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Banach Space of Bounded Linear Operators

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Summary. In this article, the basic properties of linear spaces which are defined as the set of all linear operators from one linear space to another, are described. Especially, the Banach space is introduced. This is defined by the set of all bounded linear operators.

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The notation and terminology used in this paper are introduced in the following articles: [26], [6], [24], [31], [27], [33], [32], [4], [5], [16], [23], [22], [3], [1], [2], [21], [28], [9], [7], [30], [14], [25], [17], [29], [19], [18], [8], [20], [13], [11], [12], [10], and [15].

1. Real Vector Space of Operators

Let X be a set, let Y be a non empty set, let F be a function from $[: \mathbb{R}, Y :]$ into Y, let a be a real number, and let f be a function from X into Y. Then $F^\circ(a, f)$ is an element of Y^X.

One can prove the following propositions:

(1) Let X be a non empty set and Y be a non empty loop structure. Then there exists a binary operation A_1 on (the carrier of $Y)^X$ such that for all elements f, g of (the carrier of $Y)^X$ holds $A_1(f, g) =$ (the addition of $Y)^\circ(f, g)$.

(2) Let X be a non empty set and Y be a real linear space. Then there exists a function M_1 from $[: \mathbb{R}$, (the carrier of $Y)^X :]$ into (the carrier of $Y)^X$ such that for every real number r and for every element f of (the carrier of $Y)^X$ and for every element s of X holds $M_1(\langle r, f\rangle)(s) = r \cdot f(s)$.

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Let X be a non empty set and let Y be a non empty loop structure. The functor $\mathrm{FuncAdd}(X,Y)$ yields a binary operation on (the carrier of $Y)^X$ and is defined by:

(Def. 1) For all elements f, g of (the carrier of $Y)^X$ holds $(\mathrm{FuncAdd}(X,Y))(f, g) =$ (the addition of $Y)^\circ(f, g)$.

Let X be a non empty set and let Y be a real linear space. The functor $\mathrm{FuncExtMult}(X,Y)$ yields a function from $[:\mathbb{R},$ (the carrier of $Y)^X:]$ into (the carrier of $Y)^X$ and is defined by the condition (Def. 2).

(Def. 2) Let a be a real number, f be an element of (the carrier of $Y)^X$, and x be an element of X. Then $(\mathrm{FuncExtMult}(X,Y))(\langle a, f\rangle)(x) = a \cdot f(x)$.

Let X be a set and let Y be a non empty zero structure. The functor $\mathrm{FuncZero}(X,Y)$ yielding an element of (the carrier of $Y)^X$ is defined as follows:

(Def. 3) $\mathrm{FuncZero}(X,Y) = X \longmapsto 0_Y$.

We adopt the following rules: X is a non empty set, Y is a real linear space, and f, g, h are elements of (the carrier of $Y)^X$.

The following two propositions are true:

(3) Let Y be a non empty loop structure and f, g, h be elements of (the carrier of $Y)^X$. Then $h = (\mathrm{FuncAdd}(X,Y))(f, g)$ if and only if for every element x of X holds $h(x) = f(x) + g(x)$.

(4) For every element x of X holds $(\mathrm{FuncZero}(X,Y))(x) = 0_Y$.

In the sequel a, b are real numbers.

The following propositions are true:

(5) $h = (\mathrm{FuncExtMult}(X,Y))(\langle a, f\rangle)$ iff for every element x of X holds $h(x) = a \cdot f(x)$.

(6) $(\mathrm{FuncAdd}(X,Y))(f, g) = (\mathrm{FuncAdd}(X,Y))(g, f)$.

(7) $(\mathrm{FuncAdd}(X,Y))(f, (\mathrm{FuncAdd}(X,Y))(g, h)) = (\mathrm{FuncAdd}(X,Y))((\mathrm{FuncAdd}(X,Y))(f, g), h)$.

(8) $(\mathrm{FuncAdd}(X,Y))(\mathrm{FuncZero}(X,Y), f) = f$.

(9) $(\mathrm{FuncAdd}(X,Y))(f, (\mathrm{FuncExtMult}(X,Y))(\langle -1, f\rangle)) = \mathrm{FuncZero}(X,Y)$.

(10) $(\mathrm{FuncExtMult}(X,Y))(\langle 1, f\rangle) = f$.

(11) $(\mathrm{FuncExtMult}(X,Y))(\langle a, (\mathrm{FuncExtMult}(X,Y))(\langle b, f\rangle)\rangle) = (\mathrm{FuncExtMult}(X,Y))(\langle a \cdot b, f\rangle)$.

(12) $(\mathrm{FuncAdd}(X,Y))((\mathrm{FuncExtMult}(X,Y))(\langle a, f\rangle), (\mathrm{FuncExtMult}(X,Y))(\langle b, f\rangle)) = (\mathrm{FuncExtMult}(X,Y))(\langle a + b, f\rangle)$.

(13) $\langle$(the carrier of $Y)^X, \mathrm{FuncZero}(X,Y), \mathrm{FuncAdd}(X,Y), \mathrm{FuncExtMult}(X,Y)\rangle$ is a real linear space.

Let X be a non empty set and let Y be a real linear space. The functor $\mathrm{RealVectSpace}(X,Y)$ yields a real linear space and is defined as follows:

(Def. 4) RealVectSpace$(X, Y) = \langle$(the carrier of $Y)^X$, FuncZero(X, Y), FuncAdd(X, Y), FuncExtMult$(X, Y)\rangle$.

Let X be a non empty set and let Y be a real linear space. One can check that RealVectSpace(X, Y) is strict.

Let X be a non empty set and let Y be a real linear space. Note that every vector of RealVectSpace(X, Y) is function-like and relation-like.

Let X be a non empty set, let Y be a real linear space, let f be a vector of RealVectSpace(X, Y), and let x be an element of X. Then $f(x)$ is a vector of Y.

One can prove the following propositions:

(14) Let X be a non empty set, Y be a real linear space, and f, g, h be vectors of RealVectSpace(X, Y). Then $h = f + g$ if and only if for every element x of X holds $h(x) = f(x) + g(x)$.

(15) Let X be a non empty set, Y be a real linear space, f, h be vectors of RealVectSpace(X, Y), and a be a real number. Then $h = a \cdot f$ if and only if for every element x of X holds $h(x) = a \cdot f(x)$.

(16) For every non empty set X and for every real linear space Y holds $0_{\mathrm{RealVectSpace}(X,Y)} = X \longmapsto 0_Y$.

2. Real Vector Space of Linear Operators

Let X be a non empty RLS structure, let Y be a non empty loop structure, and let I_1 be a function from X into Y. We say that I_1 is additive if and only if:

(Def. 5) For all vectors x, y of X holds $I_1(x + y) = I_1(x) + I_1(y)$.

Let X, Y be non empty RLS structures and let I_1 be a function from X into Y. We say that I_1 is homogeneous if and only if:

(Def. 6) For every vector x of X and for every real number r holds $I_1(r \cdot x) = r \cdot I_1(x)$.

Let X be a non empty RLS structure and let Y be a real linear space. Note that there exists a function from X into Y which is additive and homogeneous.

Let X, Y be real linear spaces. A linear operator from X into Y is an additive homogeneous function from X into Y.

Let X, Y be real linear spaces. The functor LinearOperators(X, Y) yields a subset of RealVectSpace(the carrier of X, Y) and is defined as follows:

(Def. 7) For every set x holds $x \in$ LinearOperators(X, Y) iff x is a linear operator from X into Y.

Let X, Y be real linear spaces. Note that LinearOperators(X, Y) is non empty.

One can prove the following propositions:

(17) For all real linear spaces X, Y holds LinearOperators(X,Y) is linearly closed.

(18) Let X, Y be real linear spaces. Then $\langle$LinearOperators(X,Y), Zero_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y)), Add_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y)), Mult_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y))$\rangle$ is a subspace of RealVectSpace(the carrier of X, Y).

Let X, Y be real linear spaces. One can verify that $\langle$LinearOperators(X,Y), Zero_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y)), Add_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y)), Mult_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y))$\rangle$ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

One can prove the following proposition

(19) Let X, Y be real linear spaces. Then $\langle$LinearOperators(X,Y), Zero_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y)), Add_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y)), Mult_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y))$\rangle$ is a real linear space.

Let X, Y be real linear spaces. The functor RVectorSpaceOfLinearOperators (X,Y) yielding a real linear space is defined as follows:

(Def. 8) RVectorSpaceOfLinearOperators$(X,Y) = \langle$LinearOperators(X,Y), Zero_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y)), Add_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y)), Mult_(LinearOperators(X,Y), RealVectSpace(the carrier of X, Y))$\rangle$.

Let X, Y be real linear spaces. Observe that RVectorSpaceOfLinearOperators (X,Y) is strict.

Let X, Y be real linear spaces. Note that every element of RVectorSpaceOfLinearOperators(X,Y) is function-like and relation-like.

Let X, Y be real linear spaces, let f be an element of RVectorSpaceOfLinearOperators(X,Y), and let v be a vector of X. Then $f(v)$ is a vector of Y.

We now state four propositions:

(20) Let X, Y be real linear spaces and f, g, h be vectors of RVectorSpaceOfLinearOperators(X,Y). Then $h = f + g$ if and only if for every vector x of X holds $h(x) = f(x) + g(x)$.

(21) Let X, Y be real linear spaces, f, h be vectors of RVectorSpaceOfLinearOperators(X,Y), and a be a real number. Then $h = a \cdot f$ if and only if for every vector x of X holds $h(x) = a \cdot f(x)$.

(22) For all real linear spaces X, Y holds $0_{\mathrm{RVectorSpaceOfLinearOperators}(X,Y)} =$ (the carrier of X) $\longmapsto 0_Y$.

(23) For all real linear spaces X, Y holds (the carrier of X) $\longmapsto 0_Y$ is a linear operator from X into Y.

3. Real Normed Linear Space of Bounded Linear Operators

One can prove the following proposition

(24) Let X be a real normed space, s_1 be a sequence of X, and g be a point of X. If s_1 is convergent and $\lim s_1 = g$, then $\|s_1\|$ is convergent and $\lim \|s_1\| = \|g\|$.

Let X, Y be real normed spaces and let I_1 be a linear operator from X into Y. We say that I_1 is bounded if and only if:

(Def. 9) There exists a real number K such that $0 \leqslant K$ and for every vector x of X holds $\|I_1(x)\| \leqslant K \cdot \|x\|$.

Next we state the proposition

(25) Let X, Y be real normed spaces and f be a linear operator from X into Y. If for every vector x of X holds $f(x) = 0_Y$, then f is bounded.

Let X, Y be real normed spaces. One can check that there exists a linear operator from X into Y which is bounded.

Let X, Y be real normed spaces. The functor BoundedLinearOperators(X, Y) yields a subset of RVectorSpaceOfLinearOperators(X, Y) and is defined by:

(Def. 10) For every set x holds $x \in$ BoundedLinearOperators(X, Y) iff x is a bounded linear operator from X into Y.

Let X, Y be real normed spaces. One can verify that BoundedLinearOperators (X, Y) is non empty.

One can prove the following two propositions:

(26) For all real normed spaces X, Y holds BoundedLinearOperators(X, Y) is linearly closed.

(27) For all real normed spaces X, Y holds $\langle$BoundedLinearOperators(X, Y), Zero_(BoundedLinearOperators(X, Y), RVectorSpaceOfLinearOperators (X, Y)), Add_(BoundedLinearOperators(X, Y), RVectorSpaceOfLinearOperators(X, Y)), Mult_(BoundedLinearOperators (X, Y), RVectorSpaceOfLinearOperators(X, Y))$\rangle$ is a subspace of RVectorSpaceOfLinearOperators(X, Y).

Let X, Y be real normed spaces.

Observe that $\langle$BoundedLinearOperators(X, Y), Zero_(BoundedLinearOperators(X, Y), RVectorSpaceOfLinearOperators(X, Y)), Add_(BoundedLinearOperators(X, Y), RVectorSpaceOfLinearOperators (X, Y)), Mult_(BoundedLinearOperators(X, Y), RVectorSpaceOfLinearOperators$(X, Y))\rangle$ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

One can prove the following proposition

(28) For all real normed spaces X, Y holds $\langle$BoundedLinearOperators(X, Y), Zero_(BoundedLinearOperators(X, Y), RVectorSpaceOfLinearOperators $(X, Y))$, Add_(BoundedLinearOperators(X, Y), RVectorSpaceOfLinearOperators$(X, Y))$, Mult_(BoundedLinearOperators(X, Y), RVectorSpaceOfLinearOperators $(X, Y))\rangle$ is a real linear space.

Let X, Y be real normed spaces.

The functor RVectorSpaceOfBoundedLinearOperators(X, Y) yields a real linear space and is defined by:

(Def. 11) RVectorSpaceOfBoundedLinearOperators$(X, Y) =$ $\langle$BoundedLinearOperators(X, Y), Zero_(BoundedLinearOperators(X, Y), RVectorSpaceOfLinearOperators$(X, Y))$, Add_(BoundedLinearOperators (X, Y), RVectorSpaceOfLinearOperators$(X, Y))$, Mult_(BoundedLinearOperators(X, Y), RVectorSpaceOfLinearOperators $(X, Y))\rangle$.

Let X, Y be real normed spaces.

Observe that RVectorSpaceOfBoundedLinearOperators(X, Y) is strict.

Let X, Y be real normed spaces. Note that every element of RVectorSpaceOfBoundedLinearOperators(X, Y) is function-like and relation-like.

Let X, Y be real normed spaces, let f be an element of RVectorSpaceOfBoundedLinearOperators(X, Y), and let v be a vector of X. Then $f(v)$ is a vector of Y.

One can prove the following propositions:

(29) Let X, Y be real normed spaces and f, g, h be vectors of RVectorSpaceOfBoundedLinearOperators(X, Y). Then $h = f + g$ if and only if for every vector x of X holds $h(x) = f(x) + g(x)$.

(30) Let X, Y be real normed spaces, f, h be vectors of RVectorSpaceOfBoundedLinearOperators(X, Y), and a be a real number. Then $h = a \cdot f$ if and only if for every vector x of X holds $h(x) = a \cdot f(x)$.

(31) For all real normed spaces X, Y holds $0_{\mathrm{RVectorSpaceOfBoundedLinearOperators}(X,Y)} = (\text{the carrier of } X) \longmapsto 0_Y$.

Let X, Y be real normed spaces and let f be a set. Let us assume that $f \in$ BoundedLinearOperators(X, Y). The functor modetrans(f, X, Y) yields a bounded linear operator from X into Y and is defined by:

(Def. 12) modetrans$(f, X, Y) = f$.

Let X, Y be real normed spaces and let u be a linear operator from X into Y. The functor $\mathrm{PreNorms}(u)$ yielding a non empty subset of $\mathbb{R}$ is defined as follows:

(Def. 13) $\mathrm{PreNorms}(u) = \{\|u(t)\|; t$ ranges over vectors of X: $\|t\| \leqslant 1\}$.

We now state three propositions:

(32) Let X, Y be real normed spaces and g be a bounded linear operator from X into Y. Then $\mathrm{PreNorms}(g)$ is non empty and upper bounded.

(33) Let X, Y be real normed spaces and g be a linear operator from X into Y. Then g is bounded if and only if $\mathrm{PreNorms}(g)$ is upper bounded.

(34) Let X, Y be real normed spaces. Then there exists a function N_1 from $\mathrm{BoundedLinearOperators}(X, Y)$ into $\mathbb{R}$ such that for every set f if $f \in \mathrm{BoundedLinearOperators}(X, Y)$, then $N_1(f) = \sup \mathrm{PreNorms}(\mathrm{modetrans}(f, X, Y))$.

Let X, Y be real normed spaces. The functor $\mathrm{BoundedLinearOperatorsNorm}(X, Y)$ yielding a function from $\mathrm{BoundedLinearOperators}(X, Y)$ into $\mathbb{R}$ is defined as follows:

(Def. 14) For every set x such that $x \in \mathrm{BoundedLinearOperators}(X, Y)$ holds $(\mathrm{BoundedLinearOperatorsNorm}(X, Y))(x) = \sup \mathrm{PreNorms}(\mathrm{modetrans}(x, X, Y))$.

The following two propositions are true:

(35) For all real normed spaces X, Y and for every bounded linear operator f from X into Y holds $\mathrm{modetrans}(f, X, Y) = f$.

(36) For all real normed spaces X, Y and for every bounded linear operator f from X into Y holds $(\mathrm{BoundedLinearOperatorsNorm}(X, Y))(f) = \sup \mathrm{PreNorms}(f)$.

Let X, Y be real normed spaces.

The functor $\mathrm{RNormSpaceOfBoundedLinearOperators}(X, Y)$ yielding a non empty normed structure is defined as follows:

(Def. 15) $\mathrm{RNormSpaceOfBoundedLinearOperators}(X, Y) = \langle \mathrm{BoundedLinearOperators}(X, Y), \mathrm{Zero\_}(\mathrm{BoundedLinearOperators}(X, Y), \mathrm{RVectorSpaceOfLinearOperators}(X, Y)), \mathrm{Add\_}(\mathrm{BoundedLinearOperators}(X, Y), \mathrm{RVectorSpaceOfLinearOperators}(X, Y)), \mathrm{Mult\_}(\mathrm{BoundedLinearOperators}(X, Y), \mathrm{RVectorSpaceOfLinearOperators}(X, Y)), \mathrm{BoundedLinearOperatorsNorm}(X, Y) \rangle$.

The following propositions are true:

(37) For all real normed spaces X, Y holds (the carrier of X) $\longmapsto 0_Y = 0_{\mathrm{RNormSpaceOfBoundedLinearOperators}(X, Y)}$.

(38) Let X, Y be real normed spaces, f be a point

of RNormSpaceOfBoundedLinearOperators(X, Y), and g be a bounded linear operator from X into Y. If $g = f$, then for every vector t of X holds $\|g(t)\| \leqslant \|f\| \cdot \|t\|$.

(39) For all real normed spaces X, Y and for every point f of RNormSpaceOfBoundedLinearOperators(X, Y) holds $0 \leqslant \|f\|$.

(40) For all real normed spaces X, Y and for every point f of RNormSpaceOfBoundedLinearOperators(X, Y) such that $f = 0_{\mathrm{RNormSpaceOfBoundedLinearOperators}(X,Y)}$ holds $0 = \|f\|$.

Let X, Y be real normed spaces. Observe that every element of RNormSpaceOfBoundedLinearOperators(X, Y) is function-like and relation-like.

Let X, Y be real normed spaces, let f be an element of RNormSpaceOfBoundedLinearOperators(X, Y), and let v be a vector of X. Then $f(v)$ is a vector of Y.

The following propositions are true:

(41) Let X, Y be real normed spaces and f, g, h be points of RNormSpaceOfBoundedLinearOperators(X, Y). Then $h = f + g$ if and only if for every vector x of X holds $h(x) = f(x) + g(x)$.

(42) Let X, Y be real normed spaces, f, h be points of RNormSpaceOfBoundedLinearOperators(X, Y), and a be a real number. Then $h = a \cdot f$ if and only if for every vector x of X holds $h(x) = a \cdot f(x)$.

(43) Let X be a real normed space, Y be a real normed space, f, g be points of RNormSpaceOfBoundedLinearOperators(X, Y), and a be a real number. Then $\|f\| = 0$ iff $f = 0_{\mathrm{RNormSpaceOfBoundedLinearOperators}(X,Y)}$ and $\|a \cdot f\| = |a| \cdot \|f\|$ and $\|f + g\| \leqslant \|f\| + \|g\|$.

(44) For all real normed spaces X, Y holds RNormSpaceOfBoundedLinearOperators(X, Y) is real normed space-like.

(45) For all real normed spaces X, Y holds RNormSpaceOfBoundedLinearOperators(X, Y) is a real normed space.

Let X, Y be real normed spaces.

Note that RNormSpaceOfBoundedLinearOperators(X, Y) is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

One can prove the following proposition

(46) Let X, Y be real normed spaces and f, g, h be points of RNormSpaceOfBoundedLinearOperators(X, Y). Then $h = f - g$ if and only if for every vector x of X holds $h(x) = f(x) - g(x)$.

4. Real Banach Space of Bounded Linear Operators

Let X be a real normed space. We say that X is complete if and only if:

(Def. 16) For every sequence s_1 of X such that s_1 is Cauchy sequence by norm holds s_1 is convergent.

Let us note that there exists a real normed space which is complete.

A real Banach space is a complete real normed space.

We now state three propositions:

(47) Let X be a real normed space and s_1 be a sequence of X. If s_1 is convergent, then $\|s_1\|$ is convergent and $\lim\|s_1\| = \|\lim s_1\|$.

(48) Let X, Y be real normed spaces. Suppose Y is complete. Let s_1 be a sequence of RNormSpaceOfBoundedLinearOperators(X, Y). If s_1 is Cauchy sequence by norm, then s_1 is convergent.

(49) For every real normed space X and for every real Banach space Y holds RNormSpaceOfBoundedLinearOperators(X, Y) is a real Banach space.

Let X be a real normed space and let Y be a real Banach space. Observe that RNormSpaceOfBoundedLinearOperators(X, Y) is complete.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[2] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(**2**):281–290, 1990.

[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[7] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.

[8] Noboru Endou, Yasumasa Suzuki, and Yasunari Shidama. Real linear space of real sequences. *Formalized Mathematics*, 11(**3**):249–253, 2003.

[9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[10] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.

[11] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.

[12] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.

[13] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[14] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[15] Henryk Oryszczyszyn and Krzysztof Prażmowski. Real functions spaces. *Formalized Mathematics*, 1(**3**):555–561, 1990.

[16] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[17] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[18] Jan Popiołek. Introduction to Banach and Hilbert spaces - part I. *Formalized Mathematics*, 2(**4**):511–516, 1991.

[19] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.

[20] Yasumasa Suzuki, Noboru Endou, and Yasunari Shidama. Banach space of absolute summable real sequences. *Formalized Mathematics*, 11(**4**):377–380, 2003.

[21] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*

[22] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[23] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[24] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.

[25] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.

[26] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[27] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

[28] Andrzej Trybulec. On the sets inhabited by numbers. *Formalized Mathematics*, 11(**4**):341–347, 2003.

[29] Wojciech A. Trybulec. Subspaces and cosets of subspaces in real linear space. *Formalized Mathematics*, 1(**2**):297–301, 1990.

[30] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[31] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[32] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[33] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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