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Upper and Lower Sequence on the Cage, Upper and Lower Arcs[1]

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The articles [25], [30], [2], [4], [3], [29], [5], [14], [27], [20], [24], [13], [1], [23], [10], [11], [8], [28], [16], [12], [21], [26], [7], [18], [19], [6], [22], [9], [15], and [17] provide the notation and terminology for this paper.

In this paper n is a natural number.

The following propositions are true:

(1) Let G be a Go-board and i_1, i_2, j_1, j_2 be natural numbers. Suppose $1 \leqslant j_1$ and $j_1 \leqslant \mathrm{width}\, G$ and $1 \leqslant j_2$ and $j_2 \leqslant \mathrm{width}\, G$ and $1 \leqslant i_1$ and $i_1 < i_2$ and $i_2 \leqslant \mathrm{len}\, G$. Then $(G \circ (i_1, j_1))_{\mathbf{1}} < (G \circ (i_2, j_2))_{\mathbf{1}}$.

(2) Let G be a Go-board and i_1, i_2, j_1, j_2 be natural numbers. Suppose $1 \leqslant i_1$ and $i_1 \leqslant \mathrm{len}\, G$ and $1 \leqslant i_2$ and $i_2 \leqslant \mathrm{len}\, G$ and $1 \leqslant j_1$ and $j_1 < j_2$ and $j_2 \leqslant \mathrm{width}\, G$. Then $(G \circ (i_1, j_1))_{\mathbf{2}} < (G \circ (i_2, j_2))_{\mathbf{2}}$.

Let f be a non empty finite sequence and let g be a finite sequence. One can verify that $f \frown g$ is non empty.

The following propositions are true:

(3) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Then $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n) -: \text{E-max}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n) :- \text{E-max}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) = \{\text{N-min}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)), \text{E-max}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))\}$.

(4) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{UpperSeq}(C,n) = ((\mathrm{Cage}(C,n))^{\text{E-max}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}_{\circlearrowleft}) :- \text{W-min}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$.

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(5) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds W-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{UpperSeq}(C,n)$ and W-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n))$.

(6) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds W-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{UpperSeq}(C,n)$ and W-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n))$.

(7) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds N-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{UpperSeq}(C,n)$ and N-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n))$.

(8) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds N-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{UpperSeq}(C,n)$ and N-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n))$.

(9) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds E-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{UpperSeq}(C,n)$ and E-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n))$.

(10) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds E-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{LowerSeq}(C,n)$ and E-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n))$.

(11) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds E-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{LowerSeq}(C,n)$ and E-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n))$.

(12) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds S-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{LowerSeq}(C,n)$ and S-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n))$.

(13) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds S-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{LowerSeq}(C,n)$ and S-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n))$.

(14) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds W-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \mathrm{rng}\,\mathrm{LowerSeq}(C,n)$ and W-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \in \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n))$.

(15) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ and N-min $Y \in X$ holds N-min $X =$ N-min Y.

(16) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ and N-max $Y \in X$ holds N-max $X =$ N-max Y.

(17) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ and E-min $Y \in X$ holds E-min $X =$ E-min Y.

(18) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ and E-max $Y \in X$ holds E-max $X =$ E-max Y.

(19) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ and S-min $Y \in X$ holds S-min $X =$ S-min Y.

(20) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $X \subseteq Y$ and $\text{S-max}\, Y \in X$ holds $\text{S-max}\, X = \text{S-max}\, Y$.

(21) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $X \subseteq Y$ and $\text{W-min}\, Y \in X$ holds $\text{W-min}\, X = \text{W-min}\, Y$.

(22) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $X \subseteq Y$ and $\text{W-max}\, Y \in X$ holds $\text{W-max}\, X = \text{W-max}\, Y$.

(23) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $\text{N-bound}\, X < \text{N-bound}\, Y$ holds $\text{N-bound}\, X \cup Y = \text{N-bound}\, Y$.

(24) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $\text{E-bound}\, X < \text{E-bound}\, Y$ holds $\text{E-bound}\, X \cup Y = \text{E-bound}\, Y$.

(25) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $\text{S-bound}\, X < \text{S-bound}\, Y$ holds $\text{S-bound}\, X \cup Y = \text{S-bound}\, X$.

(26) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $\text{W-bound}\, X < \text{W-bound}\, Y$ holds $\text{W-bound}\, X \cup Y = \text{W-bound}\, X$.

(27) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $\text{N-bound}\, X < \text{N-bound}\, Y$ holds $\text{N-min}\, X \cup Y = \text{N-min}\, Y$.

(28) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $\text{N-bound}\, X < \text{N-bound}\, Y$ holds $\text{N-max}\, X \cup Y = \text{N-max}\, Y$.

(29) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $\text{E-bound}\, X < \text{E-bound}\, Y$ holds $\text{E-min}\, X \cup Y = \text{E-min}\, Y$.

(30) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $\text{E-bound}\, X < \text{E-bound}\, Y$ holds $\text{E-max}\, X \cup Y = \text{E-max}\, Y$.

(31) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $\text{S-bound}\, X < \text{S-bound}\, Y$ holds $\text{S-min}\, X \cup Y = \text{S-min}\, X$.

(32) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $\text{S-bound}\, X < \text{S-bound}\, Y$ holds $\text{S-max}\, X \cup Y = \text{S-max}\, X$.

(33) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $\text{W-bound}\, X < \text{W-bound}\, Y$ holds $\text{W-min}\, X \cup Y = \text{W-min}\, X$.

(34) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $\text{W-bound}\, X < \text{W-bound}\, Y$ holds $\text{W-max}\, X \cup Y = \text{W-max}\, X$.

(35) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$, then $(\downharpoonleft p, f)_{\mathrm{len} \downharpoonleft p, f} = f_{\mathrm{len}\, f}$.

(36) Let f be a non constant standard special circular sequence, p, q be points of $\mathcal{E}^2_{\mathrm{T}}$, and g be a connected subset of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \mathrm{RightComp}(f)$ and $q \in \mathrm{LeftComp}(f)$ and $p \in g$ and $q \in g$, then g meets $\widetilde{\mathcal{L}}(f)$.

One can verify that there exists special sequence finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ which is non constant, standard, and s.c.c..

Next we state a number of propositions:

(37) For every S-sequence f in $\mathbb{R}^2$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \operatorname{rng} f$ holds $\downharpoonright p, f = \operatorname{mid}(f, p \hookleftarrow f, \operatorname{len} f)$.

(38) Let M be a Go-board and f be a S-sequence in $\mathbb{R}^2$. Suppose f is a sequence which elements belong to M. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \operatorname{rng} f$, then $\downharpoonleft f, p$ is a sequence which elements belong to M.

(39) Let M be a Go-board and f be a S-sequence in $\mathbb{R}^2$. Suppose f is a sequence which elements belong to M. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \operatorname{rng} f$, then $\downharpoonright p, f$ is a sequence which elements belong to M.

(40) Let G be a Go-board and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to G. Let i, j be natural numbers. If $1 \leqslant i$ and $i \leqslant \operatorname{len} G$ and $1 \leqslant j$ and $j \leqslant \operatorname{width} G$, then if $G \circ (i, j) \in \widetilde{\mathcal{L}}(f)$, then $G \circ (i, j) \in \operatorname{rng} f$.

(41) Let f be a S-sequence in $\mathbb{R}^2$ and g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that
- (i) g is unfolded, s.n.c., and one-to-one,
- (ii) $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{f_1\}$,
- (iii) $f_1 = g_{\operatorname{len} g}$,
- (iv) for every natural number i such that $1 \leqslant i$ and $i + 2 \leqslant \operatorname{len} f$ holds $\mathcal{L}(f, i) \cap \mathcal{L}(f_{\operatorname{len} f}, g_1) = \emptyset$, and
- (v) for every natural number i such that $2 \leqslant i$ and $i + 1 \leqslant \operatorname{len} g$ holds $\mathcal{L}(g, i) \cap \mathcal{L}(f_{\operatorname{len} f}, g_1) = \emptyset$.

Then $f \frown g$ is s.c.c..

(42) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Then there exists a natural number i such that $1 \leqslant i$ and $i + 1 \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $\text{W-min}\, C \in \operatorname{cell}(\operatorname{Gauge}(C, n), 1, i)$ and $\text{W-min}\, C \neq \operatorname{Gauge}(C, n) \circ (2, i)$.

(43) For every S-sequence f in $\mathbb{R}^2$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \widetilde{\mathcal{L}}(f)$ and $f(\operatorname{len} f) \in \widetilde{\mathcal{L}}(\downharpoonleft f, p)$ holds $f(\operatorname{len} f) = p$.

(44) For every non empty finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $\downharpoonleft f, p \neq \emptyset$.

(45) For every S-sequence f in $\mathbb{R}^2$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \widetilde{\mathcal{L}}(f)$ holds $(\downharpoonleft f, p)_{\operatorname{len} \downharpoonleft f, p} = p$.

(46) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n))$ and $p_{\mathbf{1}} = \text{E-bound}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$, then $p = \text{E-max}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$.

(47) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n))$ and $p_{\mathbf{1}} = \text{W-bound}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$, then $p = \text{W-min}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$.

(48) Let G be a Go-board, f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, and k be a natural number. Suppose $1 \leqslant k$ and $k < \operatorname{len} f$ and $f \frown g$ is a sequence

which elements belong to G. Then $\mathrm{left\_cell}(f \frown g, k, G) = \mathrm{left\_cell}(f, k, G)$ and $\mathrm{right\_cell}(f \frown g, k, G) = \mathrm{right\_cell}(f, k, G)$.

(49) Let D be a set, f, g be finite sequences of elements of D, and i be a natural number. If $i \leqslant \operatorname{len} f$, then $(f \frown\!\frown g){\restriction}i = f{\restriction}i$.

(50) For every set D and for all finite sequences f, g of elements of D holds $(f \frown\!\frown g){\restriction}\operatorname{len} f = f$.

(51) Let G be a Go-board, f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, and k be a natural number. Suppose $1 \leqslant k$ and $k < \operatorname{len} f$ and $f \frown\!\frown g$ is a sequence which elements belong to G. Then $\mathrm{left\_cell}(f \frown\!\frown g, k, G) = \mathrm{left\_cell}(f, k, G)$ and $\mathrm{right\_cell}(f \frown\!\frown g, k, G) = \mathrm{right\_cell}(f, k, G)$.

(52) Let G be a Go-board, f be a S-sequence in $\mathbb{R}^2$, p be a point of $\mathcal{E}^2_{\mathrm{T}}$, and k be a natural number. Suppose $1 \leqslant k$ and $k < p \leftarrow\!\!\shortmid f$ and f is a sequence which elements belong to G and $p \in \operatorname{rng} f$. Then $\mathrm{left\_cell}(\downharpoonleft f, p, k, G) = \mathrm{left\_cell}(f, k, G)$ and $\mathrm{right\_cell}(\downharpoonleft f, p, k, G) = \mathrm{right\_cell}(f, k, G)$.

(53) Let G be a Go-board, f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, p be a point of $\mathcal{E}^2_{\mathrm{T}}$, and k be a natural number. Suppose $1 \leqslant k$ and $k < p \leftarrow\!\!\shortmid f$ and f is a sequence which elements belong to G. Then $\mathrm{left\_cell}(f -: p, k, G) = \mathrm{left\_cell}(f, k, G)$ and $\mathrm{right\_cell}(f -: p, k, G) = \mathrm{right\_cell}(f, k, G)$.

(54) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that
 (i) f is unfolded, s.n.c., and one-to-one,
 (ii) g is unfolded, s.n.c., and one-to-one,
 (iii) $f_{\operatorname{len} f} = g_1$, and
 (iv) $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g_1\}$.

 Then $f \frown\!\frown g$ is s.n.c..

(55) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is one-to-one and g is one-to-one and $\operatorname{rng} f \cap \operatorname{rng} g \subseteq \{g_1\}$. Then $f \frown\!\frown g$ is one-to-one.

(56) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If f is a special sequence and $p \in \operatorname{rng} f$ and $p \neq f(1)$, then $\mathrm{Index}(p, f) + 1 = p \leftarrow\!\!\shortmid f$.

(57) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 < i$ and $i < \operatorname{len} \mathrm{Gauge}(C, n)$ and $1 \leqslant j$ and $k \leqslant \operatorname{width} \mathrm{Gauge}(C, n)$ and $\mathrm{Gauge}(C, n) \circ (i, k) \in \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C, n))$ and $\mathrm{Gauge}(C, n) \circ (i, j) \in \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C, n))$. Then $j \neq k$.

(58) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \operatorname{len} \mathrm{Gauge}(C, n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \mathrm{Gauge}(C, n)$ and $\mathcal{L}(\mathrm{Gauge}(C, n) \circ (i, j), \mathrm{Gauge}(C, n) \circ (i, k)) \cap \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C, n)) = \{\mathrm{Gauge}(C, n) \circ (i, k)\}$ and $\mathcal{L}(\mathrm{Gauge}(C, n) \circ (i, j), \mathrm{Gauge}(C, n) \circ (i, k)) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C, n)) = \{\mathrm{Gauge}(C, n) \circ (i, j)\}$. Then $\mathcal{L}(\mathrm{Gauge}(C, n) \circ (i, j), \mathrm{Gauge}(C, n) \circ (i, k))$ meets $\mathrm{LowerArc}\, C$.

(59) Let C be a simple closed curve and i, j, k be natural numbers.

Suppose $1 < i$ and $i < \operatorname{len} \mathrm{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \mathrm{Gauge}(C,n)$ and $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i,j), \mathrm{Gauge}(C,n) \circ (i,k)) \cap \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i,k)\}$ and $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i,j), \mathrm{Gauge}(C,n) \circ (i,k)) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i,j)\}$. Then $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i,j), \mathrm{Gauge}(C,n) \circ (i,k))$ meets $\mathrm{UpperArc}\, C$.

(60) Let C be a simple closed curve and i, j, k be natural numbers. Suppose that $1 < i$ and $i < \operatorname{len} \mathrm{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \mathrm{Gauge}(C,n)$ and $n > 0$ and $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i,j), \mathrm{Gauge}(C,n) \circ (i,k)) \cap \mathrm{UpperArc}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i,k)\}$ and $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i,j), \mathrm{Gauge}(C,n) \circ (i,k)) \cap \mathrm{LowerArc}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i,j)\}$. Then $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i,j), \mathrm{Gauge}(C,n) \circ (i,k))$ meets $\mathrm{LowerArc}\, C$.

(61) Let C be a simple closed curve and i, j, k be natural numbers. Suppose that $1 < i$ and $i < \operatorname{len} \mathrm{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \mathrm{Gauge}(C,n)$ and $n > 0$ and $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i,j), \mathrm{Gauge}(C,n) \circ (i,k)) \cap \mathrm{UpperArc}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i,k)\}$ and $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i,j), \mathrm{Gauge}(C,n) \circ (i,k)) \cap \mathrm{LowerArc}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i,j)\}$. Then $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i,j), \mathrm{Gauge}(C,n) \circ (i,k))$ meets $\mathrm{UpperArc}\, C$.

(62) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and j be a natural number. Suppose $\mathrm{Gauge}(C,n+1) \circ (\mathrm{Center}\, \mathrm{Gauge}(C,n+1), j) \in \mathrm{UpperArc}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n+1))$ and $1 \leqslant j$ and $j \leqslant \operatorname{width} \mathrm{Gauge}(C,n+1)$. Then $\mathcal{L}(\mathrm{Gauge}(C,1) \circ (\mathrm{Center}\, \mathrm{Gauge}(C,1),1), \mathrm{Gauge}(C,n+1) \circ (\mathrm{Center}\, \mathrm{Gauge}(C,n+1),j))$ meets $\mathrm{LowerArc}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n+1))$.

(63) Let C be a simple closed curve and j, k be natural numbers. Suppose that

(i) $1 \leqslant j$,

(ii) $j \leqslant k$,

(iii) $k \leqslant \operatorname{width} \mathrm{Gauge}(C,n+1)$,

(iv) $\mathcal{L}(\mathrm{Gauge}(C,n+1) \circ (\mathrm{Center}\, \mathrm{Gauge}(C,n+1),j), \mathrm{Gauge}(C,n+1) \circ (\mathrm{Center}\, \mathrm{Gauge}(C,n+1),k)) \cap \mathrm{UpperArc}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n+1)) = \{\mathrm{Gauge}(C,n+1) \circ (\mathrm{Center}\, \mathrm{Gauge}(C,n+1),k)\}$, and

(v) $\mathcal{L}(\mathrm{Gauge}(C,n+1) \circ (\mathrm{Center}\, \mathrm{Gauge}(C,n+1),j), \mathrm{Gauge}(C,n+1) \circ (\mathrm{Center}\, \mathrm{Gauge}(C,n+1),k)) \cap \mathrm{LowerArc}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n+1)) = \{\mathrm{Gauge}(C,n+1) \circ (\mathrm{Center}\, \mathrm{Gauge}(C,n+1),j)\}$.

Then $\mathcal{L}(\mathrm{Gauge}(C,n+1) \circ (\mathrm{Center}\, \mathrm{Gauge}(C,n+1),j), \mathrm{Gauge}(C,n+1) \circ (\mathrm{Center}\, \mathrm{Gauge}(C,n+1),k))$ meets $\mathrm{LowerArc}\, C$.

(64) Let C be a simple closed curve and j, k be natural numbers. Suppose that

(i) $1 \leqslant j$,

(ii) $j \leqslant k$,

(iii) $k \leqslant \operatorname{width} \operatorname{Gauge}(C, n+1)$,

(iv) $\mathcal{L}(\operatorname{Gauge}(C, n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C, n+1), j), \operatorname{Gauge}(C, n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C, n+1), k)) \cap \operatorname{UpperArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n+1)) = \{\operatorname{Gauge}(C, n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C, n+1), k)\}$, and

(v) $\mathcal{L}(\operatorname{Gauge}(C, n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C, n+1), j), \operatorname{Gauge}(C, n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C, n+1), k)) \cap \operatorname{LowerArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n+1)) = \{\operatorname{Gauge}(C, n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C, n+1), j)\}$.

Then $\mathcal{L}(\operatorname{Gauge}(C, n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C, n+1), j), \operatorname{Gauge}(C, n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C, n+1), k))$ meets $\operatorname{UpperArc} C$.

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Fibonacci Numbers

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Summary. We show that Fibonacci commutes with g.c.d.; we then derive the formula connecting the Fibonacci sequence with the roots of the polynomial $x^2 - x - 1$.

MML Identifier: `FIB_NUM`.

The terminology and notation used here are introduced in the following articles: [3], [9], [5], [1], [2], [4], [7], [6], and [8].

1. Fibonacci Commutes with gcd

One can prove the following three propositions:

(1) For all natural numbers m, n holds $\gcd(m, n) = \gcd(m, n + m)$.

(2) For all natural numbers k, m, n such that $\gcd(k, m) = 1$ holds $\gcd(k, m \cdot n) = \gcd(k, n)$.

(3) For every real number s such that $s > 0$ there exists a natural number n such that $n > 0$ and $0 < \frac{1}{n}$ and $\frac{1}{n} \leqslant s$.

In this article we present several logical schemes. The scheme *Fib Ind* concerns a unary predicate $\mathcal{P}$, and states that:

For every natural number k holds $\mathcal{P}[k]$

provided the following conditions are met:

- $\mathcal{P}[0]$,
- $\mathcal{P}[1]$, and
- For every natural number k such that $\mathcal{P}[k]$ and $\mathcal{P}[k + 1]$ holds $\mathcal{P}[k + 2]$.

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The scheme *Bin Ind* concerns a binary predicate $\mathcal{P}$, and states that:

For all natural numbers m, n holds $\mathcal{P}[m, n]$

provided the parameters satisfy the following conditions:

- For all natural numbers m, n such that $\mathcal{P}[m, n]$ holds $\mathcal{P}[n, m]$, and
- Let k be a natural number. Suppose that for all natural numbers m, n such that $m < k$ and $n < k$ holds $\mathcal{P}[m, n]$. Let m be a natural number. If $m \leqslant k$, then $\mathcal{P}[k, m]$.

We now state two propositions:

(4) For all natural numbers m, n holds $\mathrm{Fib}(m+(n+1)) = \mathrm{Fib}(n) \cdot \mathrm{Fib}(m) + \mathrm{Fib}(n+1) \cdot \mathrm{Fib}(m+1)$.

(5) For all natural numbers m, n holds $\gcd(\mathrm{Fib}(m), \mathrm{Fib}(n)) = \mathrm{Fib}(\gcd(m, n))$.

2. Fibonacci Numbers and the Golden Mean

Next we state the proposition

(6) Let x, a, b, c be real numbers. Suppose $a \neq 0$ and $\Delta(a, b, c) \geqslant 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c = 0$ if and only if $x = \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $x = \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

The real number τ is defined by:

(Def. 1) $\tau = \frac{1+\sqrt{5}}{2}$.

The real number $\overline{\tau}$ is defined as follows:

(Def. 2) $\overline{\tau} = \frac{1-\sqrt{5}}{2}$.

One can prove the following propositions:

(7) For every natural number n holds $\mathrm{Fib}(n) = \frac{\tau^n - \overline{\tau}^n}{\sqrt{5}}$.

(8) For every natural number n holds $|\mathrm{Fib}(n) - \frac{\tau^n}{\sqrt{5}}| < 1$.

(9) For all sequences F, G of real numbers such that for every natural number n holds $F(n) = G(n)$ holds $F = G$.

(10) For all sequences f, g, h of real numbers such that g is non-zero holds $(f/g)\,(g/h) = f/h$.

(11) For all sequences f, g of real numbers and for every natural number n holds $(f/g)(n) = \frac{f(n)}{g(n)}$ and $(f/g)(n) = f(n) \cdot g(n)^{-1}$.

(12) Let F be a sequence of real numbers. Suppose that for every natural number n holds $F(n) = \frac{\mathrm{Fib}(n+1)}{\mathrm{Fib}(n)}$. Then F is convergent and $\lim F = \tau$.

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Preparing the Internal Approximations of Simple Closed Curves[1]

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Summary. We mean by an internal approximation of a simple closed curve a special polygon disjoint with it but sufficiently close to it, i.e. such that it is clock-wise oriented and its right cells meet the curve. We prove lemmas used in the next article to construct a sequence of internal approximations.

MML Identifier: `JORDAN11`.

The articles [18], [5], [20], [11], [1], [16], [2], [21], [4], [3], [12], [17], [7], [8], [9], [10], [13], [14], [15], [6], and [19] provide the terminology and notation for this paper.

In this paper j, k, n are natural numbers and C is a subset of $\mathcal{E}^2_{\rm T}$ satisfying conditions of simple closed curve.

Let us consider C. The functor $\mathrm{ApproxIndex}\, C$ yielding a natural number is defined by:

(Def. 1) $\mathrm{ApproxIndex}\, C$ is sufficiently large for C and for every j such that j is sufficiently large for C holds $j \geqslant \mathrm{ApproxIndex}\, C$.

Next we state the proposition

(1) $\mathrm{ApproxIndex}\, C \geqslant 1$.

Let us consider C. The functor $\mathrm{Y\text{-}InitStart}\, C$ yields a natural number and is defined as follows:

(Def. 2) $\mathrm{Y\text{-}InitStart}\, C < \mathrm{width}\, \mathrm{Gauge}(C, \mathrm{ApproxIndex}\, C)$ and $\mathrm{cell}(\mathrm{Gauge}(C, \mathrm{ApproxIndex}\, C), \mathrm{X\text{-}SpanStart}(C, \mathrm{ApproxIndex}\, C) -' 1, \mathrm{Y\text{-}InitStart}\, C) \subseteq \mathrm{BDD}\, C$ and for every j such that $j < \mathrm{width}\, \mathrm{Gauge}(C, \mathrm{ApproxIndex}\, C)$ and $\mathrm{cell}(\mathrm{Gauge}(C, \mathrm{ApproxIndex}\, C), \mathrm{X\text{-}SpanStart}(C, \mathrm{ApproxIndex}\, C) -' 1, j) \subseteq \mathrm{BDD}\, C$ holds $j \geqslant \mathrm{Y\text{-}InitStart}\, C$.

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The following propositions are true:

(2) Y-InitStart $C > 1$.

(3) Y-InitStart $C + 1 <$ width Gauge(C, ApproxIndex C).

Let us consider C, n. Let us assume that n is sufficiently large for C. The functor Y-SpanStart(C, n) yields a natural number and is defined by the conditions (Def. 3).

(Def. 3)(i) Y-SpanStart$(C, n) \leqslant$ width Gauge(C, n),

(ii) for every k such that Y-SpanStart$(C, n) \leqslant k$ and $k \leqslant 2^{n -' \mathrm{ApproxIndex}\, C} \cdot$ (Y-InitStart $C -' 2) + 2$ holds cell(Gauge(C, n), X-SpanStart$(C, n) -' 1, k) \subseteq$ BDD C, and

(iii) for every j such that $j \leqslant$ width Gauge(C, n) and for every k such that $j \leqslant k$ and $k \leqslant 2^{n -' \mathrm{ApproxIndex}\, C} \cdot$ (Y-InitStart $C -' 2) + 2$ holds cell(Gauge(C, n), X-SpanStart$(C, n) -' 1, k) \subseteq$ BDD C holds $j \geqslant$ Y-SpanStart(C, n).

One can prove the following propositions:

(4) If n is sufficiently large for C, then X-SpanStart$(C, n) = 2^{n -' \mathrm{ApproxIndex}\, C} \cdot$ (X-SpanStart(C, ApproxIndex C) $- 2) + 2$.

(5) If n is sufficiently large for C, then Y-SpanStart$(C, n) \leqslant 2^{n -' \mathrm{ApproxIndex}\, C} \cdot$ (Y-InitStart $C -' 2) + 2$.

(6) If n is sufficiently large for C, then cell(Gauge(C, n), X-SpanStart$(C, n) -' 1$, Y-SpanStart$(C, n)) \subseteq$ BDD C.

(7) If n is sufficiently large for C, then $1 <$ Y-SpanStart(C, n) and Y-SpanStart$(C, n) \leqslant$ width Gauge(C, n).

(8) If n is sufficiently large for C, then $\langle$X-SpanStart(C, n), Y-SpanStart$(C, n)\rangle \in$ the indices of Gauge(C, n).

(9) If n is sufficiently large for C, then $\langle$X-SpanStart$(C, n) -' 1$, Y-SpanStart$(C, n)\rangle \in$ the indices of Gauge(C, n).

(10) If n is sufficiently large for C, then cell(Gauge(C, n), X-SpanStart$(C, n) -' 1$, Y-SpanStart$(C, n) -' 1)$ meets C.

(11) If n is sufficiently large for C, then cell(Gauge(C, n), X-SpanStart$(C, n) -' 1$, Y-SpanStart$(C, n))$ misses C.

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On the General Position of Special Polygons[1]

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Summary. In this paper we introduce the notion of general position. We also show some auxiliary theorems for proving Jordan curve theorem. The following main theorems are proved:

1. End points of a polygon are in the same component of a complement of another polygon if number of common points of these polygons is even;
2. Two points of polygon L are in the same component of a complement of polygon M if two points of polygon M are in the same component of polygon L.

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The papers [23], [6], [26], [20], [2], [18], [22], [16], [27], [1], [8], [5], [3], [25], [11], [4], [21], [19], [9], [10], [14], [15], [12], [13], [17], [24], and [7] provide the terminology and notation for this paper.

1. Preliminaries

We adopt the following rules: i, j, k, n denote natural numbers, a, b, c, x denote sets, and r denotes a real number.

The following four propositions are true:

(1) If $1 < i$, then $0 < i -' 1$.

(2) If $1 \leqslant i$, then $i -' 1 < i$.

(3) 1 is odd.

(4) Let given n, f be a finite sequence of elements of $\mathcal{E}^n_{\mathrm{T}}$, and given i. If $1 \leqslant i$ and $i + 1 \leqslant \operatorname{len} f$, then $f_i \in \operatorname{rng} f$ and $f_{i+1} \in \operatorname{rng} f$.

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Let us mention that every finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ which is s.n.c. is also s.c.c..

Next we state two propositions:

(5) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $f \frown g$ is unfolded and s.c.c. and $\operatorname{len} g \geqslant 2$, then f is unfolded and s.n.c..

(6) For all finite sequences g_1, g_2 of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\widetilde{\mathcal{L}}(g_1) \subseteq \widetilde{\mathcal{L}}(g_1 \frown g_2)$.

2. The Notion of General Position and Its Properties

Let us consider n and let f_1, f_2 be finite sequences of elements of $\mathcal{E}^n_{\mathrm{T}}$. We say that f_1 is in general position wrt f_2 if and only if:

(Def. 1) $\widetilde{\mathcal{L}}(f_1)$ misses $\operatorname{rng} f_2$ and for every i such that $1 \leqslant i$ and $i < \operatorname{len} f_2$ holds $\widetilde{\mathcal{L}}(f_1) \cap \mathcal{L}(f_2, i)$ is trivial.

Let us consider n and let f_1, f_2 be finite sequences of elements of $\mathcal{E}^n_{\mathrm{T}}$. We say that f_1 and f_2 are in general position if and only if:

(Def. 2) f_1 is in general position wrt f_2 and f_2 is in general position wrt f_1.

Let us note that the predicate f_1 and f_2 are in general position is symmetric.

The following propositions are true:

(7) Let f_1, f_2 be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f_1 and f_2 are in general position. Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $f = f_2 \restriction \operatorname{Seg} k$, then f_1 and f are in general position.

(8) Let f_1, f_2, g_1, g_2 be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $f_1 \frown f_2$ and $g_1 \frown g_2$ are in general position. Then $f_1 \frown f_2$ and g_1 are in general position.

In the sequel f, g are finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$.

The following propositions are true:

(9) For all k, f, g such that $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} g$ and f and g are in general position holds $g(k) \in (\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$ and $g(k+1) \in (\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$.

(10) Let f_1, f_2 be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f_1 and f_2 are in general position. Let given i, j. If $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f_1$ and $1 \leqslant j$ and $j+1 \leqslant \operatorname{len} f_2$, then $\mathcal{L}(f_1, i) \cap \mathcal{L}(f_2, j)$ is trivial.

(11) For all f, g holds $\{\mathcal{L}(f,i) : 1 \leqslant i \ \wedge \ i+1 \leqslant \operatorname{len} f\} \Cap \{\mathcal{L}(g,j) : 1 \leqslant j \ \wedge \ j+1 \leqslant \operatorname{len} g\}$ is finite.

(12) For all f, g such that f and g are in general position holds $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g)$ is finite.

(13) For all f, g such that f and g are in general position and for every k holds $\widetilde{\mathcal{L}}(f) \cap \mathcal{L}(g, k)$ is finite.

3. Properties of Being in the Same Component of a Complement of a Polygon

We use the following convention: f is a non constant standard special circular sequence, g is a special finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and p, p_1, p_2, q are points of $\mathcal{E}^2_{\mathrm{T}}$.

One can prove the following propositions:

(14) For all f, p_1, p_2 such that $\mathcal{L}(p_1, p_2)$ misses $\widetilde{\mathcal{L}}(f)$ there exists a subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$ and $p_1 \in C$ and $p_2 \in C$.

(15) There exists a subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$ and $a \in C$ and $b \in C$ if and only if $a \in \mathrm{RightComp}(f)$ and $b \in \mathrm{RightComp}(f)$ or $a \in \mathrm{LeftComp}(f)$ and $b \in \mathrm{LeftComp}(f)$.

(16) $a \in (\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$ and $b \in (\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$ and it is not true that there exists a subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$ and $a \in C$ and $b \in C$ if and only if $a \in \mathrm{LeftComp}(f)$ and $b \in \mathrm{RightComp}(f)$ or $a \in \mathrm{RightComp}(f)$ and $b \in \mathrm{LeftComp}(f)$.

(17) Let given f, a, b, c. Suppose that

(i) there exists a subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$ and $a \in C$ and $b \in C$, and

(ii) there exists a subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$ and $b \in C$ and $c \in C$.

Then there exists a subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$ and $a \in C$ and $c \in C$.

(18) Let given f, a, b, c. Suppose that

(i) $a \in (\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$,

(ii) $b \in (\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$,

(iii) $c \in (\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$,

(iv) it is not true that there exists a subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$ and $a \in C$ and $b \in C$, and

(v) it is not true that there exists a subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$ and $b \in C$ and $c \in C$.

Then there exists a subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$ and $a \in C$ and $c \in C$.

4. Cells Are Convex

In the sequel G denotes a Go-board.

One can prove the following propositions:

(19) If $i \leqslant \mathrm{len}\, G$, then $\mathrm{vstrip}(G, i)$ is convex.

(20) If $j \leqslant \mathrm{width}\, G$, then $\mathrm{hstrip}(G, j)$ is convex.

(21) If $i \leqslant \operatorname{len} G$ and $j \leqslant \operatorname{width} G$, then $\operatorname{cell}(G, i, j)$ is convex.

(22) For all f, k such that $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$ holds $\operatorname{leftcell}(f, k)$ is convex.

(23) For all f, k such that $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$ holds left_cell(f, k, the Go-board of f) is convex and right_cell(f, k, the Go-board of f) is convex.

5. Properties of Points Lying on the Same Line

The following propositions are true:

(24) Let given p_1, p_2, f and r be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $r \in \mathcal{L}(p_1, p_2)$ and there exists x such that $\widetilde{\mathcal{L}}(f) \cap \mathcal{L}(p_1, p_2) = \{x\}$ and $r \notin \widetilde{\mathcal{L}}(f)$. Then $\widetilde{\mathcal{L}}(f)$ misses $\mathcal{L}(p_1, r)$ or $\widetilde{\mathcal{L}}(f)$ misses $\mathcal{L}(r, p_2)$.

(25) For all points p, q, r, s of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathcal{L}(p, q)$ is vertical and $\mathcal{L}(r, s)$ is vertical and $\mathcal{L}(p, q)$ meets $\mathcal{L}(r, s)$ holds $p_{\mathbf{1}} = r_{\mathbf{1}}$.

(26) For all p, p_1, p_2 such that $p \notin \mathcal{L}(p_1, p_2)$ and $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} = p_{\mathbf{2}}$ holds $p_1 \in \mathcal{L}(p, p_2)$ or $p_2 \in \mathcal{L}(p, p_1)$.

(27) For all p, p_1, p_2 such that $p \notin \mathcal{L}(p_1, p_2)$ and $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} = p_{\mathbf{1}}$ holds $p_1 \in \mathcal{L}(p, p_2)$ or $p_2 \in \mathcal{L}(p, p_1)$.

(28) If $p \neq p_1$ and $p \neq p_2$ and $p \in \mathcal{L}(p_1, p_2)$, then $p_1 \notin \mathcal{L}(p, p_2)$.

(29) Let given p, p_1, p_2, q. Suppose $q \notin \mathcal{L}(p_1, p_2)$ and $p \in \mathcal{L}(p_1, p_2)$ and $p \neq p_1$ and $p \neq p_2$ and $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} = q_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} = q_{\mathbf{2}}$. Then $p_1 \in \mathcal{L}(q, p)$ or $p_2 \in \mathcal{L}(q, p)$.

(30) Let p_1, p_2, p_3, p_4, p be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} = (p_4)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} = (p_4)_{\mathbf{2}}$ but $\mathcal{L}(p_1, p_2) \cap \mathcal{L}(p_3, p_4) = \{p\}$. Then $p = p_1$ or $p = p_2$ or $p = p_3$.

6. The Position of the Points of a Polygon with Respect to Another Polygon

We now state several propositions:

(31) Let given p, p_1, p_2, f. Suppose $\widetilde{\mathcal{L}}(f) \cap \mathcal{L}(p_1, p_2) = \{p\}$. Let r be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) $r \notin \mathcal{L}(p_1, p_2)$,

(ii) $p_1 \notin \widetilde{\mathcal{L}}(f)$,

(iii) $p_2 \notin \widetilde{\mathcal{L}}(f)$,

(iv) $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} = r_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} = r_{\mathbf{2}}$,

(v) there exists i such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and $r \in$ right_cell(f, i, the Go-board of f) or $r \in$ left_cell(f, i, the Go-board of f) and $p \in \mathcal{L}(f, i)$, and

(vi) $r \notin \widetilde{\mathcal{L}}(f)$.

Then

(vii) there exists a subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$ and $r \in C$ and $p_1 \in C$, or

(viii) there exists a subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$ and $r \in C$ and $p_2 \in C$.

(32) Let given f, p_1, p_2, p. Suppose $\widetilde{\mathcal{L}}(f) \cap \mathcal{L}(p_1, p_2) = \{p\}$. Let r_1, r_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) $p_1 \notin \widetilde{\mathcal{L}}(f)$,

(ii) $p_2 \notin \widetilde{\mathcal{L}}(f)$,

(iii) $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} = (r_1)_{\mathbf{1}}$ and $(r_1)_{\mathbf{1}} = (r_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} = (r_1)_{\mathbf{2}}$ and $(r_1)_{\mathbf{2}} = (r_2)_{\mathbf{2}}$,

(iv) there exists i such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and $r_1 \in \text{left\_cell}(f, i, \text{the Go-board of } f)$ and $r_2 \in \text{right\_cell}(f, i, \text{the Go-board of } f)$ and $p \in \mathcal{L}(f, i)$,

(v) $r_1 \notin \widetilde{\mathcal{L}}(f)$, and

(vi) $r_2 \notin \widetilde{\mathcal{L}}(f)$.

Then it is not true that there exists a subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$ and $p_1 \in C$ and $p_2 \in C$.

(33) Let given p, f, p_1, p_2. Suppose $\widetilde{\mathcal{L}}(f) \cap \mathcal{L}(p_1, p_2) = \{p\}$ and $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ and $p_1 \notin \widetilde{\mathcal{L}}(f)$ and $p_2 \notin \widetilde{\mathcal{L}}(f)$ and $\operatorname{rng} f$ misses $\mathcal{L}(p_1, p_2)$. Then it is not true that there exists a subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$ and $p_1 \in C$ and $p_2 \in C$.

(34) Let f be a non constant standard special circular sequence and g be a special finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f and g are in general position. Let given k. Suppose $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} g$. Then $\overline{\overline{\widetilde{\mathcal{L}}(f) \cap \mathcal{L}(g, k)}}$ is an even natural number if and only if there exists a subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$ and $g(k) \in C$ and $g(k+1) \in C$.

(35) Let f_1, f_2, g_1 be special finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) $f_1 \frown f_2$ is a non constant standard special circular sequence,

(ii) $f_1 \frown f_2$ and g_1 are in general position,

(iii) $\operatorname{len} g_1 \geqslant 2$, and

(iv) g_1 is unfolded and s.n.c..

Then $\overline{\overline{\widetilde{\mathcal{L}}(f_1 \frown f_2) \cap \widetilde{\mathcal{L}}(g_1)}}$ is an even natural number if and only if there exists a subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that C is a component of $(\widetilde{\mathcal{L}}(f_1 \frown f_2))^{\mathrm{c}}$ and $g_1(1) \in C$ and $g_1(\operatorname{len} g_1) \in C$.

(36) Let f_1, f_2, g_1, g_2 be special finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) $f_1 \frown f_2$ is a non constant standard special circular sequence,

(ii) $g_1 \frown g_2$ is a non constant standard special circular sequence,

(iii) $\widetilde{\mathcal{L}}(f_1)$ misses $\widetilde{\mathcal{L}}(g_2)$,

(iv) $\widetilde{\mathcal{L}}(f_2)$ misses $\widetilde{\mathcal{L}}(g_1)$, and

(v) $f_1 \frown\!\!\frown f_2$ and $g_1 \frown\!\!\frown g_2$ are in general position.

Let p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose that $f_1(1) = p_1$ and $f_1(\mathrm{len}\, f_1) = p_2$ and $g_1(1) = q_1$ and $g_1(\mathrm{len}\, g_1) = q_2$ and $(f_1)_{\mathrm{len}\, f_1} = (f_2)_1$ and $(g_1)_{\mathrm{len}\, g_1} = (g_2)_1$ and $p_1 \neq p_2$ and $q_1 \neq q_2$ and $p_1 \in \widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2)$ and $q_1 \in \widetilde{\mathcal{L}}(g_1) \cap \widetilde{\mathcal{L}}(g_2)$ and there exists a subset C of $\mathcal{E}^2_{\rm T}$ such that C is a component of $(\widetilde{\mathcal{L}}(f_1 \frown\!\!\frown f_2))^{\rm c}$ and $q_1 \in C$ and $q_2 \in C$. Then there exists a subset C of $\mathcal{E}^2_{\rm T}$ such that C is a component of $(\widetilde{\mathcal{L}}(g_1 \frown\!\!\frown g_2))^{\rm c}$ and $p_1 \in C$ and $p_2 \in C$.

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Introducing Spans[1]

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Summary. A sequence of internal approximations of simple closed curves is introduced. They are called spans.

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The notation and terminology used here are introduced in the following papers: [23], [17], [26], [2], [18], [27], [5], [4], [1], [3], [4], [25], [11], [12], [21], [7], [9], [10], [10], [12], [14], [28], [6], [7], [19], and [23].

Let C be a non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ satisfying conditions of simple closed curve and let n be a natural number. Let us assume that n is sufficiently large for C. The functor $\mathrm{Span}(C,n)$ yielding a clockwise oriented standard non constant special circular sequence is defined by the conditions (Def. 1).

(Def. 1)(i) $\mathrm{Span}(C,n)$ is a sequence which elements belong to $\mathrm{Gauge}(C,n)$,

(ii) $(\mathrm{Span}(C,n))_1 = \mathrm{Gauge}(C,n) \circ (\text{X-SpanStart}(C,n), \text{Y-SpanStart}(C,n))$,

(iii) $(\mathrm{Span}(C,n))_2 = \mathrm{Gauge}(C,n) \circ (\text{X-SpanStart}(C,n) -' 1, \text{Y-SpanStart}(C,n))$, and

(iv) for every natural number k such that $1 \leqslant k$ and $k+2 \leqslant \mathrm{len}\,\mathrm{Span}(C,n)$ holds if front_right_cell$(\mathrm{Span}(C,n),k,\mathrm{Gauge}(C,n))$ misses C and front_left_cell$(\mathrm{Span}(C,n),k,\mathrm{Gauge}(C,n))$ misses C, then $\mathrm{Span}(C,n)$ turns left k, $\mathrm{Gauge}(C,n)$ and if front_right_cell$(\mathrm{Span}(C,n),k,\mathrm{Gauge}(C,n))$ misses C and front_left_cell$(\mathrm{Span}(C,n),k,\mathrm{Gauge}(C,n))$ meets C, then $\mathrm{Span}(C,n)$ goes straight k, $\mathrm{Gauge}(C,n)$ and if front_right_cell$(\mathrm{Span}(C,n),k,\mathrm{Gauge}(C,n))$ meets C, then $\mathrm{Span}(C,n)$ turns right k, $\mathrm{Gauge}(C,n)$.

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General Fashoda Meet Theorem for Unit Circle

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Summary. Outside and inside Fashoda theorems are proven for points in general position on unit circle. Four points must be ordered in a sense of ordering for simple closed curve. For preparation of proof, the relation between the order and condition of coordinates of points on unit circle is discussed.

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The articles [11], [9], [17], [21], [3], [4], [20], [5], [10], [1], [18], [7], [8], [12], [19], [16], [6], [2], [15], [14], and [13] provide the terminology and notation for this paper.

1. Preliminaries

In this paper x, a are real numbers.

Next we state a number of propositions:

(1) If $a \geqslant 0$ and $(x-a)\cdot(x+a) \geqslant 0$, then $-a \geqslant x$ or $x \geqslant a$.

(2) If $a \leqslant 0$ and $x < a$, then $x^{\mathbf{2}} > a^{\mathbf{2}}$.

(3) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $|p| \leqslant 1$ holds $-1 \leqslant p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leqslant 1$ and $-1 \leqslant p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leqslant 1$.

(4) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $|p| \leqslant 1$ and $p_{\mathbf{1}} \neq 0$ and $p_{\mathbf{2}} \neq 0$ holds $-1 < p_{\mathbf{1}}$ and $p_{\mathbf{1}} < 1$ and $-1 < p_{\mathbf{2}}$ and $p_{\mathbf{2}} < 1$.

(5) Let a, b, d, e, r_3 be real numbers, P_1, P_2 be non empty metric structures, x be an element of the carrier of P_1, and x_2 be an element of the carrier of P_2. Suppose $d \leqslant a$ and $a \leqslant b$ and $b \leqslant e$ and $P_1 = [a, b]_{\mathrm{M}}$ and $P_2 = [d, e]_{\mathrm{M}}$ and $x = x_2$ and $x \in$ the carrier of P_1 and $x_2 \in$ the carrier of P_2. Then $\mathrm{Ball}(x, r_3) \subseteq \mathrm{Ball}(x_2, r_3)$.

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(6) Let a, b, d, e be real numbers and B be a subset of $[d,\ e]_{\mathrm{T}}$. If $d \leqslant a$ and $a \leqslant b$ and $b \leqslant e$ and $B = [a, b]$, then $[a,\ b]_{\mathrm{T}} = [d,\ e]_{\mathrm{T}} \restriction B$.

(7) For all real numbers a, b and for every subset B of $\mathbb{I}$ such that $0 \leqslant a$ and $a \leqslant b$ and $b \leqslant 1$ and $B = [a, b]$ holds $[a,\ b]_{\mathrm{T}} = \mathbb{I} \restriction B$.

(8) Let X be a topological structure, Y, Z be non empty topological structures, f be a map from X into Y, and h be a map from Y into Z. If h is a homeomorphism and f is continuous, then $h \cdot f$ is continuous.

(9) Let X, Y, Z be topological structures, f be a map from X into Y, and h be a map from Y into Z. If h is a homeomorphism and f is one-to-one, then $h \cdot f$ is one-to-one.

(10) Let X be a topological structure, S, V be non empty topological structures, B be a non empty subset of S, f be a map from X into $S \restriction B$, g be a map from S into V, and h be a map from X into V. If $h = g \cdot f$ and f is continuous and g is continuous, then h is continuous.

(11) Let a, b, d, e, s_1, s_2, t_1, t_2 be real numbers and h be a map from $[a,\ b]_{\mathrm{T}}$ into $[d,\ e]_{\mathrm{T}}$. Suppose h is a homeomorphism and $h(s_1) = t_1$ and $h(s_2) = t_2$ and $h(a) = d$ and $h(b) = e$ and $d \leqslant e$ and $t_1 \leqslant t_2$ and $s_1 \in [a, b]$ and $s_2 \in [a, b]$. Then $s_1 \leqslant s_2$.

(12) Let a, b, d, e, s_1, s_2, t_1, t_2 be real numbers and h be a map from $[a,\ b]_{\mathrm{T}}$ into $[d,\ e]_{\mathrm{T}}$. Suppose h is a homeomorphism and $h(s_1) = t_1$ and $h(s_2) = t_2$ and $h(a) = e$ and $h(b) = d$ and $e \geqslant d$ and $t_1 \geqslant t_2$ and $s_1 \in [a, b]$ and $s_2 \in [a, b]$. Then $s_1 \leqslant s_2$.

(13) For every natural number n holds $-0_{\mathcal{E}^n_{\mathrm{T}}} = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

2. Fashoda Meet Theorems for Circle in Special Case

Next we state two propositions:

(14) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $a \neq b$ and $c \neq d$ and $f(O)_{\mathbf{1}} = a$ and $c \leqslant f(O)_{\mathbf{2}}$ and $f(O)_{\mathbf{2}} \leqslant d$ and $f(I)_{\mathbf{1}} = b$ and $c \leqslant f(I)_{\mathbf{2}}$ and $f(I)_{\mathbf{2}} \leqslant d$ and $g(O)_{\mathbf{2}} = c$ and $a \leqslant g(O)_{\mathbf{1}}$ and $g(O)_{\mathbf{1}} \leqslant b$ and $g(I)_{\mathbf{2}} = d$ and $a \leqslant g(I)_{\mathbf{1}}$ and $g(I)_{\mathbf{1}} \leqslant b$ and for every point r of $\mathbb{I}$ holds $a \geqslant f(r)_{\mathbf{1}}$ or $f(r)_{\mathbf{1}} \geqslant b$ or $c \geqslant f(r)_{\mathbf{2}}$ or $f(r)_{\mathbf{2}} \geqslant d$ but $a \geqslant g(r)_{\mathbf{1}}$ or $g(r)_{\mathbf{1}} \geqslant b$ or $c \geqslant g(r)_{\mathbf{2}}$ or $g(r)_{\mathbf{2}} \geqslant d$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(15) Let f be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is continuous and one-to-one. Then there exists a map f_2 from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$ such that $f_2(0) = f(1)$ and $f_2(1) = f(0)$ and $\operatorname{rng} f_2 = \operatorname{rng} f$ and f_2 is continuous and one-to-one.

In the sequel p, q denote points of $\mathcal{E}^2_{\mathrm{T}}$.

Next we state several propositions:

(16) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, C_0, K_1, K_2, K_3, K_4 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \leqslant 1\}$ and $K_1 = \{q_1; q_1$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_1| = 1 \wedge (q_1)_{\mathbf{2}} \leqslant (q_1)_{\mathbf{1}} \wedge (q_1)_{\mathbf{2}} \geqslant -(q_1)_{\mathbf{1}}\}$ and $K_2 = \{q_2; q_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_2| = 1 \wedge (q_2)_{\mathbf{2}} \geqslant (q_2)_{\mathbf{1}} \wedge (q_2)_{\mathbf{2}} \leqslant -(q_2)_{\mathbf{1}}\}$ and $K_3 = \{q_3; q_3$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_3| = 1 \wedge (q_3)_{\mathbf{2}} \geqslant (q_3)_{\mathbf{1}} \wedge (q_3)_{\mathbf{2}} \geqslant -(q_3)_{\mathbf{1}}\}$ and $K_4 = \{q_4; q_4$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_4| = 1 \wedge (q_4)_{\mathbf{2}} \leqslant (q_4)_{\mathbf{1}} \wedge (q_4)_{\mathbf{2}} \leqslant -(q_4)_{\mathbf{1}}\}$ and $f(O) \in K_2$ and $f(I) \in K_1$ and $g(O) \in K_3$ and $g(I) \in K_4$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(17) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, C_0, K_1, K_2, K_3, K_4 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \geqslant 1\}$ and $K_1 = \{q_1; q_1$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_1| = 1 \wedge (q_1)_{\mathbf{2}} \leqslant (q_1)_{\mathbf{1}} \wedge (q_1)_{\mathbf{2}} \geqslant -(q_1)_{\mathbf{1}}\}$ and $K_2 = \{q_2; q_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_2| = 1 \wedge (q_2)_{\mathbf{2}} \geqslant (q_2)_{\mathbf{1}} \wedge (q_2)_{\mathbf{2}} \leqslant -(q_2)_{\mathbf{1}}\}$ and $K_3 = \{q_3; q_3$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_3| = 1 \wedge (q_3)_{\mathbf{2}} \geqslant (q_3)_{\mathbf{1}} \wedge (q_3)_{\mathbf{2}} \geqslant -(q_3)_{\mathbf{1}}\}$ and $K_4 = \{q_4; q_4$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_4| = 1 \wedge (q_4)_{\mathbf{2}} \leqslant (q_4)_{\mathbf{1}} \wedge (q_4)_{\mathbf{2}} \leqslant -(q_4)_{\mathbf{1}}\}$ and $f(O) \in K_2$ and $f(I) \in K_1$ and $g(O) \in K_4$ and $g(I) \in K_3$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(18) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, C_0, K_1, K_2, K_3, K_4 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \geqslant 1\}$ and $K_1 = \{q_1; q_1$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_1| = 1 \wedge (q_1)_{\mathbf{2}} \leqslant (q_1)_{\mathbf{1}} \wedge (q_1)_{\mathbf{2}} \geqslant -(q_1)_{\mathbf{1}}\}$ and $K_2 = \{q_2; q_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_2| = 1 \wedge (q_2)_{\mathbf{2}} \geqslant (q_2)_{\mathbf{1}} \wedge (q_2)_{\mathbf{2}} \leqslant -(q_2)_{\mathbf{1}}\}$ and $K_3 = \{q_3; q_3$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_3| = 1 \wedge (q_3)_{\mathbf{2}} \geqslant (q_3)_{\mathbf{1}} \wedge (q_3)_{\mathbf{2}} \geqslant -(q_3)_{\mathbf{1}}\}$ and $K_4 = \{q_4; q_4$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_4| = 1 \wedge (q_4)_{\mathbf{2}} \leqslant (q_4)_{\mathbf{1}} \wedge (q_4)_{\mathbf{2}} \leqslant -(q_4)_{\mathbf{1}}\}$ and $f(O) \in K_2$ and $f(I) \in K_1$ and $g(O) \in K_3$ and $g(I) \in K_4$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(19) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$ and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $C_0 = \{q : |q| \geqslant 1\}$ and f is continuous and one-to-one and g is continuous and one-to-one and $f(0) = [-1, 0]$ and $f(1) = [1, 0]$ and $g(1) = [0, 1]$ and $g(0) = [0, -1]$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(20) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) $C_0 = \{p : |p| \geqslant 1\}$,

(ii) $|p_1| = 1$,

(iii) $|p_2| = 1$,

(iv) $|p_3| = 1$,

(v) $|p_4| = 1$, and

(vi) there exists a map h from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ such that h is a homeomorphism

and $h^{\circ}C_0 \subseteq C_0$ and $h(p_1) = [-1, 0]$ and $h(p_2) = [0, 1]$ and $h(p_3) = [1, 0]$ and $h(p_4) = [0, -1]$.
Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_4$ and $g(1) = p_2$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

3. Properties of Fan Morphisms

The following propositions are true:

(21) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} > 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphN(q), then $p_{\mathbf{2}} > 0$.

(22) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} \geqslant 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphN(q), then $p_{\mathbf{2}} \geqslant 0$.

(23) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} \geqslant 0$ and $\frac{q_{\mathbf{1}}}{|q|} < c_1$ and $|q| \neq 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphN(q), then $p_{\mathbf{2}} \geqslant 0$ and $p_{\mathbf{1}} < 0$.

(24) Let c_1 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $(q_1)_{\mathbf{2}} \geqslant 0$ and $(q_2)_{\mathbf{2}} \geqslant 0$ and $|q_1| \neq 0$ and $|q_2| \neq 0$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < \frac{(q_2)_{\mathbf{1}}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = c_1$-FanMorphN(q_1) and $p_2 = c_1$-FanMorphN(q_2), then $\frac{(p_1)_{\mathbf{1}}}{|p_1|} < \frac{(p_2)_{\mathbf{1}}}{|p_2|}$.

(25) Let s_3 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_3$ and $s_3 < 1$ and $q_{\mathbf{1}} > 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = s_3$-FanMorphE(q), then $p_{\mathbf{1}} > 0$.

(26) Let s_3 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_3$ and $s_3 < 1$ and $q_{\mathbf{1}} \geqslant 0$ and $\frac{q_{\mathbf{2}}}{|q|} < s_3$ and $|q| \neq 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = s_3$-FanMorphE(q), then $p_{\mathbf{1}} \geqslant 0$ and $p_{\mathbf{2}} < 0$.

(27) Let s_3 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_3$ and $s_3 < 1$ and $(q_1)_{\mathbf{1}} \geqslant 0$ and $(q_2)_{\mathbf{1}} \geqslant 0$ and $|q_1| \neq 0$ and $|q_2| \neq 0$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} < \frac{(q_2)_{\mathbf{2}}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = s_3$-FanMorphE(q_1) and $p_2 = s_3$-FanMorphE(q_2), then $\frac{(p_1)_{\mathbf{2}}}{|p_1|} < \frac{(p_2)_{\mathbf{2}}}{|p_2|}$.

(28) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} < 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphS(q), then $p_{\mathbf{2}} < 0$.

(29) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} < 0$ and $\frac{q_{\mathbf{1}}}{|q|} > c_1$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphS(q), then $p_{\mathbf{2}} < 0$ and $p_{\mathbf{1}} > 0$.

(30) Let c_1 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $(q_1)_{\mathbf{2}} \leqslant 0$ and $(q_2)_{\mathbf{2}} \leqslant 0$ and $|q_1| \neq 0$ and $|q_2| \neq 0$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < \frac{(q_2)_{\mathbf{1}}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If $p_1 = c_1$-FanMorphS(q_1) and $p_2 = c_1$-FanMorphS(q_2), then $\frac{(p_1)_{\mathbf{1}}}{|p_1|} < \frac{(p_2)_{\mathbf{1}}}{|p_2|}$.

4. Order of Points on Circle

One can prove the following propositions:

(31) For every compact non empty subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{q : |q| = 1\}$ holds W-bound $P = -1$ and E-bound $P = 1$ and S-bound $P = -1$ and N-bound $P = 1$.

(32) For every compact non empty subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{q : |q| = 1\}$ holds W-min $P = [-1, 0]$.

(33) For every compact non empty subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{q : |q| = 1\}$ holds E-max $P = [1, 0]$.

(34) For every map f from $\mathcal{E}^2_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of $\mathcal{E}^2_{\rm T}$ holds $f(p) = {\rm proj1}(p)$ holds f is continuous.

(35) For every map f from $\mathcal{E}^2_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of $\mathcal{E}^2_{\rm T}$ holds $f(p) = {\rm proj2}(p)$ holds f is continuous.

(36) For every compact non empty subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q| = 1\}$ holds UpperArc $P \subseteq P$ and LowerArc $P \subseteq P$.

(37) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q| = 1\}$. Then UpperArc $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p \in P \ \wedge\ p_{\mathbf{2}} \geqslant 0\}$.

(38) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q| = 1\}$. Then LowerArc $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p \in P \ \wedge\ p_{\mathbf{2}} \leqslant 0\}$.

(39) Let a, b, d, e be real numbers. Suppose $a \leqslant b$ and $e > 0$. Then there exists a map f from $[a, b]_{\rm T}$ into $[e \cdot a + d, e \cdot b + d]_{\rm T}$ such that f is a homeomorphism and for every real number r such that $r \in [a, b]$ holds $f(r) = e \cdot r + d$.

(40) Let a, b, d, e be real numbers. Suppose $a \leqslant b$ and $e < 0$. Then there exists a map f from $[a, b]_{\rm T}$ into $[e \cdot b + d, e \cdot a + d]_{\rm T}$ such that f is a homeomorphism and for every real number r such that $r \in [a, b]$ holds $f(r) = e \cdot r + d$.

(41) There exists a map f from $\mathbb{I}$ into $[-1, 1]_{\rm T}$ such that f is a homeomorphism and for every real number r such that $r \in [0, 1]$ holds $f(r) = (-2) \cdot r + 1$ and $f(0) = 1$ and $f(1) = -1$.

(42) There exists a map f from $\mathbb{I}$ into $[-1, 1]_{\mathrm{T}}$ such that f is a homeomorphism and for every real number r such that $r \in [0,1]$ holds $f(r) = 2 \cdot r - 1$ and $f(0) = -1$ and $f(1) = 1$.

(43) Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$. Then there exists a map f from $[-1, 1]_{\mathrm{T}}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction \mathrm{LowerArc}\, P$ such that f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in \mathrm{LowerArc}\, P$ holds $f(q_{\mathbf{1}}) = q$ and $f(-1) = \text{W-min}\, P$ and $f(1) = \text{E-max}\, P$.

(44) Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$. Then there exists a map f from $[-1, 1]_{\mathrm{T}}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction \mathrm{UpperArc}\, P$ such that f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in \mathrm{UpperArc}\, P$ holds $f(q_{\mathbf{1}}) = q$ and $f(-1) = \text{W-min}\, P$ and $f(1) = \text{E-max}\, P$.

(45) Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$. Then there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction \mathrm{LowerArc}\, P$ such that

(i) f is a homeomorphism,

(ii) for all points q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ and for all real numbers r_1, r_2 such that $f(r_1) = q_1$ and $f(r_2) = q_2$ and $r_1 \in [0,1]$ and $r_2 \in [0,1]$ holds $r_1 < r_2$ iff $(q_1)_{\mathbf{1}} > (q_2)_{\mathbf{1}}$,

(iii) $f(0) = \text{E-max}\, P$, and

(iv) $f(1) = \text{W-min}\, P$.

(46) Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$. Then there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction \mathrm{UpperArc}\, P$ such that

(i) f is a homeomorphism,

(ii) for all points q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ and for all real numbers r_1, r_2 such that $f(r_1) = q_1$ and $f(r_2) = q_2$ and $r_1 \in [0,1]$ and $r_2 \in [0,1]$ holds $r_1 < r_2$ iff $(q_1)_{\mathbf{1}} < (q_2)_{\mathbf{1}}$,

(iii) $f(0) = \text{W-min}\, P$, and

(iv) $f(1) = \text{E-max}\, P$.

(47) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. If $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_2 \in \mathrm{UpperArc}\, P$ and $\mathrm{LE}(p_1, p_2, P)$, then $p_1 \in \mathrm{UpperArc}\, P$.

(48) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $p_1 \neq p_2$ and $(p_1)_{\mathbf{1}} < 0$ and $(p_2)_{\mathbf{1}} < 0$ and $(p_1)_{\mathbf{2}} < 0$ and $(p_2)_{\mathbf{2}} < 0$. Then $(p_1)_{\mathbf{1}} > (p_2)_{\mathbf{1}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$.

(49) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $p_1 \neq p_2$ and $(p_1)_{\mathbf{1}} < 0$ and $(p_2)_{\mathbf{1}} < 0$ and $(p_1)_{\mathbf{2}} \geqslant 0$ and $(p_2)_{\mathbf{2}} \geqslant 0$.

Then $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$.

(50) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $p_1 \neq p_2$ and $(p_1)_{\mathbf{2}} \geqslant 0$ and $(p_2)_{\mathbf{2}} \geqslant 0$. Then $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$.

(51) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $p_1 \neq p_2$ and $(p_1)_{\mathbf{2}} \leqslant 0$ and $(p_2)_{\mathbf{2}} \leqslant 0$ and $p_1 \neq \text{W-min}\, P$. Then $(p_1)_{\mathbf{1}} > (p_2)_{\mathbf{1}}$.

(52) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ but $(p_2)_{\mathbf{2}} \geqslant 0$ or $(p_2)_{\mathbf{1}} \geqslant 0$ but $\mathrm{LE}(p_1, p_2, P)$. Then $(p_1)_{\mathbf{2}} \geqslant 0$ or $(p_1)_{\mathbf{1}} \geqslant 0$.

(53) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $p_1 \neq p_2$ and $(p_1)_{\mathbf{1}} \geqslant 0$ and $(p_2)_{\mathbf{1}} \geqslant 0$. Then $(p_1)_{\mathbf{2}} > (p_2)_{\mathbf{2}}$.

(54) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \in P$ and $p_2 \in P$ and $(p_1)_{\mathbf{1}} < 0$ and $(p_2)_{\mathbf{1}} < 0$ and $(p_1)_{\mathbf{2}} < 0$ and $(p_2)_{\mathbf{2}} < 0$ and $(p_1)_{\mathbf{1}} \geqslant (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} \leqslant (p_2)_{\mathbf{2}}$. Then $\mathrm{LE}(p_1, p_2, P)$.

(55) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \in P$ and $p_2 \in P$ and $(p_1)_{\mathbf{1}} > 0$ and $(p_2)_{\mathbf{1}} > 0$ and $(p_1)_{\mathbf{2}} < 0$ and $(p_2)_{\mathbf{2}} < 0$ and $(p_1)_{\mathbf{1}} \geqslant (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} \geqslant (p_2)_{\mathbf{2}}$. Then $\mathrm{LE}(p_1, p_2, P)$.

(56) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \in P$ and $p_2 \in P$ and $(p_1)_{\mathbf{1}} < 0$ and $(p_2)_{\mathbf{1}} < 0$ and $(p_1)_{\mathbf{2}} \geqslant 0$ and $(p_2)_{\mathbf{2}} \geqslant 0$ and $(p_1)_{\mathbf{1}} \leqslant (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} \leqslant (p_2)_{\mathbf{2}}$. Then $\mathrm{LE}(p_1, p_2, P)$.

(57) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \in P$ and $p_2 \in P$ and $(p_1)_{\mathbf{2}} \geqslant 0$ and $(p_2)_{\mathbf{2}} \geqslant 0$ and $(p_1)_{\mathbf{1}} \leqslant (p_2)_{\mathbf{1}}$. Then $\mathrm{LE}(p_1, p_2, P)$.

(58) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \in P$ and $p_2 \in P$ and $(p_1)_{\mathbf{1}} \geqslant 0$ and $(p_2)_{\mathbf{1}} \geqslant 0$ and $(p_1)_{\mathbf{2}} \geqslant (p_2)_{\mathbf{2}}$. Then $\mathrm{LE}(p_1, p_2, P)$.

(59) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \in P$ and $p_2 \in P$ and $(p_1)_{\mathbf{2}} \leqslant 0$ and $(p_2)_{\mathbf{2}} \leqslant 0$ and $p_2 \neq \text{W-min}\, P$ and $(p_1)_{\mathbf{1}} \geqslant (p_2)_{\mathbf{1}}$. Then $\mathrm{LE}(p_1, p_2, P)$.

(60) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} \leqslant 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1\text{-FanMorphS}(q)$, then $p_{\mathbf{2}} \leqslant 0$.

(61) Let c_1 be a real number, p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$, and P be a compact

non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $q_1 = c_1$-FanMorphS(p_1) and $q_2 = c_1$-FanMorphS(p_2). Then $\mathrm{LE}(q_1, q_2, P)$.

(62) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $\mathrm{LE}(p_2, p_3, P)$ and $\mathrm{LE}(p_3, p_4, P)$ and $(p_1)_{\mathbf{1}} < 0$ and $(p_1)_{\mathbf{2}} \geqslant 0$ and $(p_2)_{\mathbf{1}} < 0$ and $(p_2)_{\mathbf{2}} \geqslant 0$ and $(p_3)_{\mathbf{1}} < 0$ and $(p_3)_{\mathbf{2}} \geqslant 0$ and $(p_4)_{\mathbf{1}} < 0$ and $(p_4)_{\mathbf{2}} \geqslant 0$. Then there exists a map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that

f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\mathbf{1}} < 0$ and $(q_1)_{\mathbf{2}} < 0$ and $(q_2)_{\mathbf{1}} < 0$ and $(q_2)_{\mathbf{2}} < 0$ and $(q_3)_{\mathbf{1}} < 0$ and $(q_3)_{\mathbf{2}} < 0$ and $(q_4)_{\mathbf{1}} < 0$ and $(q_4)_{\mathbf{2}} < 0$ and $\mathrm{LE}(q_1, q_2, P)$ and $\mathrm{LE}(q_2, q_3, P)$ and $\mathrm{LE}(q_3, q_4, P)$.

(63) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $\mathrm{LE}(p_2, p_3, P)$ and $\mathrm{LE}(p_3, p_4, P)$ and $(p_1)_{\mathbf{2}} \geqslant 0$ and $(p_2)_{\mathbf{2}} \geqslant 0$ and $(p_3)_{\mathbf{2}} \geqslant 0$ and $(p_4)_{\mathbf{2}} > 0$. Then there exists a map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that

f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\mathbf{1}} < 0$ and $(q_1)_{\mathbf{2}} \geqslant 0$ and $(q_2)_{\mathbf{1}} < 0$ and $(q_2)_{\mathbf{2}} \geqslant 0$ and $(q_3)_{\mathbf{1}} < 0$ and $(q_3)_{\mathbf{2}} \geqslant 0$ and $(q_4)_{\mathbf{1}} < 0$ and $(q_4)_{\mathbf{2}} \geqslant 0$ and $\mathrm{LE}(q_1, q_2, P)$ and $\mathrm{LE}(q_2, q_3, P)$ and $\mathrm{LE}(q_3, q_4, P)$.

(64) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $\mathrm{LE}(p_2, p_3, P)$ and $\mathrm{LE}(p_3, p_4, P)$ and $(p_1)_{\mathbf{2}} \geqslant 0$ and $(p_2)_{\mathbf{2}} \geqslant 0$ and $(p_3)_{\mathbf{2}} \geqslant 0$ and $(p_4)_{\mathbf{2}} > 0$. Then there exists a map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that

f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\mathbf{1}} < 0$ and $(q_1)_{\mathbf{2}} < 0$ and $(q_2)_{\mathbf{1}} < 0$ and $(q_2)_{\mathbf{2}} < 0$ and $(q_3)_{\mathbf{1}} < 0$ and $(q_3)_{\mathbf{2}} < 0$ and $(q_4)_{\mathbf{1}} < 0$ and $(q_4)_{\mathbf{2}} < 0$ and $\mathrm{LE}(q_1, q_2, P)$ and $\mathrm{LE}(q_2, q_3, P)$ and $\mathrm{LE}(q_3, q_4, P)$.

(65) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $\mathrm{LE}(p_2, p_3, P)$ and $\mathrm{LE}(p_3, p_4, P)$ and $(p_1)_{\mathbf{2}} \geqslant 0$ or $(p_1)_{\mathbf{1}} \geqslant 0$ and $(p_2)_{\mathbf{2}} \geqslant 0$ or $(p_2)_{\mathbf{1}} \geqslant 0$ and $(p_3)_{\mathbf{2}} \geqslant 0$ or $(p_3)_{\mathbf{1}} \geqslant 0$ and $(p_4)_{\mathbf{2}} > 0$ or $(p_4)_{\mathbf{1}} > 0$. Then there exists a map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that

f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|f(q)| = |q|$

and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\mathbf{2}} \geqslant 0$ and $(q_2)_{\mathbf{2}} \geqslant 0$ and $(q_3)_{\mathbf{2}} \geqslant 0$ and $(q_4)_{\mathbf{2}} > 0$ and $\mathrm{LE}(q_1, q_2, P)$ and $\mathrm{LE}(q_2, q_3, P)$ and $\mathrm{LE}(q_3, q_4, P)$.

(66) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $\mathrm{LE}(p_2, p_3, P)$ and $\mathrm{LE}(p_3, p_4, P)$ and $(p_1)_{\mathbf{2}} \geqslant 0$ or $(p_1)_{\mathbf{1}} \geqslant 0$ and $(p_2)_{\mathbf{2}} \geqslant 0$ or $(p_2)_{\mathbf{1}} \geqslant 0$ and $(p_3)_{\mathbf{2}} \geqslant 0$ or $(p_3)_{\mathbf{1}} \geqslant 0$ and $(p_4)_{\mathbf{2}} > 0$ or $(p_4)_{\mathbf{1}} > 0$. Then there exists a map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that
f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\mathbf{1}} < 0$ and $(q_1)_{\mathbf{2}} < 0$ and $(q_2)_{\mathbf{1}} < 0$ and $(q_2)_{\mathbf{2}} < 0$ and $(q_3)_{\mathbf{1}} < 0$ and $(q_3)_{\mathbf{2}} < 0$ and $(q_4)_{\mathbf{1}} < 0$ and $(q_4)_{\mathbf{2}} < 0$ and $\mathrm{LE}(q_1, q_2, P)$ and $\mathrm{LE}(q_2, q_3, P)$ and $\mathrm{LE}(q_3, q_4, P)$.

(67) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_4 = \text{W-min}\, P$ and $\mathrm{LE}(p_1, p_2, P)$ and $\mathrm{LE}(p_2, p_3, P)$ and $\mathrm{LE}(p_3, p_4, P)$. Then there exists a map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that
f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\mathbf{1}} < 0$ and $(q_1)_{\mathbf{2}} < 0$ and $(q_2)_{\mathbf{1}} < 0$ and $(q_2)_{\mathbf{2}} < 0$ and $(q_3)_{\mathbf{1}} < 0$ and $(q_3)_{\mathbf{2}} < 0$ and $(q_4)_{\mathbf{1}} < 0$ and $(q_4)_{\mathbf{2}} < 0$ and $\mathrm{LE}(q_1, q_2, P)$ and $\mathrm{LE}(q_2, q_3, P)$ and $\mathrm{LE}(q_3, q_4, P)$.

(68) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $\mathrm{LE}(p_2, p_3, P)$ and $\mathrm{LE}(p_3, p_4, P)$. Then there exists a map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that
f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\mathbf{1}} < 0$ and $(q_1)_{\mathbf{2}} < 0$ and $(q_2)_{\mathbf{1}} < 0$ and $(q_2)_{\mathbf{2}} < 0$ and $(q_3)_{\mathbf{1}} < 0$ and $(q_3)_{\mathbf{2}} < 0$ and $(q_4)_{\mathbf{1}} < 0$ and $(q_4)_{\mathbf{2}} < 0$ and $\mathrm{LE}(q_1, q_2, P)$ and $\mathrm{LE}(q_2, q_3, P)$ and $\mathrm{LE}(q_3, q_4, P)$.

5. General Fashoda Theorems

One can prove the following propositions:

(69) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $\mathrm{LE}(p_2, p_3, P)$ and $\mathrm{LE}(p_3, p_4, P)$ and $p_1 \neq p_2$ and $p_2 \neq p_3$ and $p_3 \neq p_4$ and $(p_1)_{\mathbf{1}} < 0$ and $(p_2)_{\mathbf{1}} < 0$ and $(p_3)_{\mathbf{1}} < 0$ and $(p_4)_{\mathbf{1}} < 0$ and

$(p_1)_{\mathbf{2}} < 0$ and $(p_2)_{\mathbf{2}} < 0$ and $(p_3)_{\mathbf{2}} < 0$ and $(p_4)_{\mathbf{2}} < 0$. Then there exists a map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ such that f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|f(q)| = |q|$ and $[-1, 0] = f(p_1)$ and $[0, 1] = f(p_2)$ and $[1, 0] = f(p_3)$ and $[0, -1] = f(p_4)$.

(70) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $\mathrm{LE}(p_2, p_3, P)$ and $\mathrm{LE}(p_3, p_4, P)$ and $p_1 \neq p_2$ and $p_2 \neq p_3$ and $p_3 \neq p_4$. Then there exists a map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ such that f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|f(q)| = |q|$ and $[-1, 0] = f(p_1)$ and $[0, 1] = f(p_2)$ and $[1, 0] = f(p_3)$ and $[0, -1] = f(p_4)$.

(71) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $\mathrm{LE}(p_2, p_3, P)$ and $\mathrm{LE}(p_3, p_4, P)$. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \leqslant 1\}$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and $\mathrm{rng}\, f \subseteq C_0$ and $\mathrm{rng}\, g \subseteq C_0$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(72) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $\mathrm{LE}(p_2, p_3, P)$ and $\mathrm{LE}(p_3, p_4, P)$. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \leqslant 1\}$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_4$ and $g(1) = p_2$ and $\mathrm{rng}\, f \subseteq C_0$ and $\mathrm{rng}\, g \subseteq C_0$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(73) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $\mathrm{LE}(p_2, p_3, P)$ and $\mathrm{LE}(p_3, p_4, P)$. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \geqslant 1\}$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_4$ and $g(1) = p_2$ and $\mathrm{rng}\, f \subseteq C_0$ and $\mathrm{rng}\, g \subseteq C_0$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(74) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $\mathrm{LE}(p_2, p_3, P)$ and $\mathrm{LE}(p_3, p_4, P)$. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \geqslant 1\}$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and $\mathrm{rng}\, f \subseteq C_0$ and $\mathrm{rng}\, g \subseteq C_0$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

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Properties of the Internal Approximation of Jordan's Curve[1]

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The articles [19], [25], [14], [10], [1], [16], [2], [3], [24], [11], [18], [9], [26], [6], [17], [7], [8], [12], [13], [20], [15], [4], [5], [21], [23], and [22] provide the notation and terminology for this paper.

One can prove the following propositions:

(1) For every non constant standard special circular sequence f holds $\mathrm{BDD}\,\widetilde{\mathcal{L}}(f) = \mathrm{RightComp}(f)$ or $\mathrm{BDD}\,\widetilde{\mathcal{L}}(f) = \mathrm{LeftComp}(f)$.

(2) For every non constant standard special circular sequence f holds $\mathrm{UBD}\,\widetilde{\mathcal{L}}(f) = \mathrm{RightComp}(f)$ or $\mathrm{UBD}\,\widetilde{\mathcal{L}}(f) = \mathrm{LeftComp}(f)$.

(3) Let G be a Go-board, f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and k be a natural number. Suppose $1 \leqslant k$ and $k+1 \leqslant \mathrm{len}\, f$ and f is a sequence which elements belong to G. Then $\mathrm{left\_cell}(f, k, G)$ is closed.

(4) Let G be a Go-board, p be a point of $\mathcal{E}^2_{\mathrm{T}}$, and i, j be natural numbers. Suppose $1 \leqslant i$ and $i+1 \leqslant \mathrm{len}\, G$ and $1 \leqslant j$ and $j+1 \leqslant \mathrm{width}\, G$. Then $p \in \mathrm{Int}\,\mathrm{cell}(G, i, j)$ if and only if the following conditions are satisfied:

(i) $(G \circ (i, j))_{\mathbf{1}} < p_{\mathbf{1}}$,

(ii) $p_{\mathbf{1}} < (G \circ (i+1, j))_{\mathbf{1}}$,

(iii) $(G \circ (i, j))_{\mathbf{2}} < p_{\mathbf{2}}$, and

(iv) $p_{\mathbf{2}} < (G \circ (i, j+1))_{\mathbf{2}}$.

(5) For every non constant standard special circular sequence f holds $\mathrm{BDD}\,\widetilde{\mathcal{L}}(f)$ is connected.

Let f be a non constant standard special circular sequence. Observe that $\mathrm{BDD}\,\widetilde{\mathcal{L}}(f)$ is connected.

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Let C be a simple closed curve and let n be a natural number. The functor $\mathrm{SpanStart}(C,n)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 1) $\mathrm{SpanStart}(C,n) = \mathrm{Gauge}(C,n)\circ(\text{X-SpanStart}(C,n), \text{Y-SpanStart}(C,n))$.

The following four propositions are true:

(6) Let C be a simple closed curve and n be a natural number. If n is sufficiently large for C, then $(\mathrm{Span}(C,n))_1 = \mathrm{SpanStart}(C,n)$.

(7) For every simple closed curve C and for every natural number n such that n is sufficiently large for C holds $\mathrm{SpanStart}(C,n) \in \mathrm{BDD}\,C$.

(8) Let C be a simple closed curve and n, k be natural numbers. Suppose n is sufficiently large for C. Suppose $1 \leqslant k$ and $k+1 \leqslant \mathrm{len}\,\mathrm{Span}(C,n)$. Then $\mathrm{right\_cell}(\mathrm{Span}(C,n), k, \mathrm{Gauge}(C,n))$ misses C and $\mathrm{left\_cell}(\mathrm{Span}(C,n), k, \mathrm{Gauge}(C,n))$ meets C.

(9) Let C be a simple closed curve and n be a natural number. If n is sufficiently large for C, then C misses $\widetilde{\mathcal{L}}(\mathrm{Span}(C,n))$.

Let C be a simple closed curve and let n be a natural number. Observe that $\overline{\mathrm{RightComp}(\mathrm{Span}(C,n))}$ is compact.

Next we state a number of propositions:

(10) Let C be a simple closed curve and n be a natural number. If n is sufficiently large for C, then C meets $\mathrm{LeftComp}(\mathrm{Span}(C,n))$.

(11) Let C be a simple closed curve and n be a natural number. If n is sufficiently large for C, then C misses $\mathrm{RightComp}(\mathrm{Span}(C,n))$.

(12) For every simple closed curve C and for every natural number n such that n is sufficiently large for C holds $C \subseteq \mathrm{LeftComp}(\mathrm{Span}(C,n))$.

(13) For every simple closed curve C and for every natural number n such that n is sufficiently large for C holds $C \subseteq \mathrm{UBD}\,\widetilde{\mathcal{L}}(\mathrm{Span}(C,n))$.

(14) For every simple closed curve C and for every natural number n such that n is sufficiently large for C holds $\mathrm{BDD}\,\widetilde{\mathcal{L}}(\mathrm{Span}(C,n)) \subseteq \mathrm{BDD}\,C$.

(15) For every simple closed curve C and for every natural number n such that n is sufficiently large for C holds $\mathrm{UBD}\,C \subseteq \mathrm{UBD}\,\widetilde{\mathcal{L}}(\mathrm{Span}(C,n))$.

(16) For every simple closed curve C and for every natural number n such that n is sufficiently large for C holds $\mathrm{RightComp}(\mathrm{Span}(C,n)) \subseteq \mathrm{BDD}\,C$.

(17) For every simple closed curve C and for every natural number n such that n is sufficiently large for C holds $\mathrm{UBD}\,C \subseteq \mathrm{LeftComp}(\mathrm{Span}(C,n))$.

(18) Let C be a simple closed curve and n be a natural number. If n is sufficiently large for C, then $\mathrm{UBD}\,C$ misses $\mathrm{BDD}\,\widetilde{\mathcal{L}}(\mathrm{Span}(C,n))$.

(19) Let C be a simple closed curve and n be a natural number. If n is sufficiently large for C, then $\mathrm{UBD}\,C$ misses $\mathrm{RightComp}(\mathrm{Span}(C,n))$.

(20) Let C be a simple closed curve, P be a subset of $\mathcal{E}^2_{\mathrm{T}}$, and n be a natural number. Suppose n is sufficiently large for C. If P is outside component

of C, then P misses $\widetilde{\mathcal{L}}(\mathrm{Span}(C,n))$.

(21) Let C be a simple closed curve and n be a natural number. If n is sufficiently large for C, then $\mathrm{UBD}\,C$ misses $\widetilde{\mathcal{L}}(\mathrm{Span}(C,n))$.

(22) For every simple closed curve C and for every natural number n such that n is sufficiently large for C holds $\widetilde{\mathcal{L}}(\mathrm{Span}(C,n)) \subseteq \mathrm{BDD}\,C$.

(23) Let C be a simple closed curve and i, j, k, n be natural numbers. Suppose n is sufficiently large for C and $1 \leqslant k$ and $k \leqslant \mathrm{len}\,\mathrm{Span}(C,n)$ and $\langle i, j\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$ and $(\mathrm{Span}(C,n))_k = \mathrm{Gauge}(C,n) \circ (i,j)$. Then $i > 1$.

(24) Let C be a simple closed curve and i, j, k, n be natural numbers. Suppose n is sufficiently large for C and $1 \leqslant k$ and $k \leqslant \mathrm{len}\,\mathrm{Span}(C,n)$ and $\langle i, j\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$ and $(\mathrm{Span}(C,n))_k = \mathrm{Gauge}(C,n) \circ (i,j)$. Then $i < \mathrm{len}\,\mathrm{Gauge}(C,n)$.

(25) Let C be a simple closed curve and i, j, k, n be natural numbers. Suppose n is sufficiently large for C and $1 \leqslant k$ and $k \leqslant \mathrm{len}\,\mathrm{Span}(C,n)$ and $\langle i, j\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$ and $(\mathrm{Span}(C,n))_k = \mathrm{Gauge}(C,n) \circ (i,j)$. Then $j > 1$.

(26) Let C be a simple closed curve and i, j, k, n be natural numbers. Suppose n is sufficiently large for C and $1 \leqslant k$ and $k \leqslant \mathrm{len}\,\mathrm{Span}(C,n)$ and $\langle i, j\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$ and $(\mathrm{Span}(C,n))_k = \mathrm{Gauge}(C,n) \circ (i,j)$. Then $j < \mathrm{width}\,\mathrm{Gauge}(C,n)$.

(27) For every simple closed curve C and for every natural number n such that n is sufficiently large for C holds $\text{Y-SpanStart}(C,n) < \mathrm{width}\,\mathrm{Gauge}(C,n)$.

(28) Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n, m be natural numbers. If $m \geqslant n$ and $n \geqslant 1$, then $\text{X-SpanStart}(C,m) = 2^{m-'n} \cdot (\text{X-SpanStart}(C,n) - 2) + 2$.

(29) Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n, m be natural numbers. Suppose $n \leqslant m$ and n is sufficiently large for C. Then m is sufficiently large for C.

(30) Let G be a Go-board, f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i, j be natural numbers. Suppose f is a sequence which elements belong to G and special and $i \leqslant \mathrm{len}\,G$ and $j \leqslant \mathrm{width}\,G$. Then $\mathrm{cell}(G,i,j) \setminus \widetilde{\mathcal{L}}(f)$ is connected.

(31) Let C be a simple closed curve and n, k be natural numbers. Suppose n is sufficiently large for C and $\text{Y-SpanStart}(C,n) \leqslant k$ and $k \leqslant 2^{n-'\mathrm{ApproxIndex}\,C} \cdot (\text{Y-InitStart}\,C -' 2) + 2$. Then $\mathrm{cell}(\mathrm{Gauge}(C,n), \text{X-SpanStart}(C,n) -' 1, k) \setminus \widetilde{\mathcal{L}}(\mathrm{Span}(C,n)) \subseteq \mathrm{BDD}\,\widetilde{\mathcal{L}}(\mathrm{Span}(C,n))$.

(32) Let C be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and n, m, i be natural numbers. If $m \leqslant n$ and $1 < i$ and $i + 1 < \mathrm{len}\,\mathrm{Gauge}(C,m)$, then $2^{n-'m} \cdot (i-2) + 2 + 1 <$

len Gauge(C, n).

(33) Let C be a simple closed curve and n, m be natural numbers. If n is sufficiently large for C and $n \leqslant m$, then RightComp(Span(C, n)) meets RightComp(Span(C, m)).

(34) Let G be a Go-board and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to G and special. Let i, j be natural numbers. If $i \leqslant \operatorname{len} G$ and $j \leqslant \operatorname{width} G$, then $\operatorname{Int} \operatorname{cell}(G, i, j) \subseteq (\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$.

(35) Let C be a simple closed curve and n, m be natural numbers. If n is sufficiently large for C and $n \leqslant m$, then $\widetilde{\mathcal{L}}(\mathrm{Span}(C, m)) \subseteq \overline{\mathrm{LeftComp}(\mathrm{Span}(C, n))}$.

(36) Let C be a simple closed curve and n, m be natural numbers. If n is sufficiently large for C and $n \leqslant m$, then RightComp(Span(C, n)) $\subseteq$ RightComp(Span(C, m)).

(37) Let C be a simple closed curve and n, m be natural numbers. If n is sufficiently large for C and $n \leqslant m$, then LeftComp(Span(C, m)) $\subseteq$ LeftComp(Span(C, n)).

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