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# Fan Homeomorphisms in the Plane

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**Summary.** We will introduce four homeomorphisms (Fan morphisms) which give spoke-like distortion to the plane. They do not change the norms of vectors and preserve halfplanes invariant. These morphisms are used to regulate placement of points on the circle.

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The articles [14], [18], [5], [7], [1], [2], [11], [12], [10], [3], [13], [4], [9], [19], [16], [17], [15], [8], and [6] provide the notation and terminology for this paper.

## 1. Preliminaries

In this paper $x$, $a$ denote real numbers and $p$, $q$ denote points of $\mathcal{E}^2_{\rm T}$.

The following propositions are true:

(1) If $|x| < a$, then $-a < x$ and $x < a$.

(2) If $a \geqslant 0$ and $(x-a)\cdot(x+a) < 0$, then $-a < x$ and $x < a$.

(3) For every real number $s_1$ such that $-1 < s_1$ and $s_1 < 1$ holds $1+s_1 > 0$ and $1-s_1 > 0$.

(4) For every real number $a$ such that $a^{\mathbf{2}} \leqslant 1$ holds $-1 \leqslant a$ and $a \leqslant 1$.

(5) For every real number $a$ such that $a^{\mathbf{2}} < 1$ holds $-1 < a$ and $a < 1$.

(6) Let $X$ be a non empty topological structure, $g$ be a map from $X$ into $\mathbb{R}^{\mathbf{1}}$, $B$ be a subset of $X$, and $a$ be a real number. If $g$ is continuous and $B = \{p; p$ ranges over points of $X$: $\pi_p g > a\}$, then $B$ is open.

(7) Let $X$ be a non empty topological structure, $g$ be a map from $X$ into $\mathbb{R}^{\mathbf{1}}$, $B$ be a subset of $X$, and $a$ be a real number. If $g$ is continuous and $B = \{p; p$ ranges over points of $X$: $\pi_p g < a\}$, then $B$ is open.




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(8) Let $f$ be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) $f$ is continuous and one-to-one,

(ii) $\mathrm{rng}\, f = \Omega_{\mathcal{E}^2_{\mathrm{T}}}$, and

(iii) for every point $p_2$ of $\mathcal{E}^2_{\mathrm{T}}$ there exists a non empty compact subset $K$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $K = f^\circ K$ and there exists a subset $V_2$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_2 \in V_2$ and $V_2$ is open and $V_2 \subseteq K$ and $f(p_2) \in V_2$.

Then $f$ is a homeomorphism.

(9) Let $X$ be a non empty topological space, $f_1$, $f_2$ be maps from $X$ into $\mathbb{R}^{\mathbf{1}}$, and $a$, $b$ be real numbers. Suppose $f_1$ is continuous and $f_2$ is continuous and $b \neq 0$ and for every point $q$ of $X$ holds $f_2(q) \neq 0$. Then there exists a map $g$ from $X$ into $\mathbb{R}^{\mathbf{1}}$ such that

(i) for every point $p$ of $X$ and for all real numbers $r_1$, $r_2$ such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = \frac{\frac{r_1}{r_2} - a}{b}$, and

(ii) $g$ is continuous.

(10) Let $X$ be a non empty topological space, $f_1$, $f_2$ be maps from $X$ into $\mathbb{R}^{\mathbf{1}}$, and $a$, $b$ be real numbers. Suppose $f_1$ is continuous and $f_2$ is continuous and $b \neq 0$ and for every point $q$ of $X$ holds $f_2(q) \neq 0$. Then there exists a map $g$ from $X$ into $\mathbb{R}^{\mathbf{1}}$ such that

(i) for every point $p$ of $X$ and for all real numbers $r_1$, $r_2$ such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_2 \cdot \frac{\frac{r_1}{r_2} - a}{b}$, and

(ii) $g$ is continuous.

(11) Let $X$ be a non empty topological space and $f_1$ be a map from $X$ into $\mathbb{R}^{\mathbf{1}}$. Suppose $f_1$ is continuous. Then there exists a map $g$ from $X$ into $\mathbb{R}^{\mathbf{1}}$ such that for every point $p$ of $X$ and for every real number $r_1$ such that $f_1(p) = r_1$ holds $g(p) = {r_1}^{\mathbf{2}}$ and $g$ is continuous.

(12) Let $X$ be a non empty topological space and $f_1$ be a map from $X$ into $\mathbb{R}^{\mathbf{1}}$. Suppose $f_1$ is continuous. Then there exists a map $g$ from $X$ into $\mathbb{R}^{\mathbf{1}}$ such that for every point $p$ of $X$ and for every real number $r_1$ such that $f_1(p) = r_1$ holds $g(p) = |r_1|$ and $g$ is continuous.

(13) Let $X$ be a non empty topological space and $f_1$ be a map from $X$ into $\mathbb{R}^{\mathbf{1}}$. Suppose $f_1$ is continuous. Then there exists a map $g$ from $X$ into $\mathbb{R}^{\mathbf{1}}$ such that for every point $p$ of $X$ and for every real number $r_1$ such that $f_1(p) = r_1$ holds $g(p) = -r_1$ and $g$ is continuous.

(14) Let $X$ be a non empty topological space, $f_1$, $f_2$ be maps from $X$ into $\mathbb{R}^{\mathbf{1}}$, and $a$, $b$ be real numbers. Suppose $f_1$ is continuous and $f_2$ is continuous and $b \neq 0$ and for every point $q$ of $X$ holds $f_2(q) \neq 0$. Then there exists a map $g$ from $X$ into $\mathbb{R}^{\mathbf{1}}$ such that

(i) for every point $p$ of $X$ and for all real numbers $r_1$, $r_2$ such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_2 \cdot -\sqrt{|1 - (\frac{\frac{r_1}{r_2} - a}{b})^{\mathbf{2}}|}$, and

(ii) $g$ is continuous.

(15) Let $X$ be a non empty topological space, $f_1$, $f_2$ be maps from $X$ into $\mathbb{R}^{\mathbf{1}}$, and $a$, $b$ be real numbers. Suppose $f_1$ is continuous and $f_2$ is continuous and $b \neq 0$ and for every point $q$ of $X$ holds $f_2(q) \neq 0$. Then there exists a map $g$ from $X$ into $\mathbb{R}^{\mathbf{1}}$ such that

(i) for every point $p$ of $X$ and for all real numbers $r_1, r_2$ such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_2 \cdot \sqrt{|1 - (\frac{\frac{r_1}{r_2} - a}{b})^{\mathbf{2}}|}$, and

(ii) $g$ is continuous.

Let $n$ be a natural number. The functor $n\,\mathrm{NormF}$ yields a function from the carrier of $\mathcal{E}^n_{\mathrm{T}}$ into the carrier of $\mathbb{R}^{\mathbf{1}}$ and is defined by:

(Def. 1) For every point $q$ of $\mathcal{E}^n_{\mathrm{T}}$ holds $n\,\mathrm{NormF}(q) = |q|$.

Next we state several propositions:

(16) For every natural number $n$ holds $\mathrm{dom}(n\,\mathrm{NormF})$ = the carrier of $\mathcal{E}^n_{\mathrm{T}}$ and $\mathrm{dom}(n\,\mathrm{NormF}) = \mathcal{R}^n$.

(18)[1] For every natural number $n$ and for all points $p$, $q$ of $\mathcal{E}^n_{\mathrm{T}}$ holds $||p| - |q|| \leqslant |p - q|$.

(19) For every natural number $n$ and for every map $f$ from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ such that $f = n\,\mathrm{NormF}$ holds $f$ is continuous.

(20) Let $n$ be a natural number, $K_0$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$, and $f$ be a map from $(\mathcal{E}^n_{\mathrm{T}}) \restriction K_0$ into $\mathbb{R}^{\mathbf{1}}$. If for every point $p$ of $(\mathcal{E}^n_{\mathrm{T}}) \restriction K_0$ holds $f(p) = n\,\mathrm{NormF}(p)$, then $f$ is continuous.

(21) Let $n$ be a natural number, $p$ be a point of $\mathcal{E}^n$, $r$ be a real number, and $B$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$. If $B = \overline{\mathrm{Ball}}(p, r)$, then $B$ is Bounded and closed.

(22) For every point $p$ of $\mathcal{E}^2$ and for every real number $r$ and for every subset $B$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $B = \overline{\mathrm{Ball}}(p, r)$ holds $B$ is compact.

## 2. Fan Morphism for West

Let $s$ be a real number and let $q$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. The functor $\mathrm{FanW}(s, q)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 2)
$$\mathrm{FanW}(s,q) = \begin{cases} |q| \cdot [-\sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s}{1-s})^{\mathbf{2}}}, \frac{\frac{q_{\mathbf{2}}}{|q|} - s}{1-s}], & \text{if } \frac{q_{\mathbf{2}}}{|q|} \geqslant s \text{ and } q_{\mathbf{1}} < 0, \\ |q| \cdot [-\sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s}{1+s})^{\mathbf{2}}}, \frac{\frac{q_{\mathbf{2}}}{|q|} - s}{1+s}], & \text{if } \frac{q_{\mathbf{2}}}{|q|} < s \text{ and } q_{\mathbf{1}} < 0, \\ q, & \text{otherwise.} \end{cases}$$

Let $s$ be a real number. The functor $s$-FanMorphW yields a function from the carrier of $\mathcal{E}^2_{\mathrm{T}}$ into the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

[1]The proposition (17) has been removed.

(Def. 3) For every point $q$ of $\mathcal{E}^2_{\rm T}$ holds $s$-FanMorphW$(q) = $ FanW$(s, q)$.

Next we state a number of propositions:

(23) Let $s_1$ be a real number. Then

(i) if $\frac{q_{\bf 2}}{|q|} \geqslant s_1$ and $q_{\bf 1} < 0$, then $s_1$-FanMorphW$(q) = [|q| \cdot -\sqrt{1 - (\frac{\frac{q_{\bf 2}}{|q|} - s_1}{1 - s_1})^{\bf 2}}, |q| \cdot \frac{\frac{q_{\bf 2}}{|q|} - s_1}{1 - s_1}]$, and

(ii) if $q_{\bf 1} \geqslant 0$, then $s_1$-FanMorphW$(q) = q$.

(24) For every real number $s_1$ such that $\frac{q_{\bf 2}}{|q|} \leqslant s_1$ and $q_{\bf 1} < 0$ holds $s_1$-FanMorphW$(q) = [|q| \cdot -\sqrt{1 - (\frac{\frac{q_{\bf 2}}{|q|} - s_1}{1 + s_1})^{\bf 2}}, |q| \cdot \frac{\frac{q_{\bf 2}}{|q|} - s_1}{1 + s_1}]$.

(25) Let $s_1$ be a real number such that $-1 < s_1$ and $s_1 < 1$. Then

(i) if $\frac{q_{\bf 2}}{|q|} \geqslant s_1$ and $q_{\bf 1} \leqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$, then $s_1$-FanMorphW$(q) = [|q| \cdot -\sqrt{1 - (\frac{\frac{q_{\bf 2}}{|q|} - s_1}{1 - s_1})^{\bf 2}}, |q| \cdot \frac{\frac{q_{\bf 2}}{|q|} - s_1}{1 - s_1}]$, and

(ii) if $\frac{q_{\bf 2}}{|q|} \leqslant s_1$ and $q_{\bf 1} \leqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$, then $s_1$-FanMorphW$(q) = [|q| \cdot -\sqrt{1 - (\frac{\frac{q_{\bf 2}}{|q|} - s_1}{1 + s_1})^{\bf 2}}, |q| \cdot \frac{\frac{q_{\bf 2}}{|q|} - s_1}{1 + s_1}]$.

(26) Let $s_1$ be a real number, $K_1$ be a non empty subset of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_1$ into $\mathbb{R}^{\bf 1}$. Suppose that

(i) $-1 < s_1$,

(ii) $s_1 < 1$,

(iii) for every point $p$ of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $f(p) = |p| \cdot \frac{\frac{p_{\bf 2}}{|p|} - s_1}{1 - s_1}$, and

(iv) for every point $q$ of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $q_{\bf 1} \leqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$.

Then $f$ is continuous.

(27) Let $s_1$ be a real number, $K_1$ be a non empty subset of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_1$ into $\mathbb{R}^{\bf 1}$. Suppose that

(i) $-1 < s_1$,

(ii) $s_1 < 1$,

(iii) for every point $p$ of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $f(p) = |p| \cdot \frac{\frac{p_{\bf 2}}{|p|} - s_1}{1 + s_1}$, and

(iv) for every point $q$ of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $q_{\bf 1} \leqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$.

Then $f$ is continuous.

(28) Let $s_1$ be a real number, $K_1$ be a non empty subset of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_1$ into $\mathbb{R}^{\bf 1}$. Suppose that

(i) $-1 < s_1$,

(ii) $s_1 < 1$,

(iii) for every point $p$ of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $f(p) = |p| \cdot -\sqrt{1 - (\frac{\frac{p_{\mathbf{2}}}{|p|} - s_1}{1 - s_1})^{\mathbf{2}}}$, and

(iv) for every point $q$ of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $q_{\mathbf{1}} \leqslant 0$ and $\frac{q_{\mathbf{2}}}{|q|} \geqslant s_1$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$.

Then $f$ is continuous.

(29) Let $s_1$ be a real number, $K_1$ be a non empty subset of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < s_1$,

(ii) $s_1 < 1$,

(iii) for every point $p$ of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $f(p) = |p| \cdot -\sqrt{1 - (\frac{\frac{p_{\mathbf{2}}}{|p|} - s_1}{1 + s_1})^{\mathbf{2}}}$, and

(iv) for every point $q$ of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $q_{\mathbf{1}} \leqslant 0$ and $\frac{q_{\mathbf{2}}}{|q|} \leqslant s_1$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$.

Then $f$ is continuous.

(30) Let $s_1$ be a real number, $K_0$, $B_0$ be subsets of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_0$ into $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphW $\restriction K_0$ and $B_0 = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $q_{\mathbf{1}} \leqslant 0 \ \wedge \ q \neq 0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : \frac{p_{\mathbf{2}}}{|p|} \geqslant s_1 \ \wedge \ p_{\mathbf{1}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $f$ is continuous.

(31) Let $s_1$ be a real number, $K_0$, $B_0$ be subsets of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_0$ into $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphW $\restriction K_0$ and $B_0 = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $q_{\mathbf{1}} \leqslant 0 \ \wedge \ q \neq 0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : \frac{p_{\mathbf{2}}}{|p|} \leqslant s_1 \ \wedge \ p_{\mathbf{1}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $f$ is continuous.

(32) For every real number $s_1$ and for every subset $K_3$ of $\mathcal{E}^2_{\rm T}$ such that $K_3 = \{p : p_{\mathbf{2}} \geqslant s_1 \cdot |p| \ \wedge \ p_{\mathbf{1}} \leqslant 0\}$ holds $K_3$ is closed.

(33) For every real number $s_1$ and for every subset $K_3$ of $\mathcal{E}^2_{\rm T}$ such that $K_3 = \{p : p_{\mathbf{2}} \leqslant s_1 \cdot |p| \ \wedge \ p_{\mathbf{1}} \leqslant 0\}$ holds $K_3$ is closed.

(34) Let $s_1$ be a real number, $K_0$, $B_0$ be subsets of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_0$ into $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphW $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{1}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $f$ is continuous.

(35) Let $s_1$ be a real number, $K_0$, $B_0$ be subsets of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_0$ into $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphW $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{1}} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $f$ is continuous.

(36) Let $B_0$ be a subset of $\mathcal{E}^2_{\rm T}$ and $K_0$ be a subset of $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{1}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $K_0$ is

closed.

(37) Let $s_1$ be a real number, $B_0$ be a subset of $\mathcal{E}^2_{\rm T}$, $K_0$ be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$, and $f$ be a map from $(\mathcal{E}^2_{\rm T})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphW $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{1}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $f$ is continuous.

(38) Let $B_0$ be a subset of $\mathcal{E}^2_{\rm T}$ and $K_0$ be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{1}} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $K_0$ is closed.

(39) Let $s_1$ be a real number, $B_0$ be a subset of $\mathcal{E}^2_{\rm T}$, $K_0$ be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$, and $f$ be a map from $(\mathcal{E}^2_{\rm T})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphW $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{1}} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $f$ is continuous.

(40) For every real number $s_1$ and for every point $p$ of $\mathcal{E}^2_{\rm T}$ holds $|s_1$-FanMorphW$(p)| = |p|$.

(41) For every real number $s_1$ and for all sets $x$, $K_0$ such that $-1 < s_1$ and $s_1 < 1$ and $x \in K_0$ and $K_0 = \{p : p_{\mathbf{1}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$ holds $s_1$-FanMorphW$(x) \in K_0$.

(42) For every real number $s_1$ and for all sets $x$, $K_0$ such that $-1 < s_1$ and $s_1 < 1$ and $x \in K_0$ and $K_0 = \{p : p_{\mathbf{1}} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$ holds $s_1$-FanMorphW$(x) \in K_0$.

(43) Let $s_1$ be a real number and $D$ be a non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $D^{\rm c} = \{0_{\mathcal{E}^2_{\rm T}}\}$. Then there exists a map $h$ from $(\mathcal{E}^2_{\rm T})\restriction D$ into $(\mathcal{E}^2_{\rm T})\restriction D$ such that $h = s_1$-FanMorphW $\restriction D$ and $h$ is continuous.

(44) Let $s_1$ be a real number. Suppose $-1 < s_1$ and $s_1 < 1$. Then there exists a map $h$ from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ such that $h = s_1$-FanMorphW and $h$ is continuous.

(45) For every real number $s_1$ such that $-1 < s_1$ and $s_1 < 1$ holds $s_1$-FanMorphW is one-to-one.

(46) For every real number $s_1$ such that $-1 < s_1$ and $s_1 < 1$ holds $s_1$-FanMorphW is a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ and rng($s_1$-FanMorphW) = the carrier of $\mathcal{E}^2_{\rm T}$.

(47) Let $s_1$ be a real number and $p_2$ be a point of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < s_1$ and $s_1 < 1$. Then there exists a non empty compact subset $K$ of $\mathcal{E}^2_{\rm T}$ such that $K = s_1$-FanMorphW$^\circ K$ and there exists a subset $V_2$ of $\mathcal{E}^2_{\rm T}$ such that $p_2 \in V_2$ and $V_2$ is open and $V_2 \subseteq K$ and $s_1$-FanMorphW$(p_2) \in V_2$.

(48) Let $s_1$ be a real number. Suppose $-1 < s_1$ and $s_1 < 1$. Then there exists a map $f$ from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ such that $f = s_1$-FanMorphW and $f$ is a homeomorphism.

(49) Let $s_1$ be a real number and $q$ be a point of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < s_1$

and $s_1 < 1$ and $q_{\mathbf{1}} < 0$ and $\frac{q_{\mathbf{2}}}{|q|} \geqslant s_1$. Let $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = s_1$-FanMorphW$(q)$, then $p_{\mathbf{1}} < 0$ and $p_{\mathbf{2}} \geqslant 0$.

(50) Let $s_1$ be a real number and $q$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $q_{\mathbf{1}} < 0$ and $\frac{q_{\mathbf{2}}}{|q|} < s_1$. Let $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = s_1$-FanMorphW$(q)$, then $p_{\mathbf{1}} < 0$ and $p_{\mathbf{2}} < 0$.

(51) Let $s_1$ be a real number and $q_1$, $q_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $(q_1)_{\mathbf{1}} < 0$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} \geqslant s_1$ and $(q_2)_{\mathbf{1}} < 0$ and $\frac{(q_2)_{\mathbf{2}}}{|q_2|} \geqslant s_1$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} < \frac{(q_2)_{\mathbf{2}}}{|q_2|}$. Let $p_1$, $p_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = s_1$-FanMorphW$(q_1)$ and $p_2 = s_1$-FanMorphW$(q_2)$, then $\frac{(p_1)_{\mathbf{2}}}{|p_1|} < \frac{(p_2)_{\mathbf{2}}}{|p_2|}$.

(52) Let $s_1$ be a real number and $q_1$, $q_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $(q_1)_{\mathbf{1}} < 0$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} < s_1$ and $(q_2)_{\mathbf{1}} < 0$ and $\frac{(q_2)_{\mathbf{2}}}{|q_2|} < s_1$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} < \frac{(q_2)_{\mathbf{2}}}{|q_2|}$. Let $p_1$, $p_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = s_1$-FanMorphW$(q_1)$ and $p_2 = s_1$-FanMorphW$(q_2)$, then $\frac{(p_1)_{\mathbf{2}}}{|p_1|} < \frac{(p_2)_{\mathbf{2}}}{|p_2|}$.

(53) Let $s_1$ be a real number and $q_1$, $q_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $(q_1)_{\mathbf{1}} < 0$ and $(q_2)_{\mathbf{1}} < 0$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} < \frac{(q_2)_{\mathbf{2}}}{|q_2|}$. Let $p_1$, $p_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = s_1$-FanMorphW$(q_1)$ and $p_2 = s_1$-FanMorphW$(q_2)$, then $\frac{(p_1)_{\mathbf{2}}}{|p_1|} < \frac{(p_2)_{\mathbf{2}}}{|p_2|}$.

(54) Let $s_1$ be a real number and $q$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $q_{\mathbf{1}} < 0$ and $\frac{q_{\mathbf{2}}}{|q|} = s_1$. Let $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = s_1$-FanMorphW$(q)$, then $p_{\mathbf{1}} < 0$ and $p_{\mathbf{2}} = 0$.

(55) For every real number $s_1$ holds $0_{\mathcal{E}^2_{\mathrm{T}}} = s_1$-FanMorphW$(0_{\mathcal{E}^2_{\mathrm{T}}})$.

## 3. Fan Morphism for North

Let $s$ be a real number and let $q$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. The functor FanN$(s,q)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 4) $$\mathrm{FanN}(s,q) = \begin{cases} |q| \cdot [\frac{\frac{q_{\mathbf{1}}}{|q|}-s}{1-s}, \sqrt{1-(\frac{\frac{q_{\mathbf{1}}}{|q|}-s}{1-s})^{\mathbf{2}}}], & \text{if } \frac{q_{\mathbf{1}}}{|q|} \geqslant s \text{ and } q_{\mathbf{2}} > 0, \\ |q| \cdot [\frac{\frac{q_{\mathbf{1}}}{|q|}-s}{1+s}, \sqrt{1-(\frac{\frac{q_{\mathbf{1}}}{|q|}-s}{1+s})^{\mathbf{2}}}], & \text{if } \frac{q_{\mathbf{1}}}{|q|} < s \text{ and } q_{\mathbf{2}} > 0, \\ q, & \text{otherwise.} \end{cases}$$

Let $c$ be a real number. The functor $c$-FanMorphN yielding a function from the carrier of $\mathcal{E}^2_{\mathrm{T}}$ into the carrier of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 5) For every point $q$ of $\mathcal{E}^2_{\mathrm{T}}$ holds $c$-FanMorphN$(q) = \mathrm{FanN}(c,q)$.

One can prove the following propositions:

(56) Let $c_1$ be a real number. Then

(i) if $\frac{q_{\mathbf{1}}}{|q|} \geqslant c_1$ and $q_{\mathbf{2}} > 0$, then $c_1$-FanMorphN$(q) = [|q| \cdot \frac{\frac{q_{\mathbf{1}}}{|q|} - c_1}{1 - c_1}, |q| \cdot \sqrt{1 - (\frac{\frac{q_{\mathbf{1}}}{|q|} - c_1}{1 - c_1})^{\mathbf{2}}}]$, and

(ii) if $q_{\mathbf{2}} \leqslant 0$, then $c_1$-FanMorphN$(q) = q$.

(57) For every real number $c_1$ such that $\frac{q_{\mathbf{1}}}{|q|} \leqslant c_1$ and $q_{\mathbf{2}} > 0$ holds $c_1$-FanMorphN$(q) = [|q| \cdot \frac{\frac{q_{\mathbf{1}}}{|q|} - c_1}{1 + c_1}, |q| \cdot \sqrt{1 - (\frac{\frac{q_{\mathbf{1}}}{|q|} - c_1}{1 + c_1})^{\mathbf{2}}}]$.

(58) Let $c_1$ be a real number such that $-1 < c_1$ and $c_1 < 1$. Then

(i) if $\frac{q_{\mathbf{1}}}{|q|} \geqslant c_1$ and $q_{\mathbf{2}} \geqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$, then $c_1$-FanMorphN$(q) = [|q| \cdot \frac{\frac{q_{\mathbf{1}}}{|q|} - c_1}{1 - c_1}, |q| \cdot \sqrt{1 - (\frac{\frac{q_{\mathbf{1}}}{|q|} - c_1}{1 - c_1})^{\mathbf{2}}}]$, and

(ii) if $\frac{q_{\mathbf{1}}}{|q|} \leqslant c_1$ and $q_{\mathbf{2}} \geqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$, then $c_1$-FanMorphN$(q) = [|q| \cdot \frac{\frac{q_{\mathbf{1}}}{|q|} - c_1}{1 + c_1}, |q| \cdot \sqrt{1 - (\frac{\frac{q_{\mathbf{1}}}{|q|} - c_1}{1 + c_1})^{\mathbf{2}}}]$.

(59) Let $c_1$ be a real number, $K_1$ be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and $f$ be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < c_1$,

(ii) $c_1 < 1$,

(iii) for every point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $f(p) = |p| \cdot \frac{\frac{p_{\mathbf{1}}}{|p|} - c_1}{1 - c_1}$, and

(iv) for every point $q$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $q_{\mathbf{2}} \geqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$.

Then $f$ is continuous.

(60) Let $c_1$ be a real number, $K_1$ be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and $f$ be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < c_1$,

(ii) $c_1 < 1$,

(iii) for every point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $f(p) = |p| \cdot \frac{\frac{p_{\mathbf{1}}}{|p|} - c_1}{1 + c_1}$, and

(iv) for every point $q$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $q_{\mathbf{2}} \geqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$.

Then $f$ is continuous.

(61) Let $c_1$ be a real number, $K_1$ be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and $f$ be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < c_1$,

(ii) $c_1 < 1$,

(iii) for every point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $f(p) = |p| \cdot \sqrt{1 - (\frac{\frac{p_{\mathbf{1}}}{|p|} - c_1}{1 - c_1})^{\mathbf{2}}}$, and

(iv) for every point $q$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction K_1$ holds $q_{\mathbf{2}} \geqslant 0$ and $\frac{q_{\mathbf{1}}}{|q|} \geqslant c_1$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$.

Then $f$ is continuous.

(62) Let $c_1$ be a real number, $K_1$ be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and $f$ be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < c_1$,

(ii) $c_1 < 1$,

(iii) for every point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction K_1$ holds $f(p) = |p| \cdot \sqrt{1 - (\frac{\frac{p_{\mathbf{1}}}{|p|} - c_1}{1 + c_1})^{\mathbf{2}}}$, and

(iv) for every point $q$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction K_1$ holds $q_{\mathbf{2}} \geqslant 0$ and $\frac{q_{\mathbf{1}}}{|q|} \leqslant c_1$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$.

Then $f$ is continuous.

(63) Let $c_1$ be a real number, $K_0$, $B_0$ be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and $f$ be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction K_0$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphN$\restriction K_0$ and $B_0 = \{q; q$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $q_{\mathbf{2}} \geqslant 0 \wedge q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : \frac{p_{\mathbf{1}}}{|p|} \geqslant c_1 \wedge p_{\mathbf{2}} \geqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then $f$ is continuous.

(64) Let $c_1$ be a real number, $K_0$, $B_0$ be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and $f$ be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction K_0$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphN$\restriction K_0$ and $B_0 = \{q; q$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $q_{\mathbf{2}} \geqslant 0 \wedge q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : \frac{p_{\mathbf{1}}}{|p|} \leqslant c_1 \wedge p_{\mathbf{2}} \geqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then $f$ is continuous.

(65) For every real number $c_1$ and for every subset $K_3$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $K_3 = \{p : p_{\mathbf{1}} \geqslant c_1 \cdot |p| \wedge p_{\mathbf{2}} \geqslant 0\}$ holds $K_3$ is closed.

(66) For every real number $c_1$ and for every subset $K_3$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $K_3 = \{p : p_{\mathbf{1}} \leqslant c_1 \cdot |p| \wedge p_{\mathbf{2}} \geqslant 0\}$ holds $K_3$ is closed.

(67) Let $c_1$ be a real number, $K_0$, $B_0$ be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and $f$ be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction K_0$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphN$\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\mathrm{T}}) \setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : p_{\mathbf{2}} \geqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then $f$ is continuous.

(68) Let $c_1$ be a real number, $K_0$, $B_0$ be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and $f$ be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction K_0$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphN$\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\mathrm{T}}) \setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : p_{\mathbf{2}} \leqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then $f$ is continuous.

(69) Let $B_0$ be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and $K_0$ be a subset of $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$. Suppose $B_0 =$ (the carrier of $\mathcal{E}^2_{\mathrm{T}}) \setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : p_{\mathbf{2}} \geqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then $K_0$ is closed.

(70) Let $B_0$ be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and $K_0$ be a subset of $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$. Suppose $B_0 =$ (the carrier of $\mathcal{E}^2_{\mathrm{T}}) \setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : p_{\mathbf{2}} \leqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then $K_0$ is

closed.

(71) Let $c_1$ be a real number, $B_0$ be a subset of $\mathcal{E}^2_{\mathrm{T}}$, $K_0$ be a subset of $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$, and $f$ be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphN $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\mathrm{T}}$) $\setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : p_{\mathbf{2}} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then $f$ is continuous.

(72) Let $c_1$ be a real number, $B_0$ be a subset of $\mathcal{E}^2_{\mathrm{T}}$, $K_0$ be a subset of $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$, and $f$ be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphN $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\mathrm{T}}$) $\setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : p_{\mathbf{2}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then $f$ is continuous.

(73) For every real number $c_1$ and for every point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ holds $|c_1$-FanMorphN$(p)| = |p|$.

(74) For every real number $c_1$ and for all sets $x$, $K_0$ such that $-1 < c_1$ and $c_1 < 1$ and $x \in K_0$ and $K_0 = \{p : p_{\mathbf{2}} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$ holds $c_1$-FanMorphN$(x) \in K_0$.

(75) For every real number $c_1$ and for all sets $x$, $K_0$ such that $-1 < c_1$ and $c_1 < 1$ and $x \in K_0$ and $K_0 = \{p : p_{\mathbf{2}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$ holds $c_1$-FanMorphN$(x) \in K_0$.

(76) Let $c_1$ be a real number and $D$ be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $D^{\mathrm{c}} = \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then there exists a map $h$ from $(\mathcal{E}^2_{\mathrm{T}})\restriction D$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction D$ such that $h = c_1$-FanMorphN $\restriction D$ and $h$ is continuous.

(77) Let $c_1$ be a real number. Suppose $-1 < c_1$ and $c_1 < 1$. Then there exists a map $h$ from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ such that $h = c_1$-FanMorphN and $h$ is continuous.

(78) For every real number $c_1$ such that $-1 < c_1$ and $c_1 < 1$ holds $c_1$-FanMorphN is one-to-one.

(79) For every real number $c_1$ such that $-1 < c_1$ and $c_1 < 1$ holds $c_1$-FanMorphN is a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ and rng($c_1$-FanMorphN) = the carrier of $\mathcal{E}^2_{\mathrm{T}}$.

(80) Let $c_1$ be a real number and $p_2$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$. Then there exists a non empty compact subset $K$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $K = c_1$-FanMorphN$^\circ K$ and there exists a subset $V_2$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_2 \in V_2$ and $V_2$ is open and $V_2 \subseteq K$ and $c_1$-FanMorphN$(p_2) \in V_2$.

(81) Let $c_1$ be a real number. Suppose $-1 < c_1$ and $c_1 < 1$. Then there exists a map $f$ from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ such that $f = c_1$-FanMorphN and $f$ is a homeomorphism.

(82) Let $c_1$ be a real number and $q$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} > 0$ and $\frac{q_{\mathbf{1}}}{|q|} \geqslant c_1$. Let $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphN$(q)$, then $p_{\mathbf{2}} > 0$ and $p_{\mathbf{1}} \geqslant 0$.

(83) Let $c_1$ be a real number and $q$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 <$

1 and $q_{\mathbf{2}} > 0$ and $\frac{q_{\mathbf{1}}}{|q|} < c_1$. Let $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphN$(q)$, then $p_{\mathbf{2}} > 0$ and $p_{\mathbf{1}} < 0$.

(84) Let $c_1$ be a real number and $q_1$, $q_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $(q_1)_{\mathbf{2}} > 0$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} \geqslant c_1$ and $(q_2)_{\mathbf{2}} > 0$ and $\frac{(q_2)_{\mathbf{1}}}{|q_2|} \geqslant c_1$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < \frac{(q_2)_{\mathbf{1}}}{|q_2|}$. Let $p_1$, $p_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = c_1$-FanMorphN$(q_1)$ and $p_2 = c_1$-FanMorphN$(q_2)$, then $\frac{(p_1)_{\mathbf{1}}}{|p_1|} < \frac{(p_2)_{\mathbf{1}}}{|p_2|}$.

(85) Let $c_1$ be a real number and $q_1$, $q_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $(q_1)_{\mathbf{2}} > 0$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < c_1$ and $(q_2)_{\mathbf{2}} > 0$ and $\frac{(q_2)_{\mathbf{1}}}{|q_2|} < c_1$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < \frac{(q_2)_{\mathbf{1}}}{|q_2|}$. Let $p_1$, $p_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = c_1$-FanMorphN$(q_1)$ and $p_2 = c_1$-FanMorphN$(q_2)$, then $\frac{(p_1)_{\mathbf{1}}}{|p_1|} < \frac{(p_2)_{\mathbf{1}}}{|p_2|}$.

(86) Let $c_1$ be a real number and $q_1$, $q_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $(q_1)_{\mathbf{2}} > 0$ and $(q_2)_{\mathbf{2}} > 0$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < \frac{(q_2)_{\mathbf{1}}}{|q_2|}$. Let $p_1$, $p_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = c_1$-FanMorphN$(q_1)$ and $p_2 = c_1$-FanMorphN$(q_2)$, then $\frac{(p_1)_{\mathbf{1}}}{|p_1|} < \frac{(p_2)_{\mathbf{1}}}{|p_2|}$.

(87) Let $c_1$ be a real number and $q$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} > 0$ and $\frac{q_{\mathbf{1}}}{|q|} = c_1$. Let $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphN$(q)$, then $p_{\mathbf{2}} > 0$ and $p_{\mathbf{1}} = 0$.

(88) For every real number $c_1$ holds $0_{\mathcal{E}^2_{\mathrm{T}}} = c_1$-FanMorphN$(0_{\mathcal{E}^2_{\mathrm{T}}})$.

## 4. Fan Morphism for East

Let $s$ be a real number and let $q$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. The functor FanE$(s, q)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 6) $$\mathrm{FanE}(s,q) = \begin{cases} |q| \cdot [\sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s}{1-s})^{\mathbf{2}}}, \frac{\frac{q_{\mathbf{2}}}{|q|} - s}{1-s}], \text{ if } \frac{q_{\mathbf{2}}}{|q|} \geqslant s \text{ and } q_{\mathbf{1}} > 0, \\ |q| \cdot [\sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s}{1+s})^{\mathbf{2}}}, \frac{\frac{q_{\mathbf{2}}}{|q|} - s}{1+s}], \text{ if } \frac{q_{\mathbf{2}}}{|q|} < s \text{ and } q_{\mathbf{1}} > 0, \\ q, \text{ otherwise.} \end{cases}$$

Let $s$ be a real number. The functor $s$-FanMorphE yielding a function from the carrier of $\mathcal{E}^2_{\mathrm{T}}$ into the carrier of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 7) For every point $q$ of $\mathcal{E}^2_{\mathrm{T}}$ holds $s$-FanMorphE$(q) = $ FanE$(s, q)$.

Next we state a number of propositions:

(89) Let $s_1$ be a real number. Then

(i) if $\frac{q_{\mathbf{2}}}{|q|} \geqslant s_1$ and $q_{\mathbf{1}} > 0$, then $s_1$-FanMorphE$(q) = [|q| \cdot \sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1-s_1})^{\mathbf{2}}}$, $|q| \cdot \frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1-s_1}]$, and

(ii) if $q_{\mathbf{1}} \leqslant 0$, then $s_1$-FanMorphE$(q) = q$.

(90) For every real number $s_1$ such that $\frac{q_{\mathbf{2}}}{|q|} \leqslant s_1$ and $q_{\mathbf{1}} > 0$ holds $s_1$-FanMorphE$(q) = [|q| \cdot \sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1+s_1})^{\mathbf{2}}}, |q| \cdot \frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1+s_1}]$.

(91) Let $s_1$ be a real number such that $-1 < s_1$ and $s_1 < 1$. Then

(i) if $\frac{q_{\mathbf{2}}}{|q|} \geqslant s_1$ and $q_{\mathbf{1}} \geqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$, then $s_1$-FanMorphE$(q) = [|q| \cdot \sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1-s_1})^{\mathbf{2}}}, |q| \cdot \frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1-s_1}]$, and

(ii) if $\frac{q_{\mathbf{2}}}{|q|} \leqslant s_1$ and $q_{\mathbf{1}} \geqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$, then $s_1$-FanMorphE$(q) = [|q| \cdot \sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1+s_1})^{\mathbf{2}}}, |q| \cdot \frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1+s_1}]$.

(92) Let $s_1$ be a real number, $K_1$ be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and $f$ be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < s_1$,

(ii) $s_1 < 1$,

(iii) for every point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction K_1$ holds $f(p) = |p| \cdot \frac{\frac{p_{\mathbf{2}}}{|p|} - s_1}{1-s_1}$, and

(iv) for every point $q$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction K_1$ holds $q_{\mathbf{1}} \geqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$.

Then $f$ is continuous.

(93) Let $s_1$ be a real number, $K_1$ be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and $f$ be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < s_1$,

(ii) $s_1 < 1$,

(iii) for every point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction K_1$ holds $f(p) = |p| \cdot \frac{\frac{p_{\mathbf{2}}}{|p|} - s_1}{1+s_1}$, and

(iv) for every point $q$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction K_1$ holds $q_{\mathbf{1}} \geqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$.

Then $f$ is continuous.

(94) Let $s_1$ be a real number, $K_1$ be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and $f$ be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < s_1$,

(ii) $s_1 < 1$,

(iii) for every point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction K_1$ holds $f(p) = |p| \cdot \sqrt{1 - (\frac{\frac{p_{\mathbf{2}}}{|p|} - s_1}{1-s_1})^{\mathbf{2}}}$, and

(iv) for every point $q$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction K_1$ holds $q_{\mathbf{1}} \geqslant 0$ and $\frac{q_{\mathbf{2}}}{|q|} \geqslant s_1$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$.

Then $f$ is continuous.

(95) Let $s_1$ be a real number, $K_1$ be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and $f$ be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < s_1$,
(ii) $s_1 < 1$,
(iii) for every point $p$ of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $f(p) = |p| \cdot \sqrt{1 - (\frac{\frac{p_{\bf 2}}{|p|} - s_1}{1+s_1})^{\bf 2}}$, and
(iv) for every point $q$ of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $q_{\bf 1} \geqslant 0$ and $\frac{q_{\bf 2}}{|q|} \leqslant s_1$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$.
Then $f$ is continuous.

(96) Let $s_1$ be a real number, $K_0$, $B_0$ be subsets of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_0$ into $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphE$\restriction K_0$ and $B_0 = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $q_{\bf 1} \geqslant 0 \ \wedge \ q \neq 0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : \frac{p_{\bf 2}}{|p|} \geqslant s_1 \ \wedge \ p_{\bf 1} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $f$ is continuous.

(97) Let $s_1$ be a real number, $K_0$, $B_0$ be subsets of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_0$ into $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphE$\restriction K_0$ and $B_0 = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $q_{\bf 1} \geqslant 0 \ \wedge \ q \neq 0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : \frac{p_{\bf 2}}{|p|} \leqslant s_1 \ \wedge \ p_{\bf 1} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $f$ is continuous.

(98) For every real number $s_1$ and for every subset $K_3$ of $\mathcal{E}^2_{\rm T}$ such that $K_3 = \{p : p_{\bf 2} \geqslant s_1 \cdot |p| \ \wedge \ p_{\bf 1} \geqslant 0\}$ holds $K_3$ is closed.

(99) For every real number $s_1$ and for every subset $K_3$ of $\mathcal{E}^2_{\rm T}$ such that $K_3 = \{p : p_{\bf 2} \leqslant s_1 \cdot |p| \ \wedge \ p_{\bf 1} \geqslant 0\}$ holds $K_3$ is closed.

(100) Let $s_1$ be a real number, $K_0$, $B_0$ be subsets of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_0$ into $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphE$\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\bf 1} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $f$ is continuous.

(101) Let $s_1$ be a real number, $K_0$, $B_0$ be subsets of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_0$ into $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphE$\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\bf 1} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $f$ is continuous.

(102) Let $s_1$ be a real number, $B_0$ be a subset of $\mathcal{E}^2_{\rm T}$, $K_0$ be a subset of $(\mathcal{E}^2_{\rm T}) \restriction B_0$, and $f$ be a map from $(\mathcal{E}^2_{\rm T}) \restriction B_0 \restriction K_0$ into $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphE$\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\bf 1} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $f$ is continuous.

(103) Let $s_1$ be a real number, $B_0$ be a subset of $\mathcal{E}^2_{\rm T}$, $K_0$ be a subset of $(\mathcal{E}^2_{\rm T}) \restriction B_0$, and $f$ be a map from $(\mathcal{E}^2_{\rm T}) \restriction B_0 \restriction K_0$ into $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphE$\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\bf 1} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $f$ is continuous.

(104) For every real number $s_1$ and for every point $p$ of $\mathcal{E}^2_{\rm T}$ holds $|s_1$-FanMorphE$(p)| = |p|$.

(105) For every real number $s_1$ and for all sets $x$, $K_0$ such that $-1 < s_1$ and $s_1 < 1$ and $x \in K_0$ and $K_0 = \{p : p_{\mathbf{1}} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$ holds $s_1$-FanMorphE$(x) \in K_0$.

(106) For every real number $s_1$ and for all sets $x$, $K_0$ such that $-1 < s_1$ and $s_1 < 1$ and $x \in K_0$ and $K_0 = \{p : p_{\mathbf{1}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$ holds $s_1$-FanMorphE$(x) \in K_0$.

(107) Let $s_1$ be a real number and $D$ be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $D^{\mathrm{c}} = \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then there exists a map $h$ from $(\mathcal{E}^2_{\mathrm{T}})\restriction D$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction D$ such that $h = s_1$-FanMorphE $\restriction D$ and $h$ is continuous.

(108) Let $s_1$ be a real number. Suppose $-1 < s_1$ and $s_1 < 1$. Then there exists a map $h$ from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ such that $h = s_1$-FanMorphE and $h$ is continuous.

(109) For every real number $s_1$ such that $-1 < s_1$ and $s_1 < 1$ holds $s_1$-FanMorphE is one-to-one.

(110) For every real number $s_1$ such that $-1 < s_1$ and $s_1 < 1$ holds $s_1$-FanMorphE is a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ and rng($s_1$-FanMorphE) = the carrier of $\mathcal{E}^2_{\mathrm{T}}$.

(111) Let $s_1$ be a real number and $p_2$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_1$ and $s_1 < 1$. Then there exists a non empty compact subset $K$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $K = s_1$-FanMorphE$^\circ K$ and there exists a subset $V_2$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_2 \in V_2$ and $V_2$ is open and $V_2 \subseteq K$ and $s_1$-FanMorphE$(p_2) \in V_2$.

(112) Let $s_1$ be a real number. Suppose $-1 < s_1$ and $s_1 < 1$. Then there exists a map $f$ from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ such that $f = s_1$-FanMorphE and $f$ is a homeomorphism.

(113) Let $s_1$ be a real number and $q$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $q_{\mathbf{1}} > 0$ and $\frac{q_{\mathbf{2}}}{|q|} \geqslant s_1$. Let $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = s_1$-FanMorphE$(q)$, then $p_{\mathbf{1}} > 0$ and $p_{\mathbf{2}} \geqslant 0$.

(114) Let $s_1$ be a real number and $q$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $q_{\mathbf{1}} > 0$ and $\frac{q_{\mathbf{2}}}{|q|} < s_1$. Let $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = s_1$-FanMorphE$(q)$, then $p_{\mathbf{1}} > 0$ and $p_{\mathbf{2}} < 0$.

(115) Let $s_1$ be a real number and $q_1$, $q_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $(q_1)_{\mathbf{1}} > 0$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} \geqslant s_1$ and $(q_2)_{\mathbf{1}} > 0$ and $\frac{(q_2)_{\mathbf{2}}}{|q_2|} \geqslant s_1$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} < \frac{(q_2)_{\mathbf{2}}}{|q_2|}$. Let $p_1$, $p_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = s_1$-FanMorphE$(q_1)$ and $p_2 = s_1$-FanMorphE$(q_2)$, then $\frac{(p_1)_{\mathbf{2}}}{|p_1|} < \frac{(p_2)_{\mathbf{2}}}{|p_2|}$.

(116) Let $s_1$ be a real number and $q_1$, $q_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $(q_1)_{\mathbf{1}} > 0$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} < s_1$ and $(q_2)_{\mathbf{1}} > 0$ and $\frac{(q_2)_{\mathbf{2}}}{|q_2|} < s_1$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} < \frac{(q_2)_{\mathbf{2}}}{|q_2|}$. Let $p_1$, $p_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = s_1$-FanMorphE$(q_1)$ and $p_2 = s_1$-FanMorphE$(q_2)$, then $\frac{(p_1)_{\mathbf{2}}}{|p_1|} < \frac{(p_2)_{\mathbf{2}}}{|p_2|}$.

(117) Let $s_1$ be a real number and $q_1$, $q_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $(q_1)_{\mathbf{1}} > 0$ and $(q_2)_{\mathbf{1}} > 0$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} < \frac{(q_2)_{\mathbf{2}}}{|q_2|}$. Let $p_1$, $p_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = s_1$-FanMorphE$(q_1)$ and $p_2 = s_1$-FanMorphE$(q_2)$, then $\frac{(p_1)_{\mathbf{2}}}{|p_1|} < \frac{(p_2)_{\mathbf{2}}}{|p_2|}$.

(118) Let $s_1$ be a real number and $q$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $q_{\mathbf{1}} > 0$ and $\frac{q_{\mathbf{2}}}{|q|} = s_1$. Let $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = s_1$-FanMorphE$(q)$, then $p_{\mathbf{1}} > 0$ and $p_{\mathbf{2}} = 0$.

(119) For every real number $s_1$ holds $0_{\mathcal{E}^2_{\mathrm{T}}} = s_1$-FanMorphE$(0_{\mathcal{E}^2_{\mathrm{T}}})$.

## 5. Fan Morphism for South

Let $s$ be a real number and let $q$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. The functor FanS$(s, q)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 8) $$\mathrm{FanS}(s,q) = \begin{cases} |q| \cdot [\frac{\frac{q_{\mathbf{1}}}{|q|}-s}{1-s}, -\sqrt{1-(\frac{\frac{q_{\mathbf{1}}}{|q|}-s}{1-s})^{\mathbf{2}}}], & \text{if } \frac{q_{\mathbf{1}}}{|q|} \geqslant s \text{ and } q_{\mathbf{2}} < 0, \\ |q| \cdot [\frac{\frac{q_{\mathbf{1}}}{|q|}-s}{1+s}, -\sqrt{1-(\frac{\frac{q_{\mathbf{1}}}{|q|}-s}{1+s})^{\mathbf{2}}}], & \text{if } \frac{q_{\mathbf{1}}}{|q|} < s \text{ and } q_{\mathbf{2}} < 0, \\ q, & \text{otherwise.} \end{cases}$$

Let $c$ be a real number. The functor $c$-FanMorphS yielding a function from the carrier of $\mathcal{E}^2_{\mathrm{T}}$ into the carrier of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 9) For every point $q$ of $\mathcal{E}^2_{\mathrm{T}}$ holds $c$-FanMorphS$(q) = $ FanS$(c, q)$.

One can prove the following propositions:

(120) Let $c_1$ be a real number. Then

(i) if $\frac{q_{\mathbf{1}}}{|q|} \geqslant c_1$ and $q_{\mathbf{2}} < 0$, then $c_1$-FanMorphS$(q) = [|q| \cdot \frac{\frac{q_{\mathbf{1}}}{|q|}-c_1}{1-c_1}$, $|q| \cdot -\sqrt{1-(\frac{\frac{q_{\mathbf{1}}}{|q|}-c_1}{1-c_1})^{\mathbf{2}}}]$, and

(ii) if $q_{\mathbf{2}} \geqslant 0$, then $c_1$-FanMorphS$(q) = q$.

(121) For every real number $c_1$ such that $\frac{q_{\mathbf{1}}}{|q|} \leqslant c_1$ and $q_{\mathbf{2}} < 0$ holds $c_1$-FanMorphS$(q) = [|q| \cdot \frac{\frac{q_{\mathbf{1}}}{|q|}-c_1}{1+c_1}, |q| \cdot -\sqrt{1-(\frac{\frac{q_{\mathbf{1}}}{|q|}-c_1}{1+c_1})^{\mathbf{2}}}]$.

(122) Let $c_1$ be a real number such that $-1 < c_1$ and $c_1 < 1$. Then

(i) if $\frac{q_{\mathbf{1}}}{|q|} \geqslant c_1$ and $q_{\mathbf{2}} \leqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$, then $c_1$-FanMorphS$(q) = [|q| \cdot \frac{\frac{q_{\mathbf{1}}}{|q|}-c_1}{1-c_1}$, $|q| \cdot -\sqrt{1-(\frac{\frac{q_{\mathbf{1}}}{|q|}-c_1}{1-c_1})^{\mathbf{2}}}]$, and

(ii) if $\frac{q_{\mathbf{1}}}{|q|} \leqslant c_1$ and $q_{\mathbf{2}} \leqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$, then $c_1$-FanMorphS$(q) = [|q| \cdot \frac{\frac{q_{\mathbf{1}}}{|q|}-c_1}{1+c_1}$, $|q| \cdot -\sqrt{1-(\frac{\frac{q_{\mathbf{1}}}{|q|}-c_1}{1+c_1})^{\mathbf{2}}}]$.

(123) Let $c_1$ be a real number, $K_1$ be a non empty subset of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T})\restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < c_1$,

(ii) $c_1 < 1$,

(iii) for every point $p$ of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $f(p) = |p| \cdot \frac{\frac{p_{\mathbf{1}}}{|p|} - c_1}{1 - c_1}$, and

(iv) for every point $q$ of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $q_{\mathbf{2}} \leqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$.

Then $f$ is continuous.

(124) Let $c_1$ be a real number, $K_1$ be a non empty subset of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T})\restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < c_1$,

(ii) $c_1 < 1$,

(iii) for every point $p$ of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $f(p) = |p| \cdot \frac{\frac{p_{\mathbf{1}}}{|p|} - c_1}{1 + c_1}$, and

(iv) for every point $q$ of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $q_{\mathbf{2}} \leqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$.

Then $f$ is continuous.

(125) Let $c_1$ be a real number, $K_1$ be a non empty subset of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T})\restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < c_1$,

(ii) $c_1 < 1$,

(iii) for every point $p$ of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $f(p) = |p| \cdot -\sqrt{1 - (\frac{\frac{p_{\mathbf{1}}}{|p|} - c_1}{1 - c_1})^{\mathbf{2}}}$, and

(iv) for every point $q$ of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $q_{\mathbf{2}} \leqslant 0$ and $\frac{q_{\mathbf{1}}}{|q|} \geqslant c_1$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$.

Then $f$ is continuous.

(126) Let $c_1$ be a real number, $K_1$ be a non empty subset of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T})\restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < c_1$,

(ii) $c_1 < 1$,

(iii) for every point $p$ of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $f(p) = |p| \cdot -\sqrt{1 - (\frac{\frac{p_{\mathbf{1}}}{|p|} - c_1}{1 + c_1})^{\mathbf{2}}}$, and

(iv) for every point $q$ of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $q_{\mathbf{2}} \leqslant 0$ and $\frac{q_{\mathbf{1}}}{|q|} \leqslant c_1$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$.

Then $f$ is continuous.

(127) Let $c_1$ be a real number, $K_0$, $B_0$ be subsets of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f =$

$c_1$-FanMorphS $\restriction K_0$ and $B_0 = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $q_{\mathbf{2}} \leqslant 0 \wedge q \neq 0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : \frac{p_{\mathbf{1}}}{|p|} \geqslant c_1 \wedge p_{\mathbf{2}} \leqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $f$ is continuous.

(128) Let $c_1$ be a real number, $K_0$, $B_0$ be subsets of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphS $\restriction K_0$ and $B_0 = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $q_{\mathbf{2}} \leqslant 0 \wedge q \neq 0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : \frac{p_{\mathbf{1}}}{|p|} \leqslant c_1 \wedge p_{\mathbf{2}} \leqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $f$ is continuous.

(129) For every real number $c_1$ and for every subset $K_3$ of $\mathcal{E}^2_{\rm T}$ such that $K_3 = \{p : p_{\mathbf{1}} \geqslant c_1 \cdot |p| \wedge p_{\mathbf{2}} \leqslant 0\}$ holds $K_3$ is closed.

(130) For every real number $c_1$ and for every subset $K_3$ of $\mathcal{E}^2_{\rm T}$ such that $K_3 = \{p : p_{\mathbf{1}} \leqslant c_1 \cdot |p| \wedge p_{\mathbf{2}} \leqslant 0\}$ holds $K_3$ is closed.

(131) Let $c_1$ be a real number, $K_0$, $B_0$ be subsets of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphS $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{2}} \leqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $f$ is continuous.

(132) Let $c_1$ be a real number, $K_0$, $B_0$ be subsets of $\mathcal{E}^2_{\rm T}$, and $f$ be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphS $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{2}} \geqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $f$ is continuous.

(133) Let $c_1$ be a real number, $B_0$ be a subset of $\mathcal{E}^2_{\rm T}$, $K_0$ be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$, and $f$ be a map from $(\mathcal{E}^2_{\rm T})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphS $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{2}} \leqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $f$ is continuous.

(134) Let $c_1$ be a real number, $B_0$ be a subset of $\mathcal{E}^2_{\rm T}$, $K_0$ be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$, and $f$ be a map from $(\mathcal{E}^2_{\rm T})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphS $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{2}} \geqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then $f$ is continuous.

(135) For every real number $c_1$ and for every point $p$ of $\mathcal{E}^2_{\rm T}$ holds $|c_1\text{-FanMorphS}(p)| = |p|$.

(136) For every real number $c_1$ and for all sets $x$, $K_0$ such that $-1 < c_1$ and $c_1 < 1$ and $x \in K_0$ and $K_0 = \{p : p_{\mathbf{2}} \leqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$ holds $c_1\text{-FanMorphS}(x) \in K_0$.

(137) For every real number $c_1$ and for all sets $x$, $K_0$ such that $-1 < c_1$ and $c_1 < 1$ and $x \in K_0$ and $K_0 = \{p : p_{\mathbf{2}} \geqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$ holds $c_1\text{-FanMorphS}(x) \in K_0$.

(138) Let $c_1$ be a real number and $D$ be a non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $D^{\rm c} = \{0_{\mathcal{E}^2_{\rm T}}\}$. Then there exists a map $h$ from $(\mathcal{E}^2_{\rm T})\restriction D$ into $(\mathcal{E}^2_{\rm T})\restriction D$ such that $h = c_1$-FanMorphS $\restriction D$ and $h$ is continuous.

(139) Let $c_1$ be a real number. Suppose $-1 < c_1$ and $c_1 < 1$. Then there exists a map $h$ from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ such that $h = c_1$-FanMorphS and $h$ is continuous.

(140) For every real number $c_1$ such that $-1 < c_1$ and $c_1 < 1$ holds $c_1$-FanMorphS is one-to-one.

(141) For every real number $c_1$ such that $-1 < c_1$ and $c_1 < 1$ holds $c_1$-FanMorphS is a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ and $\mathrm{rng}(c_1\text{-FanMorphS})$ = the carrier of $\mathcal{E}^2_{\mathrm{T}}$.

(142) Let $c_1$ be a real number and $p_2$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$. Then there exists a non empty compact subset $K$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $K = c_1\text{-FanMorphS}^\circ K$ and there exists a subset $V_2$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_2 \in V_2$ and $V_2$ is open and $V_2 \subseteq K$ and $c_1\text{-FanMorphS}(p_2) \in V_2$.

(143) Let $c_1$ be a real number. Suppose $-1 < c_1$ and $c_1 < 1$. Then there exists a map $f$ from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ such that $f = c_1$-FanMorphS and $f$ is a homeomorphism.

(144) Let $c_1$ be a real number and $q$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} < 0$ and $\frac{q_{\mathbf{1}}}{|q|} \geqslant c_1$. Let $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1\text{-FanMorphS}(q)$, then $p_{\mathbf{2}} < 0$ and $p_{\mathbf{1}} \geqslant 0$.

(145) Let $c_1$ be a real number and $q$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} < 0$ and $\frac{q_{\mathbf{1}}}{|q|} < c_1$. Let $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1\text{-FanMorphS}(q)$, then $p_{\mathbf{2}} < 0$ and $p_{\mathbf{1}} < 0$.

(146) Let $c_1$ be a real number and $q_1$, $q_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $(q_1)_{\mathbf{2}} < 0$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} \geqslant c_1$ and $(q_2)_{\mathbf{2}} < 0$ and $\frac{(q_2)_{\mathbf{1}}}{|q_2|} \geqslant c_1$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < \frac{(q_2)_{\mathbf{1}}}{|q_2|}$. Let $p_1$, $p_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = c_1\text{-FanMorphS}(q_1)$ and $p_2 = c_1\text{-FanMorphS}(q_2)$, then $\frac{(p_1)_{\mathbf{1}}}{|p_1|} < \frac{(p_2)_{\mathbf{1}}}{|p_2|}$.

(147) Let $c_1$ be a real number and $q_1$, $q_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $(q_1)_{\mathbf{2}} < 0$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < c_1$ and $(q_2)_{\mathbf{2}} < 0$ and $\frac{(q_2)_{\mathbf{1}}}{|q_2|} < c_1$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < \frac{(q_2)_{\mathbf{1}}}{|q_2|}$. Let $p_1$, $p_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = c_1\text{-FanMorphS}(q_1)$ and $p_2 = c_1\text{-FanMorphS}(q_2)$, then $\frac{(p_1)_{\mathbf{1}}}{|p_1|} < \frac{(p_2)_{\mathbf{1}}}{|p_2|}$.

(148) Let $c_1$ be a real number and $q_1$, $q_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $(q_1)_{\mathbf{2}} < 0$ and $(q_2)_{\mathbf{2}} < 0$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < \frac{(q_2)_{\mathbf{1}}}{|q_2|}$. Let $p_1$, $p_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = c_1\text{-FanMorphS}(q_1)$ and $p_2 = c_1\text{-FanMorphS}(q_2)$, then $\frac{(p_1)_{\mathbf{1}}}{|p_1|} < \frac{(p_2)_{\mathbf{1}}}{|p_2|}$.

(149) Let $c_1$ be a real number and $q$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} < 0$ and $\frac{q_{\mathbf{1}}}{|q|} = c_1$. Let $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1\text{-FanMorphS}(q)$, then $p_{\mathbf{2}} < 0$ and $p_{\mathbf{1}} = 0$.

(150) For every real number $c_1$ holds $0_{\mathcal{E}^2_{\mathrm{T}}} = c_1\text{-FanMorphS}(0_{\mathcal{E}^2_{\mathrm{T}}})$.

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# Half Open Intervals in Real Numbers

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**Summary.** Left and right half open intervals in the real line are defined. Their properties are investigated. A class of all finite union of such intervals are, in a sense, closed by operations of union, intersection and the difference of sets.

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The terminology and notation used here are introduced in the following articles: [5], [1], [3], [4], and [2].

In this paper $s$, $g$, $h$, $r$, $p$, $p_1$, $p_2$, $q$, $q_1$, $q_2$, $x$, $y$, $z$ denote real numbers.

The following two propositions are true:

(1) $x < y$ and $x < z$ iff $x < \min(y, z)$.

(2) $y < x$ and $z < x$ iff $\max(y, z) < x$.

Let $g$, $s$ be real numbers. The functor $[g, s[$ yielding a subset of $\mathbb{R}$ is defined as follows:

(Def. 1) $[g, s[= \{r; r$ ranges over real numbers: $g \leqslant r \ \wedge \ r < s\}$.

The functor $]g, s]$ yields a subset of $\mathbb{R}$ and is defined as follows:

(Def. 2) $]g, s] = \{r; r$ ranges over real numbers: $g < r \ \wedge \ r \leqslant s\}$.

Next we state a number of propositions:

(3) $r \in [p, q[$ iff $p \leqslant r$ and $r < q$.

(4) $r \in ]p, q]$ iff $p < r$ and $r \leqslant q$.

(5) For all $g$, $s$ such that $g < s$ holds $[g, s[= ]g, s[ \cup \{g\}$.

(6) For all $g$, $s$ such that $g < s$ holds $]g, s] = ]g, s[ \cup \{s\}$.

(7) $[g, g[= \emptyset$.

(8) $]g, g] = \emptyset$.

(9) If $p \leqslant g$, then $[g, p[= \emptyset$.




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(10) If $p \leqslant g$, then $]g,p] = \emptyset$.

(11) If $g \leqslant p$ and $p \leqslant h$, then $[g,p[ \cup [p,h[ = [g,h[$.

(12) If $g \leqslant p$ and $p \leqslant h$, then $]g,p] \cup ]p,h] = ]g,h]$.

(13) If $g \leqslant p_1$ and $g \leqslant p_2$ and $p_1 \leqslant h$ and $p_2 \leqslant h$, then $[g,h] = [g,p_1[ \cup [p_1,p_2] \cup ]p_2,h]$.

(14) If $g < p_1$ and $g < p_2$ and $p_1 < h$ and $p_2 < h$, then $]g,h[ = ]g,p_1] \cup ]p_1,p_2[ \cup [p_2,h[$.

(15) $[q_1,q_2[ \cap [p_1,p_2[ = [\max(q_1,p_1), \min(q_2,p_2)[$.

(16) $]q_1,q_2] \cap ]p_1,p_2] = ]\max(q_1,p_1), \min(q_2,p_2)]$.

(17) $]p,q[ \subseteq [p,q[$ and $]p,q[ \subseteq ]p,q]$ and $[p,q[ \subseteq [p,q]$ and $]p,q] \subseteq [p,q]$.

(18) If $r \in [p,g[$ and $s \in [p,g[$, then $[r,s] \subseteq [p,g[$.

(19) If $r \in ]p,g]$ and $s \in ]p,g]$, then $[r,s] \subseteq ]p,g]$.

(20) If $p \leqslant q$ and $q \leqslant r$, then $[p,q] \cup ]q,r] = [p,r]$.

(21) If $p \leqslant q$ and $q \leqslant r$, then $[p,q[ \cup [q,r] = [p,r]$.

(22) If $[q_1,q_2[$ meets $[p_1,p_2[$, then $q_2 \geqslant p_1$.

(23) If $]q_1,q_2]$ meets $]p_1,p_2]$, then $q_2 \geqslant p_1$.

(24) If $[q_1,q_2[$ meets $[p_1,p_2[$, then $[q_1,q_2[ \cup [p_1,p_2[ = [\min(q_1,p_1), \max(q_2,p_2)[$.

(25) If $]q_1,q_2]$ meets $]p_1,p_2]$, then $]q_1,q_2] \cup ]p_1,p_2] = ]\min(q_1,p_1), \max(q_2,p_2)]$.

(26) If $[p_1,p_2[$ meets $[q_1,q_2[$, then $[p_1,p_2[ \setminus [q_1,q_2[ = [p_1,q_1[ \cup [q_2,p_2[$.

(27) If $]p_1,p_2]$ meets $]q_1,q_2]$, then $]p_1,p_2] \setminus ]q_1,q_2] = ]p_1,q_1] \cup ]q_2,p_2]$.

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# Some Remarks on Clockwise Oriented Sequences on Go-boards[1]

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**Summary.** The main goal of this paper is to present alternative characterizations of clockwise oriented sequences on Go-boards.

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The articles [8], [21], [9], [2], [3], [26], [24], [4], [16], [18], [23], [14], [20], [19], [5], [7], [13], [1], [6], [12], [28], [15], [17], [25], [27], [22], [10], and [11] provide the terminology and notation for this paper.

## 1. Preliminaries

In this paper $i$, $j$, $k$, $n$ denote natural numbers.

Next we state several propositions:

(1) For all subsets $A$, $B$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $A$ is Bounded or $B$ is Bounded holds $A \cap B$ is Bounded.

(2) For all subsets $A$, $B$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $A$ is not Bounded and $B$ is Bounded holds $A \setminus B$ is not Bounded.

(3) For every compact connected non vertical non horizontal subset $C$ of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\text{W-min}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))) \leftarrow\!\!\!\!\!\!\!\!\;\;\text{Cage}(C,n) > 1$.

(4) For every compact connected non vertical non horizontal subset $C$ of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\text{E-max}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))) \hookleftarrow \text{Cage}(C,n) > 1$.

(5) For every compact connected non vertical non horizontal subset $C$ of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\text{S-max}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))) \hookleftarrow \text{Cage}(C,n) > 1$.

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## 2. On Bounding Points of Circular Sequences

Next we state several propositions:

(6) Let $f$ be a non constant standard special circular sequence and $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \operatorname{rng} f$, then $\operatorname{leftcell}(f, p \looparrowleft f) = \operatorname{leftcell}(f^p_\circlearrowleft, 1)$.

(7) Let $f$ be a non constant standard special circular sequence and $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \operatorname{rng} f$, then $\operatorname{rightcell}(f, p \looparrowleft f) = \operatorname{rightcell}(f^p_\circlearrowleft, 1)$.

(8) For every compact connected non vertical non horizontal non empty subset $C$ of $\mathcal{E}^2_{\mathrm{T}}$ holds $\text{W-min}\, C \in \operatorname{rightcell}((\operatorname{Cage}(C, n))^{\text{W-min}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))}_\circlearrowleft, 1)$.

(9) For every compact connected non vertical non horizontal non empty subset $C$ of $\mathcal{E}^2_{\mathrm{T}}$ holds $\text{E-max}\, C \in \operatorname{rightcell}((\operatorname{Cage}(C, n))^{\text{E-max}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))}_\circlearrowleft, 1)$.

(10) For every compact connected non vertical non horizontal non empty subset $C$ of $\mathcal{E}^2_{\mathrm{T}}$ holds $\text{S-max}\, C \in \operatorname{rightcell}((\operatorname{Cage}(C, n))^{\text{S-max}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))}_\circlearrowleft, 1)$.

## 3. On Clockwise Oriented Sequences

One can prove the following propositions:

(11) Let $f$ be a clockwise oriented non constant standard special circular sequence and $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p_{\mathbf{1}} < \text{W-bound}\, \widetilde{\mathcal{L}}(f)$, then $p \in \operatorname{LeftComp}(f)$.

(12) Let $f$ be a clockwise oriented non constant standard special circular sequence and $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p_{\mathbf{1}} > \text{E-bound}\, \widetilde{\mathcal{L}}(f)$, then $p \in \operatorname{LeftComp}(f)$.

(13) Let $f$ be a clockwise oriented non constant standard special circular sequence and $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p_{\mathbf{2}} < \text{S-bound}\, \widetilde{\mathcal{L}}(f)$, then $p \in \operatorname{LeftComp}(f)$.

(14) Let $f$ be a clockwise oriented non constant standard special circular sequence and $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p_{\mathbf{2}} > \text{N-bound}\, \widetilde{\mathcal{L}}(f)$, then $p \in \operatorname{LeftComp}(f)$.

(15) Let $f$ be a clockwise oriented non constant standard special circular sequence and $G$ be a Go-board. Suppose $f$ is a sequence which elements belong to $G$. Let $i$, $j$, $k$ be natural numbers. Suppose $1 \leqslant k$ and $k + 1 \leqslant \operatorname{len} f$ and $\langle i, j\rangle \in$ the indices of $G$ and $\langle i+1, j\rangle \in$ the indices of $G$ and $f_k = G \circ (i+1, j)$ and $f_{k+1} = G \circ (i, j)$. Then $j < \operatorname{width} G$.

(16) Let $f$ be a clockwise oriented non constant standard special circular sequence and $G$ be a Go-board. Suppose $f$ is a sequence which elements belong to $G$. Let $i$, $j$, $k$ be natural numbers. Suppose $1 \leqslant k$ and $k + 1 \leqslant \operatorname{len} f$ and $\langle i, j\rangle \in$ the indices of $G$ and $\langle i, j+1\rangle \in$ the indices of $G$ and $f_k = G \circ (i, j)$ and $f_{k+1} = G \circ (i, j+1)$. Then $i < \operatorname{len} G$.

(17) Let $f$ be a clockwise oriented non constant standard special circular sequence and $G$ be a Go-board. Suppose $f$ is a sequence which elements belong to $G$. Let $i$, $j$, $k$ be natural numbers. Suppose $1 \leqslant k$ and $k + 1 \leqslant$

len $f$ and $\langle i, j\rangle \in$ the indices of $G$ and $\langle i+1, j\rangle \in$ the indices of $G$ and $f_k = G \circ (i,j)$ and $f_{k+1} = G \circ (i+1,j)$. Then $j > 1$.

(18) Let $f$ be a clockwise oriented non constant standard special circular sequence and $G$ be a Go-board. Suppose $f$ is a sequence which elements belong to $G$. Let $i$, $j$, $k$ be natural numbers. Suppose $1 \leqslant k$ and $k+1 \leqslant$ len $f$ and $\langle i, j\rangle \in$ the indices of $G$ and $\langle i, j+1\rangle \in$ the indices of $G$ and $f_k = G \circ (i,j+1)$ and $f_{k+1} = G \circ (i,j)$. Then $i > 1$.

(19) Let $f$ be a clockwise oriented non constant standard special circular sequence and $G$ be a Go-board. Suppose $f$ is a sequence which elements belong to $G$. Let $i$, $j$, $k$ be natural numbers. Suppose $1 \leqslant k$ and $k+1 \leqslant$ len $f$ and $\langle i, j\rangle \in$ the indices of $G$ and $\langle i+1, j\rangle \in$ the indices of $G$ and $f_k = G \circ (i+1,j)$ and $f_{k+1} = G \circ (i,j)$. Then $(f_k)_{\mathbf{2}} \neq$ N-bound $\widetilde{\mathcal{L}}(f)$.

(20) Let $f$ be a clockwise oriented non constant standard special circular sequence and $G$ be a Go-board. Suppose $f$ is a sequence which elements belong to $G$. Let $i$, $j$, $k$ be natural numbers. Suppose $1 \leqslant k$ and $k+1 \leqslant$ len $f$ and $\langle i, j\rangle \in$ the indices of $G$ and $\langle i, j+1\rangle \in$ the indices of $G$ and $f_k = G \circ (i,j)$ and $f_{k+1} = G \circ (i,j+1)$. Then $(f_k)_{\mathbf{1}} \neq$ E-bound $\widetilde{\mathcal{L}}(f)$.

(21) Let $f$ be a clockwise oriented non constant standard special circular sequence and $G$ be a Go-board. Suppose $f$ is a sequence which elements belong to $G$. Let $i$, $j$, $k$ be natural numbers. Suppose $1 \leqslant k$ and $k+1 \leqslant$ len $f$ and $\langle i, j\rangle \in$ the indices of $G$ and $\langle i+1, j\rangle \in$ the indices of $G$ and $f_k = G \circ (i,j)$ and $f_{k+1} = G \circ (i+1,j)$. Then $(f_k)_{\mathbf{2}} \neq$ S-bound $\widetilde{\mathcal{L}}(f)$.

(22) Let $f$ be a clockwise oriented non constant standard special circular sequence and $G$ be a Go-board. Suppose $f$ is a sequence which elements belong to $G$. Let $i$, $j$, $k$ be natural numbers. Suppose $1 \leqslant k$ and $k+1 \leqslant$ len $f$ and $\langle i, j\rangle \in$ the indices of $G$ and $\langle i, j+1\rangle \in$ the indices of $G$ and $f_k = G \circ (i,j+1)$ and $f_{k+1} = G \circ (i,j)$. Then $(f_k)_{\mathbf{1}} \neq$ W-bound $\widetilde{\mathcal{L}}(f)$.

(23) Let $f$ be a clockwise oriented non constant standard special circular sequence, $G$ be a Go-board, and $k$ be a natural number. Suppose $f$ is a sequence which elements belong to $G$ and $1 \leqslant k$ and $k+1 \leqslant$ len $f$ and $f_k =$ W-min $\widetilde{\mathcal{L}}(f)$. Then there exist natural numbers $i$, $j$ such that $\langle i, j\rangle \in$ the indices of $G$ and $\langle i, j+1\rangle \in$ the indices of $G$ and $f_k = G \circ (i,j)$ and $f_{k+1} = G \circ (i,j+1)$.

(24) Let $f$ be a clockwise oriented non constant standard special circular sequence, $G$ be a Go-board, and $k$ be a natural number. Suppose $f$ is a sequence which elements belong to $G$ and $1 \leqslant k$ and $k+1 \leqslant$ len $f$ and $f_k =$ N-min $\widetilde{\mathcal{L}}(f)$. Then there exist natural numbers $i$, $j$ such that $\langle i, j\rangle \in$ the indices of $G$ and $\langle i+1, j\rangle \in$ the indices of $G$ and $f_k = G \circ (i,j)$ and $f_{k+1} = G \circ (i+1,j)$.

(25) Let $f$ be a clockwise oriented non constant standard special circular sequence, $G$ be a Go-board, and $k$ be a natural number. Suppose $f$ is

a sequence which elements belong to $G$ and $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$ and $f_k = \text{E-max}\,\widetilde{\mathcal{L}}(f)$. Then there exist natural numbers $i$, $j$ such that $\langle i, j+1\rangle \in$ the indices of $G$ and $\langle i, j\rangle \in$ the indices of $G$ and $f_k = G \circ (i, j+1)$ and $f_{k+1} = G \circ (i, j)$.

(26) Let $f$ be a clockwise oriented non constant standard special circular sequence, $G$ be a Go-board, and $k$ be a natural number. Suppose $f$ is a sequence which elements belong to $G$ and $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$ and $f_k = \text{S-max}\,\widetilde{\mathcal{L}}(f)$. Then there exist natural numbers $i$, $j$ such that $\langle i+1, j\rangle \in$ the indices of $G$ and $\langle i, j\rangle \in$ the indices of $G$ and $f_k = G \circ (i+1, j)$ and $f_{k+1} = G \circ (i, j)$.

(27) Let $f$ be a non constant standard special circular sequence. Then $f$ is clockwise oriented if and only if $(f_{\circlearrowleft}^{\text{W-min}\,\widetilde{\mathcal{L}}(f)})_2 \in \text{W-most}\,\widetilde{\mathcal{L}}(f)$.

(28) Let $f$ be a non constant standard special circular sequence. Then $f$ is clockwise oriented if and only if $(f_{\circlearrowleft}^{\text{E-max}\,\widetilde{\mathcal{L}}(f)})_2 \in \text{E-most}\,\widetilde{\mathcal{L}}(f)$.

(29) Let $f$ be a non constant standard special circular sequence. Then $f$ is clockwise oriented if and only if $(f_{\circlearrowleft}^{\text{S-max}\,\widetilde{\mathcal{L}}(f)})_2 \in \text{S-most}\,\widetilde{\mathcal{L}}(f)$.

(30) Let $C$ be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ satisfying conditions of simple closed curve and $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{1}} = \frac{\text{W-bound}\,C + \text{E-bound}\,C}{2}$ and $i > 0$ and $1 \leqslant k$ and $k \leqslant \operatorname{width}\operatorname{Gauge}(C, i)$ and $\operatorname{Gauge}(C, i) \circ (\operatorname{Center}\operatorname{Gauge}(C, i), k) \in \operatorname{UpperArc}\widetilde{\mathcal{L}}(\operatorname{Cage}(C, i))$ and $p_{\mathbf{2}} = \sup(\mathrm{proj2}^{\circ}(\mathcal{L}(\operatorname{Gauge}(C, 1) \circ (\operatorname{Center}\operatorname{Gauge}(C, 1), 1), \operatorname{Gauge}(C, i) \circ (\operatorname{Center}\operatorname{Gauge}(C, i), k)) \cap \operatorname{LowerArc}\widetilde{\mathcal{L}}(\operatorname{Cage}(C, i))))$. Then there exists $j$ such that $1 \leqslant j$ and $j \leqslant \operatorname{len}\operatorname{Gauge}(C, i)$ and $p = \operatorname{Gauge}(C, i) \circ (\operatorname{Center}\operatorname{Gauge}(C, i), j)$.

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# Dickson's Lemma

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**Summary.** We present a Mizar formalization of the proof of Dickson's lemma following [6], chapters 4.2 and 4.3.

MML Identifier: `DICKSON`.

The papers [19], [29], [1], [7], [13], [21], [12], [8], [9], [2], [20], [26], [27], [24], [17], [18], [30], [32], [31], [28], [23], [4], [11], [5], [14], [22], [3], [15], [16], [25], and [10] provide the terminology and notation for this paper.

## 1. Preliminaries

One can prove the following two propositions:

(1) For every function $g$ and for every set $x$ such that $\operatorname{dom} g = \{x\}$ holds $g = x \dot{\longmapsto} g(x)$.

(2) For every natural number $n$ holds $n \subseteq n + 1$.

The scheme *FinSegRng2* deals with natural numbers $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(i); i$ ranges over natural numbers: $\mathcal{A} < i \ \wedge \ i \leqslant \mathcal{B} \ \wedge \ \mathcal{P}[i]\}$ is finite

for all values of the parameters.

The following proposition is true

(3) For every infinite set $X$ holds there exists a function from $\mathbb{N}$ into $X$ which is one-to-one.

Let $R$ be a relational structure and let $f$ be a sequence of $R$. We say that $f$ is ascending if and only if:




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(Def. 1) For every natural number $n$ holds $f(n+1) \neq f(n)$ and $\langle f(n), f(n+1)\rangle \in$ the internal relation of $R$.

Let $R$ be a relational structure and let $f$ be a sequence of $R$. We say that $f$ is weakly ascending if and only if:

(Def. 2) For every natural number $n$ holds $\langle f(n), f(n+1)\rangle \in$ the internal relation of $R$.

The following propositions are true:

(4) Let $R$ be a non empty transitive relational structure and $f$ be a sequence of $R$. Suppose $f$ is weakly ascending. Let $i$, $j$ be natural numbers. If $i < j$, then $f(i) \leqslant f(j)$.

(5) Let $R$ be a non empty relational structure. Then $R$ is connected if and only if the internal relation of $R$ is strongly connected in the carrier of $R$.

(6) Let $R$ be a binary relation and $X$ be a set. Then $R$ is reflexive in $X$ and connected in $X$ if and only if $R$ is strongly connected in $X$.

(7) Let $L$ be a relational structure, $Y$ be a set, and $a$ be an element of $L$. Then (the internal relation of $L$)-Seg($a$) misses $Y$ and $a \in Y$ if and only if $a$ is minimal w.r.t. $Y$, the internal relation of $L$.

(8) Let $L$ be a non empty transitive antisymmetric relational structure, $a$, $x$ be elements of $L$, and $N$ be a set. Suppose $a$ is minimal w.r.t. (the internal relation of $L$)-Seg($x$) $\cap N$, the internal relation of $L$. Then $a$ is minimal w.r.t. $N$, the internal relation of $L$.

## 2. More on Ordering Relations

Let $R$ be a relational structure. We say that $R$ is quasi ordered if and only if:

(Def. 3) $R$ is reflexive and transitive.

Let $R$ be a relational structure. Let us assume that $R$ is quasi ordered. The functor EqRel($R$) yielding an equivalence relation of the carrier of $R$ is defined as follows:

(Def. 4) EqRel($R$) = (the internal relation of $R$) $\cap$ (the internal relation of $R$)$^{\smile}$.

The following proposition is true

(9) Let $R$ be a relational structure and $x$, $y$ be elements of the carrier of $R$. If $R$ is quasi ordered, then $x \in [y]_{\mathrm{EqRel}(R)}$ iff $x \leqslant y$ and $y \leqslant x$.

Let $R$ be a relational structure. The functor $\leqslant_E R$ yielding a binary relation on Classes EqRel($R$) is defined as follows:

(Def. 5) For all sets $A$, $B$ holds $\langle A, B\rangle \in \leqslant_E R$ iff there exist elements $a$, $b$ of $R$ such that $A = [a]_{\mathrm{EqRel}(R)}$ and $B = [b]_{\mathrm{EqRel}(R)}$ and $a \leqslant b$.

We now state two propositions:

(10) For every relational structure $R$ such that $R$ is quasi ordered holds $\leqslant_E R$ partially orders Classes EqRel($R$).

(11) Let $R$ be a non empty relational structure. If $R$ is quasi ordered and connected, then $\leqslant_E R$ linearly orders Classes EqRel($R$).

Let $R$ be a binary relation. The functor $R\backslash^{\smile}$ yields a binary relation and is defined by:

(Def. 6) $R\backslash^{\smile} = R \setminus R^{\smile}$.

Let $R$ be a binary relation. Note that $R\backslash^{\smile}$ is asymmetric.

Let $X$ be a set and let $R$ be a binary relation on $X$. Then $R\backslash^{\smile}$ is a binary relation on $X$.

Let $R$ be a relational structure. The functor $R\backslash^{\smile}$ yielding a strict relational structure is defined as follows:

(Def. 7) $R\backslash^{\smile} = \langle$the carrier of $R$, the internal relation of $R\backslash^{\smile}\rangle$.

Let $R$ be a non empty relational structure. One can check that $R\backslash^{\smile}$ is non empty.

Let $R$ be a transitive relational structure. One can check that $R\backslash^{\smile}$ is transitive.

Let $R$ be a relational structure. One can check that $R\backslash^{\smile}$ is antisymmetric.

We now state several propositions:

(12) For every non empty poset $R$ and for every element $x$ of the carrier of $R$ holds $[x]_{\mathrm{EqRel}(R)} = \{x\}$.

(13) For every binary relation $R$ holds $R = R\backslash^{\smile}$ iff $R$ is asymmetric.

(14) For every binary relation $R$ such that $R$ is transitive holds $R\backslash^{\smile}$ is transitive.

(15) Let $R$ be a binary relation and $a$, $b$ be sets. If $R$ is antisymmetric, then $\langle a, b\rangle \in R\backslash^{\smile}$ iff $\langle a, b\rangle \in R$ and $a \neq b$.

(16) For every relational structure $R$ such that $R$ is well founded holds $R\backslash^{\smile}$ is well founded.

(17) For every relational structure $R$ such that $R\backslash^{\smile}$ is well founded and $R$ is antisymmetric holds $R$ is well founded.

## 3. Foundedness Properties

The following two propositions are true:

(18) Let $L$ be a relational structure, $N$ be a set, and $x$ be an element of $L\backslash^{\smile}$. Then $x$ is minimal w.r.t. $N$, the internal relation of $L\backslash^{\smile}$ if and only if $x \in N$ and for every element $y$ of $L$ such that $y \in N$ and $\langle y, x\rangle \in$ the internal relation of $L$ holds $\langle x, y\rangle \in$ the internal relation of $L$.

(19) Let $R$, $S$ be non empty relational structures and $m$ be a map from $R$ into $S$. Suppose that
  (i) $R$ is quasi ordered,
  (ii) $S$ is antisymmetric,
  (iii) $S\backslash^{\smile}$ is well founded, and
  (iv) for all elements $a$, $b$ of $R$ holds if $a \leqslant b$, then $m(a) \leqslant m(b)$ and if $m(a) = m(b)$, then $\langle a, b\rangle \in \mathrm{EqRel}(R)$.

  Then $R\backslash^{\smile}$ is well founded.

Let $R$ be a non empty relational structure and let $N$ be a subset of the carrier of $R$. The functor $\mathrm{MinClasses}\, N$ yields a family of subsets of the carrier of $R$ and is defined by the condition (Def. 8).

(Def. 8) Let $x$ be a set. Then $x \in \mathrm{MinClasses}\, N$ if and only if there exists an element $y$ of $R\backslash^{\smile}$ such that $y$ is minimal w.r.t. $N$, the internal relation of $R\backslash^{\smile}$ and $x = [y]_{\mathrm{EqRel}(R)} \cap N$.

Next we state several propositions:

(20) Let $R$ be a non empty relational structure, $N$ be a subset of the carrier of $R$, and $x$ be a set. Suppose $R$ is quasi ordered and $x \in \mathrm{MinClasses}\, N$. Let $y$ be an element of $R\backslash^{\smile}$. If $y \in x$, then $y$ is minimal w.r.t. $N$, the internal relation of $R\backslash^{\smile}$.

(21) Let $R$ be a non empty relational structure. Then $R\backslash^{\smile}$ is well founded if and only if for every subset $N$ of the carrier of $R$ such that $N \neq \emptyset$ there exists a set $x$ such that $x \in \mathrm{MinClasses}\, N$.

(22) Let $R$ be a non empty relational structure, $N$ be a subset of the carrier of $R$, and $y$ be an element of $R\backslash^{\smile}$. If $y$ is minimal w.r.t. $N$, the internal relation of $R\backslash^{\smile}$, then $\mathrm{MinClasses}\, N$ is non empty.

(23) Let $R$ be a non empty relational structure, $N$ be a subset of the carrier of $R$, and $x$ be a set. If $R$ is quasi ordered and $x \in \mathrm{MinClasses}\, N$, then $x$ is non empty.

(24) Let $R$ be a non empty relational structure. Suppose $R$ is quasi ordered. Then $R$ is connected and $R\backslash^{\smile}$ is well founded if and only if for every non empty subset $N$ of the carrier of $R$ holds $\overline{\overline{\mathrm{MinClasses}\, N}} = 1$.

(25) Let $R$ be a non empty poset. Then the following statements are equivalent
  (i) the internal relation of $R$ well orders the carrier of $R$,
  (ii) for every non empty subset $N$ of the carrier of $R$ holds $\overline{\overline{\mathrm{MinClasses}\, N}} = 1$.

Let $R$ be a relational structure, let $N$ be a subset of the carrier of $R$, and let $B$ be a set. We say that $B$ is Dickson basis of $N$, $R$ if and only if:

(Def. 9) $B \subseteq N$ and for every element $a$ of $R$ such that $a \in N$ there exists an element $b$ of $R$ such that $b \in B$ and $b \leqslant a$.

The following two propositions are true:

(26) For every relational structure $R$ holds $\emptyset$ is Dickson basis of $\emptyset_{\text{the carrier of } R}$, $R$.

(27) Let $R$ be a non empty relational structure, $N$ be a non empty subset of the carrier of $R$, and $B$ be a set. If $B$ is Dickson basis of $N$, $R$, then $B$ is non empty.

Let $R$ be a relational structure. We say that $R$ is Dickson if and only if:

(Def. 10) For every subset $N$ of the carrier of $R$ holds there exists a set which is Dickson basis of $N$, $R$ and finite.

The following two propositions are true:

(28) For every non empty relational structure $R$ such that $R\backslash^{\smile}$ is well founded and $R$ is connected holds $R$ is Dickson.

(29) Let $R$, $S$ be relational structures. Suppose that

(i) the internal relation of $R \subseteq$ the internal relation of $S$,

(ii) $R$ is Dickson, and

(iii) the carrier of $R =$ the carrier of $S$.

Then $S$ is Dickson.

Let $f$ be a function and let $b$ be a set. Let us assume that $\operatorname{dom} f = \mathbb{N}$ and $b \in \operatorname{rng} f$. The functor $f \operatorname{mindex} b$ yielding a natural number is defined by:

(Def. 11) $f(f \operatorname{mindex} b) = b$ and for every natural number $i$ such that $f(i) = b$ holds $f \operatorname{mindex} b \leqslant i$.

Let $R$ be a non empty 1-sorted structure, let $f$ be a sequence of $R$, let $b$ be a set, and let $m$ be a natural number. Let us assume that there exists a natural number $j$ such that $m < j$ and $f(j) = b$. The functor $f \operatorname{mindex}(b, m)$ yielding a natural number is defined as follows:

(Def. 12) $f(f \operatorname{mindex}(b, m)) = b$ and $m < f \operatorname{mindex}(b, m)$ and for every natural number $i$ such that $m < i$ and $f(i) = b$ holds $f \operatorname{mindex}(b, m) \leqslant i$.

Next we state several propositions:

(30) Let $R$ be a non empty relational structure. Suppose $R$ is quasi ordered and Dickson. Let $f$ be a sequence of $R$. Then there exist natural numbers $i$, $j$ such that $i < j$ and $f(i) \leqslant f(j)$.

(31) Let $R$ be a relational structure, $N$ be a subset of the carrier of $R$, and $x$ be an element of $R\backslash^{\smile}$. Suppose $R$ is quasi ordered and $x \in N$ and (the internal relation of $R$)-Seg$(x) \cap N \subseteq [x]_{\mathrm{EqRel}(R)}$. Then $x$ is minimal w.r.t. $N$, the internal relation of $R\backslash^{\smile}$.

(32) Let $R$ be a non empty relational structure. Suppose $R$ is quasi ordered and for every sequence $f$ of $R$ there exist natural numbers $i$, $j$ such that $i < j$ and $f(i) \leqslant f(j)$. Let $N$ be a non empty subset of the carrier of $R$. Then $\operatorname{MinClasses} N$ is finite and $\operatorname{MinClasses} N$ is non empty.

(33) Let $R$ be a non empty relational structure. Suppose $R$ is quasi ordered and for every non empty subset $N$ of the carrier of $R$ holds MinClasses $N$ is finite and MinClasses $N$ is non empty. Then $R$ is Dickson.

(34) For every non empty relational structure $R$ such that $R$ is quasi ordered and Dickson holds $R\backslash^{\smile}$ is well founded.

(35) Let $R$ be a non empty poset and $N$ be a non empty subset of the carrier of $R$. Suppose $R$ is Dickson. Then there exists a set $B$ such that $B$ is Dickson basis of $N$, $R$ and for every set $C$ such that $C$ is Dickson basis of $N$, $R$ holds $B \subseteq C$.

Let $R$ be a non empty relational structure and let $N$ be a subset of the carrier of $R$. Let us assume that $R$ is Dickson. The functor Dickson-Bases$(N, R)$ yields a non empty family of subsets of the carrier of $R$ and is defined as follows:

(Def. 13) For every set $B$ holds $B \in$ Dickson-Bases$(N, R)$ iff $B$ is Dickson basis of $N$, $R$.

We now state several propositions:

(36) Let $R$ be a non empty relational structure and $s$ be a sequence of $R$. If $R$ is Dickson, then there exists a sequence of $R$ which is a subsequence of $s$ and weakly ascending.

(37) For every relational structure $R$ such that $R$ is empty holds $R$ is Dickson.

(38) Let $M$, $N$ be relational structures. Suppose $M$ is Dickson and $N$ is Dickson and $M$ is quasi ordered and $N$ is quasi ordered. Then $[\!: M, N :\!]$ is quasi ordered and $[\!: M, N :\!]$ is Dickson.

(39) Let $R$, $S$ be relational structures. Suppose $R$ and $S$ are isomorphic and $R$ is Dickson and quasi ordered. Then $S$ is quasi ordered and Dickson.

(40) Let $p$ be a relational structure yielding many sorted set indexed by 1 and $z$ be an element of 1. Then $p(z)$ and $\prod p$ are isomorphic.

Let $X$ be a set, let $p$ be a relational structure yielding many sorted set indexed by $X$, and let $Y$ be a subset of $X$. Note that $p{\restriction}Y$ is relational structure yielding.

Next we state three propositions:

(41) Let $n$ be a non empty natural number and $p$ be a relational structure yielding many sorted set indexed by $n$. Then $\prod p$ is non empty if and only if $p$ is nonempty.

(42) Let $n$ be a non empty natural number, $p$ be a relational structure yielding many sorted set indexed by $n+1$, $n_1$ be a subset of $n+1$, and $n_2$ be an element of $n+1$. If $n_1 = n$ and $n_2 = n$, then $[\!: \prod(p{\restriction}n_1), p(n_2) :\!]$ and $\prod p$ are isomorphic.

(43) Let $n$ be a non empty natural number and $p$ be a relational structure yielding many sorted set indexed by $n$. Suppose that for every element

$i$ of $n$ holds $p(i)$ is Dickson and $p(i)$ is quasi ordered. Then $\prod p$ is quasi ordered and $\prod p$ is Dickson.

Let $p$ be a relational structure yielding many sorted set indexed by $\emptyset$. One can check the following observations:

* $\prod p$ is non empty,
* $\prod p$ is antisymmetric,
* $\prod p$ is quasi ordered, and
* $\prod p$ is Dickson.

The binary relation NATOrd on $\mathbb{N}$ is defined by:

(Def. 14) NATOrd $= \{\langle x, y\rangle; x$ ranges over elements of $\mathbb{N}$, $y$ ranges over elements of $\mathbb{N}$: $x \leqslant y\}$.

We now state four propositions:

(44) NATOrd is reflexive in $\mathbb{N}$.

(45) NATOrd is antisymmetric in $\mathbb{N}$.

(46) NATOrd is strongly connected in $\mathbb{N}$.

(47) NATOrd is transitive in $\mathbb{N}$.

The non empty relational structure OrderedNAT is defined as follows:

(Def. 15) OrderedNAT $= \langle \mathbb{N}, \mathrm{NATOrd}\rangle$.

One can verify the following observations:

* OrderedNAT is connected,
* OrderedNAT is Dickson,
* OrderedNAT is quasi ordered,
* OrderedNAT is antisymmetric,
* OrderedNAT is transitive, and
* OrderedNAT is well founded.

Let $n$ be a natural number. One can check the following observations:

* $\prod(n \longmapsto \mathrm{OrderedNAT})$ is non empty,
* $\prod(n \longmapsto \mathrm{OrderedNAT})$ is Dickson,
* $\prod(n \longmapsto \mathrm{OrderedNAT})$ is quasi ordered, and
* $\prod(n \longmapsto \mathrm{OrderedNAT})$ is antisymmetric.

We now state three propositions:

(48) Let $M$ be a relational structure. Suppose $M$ is Dickson and quasi ordered. Then $[: M, \mathrm{OrderedNAT} :]$ is quasi ordered and $[: M, \mathrm{OrderedNAT} :]$ is Dickson.

(49) Let $R$, $S$ be non empty relational structures. Suppose that

(i) $R$ is Dickson and quasi ordered,

(ii) $S$ is quasi ordered,

(iii) the internal relation of $R \subseteq$ the internal relation of $S$, and

(iv) the carrier of $R$ = the carrier of $S$.
Then $S\backslash^{\smile}$ is well founded.

(50) Let $R$ be a non empty relational structure. Suppose $R$ is quasi ordered. Then $R$ is Dickson if and only if for every non empty relational structure $S$ such that $S$ is quasi ordered and the internal relation of $R \subseteq$ the internal relation of $S$ and the carrier of $R$ = the carrier of $S$ holds $S\backslash^{\smile}$ is well founded.

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# On Ordering of Bags

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**Summary.** We present a Mizar formalization of chapter 4.4 of [6] devoted to special orderings in additive monoids to be used for ordering terms in multivariate polynomials. We have extended the treatment to the case of infinite number of variables. It turns out that in such case admissible orderings are not necessarily well orderings.

MML Identifier: `BAGORDER`.

The notation and terminology used here are introduced in the following papers: [21], [33], [18], [26], [7], [5], [11], [29], [8], [9], [1], [22], [30], [31], [2], [28], [24], [25], [20], [14], [34], [36], [35], [17], [16], [32], [27], [19], [4], [12], [23], [3], [15], [13], and [10].

## 1. Preliminaries

The following propositions are true:

(1) For all sets $x$, $y$, $z$ such that $z \in x$ and $z \in y$ holds $x \setminus \{z\} = y \setminus \{z\}$ iff $x = y$.

(2) For all natural numbers $n$, $k$ holds $k \in \mathrm{Seg}\, n$ iff $k-1$ is a natural number and $k - 1 < n$.

Let $f$ be a natural-yielding function and let $X$ be a set. One can verify that $f {\restriction} X$ is natural-yielding.

Let $f$ be a finite-support function and let $X$ be a set. One can check that $f {\restriction} X$ is finite-support.

Next we state three propositions:

(3) For every function $f$ and for every set $x$ such that $x \in \mathrm{dom}\, f$ holds $f \cdot \langle x \rangle = \langle f(x) \rangle$.




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(4) Let $f$, $g$, $h$ be functions. Suppose $\operatorname{dom} f = \operatorname{dom} g$ and $\operatorname{rng} f \subseteq \operatorname{dom} h$ and $\operatorname{rng} g \subseteq \operatorname{dom} h$ and $f$ and $g$ are fiberwise equipotent. Then $h \cdot f$ and $h \cdot g$ are fiberwise equipotent.

(5) For every finite sequence $f_1$ of elements of $\mathbb{N}$ holds $\sum f_1 = 0$ iff $f_1 = \operatorname{len} f_1 \mapsto 0$.

Let $n$, $i$, $j$ be natural numbers and let $b$ be a many sorted set indexed by $n$. The functor $\langle b(i), \ldots, b(j) \rangle$ yields a many sorted set indexed by $j -' i$ and is defined by:

(Def. 1) For every natural number $k$ such that $k \in j -' i$ holds $\langle b(i), \ldots, b(j) \rangle(k) = b(i + k)$.

Let $n$, $i$, $j$ be natural numbers and let $b$ be a natural-yielding many sorted set indexed by $n$. One can verify that $\langle b(i), \ldots, b(j) \rangle$ is natural-yielding.

Let $n$, $i$, $j$ be natural numbers and let $b$ be a finite-support many sorted set indexed by $n$. Note that $\langle b(i), \ldots, b(j) \rangle$ is finite-support.

One can prove the following proposition

(6) Let $n$, $i$ be natural numbers and $a$, $b$ be many sorted sets indexed by $n$. Then $a = b$ if and only if the following conditions are satisfied:

(i) $\langle a(0), \ldots, a(i + 1) \rangle = \langle b(0), \ldots, b(i + 1) \rangle$, and

(ii) $\langle a(i + 1), \ldots, a(n) \rangle = \langle b(i + 1), \ldots, b(n) \rangle$.

Let $x$ be a non empty set and let $n$ be a non empty natural number. The functor $\operatorname{Fin}(x, n)$ is defined as follows:

(Def. 2) $\operatorname{Fin}(x, n) = \{y; y$ ranges over elements of $2^x$: $y$ is finite $\wedge$ $y$ is non empty $\wedge$ $\overline{\overline{y}} \leqslant n\}$.

Let $x$ be a non empty set and let $n$ be a non empty natural number. Observe that $\operatorname{Fin}(x, n)$ is non empty.

One can prove the following propositions:

(7) Let $R$ be an antisymmetric transitive non empty relational structure and $X$ be a finite subset of the carrier of $R$. Suppose $X \neq \emptyset$. Then there exists an element $x$ of $R$ such that $x \in X$ and $x$ is maximal w.r.t. $X$, the internal relation of $R$.

(8) Let $R$ be an antisymmetric transitive non empty relational structure and $X$ be a finite subset of the carrier of $R$. Suppose $X \neq \emptyset$. Then there exists an element $x$ of $R$ such that $x \in X$ and $x$ is minimal w.r.t. $X$, the internal relation of $R$.

(9) Let $R$ be a non empty antisymmetric transitive relational structure and $f$ be a sequence of $R$. Suppose $f$ is descending. Let $j$, $i$ be natural numbers. If $i < j$, then $f(i) \neq f(j)$ and $\langle f(j), f(i) \rangle \in$ the internal relation of $R$.

Let $R$ be a non empty relational structure and let $s$ be a sequence of $R$. We say that $s$ is non-increasing if and only if:

(Def. 3) For every natural number $i$ holds $\langle s(i+1), s(i)\rangle \in$ the internal relation of $R$.

We now state three propositions:

(10) Let $R$ be a non empty transitive relational structure and $f$ be a sequence of $R$. Suppose $f$ is non-increasing. Let $j$, $i$ be natural numbers. If $i < j$, then $\langle f(j), f(i)\rangle \in$ the internal relation of $R$.

(11) Let $R$ be a non empty transitive relational structure and $s$ be a sequence of $R$. Suppose $R$ is well founded and $s$ is non-increasing. Then there exists a natural number $p$ such that for every natural number $r$ if $p \leqslant r$, then $s(p) = s(r)$.

(12) Let $X$ be a set, $a$ be an element of $X$, $A$ be a finite subset of $X$, and $R$ be an order in $X$. If $A = \{a\}$ and $R$ linearly orders $A$, then $\mathrm{SgmX}(R, A) = \langle a\rangle$.

## 2. More About Bags

Let $n$ be an ordinal number and let $b$ be a bag of $n$. The functor $\mathrm{TotDegree}\, b$ yielding a natural number is defined by:

(Def. 4) There exists a finite sequence $f$ of elements of $\mathbb{N}$ such that $\mathrm{TotDegree}\, b = \sum f$ and $f = b \cdot \mathrm{SgmX}(\subseteq_n, \mathrm{support}\, b)$.

The following propositions are true:

(13) Let $n$ be an ordinal number, $b$ be a bag of $n$, $s$ be a finite subset of $n$, and $f$, $g$ be finite sequences of elements of $\mathbb{N}$. If $f = b \cdot \mathrm{SgmX}(\subseteq_n, \mathrm{support}\, b)$ and $g = b \cdot \mathrm{SgmX}(\subseteq_n, \mathrm{support}\, b \cup s)$, then $\sum f = \sum g$.

(14) For every ordinal number $n$ and for all bags $a$, $b$ of $n$ holds $\mathrm{TotDegree}(a+b) = \mathrm{TotDegree}\, a + \mathrm{TotDegree}\, b$.

(15) For every ordinal number $n$ and for all bags $a$, $b$ of $n$ such that $b \mid a$ holds $\mathrm{TotDegree}(a -' b) = \mathrm{TotDegree}\, a - \mathrm{TotDegree}\, b$.

(16) For every ordinal number $n$ and for every bag $b$ of $n$ holds $\mathrm{TotDegree}\, b = 0$ iff $b = \mathrm{EmptyBag}\, n$.

(17) For all natural numbers $i, j, n$ holds $\langle (\mathrm{EmptyBag}\, n)(i), \ldots, (\mathrm{EmptyBag}\, n)(j)\rangle = \mathrm{EmptyBag}(j -' i)$.

(18) For all natural numbers $i$, $j$, $n$ and for all bags $a$, $b$ of $n$ holds $\langle (a+b)(i), \ldots, (a+b)(j)\rangle = \langle a(i), \ldots, a(j)\rangle + \langle b(i), \ldots, b(j)\rangle$.

(19) For every set $X$ holds $\mathrm{support}\, \mathrm{EmptyBag}\, X = \emptyset$.

(20) For every set $X$ and for every bag $b$ of $X$ such that $\mathrm{support}\, b = \emptyset$ holds $b = \mathrm{EmptyBag}\, X$.

(21) For all ordinal numbers $n$, $m$ and for every bag $b$ of $n$ such that $m \in n$ holds $b{\restriction}m$ is a bag of $m$.

(22) For every ordinal number $n$ and for all bags $a$, $b$ of $n$ such that $b \mid a$ holds $\operatorname{support} b \subseteq \operatorname{support} a$.

## 3. Some Special Orders

Let $n$ be an ordinal number and let $o$ be an order in $\operatorname{Bags} n$. We say that $o$ is admissible if and only if the conditions (Def. 5) are satisfied.

(Def. 5)(i) $o$ is strongly connected in $\operatorname{Bags} n$,

(ii) for every bag $a$ of $n$ holds $\langle \operatorname{EmptyBag} n, a\rangle \in o$, and

(iii) for all bags $a$, $b$, $c$ of $n$ such that $\langle a, b\rangle \in o$ holds $\langle a+c, b+c\rangle \in o$.

Let $n$ be an ordinal number. We introduce $\operatorname{LexOrder} n$ as a synonym of $\operatorname{BagOrder} n$.

One can prove the following propositions:

(23) For every ordinal number $n$ holds $\operatorname{LexOrder} n$ is admissible.

(24) For every infinite ordinal number $o$ holds $\operatorname{LexOrder} o$ is non well-ordering.

Let $n$ be an ordinal number. The functor $\operatorname{InvLexOrder} n$ yields an order in $\operatorname{Bags} n$ and is defined by the condition (Def. 6).

(Def. 6) Let $p$, $q$ be bags of $n$. Then $\langle p, q\rangle \in \operatorname{InvLexOrder} n$ if and only if one of the following conditions is satisfied:

(i) $p = q$, or

(ii) there exists an ordinal number $i$ such that $i \in n$ and $p(i) < q(i)$ and for every ordinal number $k$ such that $i \in k$ and $k \in n$ holds $p(k) = q(k)$.

The following propositions are true:

(25) For every ordinal number $n$ holds $\operatorname{InvLexOrder} n$ is admissible.

(26) For every ordinal number $o$ holds $\operatorname{InvLexOrder} o$ is well-ordering.

Let $n$ be an ordinal number and let $o$ be an order in $\operatorname{Bags} n$. Let us assume that for all bags $a$, $b$, $c$ of $n$ such that $\langle a, b\rangle \in o$ holds $\langle a+c, b+c\rangle \in o$. The functor $\operatorname{Graded} o$ yields an order in $\operatorname{Bags} n$ and is defined by:

(Def. 7) For all bags $a$, $b$ of $n$ holds $\langle a, b\rangle \in \operatorname{Graded} o$ iff $\operatorname{TotDegree} a < \operatorname{TotDegree} b$ or $\operatorname{TotDegree} a = \operatorname{TotDegree} b$ and $\langle a, b\rangle \in o$.

The following proposition is true

(27) Let $n$ be an ordinal number and $o$ be an order in $\operatorname{Bags} n$. Suppose for all bags $a$, $b$, $c$ of $n$ such that $\langle a, b\rangle \in o$ holds $\langle a+c, b+c\rangle \in o$ and $o$ is strongly connected in $\operatorname{Bags} n$. Then $\operatorname{Graded} o$ is admissible.

Let $n$ be an ordinal number. The functor $\operatorname{GrLexOrder} n$ yielding an order in $\operatorname{Bags} n$ is defined as follows:

(Def. 8) $\operatorname{GrLexOrder} n = \operatorname{Graded} \operatorname{LexOrder} n$.

The functor $\operatorname{GrInvLexOrder} n$ yielding an order in $\operatorname{Bags} n$ is defined by:

(Def. 9) $\operatorname{GrInvLexOrder} n = \operatorname{Graded} \operatorname{InvLexOrder} n$.

Next we state four propositions:

(28) For every ordinal number $n$ holds GrLexOrder $n$ is admissible.

(29) For every infinite ordinal number $o$ holds GrLexOrder $o$ is non well-ordering.

(30) For every ordinal number $n$ holds GrInvLexOrder $n$ is admissible.

(31) For every ordinal number $o$ holds GrInvLexOrder $o$ is well-ordering.

Let $i$, $n$ be natural numbers, let $o_1$ be an order in $\mathrm{Bags}(i+1)$, and let $o_2$ be an order in $\mathrm{Bags}(n -' (i+1))$. The functor $\mathrm{BlockOrder}(i, n, o_1, o_2)$ yielding an order in Bags $n$ is defined by the condition (Def. 10).

(Def. 10) Let $p$, $q$ be bags of $n$. Then $\langle p, q\rangle \in \mathrm{BlockOrder}(i, n, o_1, o_2)$ if and only if one of the following conditions is satisfied:

(i) $\langle p(0), \ldots, p(i+1)\rangle \neq \langle q(0), \ldots, q(i+1)\rangle$ and $\langle\langle p(0), \ldots, p(i+1)\rangle, \langle q(0), \ldots, q(i+1)\rangle\rangle \in o_1$, or

(ii) $\langle p(0), \ldots, p(i+1)\rangle = \langle q(0), \ldots, q(i+1)\rangle$ and $\langle\langle p(i+1), \ldots, p(n)\rangle, \langle q(i+1), \ldots, q(n)\rangle\rangle \in o_2$.

The following proposition is true

(32) Let $i$, $n$ be natural numbers, $o_1$ be an order in $\mathrm{Bags}(i+1)$, and $o_2$ be an order in $\mathrm{Bags}(n -' (i+1))$. If $o_1$ is admissible and $o_2$ is admissible, then $\mathrm{BlockOrder}(i, n, o_1, o_2)$ is admissible.

Let $n$ be a natural number. The functor NaivelyOrderedBags $n$ yielding a strict relational structure is defined by the conditions (Def. 11).

(Def. 11)(i) The carrier of NaivelyOrderedBags $n$ = Bags $n$, and

(ii) for all bags $x$, $y$ of $n$ holds $\langle x, y\rangle \in$ the internal relation of NaivelyOrderedBags $n$ iff $x \mid y$.

The following propositions are true:

(33) For every natural number $n$ holds the carrier of $\prod(n \longmapsto \mathrm{OrderedNAT}) = \mathrm{Bags}\, n$.

(34) For every natural number $n$ holds NaivelyOrderedBags $n = \prod(n \longmapsto \mathrm{OrderedNAT})$.

(35) Let $n$ be a natural number and $o$ be an order in Bags $n$. Suppose $o$ is admissible. Then the internal relation of NaivelyOrderedBags $n \subseteq o$ and $o$ is well-ordering.

## 4. Ordering of Finite Subsets

Let $R$ be a connected non empty poset and let $X$ be an element of Fin (the carrier of $R$). Let us assume that $X$ is non empty. The functor PosetMin $X$ yielding an element of $R$ is defined as follows:

(Def. 12) PosetMin $X \in X$ and PosetMin $X$ is minimal w.r.t. $X$, the internal relation of $R$.

The functor PosetMax $X$ yields an element of $R$ and is defined as follows:

(Def. 13) PosetMax $X \in X$ and PosetMax $X$ is maximal w.r.t. $X$, the internal relation of $R$.

Let $R$ be a connected non empty poset. The functor FinOrd-Approx $R$ yielding a function from $\mathbb{N}$ into $2^{[:\mathrm{Fin}\,(\text{the carrier of } R),\ \mathrm{Fin}\,(\text{the carrier of } R):]}$ is defined by the conditions (Def. 14).

(Def. 14)(i) dom FinOrd-Approx $R = \mathbb{N}$,

(ii) (FinOrd-Approx $R$)(0) $= \{\langle x, y\rangle; x$ ranges over elements of Fin (the carrier of $R$), $y$ ranges over elements of Fin (the carrier of $R$): $x = \emptyset \vee x \neq \emptyset \wedge y \neq \emptyset \wedge$ PosetMax $x \neq$ PosetMax $y \wedge \langle$ PosetMax $x$, PosetMax $y\rangle \in$ the internal relation of $R\}$, and

(iii) for every element $n$ of $\mathbb{N}$ holds (FinOrd-Approx $R$)($n+1$) $= \{\langle x, y\rangle; x$ ranges over elements of Fin (the carrier of $R$), $y$ ranges over elements of Fin (the carrier of $R$): $x \neq \emptyset \ \wedge\ y \neq \emptyset \ \wedge$ PosetMax $x =$ PosetMax $y \ \wedge$ $\langle x \setminus \{$PosetMax $x\}, y \setminus \{$PosetMax $y\}\rangle \in$ (FinOrd-Approx $R$)($n$)$\}$.

One can prove the following propositions:

(36) Let $R$ be a connected non empty poset and $x$, $y$ be elements of Fin (the carrier of $R$). Then $\langle x, y\rangle \in \bigcup$ rng FinOrd-Approx $R$ if and only if one of the following conditions is satisfied:

(i) $x = \emptyset$, or

(ii) $x \neq \emptyset$ and $y \neq \emptyset$ and PosetMax $x \neq$ PosetMax $y$ and $\langle$ PosetMax $x$, PosetMax $y\rangle \in$ the internal relation of $R$, or

(iii) $x \neq \emptyset$ and $y \neq \emptyset$ and PosetMax $x =$ PosetMax $y$ and $\langle x \setminus \{$PosetMax $x\}$, $y \setminus \{$PosetMax $y\}\rangle \in \bigcup$ rng FinOrd-Approx $R$.

(37) For every connected non empty poset $R$ and for every element $x$ of Fin (the carrier of $R$) such that $x \neq \emptyset$ holds $\langle x, \emptyset\rangle \notin \bigcup$ rng FinOrd-Approx $R$.

(38) Let $R$ be a connected non empty poset and $a$ be an element of Fin (the carrier of $R$). Then $a \setminus \{$PosetMax $a\}$ is an element of Fin (the carrier of $R$).

(39) For every connected non empty poset $R$ holds $\bigcup$ rng FinOrd-Approx $R$ is an order in Fin (the carrier of $R$).

Let $R$ be a connected non empty poset. The functor FinOrd $R$ yields an order in Fin (the carrier of $R$) and is defined as follows:

(Def. 15) FinOrd $R = \bigcup$ rng FinOrd-Approx $R$.

Let $R$ be a connected non empty poset. The functor FinPoset $R$ yields a poset and is defined by:

(Def. 16) FinPoset $R = \langle$Fin (the carrier of $R$), FinOrd $R\rangle$.

Let $R$ be a connected non empty poset. One can check that FinPoset $R$ is non empty.

The following proposition is true

(40) Let $R$ be a connected non empty poset and $a$, $b$ be elements of FinPoset $R$. Then $\langle a, b\rangle \in$ the internal relation of FinPoset $R$ if and only if there exist elements $x$, $y$ of Fin (the carrier of $R$) such that $a = x$ but $b = y$ but $x = \emptyset$ or $x \neq \emptyset$ and $y \neq \emptyset$ and $\mathrm{PosetMax}\, x \neq \mathrm{PosetMax}\, y$ and $\langle \mathrm{PosetMax}\, x, \mathrm{PosetMax}\, y\rangle \in$ the internal relation of $R$ or $x \neq \emptyset$ and $y \neq \emptyset$ and $\mathrm{PosetMax}\, x = \mathrm{PosetMax}\, y$ and $\langle x \setminus \{\mathrm{PosetMax}\, x\}, y \setminus \{\mathrm{PosetMax}\, y\}\rangle \in \mathrm{FinOrd}\, R$.

Let $R$ be a connected non empty poset. One can verify that FinPoset $R$ is connected.

Let $R$ be a connected non empty relational structure and let $C$ be a non empty set. Let us assume that $R$ is well founded and $C \subseteq$ the carrier of $R$. The functor $\mathrm{MinElement}(C, R)$ yields an element of $R$ and is defined by:

(Def. 17) $\mathrm{MinElement}(C, R) \in C$ and $\mathrm{MinElement}(C, R)$ is minimal w.r.t. $C$, the internal relation of $R$.

Let $R$ be a non empty relational structure, let $s$ be a sequence of $R$, and let $j$ be a natural number. The functor $\mathrm{SeqShift}(s, j)$ yields a sequence of $R$ and is defined by:

(Def. 18) For every natural number $i$ holds $(\mathrm{SeqShift}(s, j))(i) = s(i + j)$.

One can prove the following propositions:

(41) Let $R$ be a non empty relational structure, $s$ be a sequence of $R$, and $j$ be a natural number. If $s$ is descending, then $\mathrm{SeqShift}(s, j)$ is descending.

(42) For every connected non empty poset $R$ such that $R$ is well founded holds FinPoset $R$ is well founded.

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# Combining of Multi Cell Circuits

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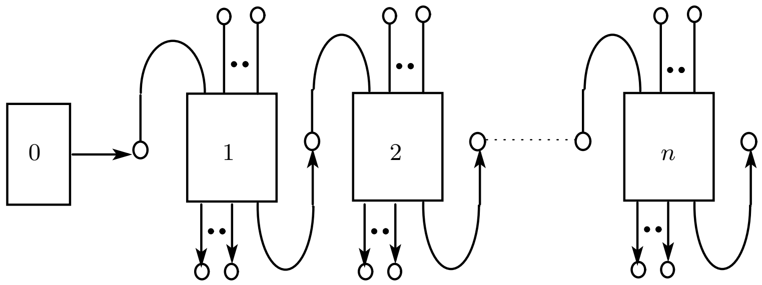
**Summary.** In this article we continue the investigations from [11] and [3] of verification of a circuit design. We concentrate on the combination of multi cell circuits from given cells (circuit modules). Namely, we formalize a design of the form




and prove its stability. The formalization proposed consists in a series of schemes which allow to define multi cells circuits and prove their properties. Our goal is to achive mathematical formalization which will allow to verify designs of real circuits.

MML Identifier: `CIRCCMB2`.

The articles [18], [2], [11], [12], [13], [3], [4], [9], [5], [6], [7], [10], [14], [16], [1], [8], [19], [20], [17], and [15] provide the terminology and notation for this paper.

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## 1. One Gate Circuits

Let $n$ be a natural number, let $f$ be a function from $Boolean^n$ into $Boolean$, and let $p$ be a finite sequence with length $n$. One can verify that 1GateCircuit$(p, f)$ is Boolean.

The following four propositions are true:

(1) Let $X$ be a finite non empty set, $n$ be a natural number, $p$ be a finite sequence with length $n$, $f$ be a function from $X^n$ into $X$, $o$ be an operation symbol of 1GateCircStr$(p, f)$, and $s$ be a state of 1GateCircuit$(p, f)$. Then $o$ depends-on-in $s = s \cdot p$.

(2) Let $X$ be a finite non empty set, $n$ be a natural number, $p$ be a finite sequence with length $n$, $f$ be a function from $X^n$ into $X$, and $s$ be a state of 1GateCircuit$(p, f)$. Then Following$(s)$ is stable.

(3) Let $S$ be a non void circuit-like non empty many sorted signature, $A$ be a non-empty circuit of $S$, and $s$ be a state of $A$. If $s$ is stable, then for every natural number $n$ holds Following$(s, n) = s$.

(4) Let $S$ be a non void circuit-like non empty many sorted signature, $A$ be a non-empty circuit of $S$, $s$ be a state of $A$, and $n_1$, $n_2$ be natural numbers. If Following$(s, n_1)$ is stable and $n_1 \leqslant n_2$, then Following$(s, n_2) =$ Following$(s, n_1)$.

## 2. Defining Multi Cell Circuit Structures

In this article we present several logical schemes. The scheme *CIRCCMB2'sch 1* deals with a non empty many sorted signature $\mathcal{A}$, a set $\mathcal{B}$, a ternary functor $\mathcal{F}$ yielding a non empty many sorted signature, and a binary functor $\mathcal{G}$ yielding a set, and states that:

There exist many sorted sets $f$, $h$ indexed by $\mathbb{N}$ such that
(i) $f(0) = \mathcal{A}$,
(ii) $h(0) = \mathcal{B}$, and
(iii) for every natural number $n$ and for every non empty many sorted signature $S$ and for every set $x$ such that $S = f(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $h(n+1) = \mathcal{G}(x, n)$

for all values of the parameters.

The scheme *CIRCCMB2'sch 2* deals with a ternary functor $\mathcal{F}$ yielding a non empty many sorted signature, a binary functor $\mathcal{G}$ yielding a set, many sorted sets $\mathcal{A}$, $\mathcal{B}$ indexed by $\mathbb{N}$, and a ternary predicate $\mathcal{P}$, and states that:

For every natural number $n$ there exists a non empty many sorted signature $S$ such that $S = \mathcal{A}(n)$ and $\mathcal{P}[S, \mathcal{B}(n), n]$

provided the parameters meet the following requirements:

- There exists a non empty many sorted signature $S$ and there exists a set $x$ such that $S = \mathcal{A}(0)$ and $x = \mathcal{B}(0)$ and $\mathcal{P}[S, x, 0]$,
- Let $n$ be a natural number, $S$ be a non empty many sorted signature, and $x$ be a set. If $S = \mathcal{A}(n)$ and $x = \mathcal{B}(n)$, then $\mathcal{A}(n+1) = \mathcal{F}(S, x, n)$ and $\mathcal{B}(n+1) = \mathcal{G}(x, n)$, and
- Let $n$ be a natural number, $S$ be a non empty many sorted signature, and $x$ be a set. If $S = \mathcal{A}(n)$ and $x = \mathcal{B}(n)$ and $\mathcal{P}[S, x, n]$, then $\mathcal{P}[\mathcal{F}(S, x, n), \mathcal{G}(x, n), n+1]$.

The scheme *CIRCCMB2'sch 3* deals with a non empty many sorted signature $\mathcal{A}$, a ternary functor $\mathcal{F}$ yielding a non empty many sorted signature, a binary functor $\mathcal{G}$ yielding a set, and many sorted sets $\mathcal{B}$, $\mathcal{C}$ indexed by $\mathbb{N}$, and states that:

For every natural number $n$ and for every set $x$ such that $x = \mathcal{C}(n)$ holds $\mathcal{C}(n+1) = \mathcal{G}(x, n)$

provided the following requirements are met:

- $\mathcal{B}(0) = \mathcal{A}$, and
- Let $n$ be a natural number, $S$ be a non empty many sorted signature, and $x$ be a set. If $S = \mathcal{B}(n)$ and $x = \mathcal{C}(n)$, then $\mathcal{B}(n+1) = \mathcal{F}(S, x, n)$ and $\mathcal{C}(n+1) = \mathcal{G}(x, n)$.

The scheme *CIRCCMB2'sch 4* deals with a non empty many sorted signature $\mathcal{A}$, a set $\mathcal{B}$, a ternary functor $\mathcal{F}$ yielding a non empty many sorted signature, a binary functor $\mathcal{G}$ yielding a set, and a natural number $\mathcal{C}$, and states that:

There exists a non empty many sorted signature $S$ and there exist many sorted sets $f$, $h$ indexed by $\mathbb{N}$ such that

(i) $S = f(\mathcal{C})$,
(ii) $f(0) = \mathcal{A}$,
(iii) $h(0) = \mathcal{B}$, and
(iv) for every natural number $n$ and for every non empty many sorted signature $S$ and for every set $x$ such that $S = f(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $h(n+1) = \mathcal{G}(x, n)$

for all values of the parameters.

The scheme *CIRCCMB2'sch 5* deals with a non empty many sorted signature $\mathcal{A}$, a set $\mathcal{B}$, a ternary functor $\mathcal{F}$ yielding a non empty many sorted signature, a binary functor $\mathcal{G}$ yielding a set, and a natural number $\mathcal{C}$, and states that:

Let $S_1$, $S_2$ be non empty many sorted signatures. Suppose that

(i) there exist many sorted sets $f$, $h$ indexed by $\mathbb{N}$ such that $S_1 = f(\mathcal{C})$ and $f(0) = \mathcal{A}$ and $h(0) = \mathcal{B}$ and for every natural number $n$ and for every non empty many sorted signature $S$ and for every set $x$ such that $S = f(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $h(n+1) = \mathcal{G}(x, n)$, and
(ii) there exist many sorted sets $f$, $h$ indexed by $\mathbb{N}$ such that $S_2 = f(\mathcal{C})$ and $f(0) = \mathcal{A}$ and $h(0) = \mathcal{B}$ and for every natural

number $n$ and for every non empty many sorted signature $S$ and for every set $x$ such that $S = f(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $h(n+1) = \mathcal{G}(x, n)$.

Then $S_1 = S_2$

for all values of the parameters.

The scheme *CIRCCMB2'sch 6* deals with a non empty many sorted signature $\mathcal{A}$, a set $\mathcal{B}$, a ternary functor $\mathcal{F}$ yielding a non empty many sorted signature, a binary functor $\mathcal{G}$ yielding a set, and a natural number $\mathcal{C}$, and states that:

(i) There exists a non empty many sorted signature $S$ and there exist many sorted sets $f$, $h$ indexed by $\mathbb{N}$ such that $S = f(\mathcal{C})$ and $f(0) = \mathcal{A}$ and $h(0) = \mathcal{B}$ and for every natural number $n$ and for every non empty many sorted signature $S$ and for every set $x$ such that $S = f(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $h(n+1) = \mathcal{G}(x, n)$, and

(ii) for all non empty many sorted signatures $S_1$, $S_2$ such that there exist many sorted sets $f$, $h$ indexed by $\mathbb{N}$ such that $S_1 = f(\mathcal{C})$ and $f(0) = \mathcal{A}$ and $h(0) = \mathcal{B}$ and for every natural number $n$ and for every non empty many sorted signature $S$ and for every set $x$ such that $S = f(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $h(n+1) = \mathcal{G}(x, n)$ and there exist many sorted sets $f$, $h$ indexed by $\mathbb{N}$ such that $S_2 = f(\mathcal{C})$ and $f(0) = \mathcal{A}$ and $h(0) = \mathcal{B}$ and for every natural number $n$ and for every non empty many sorted signature $S$ and for every set $x$ such that $S = f(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $h(n+1) = \mathcal{G}(x, n)$ holds $S_1 = S_2$

for all values of the parameters.

The scheme *CIRCCMB2'sch 7* deals with a non empty many sorted signature $\mathcal{A}$, a ternary functor $\mathcal{F}$ yielding a non empty many sorted signature, a set $\mathcal{B}$, a binary functor $\mathcal{G}$ yielding a set, and a natural number $\mathcal{C}$, and states that:

There exists an unsplit non void non empty non empty strict many sorted signature $S$ with arity held in gates and Boolean denotation held in gates and there exist many sorted sets $f$, $h$ indexed by $\mathbb{N}$ such that

(i) $S = f(\mathcal{C})$,

(ii) $f(0) = \mathcal{A}$,

(iii) $h(0) = \mathcal{B}$, and

(iv) for every natural number $n$ and for every non empty many sorted signature $S$ and for every set $x$ such that $S = f(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $h(n+1) = \mathcal{G}(x, n)$

provided the parameters meet the following requirements:

- $\mathcal{A}$ is unsplit, non void, non empty, and strict and has arity held in gates and Boolean denotation held in gates, and

- Let $S$ be an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates, $x$ be a set, and $n$ be a natural number. Then $\mathcal{F}(S, x, n)$ is unsplit, non void, non empty, and strict and has arity held in gates and Boolean denotation held in gates.

The scheme *CIRCCMB2'sch 8* deals with a non empty many sorted signature $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding an unsplit non void non empty many sorted signature with arity held in gates and Boolean denotation held in gates, a set $\mathcal{B}$, a binary functor $\mathcal{G}$ yielding a set, and a natural number $\mathcal{C}$, and states that:

There exists an unsplit non void non empty non empty strict many sorted signature $S$ with arity held in gates and Boolean denotation held in gates and there exist many sorted sets $f$, $h$ indexed by $\mathbb{N}$ such that

(i) $S = f(\mathcal{C})$,

(ii) $f(0) = \mathcal{A}$,

(iii) $h(0) = \mathcal{B}$, and

(iv) for every natural number $n$ and for every non empty many sorted signature $S$ and for every set $x$ such that $S = f(n)$ and $x = h(n)$ holds $f(n+1) = S{+}{\cdot}\mathcal{F}(x, n)$ and $h(n+1) = \mathcal{G}(x, n)$

provided the parameters meet the following requirement:

- $\mathcal{A}$ is unsplit, non void, non empty, and strict and has arity held in gates and Boolean denotation held in gates.

The scheme *CIRCCMB2'sch 9* deals with a non empty many sorted signature $\mathcal{A}$, a set $\mathcal{B}$, a ternary functor $\mathcal{F}$ yielding a non empty many sorted signature, a binary functor $\mathcal{G}$ yielding a set, and a natural number $\mathcal{C}$, and states that:

Let $S_1$, $S_2$ be unsplit non void non empty strict non empty many sorted signatures with arity held in gates and Boolean denotation held in gates. Suppose that

(i) there exist many sorted sets $f$, $h$ indexed by $\mathbb{N}$ such that $S_1 = f(\mathcal{C})$ and $f(0) = \mathcal{A}$ and $h(0) = \mathcal{B}$ and for every natural number $n$ and for every non empty many sorted signature $S$ and for every set $x$ such that $S = f(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $h(n+1) = \mathcal{G}(x, n)$, and

(ii) there exist many sorted sets $f$, $h$ indexed by $\mathbb{N}$ such that $S_2 = f(\mathcal{C})$ and $f(0) = \mathcal{A}$ and $h(0) = \mathcal{B}$ and for every natural number $n$ and for every non empty many sorted signature $S$ and for every set $x$ such that $S = f(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $h(n+1) = \mathcal{G}(x, n)$.

Then $S_1 = S_2$

for all values of the parameters.

## 3. Input of Multi Cell Circuit

We now state several propositions:

(5) For all functions $f$, $g$ such that $f \approx g$ holds $\operatorname{rng}(f{+\cdot}g) = \operatorname{rng} f \cup \operatorname{rng} g$.

(6) For all non empty many sorted signatures $S_1$, $S_2$ such that $S_1 \approx S_2$ holds $\mathrm{InputVertices}(S_1{+\cdot}S_2) = (\mathrm{InputVertices}(S_1) \setminus \mathrm{InnerVertices}(S_2)) \cup (\mathrm{InputVertices}(S_2) \setminus \mathrm{InnerVertices}(S_1))$.

(7) For every set $X$ with no pairs and for every binary relation $Y$ holds $X \setminus Y = X$.

(8) For every binary relation $X$ and for all sets $Y$, $Z$ such that $Z \subseteq Y$ and $Y \setminus Z$ has no pairs holds $X \setminus Y = X \setminus Z$.

(9) For all sets $X$, $Z$ and for every binary relation $Y$ such that $Z \subseteq Y$ and $X \setminus Z$ has no pairs holds $X \setminus Y = X \setminus Z$.

Now we present two schemes. The scheme *CIRCCMB2'sch 10* deals with an unsplit non void non empty many sorted signature $\mathcal{A}$ with arity held in gates and Boolean denotation held in gates, a unary functor $\mathcal{F}$ yielding a set, a many sorted set $\mathcal{B}$ indexed by $\mathbb{N}$, a binary functor $\mathcal{G}$ yielding an unsplit non void non empty many sorted signature with arity held in gates and Boolean denotation held in gates, and a binary functor $\mathcal{H}$ yielding a set, and states that:

> Let $n$ be a natural number. Then there exist unsplit non void non empty many sorted signatures $S_1$, $S_2$ with arity held in gates and Boolean denotation held in gates such that $S_1 = \mathcal{F}(n)$ and $S_2 = \mathcal{F}(n+1)$ and $\mathrm{InputVertices}(S_2) = \mathrm{InputVertices}(S_1) \cup (\mathrm{InputVertices}(\mathcal{G}(\mathcal{B}(n), n)) \setminus \{\mathcal{B}(n)\})$ and $\mathrm{InnerVertices}(S_1)$ is a binary relation and $\mathrm{InputVertices}(S_1)$ has no pairs

provided the following requirements are met:

- $\mathrm{InnerVertices}(\mathcal{A})$ is a binary relation,
- $\mathrm{InputVertices}(\mathcal{A})$ has no pairs,
- $\mathcal{F}(0) = \mathcal{A}$ and $\mathcal{B}(0) \in \mathrm{InnerVertices}(\mathcal{A})$,
- For every natural number $n$ and for every set $x$ holds $\mathrm{InnerVertices}(\mathcal{G}(x, n))$ is a binary relation,
- For every natural number $n$ and for every set $x$ such that $x = \mathcal{B}(n)$ holds $\mathrm{InputVertices}(\mathcal{G}(x, n)) \setminus \{x\}$ has no pairs, and
- Let $n$ be a natural number, $S$ be a non empty many sorted signature, and $x$ be a set. Suppose $S = \mathcal{F}(n)$ and $x = \mathcal{B}(n)$. Then $\mathcal{F}(n+1) = S{+\cdot}\mathcal{G}(x, n)$ and $\mathcal{B}(n+1) = \mathcal{H}(x, n)$ and $x \in \mathrm{InputVertices}(\mathcal{G}(x, n))$ and $\mathcal{H}(x, n) \in \mathrm{InnerVertices}(\mathcal{G}(x, n))$.

The scheme *CIRCCMB2'sch 11* deals with a unary functor $\mathcal{F}$ yielding an unsplit non void non empty many sorted signature with arity held in gates and Boolean denotation held in gates, a many sorted set $\mathcal{A}$ indexed by $\mathbb{N}$, a binary functor $\mathcal{G}$ yielding an unsplit non void non empty many sorted signature with arity held in gates and Boolean denotation held in gates, and a binary functor $\mathcal{H}$ yielding a set, and states that:

For every natural number $n$ holds InputVertices($\mathcal{F}(n+1)$) $=$ InputVertices($\mathcal{F}(n)$) $\cup$ (InputVertices($\mathcal{G}(\mathcal{A}(n), n)$) $\setminus \{\mathcal{A}(n)\}$) and InnerVertices($\mathcal{F}(n)$) is a binary relation and InputVertices($\mathcal{F}(n)$) has no pairs

provided the parameters meet the following requirements:

- InnerVertices($\mathcal{F}(0)$) is a binary relation,
- InputVertices($\mathcal{F}(0)$) has no pairs,
- $\mathcal{A}(0) \in$ InnerVertices($\mathcal{F}(0)$),
- For every natural number $n$ and for every set $x$ holds InnerVertices($\mathcal{G}(x, n)$) is a binary relation,
- For every natural number $n$ and for every set $x$ such that $x = \mathcal{A}(n)$ holds InputVertices($\mathcal{G}(x, n)$) $\setminus \{x\}$ has no pairs, and
- Let $n$ be a natural number, $S$ be a non empty many sorted signature, and $x$ be a set. Suppose $S = \mathcal{F}(n)$ and $x = \mathcal{A}(n)$. Then $\mathcal{F}(n+1) = S{+}{\cdot}\mathcal{G}(x, n)$ and $\mathcal{A}(n+1) = \mathcal{H}(x, n)$ and $x \in$ InputVertices($\mathcal{G}(x, n)$) and $\mathcal{H}(x, n) \in$ InnerVertices($\mathcal{G}(x, n)$).

## 4. Defining Multi Cell Circuits

Now we present several schemes. The scheme *CIRCCMB2'sch 12* deals with a non empty many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, a set $\mathcal{C}$, a ternary functor $\mathcal{F}$ yielding a non empty many sorted signature, a 4-ary functor $\mathcal{G}$ yielding a set, and a binary functor $\mathcal{H}$ yielding a set, and states that:

There exist many sorted sets $f$, $g$, $h$ indexed by $\mathbb{N}$ such that

(i) $f(0) = \mathcal{A}$,
(ii) $g(0) = \mathcal{B}$,
(iii) $h(0) = \mathcal{C}$, and
(iv) for every natural number $n$ and for every non empty many sorted signature $S$ and for every non-empty algebra $A$ over $S$ and for every set $x$ such that $S = f(n)$ and $A = g(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $g(n+1) = \mathcal{G}(S, A, x, n)$ and $h(n+1) = \mathcal{H}(x, n)$

for all values of the parameters.

The scheme *CIRCCMB2'sch 13* deals with a ternary functor $\mathcal{F}$ yielding a non empty many sorted signature, a 4-ary functor $\mathcal{G}$ yielding a set, a binary functor $\mathcal{H}$ yielding a set, many sorted sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ indexed by $\mathbb{N}$, and a 4-ary predicate $\mathcal{P}$, and states that:

Let $n$ be a natural number. Then there exists a non empty many sorted signature $S$ and there exists a non-empty algebra $A$ over $S$ such that $S = \mathcal{A}(n)$ and $A = \mathcal{B}(n)$ and $\mathcal{P}[S, A, \mathcal{C}(n), n]$

provided the following conditions are satisfied:

- There exists a non empty many sorted signature $S$ and there exists a non-empty algebra $A$ over $S$ and there exists a set $x$ such that $S = \mathcal{A}(0)$ and $A = \mathcal{B}(0)$ and $x = \mathcal{C}(0)$ and $\mathcal{P}[S, A, x, 0]$,
- Let $n$ be a natural number, $S$ be a non empty many sorted signature, $A$ be a non-empty algebra over $S$, and $x$ be a set. Suppose $S = \mathcal{A}(n)$ and $A = \mathcal{B}(n)$ and $x = \mathcal{C}(n)$. Then $\mathcal{A}(n+1) = \mathcal{F}(S, x, n)$ and $\mathcal{B}(n+1) = \mathcal{G}(S, A, x, n)$ and $\mathcal{C}(n+1) = \mathcal{H}(x, n)$,
- Let $n$ be a natural number, $S$ be a non empty many sorted signature, $A$ be a non-empty algebra over $S$, and $x$ be a set. If $S = \mathcal{A}(n)$ and $A = \mathcal{B}(n)$ and $x = \mathcal{C}(n)$ and $\mathcal{P}[S, A, x, n]$, then $\mathcal{P}[\mathcal{F}(S, x, n), \mathcal{G}(S, A, x, n), \mathcal{H}(x, n), n+1]$, and
- Let $S$ be a non empty many sorted signature, $A$ be a non-empty algebra over $S$, $x$ be a set, and $n$ be a natural number. Then $\mathcal{G}(S, A, x, n)$ is a non-empty algebra over $\mathcal{F}(S, x, n)$.

The scheme *CIRCCMB2'sch 14* deals with a ternary functor $\mathcal{F}$ yielding a non empty many sorted signature, a 4-ary functor $\mathcal{G}$ yielding a set, a binary functor $\mathcal{H}$ yielding a set, and many sorted sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ indexed by $\mathbb{N}$, and states that:

$\mathcal{A} = \mathcal{B}$ and $\mathcal{C} = \mathcal{D}$ and $\mathcal{E} = \mathcal{F}$

provided the parameters meet the following conditions:

- There exists a non empty many sorted signature $S$ and there exists a non-empty algebra $A$ over $S$ such that $S = \mathcal{A}(0)$ and $A = \mathcal{C}(0)$,
- $\mathcal{A}(0) = \mathcal{B}(0)$ and $\mathcal{C}(0) = \mathcal{D}(0)$ and $\mathcal{E}(0) = \mathcal{F}(0)$,
- Let $n$ be a natural number, $S$ be a non empty many sorted signature, $A$ be a non-empty algebra over $S$, and $x$ be a set. Suppose $S = \mathcal{A}(n)$ and $A = \mathcal{C}(n)$ and $x = \mathcal{E}(n)$. Then $\mathcal{A}(n{+}1) = \mathcal{F}(S, x, n)$ and $\mathcal{C}(n+1) = \mathcal{G}(S, A, x, n)$ and $\mathcal{E}(n+1) = \mathcal{H}(x, n)$,
- Let $n$ be a natural number, $S$ be a non empty many sorted signature, $A$ be a non-empty algebra over $S$, and $x$ be a set. Suppose $S = \mathcal{B}(n)$ and $A = \mathcal{D}(n)$ and $x = \mathcal{F}(n)$. Then $\mathcal{B}(n+1) = \mathcal{F}(S, x, n)$ and $\mathcal{D}(n+1) = \mathcal{G}(S, A, x, n)$ and $\mathcal{F}(n+1) = \mathcal{H}(x, n)$, and
- Let $S$ be a non empty many sorted signature, $A$ be a non-empty algebra over $S$, $x$ be a set, and $n$ be a natural number. Then $\mathcal{G}(S, A, x, n)$ is a non-empty algebra over $\mathcal{F}(S, x, n)$.

The scheme *CIRCCMB2'sch 15* deals with a non empty many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, a ternary functor $\mathcal{F}$ yielding a non empty many sorted signature, a 4-ary functor $\mathcal{G}$ yielding a set, a binary functor $\mathcal{H}$ yielding a set, and many sorted sets $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ indexed by $\mathbb{N}$, and states that:

Let $n$ be a natural number, $S$ be a non empty many sorted signature, and $x$ be a set. If $S = \mathcal{C}(n)$ and $x = \mathcal{E}(n)$, then

$\mathcal{C}(n+1) = \mathcal{F}(S, x, n)$ and $\mathcal{E}(n+1) = \mathcal{H}(x, n)$

provided the parameters meet the following conditions:

- $\mathcal{C}(0) = \mathcal{A}$ and $\mathcal{D}(0) = \mathcal{B}$,
- Let $n$ be a natural number, $S$ be a non empty many sorted signature, $A$ be a non-empty algebra over $S$, and $x$ be a set. Suppose $S = \mathcal{C}(n)$ and $A = \mathcal{D}(n)$ and $x = \mathcal{E}(n)$. Then $\mathcal{C}(n+1) = \mathcal{F}(S, x, n)$ and $\mathcal{D}(n+1) = \mathcal{G}(S, A, x, n)$ and $\mathcal{E}(n+1) = \mathcal{H}(x, n)$, and
- Let $S$ be a non empty many sorted signature, $A$ be a non-empty algebra over $S$, $x$ be a set, and $n$ be a natural number. Then $\mathcal{G}(S, A, x, n)$ is a non-empty algebra over $\mathcal{F}(S, x, n)$.

The scheme *CIRCCMB2'sch 16* deals with a non empty many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, a set $\mathcal{C}$, a ternary functor $\mathcal{F}$ yielding a non empty many sorted signature, a 4-ary functor $\mathcal{G}$ yielding a set, a binary functor $\mathcal{H}$ yielding a set, and a natural number $\mathcal{D}$, and states that:

There exists a non empty many sorted signature $S$ and there exists a non-empty algebra $A$ over $S$ and there exist many sorted sets $f$, $g$, $h$ indexed by $\mathbb{N}$ such that

(i) $S = f(\mathcal{D})$,
(ii) $A = g(\mathcal{D})$,
(iii) $f(0) = \mathcal{A}$,
(iv) $g(0) = \mathcal{B}$,
(v) $h(0) = \mathcal{C}$, and
(vi) for every natural number $n$ and for every non empty many sorted signature $S$ and for every non-empty algebra $A$ over $S$ and for every set $x$ such that $S = f(n)$ and $A = g(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $g(n+1) = \mathcal{G}(S, A, x, n)$ and $h(n+1) = \mathcal{H}(x, n)$

provided the following condition is satisfied:

- Let $S$ be a non empty many sorted signature, $A$ be a non-empty algebra over $S$, $x$ be a set, and $n$ be a natural number. Then $\mathcal{G}(S, A, x, n)$ is a non-empty algebra over $\mathcal{F}(S, x, n)$.

The scheme *CIRCCMB2'sch 17* deals with non empty many sorted signatures $\mathcal{A}$, $\mathcal{B}$, a non-empty algebra $\mathcal{C}$ over $\mathcal{A}$, a set $\mathcal{D}$, a ternary functor $\mathcal{F}$ yielding a non empty many sorted signature, a 4-ary functor $\mathcal{G}$ yielding a set, a binary functor $\mathcal{H}$ yielding a set, and a natural number $\mathcal{E}$, and states that:

There exists a non-empty algebra $A$ over $\mathcal{B}$ and there exist many sorted sets $f$, $g$, $h$ indexed by $\mathbb{N}$ such that

(i) $\mathcal{B} = f(\mathcal{E})$,
(ii) $A = g(\mathcal{E})$,
(iii) $f(0) = \mathcal{A}$,
(iv) $g(0) = \mathcal{C}$,
(v) $h(0) = \mathcal{D}$, and

(vi) for every natural number $n$ and for every non empty many sorted signature $S$ and for every non-empty algebra $A$ over $S$ and for every set $x$ such that $S = f(n)$ and $A = g(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $g(n+1) = \mathcal{G}(S, A, x, n)$ and $h(n+1) = \mathcal{H}(x, n)$

provided the parameters meet the following requirements:

- There exist many sorted sets $f$, $h$ indexed by $\mathbb{N}$ such that
  (i) $\mathcal{B} = f(\mathcal{E})$,
  (ii) $f(0) = \mathcal{A}$,
  (iii) $h(0) = \mathcal{D}$, and
  (iv) for every natural number $n$ and for every non empty many sorted signature $S$ and for every set $x$ such that $S = f(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $h(n+1) = \mathcal{H}(x, n)$,
  and
- Let $S$ be a non empty many sorted signature, $A$ be a non-empty algebra over $S$, $x$ be a set, and $n$ be a natural number. Then $\mathcal{G}(S, A, x, n)$ is a non-empty algebra over $\mathcal{F}(S, x, n)$.

The scheme *CIRCCMB2'sch 18* deals with non empty many sorted signatures $\mathcal{A}$, $\mathcal{B}$, a non-empty algebra $\mathcal{C}$ over $\mathcal{A}$, a set $\mathcal{D}$, a ternary functor $\mathcal{F}$ yielding a non empty many sorted signature, a 4-ary functor $\mathcal{G}$ yielding a set, a binary functor $\mathcal{H}$ yielding a set, and a natural number $\mathcal{E}$, and states that:

Let $A_1$, $A_2$ be non-empty algebras over $\mathcal{B}$. Suppose that

(i) there exist many sorted sets $f$, $g$, $h$ indexed by $\mathbb{N}$ such that $\mathcal{B} = f(\mathcal{E})$ and $A_1 = g(\mathcal{E})$ and $f(0) = \mathcal{A}$ and $g(0) = \mathcal{C}$ and $h(0) = \mathcal{D}$ and for every natural number $n$ and for every non empty many sorted signature $S$ and for every non-empty algebra $A$ over $S$ and for every set $x$ such that $S = f(n)$ and $A = g(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $g(n+1) = \mathcal{G}(S, A, x, n)$ and $h(n+1) = \mathcal{H}(x, n)$, and

(ii) there exist many sorted sets $f$, $g$, $h$ indexed by $\mathbb{N}$ such that $\mathcal{B} = f(\mathcal{E})$ and $A_2 = g(\mathcal{E})$ and $f(0) = \mathcal{A}$ and $g(0) = \mathcal{C}$ and $h(0) = \mathcal{D}$ and for every natural number $n$ and for every non empty many sorted signature $S$ and for every non-empty algebra $A$ over $S$ and for every set $x$ such that $S = f(n)$ and $A = g(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $g(n+1) = \mathcal{G}(S, A, x, n)$ and $h(n+1) = \mathcal{H}(x, n)$.

Then $A_1 = A_2$

provided the parameters meet the following condition:

- Let $S$ be a non empty many sorted signature, $A$ be a non-empty algebra over $S$, $x$ be a set, and $n$ be a natural number. Then $\mathcal{G}(S, A, x, n)$ is a non-empty algebra over $\mathcal{F}(S, x, n)$.

The scheme *CIRCCMB2'sch 19* deals with unsplit non void strict non empty many sorted signatures $\mathcal{A}$, $\mathcal{B}$ with arity held in gates and Boolean denotation held in gates, a Boolean strict circuit $\mathcal{C}$ of $\mathcal{A}$ with denotation held in gates, a ternary functor $\mathcal{F}$ yielding a non empty many sorted signature, a 4-ary functor $\mathcal{G}$ yielding a set, a set $\mathcal{D}$, a binary functor $\mathcal{H}$ yielding a set, and a natural number $\mathcal{E}$, and states that:

> There exists a Boolean strict circuit $A$ of $\mathcal{B}$ with denotation held in gates and there exist many sorted sets $f$, $g$, $h$ indexed by $\mathbb{N}$ such that
>
> (i) $\mathcal{B} = f(\mathcal{E})$,
> (ii) $A = g(\mathcal{E})$,
> (iii) $f(0) = \mathcal{A}$,
> (iv) $g(0) = \mathcal{C}$,
> (v) $h(0) = \mathcal{D}$, and
> (vi) for every natural number $n$ and for every non empty many sorted signature $S$ and for every non-empty algebra $A$ over $S$ and for every set $x$ such that $S = f(n)$ and $A = g(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $g(n+1) = \mathcal{G}(S, A, x, n)$ and $h(n+1) = \mathcal{H}(x, n)$

provided the following conditions are satisfied:

- Let $S$ be an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates, $x$ be a set, and $n$ be a natural number. Then $\mathcal{F}(S, x, n)$ is unsplit, non void, and strict and has arity held in gates and Boolean denotation held in gates,
- There exist many sorted sets $f$, $h$ indexed by $\mathbb{N}$ such that
  (i) $\mathcal{B} = f(\mathcal{E})$,
  (ii) $f(0) = \mathcal{A}$,
  (iii) $h(0) = \mathcal{D}$, and
  (iv) for every natural number $n$ and for every non empty many sorted signature $S$ and for every set $x$ such that $S = f(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $h(n+1) = \mathcal{H}(x, n)$,
- Let $S$ be a non empty many sorted signature, $A$ be a non-empty algebra over $S$, $x$ be a set, and $n$ be a natural number. Then $\mathcal{G}(S, A, x, n)$ is a non-empty algebra over $\mathcal{F}(S, x, n)$, and
- Let $S$, $S_1$ be unsplit non void strict non empty many sorted signatures with arity held in gates and Boolean denotation held in gates, $A$ be a Boolean strict circuit of $S$ with denotation held in gates, $x$ be a set, and $n$ be a natural number. Suppose $S_1 = \mathcal{F}(S, x, n)$. Then $\mathcal{G}(S, A, x, n)$ is a Boolean strict circuit of $S_1$ with denotation held in gates.

Let $S$ be a non empty many sorted signature and let $A$ be a set. Let us assume that $A$ is a non-empty algebra over $S$. The functor $\mathrm{MSAlg}(A, S)$ yielding a non-empty algebra over $S$ is defined as follows:

(Def. 1) $\mathrm{MSAlg}(A, S) = A$.

Now we present two schemes. The scheme *CIRCCMB2'sch 20* deals with unsplit non void strict non empty many sorted signatures $\mathcal{A}$, $\mathcal{B}$ with arity held in gates and Boolean denotation held in gates, a Boolean strict circuit $\mathcal{C}$ of $\mathcal{A}$ with denotation held in gates, a binary functor $\mathcal{F}$ yielding an unsplit non void non empty many sorted signature with arity held in gates and Boolean denotation held in gates, a binary functor $\mathcal{G}$ yielding a set, a set $\mathcal{D}$, a binary functor $\mathcal{H}$ yielding a set, and a natural number $\mathcal{E}$, and states that:

There exists a Boolean strict circuit $A$ of $\mathcal{B}$ with denotation held in gates and there exist many sorted sets $f$, $g$, $h$ indexed by $\mathbb{N}$ such that

(i) $\mathcal{B} = f(\mathcal{E})$,

(ii) $A = g(\mathcal{E})$,

(iii) $f(0) = \mathcal{A}$,

(iv) $g(0) = \mathcal{C}$,

(v) $h(0) = \mathcal{D}$, and

(vi) for every natural number $n$ and for every non empty many sorted signature $S$ and for every non-empty algebra $A_1$ over $S$ and for every set $x$ and for every non-empty algebra $A_2$ over $\mathcal{F}(x, n)$ such that $S = f(n)$ and $A_1 = g(n)$ and $x = h(n)$ and $A_2 = \mathcal{G}(x, n)$ holds $f(n+1) = S +\cdot \mathcal{F}(x, n)$ and $g(n+1) = A_1 +\cdot A_2$ and $h(n+1) = \mathcal{H}(x, n)$

provided the parameters meet the following requirements:

- There exist many sorted sets $f$, $h$ indexed by $\mathbb{N}$ such that
  (i) $\mathcal{B} = f(\mathcal{E})$,
  (ii) $f(0) = \mathcal{A}$,
  (iii) $h(0) = \mathcal{D}$, and
  (iv) for every natural number $n$ and for every non empty many sorted signature $S$ and for every set $x$ such that $S = f(n)$ and $x = h(n)$ holds $f(n+1) = S +\cdot \mathcal{F}(x, n)$ and $h(n+1) = \mathcal{H}(x, n)$,

  and
- Let $x$ be a set and $n$ be a natural number. Then $\mathcal{G}(x, n)$ is a Boolean strict circuit of $\mathcal{F}(x, n)$ with denotation held in gates.

The scheme *CIRCCMB2'sch 21* deals with a non empty many sorted signature $\mathcal{A}$, an unsplit non void strict non empty many sorted signature $\mathcal{B}$ with arity held in gates and Boolean denotation held in gates, a non-empty algebra $\mathcal{C}$ over $\mathcal{A}$, a set $\mathcal{D}$, a ternary functor $\mathcal{F}$ yielding a non empty many sorted signature, a 4-ary functor $\mathcal{G}$ yielding a set, a binary functor $\mathcal{H}$ yielding a set, and a natural number $\mathcal{E}$, and states that:

Let $A_1$, $A_2$ be Boolean strict circuits of $\mathcal{B}$ with denotation held in gates. Suppose that

(i) there exist many sorted sets $f$, $g$, $h$ indexed by $\mathbb{N}$ such that $\mathcal{B} = f(\mathcal{E})$ and $A_1 = g(\mathcal{E})$ and $f(0) = \mathcal{A}$ and $g(0) = \mathcal{C}$ and $h(0) = \mathcal{D}$ and for every natural number $n$ and for every non empty many sorted signature $S$ and for every non-empty algebra $A$ over $S$ and for every set $x$ such that $S = f(n)$ and $A = g(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $g(n+1) = \mathcal{G}(S, A, x, n)$ and $h(n+1) = \mathcal{H}(x, n)$, and

(ii) there exist many sorted sets $f$, $g$, $h$ indexed by $\mathbb{N}$ such that $\mathcal{B} = f(\mathcal{E})$ and $A_2 = g(\mathcal{E})$ and $f(0) = \mathcal{A}$ and $g(0) = \mathcal{C}$ and $h(0) = \mathcal{D}$ and for every natural number $n$ and for every non empty many sorted signature $S$ and for every non-empty algebra $A$ over $S$ and for every set $x$ such that $S = f(n)$ and $A = g(n)$ and $x = h(n)$ holds $f(n+1) = \mathcal{F}(S, x, n)$ and $g(n+1) = \mathcal{G}(S, A, x, n)$ and $h(n+1) = \mathcal{H}(x, n)$.

Then $A_1 = A_2$

provided the parameters have the following property:

- Let $S$ be a non empty many sorted signature, $A$ be a non-empty algebra over $S$, $x$ be a set, and $n$ be a natural number. Then $\mathcal{G}(S, A, x, n)$ is a non-empty algebra over $\mathcal{F}(S, x, n)$.

## 5. Stability of Multi Cell Circuit

One can prove the following propositions:

(10) Let $S_1$, $S_2$, $S$ be non void circuit-like non empty many sorted signatures. Suppose InnerVertices($S_1$) misses InputVertices($S_2$) and $S = S_1 {+\cdot} S_2$. Let $C_1$ be a non-empty circuit of $S_1$, $C_2$ be a non-empty circuit of $S_2$, and $C$ be a non-empty circuit of $S$. Suppose $C_1 \approx C_2$ and $C = C_1 {+\cdot} C_2$. Let $s_2$ be a state of $C_2$ and $s$ be a state of $C$. If $s_2 = s {\restriction}$the carrier of $S_2$, then Following($s_2$) = Following($s$)$\restriction$the carrier of $S_2$.

(11) Let $S_1$, $S_2$, $S$ be non void circuit-like non empty many sorted signatures. Suppose InputVertices($S_1$) misses InnerVertices($S_2$) and $S = S_1 {+\cdot} S_2$. Let $C_1$ be a non-empty circuit of $S_1$, $C_2$ be a non-empty circuit of $S_2$, and $C$ be a non-empty circuit of $S$. Suppose $C_1 \approx C_2$ and $C = C_1 {+\cdot} C_2$. Let $s_1$ be a state of $C_1$ and $s$ be a state of $C$. If $s_1 = s {\restriction}$the carrier of $S_1$, then Following($s_1$) = Following($s$)$\restriction$the carrier of $S_1$.

(12) Let $S_1$, $S_2$, $S$ be non void circuit-like non empty many sorted signatures. Suppose $S_1 \approx S_2$ and InnerVertices($S_1$) misses InputVertices($S_2$) and $S = S_1 {+\cdot} S_2$. Let $C_1$ be a non-empty circuit of $S_1$, $C_2$ be a non-empty circuit of $S_2$, and $C$ be a non-empty circuit of $S$. Suppose $C_1 \approx C_2$ and $C = C_1 {+\cdot} C_2$.

Let $s_1$ be a state of $C_1$, $s_2$ be a state of $C_2$, and $s$ be a state of $C$. Suppose $s_1 = s \restriction$the carrier of $S_1$ and $s_2 = s \restriction$the carrier of $S_2$ and $s_1$ is stable and $s_2$ is stable. Then $s$ is stable.

(13) Let $S_1$, $S_2$, $S$ be non void circuit-like non empty many sorted signatures. Suppose $S_1 \approx S_2$ and InputVertices$(S_1)$ misses InnerVertices$(S_2)$ and $S = S_1 +\cdot S_2$. Let $C_1$ be a non-empty circuit of $S_1$, $C_2$ be a non-empty circuit of $S_2$, and $C$ be a non-empty circuit of $S$. Suppose $C_1 \approx C_2$ and $C = C_1 +\cdot C_2$. Let $s_1$ be a state of $C_1$, $s_2$ be a state of $C_2$, and $s$ be a state of $C$. Suppose $s_1 = s \restriction$the carrier of $S_1$ and $s_2 = s \restriction$the carrier of $S_2$ and $s_1$ is stable and $s_2$ is stable. Then $s$ is stable.

(14) Let $S_1$, $S_2$, $S$ be non void circuit-like non empty many sorted signatures. Suppose InputVertices$(S_1)$ misses InnerVertices$(S_2)$ and $S = S_1 +\cdot S_2$. Let $A_1$ be a non-empty circuit of $S_1$, $A_2$ be a non-empty circuit of $S_2$, and $A$ be a non-empty circuit of $S$. Suppose $A_1 \approx A_2$ and $A = A_1 +\cdot A_2$. Let $s$ be a state of $A$ and $s_1$ be a state of $A_1$. Suppose $s_1 = s \restriction$the carrier of $S_1$. Let $n$ be a natural number. Then Following$(s, n) \restriction$the carrier of $S_1 =$ Following$(s_1, n)$.

(15) Let $S_1$, $S_2$, $S$ be non void circuit-like non empty many sorted signatures. Suppose InputVertices$(S_2)$ misses InnerVertices$(S_1)$ and $S = S_1 +\cdot S_2$. Let $A_1$ be a non-empty circuit of $S_1$, $A_2$ be a non-empty circuit of $S_2$, and $A$ be a non-empty circuit of $S$. Suppose $A_1 \approx A_2$ and $A = A_1 +\cdot A_2$. Let $s$ be a state of $A$ and $s_2$ be a state of $A_2$. Suppose $s_2 = s \restriction$the carrier of $S_2$. Let $n$ be a natural number. Then Following$(s, n) \restriction$the carrier of $S_2 =$ Following$(s_2, n)$.

(16) Let $S_1$, $S_2$, $S$ be non void circuit-like non empty many sorted signatures. Suppose InputVertices$(S_1)$ misses InnerVertices$(S_2)$ and $S = S_1 +\cdot S_2$. Let $A_1$ be a non-empty circuit of $S_1$, $A_2$ be a non-empty circuit of $S_2$, and $A$ be a non-empty circuit of $S$. Suppose $A_1 \approx A_2$ and $A = A_1 +\cdot A_2$. Let $s$ be a state of $A$ and $s_1$ be a state of $A_1$. Suppose $s_1 = s \restriction$the carrier of $S_1$ and $s_1$ is stable. Let $s_2$ be a state of $A_2$. If $s_2 = s \restriction$the carrier of $S_2$, then Following$(s) \restriction$the carrier of $S_2 =$ Following$(s_2)$.

(17) Let $S_1$, $S_2$, $S$ be non void circuit-like non empty many sorted signatures. Suppose $S = S_1 +\cdot S_2$. Let $A_1$ be a non-empty circuit of $S_1$, $A_2$ be a non-empty circuit of $S_2$, and $A$ be a non-empty circuit of $S$. Suppose $A_1 \approx A_2$ and $A = A_1 +\cdot A_2$. Let $s$ be a state of $A$ and $s_1$ be a state of $A_1$. Suppose $s_1 = s \restriction$the carrier of $S_1$ and $s_1$ is stable. Let $s_2$ be a state of $A_2$. If $s_2 = s \restriction$the carrier of $S_2$ and $s_2$ is stable, then $s$ is stable.

(18) Let $S_1$, $S_2$, $S$ be non void circuit-like non empty many sorted signatures. Suppose $S = S_1 +\cdot S_2$. Let $A_1$ be a non-empty circuit of $S_1$, $A_2$ be a non-empty circuit of $S_2$, and $A$ be a non-empty circuit of $S$. Suppose $A_1 \approx A_2$ and $A = A_1 +\cdot A_2$. Let $s$ be a state of $A$. Suppose $s$ is stable. Then

(i) for every state $s_1$ of $A_1$ such that $s_1 = s \restriction$the carrier of $S_1$ holds $s_1$ is stable, and

(ii) for every state $s_2$ of $A_2$ such that $s_2 = s \restriction$the carrier of $S_2$ holds $s_2$ is stable.

(19) Let $S_1$, $S_2$, $S$ be non void circuit-like non empty many sorted signatures. Suppose InputVertices($S_1$) misses InnerVertices($S_2$) and $S = S_1 {+\cdot} S_2$. Let $A_1$ be a non-empty circuit of $S_1$, $A_2$ be a non-empty circuit of $S_2$, and $A$ be a non-empty circuit of $S$. Suppose $A_1 \approx A_2$ and $A = A_1 {+\cdot} A_2$. Let $s_1$ be a state of $A_1$, $s_2$ be a state of $A_2$, and $s$ be a state of $A$. Suppose $s_1 = s \restriction$the carrier of $S_1$ and $s_2 = s \restriction$the carrier of $S_2$ and $s_1$ is stable. Let $n$ be a natural number. Then Following$(s, n) \restriction$the carrier of $S_2 =$ Following$(s_2, n)$.

(20) Let $S_1$, $S_2$, $S$ be non void circuit-like non empty many sorted signatures. Suppose InputVertices($S_1$) misses InnerVertices($S_2$) and $S = S_1 {+\cdot} S_2$. Let $A_1$ be a non-empty circuit of $S_1$, $A_2$ be a non-empty circuit of $S_2$, and $A$ be a non-empty circuit of $S$. Suppose $A_1 \approx A_2$ and $A = A_1 {+\cdot} A_2$. Let $n_1$, $n_2$ be natural numbers, $s$ be a state of $A$, $s_1$ be a state of $A_1$, and $s_2$ be a state of $A_2$. Suppose $s_1 = s \restriction$the carrier of $S_1$ and Following$(s_1, n_1)$ is stable and $s_2 =$ Following$(s, n_1) \restriction$the carrier of $S_2$ and Following$(s_2, n_2)$ is stable. Then Following$(s, n_1 + n_2)$ is stable.

(21) Let $S_1$, $S_2$, $S$ be non void circuit-like non empty many sorted signatures. Suppose InputVertices($S_1$) misses InnerVertices($S_2$) and $S = S_1 {+\cdot} S_2$. Let $A_1$ be a non-empty circuit of $S_1$, $A_2$ be a non-empty circuit of $S_2$, and $A$ be a non-empty circuit of $S$. Suppose $A_1 \approx A_2$ and $A = A_1 {+\cdot} A_2$. Let $n_1$, $n_2$ be natural numbers. Suppose for every state $s$ of $A_1$ holds Following$(s, n_1)$ is stable and for every state $s$ of $A_2$ holds Following$(s, n_2)$ is stable. Let $s$ be a state of $A$. Then Following$(s, n_1 + n_2)$ is stable.

(22) Let $S_1$, $S_2$, $S$ be non void circuit-like non empty many sorted signatures. Suppose InputVertices($S_1$) misses InnerVertices($S_2$) and InputVertices($S_2$) misses InnerVertices($S_1$) and $S = S_1 {+\cdot} S_2$. Let $A_1$ be a non-empty circuit of $S_1$, $A_2$ be a non-empty circuit of $S_2$, and $A$ be a non-empty circuit of $S$. Suppose $A_1 \approx A_2$ and $A = A_1 {+\cdot} A_2$. Let $s$ be a state of $A$ and $s_1$ be a state of $A_1$. Suppose $s_1 = s \restriction$the carrier of $S_1$. Let $s_2$ be a state of $A_2$. Suppose $s_2 = s \restriction$the carrier of $S_2$. Let $n$ be a natural number. Then Following$(s, n) =$ Following$(s_1, n) {+\cdot}$ Following$(s_2, n)$.

(23) Let $S_1$, $S_2$, $S$ be non void circuit-like non empty many sorted signatures. Suppose InputVertices($S_1$) misses InnerVertices($S_2$) and InputVertices($S_2$) misses InnerVertices($S_1$) and $S = S_1 {+\cdot} S_2$. Let $A_1$ be a non-empty circuit of $S_1$, $A_2$ be a non-empty circuit of $S_2$, and $A$ be a non-empty circuit of $S$. Suppose $A_1 \approx A_2$ and $A = A_1 {+\cdot} A_2$. Let $n_1$, $n_2$ be natural numbers, $s$ be a state of $A$, and $s_1$ be a state of $A_1$. Suppose $s_1 = s \restriction$the carrier of $S_1$. Let $s_2$ be a state of $A_2$. Suppose $s_2 = s \restriction$the carrier

of $S_2$ and Following$(s_1, n_1)$ is stable and Following$(s_2, n_2)$ is stable. Then Following$(s, \max(n_1, n_2))$ is stable.

(24) Let $S_1$, $S_2$, $S$ be non void circuit-like non empty many sorted signatures. Suppose InputVertices$(S_1)$ misses InnerVertices$(S_2)$ and InputVertices$(S_2)$ misses InnerVertices$(S_1)$ and $S = S_1 +\cdot S_2$. Let $A_1$ be a non-empty circuit of $S_1$, $A_2$ be a non-empty circuit of $S_2$, and $A$ be a non-empty circuit of $S$. Suppose $A_1 \approx A_2$ and $A = A_1 +\cdot A_2$. Let $n$ be a natural number, $s$ be a state of $A$, and $s_1$ be a state of $A_1$. Suppose $s_1 = s \restriction$the carrier of $S_1$. Let $s_2$ be a state of $A_2$. Suppose $s_2 = s \restriction$the carrier of $S_2$ but Following$(s_1, n)$ is not stable or Following$(s_2, n)$ is not stable. Then Following$(s, n)$ is not stable.

(25) Let $S_1$, $S_2$, $S$ be non void circuit-like non empty many sorted signatures. Suppose InputVertices$(S_1)$ misses InnerVertices$(S_2)$ and InputVertices$(S_2)$ misses InnerVertices$(S_1)$ and $S = S_1 +\cdot S_2$. Let $A_1$ be a non-empty circuit of $S_1$, $A_2$ be a non-empty circuit of $S_2$, and $A$ be a non-empty circuit of $S$. Suppose $A_1 \approx A_2$ and $A = A_1 +\cdot A_2$. Let $n_1$, $n_2$ be natural numbers. Suppose for every state $s$ of $A_1$ holds Following$(s, n_1)$ is stable and for every state $s$ of $A_2$ holds Following$(s, n_2)$ is stable. Let $s$ be a state of $A$. Then Following$(s, \max(n_1, n_2))$ is stable.

The scheme *CIRCCMB2'sch 22* deals with unsplit non void strict non empty many sorted signatures $\mathcal{A}$, $\mathcal{B}$ with arity held in gates and Boolean denotation held in gates, a Boolean strict circuit $\mathcal{C}$ of $\mathcal{A}$ with denotation held in gates, a Boolean strict circuit $\mathcal{D}$ of $\mathcal{B}$ with denotation held in gates, a binary functor $\mathcal{F}$ yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates, a binary functor $\mathcal{G}$ yielding a set, a many sorted set $\mathcal{E}$ indexed by $\mathbb{N}$, a set $\mathcal{F}$, a binary functor $\mathcal{H}$ yielding a set, and a unary functor $\mathcal{I}$ yielding a natural number, and states that:

For every state $s$ of $\mathcal{D}$ holds Following$(s, \mathcal{I}(0) + \mathcal{I}(2) \cdot \mathcal{I}(1))$ is stable

provided the following conditions are satisfied:

- Let $x$ be a set and $n$ be a natural number. Then $\mathcal{G}(x, n)$ is a Boolean strict circuit of $\mathcal{F}(x, n)$ with denotation held in gates,
- For every state $s$ of $\mathcal{C}$ holds Following$(s, \mathcal{I}(0))$ is stable,
- Let $n$ be a natural number, $x$ be a set, and $A$ be a non-empty circuit of $\mathcal{F}(x, n)$. If $x = \mathcal{E}(n)$ and $A = \mathcal{G}(x, n)$, then for every state $s$ of $A$ holds Following$(s, \mathcal{I}(1))$ is stable,
- There exist many sorted sets $f$, $g$ indexed by $\mathbb{N}$ such that
  - (i) $\mathcal{B} = f(\mathcal{I}(2))$,
  - (ii) $\mathcal{D} = g(\mathcal{I}(2))$,
  - (iii) $f(0) = \mathcal{A}$,
  - (iv) $g(0) = \mathcal{C}$,

(v) $\mathcal{E}(0) = \mathcal{F}$, and
(vi) for every natural number $n$ and for every non empty many sorted signature $S$ and for every non-empty algebra $A_1$ over $S$ and for every set $x$ and for every non-empty algebra $A_2$ over $\mathcal{F}(x, n)$ such that $S = f(n)$ and $A_1 = g(n)$ and $x = \mathcal{E}(n)$ and $A_2 = \mathcal{G}(x, n)$ holds $f(n+1) = S{+\cdot}\mathcal{F}(x, n)$ and $g(n+1) = A_1{+\cdot}A_2$ and $\mathcal{E}(n + 1) = \mathcal{H}(x, n)$,

- InnerVertices($\mathcal{A}$) is a binary relation and InputVertices($\mathcal{A}$) has no pairs,
- $\mathcal{E}(0) = \mathcal{F}$ and $\mathcal{F} \in$ InnerVertices($\mathcal{A}$),
- For every natural number $n$ and for every set $x$ holds InnerVertices($\mathcal{F}(x, n)$) is a binary relation,
- For every natural number $n$ and for every set $x$ such that $x = \mathcal{E}(n)$ holds InputVertices($\mathcal{F}(x, n)$) $\setminus \{x\}$ has no pairs, and
- For every natural number $n$ and for every set $x$ such that $x = \mathcal{E}(n)$ holds $\mathcal{E}(n + 1) = \mathcal{H}(x, n)$ and $x \in$ InputVertices($\mathcal{F}(x, n)$) and $\mathcal{H}(x, n) \in$ InnerVertices($\mathcal{F}(x, n)$).

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# Full Adder Circuit. Part II

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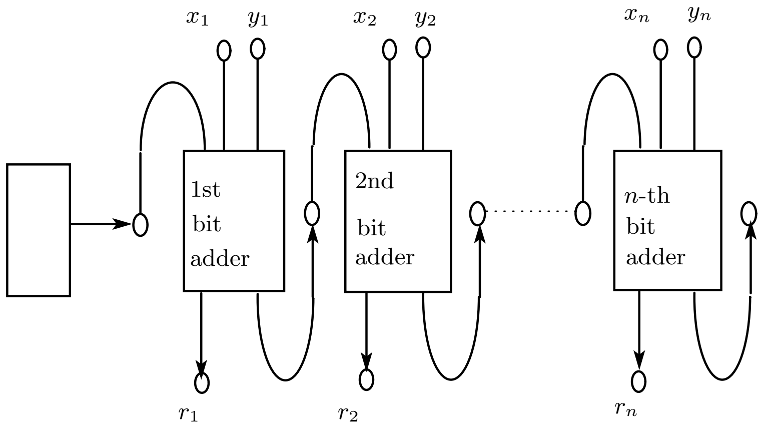
**Summary.** In this article we continue the investigations from [6] of verification of a design of adder circuit. We define it as a combination of 1-bit adders using schemes from [7]. $n$-bit adder circuit has the following structure




As the main result we prove the stability of the circuit. Further works will consist of the proof of full correctness of the circuit.

MML Identifier: FACIRC_2.

The articles [13], [20], [1], [4], [10], [11], [14], [6], [5], [8], [9], [2], [21], [16], [12], [18], [3], [17], [22], [19], and [15] provide the notation and terminology for this paper.

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The following three propositions are true:

(1) For all sets $x$, $y$, $z$ such that $x \neq z$ and $y \neq z$ holds $\{x, y\} \setminus \{z\} = \{x, y\}$.

(2) For all non empty sets $X$, $Y$ and for all natural numbers $n$, $m$ such that $n \neq m$ holds $X^n \neq Y^m$.

(3) For all sets $x$, $y$, $z$ holds $x \neq \langle\langle x, y\rangle, z\rangle$ and $y \neq \langle\langle x, y\rangle, z\rangle$.

Let us note that every many sorted signature which is void is also unsplit and has arity held in gates and Boolean denotation held in gates.

One can check that there exists a many sorted signature which is strict and void.

Let $x$ be a set. The functor SingleMSS $x$ yielding a strict void many sorted signature is defined as follows:

(Def. 1) The carrier of SingleMSS $x = \{x\}$.

Let $x$ be a set. Note that SingleMSS $x$ is non empty.

Let $x$ be a set.

(Def. 2) SingleMSA $x$ is a Boolean strict algebra over SingleMSS $x$.

We now state three propositions:

(4) For every set $x$ and for every many sorted signature $S$ holds SingleMSS $x \approx S$.

(5) Let $x$ be a set and $S$ be a non empty many sorted signature. Suppose $x \in$ the carrier of $S$. Then SingleMSS $x$+·$S$ = the many sorted signature of $S$.

(6) Let $x$ be a set, $S$ be a non empty strict many sorted signature, and $A$ be a Boolean algebra over $S$. If $x \in$ the carrier of $S$, then SingleMSA $x$+·$A$ = the algebra of $A$.

$\emptyset$ is a finite sequence with length 0. We introduce $\varepsilon$ as a synonym of $\emptyset$.

Let $n$ be a natural number and let $x$, $y$ be finite sequences. The functor $n$-BitAdderStr$(x, y)$ yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined by the condition (Def. 3).

(Def. 3) There exist many sorted sets $f$, $g$ indexed by $\mathbb{N}$ such that

(i) $n$-BitAdderStr$(x, y) = f(n)$,

(ii) $f(0) =$ 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$,

(iii) $g(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false}\rangle$, and

(iv) for every natural number $n$ and for every non empty many sorted signature $S$ and for every set $z$ such that $S = f(n)$ and $z = g(n)$ holds $f(n+1) = S$+· BitAdderWithOverflowStr$(x(n+1), y(n+1), z)$ and $g(n+1) =$ MajorityOutput$(x(n+1), y(n+1), z)$.

Let $n$ be a natural number and let $x$, $y$ be finite sequences. The functor $n$-BitAdderCirc$(x, y)$ yields a Boolean strict circuit of $n$-BitAdderStr$(x, y)$ with denotation held in gates and is defined by the condition (Def. 4).

(Def. 4) There exist many sorted sets $f$, $g$, $h$ indexed by $\mathbb{N}$ such that

(i) $n$-BitAdderStr$(x, y) = f(n)$,

(ii) $n$-BitAdderCirc$(x, y) = g(n)$,

(iii) $f(0) = $ 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$,

(iv) $g(0) = $ 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$,

(v) $h(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle$, and

(vi) for every natural number $n$ and for every non empty many sorted signature $S$ and for every non-empty algebra $A$ over $S$ and for every set $z$ such that $S = f(n)$ and $A = g(n)$ and $z = h(n)$ holds $f(n+1) = S +\cdot$ BitAdderWithOverflowStr$(x(n+1), y(n+1), z)$ and $g(n+1) = A +\cdot$ BitAdderWithOverflowCirc$(x(n+1), y(n+1), z)$ and $h(n+1) = $ MajorityOutput$(x(n+1), y(n+1), z)$.

Let $n$ be a natural number and let $x$, $y$ be finite sequences. The functor $n$-BitMajorityOutput$(x, y)$ yielding an element of InnerVertices($n$-BitAdderStr$(x, y)$) is defined by the condition (Def. 5).

(Def. 5) There exists a many sorted set $h$ indexed by $\mathbb{N}$ such that

(i) $n$-BitMajorityOutput$(x, y) = h(n)$,

(ii) $h(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle$, and

(iii) for every natural number $n$ and for every set $z$ such that $z = h(n)$ holds $h(n+1) = $ MajorityOutput$(x(n+1), y(n+1), z)$.

We now state several propositions:

(7) Let $x$, $y$ be finite sequences and $f$, $g$, $h$ be many sorted sets indexed by $\mathbb{N}$. Suppose that

(i) $f(0) = $ 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$,

(ii) $g(0) = $ 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$,

(iii) $h(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle$, and

(iv) for every natural number $n$ and for every non empty many sorted signature $S$ and for every non-empty algebra $A$ over $S$ and for every set $z$ such that $S = f(n)$ and $A = g(n)$ and $z = h(n)$ holds $f(n+1) = S +\cdot$ BitAdderWithOverflowStr$(x(n+1), y(n+1), z)$ and $g(n+1) = A +\cdot$ BitAdderWithOverflowCirc$(x(n+1), y(n+1), z)$ and $h(n+1) = $ MajorityOutput$(x(n+1), y(n+1), z)$.

Let $n$ be a natural number. Then $n$-BitAdderStr$(x, y) = f(n)$ and $n$-BitAdderCirc$(x, y) = g(n)$ and $n$-BitMajorityOutput$(x, y) = h(n)$.

(8) For all finite sequences $a$, $b$ holds 0-BitAdderStr$(a, b) = $ 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and 0-BitAdderCirc$(a, b) = $ 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ and 0-BitMajorityOutput$(a, b) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle$.

(9) Let $a$, $b$ be finite sequences and $c$ be a set. Suppose $c = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false} \rangle$. Then 1-BitAdderStr$(a, b) = $ 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false}) +\cdot$ BitAdderWithOverflowStr$(a(1), b(1), c)$ and 1-BitAdderCirc$(a, b) = $ 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false}) +\cdot$ BitAdderWithOverflowCirc$(a(1),$

$b(1), c)$ and 1-BitMajorityOutput$(a, b) =$ MajorityOutput$(a(1), b(1), c)$.

(10) For all sets $a$, $b$, $c$ such that $c = \langle\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false}\rangle$ holds 1-BitAdderStr$(\langle a\rangle, \langle b\rangle) =$ 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false})$ $+\cdot$ BitAdderWithOverflowStr$(a, b, c)$ and 1-BitAdderCirc$(\langle a\rangle, \langle b\rangle) =$ 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{false}) +\cdot$ BitAdderWithOverflowCirc$(a, b, c)$ and 1-BitMajorityOutput$(\langle a\rangle, \langle b\rangle) =$ MajorityOutput$(a, b, c)$.

(11) Let $n$ be a natural number, $p$, $q$ be finite sequences with length $n$, and $p_1$, $p_2$, $q_1$, $q_2$ be finite sequences. Then $n$-BitAdderStr$(p \frown p_1, q \frown q_1) = n$-BitAdderStr$(p \frown p_2, q \frown q_2)$ and $n$-BitAdderCirc$(p \frown p_1, q \frown q_1) = n$-BitAdderCirc$(p \frown p_2, q \frown q_2)$ and $n$-BitMajorityOutput$(p \frown p_1, q \frown q_1) = n$-BitMajorityOutput$(p \frown p_2, q \frown q_2)$.

(12) Let $n$ be a natural number, $x$, $y$ be finite sequences with length $n$, and $a$, $b$ be sets. Then $(n + 1)$-BitAdderStr$(x \frown \langle a\rangle, y \frown \langle b\rangle) = (n$-BitAdderStr$(x, y)) +\cdot$ BitAdderWithOverflowStr$(a, b,$ $n$-BitMajorityOutput$(x, y))$ and $(n + 1)$-BitAdderCirc$(x \frown \langle a\rangle, y \frown \langle b\rangle) = (n$-BitAdderCirc$(x, y)) +\cdot$ BitAdderWithOverflowCirc $(a, b, n$-BitMajorityOutput$(x, y))$ and $(n + 1)$-BitMajorityOutput$(x \frown \langle a\rangle, y \frown \langle b\rangle) =$ MajorityOutput$(a, b, n$-BitMajorityOutput$(x, y))$.

(13) Let $n$ be a natural number and $x$, $y$ be finite sequences. Then $(n + 1)$-BitAdderStr$(x, y) = (n$-BitAdderStr$(x, y)) +\cdot$ BitAdderWithOverflowStr $(x(n+1), y(n+1), n$-BitMajorityOutput$(x, y))$ and $(n+1)$-BitAdderCirc$(x, y)$ $= (n$-BitAdderCirc$(x, y)) +\cdot$ BitAdderWithOverflowCirc$(x(n+1), y(n+1),$ $n$-BitMajorityOutput$(x, y))$ and $(n + 1)$-BitMajorityOutput$(x, y) =$ MajorityOutput$(x(n + 1), y(n + 1), n$-BitMajorityOutput$(x, y))$.

(14) For all natural numbers $n$, $m$ such that $n \leqslant m$ and for all finite sequences $x$, $y$ holds InnerVertices$(n$-BitAdderStr$(x, y)) \subseteq$ InnerVertices$(m$-BitAdderStr$(x, y))$.

(15) For every natural number $n$ and for all finite sequences $x$, $y$ holds InnerVertices$((n+1)$-BitAdderStr$(x, y)) =$ InnerVertices$(n$-BitAdderStr$(x, y)) \cup$ InnerVertices(BitAdderWithOverflowStr$(x(n + 1), y(n + 1),$ $n$-BitMajorityOutput$(x, y)))$.

Let $k$, $n$ be natural numbers. Let us assume that $k \geqslant 1$ and $k \leqslant n$. Let $x$, $y$ be finite sequences. The functor $(k, n)$-BitAdderOutput$(x, y)$ yielding an element of InnerVertices$(n$-BitAdderStr$(x, y))$ is defined by:

(Def. 6) There exists a natural number $i$ such that $k = i + 1$ and $(k, n)$-BitAdderOutput$(x, y) =$ BitAdderOutput$(x(k), y(k),$ $i$-BitMajorityOutput$(x, y))$.

Next we state several propositions:

(16) For all natural numbers $n$, $k$ such that $k < n$ and for all finite sequences $x$, $y$ holds $(k + 1, n)$-BitAdderOutput$(x, y) =$ BitAdderOutput$(x(k +$

$1), y(k+1), k$-BitMajorityOutput$(x, y))$.

(17) For every natural number $n$ and for all finite sequences $x$, $y$ holds InnerVertices($n$-BitAdderStr$(x, y)$) is a binary relation.

(18) For all sets $x$, $y$, $c$ holds InnerVertices(MajorityIStr$(x, y, c)$) $= \{\langle\langle x, y\rangle, \&\rangle, \langle\langle y, c\rangle, \&\rangle, \langle\langle c, x\rangle, \&\rangle\}$.

(19) For all sets $x$, $y$, $c$ such that $x \neq \langle\langle y, c\rangle, \&\rangle$ and $y \neq \langle\langle c, x\rangle, \&\rangle$ and $c \neq \langle\langle x, y\rangle, \&\rangle$ holds InputVertices(MajorityIStr$(x, y, c)$) $= \{x, y, c\}$.

(20) For all sets $x$, $y$, $c$ holds InnerVertices(MajorityStr$(x, y, c)$) $= \{\langle\langle x, y\rangle, \&\rangle, \langle\langle y, c\rangle, \&\rangle, \langle\langle c, x\rangle, \&\rangle\} \cup \{$MajorityOutput$(x, y, c)\}$.

(21) For all sets $x$, $y$, $c$ such that $x \neq \langle\langle y, c\rangle, \&\rangle$ and $y \neq \langle\langle c, x\rangle, \&\rangle$ and $c \neq \langle\langle x, y\rangle, \&\rangle$ holds InputVertices(MajorityStr$(x, y, c)$) $= \{x, y, c\}$.

(22) For all non empty many sorted signatures $S_1$, $S_2$ such that $S_1 \approx S_2$ and InputVertices$(S_1)$ = InputVertices$(S_2)$ holds InputVertices$(S_1 +\cdot S_2)$ = InputVertices$(S_1)$.

(23) For all sets $x$, $y$, $c$ such that $x \neq \langle\langle y, c\rangle, \&\rangle$ and $y \neq \langle\langle c, x\rangle, \&\rangle$ and $c \neq \langle\langle x, y\rangle, \&\rangle$ and $c \neq \langle\langle x, y\rangle, \mathrm{xor}\rangle$ holds InputVertices(BitAdderWithOverflowStr$(x, y, c)$) $= \{x, y, c\}$.

(24) For all sets $x, y, c$ holds InnerVertices(BitAdderWithOverflowStr$(x, y, c)$) $= \{\langle\langle x, y\rangle, \mathrm{xor}\rangle$, 2GatesCircOutput$(x, y, c, \mathrm{xor})\} \cup \{\langle\langle x, y\rangle, \&\rangle, \langle\langle y, c\rangle, \&\rangle, \langle\langle c, x\rangle, \&\rangle\} \cup \{$MajorityOutput$(x, y, c)\}$.

Let us mention that every set which is empty is also non pair.

Observe that $\emptyset$ is nonpair yielding. Let $f$ be a nonpair yielding function and let $x$ be a set. Observe that $f(x)$ is non pair.

Let $n$ be a natural number and let $x$, $y$ be finite sequences. Note that $n$-BitMajorityOutput$(x, y)$ is pair.

The following propositions are true:

(25) Let $x$, $y$ be finite sequences and $n$ be a natural number. Then ($n$-BitMajorityOutput$(x, y))_{\mathbf{1}} = \varepsilon$ and ($n$-BitMajorityOutput$(x, y))_{\mathbf{2}}$ = $\underline{\underline{\mathit{Boolean}^0 \longmapsto \mathit{false}}}$ and $\pi_1$(($n$-BitMajorityOutput$(x, y))_{\mathbf{2}}) = \mathit{Boolean}^0$ or $\overline{(n\text{-BitMajorityOutput}(x, y))_{\mathbf{1}}} = 3$ and ($n$-BitMajorityOutput$(x, y))_{\mathbf{2}}$ = $\mathrm{or}_3$ and $\pi_1$(($n$-BitMajorityOutput$(x, y))_{\mathbf{2}}) = \mathit{Boolean}^3$.

(26) For every natural number $n$ and for all finite sequences $x$, $y$ and for every set $p$ holds $n$-BitMajorityOutput$(x, y) \neq \langle p, \&\rangle$ and $n$-BitMajorityOutput$(x, y) \neq \langle p, \mathrm{xor}\rangle$.

(27) Let $f$, $g$ be nonpair yielding finite sequences and $n$ be a natural number. Then InputVertices($(n+1)$-BitAdderStr$(f, g)$) = InputVertices($n$-BitAdderStr$(f, g)$) $\cup$ (InputVertices (BitAdderWithOverflowStr$(f(n+1), g(n+1), n$-BitMajorityOutput$(f, g)$)) $\setminus$ $\{n$-BitMajorityOutput$(f, g)\}$) and InnerVertices($n$-BitAdderStr$(f, g)$) is a binary relation and InputVertices($n$-BitAdderStr$(f, g)$) has no pairs.

(28) For every natural number $n$ and for all nonpair yielding finite sequences $x$, $y$ with length $n$ holds InputVertices($n$-BitAdderStr$(x, y)$) $=$ $\operatorname{rng} x \cup \operatorname{rng} y$.

(29) Let $x$, $y$, $c$ be sets, $s$ be a state of MajorityCirc$(x, y, c)$, and $a_1$, $a_2$, $a_3$ be elements of *Boolean*. If $a_1 = s(\langle\!\langle x, y\rangle, \&\rangle\!\rangle)$ and $a_2 = s(\langle\!\langle y, c\rangle, \&\rangle\!\rangle)$ and $a_3 = s(\langle\!\langle c, x\rangle, \&\rangle\!\rangle)$, then (Following$(s)$)(MajorityOutput$(x, y, c)$) $=$ $a_1 \vee a_2 \vee a_3$.

(30) Let $x$, $y$, $c$ be sets. Suppose $x \neq \langle\!\langle y, c\rangle, \&\rangle\!\rangle$ and $y \neq \langle\!\langle c, x\rangle, \&\rangle\!\rangle$ and $c \neq \langle\!\langle x, y\rangle, \&\rangle\!\rangle$ and $c \neq \langle\!\langle x, y\rangle, \mathrm{xor}\rangle\!\rangle$. Let $s$ be a state of MajorityCirc$(x, y, c)$. Then Following$(s, 2)$ is stable.

(31) Let $x$, $y$, $c$ be sets. Suppose $x \neq \langle\!\langle y, c\rangle, \&\rangle\!\rangle$ and $y \neq \langle\!\langle c, x\rangle, \&\rangle\!\rangle$ and $c \neq \langle\!\langle x, y\rangle, \&\rangle\!\rangle$ and $c \neq \langle\!\langle x, y\rangle, \mathrm{xor}\rangle\!\rangle$. Let $s$ be a state of BitAdderWithOverflowCirc$(x, y, c)$ and $a_1$, $a_2$, $a_3$ be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(c)$. Then (Following$(s, 2)$)(BitAdderOutput$(x, y, c)$) $= a_1 \oplus a_2 \oplus a_3$ and (Following$(s, 2)$)(MajorityOutput$(x, y, c)$) $= a_1 \wedge a_2 \vee a_2 \wedge a_3 \vee a_3 \wedge a_1$.

(32) Let $x$, $y$, $c$ be sets. Suppose $x \neq \langle\!\langle y, c\rangle, \&\rangle\!\rangle$ and $y \neq \langle\!\langle c, x\rangle, \&\rangle\!\rangle$ and $c \neq \langle\!\langle x, y\rangle, \&\rangle\!\rangle$ and $c \neq \langle\!\langle x, y\rangle, \mathrm{xor}\rangle\!\rangle$. Let $s$ be a state of BitAdderWithOverflowCirc$(x, y, c)$. Then Following$(s, 2)$ is stable.

(33) Let $n$ be a natural number, $x$, $y$ be nonpair yielding finite sequences with length $n$, and $s$ be a state of $n$-BitAdderCirc$(x, y)$. Then Following$(s, 1 + 2 \cdot n)$ is stable.

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# Contents

*Formaliz. Math.* 10 (**1**)

**Fan Homeomorphisms in the Plane**
By Yatsuka Nakamura .................................... **1**

**Half Open Intervals in Real Numbers**
By Yatsuka Nakamura .................................... **21**

**Some Remarks on Clockwise Oriented Sequences on Go-boards**
By Adam Naumowicz and Robert Milewski ............... **23**

**Dickson's Lemma**
By Gilbert Lee and Piotr Rudnicki ....................... **29**

**On Ordering of Bags**
By Gilbert Lee and Piotr Rudnicki ....................... **39**

**Combining of Multi Cell Circuits**
By Grzegorz Bancerek *et al.* .............................. **47**

**Full Adder Circuit. Part II**
By Grzegorz Bancerek *et al.* .............................. **65**

*Continued on inside back cover*