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Preliminaries to Automatic Generation of Mizar Documentation for Circuits

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Summary. In this paper we introduce technical notions used by a system which automatically generates Mizar documentation for specified circuits. They provide a ready for use elements needed to justify correctness of circuits' construction. We concentrate on the concept of stabilization and analyze one-gate circuits and their combinations.

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The articles [21], [26], [20], [11], [10], [27], [7], [12], [2], [3], [8], [1], [9], [14], [4], [6], [22], [25], [23], [5], [17], [16], [15], [18], [19], [13], and [24] provide the notation and terminology for this paper.

1. Stabilizing Circuits

The following proposition is true

(1) Let S be a non void circuit-like non empty many sorted signature, A be a non-empty circuit of S, s be a state of A, and x be a set. If $x \in \mathrm{InputVertices}(S)$, then for every natural number n holds $(\mathrm{Following}(s, n))(x) = s(x)$.

Let S be a non void circuit-like non empty many sorted signature, let A be a non-empty circuit of S, and let s be a state of A. We say that s is stabilizing if and only if:

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(Def. 1) There exists a natural number n such that $\mathrm{Following}(s, n)$ is stable.

Let S be a non void circuit-like non empty many sorted signature and let A be a non-empty circuit of S. We say that A is stabilizing if and only if:

(Def. 2) Every state of A is stabilizing.

We say that A has a stabilization limit if and only if:

(Def. 3) There exists a natural number n such that for every state s of A holds $\mathrm{Following}(s, n)$ is stable.

Let S be a non void circuit-like non empty many sorted signature. Note that every non-empty circuit of S which has a stabilization limit is also stabilizing.

Let S be a non void circuit-like non empty many sorted signature, let A be a non-empty circuit of S, and let s be a state of A. Let us assume that s is stabilizing. The functor $\mathrm{Result}(s)$ yields a state of A and is defined as follows:

(Def. 4) $\mathrm{Result}(s)$ is stable and there exists a natural number n such that $\mathrm{Result}(s) = \mathrm{Following}(s, n)$.

Let S be a non void circuit-like non empty many sorted signature, let A be a non-empty circuit of S, and let s be a state of A. Let us assume that s is stabilizing. The stabilization time of s is a natural number and is defined by the conditions (Def. 5).

(Def. 5)(i) $\mathrm{Following}(s, \text{the stabilization time of } s)$ is stable, and

(ii) for every natural number n such that $n <$ the stabilization time of s holds $\mathrm{Following}(s, n)$ is not stable.

The following propositions are true:

(2) Let S be a non void circuit-like non empty many sorted signature, A be a non-empty circuit of S, and s be a state of A. If s is stabilizing, then $\mathrm{Result}(s) = \mathrm{Following}(s, \text{the stabilization time of } s)$.

(3) Let S be a non void circuit-like non empty many sorted signature, A be a non-empty circuit of S, s be a state of A, and n be a natural number. If $\mathrm{Following}(s, n)$ is stable, then the stabilization time of $s \leqslant n$.

(4) Let S be a non void circuit-like non empty many sorted signature, A be a non-empty circuit of S, s be a state of A, and n be a natural number. If $\mathrm{Following}(s, n)$ is stable, then $\mathrm{Result}(s) = \mathrm{Following}(s, n)$.

(5) Let S be a non void circuit-like non empty many sorted signature, A be a non-empty circuit of S, s be a state of A, and n be a natural number. Suppose s is stabilizing and $n \geqslant$ the stabilization time of s. Then $\mathrm{Result}(s) = \mathrm{Following}(s, n)$.

(6) Let S be a non void circuit-like non empty many sorted signature, A be a non-empty circuit of S, and s be a state of A. If s is stabilizing, then for every set x such that $x \in \mathrm{InputVertices}(S)$ holds $(\mathrm{Result}(s))(x) = s(x)$.

(7) Let S_1, S be non void circuit-like non empty many sorted signatures, A_1 be a non-empty circuit of S_1, A be a non-empty circuit of S, s be a state

of A, and s_1 be a state of A_1. If $s_1 = s \restriction$the carrier of S_1, then for every vertex v_1 of S_1 holds $s_1(v_1) = s(v_1)$.

(8) Let S_1, S_2 be non void circuit-like non empty many sorted signatures. Suppose InputVertices(S_1) misses InnerVertices(S_2) and InputVertices(S_2) misses InnerVertices(S_1). Let S be a non void circuit-like non empty many sorted signature. Suppose $S = S_1 {+\cdot} S_2$. Let A_1 be a non-empty circuit of S_1 and A_2 be a non-empty circuit of S_2. Suppose $A_1 \approx A_2$. Let A be a non-empty circuit of S. Suppose $A = A_1 {+\cdot} A_2$. Let s be a state of A, s_1 be a state of A_1, and s_2 be a state of A_2. Suppose $s_1 = s \restriction$the carrier of S_1 and $s_2 = s \restriction$the carrier of S_2 and s_1 is stabilizing and s_2 is stabilizing. Then s is stabilizing.

(9) Let S_1, S_2 be non void circuit-like non empty many sorted signatures. Suppose InputVertices(S_1) misses InnerVertices(S_2) and InputVertices(S_2) misses InnerVertices(S_1). Let S be a non void circuit-like non empty many sorted signature. Suppose $S = S_1 {+\cdot} S_2$. Let A_1 be a non-empty circuit of S_1 and A_2 be a non-empty circuit of S_2. Suppose $A_1 \approx A_2$. Let A be a non-empty circuit of S. Suppose $A = A_1 {+\cdot} A_2$. Let s be a state of A and s_1 be a state of A_1. Suppose $s_1 = s \restriction$the carrier of S_1 and s_1 is stabilizing. Let s_2 be a state of A_2. Suppose $s_2 = s \restriction$the carrier of S_2 and s_2 is stabilizing. Then the stabilization time of $s = \max($the stabilization time of s_1, the stabilization time of $s_2)$.

(10) Let S_1, S_2 be non void circuit-like non empty many sorted signatures. Suppose InputVertices(S_1) misses InnerVertices(S_2). Let S be a non void circuit-like non empty many sorted signature. Suppose $S = S_1 {+\cdot} S_2$. Let A_1 be a non-empty circuit of S_1 and A_2 be a non-empty circuit of S_2. Suppose $A_1 \approx A_2$. Let A be a non-empty circuit of S. Suppose $A = A_1 {+\cdot} A_2$. Let s be a state of A and s_1 be a state of A_1. Suppose $s_1 = s \restriction$the carrier of S_1 and s_1 is stabilizing. Let s_2 be a state of A_2. Suppose $s_2 = \mathrm{Following}(s,$the stabilization time of $s_1) \restriction$the carrier of S_2 and s_2 is stabilizing. Then s is stabilizing.

(11) Let S_1, S_2 be non void circuit-like non empty many sorted signatures. Suppose InputVertices(S_1) misses InnerVertices(S_2). Let S be a non void circuit-like non empty many sorted signature. Suppose $S = S_1 {+\cdot} S_2$. Let A_1 be a non-empty circuit of S_1 and A_2 be a non-empty circuit of S_2. Suppose $A_1 \approx A_2$. Let A be a non-empty circuit of S. Suppose $A = A_1 {+\cdot} A_2$. Let s be a state of A and s_1 be a state of A_1. Suppose $s_1 = s \restriction$the carrier of S_1 and s_1 is stabilizing. Let s_2 be a state of A_2. Suppose $s_2 = \mathrm{Following}(s,$the stabilization time of $s_1) \restriction$the carrier of S_2 and s_2 is stabilizing. Then the stabilization time of $s =$ (the stabilization time of s_1) + (the stabilization time of s_2).

(12) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures.

Suppose InputVertices(S_1) misses InnerVertices(S_2) and $S = S_1 +\cdot S_2$. Let A_1 be a non-empty circuit of S_1, A_2 be a non-empty circuit of S_2, and A be a non-empty circuit of S. Suppose $A_1 \approx A_2$ and $A = A_1 +\cdot A_2$. Let s be a state of A and s_1 be a state of A_1. Suppose $s_1 = s \restriction$the carrier of S_1 and s_1 is stabilizing. Let s_2 be a state of A_2. Suppose $s_2 =$ Following(s, the stabilization time of s_1)$\restriction$the carrier of S_2 and s_2 is stabilizing. Then Result(s)$\restriction$the carrier of S_1 = Result(s_1).

2. One-gate Circuits

We now state three propositions:

(13) Let x be a set, X be a non empty finite set, n be a natural number, p be a finite sequence with length n, g be a function from X^n into X, and s be a state of 1GateCircuit(p, g). Then $s \cdot p$ is an element of X^n.

(14) For all sets x_1, x_2, x_3, x_4 holds $\mathrm{rng}\langle x_1, x_2, x_3, x_4\rangle = \{x_1, x_2, x_3, x_4\}$.

(15) For all sets x_1, x_2, x_3, x_4, x_5 holds $\mathrm{rng}\langle x_1, x_2, x_3, x_4, x_5\rangle = \{x_1, x_2, x_3, x_4, x_5\}$.

Let x_1, x_2, x_3, x_4 be sets. Then $\langle x_1, x_2, x_3, x_4\rangle$ is a finite sequence with length 4. Let x_5 be a set. Then $\langle x_1, x_2, x_3, x_4, x_5\rangle$ is a finite sequence with length 5.

Let S be a many sorted signature. We say that S is one-gate if and only if the condition (Def. 6) is satisfied.

(Def. 6) There exists a non empty finite set X and there exists a natural number n and there exists a finite sequence p with length n and there exists a function f from X^n into X such that $S =$ 1GateCircStr(p, f).

Let S be a non empty many sorted signature and let A be an algebra over S. We say that A is one-gate if and only if the condition (Def. 7) is satisfied.

(Def. 7) There exists a non empty finite set X and there exists a natural number n and there exists a finite sequence p with length n and there exists a function f from X^n into X such that $S =$ 1GateCircStr(p, f) and $A =$ 1GateCircuit(p, f).

Let p be a finite sequence and let x be a set. Observe that 1GateCircStr(p, x) is finite.

Let us note that every many sorted signature which is one-gate is also strict, non void, non empty, unsplit, and finite and has arity held in gates.

One can check that every non empty many sorted signature which is one-gate has also denotation held in gates.

Let X be a non empty finite set, let n be a natural number, let p be a finite sequence with length n, and let f be a function from X^n into X. Note that 1GateCircStr(p, f) is one-gate.

One can check that there exists a many sorted signature which is one-gate.

Let S be an one-gate many sorted signature. Observe that every circuit of S which is one-gate is also strict and non-empty.

Let X be a non empty finite set, let n be a natural number, let p be a finite sequence with length n, and let f be a function from X^n into X. One can check that 1GateCircuit(p, f) is one-gate.

Let S be an one-gate many sorted signature. Observe that there exists a circuit of S which is one-gate and non-empty.

Let S be an one-gate many sorted signature. The functor Output S yields a vertex of S and is defined as follows:

(Def. 8) Output $S = \bigcup$(the operation symbols of S).

Let S be an one-gate many sorted signature. Observe that Output S is pair.

Next we state several propositions:

(16) Let S be an one-gate many sorted signature, p be a finite sequence, and x be a set. If $S =$ 1GateCircStr(p, x), then Output $S = \langle p, x \rangle$.

(17) For every one-gate many sorted signature S holds InnerVertices$(S) = \{$Output $S\}$.

(18) Let S be an one-gate many sorted signature, A be an one-gate circuit of S, n be a natural number, X be a finite non empty set, f be a function from X^n into X, and p be a finite sequence with length n. If $A =$ 1GateCircuit(p, f), then $S =$ 1GateCircStr(p, f).

(19) Let n be a natural number, X be a finite non empty set, f be a function from X^n into X, p be a finite sequence with length n, and s be a state of 1GateCircuit(p, f). Then (Following(s))(Output 1GateCircStr$(p, f)) = f(s \cdot p)$.

(20) Let S be an one-gate many sorted signature, A be an one-gate circuit of S, and s be a state of A. Then Following(s) is stable.

Let S be a non void circuit-like non empty many sorted signature. Observe that every non-empty circuit of S which is one-gate has also a stabilization limit.

We now state two propositions:

(21) Let S be an one-gate many sorted signature, A be an one-gate circuit of S, and s be a state of A. Then Result$(s) =$ Following(s).

(22) Let S be an one-gate many sorted signature, A be an one-gate circuit of S, and s be a state of A. Then the stabilization time of $s \leqslant 1$.

In this article we present several logical schemes. The scheme *OneGate1Ex* deals with a set $\mathcal{A}$, a non empty finite set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and states that:

> There exists an one-gate many sorted signature S and there exists an one-gate circuit A of S such that InputVertices$(S) = \{\mathcal{A}\}$ and for every state s of A holds (Result(s))(Output $S) = \mathcal{F}(s(\mathcal{A}))$

for all values of the parameters.

The scheme *OneGate2Ex* deals with sets $\mathcal{A}$, $\mathcal{B}$, a non empty finite set $\mathcal{C}$, and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and states that:

> There exists an one-gate many sorted signature S and there exists an one-gate circuit A of S such that $\text{InputVertices}(S) = \{\mathcal{A}, \mathcal{B}\}$ and for every state s of A holds $(\text{Result}(s))(\text{Output}\, S) = \mathcal{F}(s(\mathcal{A}), s(\mathcal{B}))$

for all values of the parameters.

The scheme *OneGate3Ex* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, a non empty finite set $\mathcal{D}$, and a ternary functor $\mathcal{F}$ yielding an element of $\mathcal{D}$, and states that:

> There exists an one-gate many sorted signature S and there exists an one-gate circuit A of S such that $\text{InputVertices}(S) = \{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ and for every state s of A holds $(\text{Result}(s))(\text{Output}\, S)$
> $= \mathcal{F}(s(\mathcal{A}), s(\mathcal{B}), s(\mathcal{C}))$

for all values of the parameters.

The scheme *OneGate4Ex* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, a non empty finite set $\mathcal{E}$, and a 4-ary functor $\mathcal{F}$ yielding an element of $\mathcal{E}$, and states that:

> There exists an one-gate many sorted signature S and there exists an one-gate circuit A of S such that $\text{InputVertices}(S) = \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$ and for every state s of A holds $(\text{Result}(s))(\text{Output}\, S)$
> $= \mathcal{F}(s(\mathcal{A}), s(\mathcal{B}), s(\mathcal{C}), s(\mathcal{D}))$

for all values of the parameters.

The scheme *OneGate5Ex* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, a non empty finite set $\mathcal{F}$, and a 5-ary functor $\mathcal{F}$ yielding an element of $\mathcal{F}$, and states that:

> There exists an one-gate many sorted signature S and there exists an one-gate circuit A of S such that $\text{InputVertices}(S) = \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}\}$ and for every state s of A holds $(\text{Result}(s))(\text{Output}\, S)$
> $= \mathcal{F}(s(\mathcal{A}), s(\mathcal{B}), s(\mathcal{C}), s(\mathcal{D}), s(\mathcal{E}))$

for all values of the parameters.

3. Mono-sorted Circuits

One can prove the following propositions:

(23) For every constant function f holds $f = \operatorname{dom} f \longmapsto$ the value of f.

(24) For all non empty sets X, Y and for all natural numbers n, m such that $n \neq 0$ and $X^n = Y^m$ holds $X = Y$ and $n = m$.

(25) For all non empty many sorted signatures S_1, S_2 holds every vertex of S_1 is a vertex of $S_1 {+\cdot} S_2$.

(26) For all non empty many sorted signatures S_1, S_2 holds every vertex of S_2 is a vertex of $S_1 {+\cdot} S_2$.

Let X be a non empty finite set. A non void non empty unsplit many sorted signature with arity held in gates with denotation held in gates is said to be a signature over X if it satisfies the condition (Def. 9).

(Def. 9) There exists a circuit A of it such that the sorts of A are constant and the value of the sorts of $A = X$ and A has denotation held in gates.

Next we state the proposition

(27) Let n be a natural number, X be a non empty finite set, f be a function from X^n into X, and p be a finite sequence with length n. Then $1\mathrm{GateCircStr}(p, f)$ is a signature over X.

Let X be a non empty finite set. Observe that there exists a signature over X which is strict and one-gate.

Let n be a natural number, let X be a non empty finite set, let f be a function from X^n into X, and let p be a finite sequence with length n. Then $1\mathrm{GateCircStr}(p, f)$ is a strict signature over X.

Let X be a non empty finite set and let S be a signature over X. A circuit of S is called a circuit over X and S if:

(Def. 10) It has denotation held in gates and the sorts of it are constant and the value of the sorts of it $= X$.

Let X be a non empty finite set and let S be a signature over X. One can check that every circuit over X and S is non-empty and has denotation held in gates.

Next we state the proposition

(28) Let n be a natural number, X be a non empty finite set, f be a function from X^n into X, and p be a finite sequence with length n. Then $1\mathrm{GateCircuit}(p, f)$ is a circuit over X and $1\mathrm{GateCircStr}(p, f)$.

Let X be a non empty finite set and let S be an one-gate signature over X. One can check that there exists a circuit over X and S which is strict and one-gate.

Let X be a non empty finite set and let S be a signature over X. One can check that there exists a circuit over X and S which is strict.

Let n be a natural number, let X be a non empty finite set, let f be a function from X^n into X, and let p be a finite sequence with length n. Then $1\mathrm{GateCircuit}(p, f)$ is a strict circuit over X and $1\mathrm{GateCircStr}(p, f)$.

One can prove the following propositions:

(29) For every non empty finite set X and for all signatures S_1, S_2 over X holds $S_1 \approx S_2$.

(30) Let X be a non empty finite set, S_1, S_2 be signatures over X, A_1 be a circuit over X and S_1, and A_2 be a circuit over X and S_2. Then $A_1 \approx A_2$.

(31) Let X be a non empty finite set, S_1, S_2 be signatures over X, A_1 be a circuit over X and S_1, and A_2 be a circuit over X and S_2. Then $A_1 {+}\!\cdot A_2$ is a circuit of $S_1 {+}\!\cdot S_2$.

(32) Let X be a non empty finite set, S_1, S_2 be signatures over X, A_1 be a circuit over X and S_1, and A_2 be a circuit over X and S_2. Then $A_1 {+}\!\cdot A_2$

has denotation held in gates.

(33) Let X be a non empty finite set, S_1, S_2 be signatures over X, A_1 be a circuit over X and S_1, and A_2 be a circuit over X and S_2. Then the sorts of $A_1 +\cdot A_2$ are constant and the value of the sorts of $A_1 +\cdot A_2 = X$.

Let S_1, S_2 be finite non empty many sorted signatures. Note that $S_1 +\cdot S_2$ is finite.

Let X be a non empty finite set and let S_1, S_2 be signatures over X. One can verify that $S_1 +\cdot S_2$ has denotation held in gates.

Let X be a non empty finite set and let S_1, S_2 be signatures over X. Then $S_1 +\cdot S_2$ is a strict signature over X.

Let X be a non empty finite set, let S_1, S_2 be signatures over X, let A_1 be a circuit over X and S_1, and let A_2 be a circuit over X and S_2. Then $A_1 +\cdot A_2$ is a strict circuit over X and $S_1 +\cdot S_2$.

One can prove the following two propositions:

(34) For all sets x, y holds $\mathrm{rk}(x) \in \mathrm{rk}(\langle x, y \rangle)$ and $\mathrm{rk}(y) \in \mathrm{rk}(\langle x, y \rangle)$.

(35) Let S be a finite non void non empty unsplit many sorted signature with arity held in gates with denotation held in gates and A be a non-empty circuit of S such that A has denotation held in gates. Then A has a stabilization limit.

Let X be a non empty finite set and let S be a finite signature over X. One can verify that every circuit over X and S has a stabilization limit.

Now we present three schemes. The scheme *1AryDef* deals with a non empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

(i) There exists a function f from $\mathcal{A}^1$ into $\mathcal{A}$ such that for every element x of $\mathcal{A}$ holds $f(\langle x \rangle) = \mathcal{F}(x)$, and

(ii) for all functions f_1, f_2 from $\mathcal{A}^1$ into $\mathcal{A}$ such that for every element x of $\mathcal{A}$ holds $f_1(\langle x \rangle) = \mathcal{F}(x)$ and for every element x of $\mathcal{A}$ holds $f_2(\langle x \rangle) = \mathcal{F}(x)$ holds $f_1 = f_2$

for all values of the parameters.

The scheme *2AryDef* deals with a non empty set $\mathcal{A}$ and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

(i) There exists a function f from $\mathcal{A}^2$ into $\mathcal{A}$ such that for all elements x, y of $\mathcal{A}$ holds $f(\langle x, y \rangle) = \mathcal{F}(x, y)$, and

(ii) for all functions f_1, f_2 from $\mathcal{A}^2$ into $\mathcal{A}$ such that for all elements x, y of $\mathcal{A}$ holds $f_1(\langle x, y \rangle) = \mathcal{F}(x, y)$ and for all elements x, y of $\mathcal{A}$ holds $f_2(\langle x, y \rangle) = \mathcal{F}(x, y)$ holds $f_1 = f_2$

for all values of the parameters.

The scheme *3AryDef* deals with a non empty set $\mathcal{A}$ and a ternary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

(i) There exists a function f from $\mathcal{A}^3$ into $\mathcal{A}$ such that for all elements x, y, z of $\mathcal{A}$ holds $f(\langle x, y, z \rangle) = \mathcal{F}(x, y, z)$, and

(ii) for all functions f_1, f_2 from $\mathcal{A}^3$ into $\mathcal{A}$ such that for all elements x, y, z of $\mathcal{A}$ holds $f_1(\langle x, y, z\rangle) = \mathcal{F}(x, y, z)$ and for all elements x, y, z of $\mathcal{A}$ holds $f_2(\langle x, y, z\rangle) = \mathcal{F}(x, y, z)$ holds $f_1 = f_2$

for all values of the parameters.

We now state three propositions:

(36) For every function f and for every set x such that $x \in \operatorname{dom} f$ holds $f \cdot \langle x\rangle = \langle f(x)\rangle$.

(37) Let f be a function and x_1, x_2, x_3, x_4 be sets. If $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} f$ and $x_3 \in \operatorname{dom} f$ and $x_4 \in \operatorname{dom} f$, then $f \cdot \langle x_1, x_2, x_3, x_4\rangle = \langle f(x_1), f(x_2), f(x_3), f(x_4)\rangle$.

(38) Let f be a function and x_1, x_2, x_3, x_4, x_5 be sets. Suppose $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} f$ and $x_3 \in \operatorname{dom} f$ and $x_4 \in \operatorname{dom} f$ and $x_5 \in \operatorname{dom} f$. Then $f \cdot \langle x_1, x_2, x_3, x_4, x_5\rangle = \langle f(x_1), f(x_2), f(x_3), f(x_4), f(x_5)\rangle$.

Now we present several schemes. The scheme *OneGate1Result* deals with a set $\mathcal{A}$, a non empty finite set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a function $\mathcal{C}$ from $\mathcal{B}^1$ into $\mathcal{B}$, and states that:

For every state s of $\text{1GateCircuit}(\langle\mathcal{A}\rangle, \mathcal{C})$ and for every element a_1 of $\mathcal{B}$ such that $a_1 = s(\mathcal{A})$ holds $(\text{Result}(s))(\text{Output 1GateCircStr}(\langle\mathcal{A}\rangle, \mathcal{C})) = \mathcal{F}(a_1)$

provided the following requirement is met:

- For every function g from $\mathcal{B}^1$ into $\mathcal{B}$ holds $g = \mathcal{C}$ iff for every element a_1 of $\mathcal{B}$ holds $g(\langle a_1\rangle) = \mathcal{F}(a_1)$.

The scheme *OneGate2Result* deals with sets $\mathcal{A}$, $\mathcal{B}$, a non empty finite set $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a function $\mathcal{D}$ from $\mathcal{C}^2$ into $\mathcal{C}$, and states that:

For every state s of $\text{1GateCircuit}(\langle\mathcal{A}, \mathcal{B}\rangle, \mathcal{D})$ and for all elements a_1, a_2 of $\mathcal{C}$ such that $a_1 = s(\mathcal{A})$ and $a_2 = s(\mathcal{B})$ holds $(\text{Result}(s))(\text{Output 1GateCircStr}(\langle\mathcal{A}, \mathcal{B}\rangle, \mathcal{D})) = \mathcal{F}(a_1, a_2)$

provided the parameters satisfy the following condition:

- For every function g from $\mathcal{C}^2$ into $\mathcal{C}$ holds $g = \mathcal{D}$ iff for all elements a_1, a_2 of $\mathcal{C}$ holds $g(\langle a_1, a_2\rangle) = \mathcal{F}(a_1, a_2)$.

The scheme *OneGate3Result* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, a non empty finite set $\mathcal{D}$, a ternary functor $\mathcal{F}$ yielding an element of $\mathcal{D}$, and a function $\mathcal{E}$ from $\mathcal{D}^3$ into $\mathcal{D}$, and states that:

Let s be a state of $\text{1GateCircuit}(\langle\mathcal{A}, \mathcal{B}, \mathcal{C}\rangle, \mathcal{E})$ and a_1, a_2, a_3 be elements of $\mathcal{D}$. If $a_1 = s(\mathcal{A})$ and $a_2 = s(\mathcal{B})$ and $a_3 = s(\mathcal{C})$, then $(\text{Result}(s))(\text{Output 1GateCircStr}(\langle\mathcal{A}, \mathcal{B}, \mathcal{C}\rangle, \mathcal{E})) = \mathcal{F}(a_1, a_2, a_3)$

provided the following requirement is met:

- For every function g from $\mathcal{D}^3$ into $\mathcal{D}$ holds $g = \mathcal{E}$ iff for all elements a_1, a_2, a_3 of $\mathcal{D}$ holds $g(\langle a_1, a_2, a_3\rangle) = \mathcal{F}(a_1, a_2, a_3)$.

The scheme *OneGate4Result* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, a non empty finite set $\mathcal{E}$, a 4-ary functor $\mathcal{F}$ yielding an element of $\mathcal{E}$, and a function $\mathcal{F}$ from $\mathcal{E}^4$ into $\mathcal{E}$, and states that:

Let s be a state of 1GateCircuit($\langle \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \rangle, \mathcal{F}$) and a_1, a_2, a_3, a_4 be elements of $\mathcal{E}$. If $a_1 = s(\mathcal{A})$ and $a_2 = s(\mathcal{B})$ and $a_3 = s(\mathcal{C})$ and $a_4 = s(\mathcal{D})$, then (Result(s))(Output 1GateCircStr($\langle \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \rangle, \mathcal{F}$)) $= \mathcal{F}(a_1, a_2, a_3, a_4)$

provided the following condition is met:

- Let g be a function from $\mathcal{E}^4$ into $\mathcal{E}$. Then $g = \mathcal{F}$ if and only if for all elements a_1, a_2, a_3, a_4 of $\mathcal{E}$ holds $g(\langle a_1, a_2, a_3, a_4 \rangle) = \mathcal{F}(a_1, a_2, a_3, a_4)$.

The scheme *OneGate5Result* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, a non empty finite set $\mathcal{F}$, a 5-ary functor $\mathcal{F}$ yielding an element of $\mathcal{F}$, and a function $\mathcal{G}$ from $\mathcal{F}^5$ into $\mathcal{F}$, and states that:

Let s be a state of 1GateCircuit($\langle \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E} \rangle, \mathcal{G}$) and a_1, a_2, a_3, a_4, a_5 be elements of $\mathcal{F}$. Suppose $a_1 = s(\mathcal{A})$ and $a_2 = s(\mathcal{B})$ and $a_3 = s(\mathcal{C})$ and $a_4 = s(\mathcal{D})$ and $a_5 = s(\mathcal{E})$. Then (Result(s))(Output 1GateCircStr($\langle \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E} \rangle, \mathcal{G}$)) $= \mathcal{F}(a_1, a_2, a_3, a_4, a_5)$

provided the following requirement is met:

- Let g be a function from $\mathcal{F}^5$ into $\mathcal{F}$. Then $g = \mathcal{G}$ if and only if for all elements a_1, a_2, a_3, a_4, a_5 of $\mathcal{F}$ holds $g(\langle a_1, a_2, a_3, a_4, a_5 \rangle) = \mathcal{F}(a_1, a_2, a_3, a_4, a_5)$.

4. Input of a Compound Circuit

We now state a number of propositions:

(39) Let n be a natural number, X be a non empty finite set, f be a function from X^n into X, p be a finite sequence with length n, and S be a signature over X. If $\mathrm{rng}\, p \subseteq$ the carrier of S and Output 1GateCircStr(p, f) $\notin$ InputVertices(S), then InputVertices($S +\cdot$ 1GateCircStr(p, f)) = InputVertices(S).

(40) Let X_1, X_2 be sets, X be a non empty finite set, n be a natural number, f be a function from X^n into X, p be a finite sequence with length n, and S be a signature over X. Suppose $\mathrm{rng}\, p = X_1 \cup X_2$ and $X_1 \subseteq$ the carrier of S and X_2 misses InnerVertices(S) and Output 1GateCircStr(p, f) $\notin$ InputVertices(S). Then InputVertices($S +\cdot$ 1GateCircStr(p, f)) = InputVertices(S) $\cup X_2$.

(41) Let x_1 be a set, X be a non empty finite set, f be a function from X^1 into X, and S be a signature over X. If $x_1 \in$ the carrier of S and Output 1GateCircStr($\langle x_1 \rangle, f$) $\notin$ InputVertices(S), then InputVertices($S +\cdot$ 1GateCircStr($\langle x_1 \rangle, f$)) = InputVertices(S).

(42) Let x_1, x_2 be sets, X be a non empty finite set, f be a function from X^2 into X, and S be a signature over X. Suppose $x_1 \in$ the carrier of S and $x_2 \notin$ InnerVertices(S) and Output 1GateCircStr($\langle x_1, x_2\rangle, f$) $\notin$ InputVertices(S). Then InputVertices(S+· 1GateCircStr($\langle x_1, x_2\rangle, f$)) = InputVertices($S$) $\cup$ $\{x_2\}$.

(43) Let x_1, x_2 be sets, X be a non empty finite set, f be a function from X^2 into X, and S be a signature over X. Suppose $x_2 \in$ the carrier of S and $x_1 \notin$ InnerVertices(S) and Output 1GateCircStr($\langle x_1, x_2\rangle, f$) $\notin$ InputVertices(S). Then InputVertices(S+· 1GateCircStr($\langle x_1, x_2\rangle, f$)) = InputVertices($S$) $\cup$ $\{x_1\}$.

(44) Let x_1, x_2 be sets, X be a non empty finite set, f be a function from X^2 into X, and S be a signature over X. Suppose $x_1 \in$ the carrier of S and $x_2 \in$ the carrier of S and Output 1GateCircStr($\langle x_1, x_2\rangle, f$) $\notin$ InputVertices(S). Then InputVertices(S+· 1GateCircStr($\langle x_1, x_2\rangle, f$)) = InputVertices($S$).

(45) Let x_1, x_2, x_3 be sets, X be a non empty finite set, f be a function from X^3 into X, and S be a signature over X. Suppose $x_1 \in$ the carrier of S and $x_2 \notin$ InnerVertices(S) and $x_3 \notin$ InnerVertices(S) and Output 1GateCircStr($\langle x_1, x_2, x_3\rangle, f$) $\notin$ InputVertices(S). Then InputVertices(S+· 1GateCircStr($\langle x_1, x_2, x_3\rangle, f$)) = InputVertices($S$) $\cup$ $\{x_2, x_3\}$.

(46) Let x_1, x_2, x_3 be sets, X be a non empty finite set, f be a function from X^3 into X, and S be a signature over X. Suppose $x_2 \in$ the carrier of S and $x_1 \notin$ InnerVertices(S) and $x_3 \notin$ InnerVertices(S) and Output 1GateCircStr($\langle x_1, x_2, x_3\rangle, f$) $\notin$ InputVertices(S). Then InputVertices(S+· 1GateCircStr($\langle x_1, x_2, x_3\rangle, f$)) = InputVertices($S$) $\cup$ $\{x_1, x_3\}$.

(47) Let x_1, x_2, x_3 be sets, X be a non empty finite set, f be a function from X^3 into X, and S be a signature over X. Suppose $x_3 \in$ the carrier of S and $x_1 \notin$ InnerVertices(S) and $x_2 \notin$ InnerVertices(S) and Output 1GateCircStr($\langle x_1, x_2, x_3\rangle, f$) $\notin$ InputVertices(S). Then InputVertices(S+· 1GateCircStr($\langle x_1, x_2, x_3\rangle, f$)) = InputVertices($S$) $\cup$ $\{x_1, x_2\}$.

(48) Let x_1, x_2, x_3 be sets, X be a non empty finite set, f be a function from X^3 into X, and S be a signature over X. Suppose $x_1 \in$ the carrier of S and $x_2 \in$ the carrier of S and $x_3 \notin$ InnerVertices(S) and Output 1GateCircStr($\langle x_1, x_2, x_3\rangle, f$) $\notin$ InputVertices(S). Then InputVertices(S+· 1GateCircStr($\langle x_1, x_2, x_3\rangle, f$)) = InputVertices($S$) $\cup$ $\{x_3\}$.

(49) Let x_1, x_2, x_3 be sets, X be a non empty finite set, f be a function from X^3 into X, and S be a signature over X. Suppose $x_1 \in$ the

carrier of S and $x_3 \in$ the carrier of S and $x_2 \notin$ InnerVertices(S) and Output 1GateCircStr($\langle x_1, x_2, x_3 \rangle, f$) $\notin$ InputVertices(S). Then InputVertices(S+· 1GateCircStr($\langle x_1, x_2, x_3 \rangle, f$)) = InputVertices($S$) $\cup$ $\{x_2\}$.

(50) Let x_1, x_2, x_3 be sets, X be a non empty finite set, f be a function from X^3 into X, and S be a signature over X. Suppose $x_2 \in$ the carrier of S and $x_3 \in$ the carrier of S and $x_1 \notin$ InnerVertices(S) and Output 1GateCircStr($\langle x_1, x_2, x_3 \rangle, f$) $\notin$ InputVertices(S). Then InputVertices(S+· 1GateCircStr($\langle x_1, x_2, x_3 \rangle, f$)) = InputVertices($S$) $\cup$ $\{x_1\}$.

(51) Let x_1, x_2, x_3 be sets, X be a non empty finite set, f be a function from X^3 into X, and S be a signature over X. Suppose $x_1 \in$ the carrier of S and $x_2 \in$ the carrier of S and $x_3 \in$ the carrier of S and Output 1GateCircStr($\langle x_1, x_2, x_3 \rangle, f$) $\notin$ InputVertices(S). Then InputVertices(S+· 1GateCircStr($\langle x_1, x_2, x_3 \rangle, f$)) = InputVertices($S$).

5. Result of a Compound Circuit

Next we state the proposition

(52) Let X be a non empty finite set, S be a finite signature over X, A be a circuit over X and S, n be a natural number, f be a function from X^n into X, and p be a finite sequence with length n. Suppose Output 1GateCircStr(p, f) $\notin$ InputVertices(S). Let s be a state of A+· 1GateCircuit(p, f) and s' be a state of A. Suppose $s' = s\restriction$the carrier of S. Then the stabilization time of $s \leqslant 1 +$ the stabilization time of s'.

Now we present several schemes. The scheme *Comb1CircResult* deals with a set $\mathcal{A}$, a non empty finite set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, a finite signature $\mathcal{C}$ over $\mathcal{B}$, a circuit $\mathcal{D}$ over $\mathcal{B}$ and $\mathcal{C}$, and a function $\mathcal{E}$ from $\mathcal{B}^1$ into $\mathcal{B}$, and states that:

> Let s be a state of $\mathcal{D}$+· 1GateCircuit($\langle \mathcal{A} \rangle, \mathcal{E}$) and s' be a state of $\mathcal{D}$. Suppose $s' = s\restriction$the carrier of $\mathcal{C}$. Let a_1 be an element of $\mathcal{B}$. Suppose if $\mathcal{A} \in$ InnerVertices($\mathcal{C}$), then $a_1 =$ (Result(s'))($\mathcal{A}$) and if $\mathcal{A} \notin$ InnerVertices($\mathcal{C}$), then $a_1 = s(\mathcal{A})$. Then (Result(s))(Output 1GateCircStr($\langle \mathcal{A} \rangle, \mathcal{E}$)) $= \mathcal{F}(a_1)$

provided the parameters meet the following conditions:

- For every function g from $\mathcal{B}^1$ into $\mathcal{B}$ holds $g = \mathcal{E}$ iff for every element a_1 of $\mathcal{B}$ holds $g(\langle a_1 \rangle) = \mathcal{F}(a_1)$, and
- Output 1GateCircStr($\langle \mathcal{A} \rangle, \mathcal{E}$) $\notin$ InputVertices($\mathcal{C}$).

The scheme *Comb2CircResult* deals with sets $\mathcal{A}$, $\mathcal{B}$, a non empty finite set $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, a finite signature $\mathcal{D}$ over $\mathcal{C}$, a circuit $\mathcal{E}$ over $\mathcal{C}$ and $\mathcal{D}$, and a function $\mathcal{F}$ from $\mathcal{C}^2$ into $\mathcal{C}$, and states that:

Let s be a state of $\mathcal{E}$+· 1GateCircuit($\langle\mathcal{A},\mathcal{B}\rangle,\mathcal{F}$) and s' be a state of $\mathcal{E}$. Suppose $s' = s{\restriction}$the carrier of $\mathcal{D}$. Let a_1, a_2 be elements of $\mathcal{C}$. Suppose if $\mathcal{A} \in$ InnerVertices($\mathcal{D}$), then $a_1 =$ (Result(s'))($\mathcal{A}$) and if $\mathcal{A} \notin$ InnerVertices($\mathcal{D}$), then $a_1 = s(\mathcal{A})$ and if $\mathcal{B} \in$ InnerVertices($\mathcal{D}$), then $a_2 =$ (Result(s'))($\mathcal{B}$) and if $\mathcal{B} \notin$ InnerVertices($\mathcal{D}$), then $a_2 = s(\mathcal{B})$. Then (Result(s))(Output 1GateCircStr($\langle\mathcal{A},\mathcal{B}\rangle,\mathcal{F}$)) $= \mathcal{F}(a_1, a_2)$

provided the parameters meet the following requirements:

- For every function g from $\mathcal{C}^2$ into $\mathcal{C}$ holds $g = \mathcal{F}$ iff for all elements a_1, a_2 of $\mathcal{C}$ holds $g(\langle a_1, a_2\rangle) = \mathcal{F}(a_1, a_2)$, and
- Output 1GateCircStr($\langle\mathcal{A},\mathcal{B}\rangle,\mathcal{F}$) $\notin$ InputVertices($\mathcal{D}$).

The scheme *Comb3CircResult* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, a non empty finite set $\mathcal{D}$, a ternary functor $\mathcal{F}$ yielding an element of $\mathcal{D}$, a finite signature $\mathcal{E}$ over $\mathcal{D}$, a circuit $\mathcal{F}$ over $\mathcal{D}$ and $\mathcal{E}$, and a function $\mathcal{G}$ from $\mathcal{D}^3$ into $\mathcal{D}$, and states that:

Let s be a state of $\mathcal{F}$+· 1GateCircuit($\langle\mathcal{A},\mathcal{B},\mathcal{C}\rangle,\mathcal{G}$) and s' be a state of $\mathcal{F}$. Suppose $s' = s{\restriction}$the carrier of $\mathcal{E}$. Let a_1, a_2, a_3 be elements of $\mathcal{D}$. Suppose that

(i) if $\mathcal{A} \in$ InnerVertices($\mathcal{E}$), then $a_1 =$ (Result(s'))($\mathcal{A}$),
(ii) if $\mathcal{A} \notin$ InnerVertices($\mathcal{E}$), then $a_1 = s(\mathcal{A})$,
(iii) if $\mathcal{B} \in$ InnerVertices($\mathcal{E}$), then $a_2 =$ (Result(s'))($\mathcal{B}$),
(iv) if $\mathcal{B} \notin$ InnerVertices($\mathcal{E}$), then $a_2 = s(\mathcal{B})$,
(v) if $\mathcal{C} \in$ InnerVertices($\mathcal{E}$), then $a_3 =$ (Result(s'))($\mathcal{C}$), and
(vi) if $\mathcal{C} \notin$ InnerVertices($\mathcal{E}$), then $a_3 = s(\mathcal{C})$.

Then (Result(s))(Output 1GateCircStr($\langle\mathcal{A},\mathcal{B},\mathcal{C}\rangle,\mathcal{G}$)) $= \mathcal{F}(a_1, a_2, a_3)$

provided the parameters meet the following requirements:

- For every function g from $\mathcal{D}^3$ into $\mathcal{D}$ holds $g = \mathcal{G}$ iff for all elements a_1, a_2, a_3 of $\mathcal{D}$ holds $g(\langle a_1, a_2, a_3\rangle) = \mathcal{F}(a_1, a_2, a_3)$, and
- Output 1GateCircStr($\langle\mathcal{A},\mathcal{B},\mathcal{C}\rangle,\mathcal{G}$) $\notin$ InputVertices($\mathcal{E}$).

The scheme *Comb4CircResult* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, a non empty finite set $\mathcal{E}$, a 4-ary functor $\mathcal{F}$ yielding an element of $\mathcal{E}$, a finite signature $\mathcal{F}$ over $\mathcal{E}$, a circuit $\mathcal{G}$ over $\mathcal{E}$ and $\mathcal{F}$, and a function $\mathcal{H}$ from $\mathcal{E}^4$ into $\mathcal{E}$, and states that:

Let s be a state of $\mathcal{G}$+· 1GateCircuit($\langle\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\rangle,\mathcal{H}$) and s' be a state of $\mathcal{G}$. Suppose $s' = s{\restriction}$the carrier of $\mathcal{F}$. Let a_1, a_2, a_3, a_4 be elements of $\mathcal{E}$. Suppose that if $\mathcal{A} \in$ InnerVertices($\mathcal{F}$), then $a_1 =$ (Result(s'))($\mathcal{A}$) and if $\mathcal{A} \notin$ InnerVertices($\mathcal{F}$), then $a_1 = s(\mathcal{A})$ and if $\mathcal{B} \in$ InnerVertices($\mathcal{F}$), then $a_2 =$ (Result(s'))($\mathcal{B}$) and if $\mathcal{B} \notin$ InnerVertices($\mathcal{F}$), then $a_2 = s(\mathcal{B})$ and if $\mathcal{C} \in$ InnerVertices($\mathcal{F}$), then $a_3 =$ (Result(s'))($\mathcal{C}$) and if $\mathcal{C} \notin$ InnerVertices($\mathcal{F}$), then $a_3 = s(\mathcal{C})$ and if $\mathcal{D} \in$ InnerVertices($\mathcal{F}$), then $a_4 =$ (Result(s'))($\mathcal{D}$) and if $\mathcal{D} \notin$ InnerVertices($\mathcal{F}$), then $a_4 = s(\mathcal{D})$. Then (Result(s))(Output 1GateCircStr($\langle\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\rangle,\mathcal{H}$)) $= \mathcal{F}(a_1, a_2, a_3, a_4)$

provided the parameters satisfy the following conditions:

- Let g be a function from $\mathcal{E}^4$ into $\mathcal{E}$. Then $g = \mathcal{H}$ if and only if for all elements a_1, a_2, a_3, a_4 of $\mathcal{E}$ holds $g(\langle a_1, a_2, a_3, a_4\rangle) = \mathcal{F}(a_1, a_2, a_3, a_4)$, and
- Output 1GateCircStr$(\langle \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\rangle, \mathcal{H}) \notin$ InputVertices$(\mathcal{F})$.

The scheme *Comb5CircResult* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, a non empty finite set $\mathcal{F}$, a 5-ary functor $\mathcal{F}$ yielding an element of $\mathcal{F}$, a finite signature $\mathcal{G}$ over $\mathcal{F}$, a circuit $\mathcal{H}$ over $\mathcal{F}$ and $\mathcal{G}$, and a function $\mathcal{I}$ from $\mathcal{F}^5$ into $\mathcal{F}$, and states that:

> Let s be a state of $\mathcal{H}+\cdot$ 1GateCircuit$(\langle \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}\rangle, \mathcal{I})$ and s' be a state of $\mathcal{H}$. Suppose $s' = s\restriction$the carrier of $\mathcal{G}$. Let a_1, a_2, a_3, a_4, a_5 be elements of $\mathcal{F}$. Suppose that if $\mathcal{A} \in$ InnerVertices$(\mathcal{G})$, then $a_1 =$ (Result(s'))$(\mathcal{A})$ and if $\mathcal{A} \notin$ InnerVertices$(\mathcal{G})$, then $a_1 = s(\mathcal{A})$ and if $\mathcal{B} \in$ InnerVertices$(\mathcal{G})$, then $a_2 =$ (Result(s'))$(\mathcal{B})$ and if $\mathcal{B} \notin$ InnerVertices$(\mathcal{G})$, then $a_2 = s(\mathcal{B})$ and if $\mathcal{C} \in$ InnerVertices$(\mathcal{G})$, then $a_3 =$ (Result(s'))$(\mathcal{C})$ and if $\mathcal{C} \notin$ InnerVertices$(\mathcal{G})$, then $a_3 = s(\mathcal{C})$ and if $\mathcal{D} \in$ InnerVertices$(\mathcal{G})$, then $a_4 =$ (Result(s'))$(\mathcal{D})$ and if $\mathcal{D} \notin$ InnerVertices$(\mathcal{G})$, then $a_4 = s(\mathcal{D})$ and if $\mathcal{E} \in$ InnerVertices$(\mathcal{G})$, then $a_5 =$ (Result(s'))$(\mathcal{E})$ and if $\mathcal{E} \notin$ InnerVertices$(\mathcal{G})$, then $a_5 = s(\mathcal{E})$. Then (Result(s))(Output 1GateCircStr$(\langle \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}\rangle, \mathcal{I})) = \mathcal{F}(a_1, a_2, a_3, a_4, a_5)$

provided the parameters meet the following conditions:

- Let g be a function from $\mathcal{F}^5$ into $\mathcal{F}$. Then $g = \mathcal{I}$ if and only if for all elements a_1, a_2, a_3, a_4, a_5 of $\mathcal{F}$ holds $g(\langle a_1, a_2, a_3, a_4, a_5\rangle) = \mathcal{F}(a_1, a_2, a_3, a_4, a_5)$, and
- Output 1GateCircStr$(\langle \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}\rangle, \mathcal{I}) \notin$ InputVertices$(\mathcal{G})$.

6. Inputs Without Pairs

Let S be a non empty many sorted signature. We say that S has nonpair inputs if and only if:

(Def. 11) InputVertices(S) has no pairs.

Note that $\mathbb{N}$ has no pairs. Let X be a set with no pairs. Note that every subset of X has no pairs.

Let us observe that every function which is natural-yielding is also nonpair yielding.

Let us note that every finite sequence of elements of $\mathbb{N}$ is natural-yielding.

Let us observe that there exists a finite sequence which is one-to-one and natural-yielding.

Let n be a natural number. Observe that there exists a finite sequence with length n which is one-to-one and natural-yielding.

Let p be a nonpair yielding finite sequence and let f be a set. Observe that 1GateCircStr(p, f) has nonpair inputs.

One can verify that there exists an one-gate many sorted signature which has nonpair inputs. Let X be a non empty finite set. One can verify that there exists an one-gate signature over X which has nonpair inputs.

Let S be a non empty many sorted signature with nonpair inputs. One can check that InputVertices(S) has no pairs.

The following proposition is true

(53) Let S be a non empty many sorted signature with nonpair inputs and x be a vertex of S. If x is pair, then $x \in$ InnerVertices(S).

Let S be an unsplit non empty many sorted signature with arity held in gates. One can verify that InnerVertices(S) is relation-like.

Let S be an unsplit non empty non void many sorted signature with denotation held in gates. Note that InnerVertices(S) is relation-like.

Let S_1, S_2 be unsplit non empty many sorted signatures with arity held in gates with nonpair inputs. One can verify that $S_1 {+\cdot} S_2$ has nonpair inputs.

One can prove the following propositions:

(54) For every non pair set x and for every binary relation R holds $x \notin R$.

(55) Let x_1 be a set, X be a non empty finite set, f be a function from X^1 into X, and S be a signature over X with nonpair inputs. If $x_1 \in$ the carrier of S or x_1 is non pair, then $S {+\cdot}$ 1GateCircStr$(\langle x_1 \rangle, f)$ has nonpair inputs.

Let X be a non empty finite set, let S be a signature over X with nonpair inputs, let x_1 be a vertex of S, and let f be a function from X^1 into X. One can verify that $S {+\cdot}$ 1GateCircStr$(\langle x_1 \rangle, f)$ has nonpair inputs.

Let X be a non empty finite set, let S be a signature over X with nonpair inputs, let x_1 be a non pair set, and let f be a function from X^1 into X. One can verify that $S {+\cdot}$ 1GateCircStr$(\langle x_1 \rangle, f)$ has nonpair inputs.

We now state the proposition

(56) Let x_1, x_2 be sets, X be a non empty finite set, f be a function from X^2 into X, and S be a signature over X with nonpair inputs. Suppose $x_1 \in$ the carrier of S or x_1 is non pair but $x_2 \in$ the carrier of S or x_2 is non pair. Then $S {+\cdot}$ 1GateCircStr$(\langle x_1, x_2 \rangle, f)$ has nonpair inputs.

Let X be a non empty finite set, let S be a signature over X with nonpair inputs, let x_1 be a vertex of S, let n_2 be a non pair set, and let f be a function from X^2 into X. Observe that $S {+\cdot}$ 1GateCircStr$(\langle x_1, n_2 \rangle, f)$ has nonpair inputs and $S {+\cdot}$ 1GateCircStr$(\langle n_2, x_1 \rangle, f)$ has nonpair inputs.

Let X be a non empty finite set, let S be a signature over X with nonpair inputs, let x_1, x_2 be vertices of S, and let f be a function from X^2 into X. One can verify that $S {+\cdot}$ 1GateCircStr$(\langle x_1, x_2 \rangle, f)$ has nonpair inputs.

One can prove the following proposition

(57) Let x_1, x_2, x_3 be sets, X be a non empty finite set, f be a function from X^3 into X, and S be a signature over X with nonpair inputs. Suppose that

(i) $x_1 \in$ the carrier of S or x_1 is non pair,

(ii) $x_2 \in$ the carrier of S or x_2 is non pair, and

(iii) $x_3 \in$ the carrier of S or x_3 is non pair.

Then $S{+\cdot}\,$1GateCircStr$(\langle x_1, x_2, x_3\rangle, f)$ has nonpair inputs.

Let X be a non empty finite set, let S be a signature over X with nonpair inputs, let x_1, x_2 be vertices of S, let n be a non pair set, and let f be a function from X^3 into X. One can verify the following observations:

* $S{+\cdot}\,$1GateCircStr$(\langle x_1, x_2, n\rangle, f)$ has nonpair inputs,
* $S{+\cdot}\,$1GateCircStr$(\langle x_1, n, x_2\rangle, f)$ has nonpair inputs, and
* $S{+\cdot}\,$1GateCircStr$(\langle n, x_1, x_2\rangle, f)$ has nonpair inputs.

Let X be a non empty finite set, let S be a signature over X with nonpair inputs, let x be a vertex of S, let n_1, n_2 be non pair sets, and let f be a function from X^3 into X. One can check the following observations:

* $S{+\cdot}\,$1GateCircStr$(\langle x, n_1, n_2\rangle, f)$ has nonpair inputs,
* $S{+\cdot}\,$1GateCircStr$(\langle n_1, x, n_2\rangle, f)$ has nonpair inputs, and
* $S{+\cdot}\,$1GateCircStr$(\langle n_1, n_2, x\rangle, f)$ has nonpair inputs.

Let X be a non empty finite set, let S be a signature over X with nonpair inputs, let x_1, x_2, x_3 be vertices of S, and let f be a function from X^3 into X. Observe that $S{+\cdot}\,$1GateCircStr$(\langle x_1, x_2, x_3\rangle, f)$ has nonpair inputs.

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Properties of the Upper and Lower Sequence on the Cage[1]

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The terminology and notation used here are introduced in the following articles: [24], [27], [1], [3], [4], [2], [14], [12], [25], [22], [23], [11], [21], [8], [9], [6], [26], [15], [10], [18], [17], [5], [20], [19], [7], [13], and [16].

In this paper n is a natural number.

We now state a number of propositions:

(1) For all subsets A, B of $\mathcal{E}^2_{\rm T}$ such that A meets B holds $\mathrm{proj1}^\circ A$ meets $\mathrm{proj1}^\circ B$.

(2) Let A, B be subsets of $\mathcal{E}^2_{\rm T}$ and s be a real number. If A misses B and $A \subseteq \mathrm{HorizontalLine}\, s$ and $B \subseteq \mathrm{HorizontalLine}\, s$, then $\mathrm{proj1}^\circ A$ misses $\mathrm{proj1}^\circ B$.

(3) For every closed subset S of $\mathcal{E}^2_{\rm T}$ such that S is Bounded holds $\mathrm{proj1}^\circ S$ is closed.

(4) For every compact subset S of $\mathcal{E}^2_{\rm T}$ holds $\mathrm{proj1}^\circ S$ is compact.

(5) Let p, q, p_1, q_1 be points of $\mathcal{E}^2_{\rm T}$. Suppose $\mathcal{L}(p,q)$ is vertical and $\mathcal{L}(p_1,q_1)$ is vertical and $p_{\mathbf{1}} = (p_1)_{\mathbf{1}}$ and $p_{\mathbf{2}} \leqslant (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leqslant (q_1)_{\mathbf{2}}$ and $(q_1)_{\mathbf{2}} \leqslant q_{\mathbf{2}}$. Then $\mathcal{L}(p_1,q_1) \subseteq \mathcal{L}(p,q)$.

(6) Let p, q, p_1, q_1 be points of $\mathcal{E}^2_{\rm T}$. Suppose $\mathcal{L}(p,q)$ is horizontal and $\mathcal{L}(p_1,q_1)$ is horizontal and $p_{\mathbf{2}} = (p_1)_{\mathbf{2}}$ and $p_{\mathbf{1}} \leqslant (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} \leqslant (q_1)_{\mathbf{1}}$ and $(q_1)_{\mathbf{1}} \leqslant q_{\mathbf{1}}$. Then $\mathcal{L}(p_1,q_1) \subseteq \mathcal{L}(p,q)$.

(7) Let G be a Go-board and i, j, k, j_1, k_1 be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant \mathrm{len}\, G$ and $1 \leqslant j$ and $j \leqslant j_1$ and $j_1 \leqslant k_1$ and $k_1 \leqslant k$ and $k \leqslant \mathrm{width}\, G$. Then $\mathcal{L}(G \circ (i,j_1), G \circ (i,k_1)) \subseteq \mathcal{L}(G \circ (i,j), G \circ (i,k))$.

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(8) Let G be a Go-board and i, j, k, j_1, k_1 be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant \operatorname{width} G$ and $1 \leqslant j$ and $j \leqslant j_1$ and $j_1 \leqslant k_1$ and $k_1 \leqslant k$ and $k \leqslant \operatorname{len} G$. Then $\mathcal{L}(G \circ (j_1, i), G \circ (k_1, i)) \subseteq \mathcal{L}(G \circ (j, i), G \circ (k, i))$.

(9) Let G be a Go-board and j, k, j_1, k_1 be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant j_1$ and $j_1 \leqslant k_1$ and $k_1 \leqslant k$ and $k \leqslant \operatorname{width} G$. Then $\mathcal{L}(G \circ (\operatorname{Center} G, j_1), G \circ (\operatorname{Center} G, k_1)) \subseteq \mathcal{L}(G \circ (\operatorname{Center} G, j), G \circ (\operatorname{Center} G, k))$.

(10) Let G be a Go-board. Suppose $\operatorname{len} G = \operatorname{width} G$. Let j, k, j_1, k_1 be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant j_1$ and $j_1 \leqslant k_1$ and $k_1 \leqslant k$ and $k \leqslant \operatorname{len} G$. Then $\mathcal{L}(G \circ (j_1, \operatorname{Center} G), G \circ (k_1, \operatorname{Center} G)) \subseteq \mathcal{L}(G \circ (j, \operatorname{Center} G), G \circ (k, \operatorname{Center} G))$.

(11) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\operatorname{Gauge}(C, n) \circ (i, j) \in \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n))$. Then there exists a natural number j_1 such that $j \leqslant j_1$ and $j_1 \leqslant k$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (i, j_1), \operatorname{Gauge}(C, n) \circ (i, k)) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (i, j_1)\}$.

(12) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\operatorname{Gauge}(C, n) \circ (i, k) \in \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n))$. Then there exists a natural number k_1 such that $j \leqslant k_1$ and $k_1 \leqslant k$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (i, j), \operatorname{Gauge}(C, n) \circ (i, k_1)) \cap \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (i, k_1)\}$.

(13) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\operatorname{Gauge}(C, n) \circ (i, j) \in \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n))$ and $\operatorname{Gauge}(C, n) \circ (i, k) \in \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n))$. Then there exist natural numbers j_1, k_1 such that $j \leqslant j_1$ and $j_1 \leqslant k_1$ and $k_1 \leqslant k$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (i, j_1), \operatorname{Gauge}(C, n) \circ (i, k_1)) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (i, j_1)\}$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (i, j_1), \operatorname{Gauge}(C, n) \circ (i, k_1)) \cap \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (i, k_1)\}$.

(14) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\operatorname{Gauge}(C, n) \circ (j, i) \in \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n))$. Then there exists a natural number j_1 such that $j \leqslant j_1$ and $j_1 \leqslant k$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j_1, i), \operatorname{Gauge}(C, n) \circ (k, i)) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (j_1, i)\}$.

(15) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\operatorname{Gauge}(C, n) \circ (k, i) \in \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n))$. Then there exists a natural number k_1 such that $j \leqslant k_1$ and $k_1 \leqslant k$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k_1, i)) \cap$

$\widetilde{\mathcal{L}}(\text{UpperSeq}(C,n)) = \{\text{Gauge}(C,n) \circ (k_1,i)\}$.

(16) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{len}\text{Gauge}(C,n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width}\text{Gauge}(C,n)$ and $\text{Gauge}(C,n) \circ (j,i) \in \widetilde{\mathcal{L}}(\text{LowerSeq}(C,n))$ and $\text{Gauge}(C,n) \circ (k,i) \in \widetilde{\mathcal{L}}(\text{UpperSeq}(C,n))$. Then there exist natural numbers j_1, k_1 such that $j \leqslant j_1$ and $j_1 \leqslant k_1$ and $k_1 \leqslant k$ and $\mathcal{L}(\text{Gauge}(C,n) \circ (j_1,i), \text{Gauge}(C,n) \circ (k_1,i)) \cap \widetilde{\mathcal{L}}(\text{LowerSeq}(C,n)) = \{\text{Gauge}(C,n) \circ (j_1,i)\}$ and $\mathcal{L}(\text{Gauge}(C,n) \circ (j_1,i), \text{Gauge}(C,n) \circ (k_1,i)) \cap \widetilde{\mathcal{L}}(\text{UpperSeq}(C,n)) = \{\text{Gauge}(C,n) \circ (k_1,i)\}$.

(17) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant \operatorname{len}\text{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width}\text{Gauge}(C,n)$ and $\text{Gauge}(C,n) \circ (i,j) \in \widetilde{\mathcal{L}}(\text{UpperSeq}(C,n))$. Then there exists a natural number j_1 such that $j \leqslant j_1$ and $j_1 \leqslant k$ and $\mathcal{L}(\text{Gauge}(C,n) \circ (i,j_1), \text{Gauge}(C,n) \circ (i,k)) \cap \widetilde{\mathcal{L}}(\text{UpperSeq}(C,n)) = \{\text{Gauge}(C,n) \circ (i,j_1)\}$.

(18) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant \operatorname{len}\text{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width}\text{Gauge}(C,n)$ and $\text{Gauge}(C,n) \circ (i,k) \in \widetilde{\mathcal{L}}(\text{LowerSeq}(C,n))$. Then there exists a natural number k_1 such that $j \leqslant k_1$ and $k_1 \leqslant k$ and $\mathcal{L}(\text{Gauge}(C,n) \circ (i,j), \text{Gauge}(C,n) \circ (i,k_1)) \cap \widetilde{\mathcal{L}}(\text{LowerSeq}(C,n)) = \{\text{Gauge}(C,n) \circ (i,k_1)\}$.

(19) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant \operatorname{len}\text{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width}\text{Gauge}(C,n)$ and $\text{Gauge}(C,n) \circ (i,j) \in \widetilde{\mathcal{L}}(\text{UpperSeq}(C,n))$ and $\text{Gauge}(C,n) \circ (i,k) \in \widetilde{\mathcal{L}}(\text{LowerSeq}(C,n))$. Then there exist natural numbers j_1, k_1 such that $j \leqslant j_1$ and $j_1 \leqslant k_1$ and $k_1 \leqslant k$ and $\mathcal{L}(\text{Gauge}(C,n) \circ (i,j_1), \text{Gauge}(C,n) \circ (i,k_1)) \cap \widetilde{\mathcal{L}}(\text{UpperSeq}(C,n)) = \{\text{Gauge}(C,n) \circ (i,j_1)\}$ and $\mathcal{L}(\text{Gauge}(C,n) \circ (i,j_1), \text{Gauge}(C,n) \circ (i,k_1)) \cap \widetilde{\mathcal{L}}(\text{LowerSeq}(C,n)) = \{\text{Gauge}(C,n) \circ (i,k_1)\}$.

(20) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{len}\text{Gauge}(C,n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width}\text{Gauge}(C,n)$ and $\text{Gauge}(C,n) \circ (j,i) \in \widetilde{\mathcal{L}}(\text{UpperSeq}(C,n))$. Then there exists a natural number j_1 such that $j \leqslant j_1$ and $j_1 \leqslant k$ and $\mathcal{L}(\text{Gauge}(C,n) \circ (j_1,i), \text{Gauge}(C,n) \circ (k,i)) \cap \widetilde{\mathcal{L}}(\text{UpperSeq}(C,n)) = \{\text{Gauge}(C,n) \circ (j_1,i)\}$.

(21) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{len}\text{Gauge}(C,n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width}\text{Gauge}(C,n)$ and $\text{Gauge}(C,n) \circ (k,i) \in \widetilde{\mathcal{L}}(\text{LowerSeq}(C,n))$. Then there exists a natural number k_1 such that $j \leqslant k_1$ and $k_1 \leqslant k$ and $\mathcal{L}(\text{Gauge}(C,n) \circ (j,i), \text{Gauge}(C,n) \circ (k_1,i)) \cap \widetilde{\mathcal{L}}(\text{LowerSeq}(C,n)) = \{\text{Gauge}(C,n) \circ (k_1,i)\}$.

(22) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{len} \mathrm{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \mathrm{Gauge}(C, n)$ and $\mathrm{Gauge}(C, n) \circ (j, i) \in \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C, n))$ and $\mathrm{Gauge}(C, n) \circ (k, i) \in \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C, n))$. Then there exist natural numbers j_1, k_1 such that $j \leqslant j_1$ and $j_1 \leqslant k_1$ and $k_1 \leqslant k$ and $\mathcal{L}(\mathrm{Gauge}(C, n) \circ (j_1, i), \mathrm{Gauge}(C, n) \circ (k_1, i)) \cap \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C, n)) = \{\mathrm{Gauge}(C, n) \circ (j_1, i)\}$ and $\mathcal{L}(\mathrm{Gauge}(C, n) \circ (j_1, i), \mathrm{Gauge}(C, n) \circ (k_1, i)) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C, n)) = \{\mathrm{Gauge}(C, n) \circ (k_1, i)\}$.

(23) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \operatorname{len} \mathrm{Gauge}(C, n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \mathrm{Gauge}(C, n)$ and $\mathrm{Gauge}(C, n) \circ (i, k) \in \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C, n))$ and $\mathrm{Gauge}(C, n) \circ (i, j) \in \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C, n))$. Then $\mathcal{L}(\mathrm{Gauge}(C, n) \circ (i, j), \mathrm{Gauge}(C, n) \circ (i, k))$ meets $\mathrm{LowerArc}\, C$.

(24) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \operatorname{len} \mathrm{Gauge}(C, n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \mathrm{Gauge}(C, n)$ and $\mathrm{Gauge}(C, n) \circ (i, k) \in \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C, n))$ and $\mathrm{Gauge}(C, n) \circ (i, j) \in \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C, n))$. Then $\mathcal{L}(\mathrm{Gauge}(C, n) \circ (i, j), \mathrm{Gauge}(C, n) \circ (i, k))$ meets $\mathrm{UpperArc}\, C$.

(25) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \operatorname{len} \mathrm{Gauge}(C, n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \mathrm{Gauge}(C, n)$ and $n > 0$ and $\mathrm{Gauge}(C, n) \circ (i, k) \in \mathrm{UpperArc}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$ and $\mathrm{Gauge}(C, n) \circ (i, j) \in \mathrm{LowerArc}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$. Then $\mathcal{L}(\mathrm{Gauge}(C, n) \circ (i, j), \mathrm{Gauge}(C, n) \circ (i, k))$ meets $\mathrm{LowerArc}\, C$.

(26) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \operatorname{len} \mathrm{Gauge}(C, n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \mathrm{Gauge}(C, n)$ and $n > 0$ and $\mathrm{Gauge}(C, n) \circ (i, k) \in \mathrm{UpperArc}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$ and $\mathrm{Gauge}(C, n) \circ (i, j) \in \mathrm{LowerArc}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$. Then $\mathcal{L}(\mathrm{Gauge}(C, n) \circ (i, j), \mathrm{Gauge}(C, n) \circ (i, k))$ meets $\mathrm{UpperArc}\, C$.

(27) Let C be a simple closed curve and j, k be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \mathrm{Gauge}(C, n+1)$ and $\mathrm{Gauge}(C, n+1) \circ (\mathrm{Center}\, \mathrm{Gauge}(C, n+1), k) \in \mathrm{UpperArc}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n+1))$ and $\mathrm{Gauge}(C, n+1) \circ (\mathrm{Center}\, \mathrm{Gauge}(C, n+1), j) \in \mathrm{LowerArc}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n+1))$. Then $\mathcal{L}(\mathrm{Gauge}(C, n+1) \circ (\mathrm{Center}\, \mathrm{Gauge}(C, n+1), j), \mathrm{Gauge}(C, n+1) \circ (\mathrm{Center}\, \mathrm{Gauge}(C, n+1), k))$ meets $\mathrm{LowerArc}\, C$.

(28) Let C be a simple closed curve and j, k be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \mathrm{Gauge}(C, n+1)$ and $\mathrm{Gauge}(C, n+1) \circ (\mathrm{Center}\, \mathrm{Gauge}(C, n+1), k) \in \mathrm{UpperArc}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n+1))$ and $\mathrm{Gauge}(C, n+1) \circ (\mathrm{Center}\, \mathrm{Gauge}(C, n+1), j) \in \mathrm{LowerArc}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n+1))$. Then $\mathcal{L}(\mathrm{Gauge}(C, n+1) \circ (\mathrm{Center}\, \mathrm{Gauge}(C, n+1), j), \mathrm{Gauge}(C, n+1) \circ$

(Center Gauge$(C, n+1), k)$) meets UpperArc C.

(29) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and i, j, k be natural numbers. Suppose $1 < j$ and $k <$ len Gauge(C, n) and $1 \leqslant i$ and $i \leqslant$ width Gauge(C, n) and Gauge$(C, n) \circ (k, i) \in \widetilde{\mathcal{L}}$(UpperSeq$(C, n)$) and Gauge$(C, n) \circ (j, i) \in \widetilde{\mathcal{L}}$(LowerSeq$(C, n)$). Then $j \neq k$.

(30) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k <$ len Gauge(C, n) and $1 \leqslant i$ and $i \leqslant$ width Gauge(C, n) and $\mathcal{L}$(Gauge$(C, n) \circ (j, i)$, Gauge$(C, n) \circ (k, i)) \cap \widetilde{\mathcal{L}}$(UpperSeq$(C, n)) = \{$Gauge$(C, n) \circ (k, i)\}$ and $\mathcal{L}$(Gauge$(C, n) \circ (j, i)$, Gauge$(C, n) \circ (k, i)) \cap \widetilde{\mathcal{L}}$(LowerSeq$(C, n)) = \{$Gauge$(C, n) \circ (j, i)\}$. Then $\mathcal{L}$(Gauge$(C, n) \circ (j, i)$, Gauge$(C, n) \circ (k, i)$) meets LowerArc C.

(31) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k <$ len Gauge(C, n) and $1 \leqslant i$ and $i \leqslant$ width Gauge(C, n) and $\mathcal{L}$(Gauge$(C, n) \circ (j, i)$, Gauge$(C, n) \circ (k, i)) \cap \widetilde{\mathcal{L}}$(UpperSeq$(C, n)) = \{$Gauge$(C, n) \circ (k, i)\}$ and $\mathcal{L}$(Gauge$(C, n) \circ (j, i)$, Gauge$(C, n) \circ (k, i)) \cap \widetilde{\mathcal{L}}$(LowerSeq$(C, n)) = \{$Gauge$(C, n) \circ (j, i)\}$. Then $\mathcal{L}$(Gauge$(C, n) \circ (j, i)$, Gauge$(C, n) \circ (k, i)$) meets UpperArc C.

(32) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k <$ len Gauge(C, n) and $1 \leqslant i$ and $i \leqslant$ width Gauge(C, n) and Gauge$(C, n) \circ (k, i) \in \widetilde{\mathcal{L}}$(UpperSeq$(C, n)$) and Gauge$(C, n) \circ (j, i) \in \widetilde{\mathcal{L}}$(LowerSeq$(C, n)$). Then $\mathcal{L}$(Gauge$(C, n) \circ (j, i)$, Gauge$(C, n) \circ (k, i)$) meets LowerArc C.

(33) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k <$ len Gauge(C, n) and $1 \leqslant i$ and $i \leqslant$ width Gauge(C, n) and Gauge$(C, n) \circ (k, i) \in \widetilde{\mathcal{L}}$(UpperSeq$(C, n)$) and Gauge$(C, n) \circ (j, i) \in \widetilde{\mathcal{L}}$(LowerSeq$(C, n)$). Then $\mathcal{L}$(Gauge$(C, n) \circ (j, i)$, Gauge$(C, n) \circ (k, i)$) meets UpperArc C.

(34) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k <$ len Gauge(C, n) and $1 \leqslant i$ and $i \leqslant$ width Gauge(C, n) and $n > 0$ and Gauge$(C, n) \circ (k, i) \in$ UpperArc $\widetilde{\mathcal{L}}$(Cage(C, n)) and Gauge$(C, n) \circ (j, i) \in$ LowerArc $\widetilde{\mathcal{L}}$(Cage(C, n)). Then $\mathcal{L}$(Gauge$(C, n) \circ (j, i)$, Gauge$(C, n) \circ (k, i)$) meets LowerArc C.

(35) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k <$ len Gauge(C, n) and $1 \leqslant i$ and $i \leqslant$ width Gauge(C, n) and $n > 0$ and Gauge$(C, n) \circ (k, i) \in$ UpperArc $\widetilde{\mathcal{L}}$(Cage(C, n)) and Gauge$(C, n) \circ (j, i) \in$ LowerArc $\widetilde{\mathcal{L}}$(Cage(C, n)). Then $\mathcal{L}$(Gauge$(C, n) \circ (j, i)$, Gauge$(C, n) \circ (k, i)$) meets UpperArc C.

(36) Let C be a simple closed curve and j, k be natural numbers. Sup-

pose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n+1)$ and $\operatorname{Gauge}(C, n+1) \circ (k, \operatorname{Center} \operatorname{Gauge}(C, n+1)) \in \operatorname{UpperArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n+1))$ and $\operatorname{Gauge}(C, n+1) \circ (j, \operatorname{Center} \operatorname{Gauge}(C, n+1)) \in \operatorname{LowerArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n+1))$. Then $\mathcal{L}(\operatorname{Gauge}(C, n+1) \circ (j, \operatorname{Center} \operatorname{Gauge}(C, n+1)), \operatorname{Gauge}(C, n+1) \circ (k, \operatorname{Center} \operatorname{Gauge}(C, n+1)))$ meets $\operatorname{LowerArc} C$.

(37) Let C be a simple closed curve and j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n+1)$ and $\operatorname{Gauge}(C, n+1) \circ (k, \operatorname{Center} \operatorname{Gauge}(C, n+1)) \in \operatorname{UpperArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n+1))$ and $\operatorname{Gauge}(C, n+1) \circ (j, \operatorname{Center} \operatorname{Gauge}(C, n+1)) \in \operatorname{LowerArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n+1))$. Then $\mathcal{L}(\operatorname{Gauge}(C, n+1) \circ (j, \operatorname{Center} \operatorname{Gauge}(C, n+1)), \operatorname{Gauge}(C, n+1) \circ (k, \operatorname{Center} \operatorname{Gauge}(C, n+1)))$ meets $\operatorname{UpperArc} C$.

(38) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i)) \cap \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (j, i)\}$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i)) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (k, i)\}$. Then $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i))$ meets $\operatorname{LowerArc} C$.

(39) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i)) \cap \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (j, i)\}$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i)) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (k, i)\}$. Then $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i))$ meets $\operatorname{UpperArc} C$.

(40) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\operatorname{Gauge}(C, n) \circ (j, i) \in \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n))$ and $\operatorname{Gauge}(C, n) \circ (k, i) \in \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n))$. Then $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i))$ meets $\operatorname{LowerArc} C$.

(41) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\operatorname{Gauge}(C, n) \circ (j, i) \in \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n))$ and $\operatorname{Gauge}(C, n) \circ (k, i) \in \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n))$. Then $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i))$ meets $\operatorname{UpperArc} C$.

(42) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $n > 0$ and $\operatorname{Gauge}(C, n) \circ (j, i) \in \operatorname{UpperArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$ and $\operatorname{Gauge}(C, n) \circ (k, i) \in \operatorname{LowerArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$. Then $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i))$ meets $\operatorname{LowerArc} C$.

(43) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n)$

and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $n > 0$ and $\operatorname{Gauge}(C, n) \circ (j, i) \in \operatorname{UpperArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$ and $\operatorname{Gauge}(C, n) \circ (k, i) \in \operatorname{LowerArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$. Then $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i))$ meets $\operatorname{UpperArc} C$.

(44) Let C be a simple closed curve and j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n+1)$ and $\operatorname{Gauge}(C, n+1) \circ (j, \operatorname{Center} \operatorname{Gauge}(C, n+1)) \in \operatorname{UpperArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n+1))$ and $\operatorname{Gauge}(C, n+1) \circ (k, \operatorname{Center} \operatorname{Gauge}(C, n+1)) \in \operatorname{LowerArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n+1))$. Then $\mathcal{L}(\operatorname{Gauge}(C, n+1) \circ (j, \operatorname{Center} \operatorname{Gauge}(C, n+1)), \operatorname{Gauge}(C, n+1) \circ (k, \operatorname{Center} \operatorname{Gauge}(C, n+1)))$ meets $\operatorname{LowerArc} C$.

(45) Let C be a simple closed curve and j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n+1)$ and $\operatorname{Gauge}(C, n+1) \circ (j, \operatorname{Center} \operatorname{Gauge}(C, n+1)) \in \operatorname{UpperArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n+1))$ and $\operatorname{Gauge}(C, n+1) \circ (k, \operatorname{Center} \operatorname{Gauge}(C, n+1)) \in \operatorname{LowerArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n+1))$. Then $\mathcal{L}(\operatorname{Gauge}(C, n+1) \circ (j, \operatorname{Center} \operatorname{Gauge}(C, n+1)), \operatorname{Gauge}(C, n+1) \circ (k, \operatorname{Center} \operatorname{Gauge}(C, n+1)))$ meets $\operatorname{UpperArc} C$.

(46) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 \leqslant i_2$ and $i_2 < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $(\mathcal{L}(\operatorname{Gauge}(C, n) \circ (i_1, j), \operatorname{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\operatorname{Gauge}(C, n) \circ (i_1, k), \operatorname{Gauge}(C, n) \circ (i_2, k))) \cap \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (i_2, k)\}$ and $(\mathcal{L}(\operatorname{Gauge}(C, n) \circ (i_1, j), \operatorname{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\operatorname{Gauge}(C, n) \circ (i_1, k), \operatorname{Gauge}(C, n) \circ (i_2, k))) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (i_1, j)\}$. Then $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (i_1, j), \operatorname{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\operatorname{Gauge}(C, n) \circ (i_1, k), \operatorname{Gauge}(C, n) \circ (i_2, k))$ meets $\operatorname{UpperArc} C$.

(47) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 \leqslant i_2$ and $i_2 < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $(\mathcal{L}(\operatorname{Gauge}(C, n) \circ (i_1, j), \operatorname{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\operatorname{Gauge}(C, n) \circ (i_1, k), \operatorname{Gauge}(C, n) \circ (i_2, k))) \cap \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (i_2, k)\}$ and $(\mathcal{L}(\operatorname{Gauge}(C, n) \circ (i_1, j), \operatorname{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\operatorname{Gauge}(C, n) \circ (i_1, k), \operatorname{Gauge}(C, n) \circ (i_2, k))) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (i_1, j)\}$. Then $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (i_1, j), \operatorname{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\operatorname{Gauge}(C, n) \circ (i_1, k), \operatorname{Gauge}(C, n) \circ (i_2, k))$ meets $\operatorname{LowerArc} C$.

(48) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_2$ and $i_2 \leqslant i_1$ and $i_1 < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $(\mathcal{L}(\operatorname{Gauge}(C, n) \circ (i_1, j), \operatorname{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\operatorname{Gauge}(C, n) \circ (i_1, k), \operatorname{Gauge}(C, n) \circ (i_2, k))) \cap \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (i_2, k)\}$ and $(\mathcal{L}(\operatorname{Gauge}(C, n) \circ (i_1, j), \operatorname{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\operatorname{Gauge}(C, n) \circ (i_1, k), \operatorname{Gauge}(C, n) \circ (i_2, k))) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (i_1, j)\}$.

Then $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))$ meets $\mathrm{UpperArc}\, C$.

(49) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_2$ and $i_2 \leqslant i_1$ and $i_1 < \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $(\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))) \cap \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i_2,k)\}$ and $(\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i_1,j)\}$. Then $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))$ meets $\mathrm{LowerArc}\, C$.

(50) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 < \mathrm{len}\,\mathrm{Gauge}(C,n+1)$ and $1 < i_2$ and $i_2 < \mathrm{len}\,\mathrm{Gauge}(C,n+1)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n+1)$ and $\mathrm{Gauge}(C,n+1) \circ (i_1,k) \in \mathrm{UpperArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n+1))$ and $\mathrm{Gauge}(C,n+1) \circ (i_2,j) \in \mathrm{LowerArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n+1))$. Then $\mathcal{L}(\mathrm{Gauge}(C,n+1) \circ (i_2,j), \mathrm{Gauge}(C,n+1) \circ (i_2,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n+1) \circ (i_2,k), \mathrm{Gauge}(C,n+1) \circ (i_1,k))$ meets $\mathrm{UpperArc}\, C$.

(51) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 < \mathrm{len}\,\mathrm{Gauge}(C,n+1)$ and $1 < i_2$ and $i_2 < \mathrm{len}\,\mathrm{Gauge}(C,n+1)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n+1)$ and $\mathrm{Gauge}(C,n+1) \circ (i_1,k) \in \mathrm{UpperArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n+1))$ and $\mathrm{Gauge}(C,n+1) \circ (i_2,j) \in \mathrm{LowerArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n+1))$. Then $\mathcal{L}(\mathrm{Gauge}(C,n+1) \circ (i_2,j), \mathrm{Gauge}(C,n+1) \circ (i_2,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n+1) \circ (i_2,k), \mathrm{Gauge}(C,n+1) \circ (i_1,k))$ meets $\mathrm{LowerArc}\, C$.

(52) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \mathrm{len}\,\mathrm{Gauge}(C,n+1)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n+1)$ and $\mathrm{Gauge}(C,n+1) \circ (i,k) \in \mathrm{UpperArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n+1))$ and $\mathrm{Gauge}(C,n+1) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n+1),j) \in \mathrm{LowerArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n+1))$. Then $\mathcal{L}(\mathrm{Gauge}(C,n+1) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n+1),j), \mathrm{Gauge}(C,n+1) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n+1),k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n+1) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n+1),k), \mathrm{Gauge}(C,n+1) \circ (i,k))$ meets $\mathrm{UpperArc}\, C$.

(53) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \mathrm{len}\,\mathrm{Gauge}(C,n+1)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n+1)$ and $\mathrm{Gauge}(C,n+1) \circ (i,k) \in \mathrm{UpperArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n+1))$ and $\mathrm{Gauge}(C,n+1) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n+1),j) \in \mathrm{LowerArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n+1))$. Then $\mathcal{L}(\mathrm{Gauge}(C,n+1) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n+1),j), \mathrm{Gauge}(C,n+1) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n+1),k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n+1) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n+1),k), \mathrm{Gauge}(C,n+1) \circ (i,k))$ meets $\mathrm{LowerArc}\, C$.

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On the Decompositions of Intervals and Simple Closed Curves[1]

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Summary. The aim of the paper is to show that the only subcontinua of the Jordan curve are arcs, the whole curve, and singletons of its points. Additionally, it has been shown that the only subcontinua of the unit interval $\mathbb{I}$ are closed intervals.

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The articles [21], [23], [13], [24], [2], [1], [3], [25], [19], [6], [4], [20], [8], [10], [11], [15], [26], [18], [22], [14], [16], [9], [17], [5], [12], and [7] provide the terminology and notation for this paper.

1. Preliminaries

Let us note that every simple closed curve is non trivial.

Let T be a non empty topological space. One can check that there exists a subset of T which is non empty, compact, and connected.

Let us observe that every element of the carrier of $\mathbb{I}$ is real.

Next we state two propositions:

(1) Let X be a non empty set and A, B be non empty subsets of X. If $A \subset B$, then there exists an element p of X such that $p \in B$ and $A \subseteq B \setminus \{p\}$.

(2) Let X be a non empty set and A be a non empty subset of X. Then A is trivial if and only if there exists an element x of X such that $A = \{x\}$.

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Let T be a non trivial 1-sorted structure. Observe that there exists a subset of the carrier of T which is non trivial.

The following proposition is true

(3) For every non trivial set X and for every set p there exists an element q of X such that $q \neq p$.

Let X be a non trivial set. Observe that there exists a subset of X which is non trivial.

We now state a number of propositions:

(4) Let T be a non trivial set, X be a non trivial subset of T, and p be a set. Then there exists an element q of T such that $q \in X$ and $q \neq p$.

(5) Let f, g be functions and a be a set. Suppose f is one-to-one and g is one-to-one and $\operatorname{dom} f \cap \operatorname{dom} g = \{a\}$ and $\operatorname{rng} f \cap \operatorname{rng} g = \{f(a)\}$. Then $f{+}{\cdot}g$ is one-to-one.

(6) Let f, g be functions and a be a set. Suppose f is one-to-one and g is one-to-one and $\operatorname{dom} f \cap \operatorname{dom} g = \{a\}$ and $\operatorname{rng} f \cap \operatorname{rng} g = \{f(a)\}$ and $f(a) = g(a)$. Then $(f{+}{\cdot}g)^{-1} = f^{-1}{+}{\cdot}g^{-1}$.

(7) Let n be a natural number, A be a non empty subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, and p, q be points of $\mathcal{E}^n_{\mathrm{T}}$. If A is an arc from p to q, then $A \setminus \{p\}$ is non empty.

(8) For every natural number n and for all points a, b of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathcal{L}(a, b)$ is n-convex.

(9) For all real numbers s_1, s_3, s_4, l such that $s_1 \leqslant s_3$ and $s_1 < s_4$ and $0 \leqslant l$ and $l \leqslant 1$ holds $s_1 \leqslant (1 - l) \cdot s_3 + l \cdot s_4$.

(10) For every set x and for all real numbers a, b such that $a \leqslant b$ and $x \in [a, b]$ holds $x \in]a, b[$ or $x = a$ or $x = b$.

(11) For all real numbers a, b, c, d such that $]a, b[$ meets $[c, d]$ holds $b > c$.

(12) For all real numbers a, b, c, d such that $b \leqslant c$ holds $[a, b]$ misses $]c, d[$.

(13) For all real numbers a, b, c, d such that $b \leqslant c$ holds $]a, b[$ misses $[c, d]$.

(14) For all real numbers a, b, c, d such that $a < b$ and $[a, b] \subseteq [c, d]$ holds $c \leqslant a$ and $b \leqslant d$.

(15) For all real numbers a, b, c, d such that $a < b$ and $]a, b[\subseteq [c, d]$ holds $c \leqslant a$ and $b \leqslant d$.

(16) For all real numbers a, b, c, d such that $a < b$ and $]a, b[\subseteq [c, d]$ holds $[a, b] \subseteq [c, d]$.

(17) Let A be a subset of the carrier of $\mathbb{I}$ and a, b be real numbers. If $a < b$ and $A =]a, b[$, then $[a, b] \subseteq$ the carrier of $\mathbb{I}$.

(18) Let A be a subset of the carrier of $\mathbb{I}$ and a, b be real numbers. If $a < b$ and $A =]a, b]$, then $[a, b] \subseteq$ the carrier of $\mathbb{I}$.

(19) Let A be a subset of the carrier of $\mathbb{I}$ and a, b be real numbers. If $a < b$ and $A = [a, b[$, then $[a, b] \subseteq$ the carrier of $\mathbb{I}$.

(20) For all real numbers a, b such that $a \neq b$ holds $\overline{]a, b]} = [a, b]$.

(21) For all real numbers a, b such that $a \neq b$ holds $\overline{[a, b[} = [a, b]$.

(22) For every subset A of $\mathbb{I}$ and for all real numbers a, b such that $a < b$ and $A =]a, b[$ holds $\overline{A} = [a, b]$.

(23) For every subset A of the carrier of $\mathbb{I}$ and for all real numbers a, b such that $a < b$ and $A =]a, b]$ holds $\overline{A} = [a, b]$.

(24) For every subset A of the carrier of $\mathbb{I}$ and for all real numbers a, b such that $a < b$ and $A = [a, b[$ holds $\overline{A} = [a, b]$.

(25) For all real numbers a, b such that $a < b$ holds $[a, b] \neq]a, b]$.

(26) For all real numbers a, b holds $[a, b[$ misses $\{b\}$ and $]a, b]$ misses $\{a\}$.

(27) For all real numbers a, b such that $a \leqslant b$ holds $[a, b] \setminus \{a\} =]a, b]$.

(28) For all real numbers a, b such that $a \leqslant b$ holds $[a, b] \setminus \{b\} = [a, b[$.

(29) For all real numbers a, b, c such that $a < b$ and $b < c$ holds $]a, b] \cap [b, c[= \{b\}$.

(30) For all real numbers a, b, c holds $[a, b[$ misses $[b, c]$ and $[a, b]$ misses $]b, c]$.

(31) For all real numbers a, b, c such that $a \leqslant b$ and $b \leqslant c$ holds $[a, c] \setminus \{b\} = [a, b[\cup]b, c]$.

(32) Let A be a subset of the carrier of $\mathbb{I}$ and a, b be real numbers. If $a \leqslant b$ and $A = [a, b]$, then $0 \leqslant a$ and $b \leqslant 1$.

(33) Let A, B be subsets of $\mathbb{I}$ and a, b, c be real numbers. If $a < b$ and $b < c$ and $A = [a, b[$ and $B =]b, c]$, then A and B are separated.

(34) For all real numbers a, b such that $a \leqslant b$ holds $[a, b] = [a, b[\cup \{b\}$.

(35) For all real numbers a, b such that $a \leqslant b$ holds $[a, b] = \{a\} \cup]a, b]$.

(36) For all real numbers a, b, c, d such that $a \leqslant b$ and $b < c$ and $c \leqslant d$ holds $[a, d] = [a, b] \cup]b, c[\cup [c, d]$.

(37) For all real numbers a, b, c, d such that $a \leqslant b$ and $b < c$ and $c \leqslant d$ holds $[a, d] \setminus ([a, b] \cup [c, d]) =]b, c[$.

(38) For all real numbers a, b, c such that $a < b$ and $b < c$ holds $]a, b] \cup]b, c[=]a, c[$.

(39) For all real numbers a, b, c such that $a < b$ and $b < c$ holds $[b, c[\subseteq]a, c[$.

(40) For all real numbers a, b, c such that $a < b$ and $b < c$ holds $]a, b] \cup [b, c[=]a, c[$.

(41) For all real numbers a, b, c such that $a < b$ and $b < c$ holds $]a, c[\setminus]a, b] =]b, c[$.

(42) For all real numbers a, b, c such that $a < b$ and $b < c$ holds $]a, c[\setminus [b, c[=]a, b[$.

(43) For all points p_1, p_2 of $\mathbb{I}$ holds $[p_1, p_2]$ is a subset of $\mathbb{I}$.

(44) For all points a, b of $\mathbb{I}$ holds $]a, b[$ is a subset of $\mathbb{I}$.

2. Decompositions of Intervals

The following propositions are true:

(45) For every real number p holds $\{p\}$ is a closed-interval subset of $\mathbb{R}$.

(46) Let A be a non empty connected subset of $\mathbb{I}$ and a, b, c be points of $\mathbb{I}$. If $a \leqslant b$ and $b \leqslant c$ and $a \in A$ and $c \in A$, then $b \in A$.

(47) For every non empty connected subset A of $\mathbb{I}$ and for all real numbers a, b such that $a \in A$ and $b \in A$ holds $[a, b] \subseteq A$.

(48) For all real numbers a, b and for every subset A of $\mathbb{I}$ such that $a \leqslant b$ and $A = [a, b]$ holds A is closed.

(49) For all points p_1, p_2 of $\mathbb{I}$ such that $p_1 \leqslant p_2$ holds $[p_1, p_2]$ is a non empty compact connected subset of $\mathbb{I}$.

(50) Let X be a subset of the carrier of $\mathbb{I}$ and X' be a subset of $\mathbb{R}$. If $X' = X$, then X' is upper bounded and lower bounded.

(51) Let X be a subset of the carrier of $\mathbb{I}$, X' be a subset of $\mathbb{R}$, and x be a real number. If $x \in X'$ and $X' = X$, then $\inf X' \leqslant x$ and $x \leqslant \sup X'$.

(52) For every subset A of $\mathbb{R}$ and for every subset B of $\mathbb{I}$ such that $A = B$ holds A is closed iff B is closed.

(53) For every closed-interval subset C of $\mathbb{R}$ holds $\inf C \leqslant \sup C$.

(54) Let C be a non empty compact connected subset of $\mathbb{I}$ and C' be a subset of $\mathbb{R}$. If $C = C'$ and $[\inf C', \sup C'] \subseteq C'$, then $[\inf C', \sup C'] = C'$.

(55) Every non empty compact connected subset of $\mathbb{I}$ is a closed-interval subset of $\mathbb{R}$.

(56) For every non empty compact connected subset C of $\mathbb{I}$ there exist points p_1, p_2 of $\mathbb{I}$ such that $p_1 \leqslant p_2$ and $C = [p_1, p_2]$.

3. Decompositions of Simple Closed Curves

The strict non empty subspace $I(01)$ of $\mathbb{I}$ is defined as follows:

(Def. 1) The carrier of $I(01) =]0, 1[$.

One can prove the following propositions:

(57) For every subset A of $\mathbb{I}$ such that $A =$ the carrier of $I(01)$ holds $I(01) = \mathbb{I}{\restriction}A$.

(58) The carrier of $I(01) =$ (the carrier of $\mathbb{I}$) $\setminus \{0, 1\}$.

(59) $I(01)$ is an open subspace of $\mathbb{I}$.

(60) For every real number r holds $r \in$ the carrier of $I(01)$ iff $0 < r$ and $r < 1$.

(61) For all points a, b of $\mathbb{I}$ such that $a < b$ and $b \neq 1$ holds $]a, b]$ is a non empty subset of $I(01)$.

(62) For all points a, b of $\mathbb{I}$ such that $a < b$ and $a \neq 0$ holds $[a, b[$ is a non empty subset of $I(01)$.

(63) For every simple closed curve D holds $(\mathcal{E}^2_{\mathrm{T}}) \restriction \square_{\mathcal{E}^2}$ and $(\mathcal{E}^2_{\mathrm{T}}) \restriction D$ are homeomorphic.

(64) Let D be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If D is an arc from p_1 to p_2, then $I(01)$ and $(\mathcal{E}^2_{\mathrm{T}}) \restriction (D \setminus \{p_1, p_2\})$ are homeomorphic.

(65) Let D be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If D is an arc from p_1 to p_2, then $\mathbb{I}$ and $(\mathcal{E}^2_{\mathrm{T}}) \restriction D$ are homeomorphic.

(66) For all points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_1 \neq p_2$ holds $\mathbb{I}$ and $(\mathcal{E}^2_{\mathrm{T}}) \restriction \mathcal{L}(p_1, p_2)$ are homeomorphic.

(67) Let E be a subset of $I(01)$. Given points p_1, p_2 of $\mathbb{I}$ such that $p_1 < p_2$ and $E = [p_1, p_2]$. Then $\mathbb{I}$ and $I(01) \restriction E$ are homeomorphic.

(68) Let A be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, p, q be points of $\mathcal{E}^2_{\mathrm{T}}$, and a, b be points of $\mathbb{I}$. Suppose A is an arc from p to q and $a < b$. Then there exists a non empty subset E of $\mathbb{I}$ and there exists a map f from $\mathbb{I} \restriction E$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction A$ such that $E = [a, b]$ and f is a homeomorphism and $f(a) = p$ and $f(b) = q$.

(69) Let A be a topological space, B be a non empty topological space, f be a map from A into B, C be a topological space, and X be a subset of A. Suppose f is continuous and C is a subspace of B. Let h be a map from $A \restriction X$ into C. If $h = f \restriction X$, then h is continuous.

(70) For every subset X of $\mathbb{I}$ and for all points a, b of $\mathbb{I}$ such that $a \leqslant b$ and $X =]a, b[$ holds X is open.

(71) For every subset X of $I(01)$ and for all points a, b of $\mathbb{I}$ such that $a \leqslant b$ and $X =]a, b[$ holds X is open.

(72) For every non empty subset X of $I(01)$ and for every point a of $\mathbb{I}$ such that $0 < a$ and $X =]0, a]$ holds X is closed.

(73) For every non empty subset X of $I(01)$ and for every point a of $\mathbb{I}$ such that $X = [a, 1[$ holds X is closed.

(74) Let A be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, p, q be points of $\mathcal{E}^2_{\mathrm{T}}$, and a, b be points of $\mathbb{I}$. Suppose A is an arc from p to q and $a < b$ and $b \neq 1$. Then there exists a non empty subset E of $I(01)$ and there exists a map f from $I(01) \restriction E$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction (A \setminus \{p\})$ such that $E =]a, b]$ and f is a homeomorphism and $f(b) = q$.

(75) Let A be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, p, q be points of $\mathcal{E}^2_{\mathrm{T}}$, and a, b be points of $\mathbb{I}$. Suppose A is an arc from p to q and $a < b$ and $a \neq 0$. Then there exists a non empty subset E of $I(01)$ and there exists

a map f from $I(01)\restriction E$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction(A \setminus \{q\})$ such that $E = [a, b[$ and f is a homeomorphism and $f(a) = p$.

(76) Let A, B be non empty subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose A is an arc from p to q and B is an arc from q to p and $A \cap B = \{p, q\}$ and $p \neq q$. Then $I(01)$ and $(\mathcal{E}^2_{\mathrm{T}})\restriction((A \setminus \{p\}) \cup (B \setminus \{p\}))$ are homeomorphic.

(77) For every simple closed curve D and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in D$ holds $(\mathcal{E}^2_{\mathrm{T}})\restriction(D \setminus \{p\})$ and $I(01)$ are homeomorphic.

(78) Let D be a simple closed curve and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in D$ and $q \in D$, then $(\mathcal{E}^2_{\mathrm{T}})\restriction(D \setminus \{p\})$ and $(\mathcal{E}^2_{\mathrm{T}})\restriction(D \setminus \{q\})$ are homeomorphic.

(79) Let C be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and E be a subset of $I(01)$. Suppose there exist points p_1, p_2 of $\mathbb{I}$ such that $p_1 < p_2$ and $E = [p_1, p_2]$ and $I(01)\restriction E$ and $(\mathcal{E}^2_{\mathrm{T}})\restriction C$ are homeomorphic. Then there exist points s_1, s_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that C is an arc from s_1 to s_2.

(80) Let D_1 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, f be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction D_1$ into $I(01)$, and C be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a homeomorphism and $C \subseteq D_1$ and there exist points p_1, p_2 of $\mathbb{I}$ such that $p_1 < p_2$ and $f^\circ C = [p_1, p_2]$. Then there exist points s_1, s_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that C is an arc from s_1 to s_2.

(81) Let D be a simple closed curve and C be a non empty compact connected subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $C \subseteq D$. Then $C = D$ or there exist points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that C is an arc from p_1 to p_2 or there exists a point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $C = \{p\}$.

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On the Minimal Distance Between Sets in Euclidean Space[1]

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Summary. The concept of the minimal distance between two sets in a Euclidean space is introduced and some useful lemmas are proved.

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The papers [25], [26], [13], [24], [4], [27], [5], [1], [14], [17], [23], [8], [22], [15], [6], [3], [9], [10], [11], [2], [19], [21], [12], [20], [7], [16], and [18] provide the terminology and notation for this paper.

1. Preliminaries

In this paper X is a set and Y is a non empty set.

We now state several propositions:

(1) Let f be a function from X into Y. Suppose f is onto. Let y be an element of Y. Then there exists a set x such that $x \in X$ and $y = f(x)$.

(2) Let f be a function from X into Y. Suppose f is onto. Let y be an element of Y. Then there exists an element x of X such that $y = f(x)$.

(3) For every function f from X into Y and for every subset A of X such that f is onto holds $(f^\circ A)^{\rm c} \subseteq f^\circ A^{\rm c}$.

(4) For every function f from X into Y and for every subset A of X such that f is one-to-one holds $f^\circ A^{\rm c} \subseteq (f^\circ A)^{\rm c}$.

(5) For every function f from X into Y and for every subset A of X such that f is bijective holds $(f^\circ A)^{\rm c} = f^\circ A^{\rm c}$.

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2. Topological and Metrizable Spaces

One can prove the following two propositions:

(6) For every topological space T and for every subset A of T holds A is a component of $\emptyset_T$ iff A is empty.

(7) Let T be a non empty topological space and A, B, C be subsets of T. If $A \subseteq B$ and A is a component of C and B is a component of C, then $A = B$.

In the sequel n denotes a natural number.

We now state the proposition

(8) If $n \geqslant 1$, then for every subset P of $\mathcal{E}^n$ such that P is bounded holds $-P$ is not bounded.

In the sequel r is a real number and M is a non empty metric space.

Next we state a number of propositions:

(9) For every non empty subset C of $M_{\rm top}$ and for every point p of $M_{\rm top}$ holds $(\mathrm{dist}_{\min}(C))(p) \geqslant 0$.

(10) Let C be a non empty subset of $M_{\rm top}$ and p be a point of M. If for every point q of M such that $q \in C$ holds $\rho(p, q) \geqslant r$, then $(\mathrm{dist}_{\min}(C))(p) \geqslant r$.

(11) For all non empty subsets A, B of $M_{\rm top}$ holds $\mathrm{dist}^{\min}_{\min}(A, B) \geqslant 0$.

(12) For all compact subsets A, B of $M_{\rm top}$ such that A meets B holds $\mathrm{dist}^{\min}_{\min}(A, B) = 0$.

(13) Let A, B be non empty subsets of $M_{\rm top}$. Suppose that for all points p, q of M such that $p \in A$ and $q \in B$ holds $\rho(p, q) \geqslant r$. Then $\mathrm{dist}^{\min}_{\min}(A, B) \geqslant r$.

(14) Let P, Q be subsets of $\mathcal{E}^n_{\rm T}$. Suppose P is a component of $Q^{\rm c}$. Then P is inside component of Q or P is outside component of Q.

(15) If $n \geqslant 1$, then $\mathrm{BDD}\,\emptyset_{\mathcal{E}^n_{\rm T}} = \emptyset_{\mathcal{E}^n_{\rm T}}$.

(16) $\mathrm{BDD}\,\Omega_{\mathcal{E}^n_{\rm T}} = \emptyset_{\mathcal{E}^n_{\rm T}}$.

(17) If $n \geqslant 1$, then $\mathrm{UBD}\,\emptyset_{\mathcal{E}^n_{\rm T}} = \Omega_{\mathcal{E}^n_{\rm T}}$.

(18) $\mathrm{UBD}\,\Omega_{\mathcal{E}^n_{\rm T}} = \emptyset_{\mathcal{E}^n_{\rm T}}$.

(19) For every connected subset P of $\mathcal{E}^n_{\rm T}$ and for every subset Q of $\mathcal{E}^n_{\rm T}$ such that P misses Q holds $P \subseteq \mathrm{UBD}\,Q$ or $P \subseteq \mathrm{BDD}\,Q$.

3. Euclid Plane

For simplicity, we adopt the following rules: C, D are simple closed curves, n is a natural number, p, q, q_1, q_2 are points of $\mathcal{E}^2_{\rm T}$, r, s_1, s_2, t_1, t_2 are real numbers, and x, y are points of $\mathcal{E}^2$.

Next we state a number of propositions:

(20) $\rho([0, 0], r \cdot q) = |r| \cdot \rho([0, 0], q)$.

(21) $\rho(q_1 + q, q_2 + q) = \rho(q_1, q_2)$.
(22) If $p \neq q$, then $\rho(p, q) > 0$.
(23) $\rho(q_1 - q, q_2 - q) = \rho(q_1, q_2)$.
(24) $\rho(p, q) = \rho(-p, -q)$.
(25) $\rho(q - q_1, q - q_2) = \rho(q_1, q_2)$.
(26) $\rho(r \cdot p, r \cdot q) = |r| \cdot \rho(p, q)$.
(27) If $r \leqslant 1$, then $\rho(p, r \cdot p + (1 - r) \cdot q) = (1 - r) \cdot \rho(p, q)$.
(28) If $0 \leqslant r$, then $\rho(q, r \cdot p + (1 - r) \cdot q) = r \cdot \rho(p, q)$.
(29) If $p \in \mathcal{L}(q_1, q_2)$, then $\rho(q_1, p) + \rho(p, q_2) = \rho(q_1, q_2)$.
(30) If $q_1 \in \mathcal{L}(q_2, p)$ and $q_1 \neq q_2$, then $\rho(q_1, p) < \rho(q_2, p)$.
(31) If $y = [0, 0]$, then $\mathrm{Ball}(y, r) = \{q : |q| < r\}$.

4. Affine Maps

Next we state several propositions:

(32) $(\mathrm{AffineMap}(r, s_1, r, s_2))(p) = r \cdot p + [s_1, s_2]$.
(33) $(\mathrm{AffineMap}(r, q_{\mathbf{1}}, r, q_{\mathbf{2}}))(p) = r \cdot p + q$.
(34) If $s_1 > 0$ and $s_2 > 0$, then $\mathrm{AffineMap}(s_1, t_1, s_2, t_2) \cdot \mathrm{AffineMap}(\frac{1}{s_1}, -\frac{t_1}{s_1}, \frac{1}{s_2}, -\frac{t_2}{s_2}) = \mathrm{id}_{\mathcal{R}^2}$.
(35) If $y = [0, 0]$ and $x = q$ and $r > 0$, then $(\mathrm{AffineMap}(r, q_{\mathbf{1}}, r, q_{\mathbf{2}}))^{\circ}\,\mathrm{Ball}(y, 1) = \mathrm{Ball}(x, r)$.
(36) For all real numbers A, B, C, D such that $A > 0$ and $C > 0$ holds $\mathrm{AffineMap}(A, B, C, D)$ is onto.
(37) $\mathrm{Ball}(x, r)^{\mathrm{c}}$ is a connected subset of $\mathcal{E}^2_{\mathrm{T}}$.

5. Minimal Distance Between Subsets

Let us consider n and let A, B be subsets of the carrier of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{dist}_{\min}(A, B)$ yielding a real number is defined by:

(Def. 1) There exist subsets A', B' of $(\mathcal{E}^n)_{\mathrm{top}}$ such that $A = A'$ and $B = B'$ and $\mathrm{dist}_{\min}(A, B) = \mathrm{dist}^{\min}_{\min}(A', B')$.

Let M be a non empty metric space and let P, Q be non empty compact subsets of M_{top}. Let us note that the functor $\mathrm{dist}^{\min}_{\min}(P, Q)$ is commutative. Let us observe that the functor $\mathrm{dist}^{\max}_{\max}(P, Q)$ is commutative.

Let us consider n and let A, B be non empty compact subsets of $\mathcal{E}^n_{\mathrm{T}}$. Let us observe that the functor $\mathrm{dist}_{\min}(A, B)$ is commutative.

Next we state several propositions:

(38) For all non empty subsets A, B of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{dist}_{\min}(A, B) \geqslant 0$.

(39) For all compact subsets A, B of $\mathcal{E}^n_{\rm T}$ such that A meets B holds $\mathrm{dist}_{\min}(A,B) = 0$.

(40) Let A, B be non empty subsets of $\mathcal{E}^n_{\rm T}$. Suppose that for all points p, q of $\mathcal{E}^n_{\rm T}$ such that $p \in A$ and $q \in B$ holds $\rho(p,q) \geqslant r$. Then $\mathrm{dist}_{\min}(A,B) \geqslant r$.

(41) Let D be a subset of the carrier of $\mathcal{E}^n_{\rm T}$ and A, C be non empty subsets of the carrier of $\mathcal{E}^n_{\rm T}$. If $C \subseteq D$, then $\mathrm{dist}_{\min}(A,D) \leqslant \mathrm{dist}_{\min}(A,C)$.

(42) For all non empty compact subsets A, B of $\mathcal{E}^n_{\rm T}$ there exist points p, q of $\mathcal{E}^n_{\rm T}$ such that $p \in A$ and $q \in B$ and $\mathrm{dist}_{\min}(A,B) = \rho(p,q)$.

(43) For all points p, q of $\mathcal{E}^n_{\rm T}$ holds $\mathrm{dist}_{\min}(\{p\},\{q\}) = \rho(p,q)$.

Let us consider n, let p be a point of $\mathcal{E}^n_{\rm T}$, and let B be a subset of the carrier of $\mathcal{E}^n_{\rm T}$. The functor $\rho(p,B)$ yielding a real number is defined as follows:

(Def. 2) $\rho(p,B) = \mathrm{dist}_{\min}(\{p\},B)$.

Next we state several propositions:

(44) For every non empty subset A of $\mathcal{E}^n_{\rm T}$ and for every point p of $\mathcal{E}^n_{\rm T}$ holds $\rho(p,A) \geqslant 0$.

(45) For every compact subset A of $\mathcal{E}^n_{\rm T}$ and for every point p of $\mathcal{E}^n_{\rm T}$ such that $p \in A$ holds $\rho(p,A) = 0$.

(46) Let A be a non empty compact subset of $\mathcal{E}^n_{\rm T}$ and p be a point of $\mathcal{E}^n_{\rm T}$. Then there exists a point q of $\mathcal{E}^n_{\rm T}$ such that $q \in A$ and $\rho(p,A) = \rho(p,q)$.

(47) Let C be a non empty subset of the carrier of $\mathcal{E}^n_{\rm T}$ and D be a subset of the carrier of $\mathcal{E}^n_{\rm T}$. If $C \subseteq D$, then for every point q of $\mathcal{E}^n_{\rm T}$ holds $\rho(q,D) \leqslant \rho(q,C)$.

(48) Let A be a non empty subset of $\mathcal{E}^n_{\rm T}$ and p be a point of $\mathcal{E}^n_{\rm T}$. If for every point q of $\mathcal{E}^n_{\rm T}$ such that $q \in A$ holds $\rho(p,q) \geqslant r$, then $\rho(p,A) \geqslant r$.

(49) For all points p, q of $\mathcal{E}^n_{\rm T}$ holds $\rho(p,\{q\}) = \rho(p,q)$.

(50) For every non empty subset A of $\mathcal{E}^n_{\rm T}$ and for all points p, q of $\mathcal{E}^n_{\rm T}$ such that $q \in A$ holds $\rho(p,A) \leqslant \rho(p,q)$.

(51) Let A be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and B be an open subset of $\mathcal{E}^2_{\rm T}$. If $A \subseteq B$, then for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \notin B$ holds $\rho(p,B) < \rho(p,A)$.

6. BDD and UBD

The following two propositions are true:

(52) $\mathrm{UBD}\,C$ meets $\mathrm{UBD}\,D$.

(53) If $q \in \mathrm{UBD}\,C$ and $p \in \mathrm{BDD}\,C$, then $\rho(q,C) < \rho(q,p)$.

Let us consider C. Observe that $\mathrm{BDD}\,C$ is non empty.

One can prove the following three propositions:

(54) If $p \notin \mathrm{BDD}\,C$, then $\rho(p,C) \leqslant \rho(p,\mathrm{BDD}\,C)$.

(55) $C \not\subseteq \mathrm{BDD}\, D$ or $D \not\subseteq \mathrm{BDD}\, C$.

(56) If $C \subseteq \mathrm{BDD}\, D$, then $D \subseteq \mathrm{UBD}\, C$.

7. Main Definitions

We now state the proposition

(57) $\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n)) \subseteq \mathrm{UBD}\, C$.

Let us consider C. The functor LowerMiddlePoint C yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 3) $\mathrm{LowerMiddlePoint}\, C = \mathrm{FPoint}(\mathrm{LowerArc}\, C, \text{W-min}\, C, \text{E-max}\, C, \mathrm{VerticalLine}\, \frac{\text{W-bound}\, C + \text{E-bound}\, C}{2})$.

The functor UpperMiddlePoint C yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 4) $\mathrm{UpperMiddlePoint}\, C = \mathrm{FPoint}(\mathrm{UpperArc}\, C, \text{W-min}\, C, \text{E-max}\, C, \mathrm{VerticalLine}\, \frac{\text{W-bound}\, C + \text{E-bound}\, C}{2})$.

We now state several propositions:

(58) $\mathrm{LowerArc}\, C$ meets $\mathrm{VerticalLine}\, \frac{\text{W-bound}\, C + \text{E-bound}\, C}{2}$.

(59) $\mathrm{UpperArc}\, C$ meets $\mathrm{VerticalLine}\, \frac{\text{W-bound}\, C + \text{E-bound}\, C}{2}$.

(60) $(\mathrm{LowerMiddlePoint}\, C)_{\mathbf{1}} = \frac{\text{W-bound}\, C + \text{E-bound}\, C}{2}$.

(61) $(\mathrm{UpperMiddlePoint}\, C)_{\mathbf{1}} = \frac{\text{W-bound}\, C + \text{E-bound}\, C}{2}$.

(62) $\mathrm{LowerMiddlePoint}\, C \in \mathrm{LowerArc}\, C$.

(63) $\mathrm{UpperMiddlePoint}\, C \in \mathrm{UpperArc}\, C$.

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Sequences of Metric Spaces and an Abstract Intermediate Value Theorem[1]

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Summary. Relations of convergence of real sequences and convergence of metric spaces are investigated. An abstract intermediate value theorem for two closed sets in the range is presented. At the end, it is proven that an arc connecting the west minimal point and the east maximal point in a simple closed curve must be identical to the upper arc or lower arc of the closed curve.

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The notation and terminology used here are introduced in the following papers: [21], [22], [23], [3], [4], [2], [12], [18], [6], [1], [20], [7], [5], [8], [16], [14], [13], [15], [11], [19], [17], [9], and [10].

The following propositions are true:

(1) Let R be a non empty subset of $\mathbb{R}$ and r_0 be a real number. If for every real number r such that $r \in R$ holds $r \leqslant r_0$, then $\sup R \leqslant r_0$.

(2) Let X be a non empty metric space, S be a sequence of X, and F be a subset of $X_{\rm top}$. Suppose S is convergent and for every natural number n holds $S(n) \in F$ and F is closed. Then $\lim S \in F$.

(3) Let X, Y be non empty metric spaces, f be a map from $X_{\rm top}$ into $Y_{\rm top}$, and S be a sequence of X. Then $f \cdot S$ is a sequence of Y.

(4) Let X, Y be non empty metric spaces, f be a map from $X_{\rm top}$ into $Y_{\rm top}$, S be a sequence of X, and T be a sequence of Y. If S is convergent and $T = f \cdot S$ and f is continuous, then T is convergent.

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(5) For every non empty metric space X holds every function from $\mathbb{N}$ into the carrier of X is a sequence of X.

(6) Let s be a sequence of real numbers and S be a sequence of the metric space of real numbers such that $s = S$. Then

(i) s is convergent iff S is convergent, and

(ii) if s is convergent, then $\lim s = \lim S$.

(7) Let a, b be real numbers and s be a sequence of real numbers. If $\operatorname{rng} s \subseteq [a, b]$, then s is a sequence of $[a, b]_{\mathrm{M}}$.

(8) Let a, b be real numbers and S be a sequence of $[a, b]_{\mathrm{M}}$. Suppose $a \leqslant b$. Then S is a sequence of the metric space of real numbers.

(9) Let a, b be real numbers, S_1 be a sequence of $[a, b]_{\mathrm{M}}$, and S be a sequence of the metric space of real numbers such that $S = S_1$ and $a \leqslant b$. Then

(i) S is convergent iff S_1 is convergent, and

(ii) if S is convergent, then $\lim S = \lim S_1$.

(10) Let a, b be real numbers, s be a sequence of real numbers, and S be a sequence of $[a, b]_{\mathrm{M}}$. If $S = s$ and $a \leqslant b$ and s is convergent, then S is convergent and $\lim s = \lim S$.

(11) Let a, b be real numbers, s be a sequence of real numbers, and S be a sequence of $[a, b]_{\mathrm{M}}$. If $S = s$ and $a \leqslant b$ and s is non-decreasing, then S is convergent.

(12) Let a, b be real numbers, s be a sequence of real numbers, and S be a sequence of $[a, b]_{\mathrm{M}}$. If $S = s$ and $a \leqslant b$ and s is non-increasing, then S is convergent.

(13) Let s be a sequence of real numbers and r_0 be a real number. Suppose for every natural number n holds $s(n) \leqslant r_0$ and s is convergent. Then $\lim s \leqslant r_0$.

(14) Let s be a sequence of real numbers and r_0 be a real number. Suppose for every natural number n holds $s(n) \geqslant r_0$ and s is convergent. Then $\lim s \geqslant r_0$.

(15) Let R be a non empty subset of $\mathbb{R}$. Suppose R is upper bounded. Then there exists a sequence s of real numbers such that s is non-decreasing and $\operatorname{rng} s \subseteq R$ and $\lim s = \sup R$.

(16) Let R be a non empty subset of $\mathbb{R}$. Suppose R is lower bounded. Then there exists a sequence s of real numbers such that s is non-increasing and $\operatorname{rng} s \subseteq R$ and $\lim s = \inf R$.

(17) Let X be a non empty metric space, f be a map from $\mathbb{I}$ into X_{top}, F_1, F_2 be subsets of X_{top}, and r_1, r_2 be real numbers. Suppose that $0 \leqslant r_1$ and $r_2 \leqslant 1$ and $r_1 \leqslant r_2$ and $f(r_1) \in F_1$ and $f(r_2) \in F_2$ and F_1 is closed and F_2 is closed and f is continuous and $F_1 \cup F_2 =$ the carrier of X. Then there exists a real number r such that $r_1 \leqslant r$ and $r \leqslant r_2$ and $f(r) \in F_1 \cap F_2$.

(18) Let n be a natural number, p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$, and P, P_1 be non empty subsets of the carrier of $\mathcal{E}^n_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and P_1 is an arc from p_2 to p_1 and $P_1 \subseteq P$, then $P_1 = P$.

(19) Let P, P_1 be compact non empty subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is a simple closed curve and P_1 is an arc from W-min P to E-max P and $P_1 \subseteq P$. Then $P_1 = \mathrm{UpperArc}\, P$ or $P_1 = \mathrm{LowerArc}\, P$.

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On the Decomposition of a Simple Closed Curve into Two Arcs

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Summary. The purpose of the paper is to prove lemmas needed for the Jordan curve theorem. The main result is that the decomposition of a simple closed curve into two arcs with the ends p_1, p_2 is unique in the sense that every arc on the curve with the same ends must be equal to one of them.

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The articles [25], [24], [26], [14], [27], [2], [4], [8], [3], [22], [17], [21], [7], [6], [20], [1], [23], [15], [9], [5], [10], [19], [18], [11], [13], [12], and [16] provide the terminology and notation for this paper.

One can prove the following proposition

(1) Let S_1 be a finite non empty subset of $\mathbb{R}$ and e be a real number. If for every real number r such that $r \in S_1$ holds $r < e$, then $\max S_1 < e$.

For simplicity, we use the following convention: C is a simple closed curve, A, A_1, A_2 are subsets of $\mathcal{E}^2_{\rm T}$, p, p_1, p_2, q, q_1, q_2 are points of $\mathcal{E}^2_{\rm T}$, and n is a natural number.

Let us consider n. Note that there exists a subset of $\mathcal{E}^n_{\rm T}$ which is trivial.

We now state a number of propositions:

(2) For all sets a, b, c, X such that $a \in X$ and $b \in X$ and $c \in X$ holds $\{a, b, c\} \subseteq X$.

(3) $\emptyset_{\mathcal{E}^n_{\rm T}}$ is Bounded.

(4) $\mathrm{LowerArc}\, C \neq \mathrm{UpperArc}\, C$.

(5) $\mathrm{Segment}(A, p_1, p_2, q_1, q_2) \subseteq A$.

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(6) Let T be a non empty topological space and A, B be subsets of the carrier of T. If $A \subseteq B$, then $T \restriction A$ is a subspace of $T \restriction B$.

(7) If A is an arc from p_1 to p_2 and $q \in A$, then $q \in \mathrm{LSegment}(A, p_1, p_2, q)$.

(8) If A is an arc from p_1 to p_2 and $q \in A$, then $q \in \mathrm{RSegment}(A, p_1, p_2, q)$.

(9) If A is an arc from p_1 to p_2 and LE q_1, q_2, A, p_1, p_2, then $q_1 \in \mathrm{Segment}(A, p_1, p_2, q_1, q_2)$ and $q_2 \in \mathrm{Segment}(A, p_1, p_2, q_1, q_2)$.

(10) $\mathrm{Segment}(p, q, C) \subseteq C$.

(11) If $p \in C$ and $q \in C$, then $\mathrm{LE}(p, q, C)$ or $\mathrm{LE}(q, p, C)$.

(12) Let X, Y be non empty topological spaces, Y_0 be a non empty subspace of Y, f be a map from X into Y, and g be a map from X into Y_0. If $f = g$ and f is continuous, then g is continuous.

(13) Let S, T be non empty topological spaces, S_0 be a non empty subspace of S, T_0 be a non empty subspace of T, and f be a map from S into T. Suppose f is a homeomorphism. Let g be a map from S_0 into T_0. If $g = f \restriction S_0$ and g is onto, then g is a homeomorphism.

(14) Let P_1, P_2, P_3 be subsets of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P_1 is an arc from p_1 to p_2 and P_2 is an arc from p_1 to p_2 and P_3 is an arc from p_1 to p_2 and $P_2 \cap P_3 = \{p_1, p_2\}$ and $P_1 \subseteq P_2 \cup P_3$. Then $P_1 = P_2$ or $P_1 = P_3$.

(15) Let C be a simple closed curve, A_1, A_2 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose A_1 is an arc from p_1 to p_2 and A_2 is an arc from p_1 to p_2 and $A_1 \subseteq C$ and $A_2 \subseteq C$ and $A_1 \neq A_2$. Then $A_1 \cup A_2 = C$ and $A_1 \cap A_2 = \{p_1, p_2\}$.

(16) Let A_1, A_2 be subsets of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If A_1 is an arc from p_1 to p_2 and $A_1 \cap A_2 = \{q_1, q_2\}$, then $A_1 \neq A_2$.

(17) Let C be a simple closed curve, A_1, A_2 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose A_1 is an arc from p_1 to p_2 and A_2 is an arc from p_1 to p_2 and $A_1 \subseteq C$ and $A_2 \subseteq C$ and $A_1 \cap A_2 = \{p_1, p_2\}$. Then $A_1 \cup A_2 = C$.

(18) Suppose $A_1 \subseteq C$ and $A_2 \subseteq C$ and $A_1 \neq A_2$ and A_1 is an arc from p_1 to p_2 and A_2 is an arc from p_1 to p_2. Let given A. If A is an arc from p_1 to p_2 and $A \subseteq C$, then $A = A_1$ or $A = A_2$.

(19) Let C be a simple closed curve and A be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. If A is an arc from $\text{W-min}\, C$ to $\text{E-max}\, C$ and $A \subseteq C$, then $A = \mathrm{LowerArc}\, C$ or $A = \mathrm{UpperArc}\, C$.

(20) Suppose A is an arc from p_1 to p_2 and LE q_1, q_2, A, p_1, p_2. Then there exists a map g from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction A$ and there exist real numbers s_1, s_2 such that g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_1) = q_1$ and $g(s_2) = q_2$ and $0 \leqslant s_1$ and $s_1 \leqslant s_2$ and $s_2 \leqslant 1$.

(21) Suppose A is an arc from p_1 to p_2 and LE q_1, q_2, A, p_1, p_2 and $q_1 \neq q_2$. Then there exists a map g from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction A$ and there exist real numbers

s_1, s_2 such that g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_1) = q_1$ and $g(s_2) = q_2$ and $0 \leqslant s_1$ and $s_1 < s_2$ and $s_2 \leqslant 1$.

(22) If A is an arc from p_1 to p_2 and LE q_1, q_2, A, p_1, p_2, then $\mathrm{Segment}(A, p_1, p_2, q_1, q_2)$ is non empty.

(23) If $p \in C$, then $p \in \mathrm{Segment}(p, \text{W-min}\, C, C)$ and $\text{W-min}\, C \in \mathrm{Segment}(p, \text{W-min}\, C, C)$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. We say that f is continuous if and only if:

(Def. 1) f is continuous on $\mathrm{dom}\, f$.

Let f be a function from $\mathbb{R}$ into $\mathbb{R}$. Let us observe that f is continuous if and only if:

(Def. 2) f is continuous on $\mathbb{R}$.

Let a, b be real numbers. The functor $\mathrm{AffineMap}(a, b)$ yielding a function from $\mathbb{R}$ into $\mathbb{R}$ is defined by:

(Def. 3) For every real number x holds $(\mathrm{AffineMap}(a, b))(x) = a \cdot x + b$.

Let a, b be real numbers. Observe that $\mathrm{AffineMap}(a, b)$ is continuous.

Let us mention that there exists a function from $\mathbb{R}$ into $\mathbb{R}$ which is continuous.

We now state a number of propositions:

(24) Let f, g be continuous partial functions from $\mathbb{R}$ to $\mathbb{R}$. Then $g \cdot f$ is a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$.

(25) For all real numbers a, b holds $(\mathrm{AffineMap}(a, b))(0) = b$.

(26) For all real numbers a, b holds $(\mathrm{AffineMap}(a, b))(1) = a + b$.

(27) For all real numbers a, b such that $a \neq 0$ holds $\mathrm{AffineMap}(a, b)$ is one-to-one.

(28) For all real numbers a, b, x, y such that $a > 0$ and $x < y$ holds $(\mathrm{AffineMap}(a, b))(x) < (\mathrm{AffineMap}(a, b))(y)$.

(29) For all real numbers a, b, x, y such that $a < 0$ and $x < y$ holds $(\mathrm{AffineMap}(a, b))(x) > (\mathrm{AffineMap}(a, b))(y)$.

(30) For all real numbers a, b, x, y such that $a \geqslant 0$ and $x \leqslant y$ holds $(\mathrm{AffineMap}(a, b))(x) \leqslant (\mathrm{AffineMap}(a, b))(y)$.

(31) For all real numbers a, b, x, y such that $a \leqslant 0$ and $x \leqslant y$ holds $(\mathrm{AffineMap}(a, b))(x) \geqslant (\mathrm{AffineMap}(a, b))(y)$.

(32) For all real numbers a, b such that $a \neq 0$ holds $\mathrm{rng}\, \mathrm{AffineMap}(a, b) = \mathbb{R}$.

(33) For all real numbers a, b such that $a \neq 0$ holds $(\mathrm{AffineMap}(a, b))^{-1} = \mathrm{AffineMap}(a^{-1}, -\frac{b}{a})$.

(34) For all real numbers a, b such that $a > 0$ holds $(\mathrm{AffineMap}(a, b))^{\circ}[0, 1] = [b, a + b]$.

(35) For every map f from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a \neq 0$ and $f = \mathrm{AffineMap}(a, b)$ holds f is a homeomorphism.

(36) If A is an arc from p_1 to p_2 and LE q_1, q_2, A, p_1, p_2 and $q_1 \neq q_2$, then $\mathrm{Segment}(A, p_1, p_2, q_1, q_2)$ is an arc from q_1 to q_2.

(37) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P \subseteq C$ and P is an arc from p_1 to p_2 and $\text{W-min}\, C \in P$ and $\text{E-max}\, C \in P$. Then $\mathrm{UpperArc}\, C \subseteq P$ or $\mathrm{LowerArc}\, C \subseteq P$.

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The Ordering of Points on a Curve. Part III[1]

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The articles [12], [13], [1], [6], [7], [10], [4], [3], [11], [5], [8], [2], and [9] provide the notation and terminology for this paper.

We follow the rules: C, P denote simple closed curves and a, b, c, d, e denote points of $\mathcal{E}^2_{\mathrm{T}}$.

We now state several propositions:

(1) Let n be a natural number, a, p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$, and P be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $a \in P$ and P is an arc from p_1 to p_2. Then there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^n_{\mathrm{T}}) \restriction P$ and there exists a real number r such that f is a homeomorphism and $f(0) = p_1$ and $f(1) = p_2$ and $0 \leqslant r$ and $r \leqslant 1$ and $f(r) = a$.

(2) $\mathrm{LE}(\text{W-min}\, P, \text{E-max}\, P, P)$.

(3) If $\mathrm{LE}(a, \text{E-max}\, P, P)$, then $a \in \mathrm{UpperArc}\, P$.

(4) If $\mathrm{LE}(\text{E-max}\, P, a, P)$, then $a \in \mathrm{LowerArc}\, P$.

(5) If $\mathrm{LE}(a, \text{W-min}\, P, P)$, then $a \in \mathrm{LowerArc}\, P$.

(6) Let P be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $a \neq b$ and P is an arc from c to d and LE a, b, P, c, d. Then there exists e such that $a \neq e$ and $b \neq e$ and LE a, e, P, c, d and LE e, b, P, c, d.

(7) If $a \in P$, then there exists e such that $a \neq e$ and $\mathrm{LE}(a, e, P)$.

(8) If $a \neq b$ and $\mathrm{LE}(a, b, P)$, then there exists c such that $c \neq a$ and $c \neq b$ and $\mathrm{LE}(a, c, P)$ and $\mathrm{LE}(c, b, P)$.

Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and let a, b, c, d be points of $\mathcal{E}^2_{\mathrm{T}}$. We say that a, b, c, d are in this order on P if and only if:

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(Def. 1) $\mathrm{LE}(a, b, P)$ and $\mathrm{LE}(b, c, P)$ and $\mathrm{LE}(c, d, P)$ or $\mathrm{LE}(b, c, P)$ and $\mathrm{LE}(c, d, P)$ and $\mathrm{LE}(d, a, P)$ or $\mathrm{LE}(c, d, P)$ and $\mathrm{LE}(d, a, P)$ and $\mathrm{LE}(a, b, P)$ or $\mathrm{LE}(d, a, P)$ and $\mathrm{LE}(a, b, P)$ and $\mathrm{LE}(b, c, P)$.

The following propositions are true:

(9) If $a \in P$, then a, a, a, a are in this order on P.

(10) If a, b, c, d are in this order on P, then b, c, d, a are in this order on P.

(11) If a, b, c, d are in this order on P, then c, d, a, b are in this order on P.

(12) If a, b, c, d are in this order on P, then d, a, b, c are in this order on P.

(13) Suppose $a \neq b$ and a, b, c, d are in this order on P. Then there exists e such that $e \neq a$ and $e \neq b$ and a, e, b, c are in this order on P.

(14) Suppose $a \neq b$ and a, b, c, d are in this order on P. Then there exists e such that $e \neq a$ and $e \neq b$ and a, e, b, d are in this order on P.

(15) Suppose $b \neq c$ and a, b, c, d are in this order on P. Then there exists e such that $e \neq b$ and $e \neq c$ and a, b, e, c are in this order on P.

(16) Suppose $b \neq c$ and a, b, c, d are in this order on P. Then there exists e such that $e \neq b$ and $e \neq c$ and b, e, c, d are in this order on P.

(17) Suppose $c \neq d$ and a, b, c, d are in this order on P. Then there exists e such that $e \neq c$ and $e \neq d$ and a, c, e, d are in this order on P.

(18) Suppose $c \neq d$ and a, b, c, d are in this order on P. Then there exists e such that $e \neq c$ and $e \neq d$ and b, c, e, d are in this order on P.

(19) Suppose $d \neq a$ and a, b, c, d are in this order on P. Then there exists e such that $e \neq d$ and $e \neq a$ and a, b, d, e are in this order on P.

(20) Suppose $d \neq a$ and a, b, c, d are in this order on P. Then there exists e such that $e \neq d$ and $e \neq a$ and a, c, d, e are in this order on P.

(21) Suppose $a \neq c$ and $a \neq d$ and $b \neq d$ and a, b, c, d are in this order on P and b, a, c, d are in this order on P. Then $a = b$.

(22) Suppose $a \neq b$ and $b \neq c$ and $c \neq d$ and a, b, c, d are in this order on P and c, b, a, d are in this order on P. Then $a = c$.

(23) Suppose $a \neq b$ and $a \neq c$ and $b \neq d$ and a, b, c, d are in this order on P and d, b, c, a are in this order on P. Then $a = d$.

(24) Suppose $a \neq c$ and $a \neq d$ and $b \neq d$ and a, b, c, d are in this order on P and a, c, b, d are in this order on P. Then $b = c$.

(25) Suppose $a \neq b$ and $b \neq c$ and $c \neq d$ and a, b, c, d are in this order on P and a, d, c, b are in this order on P. Then $b = d$.

(26) Suppose $a \neq b$ and $a \neq c$ and $b \neq d$ and a, b, c, d are in this order on P and a, b, d, c are in this order on P. Then $c = d$.

(27) Suppose $a \in C$ and $b \in C$ and $c \in C$ and $d \in C$. Then

(i) a, b, c, d are in this order on C, or

(ii) a, b, d, c are in this order on C, or

(iii) a, c, b, d are in this order on C, or
(iv) a, c, d, b are in this order on C, or
(v) a, d, b, c are in this order on C, or
(vi) a, d, c, b are in this order on C.

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The Ordering of Points on a Curve. Part IV[1]

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The notation and terminology used in this paper are introduced in the following articles: [19], [21], [22], [2], [3], [10], [20], [13], [14], [18], [6], [17], [5], [11], [1], [7], [8], [4], [9], [16], [12], and [15].

1. Preliminaries

For simplicity, we adopt the following rules: n denotes an element of $\mathbb{N}$, V denotes a subset of the carrier of $\mathcal{E}^n_{\rm T}$, s, s_1, s_2, t, t_1, t_2 denote points of $\mathcal{E}^n_{\rm T}$, C denotes a simple closed curve, P denotes a subset of the carrier of $\mathcal{E}^2_{\rm T}$, and a, p, p_1, p_2, q, q_1, q_2 denote points of $\mathcal{E}^2_{\rm T}$.

Next we state several propositions:

(1) For all real numbers a, b holds $(a-b)^{\mathbf{2}} = (b-a)^{\mathbf{2}}$.

(2) Let S, T be non empty topological spaces, f be a map from S into T, and A be a subset of T. If f is a homeomorphism and A is connected, then $f^{-1}(A)$ is connected.

(3) Let S, T be non empty topological structures, f be a map from S into T, and A be a subset of T. If f is a homeomorphism and A is compact, then $f^{-1}(A)$ is compact.

(4) $\mathrm{proj2}^\circ\, \mathrm{NorthHalfline}\, a$ is lower bounded.

(5) $\mathrm{proj2}^\circ\, \mathrm{SouthHalfline}\, a$ is upper bounded.

(6) $\mathrm{proj1}^\circ\, \mathrm{WestHalfline}\, a$ is upper bounded.

(7) $\mathrm{proj1}^\circ\, \mathrm{EastHalfline}\, a$ is lower bounded.

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Let us consider a. One can verify the following observations:

* $\mathrm{proj2}^\circ\, \mathrm{NorthHalfline}\, a$ is non empty,
* $\mathrm{proj2}^\circ\, \mathrm{SouthHalfline}\, a$ is non empty,
* $\mathrm{proj1}^\circ\, \mathrm{WestHalfline}\, a$ is non empty, and
* $\mathrm{proj1}^\circ\, \mathrm{EastHalfline}\, a$ is non empty.

Next we state four propositions:

(8) $\inf(\mathrm{proj2}^\circ\, \mathrm{NorthHalfline}\, a) = a_{\mathbf{2}}$.

(9) $\sup(\mathrm{proj2}^\circ\, \mathrm{SouthHalfline}\, a) = a_{\mathbf{2}}$.

(10) $\sup(\mathrm{proj1}^\circ\, \mathrm{WestHalfline}\, a) = a_{\mathbf{1}}$.

(11) $\inf(\mathrm{proj1}^\circ\, \mathrm{EastHalfline}\, a) = a_{\mathbf{1}}$.

Let us consider a. One can verify the following observations:

* $\mathrm{NorthHalfline}\, a$ is closed,
* $\mathrm{SouthHalfline}\, a$ is closed,
* $\mathrm{EastHalfline}\, a$ is closed, and
* $\mathrm{WestHalfline}\, a$ is closed.

One can prove the following propositions:

(12) If $a \in \mathrm{BDD}\, P$, then $\mathrm{NorthHalfline}\, a \not\subseteq \mathrm{UBD}\, P$.

(13) If $a \in \mathrm{BDD}\, P$, then $\mathrm{SouthHalfline}\, a \not\subseteq \mathrm{UBD}\, P$.

(14) If $a \in \mathrm{BDD}\, P$, then $\mathrm{EastHalfline}\, a \not\subseteq \mathrm{UBD}\, P$.

(15) If $a \in \mathrm{BDD}\, P$, then $\mathrm{WestHalfline}\, a \not\subseteq \mathrm{UBD}\, P$.

(16) Let P be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and $q \neq p_2$, then $p_2 \notin \mathrm{LSegment}(P, p_1, p_2, q)$.

(17) Let P be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and $q \neq p_1$, then $p_1 \notin \mathrm{RSegment}(P, p_1, p_2, q)$.

(18) Let C be a simple closed curve, P be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2 and $P \subseteq C$. Then there exists a non empty subset R of $\mathcal{E}^2_{\mathrm{T}}$ such that R is an arc from p_1 to p_2 and $P \cup R = C$ and $P \cap R = \{p_1, p_2\}$.

(19) Let P be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2 and $q_1 \in P$ and $q_2 \in P$ and $q_1 \neq p_1$ and $q_1 \neq p_2$ and $q_2 \neq p_1$ and $q_2 \neq p_2$ and $q_1 \neq q_2$. Then there exists a non empty subset Q of $\mathcal{E}^2_{\mathrm{T}}$ such that Q is an arc from q_1 to q_2 and $Q \subseteq P$ and Q misses $\{p_1, p_2\}$.

2. Two Special Points on a Simple Closed Curve

Let us consider p, P. The functor $\mathrm{North\text{-}Bound}(p, P)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 1) North-Bound$(p, P) = [p_{\mathbf{1}}, \inf(\text{proj2}^\circ(P \cap \text{NorthHalfline}\, p))]$.

The functor South-Bound(p, P) yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 2) South-Bound$(p, P) = [p_{\mathbf{1}}, \sup(\text{proj2}^\circ(P \cap \text{SouthHalfline}\, p))]$.

One can prove the following propositions:

(20) $(\text{North-Bound}(p, P))_{\mathbf{1}} = p_{\mathbf{1}}$ and $(\text{South-Bound}(p, P))_{\mathbf{1}} = p_{\mathbf{1}}$.

(21) $(\text{North-Bound}(p, P))_{\mathbf{2}} = \inf(\text{proj2}^\circ(P \cap \text{NorthHalfline}\, p))$ and $(\text{South-Bound}(p, P))_{\mathbf{2}} = \sup(\text{proj2}^\circ(P \cap \text{SouthHalfline}\, p))$.

(22) For every compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathrm{BDD}\, C$ holds North-Bound$(p, C) \in C$ and North-Bound$(p, C) \in$ NorthHalfline p and South-Bound$(p, C) \in C$ and South-Bound$(p, C) \in$ SouthHalfline p.

(23) For every compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathrm{BDD}\, C$ holds $(\text{South-Bound}(p, C))_{\mathbf{2}} < p_{\mathbf{2}}$ and $p_{\mathbf{2}} < (\text{North-Bound}(p, C))_{\mathbf{2}}$.

(24) For every compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathrm{BDD}\, C$ holds $\inf(\text{proj2}^\circ(C \cap \text{NorthHalfline}\, p)) > \sup(\text{proj2}^\circ(C \cap \text{SouthHalfline}\, p))$.

(25) For every compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathrm{BDD}\, C$ holds South-Bound$(p, C) \neq$ North-Bound(p, C).

(26) For every subset C of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathcal{L}(\text{North-Bound}(p, C), \text{South-Bound}(p, C))$ is vertical.

(27) For every compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathrm{BDD}\, C$ holds $\mathcal{L}(\text{North-Bound}(p, C), \text{South-Bound}(p, C)) \cap C = \{\text{North-Bound}(p, C), \text{South-Bound}(p, C)\}$.

(28) Let C be a compact subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p \in \mathrm{BDD}\, C$ and $q \in \mathrm{BDD}\, C$ and $p_{\mathbf{1}} \neq q_{\mathbf{1}}$. Then North-Bound$(p, C)$, South-Bound$(q, C)$, North-Bound$(q, C)$, South-Bound$(p, C)$ are mutually different.

3. An Order of Points on a Simple Closed Curve

Let us consider n, V, s_1, s_2, t_1, t_2. We say that s_1, s_2 separate t_1, t_2 on V if and only if:

(Def. 3) For every subset A of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ such that A is an arc from s_1 to s_2 and $A \subseteq V$ holds A meets $\{t_1, t_2\}$.

We introduce s_1, s_2 are neighbours wrt t_1, t_2 on V as an antonym of s_1, s_2 separate t_1, t_2 on V.

We now state a number of propositions:

(29) t, t separate s_1, s_2 on V.

(30) If s_1, s_2 separate t_1, t_2 on V, then s_2, s_1 separate t_1, t_2 on V.

(31) If s_1, s_2 separate t_1, t_2 on V, then s_1, s_2 separate t_2, t_1 on V.

(32) s, t_1 separate s, t_2 on V.

(33) t_1, s separate t_2, s on V.

(34) t_1, s separate s, t_2 on V.

(35) s, t_1 separate t_2, s on V.

(36) Let p_1, p_2, q be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $q \in C$ and $p_1 \in C$ and $p_2 \in C$ and $p_1 \neq p_2$ and $p_1 \neq q$ and $p_2 \neq q$. Then p_1, p_2 are neighbours wrt q, q on C.

(37) If $p_1 \neq p_2$ and $p_1 \in C$ and $p_2 \in C$, then if p_1, p_2 separate q_1, q_2 on C, then q_1, q_2 separate p_1, p_2 on C.

(38) Suppose $p_1 \in C$ and $p_2 \in C$ and $q_1 \in C$ and $p_1 \neq p_2$ and $q_1 \neq p_1$ and $q_1 \neq p_2$ and $q_2 \neq p_1$ and $q_2 \neq p_2$. Then p_1, p_2 are neighbours wrt q_1, q_2 on C or p_1, q_1 are neighbours wrt p_2, q_2 on C.

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Order Sorted Algebras[1]

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Summary. Initial notions for order sorted algebras.

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The articles [9], [13], [14], [4], [15], [5], [8], [7], [2], [3], [1], [10], [12], [11], and [6] provide the notation and terminology for this paper.

1. Preliminaries

In this paper i is a set.

Let I be a set, let f be a many sorted set indexed by I, and let p be a finite sequence of elements of I. One can check that $f \cdot p$ is finite sequence-like.

Let S be a non empty many sorted signature. A sort symbol of S is an element of S.

Let S be a non empty many sorted signature.

(Def. 1) An element of the operation symbols of S is said to be an operation symbol of S.

Let S be a non void non empty many sorted signature and let o be an operation symbol of S. Then the result sort of o is an element of S.

Let X be a set. Then $\triangle_X$ is an order in X. We introduce $\triangle^o_X$ as a synonym of $\triangle_X$.

Let X be a set. Then $\triangle_X$ is an equivalence relation of X. We introduce $\triangle^r_X$ as a synonym of $\triangle_X$.

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We introduce overloaded many sorted signatures which are extensions of many sorted signature and are systems

⟨ a carrier, operation symbols, an overloading, an arity, a result sort ⟩,
where the carrier is a set, the operation symbols constitute a set, the overloading is an equivalence relation of the operation symbols, the arity is a function from the operation symbols into the carrier*, and the result sort is a function from the operation symbols into the carrier.

We introduce relation sorted signatures which are extensions of many sorted signature and relational structure and are systems

⟨ a carrier, an internal relation, operation symbols, an arity, a result sort ⟩,
where the carrier is a set, the internal relation is a binary relation on the carrier, the operation symbols constitute a set, the arity is a function from the operation symbols into the carrier*, and the result sort is a function from the operation symbols into the carrier.

We consider overloaded relation sorted signatures as extensions of overloaded many sorted signature and relation sorted signature as systems

⟨ a carrier, an internal relation, operation symbols, an overloading, an arity, a result sort ⟩,
where the carrier is a set, the internal relation is a binary relation on the carrier, the operation symbols constitute a set, the overloading is an equivalence relation of the operation symbols, the arity is a function from the operation symbols into the carrier*, and the result sort is a function from the operation symbols into the carrier.

For simplicity, we use the following convention: A, O are non empty sets, R is an order in A, O_1 is an equivalence relation of O, f is a function from O into A^*, and g is a function from O into A.

One can prove the following proposition

(1) $\langle A, R, O, O_1, f, g\rangle$ is non empty, non void, reflexive, transitive, and antisymmetric.

Let us consider A, R, O, O_1, f, g. One can verify that $\langle A, R, O, O_1, f, g\rangle$ is strict, non empty, reflexive, transitive, and antisymmetric.

2. The Notions: Order-Sorted, Discernable, Op-Discrete

In the sequel S is an overloaded relation sorted signature.

Let us consider S. We say that S is order-sorted if and only if:

(Def. 2) S is reflexive, transitive, and antisymmetric.

Let us note that every overloaded relation sorted signature which is order-sorted is also reflexive, transitive, and antisymmetric and there exists an overloaded relation sorted signature which is strict, non empty, non void, and order-sorted.

Let us observe that there exists an overloaded many sorted signature which is non empty and non void.

Let S be a non empty non void overloaded many sorted signature and let x, y be operation symbols of S. The predicate $x \cong y$ is defined by:

(Def. 3) $\langle x, y \rangle \in$ the overloading of S.

Let us notice that the predicate $x \cong y$ is reflexive and symmetric.

One can prove the following proposition

(2) Let S be a non empty non void overloaded many sorted signature and o, o_1, o_2 be operation symbols of S. If $o \cong o_1$ and $o_1 \cong o_2$, then $o \cong o_2$.

Let S be a non empty non void overloaded many sorted signature. We say that S is discernable if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let x, y be operation symbols of S. Suppose $x \cong y$ and $\mathrm{Arity}(x) = \mathrm{Arity}(y)$ and the result sort of $x =$ the result sort of y. Then $x = y$.

We say that S is op-discrete if and only if:

(Def. 5) The overloading of $S = \triangle^{r}_{\text{the operation symbols of } S}$.

The following two propositions are true:

(3) Let S be a non empty non void overloaded many sorted signature. Then S is op-discrete if and only if for all operation symbols x, y of S such that $x \cong y$ holds $x = y$.

(4) For every non empty non void overloaded many sorted signature S such that S is op-discrete holds S is discernable.

3. Order Sorted Signature

In the sequel S_0 is a non empty non void many sorted signature.

Let us consider S_0. The functor $\mathrm{OSSign}\, S_0$ yields a strict non empty non void order-sorted overloaded relation sorted signature and is defined by the conditions (Def. 6).

(Def. 6)(i) The carrier of $S_0 =$ the carrier of $\mathrm{OSSign}\, S_0$,

(ii) $\triangle_{\text{the carrier of } S_0} =$ the internal relation of $\mathrm{OSSign}\, S_0$,

(iii) the operation symbols of $S_0 =$ the operation symbols of $\mathrm{OSSign}\, S_0$,

(iv) $\triangle_{\text{the operation symbols of } S_0} =$ the overloading of $\mathrm{OSSign}\, S_0$,

(v) the arity of $S_0 =$ the arity of $\mathrm{OSSign}\, S_0$, and

(vi) the result sort of $S_0 =$ the result sort of $\mathrm{OSSign}\, S_0$.

Next we state the proposition

(5) $\mathrm{OSSign}\, S_0$ is discrete and op-discrete.

Let us mention that there exists a strict non empty non void order-sorted overloaded relation sorted signature which is discrete, op-discrete, and discernable.

Let us observe that every non empty non void overloaded relation sorted signature which is op-discrete is also discernable.

Let us consider S_0. Observe that OSSign S_0 is discrete and op-discrete.

An order sorted signature is a discernable non empty non void order-sorted overloaded relation sorted signature.

We use the following convention: S is a non empty poset, s_1, s_2 are elements of S, and w_1, w_2 are elements of (the carrier of S)*.

Let us consider S and let w_1, w_2 be elements of (the carrier of S)*. The predicate $w_1 \leqslant w_2$ is defined as follows:

(Def. 7) $\operatorname{len} w_1 = \operatorname{len} w_2$ and for every set i such that $i \in \operatorname{dom} w_1$ and for all s_1, s_2 such that $s_1 = w_1(i)$ and $s_2 = w_2(i)$ holds $s_1 \leqslant s_2$.

Let us note that the predicate $w_1 \leqslant w_2$ is reflexive.

We now state two propositions:

(6) For all elements w_1, w_2 of (the carrier of S)* such that $w_1 \leqslant w_2$ and $w_2 \leqslant w_1$ holds $w_1 = w_2$.

(7) If S is discrete and $w_1 \leqslant w_2$, then $w_1 = w_2$.

We follow the rules: S is an order sorted signature, o, o_1, o_2 are operation symbols of S, and w_1 is an element of (the carrier of S)*.

One can prove the following proposition

(8) If S is discrete and $o_1 \cong o_2$ and $\operatorname{Arity}(o_1) \leqslant \operatorname{Arity}(o_2)$ and the result sort of $o_1 \leqslant$ the result sort of o_2, then $o_1 = o_2$.

Let us consider S and let us consider o. We say that o is monotone if and only if:

(Def. 8) For every o_2 such that $o \cong o_2$ and $\operatorname{Arity}(o) \leqslant \operatorname{Arity}(o_2)$ holds the result sort of $o \leqslant$ the result sort of o_2.

Let us consider S. We say that S is monotone if and only if:

(Def. 9) Every operation symbol of S is monotone.

The following proposition is true

(9) If S is op-discrete, then S is monotone.

Let us observe that there exists an order sorted signature which is monotone.

Let S be a monotone order sorted signature. Observe that there exists an operation symbol of S which is monotone.

Let S be a monotone order sorted signature. One can check that every operation symbol of S is monotone.

One can check that every order sorted signature which is op-discrete is also monotone.

We now state the proposition

(10) If S is monotone and $\operatorname{Arity}(o_1) = \emptyset$ and $o_1 \cong o_2$ and $\operatorname{Arity}(o_2) = \emptyset$, then $o_1 = o_2$.

Let us consider S, o, o_1, w_1. We say that o_1 has least args for o, w_1 if and only if:

(Def. 10) $o \cong o_1$ and $w_1 \leqslant \mathrm{Arity}(o_1)$ and for every o_2 such that $o \cong o_2$ and $w_1 \leqslant \mathrm{Arity}(o_2)$ holds $\mathrm{Arity}(o_1) \leqslant \mathrm{Arity}(o_2)$.

We say that o_1 has least sort for o, w_1 if and only if:

(Def. 11) $o \cong o_1$ and $w_1 \leqslant \mathrm{Arity}(o_1)$ and for every o_2 such that $o \cong o_2$ and $w_1 \leqslant \mathrm{Arity}(o_2)$ holds the result sort of $o_1 \leqslant$ the result sort of o_2.

Let us consider S, o, o_1, w_1. We say that o_1 has least rank for o, w_1 if and only if:

(Def. 12) o_1 has least args for o, w_1 and least sort for o, w_1.

Let us consider S, o. We say that o is regular if and only if:

(Def. 13) o is monotone and for every w_1 such that $w_1 \leqslant \mathrm{Arity}(o)$ holds there exists o_1 which has least args for o, w_1.

Let S_1 be a monotone order sorted signature. We say that S_1 is regular if and only if:

(Def. 14) Every operation symbol of S_1 is regular.

In the sequel S_1 is a monotone order sorted signature, o, o_1 are operation symbols of S_1, and w_1 is an element of (the carrier of S_1)*.

We now state two propositions:

(11) S_1 is regular iff for all o, w_1 such that $w_1 \leqslant \mathrm{Arity}(o)$ holds there exists o_1 which has least rank for o, w_1.

(12) For every monotone order sorted signature S_1 such that S_1 is op-discrete holds S_1 is regular.

One can verify that there exists a monotone order sorted signature which is regular.

Let us mention that every monotone order sorted signature which is op-discrete is also regular.

Let S_2 be a regular monotone order sorted signature. One can verify that every operation symbol of S_2 is regular.

We adopt the following rules: S_2 is a regular monotone order sorted signature, o, o_3, o_4 are operation symbols of S_2, and w_1 is an element of (the carrier of S_2)*.

One can prove the following proposition

(13) If $w_1 \leqslant \mathrm{Arity}(o)$ and o_3 has least args for o, w_1 and o_4 has least args for o, w_1, then $o_3 = o_4$.

Let us consider S_2, o, w_1. Let us assume that $w_1 \leqslant \mathrm{Arity}(o)$. The functor $\mathrm{LBound}(o, w_1)$ yields an operation symbol of S_2 and is defined as follows:

(Def. 15) $\mathrm{LBound}(o, w_1)$ has least args for o, w_1.

One can prove the following proposition

(14) For every w_1 such that $w_1 \leqslant \mathrm{Arity}(o)$ holds $\mathrm{LBound}(o, w_1)$ has least rank for o, w_1.

In the sequel R denotes a non empty poset and z denotes a non empty set.

Let us consider R, z. The functor $\mathrm{ConstOSSet}(R, z)$ yielding a many sorted set indexed by the carrier of R is defined by:

(Def. 16) $\mathrm{ConstOSSet}(R, z) = (\text{the carrier of } R) \longmapsto z$.

The following proposition is true

(15) $\mathrm{ConstOSSet}(R, z)$ is non-empty and for all elements s_1, s_2 of R such that $s_1 \leqslant s_2$ holds $(\mathrm{ConstOSSet}(R, z))(s_1) \subseteq (\mathrm{ConstOSSet}(R, z))(s_2)$.

Let C be a 1-sorted structure.

(Def. 17) A many sorted set indexed by the carrier of C is said to be a many sorted set indexed by C.

Let us consider R, z. Then $\mathrm{ConstOSSet}(R, z)$ is a many sorted set indexed by R.

Let us consider R and let M be a many sorted set indexed by R. We say that M is order-sorted if and only if:

(Def. 18) For all elements s_1, s_2 of R such that $s_1 \leqslant s_2$ holds $M(s_1) \subseteq M(s_2)$.

Next we state the proposition

(16) $\mathrm{ConstOSSet}(R, z)$ is order-sorted.

Let us consider R. Observe that there exists a many sorted set indexed by R which is order-sorted.

Let us consider R, z. Then $\mathrm{ConstOSSet}(R, z)$ is an order-sorted many sorted set indexed by R.

Let R be a non empty poset. An order sorted set of R is an order-sorted many sorted set indexed by R.

Let R be a non empty poset. Observe that there exists an order sorted set of R which is non-empty.

We adopt the following convention: s_1, s_2 denote sort symbols of S, o, o_1, o_2, o_3 denote operation symbols of S, and w_1, w_2 denote elements of $(\text{the carrier of } S)^*$.

Let us consider S and let M be an algebra over S. We say that M is order-sorted if and only if:

(Def. 19) For all s_1, s_2 such that $s_1 \leqslant s_2$ holds $(\text{the sorts of } M)(s_1) \subseteq (\text{the sorts of } M)(s_2)$.

The following proposition is true

(17) For every algebra M over S holds M is order-sorted iff the sorts of M are an order sorted set of S.

In the sequel C_1 denotes a many sorted function from $(\mathrm{ConstOSSet}(S, z))^{\#} \cdot$ the arity of S into $\mathrm{ConstOSSet}(S, z) \cdot$ the result sort of S.

Let us consider S, z, C_1. The functor $\mathrm{ConstOSA}(S, z, C_1)$ yielding a strict non-empty algebra over S is defined by:

(Def. 20) The sorts of $\mathrm{ConstOSA}(S, z, C_1) = \mathrm{ConstOSSet}(S, z)$ and the characteristics of $\mathrm{ConstOSA}(S, z, C_1) = C_1$.

One can prove the following proposition

(18) $\mathrm{ConstOSA}(S, z, C_1)$ is order-sorted.

Let us consider S. One can check that there exists an algebra over S which is strict, non-empty, and order-sorted.

Let us consider S, z, C_1. One can verify that $\mathrm{ConstOSA}(S, z, C_1)$ is order-sorted.

Let us consider S. An order sorted algebra of S is an order-sorted algebra over S.

Next we state the proposition

(19) For every discrete order sorted signature S holds every algebra over S is order-sorted.

Let S be a discrete order sorted signature. Observe that every algebra over S is order-sorted.

In the sequel A denotes an order sorted algebra of S.

We now state the proposition

(20) If $w_1 \leqslant w_2$, then (the sorts of $A)^{\#}(w_1) \subseteq$ (the sorts of $A)^{\#}(w_2)$.

In the sequel M is an algebra over S_0.

Let us consider S_0, M. The functor $\mathrm{OSAlg}\, M$ yielding a strict order sorted algebra of $\mathrm{OSSign}\, S_0$ is defined as follows:

(Def. 21) The sorts of $\mathrm{OSAlg}\, M =$ the sorts of M and the characteristics of $\mathrm{OSAlg}\, M =$ the characteristics of M.

In the sequel A denotes an order sorted algebra of S.

We now state the proposition

(21) For all elements w_1, w_2, w_3 of (the carrier of $S)^*$ such that $w_1 \leqslant w_2$ and $w_2 \leqslant w_3$ holds $w_1 \leqslant w_3$.

Let us consider S, o_1, o_2. The predicate $o_1 \leqslant o_2$ is defined as follows:

(Def. 22) $o_1 \cong o_2$ and $\mathrm{Arity}(o_1) \leqslant \mathrm{Arity}(o_2)$ and the result sort of $o_1 \leqslant$ the result sort of o_2.

Let us note that the predicate $o_1 \leqslant o_2$ is reflexive.

We now state several propositions:

(22) If $o_1 \leqslant o_2$ and $o_2 \leqslant o_1$, then $o_1 = o_2$.

(23) If $o_1 \leqslant o_2$ and $o_2 \leqslant o_3$, then $o_1 \leqslant o_3$.

(24) If the result sort of $o_1 \leqslant$ the result sort of o_2, then $\mathrm{Result}(o_1, A) \subseteq \mathrm{Result}(o_2, A)$.

(25) If $\mathrm{Arity}(o_1) \leqslant \mathrm{Arity}(o_2)$, then $\mathrm{Args}(o_1, A) \subseteq \mathrm{Args}(o_2, A)$.

(26) If $o_1 \leqslant o_2$, then $\mathrm{Args}(o_1, A) \subseteq \mathrm{Args}(o_2, A)$ and $\mathrm{Result}(o_1, A) \subseteq \mathrm{Result}(o_2, A)$.

Let us consider S, A. We say that A is monotone if and only if:

(Def. 23) For all o_1, o_2 such that $o_1 \leqslant o_2$ holds $\mathrm{Den}(o_2, A) \restriction \mathrm{Args}(o_1, A) = \mathrm{Den}(o_1, A)$.

We now state two propositions:

(27) Let A be a non-empty order sorted algebra of S. Then A is monotone if and only if for all o_1, o_2 such that $o_1 \leqslant o_2$ holds $\mathrm{Den}(o_1, A) \subseteq \mathrm{Den}(o_2, A)$.

(28) If S is discrete and op-discrete, then A is monotone.

Let us consider S, z and let z_1 be an element of z. The functor $\mathrm{TrivialOSA}(S, z, z_1)$ yielding a strict order sorted algebra of S is defined by:

(Def. 24) The sorts of $\mathrm{TrivialOSA}(S, z, z_1) = \mathrm{ConstOSSet}(S, z)$ and for every o holds $\mathrm{Den}(o, \mathrm{TrivialOSA}(S, z, z_1)) = \mathrm{Args}(o, \mathrm{TrivialOSA}(S, z, z_1)) \longmapsto z_1$.

Next we state the proposition

(29) For every element z_1 of z holds $\mathrm{TrivialOSA}(S, z, z_1)$ is non-empty and $\mathrm{TrivialOSA}(S, z, z_1)$ is monotone.

Let us consider S. Note that there exists an order sorted algebra of S which is monotone, strict, and non-empty.

Let us consider S, z and let z_1 be an element of z. One can check that $\mathrm{TrivialOSA}(S, z, z_1)$ is monotone and non-empty.

In the sequel o_5, o_6 are operation symbols of S.

Let us consider S. The functor $\mathrm{OperNames}\, S$ yields a non empty family of subsets of the operation symbols of S and is defined as follows:

(Def. 25) $\mathrm{OperNames}\, S = \mathrm{Classes}\,(\text{the overloading of } S)$.

Let us consider S. One can check that every element of $\mathrm{OperNames}\, S$ is non empty.

Let us consider S. An OperName of S is an element of $\mathrm{OperNames}\, S$.

Let us consider S, o_5. The functor $\mathrm{Name}\, o_5$ yields an OperName of S and is defined by:

(Def. 26) $\mathrm{Name}\, o_5 = [o_5]_{\text{the overloading of } S}$.

Next we state three propositions:

(30) $o_5 \cong o_6$ iff $o_6 \in [o_5]_{\text{the overloading of } S}$.

(31) $o_5 \cong o_6$ iff $\mathrm{Name}\, o_5 = \mathrm{Name}\, o_6$.

(32) For every set X holds X is an OperName of S iff there exists o_5 such that $X = \mathrm{Name}\, o_5$.

Let us consider S and let o be an OperName of S. We see that the element of o is an operation symbol of S.

Next we state two propositions:

(33) For every OperName o_8 of S and for every operation symbol o_7 of S holds o_7 is an element of o_8 iff Name $o_7 = o_8$.

(34) Let S_2 be a regular monotone order sorted signature, o_5, o_6 be operation symbols of S_2, and w be an element of (the carrier of S_2)*. If $o_5 \cong o_6$ and len Arity(o_5) = len Arity(o_6) and $w \leqslant$ Arity(o_5) and $w \leqslant$ Arity(o_6), then LBound(o_5, w) = LBound(o_6, w).

Let S_2 be a regular monotone order sorted signature, let o_8 be an OperName of S_2, and let w be an element of (the carrier of S_2)*. Let us assume that there exists an element o_7 of o_8 such that $w \leqslant$ Arity(o_7). The functor LBound(o_8, w) yields an element of o_8 and is defined as follows:

(Def. 27) For every element o_7 of o_8 such that $w \leqslant$ Arity(o_7) holds LBound(o_8, w) = LBound(o_7, w).

Next we state the proposition

(35) Let S be a regular monotone order sorted signature, o be an operation symbol of S, and w_1 be an element of (the carrier of S)*. If $w_1 \leqslant$ Arity(o), then LBound$(o, w_1) \leqslant o$.

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Subalgebras of an Order Sorted Algebra. Lattice of Subalgebras[1]

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The papers [8], [5], [12], [14], [4], [7], [15], [3], [1], [6], [9], [10], [11], [2], and [13] provide the notation and terminology for this paper.

1. Auxiliary Facts about Order Sorted Sets

In this paper x denotes a set and R denotes a non empty poset.

Next we state two propositions:

(1) For all order sorted sets X, Y of R holds $X \cap Y$ is an order sorted set of R.

(2) For all order sorted sets X, Y of R holds $X \cup Y$ is an order sorted set of R.

Let R be a non empty poset and let M be an order sorted set of R. A many sorted subset indexed by M is said to be an order sorted subset of M if:

(Def. 1) It is an order sorted set of R.

Let R be a non empty poset and let M be a non-empty order sorted set of R. Note that there exists an order sorted subset of M which is non-empty.

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2. Constants of an Order Sorted Algebra

Let S be an order sorted signature and let U_0 be an order sorted algebra of S. A many sorted subset indexed by the sorts of U_0 is said to be an OSSubset of U_0 if:

(Def. 2) It is an order sorted set of S.

Let S be an order sorted signature. Note that there exists an order sorted algebra of S which is monotone, strict, and non-empty.

Let S be an order sorted signature and let U_0 be a non-empty order sorted algebra of S. One can verify that there exists an OSSubset of U_0 which is non-empty.

Next we state the proposition

(3) For every non void strict non empty many sorted signature S_0 with constant operations holds OSSign S_0 has constant operations.

Let us note that there exists an order sorted signature which is strict and has constant operations.

3. Subalgebras of an Order Sorted Algebra

The following proposition is true

(4) Let S be an order sorted signature and U_0 be an order sorted algebra of S. Then $\langle$the sorts of U_0, the characteristics of $U_0\rangle$ is order-sorted.

Let S be an order sorted signature and let U_0 be an order sorted algebra of S. One can verify that there exists a subalgebra of U_0 which is order-sorted.

Let S be an order sorted signature and let U_0 be an order sorted algebra of S. An OSSubAlgebra of U_0 is an order-sorted subalgebra of U_0.

Let S be an order sorted signature and let U_0 be an order sorted algebra of S. One can verify that there exists an OSSubAlgebra of U_0 which is strict.

Let S be an order sorted signature and let U_0 be a non-empty order sorted algebra of S. Observe that there exists an OSSubAlgebra of U_0 which is non-empty and strict.

One can prove the following proposition

(5) Let S be an order sorted signature, U_0 be an order sorted algebra of S, and U_1 be an algebra over S. Then U_1 is an OSSubAlgebra of U_0 if and only if the following conditions are satisfied:
  (i) the sorts of U_1 are an OSSubset of U_0, and
  (ii) for every OSSubset B of U_0 such that $B =$ the sorts of U_1 holds B is operations closed and the characteristics of $U_1 = \mathrm{Opers}(U_0, B)$.

In the sequel S_1 is an order sorted signature and O_0, O_1, O_2 are order sorted algebras of S_1.

The following propositions are true:

(6) O_1 is an OSSubAlgebra of O_1.

(7) If O_0 is an OSSubAlgebra of O_1 and O_1 is an OSSubAlgebra of O_2, then O_0 is an OSSubAlgebra of O_2.

(8) If O_1 is a strict OSSubAlgebra of O_2 and O_2 is a strict OSSubAlgebra of O_1, then $O_1 = O_2$.

(9) For all OSSubAlgebras O_1, O_2 of O_0 such that the sorts of $O_1 \subseteq$ the sorts of O_2 holds O_1 is an OSSubAlgebra of O_2.

(10) For all strict OSSubAlgebras O_1, O_2 of O_0 such that the sorts of $O_1 =$ the sorts of O_2 holds $O_1 = O_2$.

In the sequel s, s_1, s_2 are sort symbols of S_1.

Let us consider S_1, O_0, s. The functor $\mathrm{OSConstants}(O_0, s)$ yields a subset of (the sorts of O_0)(s) and is defined by:

(Def. 3) $\mathrm{OSConstants}(O_0, s) = \bigcup\{\mathrm{Constants}(O_0, s_2) : s_2 \leqslant s\}$.

One can prove the following proposition

(11) $\mathrm{Constants}(O_0, s) \subseteq \mathrm{OSConstants}(O_0, s)$.

Let us consider S_1 and let M be a many sorted set indexed by the carrier of S_1. The functor $\mathrm{OSCl}\, M$ yields an order sorted set of S_1 and is defined by:

(Def. 4) For every sort symbol s of S_1 holds $(\mathrm{OSCl}\, M)(s) = \bigcup\{M(s_1) : s_1 \leqslant s\}$.

Next we state three propositions:

(12) For every many sorted set M indexed by the carrier of S_1 holds $M \subseteq \mathrm{OSCl}\, M$.

(13) Let M be a many sorted set indexed by the carrier of S_1 and A be an order sorted set of S_1. If $M \subseteq A$, then $\mathrm{OSCl}\, M \subseteq A$.

(14) For every order sorted signature S and for every order sorted set X of S holds $\mathrm{OSCl}\, X = X$.

Let us consider S_1, O_0. The functor $\mathrm{OSConstants}\, O_0$ yields an OSSubset of O_0 and is defined by:

(Def. 5) For every sort symbol s of S_1 holds $(\mathrm{OSConstants}\, O_0)(s) = \mathrm{OSConstants}(O_0, s)$.

One can prove the following propositions:

(15) $\mathrm{Constants}(O_0) \subseteq \mathrm{OSConstants}\, O_0$.

(16) For every OSSubset A of O_0 such that $\mathrm{Constants}(O_0) \subseteq A$ holds $\mathrm{OSConstants}\, O_0 \subseteq A$.

(17) For every OSSubset A of O_0 holds $\mathrm{OSConstants}\, O_0 = \mathrm{OSCl}\, \mathrm{Constants}(O_0)$.

(18) For every OSSubAlgebra O_1 of O_0 holds $\mathrm{OSConstants}\, O_0$ is an OSSubset of O_1.

(19) Let S be an order sorted signature with constant operations, O_0 be a non-empty order sorted algebra of S, and O_1 be a non-empty OSSubAlgebra of O_0. Then $\mathrm{OSConstants}\, O_0$ is a non-empty OSSubset of O_1.

4. Order Sorted Subsets of an Order Sorted Algebra

Next we state the proposition

(20) Let I be a set, M be a many sorted set indexed by I, and x be a set. Then x is a many sorted subset indexed by M if and only if $x \in \prod(2^M)$.

Let R be a non empty poset and let M be an order sorted set of R. The functor OSbool M yielding a set is defined by:

(Def. 6) For every set x holds $x \in \mathrm{OSbool}\, M$ iff x is an order sorted subset of M.

Let S be an order sorted signature, let U_0 be an order sorted algebra of S, and let A be an OSSubset of U_0. The functor OSSubSort A yields a set and is defined as follows:

(Def. 7) $\mathrm{OSSubSort}\, A = \{x; x$ ranges over elements of $\mathrm{SubSorts}(A)$: x is an order sorted set of $S\}$.

We now state two propositions:

(21) For every OSSubset A of O_0 holds $\mathrm{OSSubSort}\, A \subseteq \mathrm{SubSorts}(A)$.

(22) For every OSSubset A of O_0 holds the sorts of $O_0 \in \mathrm{OSSubSort}\, A$.

Let us consider S_1, O_0 and let A be an OSSubset of O_0. One can check that OSSubSort A is non empty.

Let us consider S_1, O_0. The functor OSSubSort O_0 yielding a set is defined by:

(Def. 8) $\mathrm{OSSubSort}\, O_0 = \{x; x$ ranges over elements of $\mathrm{SubSorts}(O_0)$: x is an order sorted set of $S_1\}$.

The following proposition is true

(23) For every OSSubset A of O_0 holds $\mathrm{OSSubSort}\, A \subseteq \mathrm{OSSubSort}\, O_0$.

Let us consider S_1, O_0. One can check that OSSubSort O_0 is non empty.

Let us consider S_1, O_0 and let e be an element of OSSubSort O_0. The functor ${}^{@}e$ yielding an OSSubset of O_0 is defined by:

(Def. 9) ${}^{@}e = e$.

Next we state two propositions:

(24) For all OSSubsets A, B of O_0 holds $B \in \mathrm{OSSubSort}\, A$ iff B is operations closed and $\mathrm{OSConstants}\, O_0 \subseteq B$ and $A \subseteq B$.

(25) For every OSSubset B of O_0 holds $B \in \mathrm{OSSubSort}\, O_0$ iff B is operations closed.

Let us consider S_1, O_0, let A be an OSSubset of O_0, and let s be an element of the carrier of S_1. The functor $\mathrm{OSSubSort}(A, s)$ yields a set and is defined by:

(Def. 10) For every set x holds $x \in \mathrm{OSSubSort}(A, s)$ iff there exists an OSSubset B of O_0 such that $B \in \mathrm{OSSubSort}\, A$ and $x = B(s)$.

We now state three propositions:

(26) For every OSSubset A of O_0 and for all sort symbols s_1, s_2 of S_1 such that $s_1 \leqslant s_2$ holds $\mathrm{OSSubSort}(A, s_2)$ is coarser than $\mathrm{OSSubSort}(A, s_1)$.

(27) For every OSSubset A of O_0 and for every sort symbol s of S_1 holds $\mathrm{OSSubSort}(A, s) \subseteq \mathrm{SubSort}(A, s)$.

(28) For every OSSubset A of O_0 and for every sort symbol s of S_1 holds (the sorts of O_0)$(s) \in \mathrm{OSSubSort}(A, s)$.

Let us consider S_1, O_0, let A be an OSSubset of O_0, and let s be a sort symbol of S_1. Note that $\mathrm{OSSubSort}(A, s)$ is non empty.

Let us consider S_1, O_0 and let A be an OSSubset of O_0. The functor OSMSubSort A yields an OSSubset of O_0 and is defined by:

(Def. 11) For every sort symbol s of S_1 holds $(\mathrm{OSMSubSort}\, A)(s) = \bigcap \mathrm{OSSubSort}(A, s)$.

Let us consider S_1, O_0 and let A be an OSSubset of O_0. We say that A is os-opers closed if and only if:

(Def. 12) A is operations closed.

Let us consider S_1, O_0. One can verify that there exists an OSSubset of O_0 which is os-opers closed.

Next we state several propositions:

(29) For every OSSubset A of O_0 holds $\mathrm{OSConstants}\, O_0 \cup A \subseteq \mathrm{OSMSubSort}\, A$.

(30) For every OSSubset A of O_0 such that $\mathrm{OSConstants}\, O_0 \cup A$ is non-empty holds OSMSubSort A is non-empty.

(31) Let o be an operation symbol of S_1, A be an OSSubset of O_0, and B be an OSSubset of O_0. If $B \in \mathrm{OSSubSort}\, A$, then $((\mathrm{OSMSubSort}\, A)^{\#} \cdot$ the arity of $S_1)(o) \subseteq (B^{\#} \cdot$ the arity of $S_1)(o)$.

(32) Let o be an operation symbol of S_1, A be an OSSubset of O_0, and B be an OSSubset of O_0. Suppose $B \in \mathrm{OSSubSort}\, A$. Then $\mathrm{rng}(\mathrm{Den}(o, O_0) \restriction ((\mathrm{OSMSubSort}\, A)^{\#} \cdot$ the arity of $S_1)(o)) \subseteq (B \cdot$ the result sort of $S_1)(o)$.

(33) Let o be an operation symbol of S_1 and A be an OSSubset of O_0. Then $\mathrm{rng}(\mathrm{Den}(o, O_0) \restriction ((\mathrm{OSMSubSort}\, A)^{\#} \cdot$ the arity of $S_1)(o)) \subseteq (\mathrm{OSMSubSort}\, A \cdot$ the result sort of $S_1)(o)$.

(34) For every OSSubset A of O_0 holds OSMSubSort A is operations closed and $A \subseteq \mathrm{OSMSubSort}\, A$.

Let us consider S_1, O_0 and let A be an OSSubset of O_0. Note that OSMSubSort A is os-opers closed.

5. Operations on Subalgebras of an Order Sorted Algebra

Let us consider S_1, O_0 and let A be an os-opers closed OSSubset of O_0. Note that $O_0 \restriction A$ is order-sorted.

Let us consider S_1, O_0 and let O_1, O_2 be OSSubAlgebras of O_0. One can check that $O_1 \cap O_2$ is order-sorted.

Let us consider S_1, O_0 and let A be an OSSubset of O_0. The functor OSGen A yields a strict OSSubAlgebra of O_0 and is defined by the conditions (Def. 13).

(Def. 13)(i) A is an OSSubset of OSGen A, and

(ii) for every OSSubAlgebra O_1 of O_0 such that A is an OSSubset of O_1 holds OSGen A is an OSSubAlgebra of O_1.

We now state several propositions:

(35) For every OSSubset A of O_0 holds OSGen $A = O_0 \restriction$ OSMSubSort A and the sorts of OSGen A = OSMSubSort A.

(36) Let S be a non void non empty many sorted signature, U_0 be an algebra over S, and A be a subset of U_0. Then $\mathrm{Gen}(A) = U_0 \restriction \mathrm{MSSubSort}(A)$ and the sorts of $\mathrm{Gen}(A) = \mathrm{MSSubSort}(A)$.

(37) For every OSSubset A of O_0 holds the sorts of $\mathrm{Gen}(A) \subseteq$ the sorts of OSGen A.

(38) For every OSSubset A of O_0 holds $\mathrm{Gen}(A)$ is a subalgebra of OSGen A.

(39) Let O_0 be a strict order sorted algebra of S_1 and B be an OSSubset of O_0. If B = the sorts of O_0, then OSGen $B = O_0$.

(40) For every strict OSSubAlgebra O_1 of O_0 and for every OSSubset B of O_0 such that B = the sorts of O_1 holds OSGen $B = O_1$.

(41) For every non-empty order sorted algebra U_0 of S_1 and for every OSSubAlgebra U_1 of U_0 holds OSGen OSConstants $U_0 \cap U_1$ = OSGen OSConstants U_0.

Let us consider S_1, let U_0 be a non-empty order sorted algebra of S_1, and let U_1, U_2 be OSSubAlgebras of U_0. The functor $U_1 \sqcup_{os} U_2$ yielding a strict OSSubAlgebra of U_0 is defined by:

(Def. 14) For every OSSubset A of U_0 such that A = (the sorts of U_1) $\cup$ (the sorts of U_2) holds $U_1 \sqcup_{os} U_2$ = OSGen A.

One can prove the following propositions:

(42) Let U_0 be a non-empty order sorted algebra of S_1, U_1 be an OSSubAlgebra of U_0, and A, B be OSSubsets of U_0. If $B = A \cup$ the sorts of U_1, then OSGen $A \sqcup_{os} U_1$ = OSGen B.

(43) Let U_0 be a non-empty order sorted algebra of S_1, U_1 be an OSSubAlgebra of U_0, and B be an OSSubset of U_0. If B = the sorts of U_0, then OSGen $B \sqcup_{os} U_1$ = OSGen B.

(44) For every non-empty order sorted algebra U_0 of S_1 and for all OSSubAlgebras U_1, U_2 of U_0 holds $U_1 \sqcup_{os} U_2 = U_2 \sqcup_{os} U_1$.

(45) For every non-empty order sorted algebra U_0 of S_1 and for all strict OSSubAlgebras U_1, U_2 of U_0 holds $U_1 \cap (U_1 \sqcup_{os} U_2) = U_1$.

(46) For every non-empty order sorted algebra U_0 of S_1 and for all strict OSSubAlgebras U_1, U_2 of U_0 holds $U_1 \cap U_2 \sqcup_{os} U_2 = U_2$.

6. The Lattice of Subalgebras of an Order Sorted Algebra

Let us consider S_1, O_0. The functor $\mathrm{OSSub}\, O_0$ yields a set and is defined by:

(Def. 15) For every x holds $x \in \mathrm{OSSub}\, O_0$ iff x is a strict OSSubAlgebra of O_0.

We now state the proposition

(47) $\mathrm{OSSub}\, O_0 \subseteq \mathrm{Subalgebras}(O_0)$.

Let S be an order sorted signature and let U_0 be an order sorted algebra of S. Note that $\mathrm{OSSub}\, U_0$ is non empty.

Let us consider S_1, O_0. Then $\mathrm{OSSub}\, O_0$ is a subset of $\mathrm{Subalgebras}(O_0)$.

Let us consider S_1 and let U_0 be a non-empty order sorted algebra of S_1. The functor $\mathrm{OSAlgJoin}\, U_0$ yields a binary operation on $\mathrm{OSSub}\, U_0$ and is defined as follows:

(Def. 16) For all elements x, y of $\mathrm{OSSub}\, U_0$ and for all strict OSSubAlgebras U_1, U_2 of U_0 such that $x = U_1$ and $y = U_2$ holds $(\mathrm{OSAlgJoin}\, U_0)(x, y) = U_1 \sqcup_{os} U_2$.

Let us consider S_1 and let U_0 be a non-empty order sorted algebra of S_1. The functor $\mathrm{OSAlgMeet}\, U_0$ yields a binary operation on $\mathrm{OSSub}\, U_0$ and is defined as follows:

(Def. 17) For all elements x, y of $\mathrm{OSSub}\, U_0$ and for all strict OSSubAlgebras U_1, U_2 of U_0 such that $x = U_1$ and $y = U_2$ holds $(\mathrm{OSAlgMeet}\, U_0)(x, y) = U_1 \cap U_2$.

The following proposition is true

(48) For every non-empty order sorted algebra U_0 of S_1 and for all elements x, y of $\mathrm{OSSub}\, U_0$ holds $(\mathrm{OSAlgMeet}\, U_0)(x, y) = (\mathrm{MSAlgMeet}(U_0))(x, y)$.

In the sequel U_0 denotes a non-empty order sorted algebra of S_1.

We now state four propositions:

(49) $\mathrm{OSAlgJoin}\, U_0$ is commutative.

(50) $\mathrm{OSAlgJoin}\, U_0$ is associative.

(51) $\mathrm{OSAlgMeet}\, U_0$ is commutative.

(52) $\mathrm{OSAlgMeet}\, U_0$ is associative.

Let us consider S_1 and let U_0 be a non-empty order sorted algebra of S_1. The functor $\mathrm{OSSubAlLattice}\, U_0$ yielding a strict lattice is defined by:

(Def. 18) $\mathrm{OSSubAlLattice}\, U_0 = \langle \mathrm{OSSub}\, U_0, \mathrm{OSAlgJoin}\, U_0, \mathrm{OSAlgMeet}\, U_0 \rangle$.

Next we state the proposition

(53) For every non-empty order sorted algebra U_0 of S_1 holds $\mathrm{OSSubAlLattice}\, U_0$ is bounded.

Let us consider S_1 and let U_0 be a non-empty order sorted algebra of S_1. Note that $\mathrm{OSSubAlLattice}\,U_0$ is bounded.

The following propositions are true:

(54) For every non-empty order sorted algebra U_0 of S_1 holds $\perp_{\mathrm{OSSubAlLattice}\,U_0} = \mathrm{OSGen}\,\mathrm{OSConstants}\,U_0$.

(55) Let U_0 be a non-empty order sorted algebra of S_1 and B be an OSSubset of U_0. If $B =$ the sorts of U_0, then $\top_{\mathrm{OSSubAlLattice}\,U_0} = \mathrm{OSGen}\,B$.

(56) For every strict non-empty order sorted algebra U_0 of S_1 holds $\top_{\mathrm{OSSubAlLattice}\,U_0} = U_0$.

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Homomorphisms of Order Sorted Algebras[1]

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The notation and terminology used in this paper have been introduced in the following articles: [8], [12], [14], [15], [4], [5], [2], [1], [9], [11], [7], [10], [3], [13], and [6].

In this paper R denotes a non empty poset and S_1 denotes an order sorted signature.

Let us consider R and let F be a many sorted function indexed by the carrier of R. We say that F is order-sorted if and only if:

(Def. 1) For all elements s_1, s_2 of R such that $s_1 \leqslant s_2$ and for every set a_1 such that $a_1 \in \operatorname{dom} F(s_1)$ holds $a_1 \in \operatorname{dom} F(s_2)$ and $F(s_1)(a_1) = F(s_2)(a_1)$.

Next we state several propositions:

(1) For every set I and for every many sorted set A indexed by I holds id_A is "1-1".

(2) Let F be a many sorted function indexed by the carrier of R. Suppose F is order-sorted. Let s_1, s_2 be elements of R. If $s_1 \leqslant s_2$, then $\operatorname{dom} F(s_1) \subseteq \operatorname{dom} F(s_2)$ and $F(s_1) \subseteq F(s_2)$.

(3) Let A be an order sorted set of R, B be a non-empty order sorted set of R, and F be a many sorted function from A into B. Then F is order-sorted if and only if for all elements s_1, s_2 of R such that $s_1 \leqslant s_2$ and for every set a_1 such that $a_1 \in A(s_1)$ holds $F(s_1)(a_1) = F(s_2)(a_1)$.

(4) Let F be a many sorted function indexed by the carrier of R. Suppose F is order-sorted. Let w_1, w_2 be elements of (the carrier of R)*. If $w_1 \leqslant w_2$, then $F^{\#}(w_1) \subseteq F^{\#}(w_2)$.

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(5) For every order sorted set A of R holds id_A is order-sorted.

(6) Let A be an order sorted set of R, B, C be non-empty order sorted sets of R, F be a many sorted function from A into B, and G be a many sorted function from B into C. If F is order-sorted and G is order-sorted, then $G \circ F$ is order-sorted.

(7) Let A, B be order sorted sets of R and F be a many sorted function from A into B. If F is "1-1", onto, and order-sorted, then F^{-1} is order-sorted.

(8) Let A be an order sorted set of R and F be a many sorted function indexed by the carrier of R. If F is order-sorted, then $F^{\circ} A$ is an order sorted set of R.

Let us consider S_1 and let U_1, U_2 be order sorted algebras of S_1. We say that U_1 and U_2 are os-isomorphic if and only if:

(Def. 2) There exists a many sorted function from U_1 into U_2 which is an isomorphism of U_1 and U_2 and order-sorted.

The following propositions are true:

(9) For every order sorted algebra U_1 of S_1 holds U_1 and U_1 are os-isomorphic.

(10) Let U_1, U_2 be non-empty order sorted algebras of S_1. If U_1 and U_2 are os-isomorphic, then U_2 and U_1 are os-isomorphic.

Let us consider S_1 and let U_1, U_2 be order sorted algebras of S_1. Let us note that the predicate U_1 and U_2 are os-isomorphic is reflexive.

One can prove the following propositions:

(11) Let U_1, U_2, U_3 be non-empty order sorted algebras of S_1. Suppose U_1 and U_2 are os-isomorphic and U_2 and U_3 are os-isomorphic. Then U_1 and U_3 are os-isomorphic.

(12) Let U_1, U_2 be non-empty order sorted algebras of S_1 and F be a many sorted function from U_1 into U_2. Suppose F is order-sorted and a homomorphism of U_1 into U_2. Then $\mathrm{Im}\, F$ is order-sorted.

(13) Let U_1, U_2 be non-empty order sorted algebras of S_1 and F be a many sorted function from U_1 into U_2. Suppose F is order-sorted. Let o_1, o_2 be operation symbols of S_1. Suppose $o_1 \leqslant o_2$. Let x be an element of $\mathrm{Args}(o_1, U_1)$ and x_1 be an element of $\mathrm{Args}(o_2, U_1)$. If $x = x_1$, then $F\#x = F\#x_1$.

(14) Let U_1 be a monotone non-empty order sorted algebra of S_1, U_2 be a non-empty order sorted algebra of S_1, and F be a many sorted function from U_1 into U_2. Suppose F is order-sorted and a homomorphism of U_1 into U_2. Then $\mathrm{Im}\, F$ is order-sorted and $\mathrm{Im}\, F$ is a monotone order sorted algebra of S_1.

(15) For every monotone order sorted algebra U_1 of S_1 holds every OSSubAlgebra of U_1 is monotone.

Let us consider S_1 and let U_1 be a monotone order sorted algebra of S_1. One can check that there exists an OSSubAlgebra of U_1 which is monotone.

Let us consider S_1 and let U_1 be a monotone order sorted algebra of S_1. One can verify that every OSSubAlgebra of U_1 is monotone.

The following propositions are true:

(16) Let U_1, U_2 be non-empty order sorted algebras of S_1 and F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2 and order-sorted. Then there exists a many sorted function G from U_1 into $\operatorname{Im} F$ such that $F = G$ and G is order-sorted and an epimorphism of U_1 onto $\operatorname{Im} F$.

(17) Let U_1, U_2 be non-empty order sorted algebras of S_1 and F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2 and order-sorted. Then there exists a many sorted function F_1 from U_1 into $\operatorname{Im} F$ and there exists a many sorted function F_2 from $\operatorname{Im} F$ into U_2 such that

(i) F_1 is an epimorphism of U_1 onto $\operatorname{Im} F$,

(ii) F_2 is a monomorphism of $\operatorname{Im} F$ into U_2,

(iii) $F = F_2 \circ F_1$,

(iv) F_1 is order-sorted, and

(v) F_2 is order-sorted.

Let us consider S_1 and let U_1 be an order sorted algebra of S_1. Note that $\langle$the sorts of U_1, the characteristics of $U_1\rangle$ is order-sorted.

One can prove the following propositions:

(18) Let U_1 be an order sorted algebra of S_1. Then U_1 is monotone if and only if $\langle$the sorts of U_1, the characteristics of $U_1\rangle$ is monotone.

(19) Let U_1, U_2 be strict non-empty order sorted algebras of S_1. Suppose U_1 and U_2 are os-isomorphic. Then U_1 is monotone if and only if U_2 is monotone.

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Order Sorted Quotient Algebra[1]

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The terminology and notation used in this paper are introduced in the following papers: [7], [16], [21], [24], [4], [25], [6], [15], [9], [19], [8], [14], [5], [3], [1], [20], [17], [2], [12], [18], [10], [13], [23], [22], and [11].

1. Preliminaries

Let R be a non empty poset. One can verify that there exists an order sorted set of R which is binary relation yielding.

Let R be a non empty poset, let A, B be many sorted sets indexed by the carrier of R, and let I_1 be a many sorted relation between A and B. We say that I_1 is os-compatible if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let s_1, s_2 be elements of the carrier of R. Suppose $s_1 \leqslant s_2$. Let x, y be sets. If $x \in A(s_1)$ and $y \in B(s_1)$, then $\langle x, y \rangle \in I_1(s_1)$ iff $\langle x, y \rangle \in I_1(s_2)$.

Let R be a non empty poset and let A, B be many sorted sets indexed by the carrier of R. A many sorted relation between A and B is said to be an order sorted relation of A, B if:

(Def. 2) It is os-compatible.

The following proposition is true

(1) Let R be a non empty poset, A, B be many sorted sets indexed by the carrier of R, and O_1 be a many sorted relation between A and B. If O_1 is os-compatible, then O_1 is an order sorted set of R.

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Let R be a non empty poset and let A be a many sorted set indexed by the carrier of R. An order sorted relation of A is an order sorted relation of A, A.

Let S be an order sorted signature and let U_1 be an order sorted algebra of S. A many sorted relation indexed by U_1 is said to be an order sorted relation of U_1 if:

(Def. 3) It is os-compatible.

Let S be an order sorted signature and let U_1 be an order sorted algebra of S. One can check that there exists an order sorted relation of U_1 which is equivalence.

Let S be an order sorted signature and let U_1 be a non-empty order sorted algebra of S. Note that there exists an equivalence order sorted relation of U_1 which is MSCongruence-like.

Let S be an order sorted signature and let U_1 be a non-empty order sorted algebra of S. An order sorted congruence of U_1 is a MSCongruence-like equivalence order sorted relation of U_1.

Let R be a non empty poset. The functor $\mathrm{PathRel}\, R$ yields an equivalence relation of the carrier of R and is defined by the condition (Def. 4).

(Def. 4) Let x, y be sets. Then $\langle x, y\rangle \in \mathrm{PathRel}\, R$ if and only if the following conditions are satisfied:

(i) $x \in$ the carrier of R,

(ii) $y \in$ the carrier of R, and

(iii) there exists a finite sequence p of elements of the carrier of R such that $1 < \mathrm{len}\, p$ and $p(1) = x$ and $p(\mathrm{len}\, p) = y$ and for every natural number n such that $2 \leqslant n$ and $n \leqslant \mathrm{len}\, p$ holds $\langle p(n), p(n-1)\rangle \in$ the internal relation of R or $\langle p(n-1), p(n)\rangle \in$ the internal relation of R.

One can prove the following proposition

(2) For every non empty poset R and for all elements s_1, s_2 of the carrier of R such that $s_1 \leqslant s_2$ holds $\langle s_1, s_2\rangle \in \mathrm{PathRel}\, R$.

Let R be a non empty poset and let s_1, s_2 be elements of the carrier of R. The predicate $s_1 \cong s_2$ is defined as follows:

(Def. 5) $\langle s_1, s_2\rangle \in \mathrm{PathRel}\, R$.

Let us notice that the predicate $s_1 \cong s_2$ is reflexive and symmetric.

One can prove the following proposition

(3) For every non empty poset R and for all elements s_1, s_2, s_3 of the carrier of R such that $s_1 \cong s_2$ and $s_2 \cong s_3$ holds $s_1 \cong s_3$.

Let R be a non empty poset. The functor $\mathrm{Components}\, R$ yields a non empty family of subsets of the carrier of R and is defined by:

(Def. 6) $\mathrm{Components}\, R = \mathrm{Classes}\, \mathrm{PathRel}\, R$.

Let R be a non empty poset. Note that every element of $\mathrm{Components}\, R$ is non empty.

Let R be a non empty poset. A subset of R is called a component of R if:

(Def. 7) It $\in$ Components R.

Let R be a non empty poset and let s_1 be an element of the carrier of R. The functor $\cdot_{\mathrm{CSp}}\, s_1$ yielding a component of R is defined by:

(Def. 8) $\cdot_{\mathrm{CSp}}\, s_1 = [s_1]_{\mathrm{PathRel}\, R}$.

The following two propositions are true:

(4) For every non empty poset R and for every element s_1 of the carrier of R holds $s_1 \in \cdot_{\mathrm{CSp}}\, s_1$.

(5) For every non empty poset R and for all elements s_1, s_2 of the carrier of R such that $s_1 \leqslant s_2$ holds $\cdot_{\mathrm{CSp}}\, s_1 = \cdot_{\mathrm{CSp}}\, s_2$.

Let R be a non empty poset, let A be a many sorted set indexed by the carrier of R, and let C be a component of R. A-carrier of C is defined as follows:

(Def. 9) A-carrier of $C = \bigcup\{A(s); s$ ranges over elements of the carrier of R: $s \in C\}$.

We now state the proposition

(6) Let R be a non empty poset, A be a many sorted set indexed by the carrier of R, s be an element of the carrier of R, and x be a set. If $x \in A(s)$, then $x \in A$-carrier of $\cdot_{\mathrm{CSp}}\, s$.

Let R be a non empty poset. We say that R is locally directed if and only if:

(Def. 10) Every component of R is directed.

The following three propositions are true:

(7) For every discrete non empty poset R and for all elements x, y of the carrier of R such that $\langle x, y\rangle \in \mathrm{PathRel}\, R$ holds $x = y$.

(8) Let R be a discrete non empty poset and C be a component of R. Then there exists an element x of the carrier of R such that $C = \{x\}$.

(9) Every discrete non empty poset is locally directed.

Let us observe that there exists a non empty poset which is locally directed.

One can verify that there exists an order sorted signature which is locally directed.

Let us observe that every non empty poset which is discrete is also locally directed.

Let S be a locally directed non empty poset. Note that every component of S is directed.

One can prove the following proposition

(10) $\emptyset$ is an equivalence relation of $\emptyset$.

Let S be a locally directed order sorted signature, let A be an order sorted algebra of S, let E be an equivalence order sorted relation of A, and let C be a component of S. The functor $\mathrm{CompClass}(E, C)$ yielding an equivalence relation of (the sorts of A)-carrier of C is defined as follows:

(Def. 11) For all sets x, y holds $\langle x, y\rangle \in \mathrm{CompClass}(E, C)$ iff there exists an element s_1 of the carrier of S such that $s_1 \in C$ and $\langle x, y\rangle \in E(s_1)$.

Let S be a locally directed order sorted signature, let A be an order sorted algebra of S, let E be an equivalence order sorted relation of A, and let s_1 be an element of the carrier of S. The functor $\mathrm{OSClass}(E, s_1)$ yielding a subset of $\mathrm{Classes}\,\mathrm{CompClass}(E, \cdot_{\mathrm{CSp}}\, s_1)$ is defined by:

(Def. 12) For every set z holds $z \in \mathrm{OSClass}(E, s_1)$ iff there exists a set x such that $x \in$ (the sorts of A)(s_1) and $z = [x]_{\mathrm{CompClass}(E, \cdot_{\mathrm{CSp}}\, s_1)}$.

Let S be a locally directed order sorted signature, let A be a non-empty order sorted algebra of S, let E be an equivalence order sorted relation of A, and let s_1 be an element of the carrier of S. One can verify that $\mathrm{OSClass}(E, s_1)$ is non empty.

The following proposition is true

(11) Let S be a locally directed order sorted signature, A be an order sorted algebra of S, E be an equivalence order sorted relation of A, and s_1, s_2 be elements of the carrier of S. If $s_1 \leqslant s_2$, then $\mathrm{OSClass}(E, s_1) \subseteq \mathrm{OSClass}(E, s_2)$.

Let S be a locally directed order sorted signature, let A be an order sorted algebra of S, and let E be an equivalence order sorted relation of A. The functor $\mathrm{OSClass}\,E$ yields an order sorted set of S and is defined as follows:

(Def. 13) For every element s_1 of the carrier of S holds $(\mathrm{OSClass}\,E)(s_1) = \mathrm{OSClass}(E, s_1)$.

Let S be a locally directed order sorted signature, let A be a non-empty order sorted algebra of S, and let E be an equivalence order sorted relation of A. One can check that $\mathrm{OSClass}\,E$ is non-empty.

Let S be a locally directed order sorted signature, let U_1 be a non-empty order sorted algebra of S, let E be an equivalence order sorted relation of U_1, let s be an element of the carrier of S, and let x be an element of (the sorts of U_1)(s). The functor $\mathrm{OSClass}(E, x)$ yields an element of $\mathrm{OSClass}(E, s)$ and is defined by:

(Def. 14) $\mathrm{OSClass}(E, x) = [x]_{\mathrm{CompClass}(E, \cdot_{\mathrm{CSp}}\, s)}$.

One can prove the following three propositions:

(12) Let R be a locally directed non empty poset and x, y be elements of the carrier of R. Given an element z of the carrier of R such that $z \leqslant x$ and $z \leqslant y$. Then there exists an element u of R such that $x \leqslant u$ and $y \leqslant u$.

(13) Let S be a locally directed order sorted signature, U_1 be a non-empty order sorted algebra of S, E be an equivalence order sorted relation of U_1,

s be an element of the carrier of S, and x, y be elements of (the sorts of U_1)(s). Then $\mathrm{OSClass}(E, x) = \mathrm{OSClass}(E, y)$ if and only if $\langle x, y \rangle \in E(s)$.

(14) Let S be a locally directed order sorted signature, U_1 be a non-empty order sorted algebra of S, E be an equivalence order sorted relation of U_1, s_1, s_2 be elements of the carrier of S, and x be an element of (the sorts of U_1)(s_1). Suppose $s_1 \leqslant s_2$. Let y be an element of (the sorts of U_1)(s_2). If $y = x$, then $\mathrm{OSClass}(E, x) = \mathrm{OSClass}(E, y)$.

2. Order Sorted Quotient Algebra

In the sequel S denotes a locally directed order sorted signature and o denotes an element of the operation symbols of S.

Let us consider S, o, let A be a non-empty order sorted algebra of S, let R be an order sorted congruence of A, and let x be an element of $\mathrm{Args}(o, A)$. The functor *Rosx* yields an element of $\prod(\mathrm{OSClass}\, R \cdot \mathrm{Arity}(o))$ and is defined by the condition (Def. 15).

(Def. 15) Let n be a natural number. Suppose $n \in \mathrm{dom}\, \mathrm{Arity}(o)$. Then there exists an element y of (the sorts of A)($\mathrm{Arity}(o)_n$) such that $y = x(n)$ and $(\mathit{Rosx})(n) = \mathrm{OSClass}(R, y)$.

Let us consider S, o, let A be a non-empty order sorted algebra of S, and let R be an order sorted congruence of A. The functor $\mathrm{OSQuotRes}(R, o)$ yielding a function from ((the sorts of A) $\cdot$ (the result sort of S))(o) into ($\mathrm{OSClass}\, R \cdot$ the result sort of S)(o) is defined as follows:

(Def. 16) For every element x of (the sorts of A)(the result sort of o) holds $(\mathrm{OSQuotRes}(R, o))(x) = \mathrm{OSClass}(R, x)$.

The functor $\mathrm{OSQuotArgs}(R, o)$ yielding a function from ((the sorts of A)$^{\#}$ $\cdot$ the arity of S)(o) into (($\mathrm{OSClass}\, R$)$^{\#}$ $\cdot$ the arity of S)(o) is defined by:

(Def. 17) For every element x of $\mathrm{Args}(o, A)$ holds $(\mathrm{OSQuotArgs}(R, o))(x) = \mathit{Rosx}$.

Let us consider S, let A be a non-empty order sorted algebra of S, and let R be an order sorted congruence of A. The functor $\mathrm{OSQuotRes}\, R$ yields a many sorted function from (the sorts of A) $\cdot$ (the result sort of S) into $\mathrm{OSClass}\, R \cdot$ the result sort of S and is defined by:

(Def. 18) For every operation symbol o of S holds $(\mathrm{OSQuotRes}\, R)(o) = \mathrm{OSQuotRes}(R, o)$.

The functor $\mathrm{OSQuotArgs}\, R$ yields a many sorted function from (the sorts of A)$^{\#}$ $\cdot$ the arity of S into ($\mathrm{OSClass}\, R$)$^{\#}$ $\cdot$ the arity of S and is defined as follows:

(Def. 19) For every operation symbol o of S holds $(\mathrm{OSQuotArgs}\, R)(o) = \mathrm{OSQuotArgs}(R, o)$.

One can prove the following proposition

(15) Let A be a non-empty order sorted algebra of S, R be an order sorted congruence of A, and x be a set. Suppose $x \in ((\mathrm{OSClass}\,R)^{\#} \cdot \text{the arity of } S)(o)$. Then there exists an element a of $\mathrm{Args}(o, A)$ such that $x = \mathit{Rosa}$.

Let us consider S, o, let A be a non-empty order sorted algebra of S, and let R be an order sorted congruence of A. The functor $\mathrm{OSQuotCharact}(R, o)$ yielding a function from $((\mathrm{OSClass}\,R)^{\#} \cdot \text{the arity of } S)(o)$ into $(\mathrm{OSClass}\,R \cdot \text{the result sort of } S)(o)$ is defined as follows:

(Def. 20) For every element a of $\mathrm{Args}(o, A)$ such that $\mathit{Rosa} \in ((\mathrm{OSClass}\,R)^{\#} \cdot \text{the arity of } S)(o)$ holds $(\mathrm{OSQuotCharact}(R, o))(\mathit{Rosa}) = (\mathrm{OSQuotRes}(R, o) \cdot \mathrm{Den}(o, A))(a)$.

Let us consider S, let A be a non-empty order sorted algebra of S, and let R be an order sorted congruence of A. The functor $\mathrm{OSQuotCharact}\,R$ yielding a many sorted function from $(\mathrm{OSClass}\,R)^{\#} \cdot$ the arity of S into $\mathrm{OSClass}\,R \cdot$ the result sort of S is defined as follows:

(Def. 21) For every operation symbol o of S holds $(\mathrm{OSQuotCharact}\,R)(o) = \mathrm{OSQuotCharact}(R, o)$.

Let us consider S, let U_1 be a non-empty order sorted algebra of S, and let R be an order sorted congruence of U_1. The functor $\mathrm{QuotOSAlg}(U_1, R)$ yields an order sorted algebra of S and is defined by:

(Def. 22) $\mathrm{QuotOSAlg}(U_1, R) = \langle \mathrm{OSClass}\,R, \mathrm{OSQuotCharact}\,R \rangle$.

Let us consider S, let U_1 be a non-empty order sorted algebra of S, and let R be an order sorted congruence of U_1. One can check that $\mathrm{QuotOSAlg}(U_1, R)$ is strict and non-empty.

Let us consider S, let U_1 be a non-empty order sorted algebra of S, let R be an order sorted congruence of U_1, and let s be an element of the carrier of S. The functor $\mathrm{OSNatHom}(U_1, R, s)$ yielding a function from (the sorts of U_1)(s) into $\mathrm{OSClass}(R, s)$ is defined by:

(Def. 23) For every element x of (the sorts of U_1)(s) holds $(\mathrm{OSNatHom}(U_1, R, s))(x) = \mathrm{OSClass}(R, x)$.

Let us consider S, let U_1 be a non-empty order sorted algebra of S, and let R be an order sorted congruence of U_1. The functor $\mathrm{OSNatHom}(U_1, R)$ yielding a many sorted function from U_1 into $\mathrm{QuotOSAlg}(U_1, R)$ is defined as follows:

(Def. 24) For every element s of the carrier of S holds $(\mathrm{OSNatHom}(U_1, R))(s) = \mathrm{OSNatHom}(U_1, R, s)$.

Next we state two propositions:

(16) Let U_1 be a non-empty order sorted algebra of S and R be an order sorted congruence of U_1. Then $\mathrm{OSNatHom}(U_1, R)$ is an epimorphism of U_1 onto $\mathrm{QuotOSAlg}(U_1, R)$ and $\mathrm{OSNatHom}(U_1, R)$ is order-sorted.

(17) Let U_1, U_2 be non-empty order sorted algebras of S and F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into

U_2 and order-sorted. Then $\mathrm{Congruence}(F)$ is an order sorted congruence of U_1.

Let us consider S, let U_1, U_2 be non-empty order sorted algebras of S, and let F be a many sorted function from U_1 into U_2. Let us assume that F is a homomorphism of U_1 into U_2 and order-sorted. The functor $\mathrm{OSCng}\,F$ yielding an order sorted congruence of U_1 is defined as follows:

(Def. 25) $\mathrm{OSCng}\,F = \mathrm{Congruence}(F)$.

Let us consider S, let U_1, U_2 be non-empty order sorted algebras of S, let F be a many sorted function from U_1 into U_2, and let s be an element of the carrier of S. Let us assume that F is a homomorphism of U_1 into U_2 and order-sorted. The functor $\mathrm{OSHomQuot}(F, s)$ yields a function from (the sorts of $\mathrm{QuotOSAlg}(U_1, \mathrm{OSCng}\,F))(s)$ into (the sorts of $U_2)(s)$ and is defined as follows:

(Def. 26) For every element x of (the sorts of $U_1)(s)$ holds $(\mathrm{OSHomQuot}(F, s))(\mathrm{OSClass}(\mathrm{OSCng}\,F, x)) = F(s)(x)$.

Let us consider S, let U_1, U_2 be non-empty order sorted algebras of S, and let F be a many sorted function from U_1 into U_2. The functor $\mathrm{OSHomQuot}\,F$ yields a many sorted function from $\mathrm{QuotOSAlg}(U_1, \mathrm{OSCng}\,F)$ into U_2 and is defined by:

(Def. 27) For every element s of the carrier of S holds $(\mathrm{OSHomQuot}\,F)(s) = \mathrm{OSHomQuot}(F, s)$.

The following three propositions are true:

(18) Let U_1, U_2 be non-empty order sorted algebras of S and F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2 and order-sorted. Then $\mathrm{OSHomQuot}\,F$ is a monomorphism of $\mathrm{QuotOSAlg}(U_1, \mathrm{OSCng}\,F)$ into U_2 and $\mathrm{OSHomQuot}\,F$ is order-sorted.

(19) Let U_1, U_2 be non-empty order sorted algebras of S and F be a many sorted function from U_1 into U_2. Suppose F is an epimorphism of U_1 onto U_2 and order-sorted. Then $\mathrm{OSHomQuot}\,F$ is an isomorphism of $\mathrm{QuotOSAlg}(U_1, \mathrm{OSCng}\,F)$ and U_2.

(20) Let U_1, U_2 be non-empty order sorted algebras of S and F be a many sorted function from U_1 into U_2. Suppose F is an epimorphism of U_1 onto U_2 and order-sorted. Then $\mathrm{QuotOSAlg}(U_1, \mathrm{OSCng}\,F)$ and U_2 are isomorphic.

Let S be an order sorted signature, let U_1 be a non-empty order sorted algebra of S, and let R be an equivalence order sorted relation of U_1. We say that R is monotone if and only if the condition (Def. 28) is satisfied.

(Def. 28) Let o_1, o_2 be operation symbols of S. Suppose $o_1 \leqslant o_2$. Let x_1 be an element of $\mathrm{Args}(o_1, U_1)$ and x_2 be an element of $\mathrm{Args}(o_2, U_1)$. Suppose that for every natural number y such that $y \in \mathrm{dom}\,x_1$ holds $\langle x_1(y), x_2(y)\rangle \in$

$R(\mathrm{Arity}(o_2)_y)$. Then $\langle(\mathrm{Den}(o_1, U_1))(x_1), (\mathrm{Den}(o_2, U_1))(x_2)\rangle \in R$(the result sort of o_2).

One can prove the following two propositions:

(21) Let S be an order sorted signature and U_1 be a non-empty order sorted algebra of S. Then $[\![$the sorts of U_1, the sorts of $U_1]\!]$ is an order sorted congruence of U_1.

(22) Let S be an order sorted signature, U_1 be a non-empty order sorted algebra of S, and R be an order sorted congruence of U_1. If $R = [\![$the sorts of U_1, the sorts of $U_1]\!]$, then R is monotone.

Let S be an order sorted signature and let U_1 be a non-empty order sorted algebra of S. One can verify that there exists an order sorted congruence of U_1 which is monotone.

Let S be an order sorted signature and let U_1 be a non-empty order sorted algebra of S. Note that there exists an equivalence order sorted relation of U_1 which is monotone.

The following proposition is true

(23) Let S be an order sorted signature and U_1 be a non-empty order sorted algebra of S. Then every monotone equivalence order sorted relation of U_1 is MSCongruence-like.

Let S be an order sorted signature and let U_1 be a non-empty order sorted algebra of S. Observe that every equivalence order sorted relation of U_1 which is monotone is also MSCongruence-like.

We now state the proposition

(24) Let S be an order sorted signature and U_1 be a monotone non-empty order sorted algebra of S. Then every order sorted congruence of U_1 is monotone.

Let S be an order sorted signature and let U_1 be a monotone non-empty order sorted algebra of S. Observe that every order sorted congruence of U_1 is monotone.

Let us consider S, let U_1 be a non-empty order sorted algebra of S, and let R be a monotone order sorted congruence of U_1. Note that $\mathrm{QuotOSAlg}(U_1, R)$ is monotone.

We now state two propositions:

(25) Let given S, U_1 be a non-empty order sorted algebra of S, and R be a monotone order sorted congruence of U_1. Then $\mathrm{QuotOSAlg}(U_1, R)$ is a monotone order sorted algebra of S.

(26) Let U_1 be a non-empty order sorted algebra of S, U_2 be a monotone non-empty order sorted algebra of S, and F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2 and order-sorted. Then $\mathrm{OSCng}\, F$ is monotone.

Let us consider S, let U_1, U_2 be non-empty order sorted algebras of S, let F be a many sorted function from U_1 into U_2, let R be an order sorted congruence of U_1, and let s be an element of the carrier of S. Let us assume that F is a homomorphism of U_1 into U_2 and order-sorted and $R \subseteq \mathrm{OSCng}\, F$. The functor $\mathrm{OSHomQuot}(F, R, s)$ yields a function from (the sorts of $\mathrm{QuotOSAlg}(U_1, R))(s)$ into (the sorts of $U_2)(s)$ and is defined as follows:

(Def. 29) For every element x of (the sorts of $U_1)(s)$ holds $(\mathrm{OSHomQuot}(F, R, s))(\mathrm{OSClass}(R, x)) = F(s)(x)$.

Let us consider S, let U_1, U_2 be non-empty order sorted algebras of S, let F be a many sorted function from U_1 into U_2, and let R be an order sorted congruence of U_1. The functor $\mathrm{OSHomQuot}(F, R)$ yields a many sorted function from $\mathrm{QuotOSAlg}(U_1, R)$ into U_2 and is defined as follows:

(Def. 30) For every element s of the carrier of S holds $(\mathrm{OSHomQuot}(F, R))(s) = \mathrm{OSHomQuot}(F, R, s)$.

Next we state the proposition

(27) Let U_1, U_2 be non-empty order sorted algebras of S, F be a many sorted function from U_1 into U_2, and R be an order sorted congruence of U_1. Suppose F is a homomorphism of U_1 into U_2 and order-sorted and $R \subseteq \mathrm{OSCng}\, F$. Then $\mathrm{OSHomQuot}(F, R)$ is a homomorphism of $\mathrm{QuotOSAlg}(U_1, R)$ into U_2 and $\mathrm{OSHomQuot}(F, R)$ is order-sorted.

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Free Order Sorted Universal Algebra[1]

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Summary. Free Order Sorted Universal Algebra — the general construction for any locally directed signatures.

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The papers [21], [13], [27], [32], [33], [11], [22], [12], [7], [10], [4], [18], [2], [20], [26], [14], [5], [3], [6], [1], [8], [25], [23], [17], [24], [9], [15], [16], [29], [31], [28], [30], and [19] provide the terminology and notation for this paper.

1. Preliminaries

In this paper S is an order sorted signature.

Let S be an order sorted signature and let U_0 be an order sorted algebra of S. A subset of U_0 is called an order sorted generator set of U_0 if:

(Def. 1) For every OSSubset O of U_0 such that $O = \mathrm{OSCl}$ it holds the sorts of $\mathrm{OSGen}\, O =$ the sorts of U_0.

The following proposition is true

(1) Let S be an order sorted signature, U_0 be a strict non-empty order sorted algebra of S, and A be a subset of U_0. Then A is an order sorted generator set of U_0 if and only if for every OSSubset O of U_0 such that $O = \mathrm{OSCl}\, A$ holds $\mathrm{OSGen}\, O = U_0$.

Let us consider S, let U_0 be a monotone order sorted algebra of S, and let I_1 be an order sorted generator set of U_0. We say that I_1 is osfree if and only if the condition (Def. 2) is satisfied.

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(Def. 2) Let U_1 be a monotone non-empty order sorted algebra of S and f be a many sorted function from I_1 into the sorts of U_1. Then there exists a many sorted function h from U_0 into U_1 such that h is a homomorphism of U_0 into U_1 and order-sorted and $h \restriction I_1 = f$.

Let S be an order sorted signature and let I_1 be a monotone order sorted algebra of S. We say that I_1 is osfree if and only if:

(Def. 3) There exists an order sorted generator set of I_1 which is osfree.

2. Construction of Free Order Sorted Algebras for Given Signature

Let S be an order sorted signature and let X be a many sorted set indexed by S. The functor OSREL X yields a relation between $[\!:$ the operation symbols of S, {the carrier of S} $:\!] \cup \bigcup \mathrm{coprod}(X)$ and $([\!:$ the operation symbols of S, {the carrier of S} $:\!] \cup \bigcup \mathrm{coprod}(X))^*$ and is defined by the condition (Def. 4).

(Def. 4) Let a be an element of $[\!:$ the operation symbols of S, {the carrier of S} $:\!] \cup \bigcup \mathrm{coprod}(X)$ and b be an element of $([\!:$ the operation symbols of S, {the carrier of S} $:\!] \cup \bigcup \mathrm{coprod}(X))^*$. Then $\langle a, b \rangle \in$ OSREL X if and only if the following conditions are satisfied:

(i) $a \in [\!:$ the operation symbols of S, {the carrier of S} $:\!]$, and

(ii) for every operation symbol o of S such that $\langle o$, the carrier of $S\rangle = a$ holds $\mathrm{len}\, b = \mathrm{len}\, \mathrm{Arity}(o)$ and for every set x such that $x \in \mathrm{dom}\, b$ holds if $b(x) \in [\!:$ the operation symbols of S, {the carrier of S} $:\!]$, then for every operation symbol o_1 of S such that $\langle o_1$, the carrier of $S\rangle = b(x)$ holds the result sort of $o_1 \leqslant \mathrm{Arity}(o)_x$ and if $b(x) \in \bigcup \mathrm{coprod}(X)$, then there exists an element i of the carrier of S such that $i \leqslant \mathrm{Arity}(o)_x$ and $b(x) \in \mathrm{coprod}(i, X)$.

In the sequel S is an order sorted signature, X is a many sorted set indexed by S, o is an operation symbol of S, and b is an element of $([\!:$ the operation symbols of S, {the carrier of S} $:\!] \cup \bigcup \mathrm{coprod}(X))^*$.

One can prove the following proposition

(2) $\langle \langle o$, the carrier of $S\rangle, b\rangle \in$ OSREL X if and only if the following conditions are satisfied:

(i) $\mathrm{len}\, b = \mathrm{len}\, \mathrm{Arity}(o)$, and

(ii) for every set x such that $x \in \mathrm{dom}\, b$ holds if $b(x) \in [\!:$ the operation symbols of S, {the carrier of S} $:\!]$, then for every operation symbol o_1 of S such that $\langle o_1$, the carrier of $S\rangle = b(x)$ holds the result sort of $o_1 \leqslant \mathrm{Arity}(o)_x$ and if $b(x) \in \bigcup \mathrm{coprod}(X)$, then there exists an element i of the carrier of S such that $i \leqslant \mathrm{Arity}(o)_x$ and $b(x) \in \mathrm{coprod}(i, X)$.

Let S be an order sorted signature and let X be a many sorted set indexed by S. The functor $\mathrm{DTConOSA}\,X$ yielding a tree construction structure is defined by:

(Def. 5) $\mathrm{DTConOSA}\,X = \langle [\!:$ the operation symbols of S, {the carrier of S} $:\!] \cup \bigcup \mathrm{coprod}(X), \mathrm{OSREL}\,X\rangle$.

Let S be an order sorted signature and let X be a many sorted set indexed by S. Note that $\mathrm{DTConOSA}\,X$ is strict and non empty.

The following proposition is true

(3) Let S be an order sorted signature and X be a non-empty many sorted set indexed by S. Then the nonterminals of $\mathrm{DTConOSA}\,X$ = $[\!:$ the operation symbols of S, {the carrier of S} $:\!]$ and the terminals of $\mathrm{DTConOSA}\,X = \bigcup \mathrm{coprod}(X)$.

Let S be an order sorted signature and let X be a non-empty many sorted set indexed by S. Note that $\mathrm{DTConOSA}\,X$ has terminals, nonterminals, and useful nonterminals.

The following proposition is true

(4) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, and t be a set. Then $t \in$ the terminals of $\mathrm{DTConOSA}\,X$ if and only if there exists an element s of the carrier of S and there exists a set x such that $x \in X(s)$ and $t = \langle x, s\rangle$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let o be an operation symbol of S. The functor $\mathrm{OSSym}(o, X)$ yielding a symbol of $\mathrm{DTConOSA}\,X$ is defined as follows:

(Def. 6) $\mathrm{OSSym}(o, X) = \langle o,$ the carrier of $S\rangle$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of the carrier of S. The functor $\mathrm{ParsedTerms}(X, s)$ yielding a subset of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ is defined by the condition (Def. 7).

(Def. 7) $\mathrm{ParsedTerms}(X, s) = \{a; a$ ranges over elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$: $\bigvee_{s_1 : \text{element of the carrier of } S} \bigvee_{x : \text{set}} (s_1 \leqslant s \wedge x \in X(s_1) \wedge a =$ the root tree of $\langle x, s_1\rangle) \vee \bigvee_{o : \text{operation symbol of } S} (\langle o,$ the carrier of $S\rangle = a(\emptyset) \wedge$ the result sort of $o \leqslant s)\}$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of the carrier of S. Note that $\mathrm{ParsedTerms}(X, s)$ is non empty.

Let S be an order sorted signature and let X be a non-empty many sorted set indexed by S. The functor $\mathrm{ParsedTerms}\,X$ yields an order sorted set of S and is defined by:

(Def. 8) For every element s of the carrier of S holds $(\mathrm{ParsedTerms}\,X)(s) = \mathrm{ParsedTerms}(X, s)$.

Let S be an order sorted signature and let X be a non-empty many sorted set indexed by S. One can verify that $\mathrm{ParsedTerms}\,X$ is non-empty.

The following four propositions are true:

(5) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, o be an operation symbol of S, and x be a set. Suppose $x \in ((\mathrm{ParsedTerms}\,X)^{\#} \cdot \text{the arity of } S)(o)$. Then x is a finite sequence of elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$.

(6) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, o be an operation symbol of S, and p be a finite sequence of elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$. Then $p \in ((\mathrm{ParsedTerms}\,X)^{\#} \cdot \text{the arity of } S)(o)$ if and only if $\mathrm{dom}\,p = \mathrm{dom}\,\mathrm{Arity}(o)$ and for every natural number n such that $n \in \mathrm{dom}\,p$ holds $p(n) \in \mathrm{ParsedTerms}(X, \mathrm{Arity}(o)_n)$.

(7) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, o be an operation symbol of S, and p be a finite sequence of elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$. Then $\mathrm{OSSym}(o, X) \Rightarrow$ the roots of p if and only if $p \in ((\mathrm{ParsedTerms}\,X)^{\#} \cdot \text{the arity of } S)(o)$.

(8) For every order sorted signature S and for every non-empty many sorted set X indexed by S holds $\bigcup \mathrm{rng}\,\mathrm{ParsedTerms}\,X = \mathrm{TS}(\mathrm{DTConOSA}\,X)$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let o be an operation symbol of S. The functor $\mathrm{PTDenOp}(o, X)$ yields a function from $((\mathrm{ParsedTerms}\,X)^{\#} \cdot \text{the arity of } S)(o)$ into $(\mathrm{ParsedTerms}\,X \cdot \text{the result sort of } S)(o)$ and is defined as follows:

(Def. 9) For every finite sequence p of elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ such that $\mathrm{OSSym}(o, X) \Rightarrow$ the roots of p holds $(\mathrm{PTDenOp}(o, X))(p) = \mathrm{OSSym}(o, X)\text{-tree}(p)$.

Let S be an order sorted signature and let X be a non-empty many sorted set indexed by S. The functor $\mathrm{PTOper}\,X$ yields a many sorted function from $(\mathrm{ParsedTerms}\,X)^{\#} \cdot \text{the arity of } S$ into $\mathrm{ParsedTerms}\,X \cdot \text{the result sort of } S$ and is defined by:

(Def. 10) For every operation symbol o of S holds $(\mathrm{PTOper}\,X)(o) = \mathrm{PTDenOp}(o, X)$.

Let S be an order sorted signature and let X be a non-empty many sorted set indexed by S. The functor $\mathrm{ParsedTermsOSA}\,X$ yielding an order sorted algebra of S is defined as follows:

(Def. 11) $\mathrm{ParsedTermsOSA}\,X = \langle \mathrm{ParsedTerms}\,X, \mathrm{PTOper}\,X \rangle$.

Let S be an order sorted signature and let X be a non-empty many sorted set indexed by S. One can check that $\mathrm{ParsedTermsOSA}\,X$ is strict and non-empty.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let o be an operation symbol of S. Then $\mathrm{OSSym}(o, X)$ is a nonterminal of $\mathrm{DTConOSA}\,X$.

Next we state several propositions:

(9) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, and s be an element of the carrier of S. Then (the sorts of ParsedTermsOSA X)(s) $=$ $\{a; a$ ranges over elements of TS(DTConOSA X): $\bigvee_{s_1\,:\,\text{element of the carrier of } S}$ $\bigvee_{x\,:\,\text{set}}$ $(s_1 \leqslant s \ \wedge\ x \in X(s_1) \ \wedge\ a = \text{the root tree of } \langle x, s_1\rangle) \ \vee\ \bigvee_{o\,:\,\text{operation symbol of } S}$ $(\langle o$, the carrier of $S\rangle = a(\emptyset) \ \wedge$ the result sort of $o \leqslant s)\}$.

(10) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, s, s_1 be elements of the carrier of S, and x be a set. Suppose $x \in X(s)$. Then

(i) the root tree of $\langle x, s\rangle$ is an element of TS(DTConOSA X),

(ii) for every set z holds $\langle z$, the carrier of $S\rangle \neq$ (the root tree of $\langle x, s\rangle)(\emptyset)$, and

(iii) the root tree of $\langle x, s\rangle \in$ (the sorts of ParsedTermsOSA X)(s_1) iff $s \leqslant s_1$.

(11) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, t be an element of TS(DTConOSA X), and o be an operation symbol of S. Suppose $t(\emptyset) = \langle o$, the carrier of $S\rangle$. Then

(i) there exists a subtree sequence p joinable by OSSym(o, X) such that t = OSSym(o, X)-tree(p) and OSSym(o, X) $\Rightarrow$ the roots of p and $p \in$ Args$(o$, ParsedTermsOSA $X)$ and $t =$ (Den$(o$, ParsedTermsOSA $X))(p)$,

(ii) for every element s_2 of the carrier of S and for every set x holds $t \neq$ the root tree of $\langle x, s_2\rangle$, and

(iii) for every element s_1 of the carrier of S holds $t \in$ (the sorts of ParsedTermsOSA X)(s_1) iff the result sort of $o \leqslant s_1$.

(12) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, n_1 be a symbol of DTConOSA X, and t_1 be a finite sequence of elements of TS(DTConOSA X). Suppose $n_1 \Rightarrow$ the roots of t_1. Then

(i) $n_1 \in$ the nonterminals of DTConOSA X,

(ii) n_1-tree$(t_1) \in$ TS(DTConOSA X), and

(iii) there exists an operation symbol o of S such that $n_1 = \langle o$, the carrier of $S\rangle$ and $t_1 \in$ Args$(o$, ParsedTermsOSA $X)$ and n_1-tree(t_1) $=$ (Den$(o$, ParsedTermsOSA $X))(t_1)$ and for every element s_1 of the carrier of S holds n_1-tree$(t_1) \in$ (the sorts of ParsedTermsOSA X)(s_1) iff the result sort of $o \leqslant s_1$.

(13) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, o be an operation symbol of S, and x be a finite sequence. Then $x \in$ Args$(o$, ParsedTermsOSA $X)$ if and only if the following conditions are satisfied:

(i) x is a finite sequence of elements of TS(DTConOSA X), and

(ii) OSSym(o, X) $\Rightarrow$ the roots of x.

(14) Let S be an order sorted signature, X be a non-empty many sorted set

indexed by S, and t be an element of TS(DTConOSA X). Then there exists a sort symbol s of S such that $t \in$ (the sorts of ParsedTermsOSA X)(s) and for every element s_1 of the carrier of S such that $t \in$ (the sorts of ParsedTermsOSA X)(s_1) holds $s \leqslant s_1$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let t be an element of TS(DTConOSA X). The functor LeastSort t yields a sort symbol of S and is defined by the conditions (Def. 12).

(Def. 12)(i) $t \in$ (the sorts of ParsedTermsOSA X)(LeastSort t), and

(ii) for every element s_1 of the carrier of S such that $t \in$ (the sorts of ParsedTermsOSA X)(s_1) holds LeastSort $t \leqslant s_1$.

Let S be a non empty non void many sorted signature and let A be a non-empty algebra over S.

(Def. 13) An element of $\bigcup$(the sorts of A) is said to be an element of A.

We now state four propositions:

(15) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, and x be a set. Then x is an element of ParsedTermsOSA X if and only if x is an element of TS(DTConOSA X).

(16) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, s be an element of the carrier of S, and x be a set. If $x \in$ (the sorts of ParsedTermsOSA X)(s), then x is an element of TS(DTConOSA X).

(17) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, s be an element of the carrier of S, and x be a set. Suppose $x \in X(s)$. Let t be an element of TS(DTConOSA X). If $t =$ the root tree of $\langle x, s\rangle$, then LeastSort $t = s$.

(18) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, o be an operation symbol of S, x be an element of Args(o, ParsedTermsOSA X), and t be an element of TS(DTConOSA X). If $t =$ (Den(o, ParsedTermsOSA X))(x), then LeastSort $t =$ the result sort of o.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let o_2 be an operation symbol of S. Note that Args(o_2, ParsedTermsOSA X) is non empty.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let x be a finite sequence of elements of TS(DTConOSA X). The functor LeastSorts x yielding an element of (the carrier of S)* is defined as follows:

(Def. 14) dom LeastSorts $x =$ dom x and for every natural number y such that $y \in$ dom x there exists an element t of TS(DTConOSA X) such that $t = x(y)$ and (LeastSorts x)(y) $=$ LeastSort t.

We now state the proposition

(19) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, o be an operation symbol of S, and x be a finite sequence of elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$. Then $\mathrm{LeastSorts}\,x \leqslant \mathrm{Arity}(o)$ if and only if $x \in \mathrm{Args}(o, \mathrm{ParsedTermsOSA}\,X)$.

Let us note that there exists a monotone order sorted signature which is locally directed and regular.

Let S be a locally directed regular monotone order sorted signature, let X be a non-empty many sorted set indexed by S, let o be an operation symbol of S, and let x be a finite sequence of elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$. Let us assume that $\mathrm{OSSym}(\mathrm{LBound}(o, \mathrm{LeastSorts}\,x), X) \Rightarrow$ the roots of x. The functor $\pi_x o$ yields an element of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ and is defined by:

(Def. 15) $\pi_x o = \mathrm{OSSym}(\mathrm{LBound}(o, \mathrm{LeastSorts}\,x), X)\text{-tree}(x)$.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let t be a symbol of $\mathrm{DTConOSA}\,X$. Let us assume that there exists a finite sequence p such that $t \Rightarrow p$. The functor ${}^{@}(X, t)$ yields an operation symbol of S and is defined by:

(Def. 16) $\langle {}^{@}(X, t)$, the carrier of $S\rangle = t$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let t be a symbol of $\mathrm{DTConOSA}\,X$. Let us assume that $t \in$ the terminals of $\mathrm{DTConOSA}\,X$. The functor $\prod t$ yielding an element of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ is defined by:

(Def. 17) $\prod t =$ the root tree of t.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor $\mathrm{LCongruence}\,X$ yielding a monotone order sorted congruence of $\mathrm{ParsedTermsOSA}\,X$ is defined by:

(Def. 18) For every monotone order sorted congruence R of $\mathrm{ParsedTermsOSA}\,X$ holds $\mathrm{LCongruence}\,X \subseteq R$.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor $\mathrm{FreeOSA}\,X$ yielding a strict non-empty monotone order sorted algebra of S is defined by:

(Def. 19) $\mathrm{FreeOSA}\,X = \mathrm{QuotOSAlg}(\mathrm{ParsedTermsOSA}\,X, \mathrm{LCongruence}\,X)$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let t be a symbol of $\mathrm{DTConOSA}\,X$. The functor ${}^{@}t$ yields a subset of $[\!:\mathrm{TS}(\mathrm{DTConOSA}\,X)$, the carrier of $S:\!]$ and is defined by the condition (Def. 20).

(Def. 20) ${}^{@}t = \{\langle$the root tree of $t,\ s_1\rangle; s_1$ ranges over elements of the carrier of S: $\bigvee_{s\,:\,\text{element of the carrier of } S} \bigvee_{x\,:\,\text{set}} (x \in X(s) \ \wedge\ t = \langle x,\, s\rangle \ \wedge\ s \leqslant s_1)\}$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, let n_1 be a symbol of $\mathrm{DTConOSA}\,X$, and let x be a finite sequence of elements of $2^{[:\mathrm{TS}(\mathrm{DTConOSA}\,X),\ \text{the carrier of } S:]}$. The functor ${}^{@}(n_1, x)$ yielding a subset of $[:\mathrm{TS}(\mathrm{DTConOSA}\,X),$ the carrier of $S:]$ is defined by the condition (Def. 21).

(Def. 21) ${}^{@}(n_1, x) = \{\langle(\mathrm{Den}(o_2, \mathrm{ParsedTermsOSA}\,X))(x_2),\ s_3\rangle; o_2$ ranges over operation symbols of S, x_2 ranges over elements of $\mathrm{Args}(o_2, \mathrm{ParsedTermsOSA}\,X)$, s_3 ranges over elements of the carrier of S: $\bigvee_{o_1 : \text{operation symbol of } S} (n_1 = \langle o_1, \text{the carrier of } S\rangle \wedge o_1 \cong o_2 \wedge \mathrm{len}\,\mathrm{Arity}(o_1) = \mathrm{len}\,\mathrm{Arity}(o_2) \wedge$ the result sort of $o_1 \leqslant s_3 \wedge$ the result sort of $o_2 \leqslant s_3) \wedge \bigvee_{w_3 : \text{element of (the carrier of } S)^*} (\mathrm{dom}\, w_3 = \mathrm{dom}\, x \wedge \bigwedge_{y : \text{natural number}} (y \in \mathrm{dom}\, x \Rightarrow \langle x_2(y), (w_3)_y\rangle \in x(y)))\}$.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor PTClasses X yielding a function from $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ into $2^{[:\mathrm{TS}(\mathrm{DTConOSA}\,X),\ \text{the carrier of } S:]}$ is defined by the conditions (Def. 22).

(Def. 22)(i) For every symbol t of DTConOSA X such that $t \in$ the terminals of DTConOSA X holds (PTClasses X)(the root tree of t) $= {}^{@}t$, and

(ii) for every symbol n_1 of DTConOSA X and for every finite sequence t_1 of elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ and for every finite sequence r_1 such that $r_1 =$ the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of $2^{[:\mathrm{TS}(\mathrm{DTConOSA}\,X),\ \text{the carrier of } S:]}$ such that $x = \mathrm{PTClasses}\,X \cdot t_1$ holds (PTClasses X)(n_1-tree(t_1)) $= {}^{@}(n_1, x)$.

One can prove the following four propositions:

(20) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, and t be an element of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$. Then

(i) for every element s of the carrier of S holds $t \in$ (the sorts of ParsedTermsOSA X)(s) iff $\langle t, s\rangle \in$ (PTClasses X)(t), and

(ii) for every element s of the carrier of S and for every element y of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ such that $\langle y, s\rangle \in$ (PTClasses X)(t) holds $\langle t, s\rangle \in$ (PTClasses X)(y).

(21) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, t be an element of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$, and s be an element of the carrier of S. If there exists an element y of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ such that $\langle y, s\rangle \in$ (PTClasses X)(t), then $\langle t, s\rangle \in$ (PTClasses X)(t).

(22) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, x, y be elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$, and s_1, s_2 be elements of the carrier of S. Suppose $s_1 \leqslant s_2$ and $x \in$ (the sorts of ParsedTermsOSA X)(s_1) and $y \in$ (the sorts of ParsedTermsOSA X)(s_1). Then $\langle y, s_1\rangle \in$ (PTClasses X)(x) if and only if $\langle y, s_2\rangle \in$ (PTClasses X)(x).

(23) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, x, y, z be elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$, and s be an element of the carrier of S. If $\langle y, s\rangle \in (\mathrm{PTClasses}\,X)(x)$ and $\langle z, s\rangle \in (\mathrm{PTClasses}\,X)(y)$, then $\langle x, s\rangle \in (\mathrm{PTClasses}\,X)(z)$.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor $\mathrm{PTCongruence}\,X$ yielding an equivalence order sorted relation of $\mathrm{ParsedTermsOSA}\,X$ is defined by the condition (Def. 23).

(Def. 23) Let i be a set. Suppose $i \in$ the carrier of S. Then $(\mathrm{PTCongruence}\,X)(i) = \{\langle x, y\rangle; x$ ranges over elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$, y ranges over elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$: $\langle x, i\rangle \in (\mathrm{PTClasses}\,X)(y)\}$.

One can prove the following propositions:

(24) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, and x, y, s be sets. If $\langle x, s\rangle \in (\mathrm{PTClasses}\,X)(y)$, then $x \in \mathrm{TS}(\mathrm{DTConOSA}\,X)$ and $y \in \mathrm{TS}(\mathrm{DTConOSA}\,X)$ and $s \in$ the carrier of S.

(25) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, C be a component of S, and x, y be sets. Then $\langle x, y\rangle \in \mathrm{CompClass}(\mathrm{PTCongruence}\,X, C)$ if and only if there exists an element s_1 of the carrier of S such that $s_1 \in C$ and $\langle x, s_1\rangle \in (\mathrm{PTClasses}\,X)(y)$.

(26) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, s be an element of the carrier of S, and x be an element of (the sorts of $\mathrm{ParsedTermsOSA}\,X)(s)$. Then $\mathrm{OSClass}(\mathrm{PTCongruence}\,X, x) = \pi_1((\mathrm{PTClasses}\,X)(x))$.

(27) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, and R be a many sorted relation indexed by $\mathrm{ParsedTermsOSA}\,X$. Then $R = \mathrm{PTCongruence}\,X$ if and only if the following conditions are satisfied:

(i) for all elements s_1, s_2 of the carrier of S and for every set x such that $x \in X(s_1)$ holds if $s_1 \leqslant s_2$, then $\langle$the root tree of $\langle x, s_1\rangle$, the root tree of $\langle x, s_1\rangle\rangle \in R(s_2)$ and for every set y such that $\langle$the root tree of $\langle x, s_1\rangle$, $y\rangle \in R(s_2)$ or $\langle y$, the root tree of $\langle x, s_1\rangle\rangle \in R(s_2)$ holds $s_1 \leqslant s_2$ and $y =$ the root tree of $\langle x, s_1\rangle$, and

(ii) for all operation symbols o_1, o_2 of S and for every element x_1 of $\mathrm{Args}(o_1, \mathrm{ParsedTermsOSA}\,X)$ and for every element x_2 of $\mathrm{Args}(o_2, \mathrm{ParsedTermsOSA}\,X)$ and for every element s_3 of the carrier of S holds $\langle(\mathrm{Den}(o_1, \mathrm{ParsedTermsOSA}\,X))(x_1), (\mathrm{Den}(o_2, \mathrm{ParsedTermsOSA}\,X))(x_2)\rangle \in R(s_3)$ iff $o_1 \cong o_2$ and $\mathrm{len}\,\mathrm{Arity}(o_1) = \mathrm{len}\,\mathrm{Arity}(o_2)$ and the result sort of $o_1 \leqslant s_3$ and the result sort of $o_2 \leqslant s_3$ and there exists an element w_3 of (the carrier of $S)^*$ such that dom w_3 = dom x_1 and for every natural number y such that $y \in \operatorname{dom} w_3$ holds $\langle x_1(y), x_2(y)\rangle \in R((w_3)_y)$.

(28) Let S be a locally directed order sorted signature and X be a non-empty many sorted set indexed by S. Then PTCongruence X is monotone.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. Observe that PTCongruence X is monotone.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of the carrier of S. The functor PTVars(s, X) yields a subset of (the sorts of ParsedTermsOSA X)(s) and is defined by:

(Def. 24) For every set x holds $x \in$ PTVars(s, X) iff there exists a set a such that $a \in X(s)$ and $x =$ the root tree of $\langle a, s\rangle$.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of the carrier of S. One can check that PTVars(s, X) is non empty.

We now state the proposition

(29) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, and s be an element of the carrier of S. Then PTVars$(s, X) = \{$the root tree of t; t ranges over symbols of DTConOSA X : $t \in$ the terminals of DTConOSA $X \ \wedge\ t_{\mathbf{2}} = s\}$.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor PTVars X yielding a subset of ParsedTermsOSA X is defined by:

(Def. 25) For every element s of the carrier of S holds (PTVars X)(s) = PTVars(s, X).

The following proposition is true

(30) Let S be a locally directed order sorted signature and X be a non-empty many sorted set indexed by S. Then PTVars X is non-empty.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of the carrier of S. The functor OSFreeGen(s, X) yields a subset of (the sorts of FreeOSA X)(s) and is defined by:

(Def. 26) For every set x holds $x \in$ OSFreeGen(s, X) iff there exists a set a such that $a \in X(s)$ and $x =$ (OSNatHom(ParsedTermsOSA X, LCongruence X)) (s)(the root tree of $\langle a, s\rangle$).

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of the carrier of S. Note that OSFreeGen(s, X) is non empty.

We now state the proposition

(31) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, and s be an element of the carrier of S. Then $\mathrm{OSFreeGen}(s, X) = \{(\mathrm{OSNatHom}(\mathrm{ParsedTermsOSA}\,X, \mathrm{LCongruence}\,X))(s)(\text{the root tree of } t); t \text{ ranges over symbols of } \mathrm{DTConOSA}\,X : t \in \text{the terminals of } \mathrm{DTConOSA}\,X \ \wedge\ t_{\mathbf{2}} = s\}$.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor $\mathrm{OSFreeGen}\,X$ yielding an order sorted generator set of $\mathrm{FreeOSA}\,X$ is defined by:

(Def. 27) For every element s of the carrier of S holds $(\mathrm{OSFreeGen}\,X)(s) = \mathrm{OSFreeGen}(s, X)$.

The following proposition is true

(32) Let S be a locally directed order sorted signature and X be a non-empty many sorted set indexed by S. Then $\mathrm{OSFreeGen}\,X$ is non-empty.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. Observe that $\mathrm{OSFreeGen}\,X$ is non-empty.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, let R be an order sorted congruence of $\mathrm{ParsedTermsOSA}\,X$, and let t be an element of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$. The functor $\mathrm{OSClass}(R, t)$ yielding an element of $\mathrm{OSClass}(R, \mathrm{LeastSort}\,t)$ is defined by the condition (Def. 28).

(Def. 28) Let s be an element of the carrier of S and x be an element of (the sorts of $\mathrm{ParsedTermsOSA}\,X)(s)$. If $t = x$, then $\mathrm{OSClass}(R, t) = \mathrm{OSClass}(R, x)$.

We now state several propositions:

(33) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, R be an order sorted congruence of $\mathrm{ParsedTermsOSA}\,X$, and t be an element of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$. Then $t \in \mathrm{OSClass}(R, t)$.

(34) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, s be an element of the carrier of S, t be an element of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$, and x, x_1 be sets. Suppose $x \in X(s)$ and $t = \text{the root tree of } \langle x, s\rangle$. Then $x_1 \in \mathrm{OSClass}(\mathrm{PTCongruence}\,X, t)$ if and only if $x_1 = t$.

(35) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, R be an order sorted congruence of $\mathrm{ParsedTermsOSA}\,X$, and t_2, t_3 be elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$. Then $t_3 \in \mathrm{OSClass}(R, t_2)$ if and only if $\mathrm{OSClass}(R, t_2) = \mathrm{OSClass}(R, t_3)$.

(36) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, R_1, R_2 be order sorted congruences of $\mathrm{ParsedTermsOSA}\,X$, and t be an element of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$. If $R_1 \subseteq R_2$, then $\mathrm{OSClass}(R_1, t) \subseteq \mathrm{OSClass}(R_2, t)$.

(37) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, s be an element of the carrier of S, t be an element of TS(DTConOSA X), and x, x_1 be sets. Suppose $x \in X(s)$ and $t =$ the root tree of $\langle x, s\rangle$. Then $x_1 \in \mathrm{OSClass}(\mathrm{LCongruence}\, X, t)$ if and only if $x_1 = t$.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, let A be a non-empty many sorted set indexed by the carrier of S, let F be a many sorted function from PTVars X into A, and let t be a symbol of DTConOSA X. Let us assume that $t \in$ the terminals of DTConOSA X. The functor $\pi(F, A, t)$ yields an element of $\bigcup A$ and is defined as follows:

(Def. 29) For every function f such that $f = F(t_{\mathbf{2}})$ holds $\pi(F, A, t) = f$(the root tree of t).

Next we state the proposition

(38) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, U_1 be a monotone non-empty order sorted algebra of S, and f be a many sorted function from PTVars X into the sorts of U_1. Then there exists a many sorted function h from ParsedTermsOSA X into U_1 such that h is a homomorphism of ParsedTermsOSA X into U_1 and order-sorted and $h \restriction \mathrm{PTVars}\, X = f$.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of the carrier of S. The functor NHReverse(s, X) yields a function from OSFreeGen(s, X) into PTVars(s, X) and is defined by the condition (Def. 30).

(Def. 30) Let t be a symbol of DTConOSA X.
Suppose (OSNatHom(ParsedTermsOSA X, LCongruence X))(s)(the root tree of t) $\in$ OSFreeGen(s, X). Then (NHReverse(s, X))((OSNatHom (ParsedTermsOSA X, LCongruence X))(s)(the root tree of t)) $=$ the root tree of t.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor NHReverse X yielding a many sorted function from OSFreeGen X into PTVars X is defined as follows:

(Def. 31) For every element s of the carrier of S holds (NHReverse X)(s) $=$ NHReverse(s, X).

Next we state two propositions:

(39) Let S be a locally directed order sorted signature and X be a non-empty many sorted set indexed by S. Then OSFreeGen X is osfree.

(40) Let S be a locally directed order sorted signature and X be a non-empty many sorted set indexed by S. Then FreeOSA X is osfree.

Let S be a locally directed order sorted signature. Note that there exists a non-empty monotone order sorted algebra of S which is osfree and strict.

3. Minimal Terms

Let S be a locally directed regular monotone order sorted signature and let X be a non-empty many sorted set indexed by S. The functor $\mathrm{PTMin}\,X$ yields a function from $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ into $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ and is defined by the conditions (Def. 32).

(Def. 32)(i) For every symbol t of $\mathrm{DTConOSA}\,X$ such that $t \in$ the terminals of $\mathrm{DTConOSA}\,X$ holds $(\mathrm{PTMin}\,X)(\text{the root tree of } t) = \prod t$, and

(ii) for every symbol n_1 of $\mathrm{DTConOSA}\,X$ and for every finite sequence t_1 of elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ and for every finite sequence r_1 such that $r_1 =$ the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ such that $x = \mathrm{PTMin}\,X \cdot t_1$ holds $(\mathrm{PTMin}\,X)(n_1\text{-tree}(t_1)) = \pi_x({}^{@}(X, n_1))$.

Next we state several propositions:

(41) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, and t be an element of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$. Then

(i) $(\mathrm{PTMin}\,X)(t) \in \mathrm{OSClass}(\mathrm{PTCongruence}\,X, t)$,

(ii) $\mathrm{LeastSort}(\mathrm{PTMin}\,X)(t) \leqslant \mathrm{LeastSort}\,t$,

(iii) for every element s of the carrier of S and for every set x such that $x \in X(s)$ and $t =$ the root tree of $\langle x, s\rangle$ holds $(\mathrm{PTMin}\,X)(t) = t$, and

(iv) for every operation symbol o of S and for every finite sequence t_1 of elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ such that $\mathrm{OSSym}(o, X) \Rightarrow$ the roots of t_1 and $t = \mathrm{OSSym}(o, X)\text{-tree}(t_1)$ holds $\mathrm{LeastSorts}\,\mathrm{PTMin}\,X \cdot t_1 \leqslant \mathrm{Arity}(o)$ and $\mathrm{OSSym}(o, X) \Rightarrow$ the roots of $\mathrm{PTMin}\,X \cdot t_1$ and $\mathrm{OSSym}(\mathrm{LBound}(o, \mathrm{LeastSorts}\,\mathrm{PTMin}\,X \cdot t_1), X) \Rightarrow$ the roots of $\mathrm{PTMin}\,X \cdot t_1$ and $(\mathrm{PTMin}\,X)(t) = \mathrm{OSSym}(\mathrm{LBound}(o, \mathrm{LeastSorts}\,\mathrm{PTMin}\,X \cdot t_1), X)\text{-tree}(\mathrm{PTMin}\,X \cdot t_1)$.

(42) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, and t, t_2 be elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$. If $t_2 \in \mathrm{OSClass}(\mathrm{PTCongruence}\,X, t)$, then $(\mathrm{PTMin}\,X)(t_2) = (\mathrm{PTMin}\,X)(t)$.

(43) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, and t_2, t_3 be elements of $\mathrm{TS}(\mathrm{DTConOSA}\,X)$. Then $t_3 \in \mathrm{OSClass}(\mathrm{PTCongruence}\,X, t_2)$ if and only if $(\mathrm{PTMin}\,X)(t_3) = (\mathrm{PTMin}\,X)(t_2)$.

(44) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, and t_2 be

an element of TS(DTConOSA X). Then $(\mathrm{PTMin}\,X)((\mathrm{PTMin}\,X)(t_2)) = (\mathrm{PTMin}\,X)(t_2)$.

(45) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, R be a monotone equivalence order sorted relation of ParsedTermsOSA X, and t be an element of TS(DTConOSA X). Then $\langle t, (\mathrm{PTMin}\,X)(t)\rangle \in R(\mathrm{LeastSort}\,t)$.

(46) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, and R be a monotone equivalence order sorted relation of ParsedTermsOSA X. Then $\mathrm{PTCongruence}\,X \subseteq R$.

(47) Let S be a locally directed regular monotone order sorted signature and X be a non-empty many sorted set indexed by S. Then $\mathrm{LCongruence}\,X = \mathrm{PTCongruence}\,X$.

Let S be a locally directed regular monotone order sorted signature and let X be a non-empty many sorted set indexed by S. An element of TS(DTConOSA X) is called a minimal term of S, X if:

(Def. 33) $(\mathrm{PTMin}\,X)(\mathrm{it}) = \mathrm{it}$.

Let S be a locally directed regular monotone order sorted signature and let X be a non-empty many sorted set indexed by S. The functor MinTerms X yields a subset of TS(DTConOSA X) and is defined by:

(Def. 34) $\mathrm{MinTerms}\,X = \mathrm{rng}\,\mathrm{PTMin}\,X$.

The following proposition is true

(48) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, and x be a set. Then x is a minimal term of S, X if and only if $x \in \mathrm{MinTerms}\,X$.

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