then there exists a finite 0-sequence X_1 of $\mathbb{N}$ such that $\operatorname{dom} X_1 = \operatorname{card} X$ and for every z such that $z \in \operatorname{dom} X_1$ holds $X_1(z) = \operatorname{card}(F_3 \cdot P)(z)$ and the card intersection of F_3 wrt $1 = \sum X_1$.

(53) If $\operatorname{dom} F_3 = X$, then the card intersection of F_3 wrt $\operatorname{card} X = \overline{\overline{\mathrm{Intersection}(F_3, X \longmapsto x, x)}}$.

(54) If $F_3 = \{x\} \longmapsto X$, then the card intersection of F_3 wrt $1 = \operatorname{card} X$.

(55) Suppose $x \neq y$ and $F_3 = [x \longmapsto X, y \longmapsto Y]$. Then the card intersection of F_3 wrt $1 = \operatorname{card} X + \operatorname{card} Y$ and the card intersection of F_3 wrt $2 = \operatorname{card}(X \cap Y)$.

(56) Let given F_3, x. Suppose $\operatorname{dom} F_3$ is finite and $x \in \operatorname{dom} F_3$. Then the card intersection of F_3 wrt $1 =$ (the card intersection of $F_3 \restriction (\operatorname{dom} F_3 \setminus \{x\})$ wrt 1)$+ \operatorname{card} F_3(x)$.

(57) $\operatorname{dom} \mathrm{Intersect}(F, \operatorname{dom} F \longmapsto X') = \operatorname{dom} F$ and for every x such that $x \in \operatorname{dom} F$ holds $(\mathrm{Intersect}(F, \operatorname{dom} F \longmapsto X'))(x) = F(x) \cap X'$.

(58) $\bigcup \operatorname{rng} F \cap X' = \bigcup \operatorname{rng} \mathrm{Intersect}(F, \operatorname{dom} F \longmapsto X')$.

(59) $\mathrm{Intersection}(F, C_1, y) \cap X' = \mathrm{Intersection}(\mathrm{Intersect}(F, \operatorname{dom} F \longmapsto X'), C_1, y)$.

(60) Let F, G be finite 0-sequences. Suppose F is one-to-one and G is one-to-one and $\operatorname{rng} F$ misses $\operatorname{rng} G$. Then $F \frown G$ is one-to-one.

(61) Let given F_3, X, x, n. Suppose $\operatorname{dom} F_3 = X$ and $x \in \operatorname{dom} F_3$ and $k > 0$. Then the card intersection of F_3 wrt $k + 1 =$ (the card intersection of $F_3 \restriction (\operatorname{dom} F_3 \setminus \{x\})$ wrt $k + 1$) $+$ (the card intersection of

$\operatorname{Intersect}(F_3{\restriction}(\operatorname{dom} F_3 \setminus \{x\}), \operatorname{dom} F_3 \setminus \{x\} \longmapsto F_3(x))$ wrt k).

(62) Let F, G, b_1 be finite 0-sequences of D. Suppose that

(i) b is commutative and associative,

(ii) b has a unity or $\operatorname{len} F \geq 1$,

(iii) $\operatorname{len} F = \operatorname{len} G$,

(iv) $\operatorname{len} F = \operatorname{len} b_1$, and

(v) for every n such that $n \in \operatorname{dom} b_1$ holds $b_1(n) = b(F(n), G(n))$.

Then $b \odot F \frown G = b \odot b_1$.

Let F_4 be a finite 0-sequence of $\mathbb{Z}$. The functor $\sum F_4$ yielding an integer is defined as follows:

(Def. 5) $\sum F_4 = +_{\mathbb{Z}} \odot F_4$.

Let F_4 be a finite 0-sequence of $\mathbb{Z}$ and let us consider x. Then $F_4(x)$ is an integer.

Next we state several propositions:

(63) For every finite 0-sequence F_5 of $\mathbb{N}$ and for every finite 0-sequence F_4 of $\mathbb{Z}$ such that $F_4 = F_5$ holds $\sum F_4 = \sum F_5$.

(64) Let F, F_4 be finite 0-sequences of $\mathbb{Z}$ and i be an integer. If $\operatorname{dom} F = \operatorname{dom} F_4$ and for every n such that $n \in \operatorname{dom} F$ holds $i \cdot F(n) = F_4(n)$, then $i \cdot \sum F = \sum F_4$.

(65) If $x \in \operatorname{dom} F$, then $\bigcup \operatorname{rng} F = \bigcup \operatorname{rng}(F{\restriction}(\operatorname{dom} F \setminus \{x\})) \cup F(x)$.

(66) Let F_3 be a finite-yielding function and given X. Then there exists a finite 0-sequence X_1 of $\mathbb{Z}$ such that $\operatorname{dom} X_1 = \operatorname{card} X$ and for every n such that $n \in \operatorname{dom} X_1$ holds $X_1(n) = (-1)^n \cdot$ the card intersection of F_3 wrt $n+1$.

(67) Let F_3 be a finite-yielding function and given X. Suppose $\operatorname{dom} F_3 = X$. Let X_1 be a finite 0-sequence of $\mathbb{Z}$. Suppose $\operatorname{dom} X_1 = \operatorname{card} X$ and for every n such that $n \in \operatorname{dom} X_1$ holds $X_1(n) = (-1)^n \cdot$ the card intersection of F_3 wrt $n+1$. Then $\overline{\overline{\bigcup \operatorname{rng} F_3}} = \sum X_1$.

(68) Let given F_3, X, n, k. Suppose $\operatorname{dom} F_3 = X$. Given x, y such that $x \neq y$ and for every f such that $f \in \operatorname{Choose}(X, k, x, y)$ holds $\overline{\overline{\operatorname{Intersection}(F_3, f, x)}} = n$. Then the card intersection of F_3 wrt $k = n \cdot \binom{\operatorname{card} X}{k}$.

(69) Let given F_3, X. Suppose $\operatorname{dom} F_3 = X$. Let X_2 be a finite 0-sequence of $\mathbb{N}$. Suppose $\operatorname{dom} X_2 = \operatorname{card} X$ and for every n such that $n \in \operatorname{dom} X_2$ there exist x, y such that $x \neq y$ and for every f such that $f \in \operatorname{Choose}(X, n+1, x, y)$ holds $\overline{\overline{\operatorname{Intersection}(F_3, f, x)}} = X_2(n)$. Then there exists a finite 0-sequence F of $\mathbb{Z}$ such that $\operatorname{dom} F = \operatorname{card} X$ and $\overline{\overline{\bigcup \operatorname{rng} F_3}} = \sum F$ and for every n such that $n \in \operatorname{dom} F$ holds $F(n) = (-1)^n \cdot X_2(n) \cdot \binom{\operatorname{card} X}{n+1}$.

In the sequel g denotes a function from X into Y.

The following propositions are true:

(70) Let X, Y be finite sets. Suppose X is non empty and Y is non empty. Then there exists a finite 0-sequence F of $\mathbb{Z}$ such that $\operatorname{dom} F = \operatorname{card} Y + 1$ and $\sum F = \overline{\overline{\{g : g \text{ is onto}\}}}$ and for every n such that $n \in \operatorname{dom} F$ holds $F(n) = (-1)^n \cdot \binom{\operatorname{card} Y}{n} \cdot (\operatorname{card} Y - n)^{\operatorname{card} X}$.

(71) Let given n, k. Suppose $k \leq n$. Then there exists a finite 0-sequence F of $\mathbb{Z}$ such that $n \operatorname{block} k = \frac{1}{k!} \cdot \sum F$ and $\operatorname{dom} F = k + 1$ and for every m such that $m \in \operatorname{dom} F$ holds $F(m) = (-1)^m \cdot \binom{k}{m} \cdot (k - m)^n$.

In the sequel A, B are finite sets and f is a function from A into B.

One can prove the following proposition

(72) Let given A, B and X be a finite set. Suppose if B is empty, then A is empty and $X \subseteq A$. Let F be a function from A into B. Suppose F is one-to-one and $\operatorname{card} A = \operatorname{card} B$. Then $(\operatorname{card} A -' \operatorname{card} X)! = \overline{\overline{\{f : f \text{ is one-to-one } \wedge \operatorname{rng}(f \restriction (A \setminus X)) \subseteq F^{\circ}(A \setminus X) \wedge \bigwedge_x (x \in X \Rightarrow f(x) = F(x))\}}}$.

In the sequel F denotes a function and h denotes a function from X into $\operatorname{rng} F$.

The following proposition is true

(73) Let given F. Suppose $\operatorname{dom} F = X$ and F is one-to-one. Then there exists a finite 0-sequence X_2 of $\mathbb{Z}$ such that

(i) $\sum X_2 = \overline{\overline{\{h : h \text{ is one-to-one } \wedge \bigwedge_x (x \in X \Rightarrow h(x) \neq F(x))\}}}$,

(ii) $\operatorname{dom} X_2 = \operatorname{card} X + 1$, and

(iii) for every n such that $n \in \operatorname{dom} X_2$ holds $X_2(n) = \frac{(-1)^n \cdot (\operatorname{card} X)!}{n!}$.

In the sequel h is a function from X into X.

The following proposition is true

(74) There exists a finite 0-sequence X_2 of $\mathbb{Z}$ such that

(i) $\sum X_2 = \overline{\overline{\{h : h \text{ is one-to-one } \wedge \bigwedge_x (x \in X \Rightarrow h(x) \neq x)\}}}$,

(ii) $\operatorname{dom} X_2 = \operatorname{card} X + 1$, and

(iii) for every n such that $n \in \operatorname{dom} X_2$ holds $X_2(n) = \frac{(-1)^n \cdot (\operatorname{card} X)!}{n!}$.

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Some Equations Related to the Limit of Sequence of Subsets

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Summary. Set operations for sequences of subsets are introduced here. Some relations for these operations with the limit of sequences of subsets, also with the inferior sequence and the superior sequence of sets, and with the inferior limit and the superior limit of sets are shown.

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The articles [5], [2], [6], [1], [3], [4], and [7] provide the notation and terminology for this paper.

For simplicity, we use the following convention: n, k denote natural numbers, X denotes a set, A denotes a subset of X, and A_1, A_2 denote sequences of subsets of X.

We now state two propositions:

(1) (The inferior setsequence $A_1)(n) = \mathrm{Intersection}(A_1 \uparrow n)$.

(2) (The superior setsequence $A_1)(n) = \bigcup(A_1 \uparrow n)$.

Let us consider X and let A_1, A_2 be sequences of subsets of X. The functor $A_1 \cap A_2$ yields a sequence of subsets of X and is defined as follows:

(Def. 1) For every n holds $(A_1 \cap A_2)(n) = A_1(n) \cap A_2(n)$.

Let us note that the functor $A_1 \cap A_2$ is commutative. The functor $A_1 \cup A_2$ yielding a sequence of subsets of X is defined as follows:

(Def. 2) For every n holds $(A_1 \cup A_2)(n) = A_1(n) \cup A_2(n)$.

Let us observe that the functor $A_1 \cup A_2$ is commutative. The functor $A_1 \setminus A_2$ yielding a sequence of subsets of X is defined by:

(Def. 3) For every n holds $(A_1 \setminus A_2)(n) = A_1(n) \setminus A_2(n)$.

The functor $A_1 \dot{-} A_2$ yields a sequence of subsets of X and is defined as follows:

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(Def. 4) For every n holds $(A_1 \dot{-} A_2)(n) = A_1(n) \dot{-} A_2(n)$.

Let us note that the functor $A_1 \dot{-} A_2$ is commutative.

One can prove the following propositions:

(3) $A_1 \dot{-} A_2 = (A_1 \setminus A_2) \cup (A_2 \setminus A_1)$.

(4) $(A_1 \cap A_2) \uparrow k = A_1 \uparrow k \cap A_2 \uparrow k$.

(5) $(A_1 \cup A_2) \uparrow k = A_1 \uparrow k \cup A_2 \uparrow k$.

(6) $(A_1 \setminus A_2) \uparrow k = A_1 \uparrow k \setminus A_2 \uparrow k$.

(7) $(A_1 \dot{-} A_2) \uparrow k = A_1 \uparrow k \dot{-} A_2 \uparrow k$.

(8) $\bigcup(A_1 \cap A_2) \subseteq \bigcup A_1 \cap \bigcup A_2$.

(9) $\bigcup(A_1 \cup A_2) = \bigcup A_1 \cup \bigcup A_2$.

(10) $\bigcup A_1 \setminus \bigcup A_2 \subseteq \bigcup(A_1 \setminus A_2)$.

(11) $\bigcup A_1 \dot{-} \bigcup A_2 \subseteq \bigcup(A_1 \dot{-} A_2)$.

(12) $\mathrm{Intersection}(A_1 \cap A_2) = \mathrm{Intersection}\, A_1 \cap \mathrm{Intersection}\, A_2$.

(13) $\mathrm{Intersection}\, A_1 \cup \mathrm{Intersection}\, A_2 \subseteq \mathrm{Intersection}(A_1 \cup A_2)$.

(14) $\mathrm{Intersection}(A_1 \setminus A_2) \subseteq \mathrm{Intersection}\, A_1 \setminus \mathrm{Intersection}\, A_2$.

Let us consider X, let A_1 be a sequence of subsets of X, and let A be a subset of X. The functor $A \cap A_1$ yielding a sequence of subsets of X is defined by:

(Def. 5) For every n holds $(A \cap A_1)(n) = A \cap A_1(n)$.

The functor $A \cup A_1$ yielding a sequence of subsets of X is defined as follows:

(Def. 6) For every n holds $(A \cup A_1)(n) = A \cup A_1(n)$.

The functor $A \setminus A_1$ yields a sequence of subsets of X and is defined by:

(Def. 7) For every n holds $(A \setminus A_1)(n) = A \setminus A_1(n)$.

The functor $A_1 \setminus A$ yields a sequence of subsets of X and is defined by:

(Def. 8) For every n holds $(A_1 \setminus A)(n) = A_1(n) \setminus A$.

The functor $A \dot{-} A_1$ yielding a sequence of subsets of X is defined as follows:

(Def. 9) For every n holds $(A \dot{-} A_1)(n) = A \dot{-} A_1(n)$.

One can prove the following propositions:

(15) $A \dot{-} A_1 = (A \setminus A_1) \cup (A_1 \setminus A)$.

(16) $(A \cap A_1) \uparrow k = A \cap A_1 \uparrow k$.

(17) $(A \cup A_1) \uparrow k = A \cup A_1 \uparrow k$.

(18) $(A \setminus A_1) \uparrow k = A \setminus A_1 \uparrow k$.

(19) $(A_1 \setminus A) \uparrow k = A_1 \uparrow k \setminus A$.

(20) $(A \dot{-} A_1) \uparrow k = A \dot{-} A_1 \uparrow k$.

(21) If A_1 is non-increasing, then $A \cap A_1$ is non-increasing.

(22) If A_1 is non-decreasing, then $A \cap A_1$ is non-decreasing.

(23) If A_1 is monotone, then $A \cap A_1$ is monotone.

(24) If A_1 is non-increasing, then $A \cup A_1$ is non-increasing.

(25) If A_1 is non-decreasing, then $A \cup A_1$ is non-decreasing.

(26) If A_1 is monotone, then $A \cup A_1$ is monotone.

(27) If A_1 is non-increasing, then $A \setminus A_1$ is non-decreasing.

(28) If A_1 is non-decreasing, then $A \setminus A_1$ is non-increasing.

(29) If A_1 is monotone, then $A \setminus A_1$ is monotone.

(30) If A_1 is non-increasing, then $A_1 \setminus A$ is non-increasing.

(31) If A_1 is non-decreasing, then $A_1 \setminus A$ is non-decreasing.

(32) If A_1 is monotone, then $A_1 \setminus A$ is monotone.

(33) $\mathrm{Intersection}(A \cap A_1) = A \cap \mathrm{Intersection}\, A_1$.

(34) $\mathrm{Intersection}(A \cup A_1) = A \cup \mathrm{Intersection}\, A_1$.

(35) $\mathrm{Intersection}(A \setminus A_1) \subseteq A \setminus \mathrm{Intersection}\, A_1$.

(36) $\mathrm{Intersection}(A_1 \setminus A) = \mathrm{Intersection}\, A_1 \setminus A$.

(37) $\mathrm{Intersection}(A \dot{-} A_1) \subseteq A \dot{-} \mathrm{Intersection}\, A_1$.

(38) $\bigcup(A \cap A_1) = A \cap \bigcup A_1$.

(39) $\bigcup(A \cup A_1) = A \cup \bigcup A_1$.

(40) $A \setminus \bigcup A_1 \subseteq \bigcup(A \setminus A_1)$.

(41) $\bigcup(A_1 \setminus A) = \bigcup A_1 \setminus A$.

(42) $A \dot{-} \bigcup A_1 \subseteq \bigcup(A \dot{-} A_1)$.

(43) (The inferior setsequence $A_1 \cap A_2)(n) =$ (the inferior setsequence $A_1)(n) \cap$ (the inferior setsequence $A_2)(n)$.

(44) (The inferior setsequence $A_1)(n) \cup$(the inferior setsequence $A_2)(n) \subseteq$ (the inferior setsequence $A_1 \cup A_2)(n)$.

(45) (The inferior setsequence $A_1 \setminus A_2)(n) \subseteq$ (the inferior setsequence $A_1)(n) \setminus$ (the inferior setsequence $A_2)(n)$.

(46) (The superior setsequence $A_1 \cap A_2)(n) \subseteq$ (the superior setsequence $A_1)(n) \cap$ (the superior setsequence $A_2)(n)$.

(47) (The superior setsequence $A_1 \cup A_2)(n) =$ (the superior setsequence $A_1)(n) \cup$ (the superior setsequence $A_2)(n)$.

(48) (The superior setsequence $A_1)(n) \setminus$ (the superior setsequence $A_2)(n) \subseteq$ (the superior setsequence $A_1 \setminus A_2)(n)$.

(49) (The superior setsequence $A_1)(n) \dot{-}$(the superior setsequence $A_2)(n) \subseteq$ (the superior setsequence $A_1 \dot{-} A_2)(n)$.

(50) (The inferior setsequence $A \cap A_1)(n) = A \cap$ (the inferior setsequence $A_1)(n)$.

(51) (The inferior setsequence $A \cup A_1)(n) = A \cup$ (the inferior setsequence $A_1)(n)$.

(52) (The inferior setsequence $A \setminus A_1)(n) \subseteq A \setminus$ (the inferior setsequence $A_1)(n)$.

(53) (The inferior setsequence $A_1 \setminus A)(n) =$ (the inferior setsequence $A_1)(n) \setminus A$.

(54) (The inferior setsequence $A \dot{-} A_1)(n) \subseteq A \dot{-}$(the inferior setsequence $A_1)(n)$.

(55) (The superior setsequence $A \cap A_1)(n) = A \cap$ (the superior setsequence $A_1)(n)$.

(56) (The superior setsequence $A \cup A_1)(n) = A \cup$ (the superior setsequence $A_1)(n)$.

(57) $A \setminus$ (the superior setsequence $A_1)(n) \subseteq$ (the superior setsequence $A \setminus A_1)(n)$.

(58) (The superior setsequence $A_1 \setminus A)(n) =$ (the superior setsequence $A_1)(n) \setminus A$.

(59) $A \dot{-}$(the superior setsequence $A_1)(n) \subseteq$ (the superior setsequence $A \dot{-} A_1)(n)$.

(60) $\liminf(A_1 \cap A_2) = \liminf A_1 \cap \liminf A_2$.

(61) $\liminf A_1 \cup \liminf A_2 \subseteq \liminf(A_1 \cup A_2)$.

(62) $\liminf(A_1 \setminus A_2) \subseteq \liminf A_1 \setminus \liminf A_2$.

(63) If A_1 is convergent or A_2 is convergent, then $\liminf(A_1 \cup A_2) = \liminf A_1 \cup \liminf A_2$.

(64) If A_2 is convergent, then $\liminf(A_1 \setminus A_2) = \liminf A_1 \setminus \liminf A_2$.

(65) If A_1 is convergent or A_2 is convergent, then $\liminf(A_1 \dot{-} A_2) \subseteq \liminf A_1 \dot{-} \liminf A_2$.

(66) If A_1 is convergent and A_2 is convergent, then $\liminf(A_1 \dot{-} A_2) = \liminf A_1 \dot{-} \liminf A_2$.

(67) $\limsup(A_1 \cap A_2) \subseteq \limsup A_1 \cap \limsup A_2$.

(68) $\limsup(A_1 \cup A_2) = \limsup A_1 \cup \limsup A_2$.

(69) $\limsup A_1 \setminus \limsup A_2 \subseteq \limsup(A_1 \setminus A_2)$.

(70) $\limsup A_1 \dot{-} \limsup A_2 \subseteq \limsup(A_1 \dot{-} A_2)$.

(71) If A_1 is convergent or A_2 is convergent, then $\limsup(A_1 \cap A_2) = \limsup A_1 \cap \limsup A_2$.

(72) If A_2 is convergent, then $\limsup(A_1 \setminus A_2) = \limsup A_1 \setminus \limsup A_2$.

(73) If A_1 is convergent and A_2 is convergent, then $\limsup(A_1 \dot{-} A_2) = \limsup A_1 \dot{-} \limsup A_2$.

(74) $\liminf(A \cap A_1) = A \cap \liminf A_1$.

(75) $\liminf(A \cup A_1) = A \cup \liminf A_1$.

(76) $\liminf(A \setminus A_1) \subseteq A \setminus \liminf A_1$.

(77) $\liminf(A_1 \setminus A) = \liminf A_1 \setminus A$.

(78) $\liminf(A \dot{-} A_1) \subseteq A \dot{-} \liminf A_1$.

(79) If A_1 is convergent, then $\liminf(A \setminus A_1) = A \setminus \liminf A_1$.

(80) If A_1 is convergent, then $\liminf(A \dot{-} A_1) = A \dot{-} \liminf A_1$.

(81) $\limsup(A \cap A_1) = A \cap \limsup A_1$.

(82) $\limsup(A \cup A_1) = A \cup \limsup A_1$.

(83) $A \setminus \limsup A_1 \subseteq \limsup(A \setminus A_1)$.

(84) $\limsup(A_1 \setminus A) = \limsup A_1 \setminus A$.

(85) $A \dot{-} \limsup A_1 \subseteq \limsup(A \dot{-} A_1)$.

(86) If A_1 is convergent, then $\limsup(A \setminus A_1) = A \setminus \limsup A_1$.

(87) If A_1 is convergent, then $\limsup(A \dot{-} A_1) = A \dot{-} \limsup A_1$.

(88) If A_1 is convergent and A_2 is convergent, then $A_1 \cap A_2$ is convergent and $\lim(A_1 \cap A_2) = \lim A_1 \cap \lim A_2$.

(89) If A_1 is convergent and A_2 is convergent, then $A_1 \cup A_2$ is convergent and $\lim(A_1 \cup A_2) = \lim A_1 \cup \lim A_2$.

(90) If A_1 is convergent and A_2 is convergent, then $A_1 \setminus A_2$ is convergent and $\lim(A_1 \setminus A_2) = \lim A_1 \setminus \lim A_2$.

(91) If A_1 is convergent and A_2 is convergent, then $A_1 \dot{-} A_2$ is convergent and $\lim(A_1 \dot{-} A_2) = \lim A_1 \dot{-} \lim A_2$.

(92) If A_1 is convergent, then $A \cap A_1$ is convergent and $\lim(A \cap A_1) = A \cap \lim A_1$.

(93) If A_1 is convergent, then $A \cup A_1$ is convergent and $\lim(A \cup A_1) = A \cup \lim A_1$.

(94) If A_1 is convergent, then $A \setminus A_1$ is convergent and $\lim(A \setminus A_1) = A \setminus \lim A_1$.

(95) If A_1 is convergent, then $A_1 \setminus A$ is convergent and $\lim(A_1 \setminus A) = \lim A_1 \setminus A$.

(96) If A_1 is convergent, then $A \dot{-} A_1$ is convergent and $\lim(A \dot{-} A_1) = A \dot{-} \lim A_1$.

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On the Partial Product of Series and Related Basic Inequalities

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Summary. This article describes definition of partial product of series, introduced similarly to its related partial sum, as well as several important inequalities true for chosen special series.

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The notation and terminology used in this paper are introduced in the following articles: [1], [9], [10], [5], [2], [4], [6], [7], [8], and [3].

For simplicity, we adopt the following convention: a, b, c are positive real numbers, m, x, y, z are real numbers, n is a natural number, and s, s_1, s_2, s_3, s_4, s_5 are sequences of real numbers.

Let us consider x. Note that $|x|$ is non negative.

We now state a number of propositions:

(1) If $y > x$ and $x \geq 0$ and $m \geq 0$, then $\frac{x}{y} \leq \frac{x+m}{y+m}$.

(2) $\frac{a+b}{2} \geq \sqrt{a \cdot b}$.

(3) $\frac{b}{a} + \frac{a}{b} \geq 2$.

(4) $(\frac{x+y}{2})^{\mathbf{2}} \geq x \cdot y$.

(5) $\frac{x^{\mathbf{2}}+y^{\mathbf{2}}}{2} \geq (\frac{x+y}{2})^{\mathbf{2}}$.

(6) $x^{\mathbf{2}} + y^{\mathbf{2}} \geq 2 \cdot x \cdot y$.

(7) $\frac{x^{\mathbf{2}}+y^{\mathbf{2}}}{2} \geq x \cdot y$.

(8) $x^{\mathbf{2}} + y^{\mathbf{2}} \geq 2 \cdot |x| \cdot |y|$.

(9) $(x + y)^{\mathbf{2}} \geq 4 \cdot x \cdot y$.

(10) $x^{\mathbf{2}} + y^{\mathbf{2}} + z^{\mathbf{2}} \geq x \cdot y + y \cdot z + x \cdot z$.

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(11) $(x + y + z)^{\mathbf{2}} \geq 3 \cdot (x \cdot y + y \cdot z + x \cdot z)$.

(12) $a^3 + b^3 + c^3 \geq 3 \cdot a \cdot b \cdot c$.

(13) $\frac{a^3+b^3+c^3}{3} \geq a \cdot b \cdot c$.

(14) $(\frac{a}{b})^3 + (\frac{b}{c})^3 + (\frac{c}{a})^3 \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$.

(15) $a + b + c \geq 3 \cdot \sqrt[3]{a \cdot b \cdot c}$.

(16) $\frac{a+b+c}{3} \geq \sqrt[3]{a \cdot b \cdot c}$.

(17) If $x + y + z = 1$, then $x \cdot y + y \cdot z + x \cdot z \leq \frac{1}{3}$.

(18) If $x + y = 1$, then $x \cdot y \leq \frac{1}{4}$.

(19) If $x + y = 1$, then $x^{\mathbf{2}} + y^{\mathbf{2}} \geq \frac{1}{2}$.

(20) If $a + b = 1$, then $(1 + \frac{1}{a}) \cdot (1 + \frac{1}{b}) \geq 9$.

(21) If $x + y = 1$, then $x^3 + y^3 \geq \frac{1}{4}$.

(22) If $a + b = 1$, then $a^3 + b^3 < 1$.

(23) If $a + b = 1$, then $(a + \frac{1}{a}) \cdot (b + \frac{1}{b}) \geq \frac{25}{4}$.

(24) If $|x| \leq a$, then $x^{\mathbf{2}} \leq a^{\mathbf{2}}$.

(25) If $|x| \geq a$, then $x^{\mathbf{2}} \geq a^{\mathbf{2}}$.

(26) $||x| - |y|| \leq |x| + |y|$.

(27) If $a \cdot b \cdot c = 1$, then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}$.

(28) If $x > 0$ and $y > 0$ and $z < 0$ and $x + y + z = 0$, then $(x^2 + y^2 + z^2)^3 \geq 6 \cdot (x^3 + y^3 + z^3)^{\mathbf{2}}$.

(29) If $a \geq 1$, then $a^b + a^c \geq 2 \cdot a^{\sqrt{b \cdot c}}$.

(30) If $a \geq b$ and $b \geq c$, then $a^a \cdot b^b \cdot c^c \geq (a \cdot b \cdot c)^{\frac{a+b+c}{3}}$.

(31) $(a + b)^{n+2} \geq a^{n+2} + (n + 2) \cdot a^{n+1} \cdot b$.

(32) $\frac{a^n+b^n}{2} \geq (\frac{a+b}{2})^n$.

(33) If for every n holds $s(n) > 0$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) > 0$.

(34) If for every n holds $s(n) \geq 0$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) \geq 0$.

(35) If for every n holds $s(n) < 0$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) < 0$.

(36) If $s = s_1\, s_1$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) \geq 0$.

(37) If for every n holds $s(n) > 0$ and $s(n) > s(n-1)$, then $(n+1) \cdot s(n+1) > (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(38) If $s = s_1\, s_2$ and for every n holds $s_1(n) \geq 0$ and $s_2(n) \geq 0$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) \leq (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) \cdot (\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(39) If $s = s_1\, s_2$ and for every n holds $s_1(n) < 0$ and $s_2(n) < 0$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) \leq (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) \cdot (\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(40) For every n holds $|(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n)| \leq (\sum_{\alpha=0}^{\kappa}|s|(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(41) $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) \leq (\sum_{\alpha=0}^{\kappa}|s|(\alpha))_{\kappa\in\mathbb{N}}(n)$.

Let us consider s. The partial product of s yielding a sequence of real numbers is defined by the conditions (Def. 1).

(Def. 1)(i) (The partial product of $s)(0) = s(0)$, and

(ii) for every n holds (the partial product of $s)(n+1) =$ (the partial product of $s)(n) \cdot s(n+1)$.

We now state a number of propositions:

(42) If for every n holds $s(n) > 0$, then (the partial product of $s)(n) > 0$.

(43) If for every n holds $s(n) \geq 0$, then (the partial product of $s)(n) \geq 0$.

(44) Suppose that for every n holds $s(n) > 0$ and $s(n) < 1$. Let given n. Then (the partial product of $s)(n) > 0$ and (the partial product of $s)(n) < 1$.

(45) If for every n holds $s(n) \geq 1$, then for every n holds (the partial product of $s)(n) \geq 1$.

(46) Suppose that for every n holds $s_1(n) \geq 0$ and $s_2(n) \geq 0$. Let given n. Then (the partial product of $s_1)(n)+$(the partial product of $s_2)(n) \leq$ (the partial product of $s_1 + s_2)(n)$.

(47) If for every n holds $s(n) = \frac{2\cdot n+1}{2\cdot n+2}$, then (the partial product of $s)(n) \leq \frac{1}{\sqrt{3\cdot n+4}}$.

(48) If for every n holds $s_1(n) = 1 + s(n)$ and $s(n) > -1$ and $s(n) < 0$, then for every n holds $1 + (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) \leq$ (the partial product of $s_1)(n)$.

(49) If for every n holds $s_1(n) = 1+s(n)$ and $s(n) \geq 0$, then for every n holds $1 + (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) \leq$ (the partial product of $s_1)(n)$.

(50) If $s_3 = s_1\, s_2$ and $s_4 = s_1\, s_1$ and $s_5 = s_2\, s_2$, then for every n holds $(\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}}(n)^{\mathbf{2}} \leq (\sum_{\alpha=0}^{\kappa}(s_4)(\alpha))_{\kappa\in\mathbb{N}}(n) \cdot (\sum_{\alpha=0}^{\kappa}(s_5)(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(51) If $s_4 = s_1\, s_1$ and $s_5 = s_2\, s_2$ and for every n holds $s_1(n) \geq 0$ and $s_2(n) \geq 0$ and $s_3(n) = (s_1(n) + s_2(n))^{\mathbf{2}}$, then for every n holds $\sqrt{(\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}}(n)} \leq \sqrt{(\sum_{\alpha=0}^{\kappa}(s_4)(\alpha))_{\kappa\in\mathbb{N}}(n)} + \sqrt{(\sum_{\alpha=0}^{\kappa}(s_5)(\alpha))_{\kappa\in\mathbb{N}}(n)}$.

(52) If for every n holds $s(n) > 0$ and $s(n) > s(n-1)$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) \geq (n+1) \cdot \sqrt[n+1]{(\text{the partial product of } s)(n)}$.

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Homeomorphism between Finite Topological Spaces, Two-Dimensional Lattice Spaces and a Fixed Point Theorem

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Summary. In this paper we first introduced the notion of homeomorphism between finite topological spaces. We also gave a fixed point theorem in finite topological space. Next, we showed two 2-dimensional concrete models of lattice spaces. One was 2-dimensional linear finite topological space. Another was 2-dimensional small finite topological space.

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The articles [10], [6], [12], [1], [13], [4], [5], [2], [7], [9], [8], [3], and [11] provide the notation and terminology for this paper.

The following propositions are true:

(1) Let X be a set, Y be a non empty set, f be a function from X into Y, and A be a subset of X. If f is one-to-one, then $(f^{-1})^{\circ} f^{\circ} A = A$.

(2) For every natural number n holds $n > 0$ iff $\operatorname{Seg} n \neq \emptyset$.

Let F_1, F_2 be finite topology spaces and let h be a map from F_1 into F_2. We say that h is a homeomorphism if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) h is one-to-one and onto, and

(ii) for every element x of F_1 holds h°(the neighbour-map of F_1)$(x) =$ (the neighbour-map of F_2)$(h(x))$.

One can prove the following propositions:

(3) Let F_1, F_2 be non empty finite topology spaces and h be a map from F_1 into F_2. Suppose h is a homeomorphism. Then there exists a map g from F_2 into F_1 such that $g = h^{-1}$ and g is a homeomorphism.

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(4) Let F_1, F_2 be non empty finite topology spaces, h be a map from F_1 into F_2, n be a natural number, x be an element of F_1, and y be an element of F_2. Suppose h is a homeomorphism and $y = h(x)$. Let z be an element of F_1. Then $z \in U(x, n)$ if and only if $h(z) \in U(y, n)$.

(5) Let F_1, F_2 be non empty finite topology spaces, h be a map from F_1 into F_2, n be a natural number, x be an element of F_1, and y be an element of F_2. Suppose h is a homeomorphism and $y = h(x)$. Let v be an element of F_2. Then $h^{-1}(v) \in U(x, n)$ if and only if $v \in U(y, n)$.

(6) Let n be a non zero natural number and f be a map from FTSL1(n) into FTSL1(n). If f is continuous 0, then there exists an element p of FTSL1(n) such that $f(p) \in U(p, 0)$.

(7) Let T be a non empty finite topology space, p be an element of T, and k be a natural number. If T is filled, then $U(p, k) \subseteq U(p, k + 1)$.

(8) Let T be a non empty finite topology space, p be an element of T, and k be a natural number. If T is filled, then $U(p, 0) \subseteq U(p, k)$.

(9) Let n be a non zero natural number, j_1, j, k be natural numbers, and p be an element of FTSL1(n). If $p = j_1$, then $j \in U(p, k)$ iff $j \in \mathrm{Seg}\, n$ and $|j_1 - j| \leq k + 1$.

(10) Let k_1, k_2 be natural numbers, n be a non zero natural number, and f be a map from FTSL1(n) into FTSL1(n). Suppose f is continuous k_1 and $k_2 = \lceil \frac{k_1}{2} \rceil$. Then there exists an element p of FTSL1(n) such that $f(p) \in U(p, k_2)$.

Let n, m be natural numbers. The functor Nbdl2(n, m) yields a function from $[\!:\mathrm{Seg}\, n, \mathrm{Seg}\, m:\!]$ into $2^{[:\mathrm{Seg}\, n, \mathrm{Seg}\, m:]}$ and is defined by:

(Def. 2) For every set x such that $x \in [\!:\mathrm{Seg}\, n, \mathrm{Seg}\, m:\!]$ and for all natural numbers i, j such that $x = \langle i, j \rangle$ holds $(\mathrm{Nbdl2}(n, m))(x) = [\!:(\mathrm{Nbdl1}(n))(i), (\mathrm{Nbdl1}(m))(j):\!]$.

Let n, m be natural numbers. The functor FTSL2(n, m) yielding a strict finite topology space is defined as follows:

(Def. 3) $\mathrm{FTSL2}(n, m) = \langle [\!:\mathrm{Seg}\, n, \mathrm{Seg}\, m:\!], \mathrm{Nbdl2}(n, m) \rangle$.

Let n, m be non zero natural numbers. One can verify that FTSL2(n, m) is non empty.

We now state three propositions:

(11) For all non zero natural numbers n, m holds FTSL2(n, m) is filled.

(12) For all non zero natural numbers n, m holds FTSL2(n, m) is symmetric.

(13) For every non zero natural number n holds there exists a map from FTSL2($n, 1$) into FTSL1(n) which is a homeomorphism.

Let n, m be natural numbers. The functor Nbds2(n, m) yielding a function from $[\!:\mathrm{Seg}\, n, \mathrm{Seg}\, m:\!]$ into $2^{[:\mathrm{Seg}\, n, \mathrm{Seg}\, m:]}$ is defined by:

(Def. 4) For every set x such that $x \in [:\mathrm{Seg}\, n, \mathrm{Seg}\, m:]$ and for all natural numbers i, j such that $x = \langle i, j \rangle$ holds $(\mathrm{Nbds2}(n, m))(x) = [:\{i\}, (\mathrm{Nbdl1}(m))(j):] \cup [:(\mathrm{Nbdl1}(n))(i), \{j\}:]$.

Let n, m be natural numbers. The functor $\mathrm{FTSS2}(n, m)$ yielding a strict finite topology space is defined as follows:

(Def. 5) $\mathrm{FTSS2}(n, m) = \langle [:\mathrm{Seg}\, n, \mathrm{Seg}\, m:], \mathrm{Nbds2}(n, m) \rangle$.

Let n, m be non zero natural numbers. Note that $\mathrm{FTSS2}(n, m)$ is non empty.

One can prove the following propositions:

(14) For all non zero natural numbers n, m holds $\mathrm{FTSS2}(n, m)$ is filled.

(15) For all non zero natural numbers n, m holds $\mathrm{FTSS2}(n, m)$ is symmetric.

(16) For every non zero natural number n holds there exists a map from $\mathrm{FTSS2}(n, 1)$ into $\mathrm{FTSL1}(n)$ which is a homeomorphism.

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The Maclaurin Expansions

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Summary. A concept of the Maclaurin expansions is defined here. This article contains the definition of the Maclaurin expansion and expansions of exp, sin and cos functions.

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The papers [15], [16], [4], [12], [2], [14], [5], [1], [3], [7], [6], [10], [11], [8], [9], [17], and [13] provide the notation and terminology for this paper.

The following proposition is true

(1) For every real number x and for every natural number n holds $|x^n| = |x|^n$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, let Z be a subset of $\mathbb{R}$, and let a be a real number. The functor Maclaurin(f, Z, a) yields a sequence of real numbers and is defined by:

(Def. 1) Maclaurin$(f, Z, a) = \text{Taylor}(f, Z, 0, a)$.

The following propositions are true:

(2) Let n be a natural number, f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and r be a real number. Suppose $0 < r$ and f is differentiable $n+1$ times on $]-r, r[$. Let x be a real number. Suppose $x \in]-r, r[$. Then there exists a real number s such that $0 < s$ and $s < 1$ and $f(x) = (\sum_{\alpha=0}^{\kappa}(\text{Maclaurin}(f,]-r, r[, x))(\alpha))_{\kappa \in \mathbb{N}}(n) + \frac{f'(]-r,r[)(n+1)(s \cdot x) \cdot x^{n+1}}{(n+1)!}$.

(3) Let n be a natural number, f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and x_0, r be real numbers. Suppose $0 < r$ and f is differentiable $n+1$ times on $]x_0 - r, x_0 + r[$. Let x be a real number. Suppose $x \in]x_0 - r, x_0 + r[$. Then there exists a real number s such that $0 < s$ and $s < 1$ and $|f(x) - (\sum_{\alpha=0}^{\kappa}(\text{Taylor}(f,]x_0 - r, x_0 + r[, x_0, x))(\alpha))_{\kappa \in \mathbb{N}}(n)| = |\frac{f'(]x_0-r,x_0+r[)(n+1)(x_0+s\cdot(x-x_0))\cdot(x-x_0)^{n+1}}{(n+1)!}|$.

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(4) Let n be a natural number, f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and r be a real number. Suppose $0 < r$ and f is differentiable $n+1$ times on $]-r, r[$. Let x be a real number. Suppose $x \in]-r, r[$. Then there exists a real number s such that $0 < s$ and $s < 1$ and $|f(x) - (\sum_{\alpha=0}^{\kappa}(\mathrm{Maclaurin}(f,]-r, r[, x))(\alpha))_{\kappa \in \mathbb{N}}(n)| = |\frac{f'(]-r,r[)(n+1)(s \cdot x) \cdot x^{n+1}}{(n+1)!}|$.

(5) For every real number r holds $\exp'_{\restriction]-r,r[} = \exp \restriction]-r, r[$ and $\mathrm{dom}(\exp \restriction]-r, r[) =]-r, r[$.

(6) For every natural number n and for every real number r holds $\exp'(]-r, r[)(n) = \exp \restriction]-r, r[$.

(7) For every natural number n and for all real numbers r, x such that $x \in]-r, r[$ holds $\exp'(]-r, r[)(n)(x) = \exp(x)$.

(8) For every natural number n and for all real numbers r, x such that $0 < r$ holds $(\mathrm{Maclaurin}(\exp,]-r, r[, x))(n) = \frac{x^n}{n!}$.

(9) Let n be a natural number and r, x, s be real numbers. Suppose $x \in]-r, r[$ and $0 < s$ and $s < 1$. Then $|\frac{\exp'(]-r,r[)(n+1)(s \cdot x) \cdot x^{n+1}}{(n+1)!}| \leq \frac{|\exp(s \cdot x)| \cdot |x|^{n+1}}{(n+1)!}$.

(10) For every real number r and for every natural number n holds exp is differentiable n times on $]-r, r[$.

(11) Let r be a real number. Suppose $0 < r$. Then there exist real numbers M, L such that
  (i) $0 \leq M$,
  (ii) $0 \leq L$, and
  (iii) for every natural number n and for all real numbers x, s such that $x \in]-r, r[$ and $0 < s$ and $s < 1$ holds $|\frac{\exp'(]-r,r[)(n)(s \cdot x) \cdot x^n}{n!}| \leq \frac{M \cdot L^n}{n!}$.

(12) Let M, L be real numbers. Suppose $M \geq 0$ and $L \geq 0$. Let e be a real number. Suppose $e > 0$. Then there exists a natural number n such that for every natural number m if $n \leq m$, then $\frac{M \cdot L^m}{m!} < e$.

(13) Let r, e be real numbers. Suppose $0 < r$ and $0 < e$. Then there exists a natural number n such that for every natural number m if $n \leq m$, then for all real numbers x, s such that $x \in]-r, r[$ and $0 < s$ and $s < 1$ holds $|\frac{\exp'(]-r,r[)(m)(s \cdot x) \cdot x^m}{m!}| < e$.

(14) Let r, e be real numbers. Suppose $0 < r$ and $0 < e$. Then there exists a natural number n such that for every natural number m if $n \leq m$, then for every real number x such that $x \in]-r, r[$ holds $|\exp(x) - (\sum_{\alpha=0}^{\kappa}(\mathrm{Maclaurin}(\exp,]-r, r[, x))(\alpha))_{\kappa \in \mathbb{N}}(m)| < e$.

(15) For every real number x holds $x\,\mathrm{ExpSeq}$ is absolutely summable.

(16) For all real numbers r, x such that $0 < r$ holds $\mathrm{Maclaurin}(\exp,]-r, r[, x) = x\,\mathrm{ExpSeq}$ and $\mathrm{Maclaurin}(\exp,]-r, r[, x)$ is absolutely summable and $\exp(x) = \sum \mathrm{Maclaurin}(\exp,]-r, r[, x)$.

(17) Let r be a real number. Then

(i) (the function sin)$'_{\restriction]-r,r[}$ = (the function cos)$\restriction]-r,r[$,

(ii) (the function cos)$'_{\restriction]-r,r[}$ = (−the function sin)$\restriction]-r,r[$,

(iii) dom((the function sin)$\restriction]-r,r[$) $=]-r,r[$, and

(iv) dom((the function cos)$\restriction]-r,r[$) $=]-r,r[$.

(18) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and Z be a subset of $\mathbb{R}$. If f is differentiable on Z, then $(-f)'_{\restriction Z} = -f'_{\restriction Z}$.

(19) Let r be a real number and n be a natural number. Then

(i) (the function sin)$'(]-r,r[)(2 \cdot n) = (-1)^n$ ((the function sin)$\restriction]-r,r[$),

(ii) (the function sin)$'(]-r,r[)(2 \cdot n+1) = (-1)^n$ ((the function cos)$\restriction]-r,r[$),

(iii) (the function cos)$'(]-r,r[)(2 \cdot n) = (-1)^n$ ((the function cos)$\restriction]-r,r[$), and

(iv) (the function cos)$'(]-r,r[)(2 \cdot n + 1) = (-1)^{n+1}$ ((the function sin)$\restriction]-r,r[$).

(20) Let n be a natural number and r, x be real numbers. Suppose $r > 0$. Then

(i) (Maclaurin(the function sin, $]-r,r[$, x))$(2 \cdot n) = 0$,

(ii) (Maclaurin(the function sin, $]-r,r[$, x))$(2 \cdot n + 1) = \frac{(-1)^n \cdot x^{2 \cdot n+1}}{(2 \cdot n+1)!}$,

(iii) (Maclaurin(the function cos, $]-r,r[$, x))$(2 \cdot n) = \frac{(-1)^n \cdot x^{2 \cdot n}}{(2 \cdot n)!}$, and

(iv) (Maclaurin(the function cos, $]-r,r[$, x))$(2 \cdot n + 1) = 0$.

(21) Let r be a real number and n be a natural number. Then the function sin is differentiable n times on $]-r,r[$ and the function cos is differentiable n times on $]-r,r[$.

(22) Let r be a real number. Suppose $r > 0$. Then there exist real numbers r_1, r_2 such that

(i) $r_1 \geq 0$,

(ii) $r_2 \geq 0$, and

(iii) for every natural number n and for all real numbers x, s such that $x \in]-r,r[$ and $0 < s$ and $s < 1$ holds $|\frac{(\text{the function sin})'(]-r,r[)(n)(s \cdot x) \cdot x^n}{n!}| \leq \frac{r_1 \cdot r_2{}^n}{n!}$ and $|\frac{(\text{the function cos})'(]-r,r[)(n)(s \cdot x) \cdot x^n}{n!}| \leq \frac{r_1 \cdot r_2{}^n}{n!}$.

(23) Let r, e be real numbers. Suppose $0 < r$ and $0 < e$. Then there exists a natural number n such that for every natural number m if $n \leq m$, then for all real numbers x, s such that $x \in]-r,r[$ and $0 < s$ and $s < 1$ holds $|\frac{(\text{the function sin})'(]-r,r[)(m)(s \cdot x) \cdot x^m}{m!}| < e$ and $|\frac{(\text{the function cos})'(]-r,r[)(m)(s \cdot x) \cdot x^m}{m!}| < e$.

(24) Let r, e be real numbers. Suppose $0 < r$ and $0 < e$. Then there exists a natural number n such that for every natural number m if $n \leq m$, then for every real number x such that $x \in]-r,r[$ holds |(the function sin)(x) − ($\sum_{\alpha=0}^{\kappa}$(Maclaurin(the func-

tion $\sin,]-r, r[, x))(\alpha))_{\kappa \in \mathbb{N}}(m)| < e$ and $|(\text{the function } \cos)(x) - (\sum_{\alpha=0}^{\kappa}(\text{Maclaurin}(\text{the function } \cos,]-r, r[, x))(\alpha))_{\kappa \in \mathbb{N}}(m)| < e$.

(25) Let r, x be real numbers and m be a natural number. Suppose $0 < r$. Then $(\sum_{\alpha=0}^{\kappa}(\text{Maclaurin}(\text{the function } \sin,]-r, r[, x))(\alpha))_{\kappa \in \mathbb{N}}(2 \cdot m + 1) = (\sum_{\alpha=0}^{\kappa} x \,\text{P\_sin}(\alpha))_{\kappa \in \mathbb{N}}(m)$ and $(\sum_{\alpha=0}^{\kappa}(\text{Maclaurin}(\text{the function } \cos,]-r, r[, x))(\alpha))_{\kappa \in \mathbb{N}}(2 \cdot m + 1) = (\sum_{\alpha=0}^{\kappa} x \,\text{P\_cos}(\alpha))_{\kappa \in \mathbb{N}}(m)$.

(26) Let r, x be real numbers and m be a natural number. Suppose $0 < r$ and $m > 0$. Then $(\sum_{\alpha=0}^{\kappa}(\text{Maclaurin}(\text{the function } \sin,]-r, r[, x))(\alpha))_{\kappa \in \mathbb{N}}(2 \cdot m) = (\sum_{\alpha=0}^{\kappa} x \,\text{P\_sin}(\alpha))_{\kappa \in \mathbb{N}}(m - 1)$ and $(\sum_{\alpha=0}^{\kappa}(\text{Maclaurin}(\text{the function } \cos,]-r, r[, x))(\alpha))_{\kappa \in \mathbb{N}}(2 \cdot m) = (\sum_{\alpha=0}^{\kappa} x \,\text{P\_cos}(\alpha))_{\kappa \in \mathbb{N}}(m)$.

(27) Let r, x be real numbers and m be a natural number. If $0 < r$, then $(\sum_{\alpha=0}^{\kappa}(\text{Maclaurin}(\text{the function } \cos,]-r, r[, x))(\alpha))_{\kappa \in \mathbb{N}}(2 \cdot m) = (\sum_{\alpha=0}^{\kappa} x \,\text{P\_cos}(\alpha))_{\kappa \in \mathbb{N}}(m)$.

(28) Let r, x be real numbers. Suppose $r > 0$. Then

- (i) $(\sum_{\alpha=0}^{\kappa}(\text{Maclaurin}(\text{the function } \sin,]-r, r[, x))(\alpha))_{\kappa \in \mathbb{N}}$ is convergent,
- (ii) $(\text{the function } \sin)(x) = \sum \text{Maclaurin}(\text{the function } \sin,]-r, r[, x)$,
- (iii) $(\sum_{\alpha=0}^{\kappa}(\text{Maclaurin}(\text{the function } \cos,]-r, r[, x))(\alpha))_{\kappa \in \mathbb{N}}$ is convergent, and
- (iv) $(\text{the function } \cos)(x) = \sum \text{Maclaurin}(\text{the function } \cos,]-r, r[, x)$.

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Several Differentiable Formulas of Special Functions

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Summary. In this article, we give several differentiable formulas of special functions. There are some specific composite functions consisting of rational functions, irrational functions, trigonometric functions, exponential functions or logarithmic functions.

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The notation and terminology used in this paper have been introduced in the following articles: [13], [15], [16], [1], [4], [10], [12], [3], [6], [9], [7], [8], [11], [17], [5], [14], and [2].

For simplicity, we follow the rules: x, a, b, c denote real numbers, n denotes a natural number, Z denotes an open subset of $\mathbb{R}$, and f, f_1, f_2 denote partial functions from $\mathbb{R}$ to $\mathbb{R}$.

One can prove the following propositions:

(1) Suppose $Z \subseteq \mathrm{dom}(\log\_(e) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) > 0$. Then $\log\_(e) \cdot f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot f)'_{\restriction Z}(x) = \frac{1}{a+x}$.

(2) Suppose $Z \subseteq \mathrm{dom}(\log\_(e) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = x - a$ and $f(x) > 0$. Then $\log\_(e) \cdot f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot f)'_{\restriction Z}(x) = \frac{1}{x-a}$.

(3) Suppose $Z \subseteq \mathrm{dom}(-\log\_(e) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) > 0$. Then $-\log\_(e) \cdot f$ is differentiable on Z and for every x such that $x \in Z$ holds $(-\log\_(e) \cdot f)'_{\restriction Z}(x) = \frac{1}{a-x}$.

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(4) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z - a\, f)$ and $f = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = a + x$ and $f_1(x) > 0$. Then $\mathrm{id}_Z - a\, f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\mathrm{id}_Z - a\, f)'_{\restriction Z}(x) = \frac{x}{a+x}$.

(5) Suppose $Z \subseteq \mathrm{dom}((2 \cdot a)\, f - \mathrm{id}_Z)$ and $f = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = a + x$ and $f_1(x) > 0$. Then $(2 \cdot a)\, f - \mathrm{id}_Z$ is differentiable on Z and for every x such that $x \in Z$ holds $((2 \cdot a)\, f - \mathrm{id}_Z)'_{\restriction Z}(x) = \frac{a-x}{a+x}$.

(6) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z - (2 \cdot a)\, f)$ and $f = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x + a$ and $f_1(x) > 0$. Then $\mathrm{id}_Z - (2 \cdot a)\, f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\mathrm{id}_Z - (2 \cdot a)\, f)'_{\restriction Z}(x) = \frac{x-a}{x+a}$.

(7) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z + (2 \cdot a)\, f)$ and $f = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x - a$ and $f_1(x) > 0$. Then $\mathrm{id}_Z + (2 \cdot a)\, f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\mathrm{id}_Z + (2 \cdot a)\, f)'_{\restriction Z}(x) = \frac{x+a}{x-a}$.

(8) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z + (a - b)\, f)$ and $f = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x + b$ and $f_1(x) > 0$. Then $\mathrm{id}_Z + (a - b)\, f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\mathrm{id}_Z + (a - b)\, f)'_{\restriction Z}(x) = \frac{x+a}{x+b}$.

(9) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z + (a + b)\, f)$ and $f = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x - b$ and $f_1(x) > 0$. Then $\mathrm{id}_Z + (a + b)\, f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\mathrm{id}_Z + (a + b)\, f)'_{\restriction Z}(x) = \frac{x+a}{x-b}$.

(10) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z - (a + b)\, f)$ and $f = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x + b$ and $f_1(x) > 0$. Then $\mathrm{id}_Z - (a + b)\, f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\mathrm{id}_Z - (a + b)\, f)'_{\restriction Z}(x) = \frac{x-a}{x+b}$.

(11) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z + (b - a)\, f)$ and $f = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x - b$ and $f_1(x) > 0$. Then $\mathrm{id}_Z + (b - a)\, f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\mathrm{id}_Z + (b - a)\, f)'_{\restriction Z}(x) = \frac{x-a}{x-b}$.

(12) Suppose $Z \subseteq \mathrm{dom}(f_1 + c\, f_2)$ and for every x such that $x \in Z$ holds $f_1(x) = a + b \cdot x$ and $f_2 = \frac{2}{\mathbb{Z}}$. Then $f_1 + c\, f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $(f_1 + c\, f_2)'_{\restriction Z}(x) = b + 2 \cdot c \cdot x$.

(13) Suppose $Z \subseteq \mathrm{dom}(\log\_(e) \cdot (f_1 + c\, f_2))$ and $f_2 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f_1(x) = a + b \cdot x$ and $(f_1 + c\, f_2)(x) > 0$. Then $\log\_(e) \cdot (f_1 + c\, f_2)$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot (f_1 + c\, f_2))'_{\restriction Z}(x) = \frac{b+2 \cdot c \cdot x}{a+b \cdot x+c \cdot x^2}$.

(14) Suppose $Z \subseteq \mathrm{dom}\, f$ and for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) \neq 0$. Then $\frac{1}{f}$ is differentiable on Z and for every x such that

$x \in Z$ holds $(\frac{1}{f})'_{\restriction Z}(x) = -\frac{1}{(a+x)^{\mathbf{2}}}$.

(15) Suppose $Z \subseteq \mathrm{dom}((-1)\,\frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) \neq 0$. Then $(-1)\,\frac{1}{f}$ is differentiable on Z and for every x such that $x \in Z$ holds $((-1)\,\frac{1}{f})'_{\restriction Z}(x) = \frac{1}{(a+x)^{\mathbf{2}}}$.

(16) Suppose $Z \subseteq \mathrm{dom}\, f$ and for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) \neq 0$. Then $\frac{1}{f}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{f})'_{\restriction Z}(x) = \frac{1}{(a-x)^{\mathbf{2}}}$.

(17) Suppose $Z \subseteq \mathrm{dom}(f_1 + f_2)$ and for every x such that $x \in Z$ holds $f_1(x) = a^{\mathbf{2}}$ and $f_2 = \frac{2}{\mathbb{Z}}$. Then $f_1 + f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $(f_1 + f_2)'_{\restriction Z}(x) = 2 \cdot x$.

(18) Suppose $Z \subseteq \mathrm{dom}(\log\_(e) \cdot (f_1 + f_2))$ and $f_2 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f_1(x) = a^{\mathbf{2}}$ and $(f_1 + f_2)(x) > 0$. Then $\log\_(e) \cdot (f_1 + f_2)$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot (f_1 + f_2))'_{\restriction Z}(x) = \frac{2 \cdot x}{a^{\mathbf{2}} + x^2}$.

(19) Suppose $Z \subseteq \mathrm{dom}(-\log\_(e) \cdot (f_1 - f_2))$ and $f_2 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f_1(x) = a^{\mathbf{2}}$ and $(f_1 - f_2)(x) > 0$. Then $-\log\_(e) \cdot (f_1 - f_2)$ is differentiable on Z and for every x such that $x \in Z$ holds $(-\log\_(e) \cdot (f_1 - f_2))'_{\restriction Z}(x) = \frac{2 \cdot x}{a^{\mathbf{2}} - x^2}$.

(20) Suppose $Z \subseteq \mathrm{dom}(f_1 + f_2)$ and for every x such that $x \in Z$ holds $f_1(x) = a$ and $f_2 = \frac{3}{\mathbb{Z}}$. Then $f_1 + f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $(f_1 + f_2)'_{\restriction Z}(x) = 3 \cdot x^2$.

(21) Suppose $Z \subseteq \mathrm{dom}(\log\_(e) \cdot (f_1 + f_2))$ and $f_2 = \frac{3}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f_1(x) = a$ and $(f_1 + f_2)(x) > 0$. Then $\log\_(e) \cdot (f_1 + f_2)$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot (f_1 + f_2))'_{\restriction Z}(x) = \frac{3 \cdot x^2}{a + x^3}$.

(22) Suppose $Z \subseteq \mathrm{dom}(\log\_(e) \cdot \frac{f_1}{f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = a + x$ and $f_1(x) > 0$ and $f_2(x) = a - x$ and $f_2(x) > 0$. Then $\log\_(e) \cdot \frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot \frac{f_1}{f_2})'_{\restriction Z}(x) = \frac{2 \cdot a}{a^{\mathbf{2}} - x^{\mathbf{2}}}$.

(23) Suppose $Z \subseteq \mathrm{dom}(\log\_(e) \cdot \frac{f_1}{f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = x - a$ and $f_1(x) > 0$ and $f_2(x) = x + a$ and $f_2(x) > 0$. Then $\log\_(e) \cdot \frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot \frac{f_1}{f_2})'_{\restriction Z}(x) = \frac{2 \cdot a}{x^{\mathbf{2}} - a^{\mathbf{2}}}$.

(24) Suppose $Z \subseteq \mathrm{dom}(\log\_(e) \cdot \frac{f_1}{f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = x - a$ and $f_1(x) > 0$ and $f_2(x) = x - b$ and $f_2(x) > 0$. Then $\log\_(e) \cdot \frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot \frac{f_1}{f_2})'_{\restriction Z}(x) = \frac{a - b}{(x-a) \cdot (x-b)}$.

(25) Suppose $Z \subseteq \mathrm{dom}(\frac{1}{a-b}\, f)$ and $f = \log\_(e) \cdot \frac{f_1}{f_2}$ and for every x such that

$x \in Z$ holds $f_1(x) = x - a$ and $f_1(x) > 0$ and $f_2(x) = x - b$ and $f_2(x) > 0$ and $a - b \neq 0$. Then $\frac{1}{a-b}\, f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{a-b}\, f)'_{\restriction Z}(x) = \frac{1}{(x-a)\cdot(x-b)}$.

(26) Suppose $Z \subseteq \mathrm{dom}(\log\_(e) \cdot \frac{f_1}{f_2})$ and $f_2 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f_1(x) = x - a$ and $f_1(x) > 0$ and $f_2(x) > 0$ and $x \neq 0$. Then $\log\_(e) \cdot \frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot \frac{f_1}{f_2})'_{\restriction Z}(x) = \frac{2\cdot a - x}{x\cdot(x-a)}$.

(27) Suppose $Z \subseteq \mathrm{dom}((\frac{3}{2}_{\mathbb{R}}) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) > 0$. Then $(\frac{3}{2}_{\mathbb{R}}) \cdot f$ is differentiable on Z and for every x such that $x \in Z$ holds $((\frac{3}{2}_{\mathbb{R}}) \cdot f)'_{\restriction Z}(x) = \frac{3}{2} \cdot (a + x)^{\frac{1}{2}}_{\mathbb{R}}$.

(28) Suppose $Z \subseteq \mathrm{dom}(\frac{2}{3}\, ((\frac{3}{2}_{\mathbb{R}}) \cdot f))$ and for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) > 0$. Then $\frac{2}{3}\, ((\frac{3}{2}_{\mathbb{R}}) \cdot f)$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{2}{3}\, ((\frac{3}{2}_{\mathbb{R}}) \cdot f))'_{\restriction Z}(x) = (a + x)^{\frac{1}{2}}_{\mathbb{R}}$.

(29) Suppose $Z \subseteq \mathrm{dom}((-\frac{2}{3})\, ((\frac{3}{2}_{\mathbb{R}}) \cdot f))$ and for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) > 0$. Then $(-\frac{2}{3})\, ((\frac{3}{2}_{\mathbb{R}}) \cdot f)$ is differentiable on Z and for every x such that $x \in Z$ holds $((-\frac{2}{3})\, ((\frac{3}{2}_{\mathbb{R}}) \cdot f))'_{\restriction Z}(x) = (a - x)^{\frac{1}{2}}_{\mathbb{R}}$.

(30) Suppose $Z \subseteq \mathrm{dom}(2\, ((\frac{1}{2}_{\mathbb{R}}) \cdot f))$ and for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) > 0$. Then $2\, ((\frac{1}{2}_{\mathbb{R}}) \cdot f)$ is differentiable on Z and for every x such that $x \in Z$ holds $(2\, ((\frac{1}{2}_{\mathbb{R}}) \cdot f))'_{\restriction Z}(x) = (a + x)^{-\frac{1}{2}}_{\mathbb{R}}$.

(31) Suppose $Z \subseteq \mathrm{dom}((-2)\, ((\frac{1}{2}_{\mathbb{R}}) \cdot f))$ and for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) > 0$. Then $(-2)\, ((\frac{1}{2}_{\mathbb{R}}) \cdot f)$ is differentiable on Z and for every x such that $x \in Z$ holds $((-2)\, ((\frac{1}{2}_{\mathbb{R}}) \cdot f))'_{\restriction Z}(x) = (a - x)^{-\frac{1}{2}}_{\mathbb{R}}$.

(32) Suppose $Z \subseteq \mathrm{dom}(\frac{2}{3\cdot b}\, ((\frac{3}{2}_{\mathbb{R}}) \cdot f))$ and for every x such that $x \in Z$ holds $f(x) = a + b \cdot x$ and $b \neq 0$ and $f(x) > 0$. Then $\frac{2}{3\cdot b}\, ((\frac{3}{2}_{\mathbb{R}}) \cdot f)$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{2}{3\cdot b}\, ((\frac{3}{2}_{\mathbb{R}}) \cdot f))'_{\restriction Z}(x) = (a + b \cdot x)^{\frac{1}{2}}_{\mathbb{R}}$.

(33) Suppose $Z \subseteq \mathrm{dom}((-\frac{2}{3\cdot b})\, ((\frac{3}{2}_{\mathbb{R}}) \cdot f))$ and for every x such that $x \in Z$ holds $f(x) = a - b \cdot x$ and $b \neq 0$ and $f(x) > 0$. Then $(-\frac{2}{3\cdot b})\, ((\frac{3}{2}_{\mathbb{R}}) \cdot f)$ is differentiable on Z and for every x such that $x \in Z$ holds $((-\frac{2}{3\cdot b})\, ((\frac{3}{2}_{\mathbb{R}}) \cdot f))'_{\restriction Z}(x) = (a - b \cdot x)^{\frac{1}{2}}_{\mathbb{R}}$.

(34) Suppose $Z \subseteq \mathrm{dom}((\frac{1}{2}_{\mathbb{R}}) \cdot f)$ and $f = f_1 + f_2$ and $f_2 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f_1(x) = a^{\mathbf{2}}$ and $f(x) > 0$. Then $(\frac{1}{2}_{\mathbb{R}}) \cdot f$ is differentiable on

Z and for every x such that $x \in Z$ holds $((\frac{1}{2}_{\mathbb{R}}) \cdot f)'_{\restriction Z}(x) = x \cdot (a^{\mathbf{2}} + x^2)_{\mathbb{R}}^{-\frac{1}{2}}$.

(35) Suppose $Z \subseteq \mathrm{dom}(-(\frac{1}{2}_{\mathbb{R}}) \cdot f)$ and $f = f_1 - f_2$ and $f_2 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f_1(x) = a^{\mathbf{2}}$ and $f(x) > 0$. Then $-(\frac{1}{2}_{\mathbb{R}}) \cdot f$ is differentiable on Z and for every x such that $x \in Z$ holds $(-(\frac{1}{2}_{\mathbb{R}}) \cdot f)'_{\restriction Z}(x) = x \cdot (a^{\mathbf{2}} - x^2)_{\mathbb{R}}^{-\frac{1}{2}}$.

(36) Suppose $Z \subseteq \mathrm{dom}(2\,((\frac{1}{2}_{\mathbb{R}}) \cdot f))$ and $f = f_1 + f_2$ and $f_2 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f_1(x) = x$ and $f(x) > 0$. Then $2\,((\frac{1}{2}_{\mathbb{R}}) \cdot f)$ is differentiable on Z and for every x such that $x \in Z$ holds $(2\,((\frac{1}{2}_{\mathbb{R}}) \cdot f))'_{\restriction Z}(x) = (2 \cdot x + 1) \cdot (x^2 + x)_{\mathbb{R}}^{-\frac{1}{2}}$.

(37) Suppose $Z \subseteq \mathrm{dom}((\text{the function sin}) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x + b$. Then

(i) (the function sin) $\cdot f$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function sin}) \cdot f)'_{\restriction Z}(x) = a \cdot (\text{the function cos})(a \cdot x + b)$.

(38) Suppose $Z \subseteq \mathrm{dom}((\text{the function cos}) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x + b$. Then

(i) (the function cos) $\cdot f$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function cos}) \cdot f)'_{\restriction Z}(x) = -a \cdot (\text{the function sin})(a \cdot x + b)$.

(39) Suppose that for every x such that $x \in Z$ holds $(\text{the function cos})(x) \neq 0$. Then

(i) $\frac{1}{\text{the function cos}}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{\text{the function cos}})'_{\restriction Z}(x) = \frac{(\text{the function sin})(x)}{(\text{the function cos})(x)^{\mathbf{2}}}$.

(40) Suppose that for every x such that $x \in Z$ holds $(\text{the function sin})(x) \neq 0$. Then

(i) $\frac{1}{\text{the function sin}}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{\text{the function sin}})'_{\restriction Z}(x) = -\frac{(\text{the function cos})(x)}{(\text{the function sin})(x)^{\mathbf{2}}}$.

(41) Suppose $Z \subseteq \mathrm{dom}((\text{the function sin})\ (\text{the function cos}))$. Then

(i) (the function sin) (the function cos) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function sin})\ (\text{the function cos}))'_{\restriction Z}(x) = \cos(2 \cdot x)$.

(42) Suppose $Z \subseteq \mathrm{dom}(\log\_(e) \cdot (\text{the function cos}))$ and for every x such that $x \in Z$ holds $(\text{the function cos})(x) > 0$. Then $\log\_(e) \cdot (\text{the function cos})$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot (\text{the function cos}))'_{\restriction Z}(x) = -\tan x$.

(43) Suppose $Z \subseteq \mathrm{dom}(\log\_(e) \cdot (\text{the function sin}))$ and for every x such that $x \in Z$ holds (the function sin)$(x) > 0$. Then $\log\_(e) \cdot$ (the function sin) is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot (\text{the function sin}))'_{\restriction Z}(x) = \cot x$.

(44) Suppose $Z \subseteq \mathrm{dom}((-\mathrm{id}_Z)\,(\text{the function cos}))$. Then
 (i) $(-\mathrm{id}_Z)$ (the function cos) is differentiable on Z, and
 (ii) for every x such that $x \in Z$ holds $((-\mathrm{id}_Z)\,(\text{the function cos}))'_{\restriction Z}(x) = -(\text{the function cos})(x) + x \cdot (\text{the function sin})(x)$.

(45) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z\,(\text{the function sin}))$. Then
 (i) id_Z (the function sin) is differentiable on Z, and
 (ii) for every x such that $x \in Z$ holds $(\mathrm{id}_Z\,(\text{the function sin}))'_{\restriction Z}(x) = (\text{the function sin})(x) + x \cdot (\text{the function cos})(x)$.

(46) Suppose $Z \subseteq \mathrm{dom}((-\mathrm{id}_Z)\,(\text{the function cos}) + \text{the function sin})$. Then
 (i) $(-\mathrm{id}_Z)$ (the function cos)+the function sin is differentiable on Z, and
 (ii) for every x such that $x \in Z$ holds $((-\mathrm{id}_Z)\,(\text{the function cos}) + \text{the function sin})'_{\restriction Z}(x) = x \cdot (\text{the function sin})(x)$.

(47) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z\,(\text{the function sin}) + \text{the function cos})$. Then
 (i) id_Z (the function sin)+the function cos is differentiable on Z, and
 (ii) for every x such that $x \in Z$ holds $(\mathrm{id}_Z\,(\text{the function sin}) + \text{the function cos})'_{\restriction Z}(x) = x \cdot (\text{the function cos})(x)$.

(48) Suppose $Z \subseteq \mathrm{dom}(2\,((^{\frac{1}{2}}_{\mathbb{R}}) \cdot (\text{the function sin})))$ and for every x such that $x \in Z$ holds (the function sin)$(x) > 0$. Then
 (i) $2\,((^{\frac{1}{2}}_{\mathbb{R}}) \cdot (\text{the function sin}))$ is differentiable on Z, and
 (ii) for every x such that $x \in Z$ holds $(2\,((^{\frac{1}{2}}_{\mathbb{R}}) \cdot (\text{the function sin})))'_{\restriction Z}(x) = (\text{the function cos})(x) \cdot (\text{the function sin})(x)_{\mathbb{R}}^{-\frac{1}{2}}$.

(49) Suppose $Z \subseteq \mathrm{dom}(\frac{1}{2}\,((^{2}_{\mathbb{Z}}) \cdot (\text{the function sin})))$. Then
 (i) $\frac{1}{2}\,((^{2}_{\mathbb{Z}}) \cdot (\text{the function sin}))$ is differentiable on Z, and
 (ii) for every x such that $x \in Z$ holds $(\frac{1}{2}\,((^{2}_{\mathbb{Z}}) \cdot (\text{the function sin})))'_{\restriction Z}(x) = (\text{the function sin})(x) \cdot (\text{the function cos})(x)$.

(50) Suppose that
 (i) $Z \subseteq \mathrm{dom}((\text{the function sin}) + \frac{1}{2}\,((^{2}_{\mathbb{Z}}) \cdot (\text{the function sin})))$, and
 (ii) for every x such that $x \in Z$ holds (the function sin)$(x) > 0$ and (the function sin)$(x) < 1$.

 Then
 (iii) $(\text{the function sin}) + \frac{1}{2}\,((^{2}_{\mathbb{Z}}) \cdot (\text{the function sin}))$ is differentiable on Z, and
 (iv) for every x such that $x \in Z$ holds $((\text{the function sin}) + \frac{1}{2}\,((^{2}_{\mathbb{Z}}) \cdot (\text{the function sin})))'_{\restriction Z}(x) = \frac{(\text{the function cos})(x)^3}{1 - (\text{the function sin})(x)}$.

(51) Suppose that
 (i) $Z \subseteq \mathrm{dom}(\frac{1}{2}\,((^{2}_{\mathbb{Z}}) \cdot (\text{the function sin})) - \text{the function cos})$, and

(ii) for every x such that $x \in Z$ holds (the function sin)$(x) > 0$ and (the function cos)$(x) < 1$.

Then

(iii) $\frac{1}{2}\,((\frac{2}{\mathbb{Z}})\cdot$ (the function sin))$-$the function cos is differentiable on Z, and

(iv) for every x such that $x \in Z$ holds $(\frac{1}{2}\,((\frac{2}{\mathbb{Z}})\cdot$ (the function sin))$-$the function cos$)'_{\restriction Z}(x) = \frac{(\text{the function sin})(x)^3}{1-(\text{the function cos})(x)}$.

(52) Suppose that

(i) $Z \subseteq \mathrm{dom}($(the function sin)$-\frac{1}{2}\,((\frac{2}{\mathbb{Z}})\cdot$ (the function sin))), and

(ii) for every x such that $x \in Z$ holds (the function sin)$(x) > 0$ and (the function sin)$(x) > -1$.

Then

(iii) (the function sin)$-\frac{1}{2}\,((\frac{2}{\mathbb{Z}})\cdot$(the function sin)) is differentiable on Z, and

(iv) for every x such that $x \in Z$ holds ((the function sin)$-\frac{1}{2}\,((\frac{2}{\mathbb{Z}})\cdot$ (the function sin)))$'_{\restriction Z}(x) = \frac{(\text{the function cos})(x)^3}{1+(\text{the function sin})(x)}$.

(53) Suppose that

(i) $Z \subseteq \mathrm{dom}(-$the function cos $-\ \frac{1}{2}\,((\frac{2}{\mathbb{Z}})\cdot$ (the function sin))), and

(ii) for every x such that $x \in Z$ holds (the function sin)$(x) > 0$ and (the function cos)$(x) > -1$.

Then

(iii) $-$the function cos $-\ \frac{1}{2}\,((\frac{2}{\mathbb{Z}})\cdot$ (the function sin)) is differentiable on Z, and

(iv) for every x such that $x \in Z$ holds ($-$the function cos $-\ \frac{1}{2}\,((\frac{2}{\mathbb{Z}})\cdot$ (the function sin)))$'_{\restriction Z}(x) = \frac{(\text{the function sin})(x)^3}{1+(\text{the function cos})(x)}$.

(54) Suppose $Z \subseteq \mathrm{dom}(\frac{1}{n}\,((\frac{n}{\mathbb{Z}})\cdot$ (the function sin))) and $n > 0$. Then

(i) $\frac{1}{n}\,((\frac{n}{\mathbb{Z}})\cdot$ (the function sin)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{n}\,((\frac{n}{\mathbb{Z}})\cdot$ (the function sin)))$'_{\restriction Z}(x) =$ ((the function sin)$(x)^{n-1}_{\mathbb{Z}}) \cdot$ (the function cos)(x).

(55) Suppose $Z \subseteq \mathrm{dom}(\exp f)$ and for every x such that $x \in Z$ holds $f(x) = x - 1$. Then $\exp f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\exp f)'_{\restriction Z}(x) = x \cdot \exp(x)$.

(56) Suppose $Z \subseteq \mathrm{dom}(\log\_(e) \cdot \frac{\exp}{\exp + f})$ and for every x such that $x \in Z$ holds $f(x) = 1$. Then $\log\_(e) \cdot \frac{\exp}{\exp + f}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot \frac{\exp}{\exp + f})'_{\restriction Z}(x) = \frac{1}{\exp(x)+1}$.

(57) Suppose $Z \subseteq \mathrm{dom}(\log\_(e) \cdot \frac{\exp - f}{\exp})$ and for every x such that $x \in Z$ holds $f(x) = 1$ and $(\exp - f)(x) > 0$. Then $\log\_(e) \cdot \frac{\exp - f}{\exp}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot \frac{\exp - f}{\exp})'_{\restriction Z}(x) = \frac{1}{\exp(x)-1}$.

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