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Improvement of Radix-2^k Signed-Digit Number for High Speed Circuit

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Summary. In this article, a new radix-2^k signed-digit number (Radix-2^k sub signed-digit number) is defined and its properties for hardware realization are discussed.

Until now, high speed calculation method with Radix-2^k signed-digit numbers is proposed, but this method used "Compares With 2" to calculate carry. "Compares with 2" is a very simple method, but it needs very complicated hardware especially when the value of k becomes large. In this article, we propose a subset of Radix-2^k signed-digit, named Radix-2^k sub signed-digit numbers. Radix-2^k sub signed-digit was designed so that the carry calculation use "bit compare" to hardware-realization simplifies more.

In the first section of this article, we defined the concept of Radix-2^k sub signed-digit numbers and proved some of their properties. In the second section, we defined the new carry calculation method in consideration of hardware-realization, and proved some of their properties. In the third section, we provide some functions for generating Radix-2^k sub signed-digit numbers from Radix-2^k signed-digit numbers. In the last section, we defined some functions for generation natural numbers from Radix-2^k sub signed-digit, and we clarified its correctness.

MML Identifier: `RADIX_3`.

The articles [11], [14], [8], [12], [1], [4], [3], [13], [10], [7], [2], [9], [5], and [6] provide the notation and terminology for this paper.

1. Definition for Radix-2^k Sub Signed-Digit Number

We adopt the following convention: i, n, m, k, x are natural numbers and i_1, i_2 are integers.

Next we state the proposition

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(1) $((\mathrm{Radix}\,k)^n_{\mathbb{N}}) \cdot \mathrm{Radix}\,k = (\mathrm{Radix}\,k)^{n+1}_{\mathbb{N}}$.

Let us consider k. The functor k –SD_Sub_S is defined as follows:

(Def. 1) k –SD_Sub_S $= \{e; e$ ranges over elements of $\mathbb{Z}$: $e \geqslant -\mathrm{Radix}(k -' 1) \wedge e \leqslant \mathrm{Radix}(k -' 1) - 1\}$.

Let us consider k. The functor k –SD_Sub is defined by:

(Def. 2) k –SD_Sub $= \{e; e$ ranges over elements of $\mathbb{Z}$: $e \geqslant -\mathrm{Radix}(k -' 1) - 1 \wedge e \leqslant \mathrm{Radix}(k -' 1)\}$.

The following propositions are true:

(2) If $i_1 \in k$ –SD_Sub, then $-\mathrm{Radix}(k -' 1) - 1 \leqslant i_1$ and $i_1 \leqslant \mathrm{Radix}(k -' 1)$.

(3) For every natural number k holds k –SD_Sub_S $\subseteq k$ –SD_Sub.

(4) k –SD_Sub_S $\subseteq (k+1)$ –SD_Sub_S.

(5) For every natural number k such that $2 \leqslant k$ holds k –SD_Sub $\subseteq k$ –SD.

(6) $0 \in 0$ –SD_Sub_S.

(7) $0 \in k$ –SD_Sub_S.

(8) $0 \in k$ –SD_Sub.

(9) For every set e such that $e \in k$ –SD_Sub holds e is an integer.

(10) k –SD_Sub $\subseteq \mathbb{Z}$.

(11) k –SD_Sub_S $\subseteq \mathbb{Z}$.

Let us consider k. One can verify that k –SD_Sub_S is non empty.

Let us consider k. Note that k –SD_Sub is non empty.

Let us consider k. Then k –SD_Sub_S is a non empty subset of $\mathbb{Z}$.

Let us consider k. Then k –SD_Sub is a non empty subset of $\mathbb{Z}$.

In the sequel a denotes a n-tuple of k –SD and a_1 denotes a n-tuple of k –SD_Sub.

One can prove the following proposition

(12) If $i \in \mathrm{Seg}\,n$, then $a_1(i)$ is an element of k –SD_Sub.

2. Definition for New Carry Calculation Method

Let x be an integer and let k be a natural number.

The functor SDSubAddCarry(x,k) yields an integer and is defined as follows:

(Def. 3) $$\mathrm{SDSubAddCarry}(x,k) = \begin{cases} 1, & \text{if } \mathrm{Radix}(k -' 1) \leqslant x, \\ -1, & \text{if } x < -\mathrm{Radix}(k -' 1), \\ 0, & \text{otherwise.} \end{cases}$$

Let x be an integer and let k be a natural number.

The functor SDSubAddData(x,k) yields an integer and is defined as follows:

(Def. 4) $\mathrm{SDSubAddData}(x,k) = x - \mathrm{Radix}\,k \cdot \mathrm{SDSubAddCarry}(x,k)$.

One can prove the following propositions:

(13) For every integer x and for every natural number k such that $2 \leqslant k$ holds $-1 \leqslant \mathrm{SDSubAddCarry}(x,k)$ and $\mathrm{SDSubAddCarry}(x,k) \leqslant 1$.

(14) If $2 \leqslant k$ and $i_1 \in k\,$–SD, then $\mathrm{SDSubAddData}(i_1,k) \geqslant -\mathrm{Radix}(k -' 1)$ and $\mathrm{SDSubAddData}(i_1,k) \leqslant \mathrm{Radix}(k -' 1) - 1$.

(15) For every integer x and for every natural number k such that $2 \leqslant k$ holds $\mathrm{SDSubAddCarry}(x,k) \in k\,$–SD_Sub_S .

(16) If $2 \leqslant k$ and $i_1 \in k\,$–SD and $i_2 \in k\,$–SD, then $\mathrm{SDSubAddData}(i_1,k) + \mathrm{SDSubAddCarry}(i_2,k) \in k\,$–SD_Sub .

(17) If $2 \leqslant k$, then $\mathrm{SDSubAddCarry}(0,k) = 0$.

3. Definition for Translation from Radix-2^k Signed-Digit Number

Let i, k, n be natural numbers and let x be a n-tuple of $k\,$–SD_Sub. The functor DigA_SDSub(x,i) yields an integer and is defined as follows:

(Def. 5)(i) DigA_SDSub$(x,i) = x(i)$ if $i \in \mathrm{Seg}\,n$,

(ii) DigA_SDSub$(x,i) = 0$ if $i = 0$.

Let i, k, n be natural numbers and let x be a n-tuple of $k\,$–SD. The functor SD2SDSubDigit(x,i,k) yields an integer and is defined by:

(Def. 6)
$$\mathrm{SD2SDSubDigit}(x,i,k) = \begin{cases} \text{(i) } \mathrm{SDSubAddData}(\mathrm{DigA}(x,i),k) + \\ \quad \mathrm{SDSubAddCarry}(\mathrm{DigA}(x,i -' 1),k), \\ \quad \text{if } i \in \mathrm{Seg}\,n, \\ \text{(ii) } \mathrm{SDSubAddCarry}(\mathrm{DigA}(x,i -' 1),k), \\ \quad \text{if } i = n+1, \\ 0, \text{ otherwise.} \end{cases}$$

We now state the proposition

(18) If $2 \leqslant k$ and $i \in \mathrm{Seg}(n+1)$, then SD2SDSubDigit(a,i,k) is an element of $k\,$–SD_Sub.

Let i, k, n be natural numbers and let x be a n-tuple of $k\,$–SD. Let us assume that $2 \leqslant k$ and $i \in \mathrm{Seg}(n+1)$. The functor SD2SDSubDigitS(x,i,k) yielding an element of $k\,$–SD_Sub is defined by:

(Def. 7) SD2SDSubDigitS(x,i,k) = SD2SDSubDigit(x,i,k).

Let n, k be natural numbers and let x be a n-tuple of $k\,$–SD. The functor SD2SDSub x yielding a $n+1$-tuple of $k\,$–SD_Sub is defined by:

(Def. 8) For every natural number i such that $i \in \mathrm{Seg}(n+1)$ holds DigA_SDSub(SD2SDSub x, i) = SD2SDSubDigitS(x,i,k).

Next we state two propositions:

(19) If $i \in \mathrm{Seg}\,n$, then DigA_SDSub(a_1,i) is an element of $k\,$–SD_Sub.

(20) If $2 \leqslant k$ and $i_1 \in k\,$–SD and $i_2 \in k\,$–SD_Sub, then $\mathrm{SDSubAddData}(i_1 + i_2,k) \in k\,$–SD_Sub_S .

4. Definiton for Translation from Radix-2^k Sub Signed-Digit Number to INT

Let i, k, n be natural numbers and let x be a n-tuple of k −SD_Sub. The functor DigB_SDSub(x, i) yielding an element of $\mathbb{Z}$ is defined by:

(Def. 9) DigB_SDSub$(x, i) =$ DigA_SDSub(x, i).

Let i, k, n be natural numbers and let x be a n-tuple of k −SD_Sub. The functor SDSub2INTDigit(x, i, k) yielding an element of $\mathbb{Z}$ is defined as follows:

(Def. 10) $\mathrm{SDSub2INTDigit}(x, i, k) = ((\mathrm{Radix}\, k)_{\mathbb{N}}^{i -' 1}) \cdot \mathrm{DigB\_SDSub}(x, i)$.

Let n, k be natural numbers and let x be a n-tuple of k −SD_Sub. The functor SDSub2INT x yields a n-tuple of $\mathbb{Z}$ and is defined as follows:

(Def. 11) For every natural number i such that $i \in \mathrm{Seg}\, n$ holds $(\mathrm{SDSub2INT}\, x)_i =$ SDSub2INTDigit(x, i, k).

Let n, k be natural numbers and let x be a n-tuple of k −SD_Sub. The functor SDSub2IntOut x yields an integer and is defined as follows:

(Def. 12) SDSub2IntOut $x = \sum$ SDSub2INT x.

Next we state two propositions:

(21) For every i such that $i \in \mathrm{Seg}\, n$ holds if $2 \leqslant k$, then DigA_SDSub(SD2SDSub DecSD$(m, n + 1, k), i) =$ DigA_SDSub(SD2SDSub DecSD$(m \bmod (\mathrm{Radix}\, k)_{\mathbb{N}}^{n}, n, k), i)$.

(22) For every n such that $n \geqslant 1$ and for all k, x such that $k \geqslant 2$ and x is represented by n, k holds $x =$ SDSub2IntOut SD2SDSub DecSD(x, n, k).

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High Speed Adder Algorithm with Radix-2^k Sub Signed-Digit Number

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Summary. In this article, a new adder algorithm using Radix-2^k sub signed-digit numbers is defined and properties for the hardware-realization is discussed.

Until now, we proposed Radix-2^k sub signed-digit numbers in consideration of the hardware realization. In this article, we proposed High Speed Adder Algorithm using this Radix-2^k sub signed-digit numbers. This method has two ways to speed up at hardware-realization. One is 'bit compare' at carry calculation, it is proposed in another article. Other is carry calculation between two numbers. We proposed that n digits Radix-2^k signed-digit numbers is expressed in $n+1$ digits Radix-2^k sub signed-digit numbers, and addition result of two $n+1$ digits Radix-2^k sub signed-digit numbers is expressed in $n+1$ digits. In this way, carry operation between two Radix-2^k sub signed-digit numbers can be processed at $n+1$ digit adder circuit and additional circuit to operate carry is not needed.

In the first section of this article, we prepared some useful theorems for operation of Radix-2^k numbers. In the second section, we proved some properties about carry on Radix-2^k sub signed-digit numbers. In the last section, we defined the new addition operation using Radix-2^k sub signed-digit numbers, and we clarified its correctness.

MML Identifier: `RADIX_4`.

The terminology and notation used here are introduced in the following articles: [11], [13], [12], [1], [4], [3], [10], [7], [2], [8], [5], [6], and [9].

1. Preliminaries

In this paper i, n, m, k, x, y are natural numbers.

The following proposition is true

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(1) For every natural number k such that $2 \leqslant k$ holds $2 < \mathrm{Radix}\,k$.

2. Carry Operation at $n+1$ Digits Radix-2^k Sub Signed-Digit Number

The following propositions are true:

(2) For all integers x, y and for every natural number k such that $3 \leqslant k$ holds $\mathrm{SDSubAddCarry}(\mathrm{SDSubAddCarry}(x,k) + \mathrm{SDSubAddCarry}(y,k),k) = 0$.

(3) If $2 \leqslant k$, then $\mathrm{DigA\_SDSub}(\mathrm{SD2SDSub\,DecSD}(m,n,k),n+1) = \mathrm{SDSubAddCarry}(\mathrm{DigA}(\mathrm{DecSD}(m,n,k),n),k)$.

(4) If $2 \leqslant k$ and m is represented by 1, k, then $\mathrm{DigA\_SDSub}(\mathrm{SD2SDSub\,DecSD}(m,1,k),1+1) = \mathrm{SDSubAddCarry}(m,k)$.

(5) Let k, x, n be natural numbers. Suppose $n \geqslant 1$ and $k \geqslant 3$ and x is represented by $n+1$, k. Then $\mathrm{DigA\_SDSub}(\mathrm{SD2SDSub\,DecSD}(x \bmod (\mathrm{Radix}\,k)^n_{\mathbb{N}},n,k),n+1) = \mathrm{SDSubAddCarry}(\mathrm{DigA}(\mathrm{DecSD}(x,n,k),n),k)$.

(6) If $2 \leqslant k$ and m is represented by 1, k, then $\mathrm{DigA\_SDSub}(\mathrm{SD2SDSub\,DecSD}(m,1,k),1) = m - \mathrm{SDSubAddCarry}(m,k) \cdot \mathrm{Radix}\,k$.

(7) Let k, x, n be natural numbers. Suppose $n \geqslant 1$ and $k \geqslant 2$ and x is represented by $n+1$, k. Then $((\mathrm{Radix}\,k)^n_{\mathbb{N}}) \cdot \mathrm{DigA\_SDSub}(\mathrm{SD2SDSub\,DecSD}(x,n+1,k),n+1) = (((\mathrm{Radix}\,k)^n_{\mathbb{N}}) \cdot \mathrm{DigA}(\mathrm{DecSD}(x,n+1,k),n+1) - ((\mathrm{Radix}\,k)^{n+1}_{\mathbb{N}}) \cdot \mathrm{SDSubAddCarry}(\mathrm{DigA}(\mathrm{DecSD}(x,n+1,k),n+1),k)) + ((\mathrm{Radix}\,k)^n_{\mathbb{N}}) \cdot \mathrm{SDSubAddCarry}(\mathrm{DigA}(\mathrm{DecSD}(x,n+1,k),n),k)$.

3. Definition for Adder Operation on Radix-2^k Sub Signed-Digit Number

Let i, n, k be natural numbers, let x be a n-tuple of k −SD_Sub, and let y be a n-tuple of k −SD_Sub. Let us assume that $i \in \mathrm{Seg}\,n$ and $k \geqslant 2$. The functor $\mathrm{SDSubAddDigit}(x,y,i,k)$ yields an element of k −SD_Sub and is defined as follows:

(Def. 1) $\mathrm{SDSubAddDigit}(x,y,i,k) = \mathrm{SDSubAddData}(\mathrm{DigA\_SDSub}(x,i) + \mathrm{DigA\_SDSub}(y,i),k) + \mathrm{SDSubAddCarry}(\mathrm{DigA\_SDSub}(x,i -' 1) + \mathrm{DigA\_SDSub}(y,i -' 1),k)$.

Let n, k be natural numbers and let x, y be n-tuples of k −SD_Sub. The functor $x' +' y$ yields a n-tuple of k −SD_Sub and is defined by:

(Def. 2) For every i such that $i \in \mathrm{Seg}\,n$ holds $\mathrm{DigA\_SDSub}(x' +' y,i) = \mathrm{SDSubAddDigit}(x,y,i,k)$.

Next we state two propositions:

(8) For every i such that $i \in \operatorname{Seg} n$ holds if $2 \leqslant k$, then $\mathrm{SDSubAddDigit}(\mathrm{SD2SDSub\,DecSD}(x, n+1, k), \mathrm{SD2SDSub\,DecSD}(y, n+1, k), i, k) = \mathrm{SDSubAddDigit}(\mathrm{SD2SDSub\,DecSD}(x \bmod (\mathrm{Radix}\,k)^n_{\mathbb{N}}, n, k), \mathrm{SD2SDSub\,DecSD}(y \bmod (\mathrm{Radix}\,k)^n_{\mathbb{N}}, n, k), i, k)$.

(9) Let given n. Suppose $n \geqslant 1$. Let given k, x, y. Suppose $k \geqslant 3$ and x is represented by n, k and y is represented by n, k. Then $x + y = \mathrm{SDSub2IntOut}\,\mathrm{SD2SDSub\,DecSD}(x, n, k)' +' \mathrm{SD2SDSub\,DecSD}(y, n, k)$.

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The Underlying Principle of Dijkstra's Shortest Path Algorithm

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Summary. A path from a source vertex v to a target vertex u is said to be a shortest path if its total cost is minimum among all v-to-u paths. Dijkstra's algorithm is a classic shortest path algorithm, which is described in many textbooks. To justify its correctness (whose rigorous proof will be given in the next article), it is necessary to clarify its underlying principle. For this purpose, the article justifies the following basic facts, which are the core of Dijkstra's algorithm.

- A graph is given, its vertex set is denoted by V. Assume U is the subset of V, and if a path p from s to t is the shortest among the set of paths, each of which passes through only the vertices in U, except the source and sink, and its source and sink is s and in V, respectively, then p is a shortest path from s to t in the graph, and for any subgraph which contains at least U, it is also the shortest.
- Let $p(s,x,U)$ denote the shortest path from s to x in a subgraph whose the vertex set is the union of $\{s,x\}$ and U, and cost (p) denote the cost of path $p(s,x,U)$, $\mathrm{cost}(x,y)$ the cost of the edge from x to y. Give $p(s,x,U)$, $q(s,y,U)$ and $r(s,y,U \cup \{x\})$. If $\mathrm{cost}(p) = \min\{\mathrm{cost}(w) : w(s,t,U) \wedge t \in V\}$, then we have

$$\mathrm{cost}(r) = \min(\mathrm{cost}(p) + \mathrm{cost}(x,y), \mathrm{cost}(q)).$$

This is the well-known triangle comparison of Dijkstra's algorithm.

MML Identifier: `GRAPH_5`.

The articles [14], [16], [13], [17], [5], [3], [4], [15], [1], [8], [9], [2], [10], [6], [12], [7], and [11] provide the terminology and notation for this paper.

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1. Preliminaries

We follow the rules: n, m, i, j, k denote natural numbers, x, y, e, X, V, U denote sets, and W, f, g denote functions.

Let f be a finite function. Observe that $\operatorname{rng} f$ is finite.

One can prove the following two propositions:

(1) For every finite function f holds $\operatorname{card}\operatorname{rng} f \leqslant \operatorname{card}\operatorname{dom} f$.

(2) If $\operatorname{rng} f \subseteq \operatorname{rng} g$ and $x \in \operatorname{dom} f$, then there exists y such that $y \in \operatorname{dom} g$ and $f(x) = g(y)$.

The scheme *LambdaAB* deals with sets $\mathcal{A}$, $\mathcal{B}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

> There exists a function f such that $\operatorname{dom} f = \mathcal{A}$ and for every element x of $\mathcal{B}$ such that $x \in \mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

for all values of the parameters.

The following propositions are true:

(3) Let D be a finite set, n be a natural number, and X be a set. If $X = \{x; x$ ranges over elements of D^*: $1 \leqslant \operatorname{len} x \ \wedge\ \operatorname{len} x \leqslant n\}$, then X is finite.

(4) Let D be a finite set, n be a natural number, and X be a set. If $X = \{x; x$ ranges over elements of D^*: $\operatorname{len} x \leqslant n\}$, then X is finite.

(5) For every finite set D holds $\operatorname{card} D \neq 0$ iff $D \neq \emptyset$.

(6) Let D be a finite set and k be a natural number. Suppose $\operatorname{card} D = k+1$. Then there exists an element x of D and there exists a subset C of D such that $D = C \cup \{x\}$ and $\operatorname{card} C = k$.

(7) For every finite set D such that $\operatorname{card} D = 1$ there exists an element x of D such that $D = \{x\}$.

The scheme *MinValue* deals with a non empty finite set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a real number, and states that:

> There exists an element x of $\mathcal{A}$ such that for every element y of $\mathcal{A}$ holds $\mathcal{F}(x) \leqslant \mathcal{F}(y)$

for all values of the parameters.

Let D be a set and let X be a non empty subset of D^*. We see that the element of X is a finite sequence of elements of D.

2. Additional Properties of Finite Sequences

In the sequel p, q are finite sequences.

One can prove the following propositions:

(8) $p \neq \emptyset$ iff $\operatorname{len} p \geqslant 1$.

(9) For all n, m such that $1 \leqslant n$ and $n < m$ and $m \leqslant \operatorname{len} p$ holds $p(n) \neq p(m)$ iff p is one-to-one.

(10) For all n, m such that $1 \leqslant n$ and $n < m$ and $m \leqslant \operatorname{len} p$ holds $p(n) \neq p(m)$ iff $\operatorname{card} \operatorname{rng} p = \operatorname{len} p$.

In the sequel G denotes a graph and p_1, q_1 denote finite sequences of elements of the edges of G.

Next we state two propositions:

(11) If $i \in \operatorname{dom} p_1$, then (the source of G)($p_1(i)$) $\in$ the vertices of G and (the target of G)($p_1(i)$) $\in$ the vertices of G.

(12) If $q \frown \langle x \rangle$ is one-to-one and $\operatorname{rng}(q \frown \langle x \rangle) \subseteq \operatorname{rng} p$, then there exist finite sequences p_2, p_3 such that $p = p_2 \frown \langle x \rangle \frown p_3$ and $\operatorname{rng} q \subseteq \operatorname{rng}(p_2 \frown p_3)$.

3. Additional Properties of Chains and Oriented Paths

One can prove the following three propositions:

(13) If $p \frown q$ is a chain of G, then p is a chain of G and q is a chain of G.

(14) If $p \frown q$ is an oriented chain of G, then p is an oriented chain of G and q is an oriented chain of G.

(15) Let p, q be oriented chains of G. Suppose (the target of G)($p(\operatorname{len} p)$) = (the source of G)($q(1)$). Then $p \frown q$ is an oriented chain of G.

4. Additional Properties of Acyclic Oriented Paths

The following propositions are true:

(16) $\emptyset$ is a Simple oriented chain of G.

(17) Suppose $p \frown q$ is a Simple oriented chain of G. Then p is a Simple oriented chain of G and q is a Simple oriented chain of G.

(18) If $\operatorname{len} p_1 = 1$, then p_1 is a Simple oriented chain of G.

(19) Let p be a Simple oriented chain of G and q be a finite sequence of elements of the edges of G. Suppose that
  (i) $\operatorname{len} p \geqslant 1$,
  (ii) $\operatorname{len} q = 1$,
  (iii) (the source of G)($q(1)$) = (the target of G)($p(\operatorname{len} p)$),
  (iv) (the source of G)($p(1)$) $\neq$ (the target of G)($p(\operatorname{len} p)$), and
  (v) it is not true that there exists k such that $1 \leqslant k$ and $k \leqslant \operatorname{len} p$ and (the target of G)($p(k)$) = (the target of G)($q(1)$).

  Then $p \frown q$ is a Simple oriented chain of G.

(20) Every Simple oriented chain of G is one-to-one.

5. The Set of the Vertices On a Path or an Edge

Let G be a graph and let e be an element of the edges of G. The functor vertices e is defined as follows:

(Def. 1) vertices $e = \{(\text{the source of } G)(e), (\text{the target of } G)(e)\}$.

Let us consider G, p_1. The functor vertices p_1 yields a subset of the vertices of G and is defined by:

(Def. 2) vertices $p_1 = \{v; v$ ranges over vertices of G: $\bigvee_i$ $(i \in \operatorname{dom} p_1 \ \wedge \ v \in \operatorname{vertices}((p_1)_i))\}$.

We now state several propositions:

(21) Let p be a Simple oriented chain of G. Suppose $p = p_1 \frown q_1$ and $\operatorname{len} p_1 \geqslant 1$ and $\operatorname{len} q_1 \geqslant 1$ and $(\text{the source of } G)(p(1)) \neq (\text{the target of } G)(p(\operatorname{len} p))$. Then $(\text{the source of } G)(p(1)) \notin$ vertices q_1 and $(\text{the target of } G)(p(\operatorname{len} p)) \notin$ vertices p_1.

(22) vertices $p_1 \subseteq V$ iff for every i such that $i \in \operatorname{dom} p_1$ holds $\operatorname{vertices}((p_1)_i) \subseteq V$.

(23) Suppose vertices $p_1 \not\subseteq V$. Then there exists a natural number i and there exist finite sequences q, r of elements of the edges of G such that $i + 1 \leqslant \operatorname{len} p_1$ and $\operatorname{vertices}((p_1)_{i+1}) \not\subseteq V$ and $\operatorname{len} q = i$ and $p_1 = q \frown r$ and vertices $q \subseteq V$.

(24) If $\operatorname{rng} q_1 \subseteq \operatorname{rng} p_1$, then vertices $q_1 \subseteq$ vertices p_1.

(25) If $\operatorname{rng} q_1 \subseteq \operatorname{rng} p_1$ and vertices $p_1 \setminus X \subseteq V$, then vertices $q_1 \setminus X \subseteq V$.

(26) If $\operatorname{vertices}(p_1 \frown q_1) \setminus X \subseteq V$, then vertices $p_1 \setminus X \subseteq V$ and vertices $q_1 \setminus X \subseteq V$.

In the sequel v, v_1, v_2, v_3 denote elements of the vertices of G.

One can prove the following four propositions:

(27) For every element e of the edges of G such that $v = (\text{the source of } G)(e)$ or $v = (\text{the target of } G)(e)$ holds $v \in$ vertices e.

(28) If $i \in \operatorname{dom} p_1$ and if $v = (\text{the source of } G)(p_1(i))$ or $v = (\text{the target of } G)(p_1(i))$, then $v \in$ vertices p_1.

(29) If $\operatorname{len} p_1 = 1$, then vertices $p_1 = \operatorname{vertices}((p_1)_1)$.

(30) vertices $p_1 \subseteq \operatorname{vertices}(p_1 \frown q_1)$ and vertices $q_1 \subseteq \operatorname{vertices}(p_1 \frown q_1)$.

In the sequel p, q are oriented chains of G.

Next we state two propositions:

(31) If $p = q \frown p_1$ and $\operatorname{len} q \geqslant 1$ and $\operatorname{len} p_1 = 1$, then vertices $p =$ vertices $q \cup \{(\text{the target of } G)(p_1(1))\}$.

(32) If $v \neq (\text{the source of } G)(p(1))$ and $v \in$ vertices p, then there exists i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} p$ and $v = (\text{the target of } G)(p(i))$.

6. Directed Paths between Two Vertices

Let us consider G, p, v_1, v_2. We say that p is oriented path from v_1 to v_2 if and only if:

(Def. 3) $p \neq \emptyset$ and (the source of G)$(p(1)) = v_1$ and (the target of G)$(p(\operatorname{len} p)) = v_2$.

Let us consider G, v_1, v_2, p, V. We say that p is oriented path from v_1 to v_2 in V if and only if:

(Def. 4) p is oriented path from v_1 to v_2 and $\operatorname{vertices} p \setminus \{v_2\} \subseteq V$.

Let G be a graph and let v_1, v_2 be elements of the vertices of G. The functor $\operatorname{OrientedPaths}(v_1, v_2)$ yields a subset of (the edges of G)* and is defined by:

(Def. 5) $\operatorname{OrientedPaths}(v_1, v_2) = \{p; p$ ranges over oriented chains of G: p is oriented path from v_1 to $v_2\}$.

Next we state several propositions:

(33) If p is oriented path from v_1 to v_2, then $v_1 \in \operatorname{vertices} p$ and $v_2 \in \operatorname{vertices} p$.

(34) $x \in \operatorname{OrientedPaths}(v_1, v_2)$ iff there exists p such that $p = x$ and p is oriented path from v_1 to v_2.

(35) If p is oriented path from v_1 to v_2 in V and $v_1 \neq v_2$, then $v_1 \in V$.

(36) If p is oriented path from v_1 to v_2 in V and $V \subseteq U$, then p is oriented path from v_1 to v_2 in U.

(37) Suppose $\operatorname{len} p \geqslant 1$ and p is oriented path from v_1 to v_2 and $p_1(1)$ orientedly joins v_2, v_3 and $\operatorname{len} p_1 = 1$. Then there exists q such that $q = p \frown p_1$ and q is oriented path from v_1 to v_3.

(38) Suppose $q = p \frown p_1$ and $\operatorname{len} p \geqslant 1$ and $\operatorname{len} p_1 = 1$ and p is oriented path from v_1 to v_2 in V and $p_1(1)$ orientedly joins v_2, v_3. Then q is oriented path from v_1 to v_3 in $V \cup \{v_2\}$.

7. Acyclic (or Simple) Paths

Let G be a graph, let p be an oriented chain of G, and let v_1, v_2 be elements of the vertices of G. We say that p is acyclic path from v_1 to v_2 if and only if:

(Def. 6) p is Simple and oriented path from v_1 to v_2.

Let G be a graph, let p be an oriented chain of G, let v_1, v_2 be elements of the vertices of G, and let V be a set. We say that p is acyclic path from v_1 to v_2 in V if and only if:

(Def. 7) p is Simple and oriented path from v_1 to v_2 in V.

Let G be a graph and let v_1, v_2 be elements of the vertices of G. The functor $\operatorname{AcyclicPaths}(v_1, v_2)$ yielding a subset of (the edges of G)* is defined as follows:

(Def. 8) $\mathrm{AcyclicPaths}(v_1, v_2) = \{p; p$ ranges over Simple oriented chains of G: p is acyclic path from v_1 to $v_2\}$.

Let G be a graph, let v_1, v_2 be elements of the vertices of G, and let V be a set. The functor $\mathrm{AcyclicPaths}(v_1, v_2, V)$ yielding a subset of (the edges of G)* is defined as follows:

(Def. 9) $\mathrm{AcyclicPaths}(v_1, v_2, V) = \{p; p$ ranges over Simple oriented chains of G: p is acyclic path from v_1 to v_2 in $V\}$.

Let G be a graph and let p be an oriented chain of G. The functor $\mathrm{AcyclicPaths}(p)$ yielding a subset of (the edges of G)* is defined by the condition (Def. 10).

(Def. 10) $\mathrm{AcyclicPaths}(p) = \{q; q$ ranges over Simple oriented chains of G: $q \neq \emptyset \wedge$ (the source of G)$(q(1)) =$ (the source of G)$(p(1)) \wedge$ (the target of G)$(q(\mathrm{len}\, q)) =$ (the target of G)$(p(\mathrm{len}\, p)) \wedge \mathrm{rng}\, q \subseteq \mathrm{rng}\, p\}$.

Let G be a graph. The functor $\mathrm{AcyclicPaths}(G)$ yields a subset of (the edges of G)* and is defined as follows:

(Def. 11) $\mathrm{AcyclicPaths}(G) = \{q : q$ ranges over Simple oriented chains of $G\}$.

The following propositions are true:

(39) If $p = \emptyset$, then p is not acyclic path from v_1 to v_2.

(40) If p is acyclic path from v_1 to v_2, then p is oriented path from v_1 to v_2.

(41) $\mathrm{AcyclicPaths}(v_1, v_2) \subseteq \mathrm{OrientedPaths}(v_1, v_2)$.

(42) $\mathrm{AcyclicPaths}(p) \subseteq \mathrm{AcyclicPaths}(G)$.

(43) $\mathrm{AcyclicPaths}(v_1, v_2) \subseteq \mathrm{AcyclicPaths}(G)$.

(44) If p is oriented path from v_1 to v_2, then $\mathrm{AcyclicPaths}(p) \subseteq \mathrm{AcyclicPaths}(v_1, v_2)$.

(45) If p is oriented path from v_1 to v_2 in V, then $\mathrm{AcyclicPaths}(p) \subseteq \mathrm{AcyclicPaths}(v_1, v_2, V)$.

(46) If $q \in \mathrm{AcyclicPaths}(p)$, then $\mathrm{len}\, q \leqslant \mathrm{len}\, p$.

(47) If p is oriented path from v_1 to v_2, then $\mathrm{AcyclicPaths}(p) \neq \emptyset$ and $\mathrm{AcyclicPaths}(v_1, v_2) \neq \emptyset$.

(48) If p is oriented path from v_1 to v_2 in V, then $\mathrm{AcyclicPaths}(p) \neq \emptyset$ and $\mathrm{AcyclicPaths}(v_1, v_2, V) \neq \emptyset$.

(49) $\mathrm{AcyclicPaths}(v_1, v_2, V) \subseteq \mathrm{AcyclicPaths}(G)$.

8. Weight Graphs and Their Basic Properties

The subset $\mathbb{R}_{\geqslant 0}$ of $\mathbb{R}$ is defined by:

(Def. 12) $\mathbb{R}_{\geqslant 0} = \{r; r$ ranges over real numbers: $r \geqslant 0\}$.

Let us mention that $\mathbb{R}_{\geqslant 0}$ is non empty.

Let G be a graph and let W be a function. We say that W is nonnegative weight of G if and only if:

(Def. 13) W is a function from the edges of G into $\mathbb{R}_{\geqslant 0}$.

Let G be a graph and let W be a function. We say that W is weight of G if and only if:

(Def. 14) W is a function from the edges of G into $\mathbb{R}$.

Let G be a graph, let p be a finite sequence of elements of the edges of G, and let W be a function. Let us assume that W is weight of G. The functor $\mathrm{RealSequence}(p, W)$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

(Def. 15) $\operatorname{dom} p = \operatorname{dom} \mathrm{RealSequence}(p, W)$ and for every natural number i such that $i \in \operatorname{dom} p$ holds $(\mathrm{RealSequence}(p, W))(i) = W(p(i))$.

Let G be a graph, let p be a finite sequence of elements of the edges of G, and let W be a function. The functor $\mathrm{cost}(p, W)$ yields a real number and is defined as follows:

(Def. 16) $\mathrm{cost}(p, W) = \sum \mathrm{RealSequence}(p, W)$.

Next we state a number of propositions:

(50) If W is nonnegative weight of G, then W is weight of G.

(51) Let f be a finite sequence of elements of $\mathbb{R}$. Suppose W is nonnegative weight of G and $f = \mathrm{RealSequence}(p_1, W)$. Let given i. If $i \in \operatorname{dom} f$, then $f(i) \geqslant 0$.

(52) If $\operatorname{rng} q_1 \subseteq \operatorname{rng} p_1$ and W is weight of G and $i \in \operatorname{dom} q_1$, then there exists j such that $j \in \operatorname{dom} p_1$ and $(\mathrm{RealSequence}(p_1, W))(j) = (\mathrm{RealSequence}(q_1, W))(i)$.

(53) If $\operatorname{len} q_1 = 1$ and $\operatorname{rng} q_1 \subseteq \operatorname{rng} p_1$ and W is nonnegative weight of G, then $\mathrm{cost}(q_1, W) \leqslant \mathrm{cost}(p_1, W)$.

(54) If W is nonnegative weight of G, then $\mathrm{cost}(p_1, W) \geqslant 0$.

(55) If $p_1 = \emptyset$ and W is weight of G, then $\mathrm{cost}(p_1, W) = 0$.

(56) Let D be a non empty finite subset of (the edges of $G)^*$. If $D = \mathrm{AcyclicPaths}(v_1, v_2)$, then there exists p_1 such that $p_1 \in D$ and for every q_1 such that $q_1 \in D$ holds $\mathrm{cost}(p_1, W) \leqslant \mathrm{cost}(q_1, W)$.

(57) Let D be a non empty finite subset of (the edges of $G)^*$. If $D = \mathrm{AcyclicPaths}(v_1, v_2, V)$, then there exists p_1 such that $p_1 \in D$ and for every q_1 such that $q_1 \in D$ holds $\mathrm{cost}(p_1, W) \leqslant \mathrm{cost}(q_1, W)$.

(58) If W is weight of G, then $\mathrm{cost}(p_1 \frown q_1, W) = \mathrm{cost}(p_1, W) + \mathrm{cost}(q_1, W)$.

(59) If q_1 is one-to-one and $\operatorname{rng} q_1 \subseteq \operatorname{rng} p_1$ and W is nonnegative weight of G, then $\mathrm{cost}(q_1, W) \leqslant \mathrm{cost}(p_1, W)$.

(60) If $p_1 \in \text{AcyclicPaths}(p)$ and W is nonnegative weight of G, then $\text{cost}(p_1, W) \leqslant \text{cost}(p, W)$.

9. Shortest Paths and Their Basic Properties

Let G be a graph, let v_1, v_2 be vertices of G, let p be an oriented chain of G, and let W be a function. We say that p is shortest path from v_1 to v_2 in W if and only if the conditions (Def. 17) are satisfied.

(Def. 17)(i) p is oriented path from v_1 to v_2, and

(ii) for every oriented chain q of G such that q is oriented path from v_1 to v_2 holds $\text{cost}(p, W) \leqslant \text{cost}(q, W)$.

Let G be a graph, let v_1, v_2 be vertices of G, let p be an oriented chain of G, let V be a set, and let W be a function. We say that p is shortest path from v_1 to v_2 in V w.r.t. W if and only if the conditions (Def. 18) are satisfied.

(Def. 18)(i) p is oriented path from v_1 to v_2 in V, and

(ii) for every oriented chain q of G such that q is oriented path from v_1 to v_2 in V holds $\text{cost}(p, W) \leqslant \text{cost}(q, W)$.

10. Basic Properties of a Graph with Finite Vertices

For simplicity, we adopt the following rules: G is a finite graph, p_4 is a Simple oriented chain of G, P, Q are oriented chains of G, v_1, v_2, v_3 are elements of the vertices of G, and p_1, q_1 are finite sequences of elements of the edges of G.

One can prove the following two propositions:

(61) $\text{len}\, p_4 \leqslant$ the number of vertices of G.

(62) $\text{len}\, p_4 \leqslant$ the number of edges of G.

Let us consider G. Note that $\text{AcyclicPaths}(G)$ is finite.

Let us consider G, P. Note that $\text{AcyclicPaths}(P)$ is finite.

Let us consider G, v_1, v_2. One can verify that $\text{AcyclicPaths}(v_1, v_2)$ is finite.

Let us consider G, v_1, v_2, V. Observe that $\text{AcyclicPaths}(v_1, v_2, V)$ is finite.

We now state four propositions:

(63) If $\text{AcyclicPaths}(v_1, v_2) \neq \emptyset$, then there exists p_1 such that $p_1 \in \text{AcyclicPaths}(v_1, v_2)$ and for every q_1 such that $q_1 \in \text{AcyclicPaths}(v_1, v_2)$ holds $\text{cost}(p_1, W) \leqslant \text{cost}(q_1, W)$.

(64) If $\text{AcyclicPaths}(v_1, v_2, V) \neq \emptyset$, then there exists p_1 such that $p_1 \in \text{AcyclicPaths}(v_1, v_2, V)$ and for every q_1 such that $q_1 \in \text{AcyclicPaths}(v_1, v_2, V)$ holds $\text{cost}(p_1, W) \leqslant \text{cost}(q_1, W)$.

(65) If P is oriented path from v_1 to v_2 and W is nonnegative weight of G, then there exists a Simple oriented chain of G which is shortest path from v_1 to v_2 in W.

(66) Suppose P is oriented path from v_1 to v_2 in V and W is nonnegative weight of G. Then there exists a Simple oriented chain of G which is shortest path from v_1 to v_2 in V w.r.t. W.

11. Three Basic Theorems for Dijkstra's Shortest Path Algorithm

We now state two propositions:

(67) Suppose that
(i) W is nonnegative weight of G,
(ii) P is shortest path from v_1 to v_2 in V w.r.t. W,
(iii) $v_1 \neq v_2$, and
(iv) for all Q, v_3 such that $v_3 \notin V$ and Q is shortest path from v_1 to v_3 in V w.r.t. W holds $\mathrm{cost}(P, W) \leqslant \mathrm{cost}(Q, W)$.
Then P is shortest path from v_1 to v_2 in W.

(68) Suppose that
(i) W is nonnegative weight of G,
(ii) P is shortest path from v_1 to v_2 in V w.r.t. W,
(iii) $v_1 \neq v_2$,
(iv) $V \subseteq U$, and
(v) for all Q, v_3 such that $v_3 \notin V$ and Q is shortest path from v_1 to v_3 in V w.r.t. W holds $\mathrm{cost}(P, W) \leqslant \mathrm{cost}(Q, W)$.
Then P is shortest path from v_1 to v_2 in U w.r.t. W.

Let G be a graph, let p_1 be a finite sequence of elements of the edges of G, let V be a set, let v_1 be a vertex of G, and let W be a function. We say that p_1 is longest in shortest path from v_1 in V w.r.t. W if and only if the condition (Def. 19) is satisfied.

(Def. 19) Let v be a vertex of G. Suppose $v \in V$ and $v \neq v_1$. Then there exists an oriented chain q of G such that q is shortest path from v_1 to v in V w.r.t. W and $\mathrm{cost}(q, W) \leqslant \mathrm{cost}(p_1, W)$.

One can prove the following proposition

(69) Let G be a finite oriented graph, P, Q, R be oriented chains of G, and v_1, v_2, v_3 be elements of the vertices of G such that $e \in$ the edges of G and W is nonnegative weight of G and $\mathrm{len}\, P \geqslant 1$ and P is shortest path from v_1 to v_2 in V w.r.t. W and $v_1 \neq v_2$ and $v_1 \neq v_3$ and $R = P \frown \langle e \rangle$ and Q is shortest path from v_1 to v_3 in V w.r.t. W and e orientedly joins v_2, v_3 and P is longest in shortest path from v_1 in V w.r.t. W. Then
(i) if $\mathrm{cost}(Q, W) \leqslant \mathrm{cost}(R, W)$, then Q is shortest path from v_1 to v_3 in $V \cup \{v_2\}$ w.r.t. W, and
(ii) if $\mathrm{cost}(Q, W) \geqslant \mathrm{cost}(R, W)$, then R is shortest path from v_1 to v_3 in $V \cup \{v_2\}$ w.r.t. W.

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On the Hausdorff Distance Between Compact Subsets[1]

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Summary. In [1] the pseudo-metric $\mathrm{dist}^{\max}_{\min}$ on compact subsets A and B of a topological space generated from arbitrary metric space is defined. Using this notion we define the Hausdorff distance (see e.g. [5]) of A and B as a maximum of the two pseudo-distances: from A to B and from B to A. We justify its distance properties. At the end we define some special notions which enable to apply the Hausdorff distance operator "HausDist" to the subsets of the Euclidean topological space $\mathcal{E}^n_T$.

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The papers [16], [18], [15], [10], [17], [19], [3], [14], [6], [9], [8], [11], [2], [7], [4], [1], [13], and [12] provide the terminology and notation for this paper.

1. Preliminaries

Let r be a real number. Then $\{r\}$ is a subset of $\mathbb{R}$.

Let M be a non empty metric space. One can verify that $M_{\rm top}$ is T_2.

Next we state a number of propositions:

(1) For all real numbers x, y such that $x \geqslant 0$ and $y \geqslant 0$ and $\max(x, y) = 0$ holds $x = 0$.

(2) For every non empty metric space M and for every point x of M holds $(\mathrm{dist}(x))(x) = 0$.

(3) For every non empty metric space M and for every subset P of $M_{\rm top}$ and for every point x of M such that $x \in P$ holds $0 \in (\mathrm{dist}(x))^{\circ}P$.

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(4) Let M be a non empty metric space, P be a subset of M_{top}, x be a point of M, and y be a real number. If $y \in (\mathrm{dist}(x))^{\circ}P$, then $y \geqslant 0$.

(5) For every non empty metric space M and for every subset P of M_{top} and for every set x such that $x \in P$ holds $(\mathrm{dist}_{\mathrm{min}}(P))(x) = 0$.

(6) Let M be a non empty metric space, p be a point of M, q be a point of M_{top}, and r be a real number. If $p = q$ and $r > 0$, then $\mathrm{Ball}(p, r)$ is a neighbourhood of q.

(7) Let M be a non empty metric space, A be a subset of M_{top}, and p be a point of M. Then $p \in \overline{A}$ if and only if for every real number r such that $r > 0$ holds $\mathrm{Ball}(p, r)$ meets A.

(8) Let M be a non empty metric space, p be a point of M, and A be a subset of M_{top}. Then $p \in \overline{A}$ if and only if for every real number r such that $r > 0$ there exists a point q of M such that $q \in A$ and $\rho(p, q) < r$.

(9) Let M be a non empty metric space, P be a non empty subset of M_{top}, and x be a point of M. Then $(\mathrm{dist}_{\mathrm{min}}(P))(x) = 0$ if and only if for every real number r such that $r > 0$ there exists a point p of M such that $p \in P$ and $\rho(x, p) < r$.

(10) Let M be a non empty metric space, P be a non empty subset of M_{top}, and x be a point of M. Then $x \in \overline{P}$ if and only if $(\mathrm{dist}_{\mathrm{min}}(P))(x) = 0$.

(11) Let M be a non empty metric space, P be a non empty closed subset of M_{top}, and x be a point of M. Then $x \in P$ if and only if $(\mathrm{dist}_{\mathrm{min}}(P))(x) = 0$.

(12) For every non empty subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ there exists a non empty subset X of $\mathbb{R}$ such that $A = X$ and $\inf A = \inf X$.

(13) For every non empty subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ there exists a non empty subset X of $\mathbb{R}$ such that $A = X$ and $\sup A = \sup X$.

(14) Let M be a non empty metric space, P be a non empty subset of M_{top}, x be a point of M, and X be a subset of $\mathbb{R}$. If $X = (\mathrm{dist}(x))^{\circ}P$, then X is lower bounded.

(15) Let M be a non empty metric space, P be a non empty subset of M_{top}, and x, y be points of M. If $y \in P$, then $(\mathrm{dist}_{\mathrm{min}}(P))(x) \leqslant \rho(x, y)$.

(16) Let M be a non empty metric space, P be a non empty subset of M_{top}, r be a real number, and x be a point of M. If for every point y of M such that $y \in P$ holds $\rho(x, y) \geqslant r$, then $(\mathrm{dist}_{\mathrm{min}}(P))(x) \geqslant r$.

(17) Let M be a non empty metric space, P be a non empty subset of M_{top}, and x, y be points of M. Then $(\mathrm{dist}_{\mathrm{min}}(P))(x) \leqslant \rho(x, y) + (\mathrm{dist}_{\mathrm{min}}(P))(y)$.

(18) Let M be a non empty metric space, P be a subset of the carrier of M_{top}, and Q be a non empty subset of the carrier of M. If $P = Q$, then $M_{\mathrm{top}} \restriction P = (M \restriction Q)_{\mathrm{top}}$.

(19) Let M be a non empty metric space, A be a subset of M, B be a non empty subset of the carrier of M, and C be a subset of $M \restriction B$. If $A \subseteq B$

and $A = C$ and C is bounded, then A is bounded.

(20) Let M be a non empty metric space, B be a subset of M, and A be a subset of $M_{\rm top}$. If $A = B$ and A is compact, then B is bounded.

(21) Let M be a non empty metric space, P be a non empty subset of $M_{\rm top}$, and z be a point of M. Then there exists a point w of M such that $w \in P$ and $({\rm dist}_{\rm min}(P))(z) \leqslant \rho(w, z)$.

Let M be a non empty metric space and let x be a point of M. Note that ${\rm dist}(x)$ is continuous.

Let M be a non empty metric space and let X be a compact non empty subset of $M_{\rm top}$. One can check that ${\rm dist}_{\rm max}(X)$ is continuous and ${\rm dist}_{\rm min}(X)$ is continuous.

One can prove the following propositions:

(22) Let M be a non empty metric space, P be a non empty subset of $M_{\rm top}$, and x, y be points of M. If $y \in P$ and P is compact, then $({\rm dist}_{\rm max}(P))(x) \geqslant \rho(x, y)$.

(23) Let M be a non empty metric space, P be a non empty subset of $M_{\rm top}$, and z be a point of M. If P is compact, then there exists a point w of M such that $w \in P$ and $({\rm dist}_{\rm max}(P))(z) \geqslant \rho(w, z)$.

(24) Let M be a non empty metric space, P, Q be non empty subsets of $M_{\rm top}$, and z be a point of M. If P is compact and Q is compact and $z \in Q$, then $({\rm dist}_{\rm min}(P))(z) \leqslant {\rm dist}^{\rm max}_{\rm max}(P, Q)$.

(25) Let M be a non empty metric space, P, Q be non empty subsets of $M_{\rm top}$, and z be a point of M. If P is compact and Q is compact and $z \in Q$, then $({\rm dist}_{\rm max}(P))(z) \leqslant {\rm dist}^{\rm max}_{\rm max}(P, Q)$.

(26) Let M be a non empty metric space, P, Q be non empty subsets of $M_{\rm top}$, and X be a subset of $\mathbb{R}$. If $X = ({\rm dist}_{\rm max}(P))^{\circ}Q$ and P is compact and Q is compact, then X is upper bounded.

(27) Let M be a non empty metric space, P, Q be non empty subsets of $M_{\rm top}$, and X be a subset of $\mathbb{R}$. If $X = ({\rm dist}_{\rm min}(P))^{\circ}Q$ and P is compact and Q is compact, then X is upper bounded.

(28) Let M be a non empty metric space, P be a non empty subset of $M_{\rm top}$, and z be a point of M. If P is compact, then $({\rm dist}_{\rm min}(P))(z) \leqslant ({\rm dist}_{\rm max}(P))(z)$.

(29) For every non empty metric space M and for every non empty subset P of $M_{\rm top}$ holds $({\rm dist}_{\rm min}(P))^{\circ}P = \{0\}$.

(30) Let M be a non empty metric space and P, Q be non empty subsets of $M_{\rm top}$. If P is compact and Q is compact, then ${\rm dist}^{\rm max}_{\rm min}(P, Q) \geqslant 0$.

(31) For every non empty metric space M and for every non empty subset P of $M_{\rm top}$ holds ${\rm dist}^{\rm max}_{\rm min}(P, P) = 0$.

(32) Let M be a non empty metric space and P, Q be non empty subsets of $M_{\rm top}$. If P is compact and Q is compact, then $\operatorname{dist}^{\min}_{\max}(P,Q) \geqslant 0$.

(33) Let M be a non empty metric space, Q, R be non empty subsets of $M_{\rm top}$, and y be a point of M. If Q is compact and R is compact and $y \in Q$, then $(\operatorname{dist}_{\min}(R))(y) \leqslant \operatorname{dist}^{\max}_{\min}(R,Q)$.

2. The Hausdorff Distance

Let M be a non empty metric space and let P, Q be subsets of $M_{\rm top}$. The functor $\operatorname{HausDist}(P,Q)$ yields a real number and is defined by:

(Def. 1) $\operatorname{HausDist}(P,Q) = \max(\operatorname{dist}^{\max}_{\min}(P,Q), \operatorname{dist}^{\max}_{\min}(Q,P))$.

Let us notice that the functor $\operatorname{HausDist}(P,Q)$ is commutative.

The following propositions are true:

(34) Let M be a non empty metric space, Q, R be non empty subsets of $M_{\rm top}$, and y be a point of M. If Q is compact and R is compact and $y \in Q$, then $(\operatorname{dist}_{\min}(R))(y) \leqslant \operatorname{HausDist}(Q,R)$.

(35) Let M be a non empty metric space and P, Q, R be non empty subsets of $M_{\rm top}$. If P is compact and Q is compact and R is compact, then $\operatorname{dist}^{\max}_{\min}(P,R) \leqslant \operatorname{HausDist}(P,Q) + \operatorname{HausDist}(Q,R)$.

(36) Let M be a non empty metric space and P, Q, R be non empty subsets of $M_{\rm top}$. If P is compact and Q is compact and R is compact, then $\operatorname{dist}^{\max}_{\min}(R,P) \leqslant \operatorname{HausDist}(P,Q) + \operatorname{HausDist}(Q,R)$.

(37) Let M be a non empty metric space and P, Q be non empty subsets of $M_{\rm top}$. If P is compact and Q is compact, then $\operatorname{HausDist}(P,Q) \geqslant 0$.

(38) For every non empty metric space M and for every non empty subset P of $M_{\rm top}$ holds $\operatorname{HausDist}(P,P) = 0$.

(39) Let M be a non empty metric space and P, Q be non empty subsets of $M_{\rm top}$. If P is compact and Q is compact and $\operatorname{HausDist}(P,Q) = 0$, then $P = Q$.

(40) Let M be a non empty metric space and P, Q, R be non empty subsets of $M_{\rm top}$. If P is compact and Q is compact and R is compact, then $\operatorname{HausDist}(P,R) \leqslant \operatorname{HausDist}(P,Q) + \operatorname{HausDist}(Q,R)$.

Let n be a natural number and let P, Q be subsets of the carrier of $\mathcal{E}^n_{\rm T}$. The functor $\operatorname{dist}^{\max}_{\min}(P,Q)$ yields a real number and is defined by:

(Def. 2) There exist subsets P', Q' of $(\mathcal{E}^n)_{\rm top}$ such that $P = P'$ and $Q = Q'$ and $\operatorname{dist}^{\max}_{\min}(P,Q) = \operatorname{dist}^{\max}_{\min}(P',Q')$.

Let n be a natural number and let P, Q be subsets of the carrier of $\mathcal{E}^n_{\rm T}$. The functor $\operatorname{HausDist}(P,Q)$ yields a real number and is defined by:

(Def. 3) There exist subsets P', Q' of $(\mathcal{E}^n)_{\rm top}$ such that $P = P'$ and $Q = Q'$ and $\operatorname{HausDist}(P,Q) = \operatorname{HausDist}(P',Q')$.

Let us note that the functor $\mathrm{HausDist}(P,Q)$ is commutative.

In the sequel n denotes a natural number.

Next we state four propositions:

(41) For all non empty subsets P, Q of $\mathcal{E}^n_{\mathrm{T}}$ such that P is compact and Q is compact holds $\mathrm{HausDist}(P,Q) \geqslant 0$.

(42) For every non empty subset P of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{HausDist}(P,P) = 0$.

(43) For all non empty subsets P, Q of $\mathcal{E}^n_{\mathrm{T}}$ such that P is compact and Q is compact and $\mathrm{HausDist}(P,Q) = 0$ holds $P = Q$.

(44) For all non empty subsets P, Q, R of $\mathcal{E}^n_{\mathrm{T}}$ such that P is compact and Q is compact and R is compact holds $\mathrm{HausDist}(P,R) \leqslant \mathrm{HausDist}(P,Q) + \mathrm{HausDist}(Q,R)$.

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Chains on a Grating in Euclidean Space[1]

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The notation and terminology used here are introduced in the following papers: [20], [10], [22], [23], [18], [8], [12], [9], [17], [1], [19], [14], [3], [6], [13], [16], [2], [11], [4], [7], [21], and [5].

1. Preliminaries

We use the following convention: X, x, y, z are sets and n, m, k, k', d' are natural numbers.

The following two propositions are true:

(1) For all real numbers x, y such that $x < y$ there exists a real number z such that $x < z$ and $z < y$.

(2) For all real numbers x, y there exists a real number z such that $x < z$ and $y < z$.

The scheme *FrSet 1 2* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

$$\{\mathcal{F}(x,y); x \text{ ranges over elements of } \mathcal{B}, y \text{ ranges over elements of } \mathcal{B} : \mathcal{P}[x,y]\} \subseteq \mathcal{A}$$

for all values of the parameters.

Let B be a set and let A be a subset of B. Then 2^A is a subset of 2^B.

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Let X be a set. A subset of X is an element of 2^X.

Let d be a real natural number. Let us observe that d is zero if and only if:

(Def. 1) $d \not> 0$.

Let d be a natural number. Let us observe that d is zero if and only if:

(Def. 2) $d \not\geq 1$.

Let us note that there exists a natural number which is non zero.

In the sequel d denotes a non zero natural number.

Let us consider d. Observe that $\operatorname{Seg} d$ is non empty.

In the sequel i, i_0 denote elements of $\operatorname{Seg} d$.

Let us consider X. Let us observe that X is trivial if and only if:

(Def. 3) For all x, y such that $x \in X$ and $y \in X$ holds $x = y$.

Next we state the proposition

(4)[2] $\{x, y\}$ is trivial iff $x = y$.

Let us observe that there exists a set which is non trivial and finite.

Let X be a non trivial set and let Y be a set. Note that $X \cup Y$ is non trivial and $Y \cup X$ is non trivial.

Let us observe that $\mathbb{R}$ is non trivial.

Let X be a non trivial set. Observe that there exists a subset of X which is non trivial and finite.

The following proposition is true

(5) If X is trivial and $X \cup \{y\}$ is non trivial, then there exists x such that $X = \{x\}$.

Now we present two schemes. The scheme *NonEmptyFinite* deals with a non empty set $\mathcal{A}$, a non empty finite subset $\mathcal{B}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the following requirements are met:

- For every element x of $\mathcal{A}$ such that $x \in \mathcal{B}$ holds $\mathcal{P}[\{x\}]$, and
- Let x be an element of $\mathcal{A}$ and B be a non empty finite subset of $\mathcal{A}$. If $x \in \mathcal{B}$ and $B \subseteq \mathcal{B}$ and $x \notin B$ and $\mathcal{P}[B]$, then $\mathcal{P}[B \cup \{x\}]$.

The scheme *NonTrivialFinite* deals with a non trivial set $\mathcal{A}$, a non trivial finite subset $\mathcal{B}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the following conditions are met:

- For all elements x, y of $\mathcal{A}$ such that $x \in \mathcal{B}$ and $y \in \mathcal{B}$ and $x \neq y$ holds $\mathcal{P}[\{x, y\}]$, and
- Let x be an element of $\mathcal{A}$ and B be a non trivial finite subset of $\mathcal{A}$. If $x \in \mathcal{B}$ and $B \subseteq \mathcal{B}$ and $x \notin B$ and $\mathcal{P}[B]$, then $\mathcal{P}[B \cup \{x\}]$.

Next we state the proposition

[2]The proposition (3) has been removed.

(6) $\overline{\overline{X}} = 2$ iff there exist x, y such that $x \in X$ and $y \in X$ and $x \neq y$ and for every z such that $z \in X$ holds $z = x$ or $z = y$.

Let X, Y be finite sets. Note that $X \dot{-} Y$ is finite.

We now state three propositions:

(7) m is even iff n is even iff $m + n$ is even.

(8) Let X, Y be finite sets. Suppose X misses Y. Then $\operatorname{card} X$ is even iff $\operatorname{card} Y$ is even if and only if $\operatorname{card}(X \cup Y)$ is even.

(9) For all finite sets X, Y holds $\operatorname{card} X$ is even iff $\operatorname{card} Y$ is even iff $\operatorname{card}(X \dot{-} Y)$ is even.

Let us consider n. Then $\mathcal{R}^n$ can be characterized by the condition:

(Def. 4) For every x holds $x \in \mathcal{R}^n$ iff x is a function from $\operatorname{Seg} n$ into $\mathbb{R}$.

We adopt the following rules: l, r, l', r', x are elements of $\mathcal{R}^d$, G_1 is a non trivial finite subset of $\mathbb{R}$, and l_1, r_1, l'_1, r'_1, x_1 are real numbers.

Let us consider d, x, i. Then $x(i)$ is a real number.

2. Gratings, Cells, Chains, Cycles

Let us consider d. A function from $\operatorname{Seg} d$ into $2^{\mathbb{R}}$ is said to be a d-dimensional grating if:

(Def. 5) For every i holds it(i) is non trivial and finite.

In the sequel G is a d-dimensional grating.

Let us consider d, G, i. Then $G(i)$ is a non trivial finite subset of $\mathbb{R}$.

The following propositions are true:

(10) $x \in \prod G$ iff for every i holds $x(i) \in G(i)$.

(11) $\prod G$ is finite.

(12) For every non empty finite subset X of $\mathbb{R}$ there exists r_1 such that $r_1 \in X$ and for every x_1 such that $x_1 \in X$ holds $r_1 \geqslant x_1$.

(13) For every non empty finite subset X of $\mathbb{R}$ there exists l_1 such that $l_1 \in X$ and for every x_1 such that $x_1 \in X$ holds $l_1 \leqslant x_1$.

(14) There exist l_1, r_1 such that $l_1 \in G_1$ and $r_1 \in G_1$ and $l_1 < r_1$ and for every x_1 such that $x_1 \in G_1$ holds $l_1 \not< x_1$ or $x_1 \not< r_1$.

(15) There exist l_1, r_1 such that $l_1 \in G_1$ and $r_1 \in G_1$ and $r_1 < l_1$ and for every x_1 such that $x_1 \in G_1$ holds $x_1 \not< r_1$ and $l_1 \not< x_1$.

Let us consider G_1. An element of $[\!:\mathbb{R}, \mathbb{R}:\!]$ is called a gap of G_1 if it satisfies the condition (Def. 6).

(Def. 6) There exist l_1, r_1 such that

(i) it $= \langle l_1, r_1 \rangle$,

(ii) $l_1 \in G_1$,

(iii) $r_1 \in G_1$, and

(iv) $l_1 < r_1$ and for every x_1 such that $x_1 \in G_1$ holds $l_1 \not< x_1$ or $x_1 \not< r_1$ or $r_1 < l_1$ and for every x_1 such that $x_1 \in G_1$ holds $l_1 \not< x_1$ and $x_1 \not< r_1$.

The following propositions are true:

(16) $\langle l_1, r_1 \rangle$ is a gap of G_1 if and only if the following conditions are satisfied:

(i) $l_1 \in G_1$,

(ii) $r_1 \in G_1$, and

(iii) $l_1 < r_1$ and for every x_1 such that $x_1 \in G_1$ holds $l_1 \not< x_1$ or $x_1 \not< r_1$ or $r_1 < l_1$ and for every x_1 such that $x_1 \in G_1$ holds $l_1 \not< x_1$ and $x_1 \not< r_1$.

(17) If $G_1 = \{l_1, r_1\}$, then $\langle l'_1, r'_1 \rangle$ is a gap of G_1 iff $l'_1 = l_1$ and $r'_1 = r_1$ or $l'_1 = r_1$ and $r'_1 = l_1$.

(18) If $x_1 \in G_1$, then there exists r_1 such that $\langle x_1, r_1 \rangle$ is a gap of G_1.

(19) If $x_1 \in G_1$, then there exists l_1 such that $\langle l_1, x_1 \rangle$ is a gap of G_1.

(20) If $\langle l_1, r_1 \rangle$ is a gap of G_1 and $\langle l_1, r'_1 \rangle$ is a gap of G_1, then $r_1 = r'_1$.

(21) If $\langle l_1, r_1 \rangle$ is a gap of G_1 and $\langle l'_1, r_1 \rangle$ is a gap of G_1, then $l_1 = l'_1$.

(22) If $r_1 < l_1$ and $\langle l_1, r_1 \rangle$ is a gap of G_1 and $r'_1 < l'_1$ and $\langle l'_1, r'_1 \rangle$ is a gap of G_1, then $l_1 = l'_1$ and $r_1 = r'_1$.

Let us consider d, l, r. The functor $\mathrm{cell}(l, r)$ yielding a non empty subset of $\mathcal{R}^d$ is defined as follows:

(Def. 7) $\mathrm{cell}(l, r) = \{x : \bigwedge_i (l(i) \leqslant x(i) \wedge x(i) \leqslant r(i)) \vee \bigvee_i (r(i) < l(i) \wedge (x(i) \leqslant r(i) \ \vee \ l(i) \leqslant x(i)))\}$.

We now state several propositions:

(23) $x \in \mathrm{cell}(l, r)$ iff for every i holds $l(i) \leqslant x(i)$ and $x(i) \leqslant r(i)$ or there exists i such that $r(i) < l(i)$ but $x(i) \leqslant r(i)$ or $l(i) \leqslant x(i)$.

(24) If for every i holds $l(i) \leqslant r(i)$, then $x \in \mathrm{cell}(l, r)$ iff for every i holds $l(i) \leqslant x(i)$ and $x(i) \leqslant r(i)$.

(25) If there exists i such that $r(i) < l(i)$, then $x \in \mathrm{cell}(l, r)$ iff there exists i such that $r(i) < l(i)$ but $x(i) \leqslant r(i)$ or $l(i) \leqslant x(i)$.

(26) $l \in \mathrm{cell}(l, r)$ and $r \in \mathrm{cell}(l, r)$.

(27) $\mathrm{cell}(x, x) = \{x\}$.

(28) If for every i holds $l'(i) \leqslant r'(i)$, then $\mathrm{cell}(l, r) \subseteq \mathrm{cell}(l', r')$ iff for every i holds $l'(i) \leqslant l(i)$ and $l(i) \leqslant r(i)$ and $r(i) \leqslant r'(i)$.

(29) If for every i holds $r(i) < l(i)$, then $\mathrm{cell}(l, r) \subseteq \mathrm{cell}(l', r')$ iff for every i holds $r(i) \leqslant r'(i)$ and $r'(i) < l'(i)$ and $l'(i) \leqslant l(i)$.

(30) Suppose for every i holds $l(i) \leqslant r(i)$ and for every i holds $r'(i) < l'(i)$. Then $\mathrm{cell}(l, r) \subseteq \mathrm{cell}(l', r')$ if and only if there exists i such that $r(i) \leqslant r'(i)$ or $l'(i) \leqslant l(i)$.

(31) If for every i holds $l(i) \leqslant r(i)$ or for every i holds $l(i) > r(i)$, then $\mathrm{cell}(l, r) = \mathrm{cell}(l', r')$ iff $l = l'$ and $r = r'$.

Let us consider d, G, k. Let us assume that $k \leqslant d$. The functor k-cells(G) yields a finite non empty subset of $2^{\mathcal{R}^d}$ and is defined by the condition (Def. 8).

(Def. 8) $k\text{-cells}(G) = \{\text{cell}(l, r) : \bigvee_{X:\text{subset of Seg}\,d} (\text{card}\,X = k \ \wedge\ \bigwedge_i (i \in X \ \wedge\ l(i) < r(i) \ \wedge\ \langle l(i), r(i)\rangle \text{ is a gap of } G(i) \ \vee\ i \notin X \ \wedge\ l(i) = r(i) \ \wedge\ l(i) \in G(i))) \ \vee\ k = d \ \wedge\ \bigwedge_i (r(i) < l(i) \ \wedge\ \langle l(i), r(i)\rangle \text{ is a gap of } G(i))\}$.

We now state a number of propositions:

(32) Suppose $k \leqslant d$. Let A be a subset of $\mathcal{R}^d$. Then $A \in k$-cells(G) if and only if there exist l, r such that $A = \text{cell}(l, r)$ but there exists a subset X of Seg d such that card $X = k$ and for every i holds $i \in X$ and $l(i) < r(i)$ and $\langle l(i), r(i)\rangle$ is a gap of $G(i)$ or $i \notin X$ and $l(i) = r(i)$ and $l(i) \in G(i)$ or $k = d$ and for every i holds $r(i) < l(i)$ and $\langle l(i), r(i)\rangle$ is a gap of $G(i)$.

(33) Suppose $k \leqslant d$. Then cell$(l, r) \in k$-cells(G) if and only if one of the following conditions is satisfied:

(i) there exists a subset X of Seg d such that card $X = k$ and for every i holds $i \in X$ and $l(i) < r(i)$ and $\langle l(i), r(i)\rangle$ is a gap of $G(i)$ or $i \notin X$ and $l(i) = r(i)$ and $l(i) \in G(i)$, or

(ii) $k = d$ and for every i holds $r(i) < l(i)$ and $\langle l(i), r(i)\rangle$ is a gap of $G(i)$.

(34) Suppose $k \leqslant d$ and cell$(l, r) \in k$-cells(G). Then

(i) for every i holds $l(i) < r(i)$ and $\langle l(i), r(i)\rangle$ is a gap of $G(i)$ or $l(i) = r(i)$ and $l(i) \in G(i)$, or

(ii) for every i holds $r(i) < l(i)$ and $\langle l(i), r(i)\rangle$ is a gap of $G(i)$.

(35) If $k \leqslant d$ and cell$(l, r) \in k$-cells(G), then for every i holds $l(i) \in G(i)$ and $r(i) \in G(i)$.

(36) If $k \leqslant d$ and cell$(l, r) \in k$-cells(G), then for every i holds $l(i) \leqslant r(i)$ or for every i holds $r(i) < l(i)$.

(37) For every subset A of $\mathcal{R}^d$ holds $A \in 0$-cells(G) iff there exists x such that $A = \text{cell}(x, x)$ and for every i holds $x(i) \in G(i)$.

(38) cell$(l, r) \in 0$-cells(G) iff $l = r$ and for every i holds $l(i) \in G(i)$.

(39) Let A be a subset of $\mathcal{R}^d$. Then $A \in d$-cells(G) if and only if there exist l, r such that $A = \text{cell}(l, r)$ but for every i holds $\langle l(i), r(i)\rangle$ is a gap of $G(i)$ but for every i holds $l(i) < r(i)$ or for every i holds $r(i) < l(i)$.

(40) cell$(l, r) \in d$-cells(G) iff for every i holds $\langle l(i), r(i)\rangle$ is a gap of $G(i)$ but for every i holds $l(i) < r(i)$ or for every i holds $r(i) < l(i)$.

(41) Suppose $d = d' + 1$. Let A be a subset of $\mathcal{R}^d$. Then $A \in d'$-cells(G) if and only if there exist l, r, i_0 such that $A = \text{cell}(l, r)$ and $l(i_0) = r(i_0)$ and $l(i_0) \in G(i_0)$ and for every i such that $i \neq i_0$ holds $l(i) < r(i)$ and $\langle l(i), r(i)\rangle$ is a gap of $G(i)$.

(42) Suppose $d = d' + 1$. Then cell$(l, r) \in d'$-cells(G) if and only if there exists i_0 such that $l(i_0) = r(i_0)$ and $l(i_0) \in G(i_0)$ and for every i such that $i \neq i_0$ holds $l(i) < r(i)$ and $\langle l(i), r(i)\rangle$ is a gap of $G(i)$.

(43) Let A be a subset of $\mathcal{R}^d$. Then $A \in 1\text{-cells}(G)$ if and only if there exist l, r, i_0 such that $A = \mathrm{cell}(l, r)$ and $l(i_0) < r(i_0)$ or $d = 1$ and $r(i_0) < l(i_0)$ and $\langle l(i_0), r(i_0)\rangle$ is a gap of $G(i_0)$ and for every i such that $i \neq i_0$ holds $l(i) = r(i)$ and $l(i) \in G(i)$.

(44) $\mathrm{cell}(l, r) \in 1\text{-cells}(G)$ if and only if there exists i_0 such that $l(i_0) < r(i_0)$ or $d = 1$ and $r(i_0) < l(i_0)$ but $\langle l(i_0), r(i_0)\rangle$ is a gap of $G(i_0)$ but for every i such that $i \neq i_0$ holds $l(i) = r(i)$ and $l(i) \in G(i)$.

(45) Suppose $k \leqslant d$ and $k' \leqslant d$ and $\mathrm{cell}(l, r) \in k\text{-cells}(G)$ and $\mathrm{cell}(l', r') \in k'\text{-cells}(G)$ and $\mathrm{cell}(l, r) \subseteq \mathrm{cell}(l', r')$. Let given i. Then

(i) $l(i) = l'(i)$ and $r(i) = r'(i)$, or

(ii) $l(i) = l'(i)$ and $r(i) = l'(i)$, or

(iii) $l(i) = r'(i)$ and $r(i) = r'(i)$, or

(iv) $l(i) \leqslant r(i)$ and $r'(i) < l'(i)$ and $r'(i) \leqslant l(i)$ and $r(i) \leqslant l'(i)$.

(46) Suppose $k < k'$ and $k' \leqslant d$ and $\mathrm{cell}(l, r) \in k\text{-cells}(G)$ and $\mathrm{cell}(l', r') \in k'\text{-cells}(G)$ and $\mathrm{cell}(l, r) \subseteq \mathrm{cell}(l', r')$. Then there exists i such that $l(i) = l'(i)$ and $r(i) = l'(i)$ or $l(i) = r'(i)$ and $r(i) = r'(i)$.

(47) Let X, X' be subsets of $\mathrm{Seg}\, d$. Suppose that

(i) $\mathrm{cell}(l, r) \subseteq \mathrm{cell}(l', r')$,

(ii) for every i holds $i \in X$ and $l(i) < r(i)$ and $\langle l(i), r(i)\rangle$ is a gap of $G(i)$ or $i \notin X$ and $l(i) = r(i)$ and $l(i) \in G(i)$, and

(iii) for every i holds $i \in X'$ and $l'(i) < r'(i)$ and $\langle l'(i), r'(i)\rangle$ is a gap of $G(i)$ or $i \notin X'$ and $l'(i) = r'(i)$ and $l'(i) \in G(i)$.

Then

(iv) $X \subseteq X'$,

(v) for every i such that $i \in X$ or $i \notin X'$ holds $l(i) = l'(i)$ and $r(i) = r'(i)$, and

(vi) for every i such that $i \notin X$ and $i \in X'$ holds $l(i) = l'(i)$ and $r(i) = l'(i)$ or $l(i) = r'(i)$ and $r(i) = r'(i)$.

Let us consider d, G, k. A k-cell of G is an element of $k\text{-cells}(G)$.

Let us consider d, G, k. A k-chain of G is a subset of $k\text{-cells}(G)$.

Let us consider d, G, k. The functor $0_k G$ yields a k-chain of G and is defined as follows:

(Def. 9) $0_k G = \emptyset$.

Let us consider d, G. The functor ΩG yielding a d-chain of G is defined as follows:

(Def. 10) $\Omega G = d\text{-cells}(G)$.

Let us consider d, G, k and let C_1, C_2 be k-chains of G. Then $C_1 \dot{-} C_2$ is a k-chain of G. We introduce $C_1 + C_2$ as a synonym of $C_1 \dot{-} C_2$.

Let us consider d, G. The infinite cell of G yielding a d-cell of G is defined by:

(Def. 11) There exist l, r such that the infinite cell of $G = \mathrm{cell}(l,r)$ and for every i holds $r(i) < l(i)$ and $\langle l(i), r(i)\rangle$ is a gap of $G(i)$.

We now state two propositions:

(48) If $\mathrm{cell}(l,r)$ is a d-cell of G, then $\mathrm{cell}(l,r) =$ the infinite cell of G iff for every i holds $r(i) < l(i)$.

(49) $\mathrm{cell}(l,r) =$ the infinite cell of G iff for every i holds $r(i) < l(i)$ and $\langle l(i), r(i)\rangle$ is a gap of $G(i)$.

The scheme *ChainInd* deals with a non zero natural number $\mathcal{A}$, a $\mathcal{A}$-dimensional grating $\mathcal{B}$, a natural number $\mathcal{C}$, a $\mathcal{C}$-chain $\mathcal{D}$ of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{D}]$

provided the parameters have the following properties:

- $\mathcal{P}[0_{\mathcal{C}}\mathcal{B}]$,
- For every $\mathcal{C}$-cell A of $\mathcal{B}$ such that $A \in \mathcal{D}$ holds $\mathcal{P}[\{A\}]$, and
- For all $\mathcal{C}$-chains C_1, C_2 of $\mathcal{B}$ such that $C_1 \subseteq \mathcal{D}$ and $C_2 \subseteq \mathcal{D}$ and $\mathcal{P}[C_1]$ and $\mathcal{P}[C_2]$ holds $\mathcal{P}[C_1 + C_2]$.

Let us consider d, G, k and let A be a k-cell of G. The functor $A^\star$ yields a $k+1$-chain of G and is defined by:

(Def. 12) $A^\star = \{B; B$ ranges over $k+1$-cells of G: $A \subseteq B\}$.

Next we state the proposition

(50) For every k-cell A of G and for every $k+1$-cell B of G holds $B \in A^\star$ iff $A \subseteq B$.

Let us consider d, G, k and let C be a $k+1$-chain of G. The functor ∂C yielding a k-chain of G is defined as follows:

(Def. 13) $\partial C = \{A; A$ ranges over k-cells of G: $k+1 \leqslant d \ \wedge\ \mathrm{card}(A^\star \cap C)$is odd$\}$.

We introduce $\dot{C}$ as a synonym of ∂C.

Let us consider d, G, k, let C be a $k+1$-chain of G, and let C' be a k-chain of G. We say that C' bounds C if and only if:

(Def. 14) $C' = \partial C$.

The following propositions are true:

(51) For every k-cell A of G and for every $k+1$-chain C of G holds $A \in \partial C$ iff $k+1 \leqslant d$ and $\mathrm{card}(A^\star \cap C)$ is odd.

(52) If $k+1 > d$, then for every $k+1$-chain C of G holds $\partial C = 0_k G$.

(53) If $k+1 \leqslant d$, then for every k-cell A of G and for every $k+1$-cell B of G holds $A \in \partial\{B\}$ iff $A \subseteq B$.

(54) If $d = d' + 1$, then for every d'-cell A of G holds $\mathrm{card}\, A^\star = 2$.

(55) For every d-dimensional grating G and for every $0+1$-cell B of G holds $\mathrm{card}\, \partial\{B\} = 2$.

(56) $\Omega G = (0_d G)^{\mathrm{c}}$ and $0_d G = (\Omega G)^{\mathrm{c}}$.

(57) For every k-chain C of G holds $C + 0_k G = C$.

(58) For every k-chain C of G holds $C + C = 0_k G$.

(59) For every d-chain C of G holds $C^{\mathrm{c}} = C + \Omega G$.

(60) $\partial 0_{k+1} G = 0_k G$.

(61) For every $d' + 1$-dimensional grating G holds $\partial \Omega G = 0_{d'} G$.

(62) For all $k + 1$-chains C_1, C_2 of G holds $\partial(C_1 + C_2) = \partial C_1 + \partial C_2$.

(63) For every $d' + 1$-dimensional grating G and for every $d' + 1$-chain C of G holds $\partial(C^{\mathrm{c}}) = \partial C$.

(64) For every $k + 1 + 1$-chain C of G holds $\partial\partial C = 0_k G$.

Let us consider d, G, k. A k-chain of G is called a k-cycle of G if:

(Def. 15) $k = 0$ and card it is even or there exists k' such that $k = k' + 1$ and there exists a $k' + 1$-chain C of G such that $C =$ it and $\partial C = 0_{k'} G$.

One can prove the following propositions:

(65) For every $k + 1$-chain C of G holds C is a $k + 1$-cycle of G iff $\partial C = 0_k G$.

(66) If $k > d$, then every k-chain of G is a k-cycle of G.

(67) For every 0-chain C of G holds C is a 0-cycle of G iff $\operatorname{card} C$ is even.

Let us consider d, G, k and let C be a $k + 1$-cycle of G. Then ∂C can be characterized by the condition:

(Def. 16) $\partial C = 0_k G$.

Let us consider d, G, k. Then $0_k G$ is a k-cycle of G.

Let us consider d, G. Then ΩG is a d-cycle of G.

Let us consider d, G, k and let C_1, C_2 be k-cycles of G. Then $C_1 \dot{-} C_2$ is a k-cycle of G. We introduce $C_1 + C_2$ as a synonym of $C_1 \dot{-} C_2$.

We now state the proposition

(68) For every d-cycle C of G holds C^{c} is a d-cycle of G.

Let us consider d, G, k and let C be a $k + 1$-chain of G. Then ∂C is a k-cycle of G.

3. Groups and Homomorphisms

Let us consider d, G, k. The functor k-Chains(G) yields a strict Abelian group and is defined by the conditions (Def. 17).

(Def. 17)(i) The carrier of k-Chains$(G) = 2^{k\text{-}\mathrm{cells}(G)}$,

(ii) $0_{k\text{-}\mathrm{Chains}(G)} = 0_k G$, and

(iii) for all elements A, B of k-Chains(G) and for all k-chains A', B' of G such that $A = A'$ and $B = B'$ holds $A + B = A' + B'$.

Let us consider d, G, k. A k-grchain of G is an element of k-Chains(G).

One can prove the following proposition

(69) For every set x holds x is a k-chain of G iff x is a k-grchain of G.

Let us consider d, G, k. The functor ∂ yielding a homomorphism from $(k+1)$-Chains(G) to k-Chains(G) is defined by:

(Def. 18) For every element A of $(k+1)$-Chains(G) and for every $k+1$-chain A' of G such that $A = A'$ holds $\partial(A) = \partial A'$.

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Bessel's Inequality

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Summary. In this article we defined the operation of a set and proved Bessel's inequality. In the first section, we defined the sum of all results of an operation, in which the results are given by taking each element of a set. In the second section, we defined Orthogonal Family and Orthonormal Family. In the last section, we proved some properties of operation of set and Bessel's inequality.

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The articles [12], [16], [10], [7], [5], [6], [17], [15], [9], [13], [3], [8], [1], [11], [4], [2], and [14] provide the terminology and notation for this paper.

1. Sum of the Result of Operation with Each Element of a Set

For simplicity, we adopt the following convention: X denotes a real unitary space, x, y, y_1, y_2 denote points of X, i, j denote natural numbers, D_1 denotes a non empty set, and p_1, p_2 denote finite sequences of elements of D_1.

Next we state the proposition

(1) Suppose p_1 is one-to-one and p_2 is one-to-one and $\operatorname{rng} p_1 = \operatorname{rng} p_2$. Then $\operatorname{dom} p_1 = \operatorname{dom} p_2$ and there exists a permutation P of $\operatorname{dom} p_1$ such that $p_2 = p_1 \cdot P$ and $\operatorname{dom} P = \operatorname{dom} p_1$ and $\operatorname{rng} P = \operatorname{dom} p_1$.

Let D_1 be a non empty set and let f be a binary operation on D_1. Let us assume that f is commutative and associative and has a unity. Let Y be a finite subset of D_1. The functor $f \oplus Y$ yields an element of D_1 and is defined as follows:

(Def. 1) There exists a finite sequence p of elements of D_1 such that p is one-to-one and $\operatorname{rng} p = Y$ and $f \oplus Y = f \odot p$.

Let us consider X and let Y be a finite subset of the carrier of X. The functor $\mathrm{SetopSum}(Y, X)$ is defined as follows:

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(Def. 2) $\text{SetopSum}(Y, X) = \begin{cases} (\text{the addition of } X) \oplus Y, & \text{if } Y \neq \emptyset, \\ 0_X, & \text{otherwise.} \end{cases}$

Let us consider X, x, let p be a finite sequence, and let us consider i. The functor $\text{PO}(i, p, x)$ is defined by:

(Def. 3) $\text{PO}(i, p, x) = (\text{the scalar product of } X)(\langle x, p(i) \rangle)$.

Let D_2, D_1 be non empty sets, let F be a function from D_1 into D_2, and let p be a finite sequence of elements of D_1. The functor $\text{FuncSeq}(F, p)$ yielding a finite sequence of elements of D_2 is defined as follows:

(Def. 4) $\text{FuncSeq}(F, p) = F \cdot p$.

Let D_2, D_1 be non empty sets and let f be a binary operation on D_2. Let us assume that f is commutative and associative and has a unity. Let Y be a finite subset of D_1 and let F be a function from D_1 into D_2. Let us assume that $Y \subseteq \text{dom}\, F$. The functor $\text{setopfunc}(Y, D_1, D_2, F, f)$ yielding an element of D_2 is defined by:

(Def. 5) There exists a finite sequence p of elements of D_1 such that p is one-to-one and $\text{rng}\, p = Y$ and $\text{setopfunc}(Y, D_1, D_2, F, f) = f \odot \text{FuncSeq}(F, p)$.

Let us consider X, x and let Y be a finite subset of the carrier of X. The functor $\text{SetopPreProd}(x, Y, X)$ yields a real number and is defined by the condition (Def. 6).

(Def. 6) There exists a finite sequence p of elements of the carrier of X such that

(i) p is one-to-one,

(ii) $\text{rng}\, p = Y$, and

(iii) there exists a finite sequence q of elements of $\mathbb{R}$ such that $\text{dom}\, q = \text{dom}\, p$ and for every i such that $i \in \text{dom}\, q$ holds $q(i) = \text{PO}(i, p, x)$ and $\text{SetopPreProd}(x, Y, X) = +_{\mathbb{R}} \odot q$.

Let us consider X, x and let Y be a finite subset of the carrier of X. The functor $\text{SetopProd}(x, Y, X)$ yielding a real number is defined as follows:

(Def. 7) $\text{SetopProd}(x, Y, X) = \begin{cases} \text{SetopPreProd}(x, Y, X), & \text{if } Y \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$

2. Orthogonal Family and Orthonormal Family

Let us consider X. A subset of the carrier of X is said to be an orthogonal family of X if:

(Def. 8) For all x, y such that $x \in$ it and $y \in$ it and $x \neq y$ holds $(x|y) = 0$.

The following proposition is true

(2) $\emptyset$ is an orthogonal family of X.

Let us consider X. Observe that there exists an orthogonal family of X which is finite.

Let us consider X. A subset of the carrier of X is said to be an orthonormal family of X if:

(Def. 9) It is an orthogonal family of X and for every x such that $x \in$ it holds $(x|x) = 1$.

One can prove the following proposition

(3) $\emptyset$ is an orthonormal family of X.

Let us consider X. One can check that there exists an orthonormal family of X which is finite.

The following proposition is true

(4) $x = 0_X$ iff for every y holds $(x|y) = 0$.

3. Bessel's Inequality

We now state a number of propositions:

(5) $\|x+y\|^{\mathbf{2}} + \|x-y\|^{\mathbf{2}} = 2 \cdot \|x\|^{\mathbf{2}} + 2 \cdot \|y\|^{\mathbf{2}}$.

(6) If x, y are orthogonal, then $\|x+y\|^{\mathbf{2}} = \|x\|^{\mathbf{2}} + \|y\|^{\mathbf{2}}$.

(7) Let p be a finite sequence of elements of the carrier of X. Suppose $\operatorname{len} p \geqslant 1$ and for all i, j such that $i \in \operatorname{dom} p$ and $j \in \operatorname{dom} p$ and $i \neq j$ holds (the scalar product of X)($\langle p(i), p(j) \rangle$) $= 0$. Let q be a finite sequence of elements of $\mathbb{R}$. Suppose $\operatorname{dom} p = \operatorname{dom} q$ and for every i such that $i \in \operatorname{dom} q$ holds $q(i) =$ (the scalar product of X)($\langle p(i), p(i) \rangle$). Then ((the addition of $X \odot p$)|(the addition of $X \odot p$)) $= +_{\mathbb{R}} \odot q$.

(8) Let p be a finite sequence of elements of the carrier of X. Suppose $\operatorname{len} p \geqslant 1$. Let q be a finite sequence of elements of $\mathbb{R}$. Suppose $\operatorname{dom} p = \operatorname{dom} q$ and for every i such that $i \in \operatorname{dom} q$ holds $q(i) =$ (the scalar product of X)($\langle x, p(i) \rangle$). Then (x|(the addition of $X \odot p$)) $= +_{\mathbb{R}} \odot q$.

(9) Let S be a finite non empty subset of the carrier of X and F be a function from the carrier of X into the carrier of X. Suppose $S \subseteq \operatorname{dom} F$ and for all y_1, y_2 such that $y_1 \in S$ and $y_2 \in S$ and $y_1 \neq y_2$ holds (the scalar product of X)($\langle F(y_1), F(y_2) \rangle$) $= 0$. Let H be a function from the carrier of X into $\mathbb{R}$. Suppose $S \subseteq \operatorname{dom} H$ and for every y such that $y \in S$ holds $H(y) =$ (the scalar product of X)($\langle F(y), F(y) \rangle$). Let p be a finite sequence of elements of the carrier of X. Suppose p is one-to-one and $\operatorname{rng} p = S$. Then (the scalar product of X)($\langle$the addition of $X \odot \mathrm{FuncSeq}(F,p)$, the addition of $X \odot \mathrm{FuncSeq}(F,p) \rangle$) $= +_{\mathbb{R}} \odot \mathrm{FuncSeq}(H,p)$.

(10) Let S be a finite non empty subset of the carrier of X and F be a function from the carrier of X into the carrier of X. Suppose $S \subseteq \operatorname{dom} F$. Let H be a function from the carrier of X into $\mathbb{R}$. Suppose $S \subseteq \operatorname{dom} H$ and for every y such that $y \in S$ holds $H(y) =$ (the scalar product of X)($\langle x, F(y) \rangle$). Let

p be a finite sequence of elements of the carrier of X. Suppose p is one-to-one and $\operatorname{rng} p = S$. Then (the scalar product of X)($\langle x$, the addition of $X \odot \mathrm{FuncSeq}(F, p)\rangle$) $= +_{\mathbb{R}} \odot \mathrm{FuncSeq}(H, p)$.

(11) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let given x and S be a finite orthonormal family of X. Suppose S is non empty. Let H be a function from the carrier of X into $\mathbb{R}$. Suppose $S \subseteq \operatorname{dom} H$ and for every y such that $y \in S$ holds $H(y) = (x|y)^{\mathbf{2}}$. Let F be a function from the carrier of X into the carrier of X. Suppose $S \subseteq \operatorname{dom} F$ and for every y such that $y \in S$ holds $F(y) = (x|y) \cdot y$. Then $(x|$ setopfunc(S, the carrier of X, the carrier of X, F, the addition of X)) = setopfunc(S, the carrier of X, $\mathbb{R}, H, +_{\mathbb{R}}$).

(12) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let given x and S be a finite orthonormal family of X. Suppose S is non empty. Let F be a function from the carrier of X into the carrier of X. Suppose $S \subseteq \operatorname{dom} F$ and for every y such that $y \in S$ holds $F(y) = (x|y) \cdot y$. Let H be a function from the carrier of X into $\mathbb{R}$. Suppose $S \subseteq \operatorname{dom} H$ and for every y such that $y \in S$ holds $H(y) = (x|y)^{\mathbf{2}}$. Then (setopfunc($S$, the carrier of X, the carrier of X, F, the addition of X)$|$ setopfunc(S, the carrier of X, the carrier of X, F, the addition of X)) = setopfunc(S, the carrier of X, $\mathbb{R}, H, +_{\mathbb{R}}$).

(13) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let given x and S be a finite orthonormal family of X. Suppose S is non empty. Let H be a function from the carrier of X into $\mathbb{R}$. Suppose $S \subseteq \operatorname{dom} H$ and for every y such that $y \in S$ holds $H(y) = (x|y)^{\mathbf{2}}$. Then setopfunc($S$, the carrier of X, $\mathbb{R}, H, +_{\mathbb{R}}$) $\leqslant \|x\|^{\mathbf{2}}$.

(14) Let D_2, D_1 be non empty sets and f be a binary operation on D_2. Suppose f is commutative and associative and has a unity. Let Y_1, Y_2 be finite subsets of D_1. Suppose Y_1 misses Y_2. Let F be a function from D_1 into D_2. Suppose $Y_1 \subseteq \operatorname{dom} F$ and $Y_2 \subseteq \operatorname{dom} F$. Let Z be a finite subset of D_1. If $Z = Y_1 \cup Y_2$, then $\mathrm{setopfunc}(Z, D_1, D_2, F, f) = f(\mathrm{setopfunc}(Y_1, D_1, D_2, F, f), \mathrm{setopfunc}(Y_2, D_1, D_2, F, f))$.

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A Representation of Integers by Binary Arithmetics and Addition of Integers

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Summary. In this article, we introduce the new concept of 2's complement representation. Natural numbers that are congruent mod n can be represented by the same n bits binary. Using the concept introduced here, negative numbers that are congruent mod n also can be represented by the same n bit binary. We also show some properties of addition of integers using this concept.

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The articles [16], [20], [2], [3], [12], [11], [10], [9], [17], [13], [14], [6], [7], [1], [15], [18], [4], [21], [8], [5], and [19] provide the notation and terminology for this paper.

1. Preliminaries

We follow the rules: n denotes a non empty natural number, j, k, l, m denote natural numbers, and g, h, i denote integers.

We now state a number of propositions:

(1) If $m > 0$, then $m \cdot 2 \geqslant m + 1$.

(2) For every natural number m holds $2^m \geqslant m$.

(3) For every natural number m holds $\langle \underbrace{0, \ldots, 0}_{m} \rangle + \langle \underbrace{0, \ldots, 0}_{m} \rangle = \langle \underbrace{0, \ldots, 0}_{m} \rangle$.

(4) For every natural number k such that $k \leqslant l$ and $l \leqslant m$ holds $k = l$ or $k + 1 \leqslant l$ and $l \leqslant m$.

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(5) For every non empty natural number n and for all n-tuples x, y of *Boolean* such that $x = \langle\underbrace{0,\ldots,0}_{n}\rangle$ and $y = \langle\underbrace{0,\ldots,0}_{n}\rangle$ holds $\mathrm{carry}(x,y) = \langle\underbrace{0,\ldots,0}_{n}\rangle$.

(6) For every non empty natural number n and for all n-tuples x, y of *Boolean* such that $x = \langle\underbrace{0,\ldots,0}_{n}\rangle$ and $y = \langle\underbrace{0,\ldots,0}_{n}\rangle$ holds $x+y = \langle\underbrace{0,\ldots,0}_{n}\rangle$.

(7) For every non empty natural number n and for every n-tuple F of *Boolean* such that $F = \langle\underbrace{0,\ldots,0}_{n}\rangle$ holds $\mathrm{Intval}(F) = 0$.

(8) If $l + m \leqslant k - 1$, then $l < k$ and $m < k$.

(9) If $g \leqslant h + i$ and $h < 0$ and $i < 0$, then $g < h$ and $g < i$.

(10) If $l + m \leqslant 2^n - 1$, then add_ovfl(n-BinarySequence(l), n-BinarySequence(m)) = *false*.

(11) For every non empty natural number n and for all natural numbers l, m such that $l + m \leqslant 2^n - 1$ holds Absval((n-BinarySequence(l)) + (n-BinarySequence(m))) = $l + m$.

(12) For every non empty natural number n and for every n-tuple z of *Boolean* such that $z_n =$ *true* holds $\mathrm{Absval}(z) \geqslant 2^{n-'1}$.

(13) If $l + m \leqslant 2^{n-'1} - 1$, then (carry($n$-BinarySequence($l$), n-BinarySequence(m)))$_n$ = *false*.

(14) For every non empty natural number n such that $l+m \leqslant 2^{n-'1} - 1$ holds Intval((n-BinarySequence(l)) + (n-BinarySequence(m))) = $l + m$.

(15) For every 1-tuple z of *Boolean* such that $z = \langle$*true*$\rangle$ holds $\mathrm{Intval}(z) = -1$.

(16) For every 1-tuple z of *Boolean* such that $z = \langle$*false*$\rangle$ holds $\mathrm{Intval}(z) = 0$.

(17) For every boolean set x holds *true* $\vee\, x =$ *true*.

(18) For every non empty natural number n holds $0 \leqslant 2^{n-'1} - 1$ and $-2^{n-'1} \leqslant 0$.

(19) For all n-tuples x, y of *Boolean* such that $x = \langle\underbrace{0,\ldots,0}_{n}\rangle$ and $y = \langle\underbrace{0,\ldots,0}_{n}\rangle$ holds x and y are summable.

(20) $i \cdot n \bmod n = 0$.

2. Majorant Power

Let m, j be natural numbers. The functor $\mathrm{MajP}(m,j)$ yielding a natural number is defined as follows:

(Def. 1) $2^{\mathrm{MajP}(m,j)} \geqslant j$ and $\mathrm{MajP}(m,j) \geqslant m$ and for every natural number k such that $2^k \geqslant j$ and $k \geqslant m$ holds $k \geqslant \mathrm{MajP}(m,j)$.

One can prove the following propositions:

(21) If $j \geqslant k$, then $\mathrm{MajP}(m,j) \geqslant \mathrm{MajP}(m,k)$.

(22) If $l \geqslant m$, then $\mathrm{MajP}(l,j) \geqslant \mathrm{MajP}(m,j)$.

(23) If $m \geqslant 1$, then $\mathrm{MajP}(m,1) = m$.

(24) If $j \leqslant 2^m$, then $\mathrm{MajP}(m,j) = m$.

(25) If $j > 2^m$, then $\mathrm{MajP}(m,j) > m$.

3. 2's Complement

Let m be a natural number and let i be an integer.

The functor $\mathrm{2sComplement}(m,i)$ yields a m-tuple of *Boolean* and is defined by:

(Def. 2) $$\mathrm{2sComplement}(m,i) = \begin{cases} m\text{-BinarySequence}(|2^{\mathrm{MajP}(m,|i|)} + i|), & \text{if } i < 0, \\ m\text{-BinarySequence}(|i|), & \text{otherwise.} \end{cases}$$

The following propositions are true:

(26) For every natural number m holds $\mathrm{2sComplement}(m,0) = \langle \underbrace{0,\ldots,0}_{m} \rangle$.

(27) For every integer i such that $i \leqslant 2^{n-'1} - 1$ and $-2^{n-'1} \leqslant i$ holds $\mathrm{Intval}(\mathrm{2sComplement}(n,i)) = i$.

(28) For all integers h, i such that $h \geqslant 0$ and $i \geqslant 0$ or $h < 0$ and $i < 0$ but $h \bmod 2^n = i \bmod 2^n$ holds $\mathrm{2sComplement}(n,h) = \mathrm{2sComplement}(n,i)$.

(29) For all integers h, i such that $h \geqslant 0$ and $i \geqslant 0$ or $h < 0$ and $i < 0$ but $h \equiv i (\bmod 2^n)$ holds $\mathrm{2sComplement}(n,h) = \mathrm{2sComplement}(n,i)$.

(30) For all natural numbers l, m such that $l \bmod 2^n = m \bmod 2^n$ holds $n\text{-BinarySequence}(l) = n\text{-BinarySequence}(m)$.

(31) For all natural numbers l, m such that $l \equiv m (\bmod 2^n)$ holds $n\text{-BinarySequence}(l) = n\text{-BinarySequence}(m)$.

(32) For every natural number j such that $1 \leqslant j$ and $j \leqslant n$ holds $(\mathrm{2sComplement}(n+1,i))_j = (\mathrm{2sComplement}(n,i))_j$.

(33) There exists an element x of *Boolean* such that $\mathrm{2sComplement}(m+1,i) = (\mathrm{2sComplement}(m,i)) \frown \langle x \rangle$.

(34) There exists an element x of *Boolean* such that $(m+1)\text{-BinarySequence}(l) = (m\text{-BinarySequence}(l)) \frown \langle x \rangle$.

(35) Let n be a non empty natural number. Suppose $-2^n \leqslant h + i$ and $h < 0$ and $i < 0$ and $-2^{n-'1} \leqslant h$ and $-2^{n-'1} \leqslant i$. Then $(\mathrm{carry}(\mathrm{2sComplement}(n+1,h), \mathrm{2sComplement}(n+1,i)))_{n+1} = \mathit{true}$.

(36) For every non empty natural number n such that $-2^{n-'1} \leqslant h+i$ and $h+i \leqslant 2^{n-'1}-1$ and $h \geqslant 0$ and $i \geqslant 0$ holds $\mathrm{Intval}(\mathrm{2sComplement}(n,h)+\mathrm{2sComplement}(n,i)) = h+i$.

(37) Let n be a non empty natural number. Suppose $-2^{(n+1)-'1} \leqslant h+i$ and $h+i \leqslant 2^{(n+1)-'1}-1$ and $h<0$ and $i<0$ and $-2^{n-'1} \leqslant h$ and $-2^{n-'1} \leqslant i$. Then $\mathrm{Intval}(\mathrm{2sComplement}(n+1,h)+\mathrm{2sComplement}(n+1,i)) = h+i$.

(38) Let n be a non empty natural number. Suppose that $-2^{n-'1} \leqslant h$ and $h \leqslant 2^{n-'1}-1$ and $-2^{n-'1} \leqslant i$ and $i \leqslant 2^{n-'1}-1$ and $-2^{n-'1} \leqslant h+i$ and $h+i \leqslant 2^{n-'1}-1$ and $h \geqslant 0$ and $i<0$ or $h<0$ and $i \geqslant 0$. Then $\mathrm{Intval}(\mathrm{2sComplement}(n,h)+\mathrm{2sComplement}(n,i)) = h+i$.

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The Inner Product of Finite Sequences and of Points of n-dimensional Topological Space

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Summary. First, we define the inner product to finite sequences of real value. Next, we extend it to points of n-dimensional topological space $\mathcal{E}^n_{\mathrm{T}}$. At the end, orthogonality is introduced to this space.

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The notation and terminology used in this paper are introduced in the following articles: [11], [3], [9], [7], [1], [2], [6], [8], [4], [5], and [10].

1. Preliminaries

For simplicity, we use the following convention: i, n denote natural numbers, x, y, a denote real numbers, v denotes an element of $\mathbb{R}^n$, and p, p_1, p_2, p_3, q, q_1, q_2 denote points of $\mathcal{E}^n_{\mathrm{T}}$.

We now state several propositions:

(1) For every i such that $i \in \operatorname{Seg} n$ holds $(v \bullet \langle \underbrace{0, \ldots, 0}_{n} \rangle)(i) = 0$.

(2) $v \bullet \langle \underbrace{0, \ldots, 0}_{n} \rangle = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(3) For every finite sequence x of elements of $\mathbb{R}$ holds $(-1) \cdot x = -x$.

(4) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $x - y = x + -y$.

(5) For every finite sequence x of elements of $\mathbb{R}$ holds $\operatorname{len}(-x) = \operatorname{len} x$.

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(6) For all finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $\operatorname{len}(x_1 + x_2) = \operatorname{len} x_1$.

(7) For all finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $\operatorname{len}(x_1 - x_2) = \operatorname{len} x_1$.

(8) For every real number a and for every finite sequence x of elements of $\mathbb{R}$ holds $\operatorname{len}(a \cdot x) = \operatorname{len} x$.

(9) For all finite sequences x, y, z of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $(x + y) \bullet z = x \bullet z + y \bullet z$.

2. Inner Product of Finite Sequences

Let x_1, x_2 be finite sequences of elements of $\mathbb{R}$. The functor $|(x_1, x_2)|$ yielding a real number is defined as follows:

(Def. 1) $|(x_1, x_2)| = \sum(x_1 \bullet x_2)$.

Let us observe that the functor $|(x_1, x_2)|$ is commutative.

We now state a number of propositions:

(10) Let y_1, y_2 be finite sequences of elements of $\mathbb{R}$ and x_1, x_2 be elements of $\mathcal{R}^n$. If $x_1 = y_1$ and $x_2 = y_2$, then $|(y_1, y_2)| = \frac{1}{4} \cdot (|x_1 + x_2|^{\mathbf{2}} - |x_1 - x_2|^{\mathbf{2}})$.

(11) For every finite sequence x of elements of $\mathbb{R}$ holds $|(x, x)| \geqslant 0$.

(12) For every finite sequence x of elements of $\mathbb{R}$ holds $|x|^{\mathbf{2}} = |(x, x)|$.

(13) For every finite sequence x of elements of $\mathbb{R}$ holds $|x| = \sqrt{|(x, x)|}$.

(14) For every finite sequence x of elements of $\mathbb{R}$ holds $0 \leqslant |x|$.

(15) For every finite sequence x of elements of $\mathbb{R}$ holds $|(x, x)| = 0$ iff $x = \langle \underbrace{0, \ldots, 0}_{\operatorname{len} x} \rangle$.

(16) For every finite sequence x of elements of $\mathbb{R}$ holds $|(x, x)| = 0$ iff $|x| = 0$.

(17) For every finite sequence x of elements of $\mathbb{R}$ holds $|(x, \langle \underbrace{0, \ldots, 0}_{\operatorname{len} x} \rangle)| = 0$.

(18) For every finite sequence x of elements of $\mathbb{R}$ holds $|(\langle \underbrace{0, \ldots, 0}_{\operatorname{len} x} \rangle, x)| = 0$.

(19) For all finite sequences x, y, z of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $|(x + y, z)| = |(x, z)| + |(y, z)|$.

(20) For all finite sequences x, y of elements of $\mathbb{R}$ and for every real number a such that $\operatorname{len} x = \operatorname{len} y$ holds $|(a \cdot x, y)| = a \cdot |(x, y)|$.

(21) For all finite sequences x, y of elements of $\mathbb{R}$ and for every real number a such that $\operatorname{len} x = \operatorname{len} y$ holds $|(x, a \cdot y)| = a \cdot |(x, y)|$.

(22) For all finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $|(-x_1, x_2)| = -|(x_1, x_2)|$.

(23) For all finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $|(x_1, -x_2)| = -|(x_1, x_2)|$.

(24) For all finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $|(-x_1, -x_2)| = |(x_1, x_2)|$.

(25) For all finite sequences x_1, x_2, x_3 of elements of $\mathbb{R}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_2 = \operatorname{len} x_3$ holds $|(x_1 - x_2, x_3)| = |(x_1, x_3)| - |(x_2, x_3)|$.

(26) Let x, y be real numbers and x_1, x_2, x_3 be finite sequences of elements of $\mathbb{R}$. If $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_2 = \operatorname{len} x_3$, then $|(x \cdot x_1 + y \cdot x_2, x_3)| = x \cdot |(x_1, x_3)| + y \cdot |(x_2, x_3)|$.

(27) For all finite sequences x, y_1, y_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y_1$ and $\operatorname{len} y_1 = \operatorname{len} y_2$ holds $|(x, y_1 + y_2)| = |(x, y_1)| + |(x, y_2)|$.

(28) For all finite sequences x, y_1, y_2 of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y_1$ and $\operatorname{len} y_1 = \operatorname{len} y_2$ holds $|(x, y_1 - y_2)| = |(x, y_1)| - |(x, y_2)|$.

(29) Let x_1, x_2, y_1, y_2 be finite sequences of elements of $\mathbb{R}$. If $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_2 = \operatorname{len} y_1$ and $\operatorname{len} y_1 = \operatorname{len} y_2$, then $|(x_1+x_2, y_1+y_2)| = |(x_1, y_1)| + |(x_1, y_2)| + |(x_2, y_1)| + |(x_2, y_2)|$.

(30) Let x_1, x_2, y_1, y_2 be finite sequences of elements of $\mathbb{R}$. If $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_2 = \operatorname{len} y_1$ and $\operatorname{len} y_1 = \operatorname{len} y_2$, then $|(x_1 - x_2, y_1 - y_2)| = (|(x_1, y_1)| - |(x_1, y_2)| - |(x_2, y_1)|) + |(x_2, y_2)|$.

(31) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(x + y, x + y)| = |(x, x)| + 2 \cdot |(x, y)| + |(y, y)|$.

(32) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(x - y, x - y)| = (|(x, x)| - 2 \cdot |(x, y)|) + |(y, y)|$.

(33) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|x + y|^{\mathbf{2}} = |x|^{\mathbf{2}} + 2 \cdot |(y, x)| + |y|^{\mathbf{2}}$.

(34) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|x - y|^{\mathbf{2}} = (|x|^{\mathbf{2}} - 2 \cdot |(y, x)|) + |y|^{\mathbf{2}}$.

(35) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|x + y|^{\mathbf{2}} + |x - y|^{\mathbf{2}} = 2 \cdot (|x|^{\mathbf{2}} + |y|^{\mathbf{2}})$.

(36) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|x + y|^{\mathbf{2}} - |x - y|^{\mathbf{2}} = 4 \cdot |(x, y)|$.

(37) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $||(x, y)|| \leqslant |x| \cdot |y|$.

(38) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|x + y| \leqslant |x| + |y|$.

3. Inner Product of Points of $\mathcal{E}^n_{\mathrm{T}}$

Let us consider n and let p, q be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $|(p, q)|$ yielding a real number is defined as follows:

(Def. 2) There exist finite sequences f, g of elements of $\mathbb{R}$ such that $f = p$ and $g = q$ and $|(p, q)| = |(f, g)|$.

Let us observe that the functor $|(p, q)|$ is commutative.

We now state a number of propositions:

(39) For every natural number n and for all points p_1, p_2 of $\mathcal{E}^n_{\mathrm{T}}$ holds $|(p_1, p_2)| = \frac{1}{4} \cdot (|p_1 + p_2|^{\mathbf{2}} - |p_1 - p_2|^{\mathbf{2}})$.

(40) $|(p_1 + p_2, p_3)| = |(p_1, p_3)| + |(p_2, p_3)|$.

(41) For every real number x holds $|(x \cdot p_1, p_2)| = x \cdot |(p_1, p_2)|$.

(42) For every real number x holds $|(p_1, x \cdot p_2)| = x \cdot |(p_1, p_2)|$.

(43) $|(-p_1, p_2)| = -|(p_1, p_2)|$.

(44) $|(p_1, -p_2)| = -|(p_1, p_2)|$.

(45) $|(-p_1, -p_2)| = |(p_1, p_2)|$.

(46) $|(p_1 - p_2, p_3)| = |(p_1, p_3)| - |(p_2, p_3)|$.

(47) $|(x \cdot p_1 + y \cdot p_2, p_3)| = x \cdot |(p_1, p_3)| + y \cdot |(p_2, p_3)|$.

(48) $|(p, q_1 + q_2)| = |(p, q_1)| + |(p, q_2)|$.

(49) $|(p, q_1 - q_2)| = |(p, q_1)| - |(p, q_2)|$.

(50) $|(p_1 + p_2, q_1 + q_2)| = |(p_1, q_1)| + |(p_1, q_2)| + |(p_2, q_1)| + |(p_2, q_2)|$.

(51) $|(p_1 - p_2, q_1 - q_2)| = (|(p_1, q_1)| - |(p_1, q_2)| - |(p_2, q_1)|) + |(p_2, q_2)|$.

(52) $|(p + q, p + q)| = |(p, p)| + 2 \cdot |(p, q)| + |(q, q)|$.

(53) $|(p - q, p - q)| = (|(p, p)| - 2 \cdot |(p, q)|) + |(q, q)|$.

(54) $|(p, 0_{\mathcal{E}^n_{\mathrm{T}}})| = 0$.

(55) $|(0_{\mathcal{E}^n_{\mathrm{T}}}, p)| = 0$.

(56) $|(0_{\mathcal{E}^n_{\mathrm{T}}}, 0_{\mathcal{E}^n_{\mathrm{T}}})| = 0$.

(57) $|(p, p)| \geqslant 0$.

(58) $|(p, p)| = |p|^{\mathbf{2}}$.

(59) $|p| = \sqrt{|(p, p)|}$.

(60) $0 \leqslant |p|$.

(61) $|0_{\mathcal{E}^n_{\mathrm{T}}}| = 0$.

(62) $|(p, p)| = 0$ iff $|p| = 0$.

(63) $|(p, p)| = 0$ iff $p = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

(64) $|p| = 0$ iff $p = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

(65) $p \neq 0_{\mathcal{E}^n_{\mathrm{T}}}$ iff $|(p, p)| > 0$.

(66) $p \neq 0_{\mathcal{E}^n_{\mathrm{T}}}$ iff $|p| > 0$.

(67) $|p + q|^{\mathbf{2}} = |p|^{\mathbf{2}} + 2 \cdot |(q, p)| + |q|^{\mathbf{2}}$.

(68) $|p - q|^{\mathbf{2}} = (|p|^{\mathbf{2}} - 2 \cdot |(q, p)|) + |q|^{\mathbf{2}}$.

(69) $|p + q|^{\mathbf{2}} + |p - q|^{\mathbf{2}} = 2 \cdot (|p|^{\mathbf{2}} + |q|^{\mathbf{2}})$.

(70) $|p + q|^{\mathbf{2}} - |p - q|^{\mathbf{2}} = 4 \cdot |(p, q)|$.

(71) $|(p, q)| = \frac{1}{4} \cdot (|p + q|^{\mathbf{2}} - |p - q|^{\mathbf{2}})$.

(72) $|(p, q)| \leqslant |(p, p)| + |(q, q)|$.

(73) For all points p, q of $\mathcal{E}^n_{\mathrm{T}}$ holds $||(p, q)|| \leqslant |p| \cdot |q|$.

(74) $|p + q| \leqslant |p| + |q|$.

Let us consider n, p, q. We say that p, q are orthogonal if and only if:

(Def. 3) $|(p, q)| = 0$.

Let us note that the predicate p, q are orthogonal is symmetric.

The following propositions are true:

(75) p, $0_{\mathcal{E}^n_{\mathrm{T}}}$ are orthogonal.

(76) $0_{\mathcal{E}^n_{\mathrm{T}}}$, p are orthogonal.

(77) p, p are orthogonal iff $p = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

(78) If p, q are orthogonal, then $a \cdot p$, q are orthogonal.

(79) If p, q are orthogonal, then p, $a \cdot q$ are orthogonal.

(80) If for every q holds p, q are orthogonal, then $p = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

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Solving Roots of Polynomial Equation of Degree 4 with Real Coefficients

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Summary. In this paper, we describe the definition of the fourth degree algebraic equations and their properties. We clarify the relation between the four roots of this equation and its coefficient. Also, the form of these roots for various conditions is discussed. This solution is known as the Cardano solution.

MML Identifier: POLYEQ_2.

The articles [3], [4], [1], and [2] provide the notation and terminology for this paper.

Let a, b, c, d, e, x be real numbers. The functor $\mathrm{Four}(a,b,c,d,e,x)$ is defined by:

(Def. 1) $\mathrm{Four}(a,b,c,d,e,x) = a \cdot x^4 + b \cdot x^3 + c \cdot x^{\mathbf{2}} + d \cdot x + e$.

Let a, b, c, d, e, x be real numbers. Note that $\mathrm{Four}(a,b,c,d,e,x)$ is real.

One can prove the following propositions:

(1) Let a, c, e, x be real numbers. Suppose $a \neq 0$ and $e \neq 0$ and $c^{\mathbf{2}} - 4 \cdot a \cdot e > 0$. Suppose $\mathrm{Four}(a,0,c,0,e,x) = 0$. Then $x \neq 0$ but $x = \sqrt{\frac{-c+\sqrt{\Delta(a,c,e)}}{2 \cdot a}}$ or $x = \sqrt{\frac{-c-\sqrt{\Delta(a,c,e)}}{2 \cdot a}}$ or $x = -\sqrt{\frac{-c+\sqrt{\Delta(a,c,e)}}{2 \cdot a}}$ or $x = -\sqrt{\frac{-c-\sqrt{\Delta(a,c,e)}}{2 \cdot a}}$.

(2) Let a, b, c, x, y be real numbers. Suppose $a \neq 0$ and $y = x + \frac{1}{x}$. If $\mathrm{Four}(a,b,c,b,a,x) = 0$, then $x \neq 0$ and $(a \cdot y^{\mathbf{2}} + b \cdot y + c) - 2 \cdot a = 0$.

(3) Let a, b, c, x, y be real numbers. Suppose $a \neq 0$ and $(b^{\mathbf{2}} - 4 \cdot a \cdot c) + 8 \cdot a^{\mathbf{2}} > 0$ and $y = x + \frac{1}{x}$. Suppose $\mathrm{Four}(a,b,c,b,a,x) = 0$. Let y_1, y_2 be real numbers. Suppose $y_1 = \frac{-b+\sqrt{(b^{\mathbf{2}}-4 \cdot a \cdot c)+8 \cdot a^{\mathbf{2}}}}{2 \cdot a}$ and $y_2 = \frac{-b-\sqrt{(b^{\mathbf{2}}-4 \cdot a \cdot c)+8 \cdot a^{\mathbf{2}}}}{2 \cdot a}$. Then $x \neq 0$ but $x = \frac{y_1+\sqrt{\Delta(1,-y_1,1)}}{2}$ or $x = \frac{y_2+\sqrt{\Delta(1,-y_2,1)}}{2}$ or $x = \frac{y_1-\sqrt{\Delta(1,-y_1,1)}}{2}$ or $x = \frac{y_2-\sqrt{\Delta(1,-y_2,1)}}{2}$.

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(4) For every real number x holds $x^3 = x^{\mathbf{2}} \cdot x$ and $x^3 \cdot x = x^4$ and $x^{\mathbf{2}} \cdot x^{\mathbf{2}} = x^4$.

(5) For all real numbers x, y such that $x + y \neq 0$ holds $(x+y)^4 = (x^3 + (3 \cdot y \cdot x^{\mathbf{2}} + 3 \cdot y^{\mathbf{2}} \cdot x) + y^3) \cdot x + (x^3 + (3 \cdot y \cdot x^{\mathbf{2}} + 3 \cdot y^{\mathbf{2}} \cdot x) + y^3) \cdot y$.

(6) For all real numbers x, y such that $x + y \neq 0$ holds $(x+y)^4 = x^4 + (4 \cdot y \cdot x^3 + 6 \cdot y^{\mathbf{2}} \cdot x^{\mathbf{2}} + 4 \cdot y^3 \cdot x) + y^4$.

(7) Let a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5 be real numbers. Suppose that for every real number x holds $\mathrm{Four}(a_1, a_2, a_3, a_4, a_5, x) = \mathrm{Four}(b_1, b_2, b_3, b_4, b_5, x)$. Then $a_5 = b_5$ and $((a_1 - a_2) + a_3) - a_4 = ((b_1 - b_2) + b_3) - b_4$ and $a_1 + a_2 + a_3 + a_4 = b_1 + b_2 + b_3 + b_4$.

(8) Let a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5 be real numbers. Suppose that for every real number x holds $\mathrm{Four}(a_1, a_2, a_3, a_4, a_5, x) = \mathrm{Four}(b_1, b_2, b_3, b_4, b_5, x)$. Then $a_1 - b_1 = b_3 - a_3$ and $a_2 - b_2 = b_4 - a_4$.

(9) Let a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5 be real numbers. Suppose that for every real number x holds $\mathrm{Four}(a_1, a_2, a_3, a_4, a_5, x) = \mathrm{Four}(b_1, b_2, b_3, b_4, b_5, x)$. Then $a_1 = b_1$ and $a_2 = b_2$ and $a_3 = b_3$ and $a_4 = b_4$ and $a_5 = b_5$.

Let $a_1, x_1, x_2, x_3, x_4, x$ be real numbers. The functor $\mathrm{Four0}(a_1, x_1, x_2, x_3, x_4, x)$ is defined by:

(Def. 2) $\mathrm{Four0}(a_1, x_1, x_2, x_3, x_4, x) = a_1 \cdot ((x - x_1) \cdot (x - x_2) \cdot (x - x_3) \cdot (x - x_4))$.

Let a_1, x_1, x_2, x_3, x_4, x be real numbers.

One can verify that $\mathrm{Four0}(a_1, x_1, x_2, x_3, x_4, x)$ is real.

One can prove the following propositions:

(10) Let a_1, a_2, a_3, a_4, a_5, x, x_1, x_2, x_3, x_4 be real numbers. Suppose $a_1 \neq 0$. Suppose that for every real number x holds $\mathrm{Four}(a_1, a_2, a_3, a_4, a_5, x) = \mathrm{Four0}(a_1, x_1, x_2, x_3, x_4, x)$. Then $\frac{a_1 \cdot x^4 + a_2 \cdot x^3 + a_3 \cdot x^{\mathbf{2}} + a_4 \cdot x + a_5}{a_1} = ((x^{\mathbf{2}} \cdot x^{\mathbf{2}} - (x_1 + x_2 + x_3) \cdot (x^{\mathbf{2}} \cdot x)) + (x_1 \cdot x_3 + x_2 \cdot x_3 + x_1 \cdot x_2) \cdot x^{\mathbf{2}}) - x_1 \cdot x_2 \cdot x_3 \cdot x - (x - x_1) \cdot (x - x_2) \cdot (x - x_3) \cdot x_4$.

(11) Let a_1, a_2, a_3, a_4, a_5, x, x_1, x_2, x_3, x_4 be real numbers. Suppose $a_1 \neq 0$. Suppose that for every real number x holds $\mathrm{Four}(a_1, a_2, a_3, a_4, a_5, x) = \mathrm{Four0}(a_1, x_1, x_2, x_3, x_4, x)$. Then $\frac{a_1 \cdot x^4 + a_2 \cdot x^3 + a_3 \cdot x^{\mathbf{2}} + a_4 \cdot x + a_5}{a_1} = (((x^4 - (x_1 + x_2 + x_3 + x_4) \cdot x^3) + (x_1 \cdot x_2 + x_1 \cdot x_3 + x_1 \cdot x_4 + (x_2 \cdot x_3 + x_2 \cdot x_4) + x_3 \cdot x_4) \cdot x^{\mathbf{2}}) - (x_1 \cdot x_2 \cdot x_3 + x_1 \cdot x_2 \cdot x_4 + x_1 \cdot x_3 \cdot x_4 + x_2 \cdot x_3 \cdot x_4) \cdot x) + x_1 \cdot x_2 \cdot x_3 \cdot x_4$.

(12) Let a_1, a_2, a_3, a_4, a_5, x_1, x_2, x_3, x_4 be real numbers. Suppose $a_1 \neq 0$ and for every real number x holds $\mathrm{Four}(a_1, a_2, a_3, a_4, a_5, x) = \mathrm{Four0}(a_1, x_1, x_2, x_3, x_4, x)$. Then $\frac{a_2}{a_1} = -(x_1 + x_2 + x_3 + x_4)$ and $\frac{a_3}{a_1} = x_1 \cdot x_2 + x_1 \cdot x_3 + x_1 \cdot x_4 + (x_2 \cdot x_3 + x_2 \cdot x_4) + x_3 \cdot x_4$ and $\frac{a_4}{a_1} = -(x_1 \cdot x_2 \cdot x_3 + x_1 \cdot x_2 \cdot x_4 + x_1 \cdot x_3 \cdot x_4 + x_2 \cdot x_3 \cdot x_4)$ and $\frac{a_5}{a_1} = x_1 \cdot x_2 \cdot x_3 \cdot x_4$.

(13) Let a, k, y be real numbers. Suppose $a \neq 0$. Suppose that for every real number x holds $x^4 + a^4 = k \cdot a \cdot x \cdot (x^{\mathbf{2}} + a^{\mathbf{2}})$. Then $(y^4 - k \cdot y^3 - k \cdot y) + 1 = 0$.

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Morphisms Into Chains. Part I

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Summary. This work is the continuation of formalization of [10]. Items from 2.1 to 2.8 of Chapter 4 are proved.

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The papers [16], [7], [19], [15], [4], [17], [18], [14], [1], [20], [22], [21], [5], [6], [2], [12], [13], [23], [3], [8], [11], and [9] provide the notation and terminology for this paper.

1. Preliminaries

Let X be a set. One can verify that there exists a subset of X which is trivial.

Let X be a trivial set. Note that every subset of X is trivial.

Let L be a 1-sorted structure. One can check that there exists a subset of L which is trivial.

Let L be a relational structure. Note that there exists a subset of L which is trivial.

Let L be a non empty 1-sorted structure. One can check that there exists a subset of L which is non empty and trivial.

Let L be a non empty relational structure. Note that there exists a subset of L which is non empty and trivial.

Next we state three propositions:

(1) For every set X holds $\subseteq_X$ is reflexive in X.

(2) For every set X holds $\subseteq_X$ is transitive in X.

(3) For every set X holds $\subseteq_X$ is antisymmetric in X.

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2. Main Part

Let L be a relational structure. Observe that there exists a binary relation on L which is auxiliary(i).

Let L be a transitive relational structure. Observe that there exists a binary relation on L which is auxiliary(i) and auxiliary(ii).

Let L be an antisymmetric relational structure with l.u.b.'s. Observe that there exists a binary relation on L which is auxiliary(iii).

Let L be a non empty lower-bounded antisymmetric relational structure. Note that there exists a binary relation on L which is auxiliary(iv).

Let L be a non empty relational structure and let R be a binary relation on L. We say that R is extra-order if and only if:

(Def. 1) R is auxiliary(i), auxiliary(ii), and auxiliary(iv).

Let L be a non empty relational structure. One can verify that every binary relation on L which is extra-order is also auxiliary(i), auxiliary(ii), and auxiliary(iv) and every binary relation on L which is auxiliary(i), auxiliary(ii), and auxiliary(iv) is also extra-order.

Let L be a non empty relational structure. One can verify that every binary relation on L which is extra-order and auxiliary(iii) is also auxiliary and every binary relation on L which is auxiliary is also extra-order.

Let L be a lower-bounded antisymmetric transitive non empty relational structure. One can check that there exists a binary relation on L which is extra-order.

Let L be a lower-bounded poset with l.u.b.'s and let R be an auxiliary(ii) binary relation on L. The functor $R-$LowerMap yields a map from L into $\langle \mathrm{LOWER}\, L, \subseteq \rangle$ and is defined as follows:

(Def. 2) For every element x of the carrier of L holds $R-\mathrm{LowerMap}(x) = {\downarrow_R} x$.

Let L be a lower-bounded poset with l.u.b.'s and let R be an auxiliary(ii) binary relation on L. One can verify that $R-$LowerMap is monotone.

Let L be a 1-sorted structure and let R be a binary relation on the carrier of L. A subset of L is called a strict chain of R if:

(Def. 3) For all sets x, y such that $x \in$ it and $y \in$ it holds $\langle x, y \rangle \in R$ or $x = y$ or $\langle y, x \rangle \in R$.

The following proposition is true

(4) Let L be a 1-sorted structure, C be a trivial subset of L, and R be a binary relation on the carrier of L. Then C is a strict chain of R.

Let L be a non empty 1-sorted structure and let R be a binary relation on the carrier of L. One can check that there exists a strict chain of R which is non empty and trivial.

One can prove the following four propositions:

(5) Let L be an antisymmetric relational structure, R be an auxiliary(i) binary relation on L, C be a strict chain of R, and x, y be elements of the carrier of L. If $x \in C$ and $y \in C$ and $x < y$, then $\langle x, y \rangle \in R$.

(6) Let L be an antisymmetric relational structure, R be an auxiliary(i) binary relation on L, and x, y be elements of the carrier of L. If $\langle x, y \rangle \in R$ and $\langle y, x \rangle \in R$, then $x = y$.

(7) Let L be a non empty lower-bounded antisymmetric relational structure, x be an element of the carrier of L, and R be an auxiliary(iv) binary relation on L. Then $\{\perp_L, x\}$ is a strict chain of R.

(8) Let L be a non empty lower-bounded antisymmetric relational structure, R be an auxiliary(iv) binary relation on L, and C be a strict chain of R. Then $C \cup \{\perp_L\}$ is a strict chain of R.

Let L be a 1-sorted structure, let R be a binary relation on the carrier of L, and let C be a strict chain of R. We say that C is maximal if and only if:

(Def. 4) For every strict chain D of R such that $C \subseteq D$ holds $C = D$.

Let L be a 1-sorted structure, let R be a binary relation on the carrier of L, and let C be a set. The functor StrictChains(R, C) is defined by:

(Def. 5) For every set x holds $x \in$ StrictChains(R, C) iff x is a strict chain of R and $C \subseteq x$.

Let L be a 1-sorted structure, let R be a binary relation on the carrier of L, and let C be a strict chain of R. Note that StrictChains(R, C) is non empty.

Let R be a binary relation and let X be a set. We introduce X is inductive w.r.t. R as a synonym of X has the upper Zorn property w.r.t. R.

Next we state several propositions:

(9) Let L be a 1-sorted structure, R be a binary relation on the carrier of L, and C be a strict chain of R. Then StrictChains(R, C) is inductive w.r.t. $\subseteq_{\mathrm{StrictChains}(R,C)}$ and there exists a set D such that D is maximal in $\subseteq_{\mathrm{StrictChains}(R,C)}$ and $C \subseteq D$.

(10) Let L be a non empty transitive relational structure, C be a non empty subset of the carrier of L, and X be a subset of C. Suppose sup X exists in L and $\bigsqcup_L X \in C$. Then sup X exists in sub(C) and $\bigsqcup_L X = \bigsqcup_{\mathrm{sub}(C)} X$.

(11) Let L be a non empty poset, R be an auxiliary(i) auxiliary(ii) binary relation on L, C be a non empty strict chain of R, and X be a subset of C. If sup X exists in L and C is maximal, then sup X exists in sub(C).

(12) Let L be a non empty poset, R be an auxiliary(i) auxiliary(ii) binary relation on L, C be a non empty strict chain of R, and X be a subset of C. Suppose inf $\uparrow\bigsqcup_L X \cap C$ exists in L and sup X exists in L and C is maximal. Then $\bigsqcup_{\mathrm{sub}(C)} X = \bigsqcap_L(\uparrow\bigsqcup_L X \cap C)$ and if $\bigsqcup_L X \notin C$, then $\bigsqcup_L X < \bigsqcap_L(\uparrow\bigsqcup_L X \cap C)$.

(13) Let L be a complete non empty poset, R be an auxiliary(i) auxiliary(ii)

binary relation on L, and C be a non empty strict chain of R. If C is maximal, then $\mathrm{sub}(C)$ is complete.

(14) Let L be a non empty lower-bounded antisymmetric relational structure, R be an auxiliary(iv) binary relation on L, and C be a strict chain of R. If C is maximal, then $\bot_L \in C$.

(15) Let L be a non empty upper-bounded poset, R be an auxiliary(ii) binary relation on L, C be a strict chain of R, and m be an element of the carrier of L. Suppose C is maximal and m is a maximum of C and $\langle m, \top_L\rangle \in R$. Then $\langle \top_L, \top_L\rangle \in R$ and $m = \top_L$.

Let L be a relational structure, let C be a set, and let R be a binary relation on the carrier of L. We say that R satisfies SIC on C if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let x, z be elements of the carrier of L. Suppose $x \in C$ and $z \in C$ and $\langle x, z\rangle \in R$ and $x \neq z$. Then there exists an element y of L such that $y \in C$ and $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$ and $x \neq y$.

Let L be a relational structure, let R be a binary relation on the carrier of L, and let C be a strict chain of R. We say that C satisfies SIC if and only if:

(Def. 7) R satisfies SIC on C.

We introduce C satisfies the interpolation property and C satisfies the interpolation property as synonyms of C satisfies SIC.

The following proposition is true

(16) Let L be a relational structure, C be a set, and R be an auxiliary(i) binary relation on L. Suppose R satisfies SIC on C. Let x, z be elements of the carrier of L. Suppose $x \in C$ and $z \in C$ and $\langle x, z\rangle \in R$ and $x \neq z$. Then there exists an element y of L such that $y \in C$ and $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$ and $x < y$.

Let L be a relational structure and let R be a binary relation on the carrier of L. Note that every strict chain of R which is trivial satisfies also SIC.

Let L be a non empty relational structure and let R be a binary relation on the carrier of L. One can check that there exists a strict chain of R which is non empty and trivial.

Next we state the proposition

(17) Let L be a lower-bounded poset with l.u.b.'s, R be an auxiliary(i) auxiliary(ii) binary relation on L, and C be a strict chain of R. Suppose C is maximal and R satisfies strong interpolation property. Then R satisfies SIC on C.

Let R be a binary relation and let C, y be sets. The functor $\mathrm{SetBelow}(R, C, y)$ is defined as follows:

(Def. 8) $\mathrm{SetBelow}(R, C, y) = R^{-1}(\{y\}) \cap C$.

The following proposition is true

(18) For every binary relation R and for all sets C, x, y holds $x \in \mathrm{SetBelow}(R, C, y)$ iff $\langle x, y\rangle \in R$ and $x \in C$.

Let L be a 1-sorted structure, let R be a binary relation on the carrier of L, and let C, y be sets. Then $\mathrm{SetBelow}(R, C, y)$ is a subset of L.

Next we state three propositions:

(19) Let L be a relational structure, R be an auxiliary(i) binary relation on L, C be a set, and y be an element of the carrier of L. Then $\mathrm{SetBelow}(R, C, y) \leqslant y$.

(20) Let L be a reflexive transitive relational structure, R be an auxiliary(ii) binary relation on L, C be a subset of the carrier of L, and x, y be elements of the carrier of L. If $x \leqslant y$, then $\mathrm{SetBelow}(R, C, x) \subseteq \mathrm{SetBelow}(R, C, y)$.

(21) Let L be a relational structure, R be an auxiliary(i) binary relation on L, C be a set, and x be an element of the carrier of L. If $x \in C$ and $\langle x, x\rangle \in R$ and sup $\mathrm{SetBelow}(R, C, x)$ exists in L, then $x = \sup \mathrm{SetBelow}(R, C, x)$.

Let L be a relational structure and let C be a subset of L. We say that C is sup-closed if and only if:

(Def. 9) For every subset X of C such that sup X exists in L holds $\bigsqcup_L X = \bigsqcup_{\mathrm{sub}(C)} X$.

Next we state three propositions:

(22) Let L be a complete non empty poset, R be an extra-order binary relation on L, C be a strict chain of R satisfying SIC, and p, q be elements of the carrier of L. Suppose $p \in C$ and $q \in C$ and $p < q$. Then there exists an element y of L such that $p < y$ and $\langle y, q\rangle \in R$ and $y = \sup \mathrm{SetBelow}(R, C, y)$.

(23) Let L be a lower-bounded non empty poset, R be an extra-order binary relation on L, and C be a non empty strict chain of R. Suppose that

(i) C is sup-closed,

(ii) for every element c of the carrier of L such that $c \in C$ holds sup $\mathrm{SetBelow}(R, C, c)$ exists in L, and

(iii) R satisfies SIC on C.

Let c be an element of the carrier of L. If $c \in C$, then $c = \sup \mathrm{SetBelow}(R, C, c)$.

(24) Let L be a non empty reflexive antisymmetric relational structure, R be an auxiliary(i) binary relation on L, and C be a strict chain of R. Suppose that for every element c of the carrier of L such that $c \in C$ holds sup $\mathrm{SetBelow}(R, C, c)$ exists in L and $c = \sup \mathrm{SetBelow}(R, C, c)$. Then R satisfies SIC on C.

Let L be a non empty relational structure, let R be a binary relation on the carrier of L, and let C be a set. The functor $\mathrm{SupBelow}(R, C)$ is defined by:

(Def. 10) For every set y holds $y \in \mathrm{SupBelow}(R, C)$ iff $y = \sup \mathrm{SetBelow}(R, C, y)$.

Let L be a non empty relational structure, let R be a binary relation on the carrier of L, and let C be a set. Then SupBelow(R, C) is a subset of L.

One can prove the following propositions:

(25) Let L be a non empty reflexive transitive relational structure, R be an auxiliary(i) auxiliary(ii) binary relation on L, and C be a strict chain of R. Suppose that for every element c of L holds sup SetBelow(R, C, c) exists in L. Then SupBelow(R, C) is a strict chain of R.

(26) Let L be a non empty poset, R be an auxiliary(i) auxiliary(ii) binary relation on L, and C be a subset of the carrier of L. Suppose that for every element c of L holds sup SetBelow(R, C, c) exists in L. Then SupBelow(R, C) is sup-closed.

(27) Let L be a complete non empty poset, R be an extra-order binary relation on L, C be a strict chain of R satisfying SIC, and d be an element of the carrier of L. Suppose $d \in$ SupBelow(R, C). Then $d = \bigsqcup_L \{b; b$ ranges over elements of the carrier of L: $b \in$ SupBelow$(R, C) \ \wedge \ \langle b, d \rangle \in R\}$.

(28) Let L be a complete non empty poset, R be an extra-order binary relation on L, and C be a strict chain of R satisfying SIC. Then R satisfies SIC on SupBelow(R, C).

(29) Let L be a complete non empty poset, R be an extra-order binary relation on L, C be a strict chain of R satisfying SIC, and a, b be elements of the carrier of L. Suppose $a \in C$ and $b \in C$ and $a < b$. Then there exists an element d of L such that $d \in$ SupBelow(R, C) and $a < d$ and $\langle d, b \rangle \in R$.

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Propositional Calculus for Boolean Valued Functions. Part VII

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Summary. In this paper, we proved some elementary propositional calculus formulae for Boolean valued functions.

MML Identifier: `BVFUNC25`.

The notation and terminology used in this paper have been introduced in the following articles: [4], [3], [2], and [1].

We use the following convention: Y is a non empty set and a, b, c, d are elements of $Boolean^Y$.

The following propositions are true:

(1) $\neg(a \Rightarrow b) = a \wedge \neg b$.

(2) $\neg b \Rightarrow \neg a \Rightarrow a \Rightarrow b = true(Y)$.

(3) $a \Rightarrow b = \neg b \Rightarrow \neg a$.

(4) $a \Leftrightarrow b = \neg a \Leftrightarrow \neg b$.

(5) $a \Rightarrow b = a \Rightarrow a \wedge b$.

(6) $a \Leftrightarrow b = a \vee b \Rightarrow a \wedge b$.

(7) $a \Leftrightarrow \neg a = false(Y)$.

(8) $a \Rightarrow b \Rightarrow c = b \Rightarrow a \Rightarrow c$.

(9) $a \Rightarrow b \Rightarrow c = a \Rightarrow b \Rightarrow a \Rightarrow c$.

(10) $a \Leftrightarrow b = a \oplus \neg b$.

(11) $a \wedge (b \oplus c) = a \wedge b \oplus a \wedge c$.

(12) $a \Leftrightarrow b = \neg(a \oplus b)$.

(13) $a \oplus a = false(Y)$.

(14) $a \oplus \neg a = true(Y)$.

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(15) $a \Rightarrow b \Rightarrow b \Rightarrow a = b \Rightarrow a$.

(16) $(a \vee b) \wedge (\neg a \vee \neg b) = \neg a \wedge b \vee a \wedge \neg b$.

(17) $a \wedge b \vee \neg a \wedge \neg b = (\neg a \vee b) \wedge (a \vee \neg b)$.

(18) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(19) $a \Leftrightarrow b \Leftrightarrow c = a \Leftrightarrow b \Leftrightarrow c$.

(20) $\neg\neg a \Rightarrow a = \mathit{true}(Y)$.

(21) $(a \Rightarrow b) \wedge a \Rightarrow b = \mathit{true}(Y)$.

(22) $a \Rightarrow \neg a \Rightarrow a = \mathit{true}(Y)$.

(23) $\neg a \Rightarrow a \Leftrightarrow a = \mathit{true}(Y)$.

(24) $a \vee (a \Rightarrow b) = \mathit{true}(Y)$.

(25) $(a \Rightarrow b) \vee (c \Rightarrow a) = \mathit{true}(Y)$.

(26) $(a \Rightarrow b) \vee (\neg a \Rightarrow b) = \mathit{true}(Y)$.

(27) $(a \Rightarrow b) \vee (a \Rightarrow \neg b) = \mathit{true}(Y)$.

(28) $\neg a \Rightarrow \neg b \Leftrightarrow b \Rightarrow a = \mathit{true}(Y)$.

(29) $a \Rightarrow b \Rightarrow a \Rightarrow c \Rightarrow b \Rightarrow b = \mathit{true}(Y)$.

(30) $a \Rightarrow b = a \Leftrightarrow a \wedge b$.

(31) $a \Rightarrow b = \mathit{true}(Y)$ and $b \Rightarrow a = \mathit{true}(Y)$ iff $a = b$.

(32) $a = \neg a \Rightarrow a$.

(33) $a \Rightarrow a \Rightarrow b \Rightarrow a = \mathit{true}(Y)$.

(34) $a = a \Rightarrow b \Rightarrow a$.

(35) $a = (b \Rightarrow a) \wedge (\neg b \Rightarrow a)$.

(36) $a \wedge b = \neg(a \Rightarrow \neg b)$.

(37) $a \vee b = \neg a \Rightarrow b$.

(38) $a \vee b = a \Rightarrow b \Rightarrow b$.

(39) $a \Rightarrow b \Rightarrow a \Rightarrow a = \mathit{true}(Y)$.

(40) $a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow b \Rightarrow a \Rightarrow d \Rightarrow c = \mathit{true}(Y)$.

(41) $(a \Rightarrow b) \wedge a \wedge c \Rightarrow b = \mathit{true}(Y)$.

(42) $b \Rightarrow c \Rightarrow a \wedge b \Rightarrow c = \mathit{true}(Y)$.

(43) $a \wedge b \Rightarrow c \Rightarrow a \wedge b \Rightarrow c \wedge b = \mathit{true}(Y)$.

(44) $a \Rightarrow b \Rightarrow a \wedge c \Rightarrow b \wedge c = \mathit{true}(Y)$.

(45) $(a \Rightarrow b) \wedge (a \wedge c) \Rightarrow b \wedge c = \mathit{true}(Y)$.

(46) $a \wedge (a \Rightarrow b) \wedge (b \Rightarrow c) \Subset c$.

(47) $(a \vee b) \wedge (a \Rightarrow c) \wedge (b \Rightarrow c) \Subset \neg a \Rightarrow b \vee c$.

Acknowledgments

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Basic Notions and Properties of Orthoposets[1]

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Summary. Orthoposets are defined. The approach is the standard one via order relation similar to common text books on algebra like [8].

MML Identifier: `OPOSET_1`.

The terminology and notation used in this paper are introduced in the following papers: [11], [13], [5], [3], [4], [15], [14], [16], [12], [9], [7], [10], [2], [6], and [1].

1. General Notions and Properties

In this paper S, X denote non empty sets and R denotes a binary relation on X.

We consider orthorelational structures, extensions of relational structure and ComplStr, as systems

⟨ a carrier, an internal relation, a complement operation ⟩,

where the carrier is a set, the internal relation is a binary relation on the carrier, and the complement operation is a unary operation on the carrier.

Let A, B be sets. The functor $\emptyset_{A,B}$ yields a relation between A and B and is defined as follows:

(Def. 1) $\emptyset_{A,B} = \emptyset$.

The functor $\Omega_B(A)$ yields a relation between A and B and is defined by:

(Def. 2) $\Omega_B(A) = [\!: A, B :\!]$.

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[2]This paper was worked out while the author was visiting the University of Białystok in autumn 2002.

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We now state several propositions:

(1) $\mathrm{field}(\mathrm{id}_X) = X$.

(2) $\mathrm{id}_{\{\emptyset\}} = \{\langle\emptyset, \emptyset\rangle\}$.

(3) $\mathrm{op}_1 = \{\langle\emptyset, \emptyset\rangle\}$.

(4) Let L be a non empty reflexive antisymmetric relational structure and x, y be elements of L. If $x \leqslant y$, then $\sup\{x, y\} = y$ and $\inf\{x, y\} = x$.

(5) $\mathrm{dom}\, R \subseteq \mathrm{field}\, R$ and $\mathrm{rng}\, R \subseteq \mathrm{field}\, R$.

(6) For all sets A, B holds $\mathrm{field}(\emptyset_{A,B}) = \emptyset$.

Let Y be a set. Note that there exists a binary relation on Y which is antisymmetric.

We now state a number of propositions:

(7) If R is reflexive in X, then R is reflexive and $\mathrm{field}\, R = X$.

(8) If R is symmetric in X, then R is symmetric.

(9) If R is symmetric and $\mathrm{field}\, R \subseteq S$, then R is symmetric in S.

(10) If R is antisymmetric and $\mathrm{field}\, R \subseteq S$, then R is antisymmetric in S.

(11) If R is antisymmetric in X, then R is antisymmetric.

(12) If R is transitive and $\mathrm{field}\, R \subseteq S$, then R is transitive in S.

(13) If R is transitive in X, then R is transitive.

(14) If R is asymmetric and $\mathrm{field}\, R \subseteq S$, then R is asymmetric in S.

(15) If R is asymmetric in X, then R is asymmetric.

(16) If R is irreflexive and $\mathrm{field}\, R \subseteq S$, then R is irreflexive in S.

(17) If R is irreflexive in X, then R is irreflexive.

Let X be a set. Observe that every binary relation on X which is equivalence relation-like is also reflexive, symmetric, and transitive.

Let us consider X. One can check that there exists a binary relation on X which is equivalence relation-like.

Let X be a set. Note that there exists a binary relation on X which is irreflexive, asymmetric, and transitive.

The following proposition is true

(18) $\triangle_\emptyset$ is antisymmetric.

Let us consider X, R and let C be a unary operation on X. Note that $\langle X, R, C\rangle$ is non empty.

Let us mention that there exists a orthorelational structure which is non empty and strict.

Let us consider X and let f be a unary operation on X. We say that f is dneg if and only if:

(Def. 3) For every element x of X holds $f(f(x)) = x$.

We introduce f is involutive as a synonym of f is dneg.

One can prove the following two propositions:

(19) op_1 is dneg.

(20) id_X is dneg.

Let O be a non empty orthorelational structure and let f be a map from O into O. We say that f is DNeg if and only if:

(Def. 4) f is dneg.

Let O be a non empty orthorelational structure. Observe that there exists a map from O into O which is DNeg.

The strict orthorelational structure TrivOrthoRelStr is defined as follows:

(Def. 5) $\mathrm{TrivOrthoRelStr} = \langle \{\emptyset\}, \mathrm{id}_{\{\emptyset\}}, \mathrm{op}_1 \rangle$.

We introduce TrivPoset as a synonym of TrivOrthoRelStr.

Let us mention that TrivOrthoRelStr is non empty.

The strict orthorelational structure TrivAsymOrthoRelStr is defined by:

(Def. 6) $\mathrm{TrivAsymOrthoRelStr} = \langle \{\emptyset\}, \emptyset_{\{\emptyset\},\{\emptyset\}}, \mathrm{op}_1 \rangle$.

Let us mention that TrivAsymOrthoRelStr is non empty.

Let O be a non empty orthorelational structure. We say that O is Dneg if and only if:

(Def. 7) There exists a map f from O into O such that $f =$ the complement operation of O and f is DNeg.

One can prove the following proposition

(21) TrivOrthoRelStr is Dneg.

Let us note that TrivOrthoRelStr is Dneg.

Let us observe that there exists a non empty orthorelational structure which is Dneg.

In the sequel O is a non empty orthorelational structure.

Let R_1, R_2 be relational structures and let f be a map from R_1 into R_2. We say that f is Antitone on R_1, R_2 if and only if:

(Def. 8) f is antitone.

Let R be a relational structure and let f be a map from R into R. We say that f is Antitone on R if and only if:

(Def. 9) f is Antitone on R, R.

Let us consider O. We say that O is SubReFlexive if and only if:

(Def. 10) The internal relation of O is reflexive.

Let us consider O. We say that O is ReFlexive if and only if:

(Def. 11) The internal relation of O is reflexive in the carrier of O.

We now state two propositions:

(22) If O is ReFlexive, then O is SubReFlexive.

(23) TrivOrthoRelStr is ReFlexive.

Let us observe that TrivOrthoRelStr is ReFlexive.

One can verify that there exists a non empty orthorelational structure which is ReFlexive and strict.

Let us consider O. We say that O is SubIrreFlexive if and only if:

(Def. 12) The internal relation of O is irreflexive.

We say that O is IrreFlexive if and only if:

(Def. 13) The internal relation of O is irreflexive in the carrier of O.

We now state two propositions:

(24) If O is IrreFlexive, then O is SubIrreFlexive.

(25) TrivAsymOrthoRelStr is IrreFlexive.

Let us note that every non empty orthorelational structure which is IrreFlexive is also SubIrreFlexive.

Let us observe that TrivAsymOrthoRelStr is IrreFlexive.

Let us note that there exists a non empty orthorelational structure which is IrreFlexive and strict.

Let us consider O. We say that O is SubSymmetric if and only if:

(Def. 14) The internal relation of O is a symmetric binary relation on the carrier of O.

Let us consider O. We say that O is Symmetric if and only if:

(Def. 15) The internal relation of O is symmetric in the carrier of O.

We now state two propositions:

(26) If O is Symmetric, then O is SubSymmetric.

(27) TrivOrthoRelStr is Symmetric.

Let us observe that every non empty orthorelational structure which is Symmetric is also SubSymmetric.

Let us note that there exists a non empty orthorelational structure which is Symmetric and strict.

Let us consider O. We say that O is SubAntisymmetric if and only if:

(Def. 16) The internal relation of O is an antisymmetric binary relation on the carrier of O.

Let us consider O. We say that O is Antisymmetric if and only if:

(Def. 17) The internal relation of O is antisymmetric in the carrier of O.

Next we state two propositions:

(28) If O is Antisymmetric, then O is SubAntisymmetric.

(29) TrivOrthoRelStr is Antisymmetric.

Let us observe that every non empty orthorelational structure which is Antisymmetric is also SubAntisymmetric.

One can verify that TrivOrthoRelStr is Symmetric and Antisymmetric.

One can check that there exists a non empty orthorelational structure which is Symmetric, Antisymmetric, and strict.

Let us consider O. We say that O is SubAsymmetric if and only if:

(Def. 18) The internal relation of O is an asymmetric binary relation on the carrier of O.

Let us consider O. We say that O is Asymmetric if and only if:

(Def. 19) The internal relation of O is asymmetric in the carrier of O.

One can prove the following two propositions:

(30) If O is Asymmetric, then O is SubAsymmetric.

(31) TrivAsymOrthoRelStr is Asymmetric.

Let us mention that every non empty orthorelational structure which is Asymmetric is also SubAsymmetric.

One can check that TrivAsymOrthoRelStr is Asymmetric.

Let us observe that there exists a non empty orthorelational structure which is Asymmetric and strict.

Let us consider O. We say that O is SubTransitive if and only if:

(Def. 20) The internal relation of O is a transitive binary relation on the carrier of O.

Let us consider O. We say that O is Transitive if and only if:

(Def. 21) The internal relation of O is transitive in the carrier of O.

Next we state two propositions:

(32) If O is Transitive, then O is SubTransitive.

(33) TrivOrthoRelStr is Transitive.

Let us observe that every non empty orthorelational structure which is Transitive is also SubTransitive.

Let us observe that TrivOrthoRelStr is Transitive.

Let us observe that there exists a non empty orthorelational structure which is ReFlexive, Symmetric, Antisymmetric, Transitive, and strict.

Next we state the proposition

(34) TrivAsymOrthoRelStr is Transitive.

Let us mention that TrivAsymOrthoRelStr is IrreFlexive, Asymmetric, and Transitive.

Let us observe that there exists a non empty orthorelational structure which is IrreFlexive, Asymmetric, Transitive, and strict.

Next we state four propositions:

(35) If O is SubSymmetric and SubTransitive, then O is SubReFlexive.

(36) If O is SubIrreFlexive and SubTransitive, then O is SubAsymmetric.

(37) If O is SubAsymmetric, then O is SubIrreFlexive.

(38) If O is ReFlexive and SubSymmetric, then O is Symmetric.

One can check that every non empty orthorelational structure which is ReFlexive and SubSymmetric is also Symmetric.

Next we state the proposition

(39) If O is ReFlexive and SubAntisymmetric, then O is Antisymmetric.

Let us note that every non empty orthorelational structure which is ReFlexive and SubAntisymmetric is also Antisymmetric.

The following proposition is true

(40) If O is ReFlexive and SubTransitive, then O is Transitive.

Let us note that every non empty orthorelational structure which is ReFlexive and SubTransitive is also Transitive.

One can prove the following proposition

(41) If O is IrreFlexive and SubTransitive, then O is Transitive.

Let us observe that every non empty orthorelational structure which is IrreFlexive and SubTransitive is also Transitive.

Next we state the proposition

(42) If O is IrreFlexive and SubAsymmetric, then O is Asymmetric.

Let us note that every non empty orthorelational structure which is IrreFlexive and SubAsymmetric is also Asymmetric.

2. Basic Poset Notions

Let us consider O. We say that O is SubQuasiOrdered if and only if:

(Def. 22) O is SubReFlexive and SubTransitive.

We introduce O is SubQuasiordered, O is SubPreOrdered, O is SubPreordered, and O is Subpreordered as synonyms of O is SubQuasiOrdered.

Let us consider O. We say that O is QuasiOrdered if and only if:

(Def. 23) O is ReFlexive and Transitive.

We introduce O is Quasiordered, O is PreOrdered, and O is Preordered as synonyms of O is QuasiOrdered.

The following proposition is true

(43) If O is QuasiOrdered, then O is SubQuasiOrdered.

Let us observe that every non empty orthorelational structure which is QuasiOrdered is also SubQuasiOrdered.

Let us note that TrivOrthoRelStr is QuasiOrdered.

Let us consider O. We say that O is QuasiPure if and only if:

(Def. 24) O is Dneg and QuasiOrdered.

Let us mention that there exists a non empty orthorelational structure which is QuasiPure, Dneg, QuasiOrdered, and strict.

Let us note that TrivOrthoRelStr is QuasiPure.

A QuasiPureOrthoRelStr is a QuasiPure non empty orthorelational structure.

Let us consider O. We say that O is SubPartialOrdered if and only if:

(Def. 25) O is ReFlexive, SubAntisymmetric, and SubTransitive.

We introduce O is SubPartialordered as a synonym of O is SubPartialOrdered.

Let us consider O. We say that O is PartialOrdered if and only if:

(Def. 26) O is ReFlexive, Antisymmetric, and Transitive.

We introduce O is Partialordered as a synonym of O is PartialOrdered.

We now state the proposition

(44) O is SubPartialOrdered iff O is PartialOrdered.

Let us note that every non empty orthorelational structure which is SubPartialOrdered is also PartialOrdered and every non empty orthorelational structure which is PartialOrdered is also SubPartialOrdered.

Let us observe that every non empty orthorelational structure which is PartialOrdered is also ReFlexive, Antisymmetric, and Transitive and every non empty orthorelational structure which is ReFlexive, Antisymmetric, and Transitive is also PartialOrdered.

Let us consider O. We say that O is Pure if and only if:

(Def. 27) O is Dneg and PartialOrdered.

Let us mention that there exists a non empty orthorelational structure which is Pure, Dneg, PartialOrdered, and strict.

One can check that TrivOrthoRelStr is Pure.

A PureOrthoRelStr is a Pure non empty orthorelational structure.

Let us consider O. We say that O is SubStrictPartialOrdered if and only if:

(Def. 28) O is SubAsymmetric and SubTransitive.

Let us consider O. We say that O is StrictPartialOrdered if and only if:

(Def. 29) O is Asymmetric and Transitive.

We introduce O is Strictpartialordered, O is StrictOrdered, and O is Strictordered as synonyms of O is StrictPartialOrdered.

The following proposition is true

(45) If O is StrictPartialOrdered, then O is SubStrictPartialOrdered.

Let us note that every non empty orthorelational structure which is StrictPartialOrdered is also SubStrictPartialOrdered.

One can prove the following proposition

(46) If O is SubStrictPartialOrdered, then O is SubIrreFlexive.

Let us note that every non empty orthorelational structure which is SubStrictPartialOrdered is also SubIrreFlexive.

Next we state the proposition

(47) If O is IrreFlexive and SubStrictPartialOrdered, then O is StrictPartialOrdered.

Let us mention that every non empty orthorelational structure which is IrreFlexive and SubStrictPartialOrdered is also StrictPartialOrdered.

We now state the proposition

(48) If O is StrictPartialOrdered, then O is IrreFlexive.

Let us note that every non empty orthorelational structure which is StrictPartialOrdered is also IrreFlexive.

One can check that TrivAsymOrthoRelStr is IrreFlexive and StrictPartialOrdered.

Let us mention that there exists a non empty strict orthorelational structure which is IrreFlexive and StrictPartialOrdered.

In the sequel P_1 denotes a PartialOrdered non empty orthorelational structure and Q_1 denotes a QuasiOrdered non empty orthorelational structure.

We now state the proposition

(49) If Q_1 is SubAntisymmetric, then Q_1 is PartialOrdered.

Let P_1 be a PartialOrdered non empty orthorelational structure. Note that the internal relation of P_1 is ordering.

One can prove the following proposition

(50) P_1 is a poset.

Let us note that every non empty orthorelational structure which is PartialOrdered is also reflexive, transitive, and antisymmetric.

Let P_2, P_3 be PartialOrdered non empty orthorelational structures and let f be a map from P_2 into P_3. We say that f is Antitone on P_2, P_3 if and only if:

(Def. 30) f is antitone.

Let P_1 be a PartialOrdered non empty orthorelational structure and let f be a map from P_1 into P_1. We say that f is Antitone on P_1 if and only if:

(Def. 31) f is Antitone on P_1, P_1.

Let P_2, P_3 be PartialOrdered non empty orthorelational structures and let f be a map from P_2 into P_3. We say that f is Antitone if and only if:

(Def. 32) f is Antitone on P_2, P_3.

Let P_1 be a PartialOrdered non empty orthorelational structure. Note that there exists a map from P_1 into P_1 which is Antitone.

Let us consider P_1 and let f be a unary operation on the carrier of P_1. We say that f is Orderinvolutive if and only if:

(Def. 33) f is a DNeg map from P_1 into P_1 and an Antitone map from P_1 into P_1.

Let us consider P_1. We say that P_1 is OrderInvolutive if and only if:

(Def. 34) There exists a map f from P_1 into P_1 such that $f =$ the complement operation of P_1 and f is Orderinvolutive.

Next we state the proposition

(51) The complement operation of TrivOrthoRelStr is Orderinvolutive.

Let us observe that TrivOrthoRelStr is OrderInvolutive.

One can check that there exists a PartialOrdered non empty orthorelational structure which is OrderInvolutive and Pure.

A PreOrthoPoset is an OrderInvolutive Pure PartialOrdered non empty orthorelational structure.

Let us consider P_1 and let f be a unary operation on the carrier of P_1. We say that f is QuasiOrthoComplement on P_1 if and only if:

(Def. 35) f is Orderinvolutive and for every element y of P_1 holds sup $\{y, f(y)\}$ exists in P_1 and inf $\{y, f(y)\}$ exists in P_1.

Let us consider P_1. We say that P_1 is QuasiOrthocomplemented if and only if:

(Def. 36) There exists a map f from P_1 into P_1 such that $f =$ the complement operation of P_1 and f is QuasiOrthoComplement on P_1.

Next we state the proposition

(52) TrivOrthoRelStr is QuasiOrthocomplemented.

Let us consider P_1 and let f be a unary operation on the carrier of P_1. We say that f is OrthoComplement on P_1 if and only if the conditions (Def. 37) are satisfied.

(Def. 37)(i) f is Orderinvolutive, and

(ii) for every element y of P_1 holds sup $\{y, f(y)\}$ exists in P_1 and inf $\{y, f(y)\}$ exists in P_1 and $\bigsqcup_{P_1}\{y, f(y)\}$ is a maximum of the carrier of P_1 and $\bigsqcap_{P_1}\{y, f(y)\}$ is a minimum of the carrier of P_1.

We introduce f is OCompl on P_1 as a synonym of f is OrthoComplement on P_1.

Let us consider P_1. We say that P_1 is Orthocomplemented if and only if:

(Def. 38) There exists a map f from P_1 into P_1 such that $f =$ the complement operation of P_1 and f is OrthoComplement on P_1.

We introduce P_1 is Ocompl as a synonym of P_1 is Orthocomplemented.

Next we state two propositions:

(53) Let f be a unary operation on the carrier of P_1. If f is OrthoComplement on P_1, then f is QuasiOrthoComplement on P_1.

(54) TrivOrthoRelStr is Orthocomplemented.

One can check that TrivOrthoRelStr is QuasiOrthocomplemented and Orthocomplemented.

Let us mention that there exists a PartialOrdered non empty orthorelational structure which is Orthocomplemented and QuasiOrthocomplemented.

A QuasiOrthoPoset is a QuasiOrthocomplemented PartialOrdered non empty orthorelational structure. An orthoposet is an Orthocomplemented PartialOrdered non empty orthorelational structure.

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