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On the Instructions of SCM[1]

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The articles [15], [8], [9], [10], [14], [11], [18], [2], [4], [6], [7], [5], [16], [1], [3], [19], [20], [12], [17], and [13] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: a, b are data-locations, i_1, i_2, i_3 are instruction-locations of **SCM**, s_1, s_2 are states of **SCM**, T is an instruction type of **SCM**, and k is a natural number.

We now state a number of propositions:

(1) $a \notin$ the instruction locations of **SCM**.

(2) $\text{Data-Loc}_{\text{SCM}} \neq$ the instruction locations of **SCM**.

(3) For every object o of **SCM** holds $o = \mathbf{IC}_{\mathbf{SCM}}$ or $o \in$ the instruction locations of **SCM** or o is a data-location.

(4) If $i_2 \neq i_3$, then $\text{Next}(i_2) \neq \text{Next}(i_3)$.

(5) If s_1 and s_2 are equal outside the instruction locations of **SCM**, then $s_1(a) = s_2(a)$.

(6) Let N be a set with non empty elements, S be a realistic IC-Ins-separated definite non empty non void AMI over N, t, u be states of S, i_1 be an instruction-location of S, e be an element of $\text{ObjectKind}(\mathbf{IC}_S)$, and I be an element of $\text{ObjectKind}(i_1)$. If $e = i_1$ and $u = t{+\cdot}[\mathbf{IC}_S \longmapsto e, i_1 \longmapsto I]$, then $u(i_1) = I$ and $\mathbf{IC}_u = i_1$ and $\mathbf{IC}_{\text{Following}(u)} = (\text{Exec}(u(\mathbf{IC}_u), u))(\mathbf{IC}_S)$.

(7) $\text{AddressPart}(\mathbf{halt}_{\mathbf{SCM}}) = \emptyset$.

(8) $\text{AddressPart}(a{:=}b) = \langle a, b \rangle$.

(9) $\text{AddressPart}(\text{AddTo}(a, b)) = \langle a, b \rangle$.

(10) $\text{AddressPart}(\text{SubFrom}(a, b)) = \langle a, b \rangle$.

(11) $\text{AddressPart}(\text{MultBy}(a, b)) = \langle a, b \rangle$.

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(12) $\mathrm{AddressPart}(\mathrm{Divide}(a, b)) = \langle a, b\rangle$.

(13) $\mathrm{AddressPart}(\mathrm{goto}\ i_2) = \langle i_2\rangle$.

(14) $\mathrm{AddressPart}(\mathbf{if}\ a = 0\ \mathbf{goto}\ i_2) = \langle i_2, a\rangle$.

(15) $\mathrm{AddressPart}(\mathbf{if}\ a > 0\ \mathbf{goto}\ i_2) = \langle i_2, a\rangle$.

(16) If $T = 0$, then $\mathrm{AddressParts}\, T = \{0\}$.

Let us consider T. One can check that $\mathrm{AddressParts}\, T$ is non empty.

The following propositions are true:

(17) If $T = 1$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\, T} = \{1, 2\}$.

(18) If $T = 2$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\, T} = \{1, 2\}$.

(19) If $T = 3$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\, T} = \{1, 2\}$.

(20) If $T = 4$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\, T} = \{1, 2\}$.

(21) If $T = 5$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\, T} = \{1, 2\}$.

(22) If $T = 6$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\, T} = \{1\}$.

(23) If $T = 7$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\, T} = \{1, 2\}$.

(24) If $T = 8$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\, T} = \{1, 2\}$.

(25) $\prod_{\mathrm{AddressParts}\, \mathrm{InsCode}(a := b)}(1) = \mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$.

(26) $\prod_{\mathrm{AddressParts}\, \mathrm{InsCode}(a := b)}(2) = \mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$.

(27) $\prod_{\mathrm{AddressParts}\, \mathrm{InsCode}(\mathrm{AddTo}(a, b))}(1) = \mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$.

(28) $\prod_{\mathrm{AddressParts}\, \mathrm{InsCode}(\mathrm{AddTo}(a, b))}(2) = \mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$.

(29) $\prod_{\mathrm{AddressParts}\, \mathrm{InsCode}(\mathrm{SubFrom}(a, b))}(1) = \mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$.

(30) $\prod_{\mathrm{AddressParts}\, \mathrm{InsCode}(\mathrm{SubFrom}(a, b))}(2) = \mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$.

(31) $\prod_{\mathrm{AddressParts}\, \mathrm{InsCode}(\mathrm{MultBy}(a, b))}(1) = \mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$.

(32) $\prod_{\mathrm{AddressParts}\, \mathrm{InsCode}(\mathrm{MultBy}(a, b))}(2) = \mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$.

(33) $\prod_{\mathrm{AddressParts}\, \mathrm{InsCode}(\mathrm{Divide}(a, b))}(1) = \mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$.

(34) $\prod_{\mathrm{AddressParts}\, \mathrm{InsCode}(\mathrm{Divide}(a, b))}(2) = \mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$.

(35) $\prod_{\mathrm{AddressParts}\, \mathrm{InsCode}(\mathrm{goto}\ i_2)}(1) =$ the instruction locations of **SCM**.

(36) $\prod_{\mathrm{AddressParts}\, \mathrm{InsCode}(\mathbf{if}\ a=0\ \mathbf{goto}\ i_2)}(1) =$ the instruction locations of **SCM**.

(37) $\prod_{\mathrm{AddressParts}\, \mathrm{InsCode}(\mathbf{if}\ a=0\ \mathbf{goto}\ i_2)}(2) = \mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$.

(38) $\prod_{\mathrm{AddressParts}\, \mathrm{InsCode}(\mathbf{if}\ a>0\ \mathbf{goto}\ i_2)}(1) =$ the instruction locations of **SCM**.

(39) $\prod_{\mathrm{AddressParts}\, \mathrm{InsCode}(\mathbf{if}\ a>0\ \mathbf{goto}\ i_2)}(2) = \mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$.

(40) $\mathrm{NIC}(\mathbf{halt}_{\mathbf{SCM}}, i_1) = \{i_1\}$.

Let us note that $\mathrm{JUMP}(\mathbf{halt}_{\mathbf{SCM}})$ is empty.

One can prove the following proposition

(41) $\mathrm{NIC}(a := b, i_1) = \{\mathrm{Next}(i_1)\}$.

Let us consider a, b. One can verify that JUMP(a:=b) is empty.

Next we state the proposition

(42) $\mathrm{NIC}(\mathrm{AddTo}(a,b),i_1) = \{\mathrm{Next}(i_1)\}$.

Let us consider a, b. Note that JUMP(AddTo(a,b)) is empty.

The following proposition is true

(43) $\mathrm{NIC}(\mathrm{SubFrom}(a,b),i_1) = \{\mathrm{Next}(i_1)\}$.

Let us consider a, b. One can check that JUMP(SubFrom(a,b)) is empty.

Next we state the proposition

(44) $\mathrm{NIC}(\mathrm{MultBy}(a,b),i_1) = \{\mathrm{Next}(i_1)\}$.

Let us consider a, b. Observe that JUMP(MultBy(a,b)) is empty.

The following proposition is true

(45) $\mathrm{NIC}(\mathrm{Divide}(a,b),i_1) = \{\mathrm{Next}(i_1)\}$.

Let us consider a, b. Note that JUMP(Divide(a,b)) is empty.

We now state two propositions:

(46) $\mathrm{NIC}(\mathrm{goto}\ i_2,i_1) = \{i_2\}$.

(47) $\mathrm{JUMP}(\mathrm{goto}\ i_2) = \{i_2\}$.

Let us consider i_2. One can check that JUMP(goto i_2) is non empty and trivial.

The following two propositions are true:

(48) $i_2 \in \mathrm{NIC}(\mathbf{if}\ a = 0\ \mathbf{goto}\ i_2,i_1)$ and $\mathrm{NIC}(\mathbf{if}\ a = 0\ \mathbf{goto}\ i_2,i_1) \subseteq \{i_2,\mathrm{Next}(i_1)\}$.

(49) $\mathrm{JUMP}(\mathbf{if}\ a = 0\ \mathbf{goto}\ i_2) = \{i_2\}$.

Let us consider a, i_2. Note that JUMP(**if** $a = 0$ **goto** i_2) is non empty and trivial.

One can prove the following propositions:

(50) $i_2 \in \mathrm{NIC}(\mathbf{if}\ a > 0\ \mathbf{goto}\ i_2,i_1)$ and $\mathrm{NIC}(\mathbf{if}\ a > 0\ \mathbf{goto}\ i_2,i_1) \subseteq \{i_2,\mathrm{Next}(i_1)\}$.

(51) $\mathrm{JUMP}(\mathbf{if}\ a > 0\ \mathbf{goto}\ i_2) = \{i_2\}$.

Let us consider a, i_2. One can check that JUMP(**if** $a > 0$ **goto** i_2) is non empty and trivial.

Next we state two propositions:

(52) $\mathrm{SUCC}(i_1) = \{i_1,\mathrm{Next}(i_1)\}$.

(53) Let f be a function from $\mathbb{N}$ into the instruction locations of **SCM**. Suppose that for every natural number k holds $f(k) = \mathbf{i}_k$. Then

(i) f is bijective, and

(ii) for every natural number k holds $f(k+1) \in \mathrm{SUCC}(f(k))$ and for every natural number j such that $f(j) \in \mathrm{SUCC}(f(k))$ holds $k \leqslant j$.

Let us note that **SCM** is standard.

One can prove the following three propositions:

(54) $\mathrm{il}_{\mathbf{SCM}}(k) = \mathbf{i}_k$.

(55) $\mathrm{Next}(\mathrm{il}_{\mathbf{SCM}}(k)) = \mathrm{il}_{\mathbf{SCM}}(k+1)$.

(56) $\mathrm{Next}(i_1) = \mathrm{NextLoc}\, i_1$.

Let us observe that InsCode($\mathbf{halt}_{\mathbf{SCM}}$) is jump-only.

Let us observe that $\mathbf{halt}_{\mathbf{SCM}}$ is jump-only.

Let us consider i_2. Observe that InsCode(goto i_2) is jump-only.

Let us consider i_2. Note that goto i_2 is jump-only non sequential and non instruction location free.

Let us consider a, i_2. One can verify that InsCode(**if** $a = 0$ **goto** i_2) is jump-only and InsCode(**if** $a > 0$ **goto** i_2) is jump-only.

Let us consider a, i_2. One can verify that **if** $a = 0$ **goto** i_2 is jump-only non sequential and non instruction location free and **if** $a > 0$ **goto** i_2 is jump-only non sequential and non instruction location free.

Let us consider a, b. One can verify the following observations:

* InsCode(a:=b) is non jump-only,
* InsCode(AddTo(a, b)) is non jump-only,
* InsCode(SubFrom(a, b)) is non jump-only,
* InsCode(MultBy(a, b)) is non jump-only, and
* InsCode(Divide(a, b)) is non jump-only.

Let us consider a, b. One can check the following observations:

* a:=b is non jump-only and sequential,
* AddTo(a, b) is non jump-only and sequential,
* SubFrom(a, b) is non jump-only and sequential,
* MultBy(a, b) is non jump-only and sequential, and
* Divide(a, b) is non jump-only and sequential.

Let us note that **SCM** is homogeneous and has explicit jumps and no implicit jumps.

Let us observe that **SCM** is regular.

We now state three propositions:

(57) IncAddr(goto i_2, k) = goto $\mathrm{il}_{\mathbf{SCM}}(\mathrm{locnum}(i_2) + k)$.

(58) IncAddr(**if** $a = 0$ **goto** i_2, k) = **if** $a = 0$ **goto** $\mathrm{il}_{\mathbf{SCM}}(\mathrm{locnum}(i_2) + k)$.

(59) IncAddr(**if** $a > 0$ **goto** i_2, k) = **if** $a > 0$ **goto** $\mathrm{il}_{\mathbf{SCM}}(\mathrm{locnum}(i_2) + k)$.

Let us note that **SCM** is IC-good and Exec-preserving.

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Input and Output of Instructions[1]

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The terminology and notation used here are introduced in the following articles: [10], [5], [9], [6], [13], [1], [7], [4], [2], [11], [3], [12], and [8].

1. Preliminaries

In this paper N is a set with non empty elements.

One can prove the following propositions:

(1) For all sets x, y, z such that $x \neq y$ and $x \neq z$ holds $\{x, y, z\} \setminus \{x\} = \{y, z\}$.

(2) For every non empty non void AMI A over N and for every state s of A and for every object o of A holds $s(o) \in \mathrm{ObjectKind}(o)$.

(3) Let A be a realistic IC-Ins-separated definite non empty non void AMI over N, s be a state of A, f be an instruction-location of A, and w be an element of $\mathrm{ObjectKind}(\mathbf{IC}_A)$. Then $(s +\cdot (\mathbf{IC}_A, w))(f) = s(f)$.

Let N be a set with non empty elements, let A be an IC-Ins-separated definite non empty non void AMI over N, let s be a state of A, let o be an object of A, and let a be an element of $\mathrm{ObjectKind}(o)$. Then $s +\cdot (o, a)$ is a state of A.

We now state several propositions:

(4) Let A be a steady-programmed IC-Ins-separated definite non empty non void AMI over N, s be a state of A, o be an object of A, f be an instruction-location of A, I be an instruction of A, and w be an element of $\mathrm{ObjectKind}(o)$. If $f \neq o$, then $(\mathrm{Exec}(I, s))(f) = (\mathrm{Exec}(I, s +\cdot (o, w)))(f)$.

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(5) Let A be an IC-Ins-separated definite non empty non void AMI over N, s be a state of A, o be an object of A, and w be an element of ObjectKind(o). If $o \neq \mathbf{IC}_A$, then $\mathbf{IC}_s = \mathbf{IC}_{s+\cdot(o,w)}$.

(6) Let A be a standard IC-Ins-separated definite non empty non void AMI over N, I be an instruction of A, s be a state of A, o be an object of A, and w be an element of ObjectKind(o). If I is sequential and $o \neq \mathbf{IC}_A$, then $\mathbf{IC}_{\mathrm{Exec}(I,s)} = \mathbf{IC}_{\mathrm{Exec}(I,s+\cdot(o,w))}$.

(7) Let A be a standard IC-Ins-separated definite non empty non void AMI over N, I be an instruction of A, s be a state of A, o be an object of A, and w be an element of ObjectKind(o). If I is sequential and $o \neq \mathbf{IC}_A$, then $\mathbf{IC}_{\mathrm{Exec}(I,s+\cdot(o,w))} = \mathbf{IC}_{\mathrm{Exec}(I,s)+\cdot(o,w)}$.

(8) Let A be a standard steady-programmed IC-Ins-separated definite non empty non void AMI over N, I be an instruction of A, s be a state of A, o be an object of A, w be an element of ObjectKind(o), and i be an instruction-location of A. Then $(\mathrm{Exec}(I, s +\cdot (o, w)))(i) = (\mathrm{Exec}(I, s) +\cdot (o, w))(i)$.

2. Input and Output of Instructions

Let N be a set and let A be an AMI over N. We say that A has non trivial instruction set if and only if:

(Def. 1) The instructions of A are non trivial.

Let N be a set and let A be a non empty AMI over N. We say that A has non trivial ObjectKinds if and only if:

(Def. 2) For every object o of A holds ObjectKind(o) is non trivial.

Let N be a set with non empty elements. One can verify that STC(N) has non trivial ObjectKinds.

Let N be a set with non empty elements. Observe that there exists a regular standard IC-Ins-separated definite non empty non void AMI over N which is halting, realistic, steady-programmed, programmable, IC-good, and Exec-preserving and has explicit jumps, no implicit jumps, non trivial ObjectKinds, and non trivial instruction set.

Let N be a set with non empty elements. Note that every definite non empty non void AMI over N which has non trivial ObjectKinds has also non trivial instruction set.

Let N be a set with non empty elements. One can check that every IC-Ins-separated non empty AMI over N which has non trivial ObjectKinds has also non trivial instruction locations.

Let N be a set with non empty elements, let A be a non empty AMI over N with non trivial ObjectKinds, and let o be an object of A. Observe that ObjectKind(o) is non trivial.

Let N be a set with non empty elements and let A be an AMI over N with non trivial instruction set. Note that the instructions of A is non trivial.

Let N be a set with non empty elements and let A be an IC-Ins-separated non empty AMI over N with non trivial instruction locations. Note that ObjectKind$(\mathbf{IC}_A)$ is non trivial.

Let N be a set with non empty elements, let A be a non empty non void AMI over N, and let I be an instruction of A. The functor Output I yielding a subset of the carrier of A is defined as follows:

(Def. 3) For every object o of A holds $o \in \mathrm{Output}\, I$ iff there exists a state s of A such that $s(o) \neq (\mathrm{Exec}(I, s))(o)$.

Let N be a set with non empty elements, let A be an IC-Ins-separated definite non empty non void AMI over N, and let I be an instruction of A. The functor IODiff I yielding a subset of the carrier of A is defined by the condition (Def. 4).

(Def. 4) Let o be an object of A. Then $o \in \mathrm{IODiff}\, I$ if and only if for every state s of A and for every element a of ObjectKind(o) holds $\mathrm{Exec}(I, s) = \mathrm{Exec}(I, s +\cdot (o, a))$.

The functor IOSum I yielding a subset of the carrier of A is defined by the condition (Def. 5).

(Def. 5) Let o be an object of A. Then $o \in \mathrm{IOSum}\, I$ if and only if there exists a state s of A and there exists an element a of ObjectKind(o) such that $\mathrm{Exec}(I, s +\cdot (o, a)) \neq \mathrm{Exec}(I, s) +\cdot (o, a)$.

Let N be a set with non empty elements, let A be an IC-Ins-separated definite non empty non void AMI over N, and let I be an instruction of A. The functor Input I yielding a subset of the carrier of A is defined as follows:

(Def. 6) $\mathrm{Input}\, I = \mathrm{IOSum}\, I \setminus \mathrm{IODiff}\, I$.

The following propositions are true:

(9) Let A be an IC-Ins-separated definite non empty non void AMI over N and I be an instruction of A. Then IODiff I misses Input I.

(10) Let A be an IC-Ins-separated definite non empty non void AMI over N with non trivial ObjectKinds and I be an instruction of A. Then $\mathrm{IODiff}\, I \subseteq \mathrm{Output}\, I$.

(11) For every IC-Ins-separated definite non empty non void AMI A over N and for every instruction I of A holds $\mathrm{Output}\, I \subseteq \mathrm{IOSum}\, I$.

(12) For every IC-Ins-separated definite non empty non void AMI A over N and for every instruction I of A holds $\mathrm{Input}\, I \subseteq \mathrm{IOSum}\, I$.

(13) Let A be an IC-Ins-separated definite non empty non void AMI over N with non trivial ObjectKinds and I be an instruction of A. Then $\mathrm{IODiff}\, I = \mathrm{Output}\, I \setminus \mathrm{Input}\, I$.

(14) Let A be an IC-Ins-separated definite non empty non void AMI over N with non trivial ObjectKinds and I be an instruction of A. Then $\mathrm{IOSum}\, I = \mathrm{Output}\, I \cup \mathrm{Input}\, I$.

(15) Let A be an IC-Ins-separated definite non empty non void AMI over N, I be an instruction of A, and o be an object of A. If $\mathrm{ObjectKind}(o)$ is trivial, then $o \notin \mathrm{IOSum}\, I$.

(16) Let A be an IC-Ins-separated definite non empty non void AMI over N, I be an instruction of A, and o be an object of A. If $\mathrm{ObjectKind}(o)$ is trivial, then $o \notin \mathrm{Input}\, I$.

(17) Let A be an IC-Ins-separated definite non empty non void AMI over N, I be an instruction of A, and o be an object of A. If $\mathrm{ObjectKind}(o)$ is trivial, then $o \notin \mathrm{Output}\, I$.

(18) Let A be an IC-Ins-separated definite non empty non void AMI over N and I be an instruction of A. Then I is halting if and only if $\mathrm{Output}\, I$ is empty.

(19) Let A be an IC-Ins-separated definite non empty non void AMI over N with non trivial ObjectKinds and I be an instruction of A. If I is halting, then $\mathrm{IODiff}\, I$ is empty.

(20) Let A be an IC-Ins-separated definite non empty non void AMI over N and I be an instruction of A. If I is halting, then $\mathrm{IOSum}\, I$ is empty.

(21) Let A be an IC-Ins-separated definite non empty non void AMI over N and I be an instruction of A. If I is halting, then $\mathrm{Input}\, I$ is empty.

Let N be a set with non empty elements, let A be a halting IC-Ins-separated definite non empty non void AMI over N, and let I be a halting instruction of A. One can verify the following observations:

* $\mathrm{Input}\, I$ is empty,
* $\mathrm{Output}\, I$ is empty, and
* $\mathrm{IOSum}\, I$ is empty.

Let N be a set with non empty elements, let A be a halting IC-Ins-separated definite non empty non void AMI over N with non trivial ObjectKinds, and let I be a halting instruction of A. Note that $\mathrm{IODiff}\, I$ is empty.

The following propositions are true:

(22) Let A be a steady-programmed IC-Ins-separated definite non empty non void AMI over N with non trivial instruction set, f be an instruction-location of A, and I be an instruction of A. Then $f \notin \mathrm{IODiff}\, I$.

(23) Let A be a standard IC-Ins-separated definite non empty non void AMI over N and I be an instruction of A. If I is sequential, then $\mathbf{IC}_A \notin \mathrm{IODiff}\, I$.

(24) Let A be an IC-Ins-separated definite non empty non void AMI over N and I be an instruction of A. If there exists a state s of A such that $(\mathrm{Exec}(I, s))(\mathbf{IC}_A) \neq \mathbf{IC}_s$, then $\mathbf{IC}_A \in \mathrm{Output}\, I$.

(25) Let A be a standard IC-Ins-separated definite non empty non void AMI over N and I be an instruction of A. If I is sequential, then $\mathbf{IC}_A \in \mathrm{Output}\, I$.

(26) Let A be an IC-Ins-separated definite non empty non void AMI over N and I be an instruction of A. If there exists a state s of A such that $(\mathrm{Exec}(I, s))(\mathbf{IC}_A) \neq \mathbf{IC}_s$, then $\mathbf{IC}_A \in \mathrm{IOSum}\, I$.

(27) Let A be a standard IC-Ins-separated definite non empty non void AMI over N and I be an instruction of A. If I is sequential, then $\mathbf{IC}_A \in \mathrm{IOSum}\, I$.

(28) Let A be an IC-Ins-separated definite non empty non void AMI over N, f be an instruction-location of A, and I be an instruction of A. Suppose that for every state s of A and for every programmed finite partial state p of A holds $\mathrm{Exec}(I, s{+\cdot}p) = \mathrm{Exec}(I, s){+\cdot}p$. Then $f \notin \mathrm{IOSum}\, I$.

(29) Let A be an IC-Ins-separated definite non empty non void AMI over N, I be an instruction of A, and o be an object of A. If I is jump-only, then if $o \in \mathrm{Output}\, I$, then $o = \mathbf{IC}_A$.

3. Input and Output of the Instructions of **SCM**

In the sequel a, b are data-locations, f is an instruction-location of **SCM**, and I is an instruction of **SCM**.

We now state two propositions:

(30) For every state s of **SCM** and for every element w of $\mathrm{ObjectKind}(\mathbf{IC}_{\mathbf{SCM}})$ holds $(s +\cdot (\mathbf{IC}_{\mathbf{SCM}}, w))(a) = s(a)$.

(31) $f \neq \mathrm{Next}(f)$.

Let s be a state of **SCM**, let d_1 be a data-location, and let k be an integer. Then $s +\cdot (d_1, k)$ is a state of **SCM**.

Let us observe that **SCM** has non trivial ObjectKinds.

Next we state a number of propositions:

(32) $\mathrm{IODiff}(a{:=}a) = \emptyset$.

(33) If $a \neq b$, then $\mathrm{IODiff}(a{:=}b) = \{a\}$.

(34) $\mathrm{IODiff}\,\mathrm{AddTo}(a, b) = \emptyset$.

(35) $\mathrm{IODiff}\,\mathrm{SubFrom}(a, a) = \{a\}$.

(36) If $a \neq b$, then $\mathrm{IODiff}\,\mathrm{SubFrom}(a, b) = \emptyset$.

(37) $\mathrm{IODiff}\,\mathrm{MultBy}(a, b) = \emptyset$.

(38) IODiff Divide$(a, a) = \{a\}$.
(39) If $a \neq b$, then IODiff Divide$(a, b) = \emptyset$.
(40) IODiff goto $f = \{\mathbf{IC}_{\mathbf{SCM}}\}$.
(41) IODiff(**if** $a = 0$ **goto** $f) = \emptyset$.
(42) IODiff(**if** $a > 0$ **goto** $f) = \emptyset$.
(43) Output$(a{:=}a) = \{\mathbf{IC}_{\mathbf{SCM}}\}$.
(44) If $a \neq b$, then Output$(a{:=}b) = \{a, \mathbf{IC}_{\mathbf{SCM}}\}$.
(45) Output AddTo$(a, b) = \{a, \mathbf{IC}_{\mathbf{SCM}}\}$.
(46) Output SubFrom$(a, b) = \{a, \mathbf{IC}_{\mathbf{SCM}}\}$.
(47) Output MultBy$(a, b) = \{a, \mathbf{IC}_{\mathbf{SCM}}\}$.
(48) Output Divide$(a, b) = \{a, b, \mathbf{IC}_{\mathbf{SCM}}\}$.
(49) Output goto $f = \{\mathbf{IC}_{\mathbf{SCM}}\}$.
(50) Output(**if** $a = 0$ **goto** $f) = \{\mathbf{IC}_{\mathbf{SCM}}\}$.
(51) Output(**if** $a > 0$ **goto** $f) = \{\mathbf{IC}_{\mathbf{SCM}}\}$.
(52) $f \notin$ IOSum I.
(53) IOSum$(a{:=}a) = \{\mathbf{IC}_{\mathbf{SCM}}\}$.
(54) If $a \neq b$, then IOSum$(a{:=}b) = \{a, b, \mathbf{IC}_{\mathbf{SCM}}\}$.
(55) IOSum AddTo$(a, b) = \{a, b, \mathbf{IC}_{\mathbf{SCM}}\}$.
(56) IOSum SubFrom$(a, b) = \{a, b, \mathbf{IC}_{\mathbf{SCM}}\}$.
(57) IOSum MultBy$(a, b) = \{a, b, \mathbf{IC}_{\mathbf{SCM}}\}$.
(58) IOSum Divide$(a, b) = \{a, b, \mathbf{IC}_{\mathbf{SCM}}\}$.
(59) IOSum goto $f = \{\mathbf{IC}_{\mathbf{SCM}}\}$.
(60) IOSum(**if** $a = 0$ **goto** $f) = \{a, \mathbf{IC}_{\mathbf{SCM}}\}$.
(61) IOSum(**if** $a > 0$ **goto** $f) = \{a, \mathbf{IC}_{\mathbf{SCM}}\}$.
(62) Input$(a{:=}a) = \{\mathbf{IC}_{\mathbf{SCM}}\}$.
(63) If $a \neq b$, then Input$(a{:=}b) = \{b, \mathbf{IC}_{\mathbf{SCM}}\}$.
(64) Input AddTo$(a, b) = \{a, b, \mathbf{IC}_{\mathbf{SCM}}\}$.
(65) Input SubFrom$(a, a) = \{\mathbf{IC}_{\mathbf{SCM}}\}$.
(66) If $a \neq b$, then Input SubFrom$(a, b) = \{a, b, \mathbf{IC}_{\mathbf{SCM}}\}$.
(67) Input MultBy$(a, b) = \{a, b, \mathbf{IC}_{\mathbf{SCM}}\}$.
(68) Input Divide$(a, a) = \{\mathbf{IC}_{\mathbf{SCM}}\}$.
(69) If $a \neq b$, then Input Divide$(a, b) = \{a, b, \mathbf{IC}_{\mathbf{SCM}}\}$.
(70) Input goto $f = \emptyset$.
(71) Input(**if** $a = 0$ **goto** $f) = \{a, \mathbf{IC}_{\mathbf{SCM}}\}$.
(72) Input(**if** $a > 0$ **goto** $f) = \{a, \mathbf{IC}_{\mathbf{SCM}}\}$.

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On the Instructions of $\mathbf{SCM}_{\mathrm{FSA}}$[1]

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The articles [18], [10], [11], [12], [22], [5], [14], [3], [6], [20], [7], [8], [9], [4], [19], [1], [2], [23], [24], [17], [16], [13], [21], and [15] provide the terminology and notation for this paper.

For simplicity, we use the following convention: a, b are integer locations, f is a finite sequence location, i_1, i_2, i_3 are instruction-locations of $\mathbf{SCM}_{\mathrm{FSA}}$, T is an instruction type of $\mathbf{SCM}_{\mathrm{FSA}}$, and k is a natural number.

Next we state two propositions:

(1) For every function f and for all sets a, A, b, B, c, C such that $a \neq b$ and $a \neq c$ holds $(f {+\cdot} (a \dot{\longmapsto} A) {+\cdot} (b \dot{\longmapsto} B) {+\cdot} (c \dot{\longmapsto} C))(a) = A$.

(2) For all sets a, b holds $\langle a \rangle +\cdot (1, b) = \langle b \rangle$.

Let l_1, l_2 be integer locations and let a, b be integers. Then $[l_1 \longmapsto a, l_2 \longmapsto b]$ is a finite partial state of $\mathbf{SCM}_{\mathrm{FSA}}$.

One can prove the following propositions:

(3) $a \notin$ the instruction locations of $\mathbf{SCM}_{\mathrm{FSA}}$.

(4) $f \notin$ the instruction locations of $\mathbf{SCM}_{\mathrm{FSA}}$.

(5) $\text{Data-Loc}_{\mathrm{SCM}_{\mathrm{FSA}}} \neq$ the instruction locations of $\mathbf{SCM}_{\mathrm{FSA}}$.

(6) $\text{Data}^*\text{-Loc}_{\mathrm{SCM}_{\mathrm{FSA}}} \neq$ the instruction locations of $\mathbf{SCM}_{\mathrm{FSA}}$.

(7) Let o be an object of $\mathbf{SCM}_{\mathrm{FSA}}$. Then
 (i) $o = \mathbf{IC}_{\mathbf{SCM}_{\mathrm{FSA}}}$, or
 (ii) $o \in$ the instruction locations of $\mathbf{SCM}_{\mathrm{FSA}}$, or
 (iii) o is an integer location or a finite sequence location.

(8) If $i_2 \neq i_3$, then $\mathrm{Next}(i_2) \neq \mathrm{Next}(i_3)$.

(9) $a{:=}b = \langle\!\langle 1, \langle a, b \rangle \rangle\!\rangle$.

(10) $\mathrm{AddTo}(a, b) = \langle\!\langle 2, \langle a, b \rangle \rangle\!\rangle$.

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(11) $\mathrm{SubFrom}(a,b) = \langle\!\langle 3, \langle a,b\rangle\rangle\!\rangle$.

(12) $\mathrm{MultBy}(a,b) = \langle\!\langle 4, \langle a,b\rangle\rangle\!\rangle$.

(13) $\mathrm{Divide}(a,b) = \langle\!\langle 5, \langle a,b\rangle\rangle\!\rangle$.

(14) $\mathrm{goto}\ i_1 = \langle\!\langle 6, \langle i_1\rangle\rangle\!\rangle$.

(15) $\mathbf{if}\ a = 0\ \mathbf{goto}\ i_1 = \langle\!\langle 7, \langle i_1,a\rangle\rangle\!\rangle$.

(16) $\mathbf{if}\ a > 0\ \mathbf{goto}\ i_1 = \langle\!\langle 8, \langle i_1,a\rangle\rangle\!\rangle$.

(17) $\mathrm{AddressPart}(\mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}}) = \emptyset$.

(18) $\mathrm{AddressPart}(a{:=}b) = \langle a,b\rangle$.

(19) $\mathrm{AddressPart}(\mathrm{AddTo}(a,b)) = \langle a,b\rangle$.

(20) $\mathrm{AddressPart}(\mathrm{SubFrom}(a,b)) = \langle a,b\rangle$.

(21) $\mathrm{AddressPart}(\mathrm{MultBy}(a,b)) = \langle a,b\rangle$.

(22) $\mathrm{AddressPart}(\mathrm{Divide}(a,b)) = \langle a,b\rangle$.

(23) $\mathrm{AddressPart}(\mathrm{goto}\ i_2) = \langle i_2\rangle$.

(24) $\mathrm{AddressPart}(\mathbf{if}\ a = 0\ \mathbf{goto}\ i_2) = \langle i_2,a\rangle$.

(25) $\mathrm{AddressPart}(\mathbf{if}\ a > 0\ \mathbf{goto}\ i_2) = \langle i_2,a\rangle$.

(26) $\mathrm{AddressPart}(b{:=}f_a) = \langle b,f,a\rangle$.

(27) $\mathrm{AddressPart}(f_a{:=}b) = \langle b,f,a\rangle$.

(28) $\mathrm{AddressPart}(a{:=}\mathrm{len} f) = \langle a,f\rangle$.

(29) $\mathrm{AddressPart}(f{:=}\langle\underbrace{0,\ldots,0}_{a}\rangle) = \langle a,f\rangle$.

(30) If $T = 0$, then $\mathrm{AddressParts}\,T = \{0\}$.

Let us consider T. Observe that $\mathrm{AddressParts}\,T$ is non empty.

Next we state a number of propositions:

(31) If $T = 1$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\,T} = \{1,2\}$.

(32) If $T = 2$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\,T} = \{1,2\}$.

(33) If $T = 3$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\,T} = \{1,2\}$.

(34) If $T = 4$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\,T} = \{1,2\}$.

(35) If $T = 5$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\,T} = \{1,2\}$.

(36) If $T = 6$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\,T} = \{1\}$.

(37) If $T = 7$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\,T} = \{1,2\}$.

(38) If $T = 8$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\,T} = \{1,2\}$.

(39) If $T = 9$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\,T} = \{1,2,3\}$.

(40) If $T = 10$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\,T} = \{1,2,3\}$.

(41) If $T = 11$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\,T} = \{1,2\}$.

(42) If $T = 12$, then $\mathrm{dom} \prod_{\mathrm{AddressParts}\,T} = \{1,2\}$.

(43) $\prod_{\mathrm{AddressParts}\,\mathrm{InsCode}(a:=b)}(1) = \mathrm{Data\text{-}Loc}_{\mathrm{SCM}_{\mathrm{FSA}}}$.

(44) $\prod_{\mathrm{AddressParts}\,\mathrm{InsCode}(a:=b)}(2) = \mathrm{Data\text{-}Loc}_{\mathrm{SCM}_{\mathrm{FSA}}}$.

(45) $\prod_{\mathrm{AddressParts\,InsCode(AddTo}(a,b))}(1) = \text{Data-Loc}_{\mathrm{SCM_{FSA}}}$.

(46) $\prod_{\mathrm{AddressParts\,InsCode(AddTo}(a,b))}(2) = \text{Data-Loc}_{\mathrm{SCM_{FSA}}}$.

(47) $\prod_{\mathrm{AddressParts\,InsCode(SubFrom}(a,b))}(1) = \text{Data-Loc}_{\mathrm{SCM_{FSA}}}$.

(48) $\prod_{\mathrm{AddressParts\,InsCode(SubFrom}(a,b))}(2) = \text{Data-Loc}_{\mathrm{SCM_{FSA}}}$.

(49) $\prod_{\mathrm{AddressParts\,InsCode(MultBy}(a,b))}(1) = \text{Data-Loc}_{\mathrm{SCM_{FSA}}}$.

(50) $\prod_{\mathrm{AddressParts\,InsCode(MultBy}(a,b))}(2) = \text{Data-Loc}_{\mathrm{SCM_{FSA}}}$.

(51) $\prod_{\mathrm{AddressParts\,InsCode(Divide}(a,b))}(1) = \text{Data-Loc}_{\mathrm{SCM_{FSA}}}$.

(52) $\prod_{\mathrm{AddressParts\,InsCode(Divide}(a,b))}(2) = \text{Data-Loc}_{\mathrm{SCM_{FSA}}}$.

(53) $\prod_{\mathrm{AddressParts\,InsCode(goto}\ i_2)}(1) =$ the instruction locations of $\mathbf{SCM}_{\mathrm{FSA}}$.

(54) $\prod_{\mathrm{AddressParts\,InsCode}(\mathbf{if}\ a=0\ \mathbf{goto}\ i_2)}(1) =$ the instruction locations of $\mathbf{SCM}_{\mathrm{FSA}}$.

(55) $\prod_{\mathrm{AddressParts\,InsCode}(\mathbf{if}\ a=0\ \mathbf{goto}\ i_2)}(2) = \text{Data-Loc}_{\mathrm{SCM_{FSA}}}$.

(56) $\prod_{\mathrm{AddressParts\,InsCode}(\mathbf{if}\ a>0\ \mathbf{goto}\ i_2)}(1) =$ the instruction locations of $\mathbf{SCM}_{\mathrm{FSA}}$.

(57) $\prod_{\mathrm{AddressParts\,InsCode}(\mathbf{if}\ a>0\ \mathbf{goto}\ i_2)}(2) = \text{Data-Loc}_{\mathrm{SCM_{FSA}}}$.

(58) $\prod_{\mathrm{AddressParts\,InsCode}(b:=f_a)}(1) = \text{Data-Loc}_{\mathrm{SCM_{FSA}}}$.

(59) $\prod_{\mathrm{AddressParts\,InsCode}(b:=f_a)}(2) = \text{Data}^*\text{-Loc}_{\mathrm{SCM_{FSA}}}$.

(60) $\prod_{\mathrm{AddressParts\,InsCode}(b:=f_a)}(3) = \text{Data-Loc}_{\mathrm{SCM_{FSA}}}$.

(61) $\prod_{\mathrm{AddressParts\,InsCode}(f_a:=b)}(1) = \text{Data-Loc}_{\mathrm{SCM_{FSA}}}$.

(62) $\prod_{\mathrm{AddressParts\,InsCode}(f_a:=b)}(2) = \text{Data}^*\text{-Loc}_{\mathrm{SCM_{FSA}}}$.

(63) $\prod_{\mathrm{AddressParts\,InsCode}(f_a:=b)}(3) = \text{Data-Loc}_{\mathrm{SCM_{FSA}}}$.

(64) $\prod_{\mathrm{AddressParts\,InsCode}(a:=\mathrm{len}f)}(1) = \text{Data-Loc}_{\mathrm{SCM_{FSA}}}$.

(65) $\prod_{\mathrm{AddressParts\,InsCode}(a:=\mathrm{len}f)}(2) = \text{Data}^*\text{-Loc}_{\mathrm{SCM_{FSA}}}$.

(66) $\prod_{\mathrm{AddressParts\,InsCode}(f:=\langle\underbrace{0,\ldots,0}_{a}\rangle)}(1) = \text{Data-Loc}_{\mathrm{SCM_{FSA}}}$.

(67) $\prod_{\mathrm{AddressParts\,InsCode}(f:=\langle\underbrace{0,\ldots,0}_{a}\rangle)}(2) = \text{Data}^*\text{-Loc}_{\mathrm{SCM_{FSA}}}$.

(68) $\mathrm{NIC}(\mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}}, i_1) = \{i_1\}$.

One can verify that $\mathrm{JUMP}(\mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}})$ is empty.

We now state the proposition

(69) $\mathrm{NIC}(a{:=}b, i_1) = \{\mathrm{Next}(i_1)\}$.

Let us consider a, b. Note that $\mathrm{JUMP}(a{:=}b)$ is empty.

One can prove the following proposition

(70) $\mathrm{NIC}(\mathrm{AddTo}(a,b), i_1) = \{\mathrm{Next}(i_1)\}$.

Let us consider a, b. Note that $\mathrm{JUMP}(\mathrm{AddTo}(a,b))$ is empty.

Next we state the proposition

(71) $\mathrm{NIC}(\mathrm{SubFrom}(a,b), i_1) = \{\mathrm{Next}(i_1)\}$.

Let us consider a, b. Note that JUMP(SubFrom(a, b)) is empty.

One can prove the following proposition

(72) NIC(MultBy$(a, b), i_1) = \{\text{Next}(i_1)\}$.

Let us consider a, b. Note that JUMP(MultBy(a, b)) is empty.

Next we state the proposition

(73) NIC(Divide$(a, b), i_1) = \{\text{Next}(i_1)\}$.

Let us consider a, b. One can verify that JUMP(Divide(a, b)) is empty.

We now state two propositions:

(74) NIC(goto $i_2, i_1) = \{i_2\}$.

(75) JUMP(goto $i_2) = \{i_2\}$.

Let us consider i_2. One can verify that JUMP(goto i_2) is non empty and trivial.

We now state two propositions:

(76) $i_2 \in$ NIC(**if** $a = 0$ **goto** i_2, i_1) and NIC(**if** $a = 0$ **goto** $i_2, i_1) \subseteq \{i_2, \text{Next}(i_1)\}$.

(77) JUMP(**if** $a = 0$ **goto** $i_2) = \{i_2\}$.

Let us consider a, i_2. One can check that JUMP(**if** $a = 0$ **goto** i_2) is non empty and trivial.

One can prove the following two propositions:

(78) $i_2 \in$ NIC(**if** $a > 0$ **goto** i_2, i_1) and NIC(**if** $a > 0$ **goto** $i_2, i_1) \subseteq \{i_2, \text{Next}(i_1)\}$.

(79) JUMP(**if** $a > 0$ **goto** $i_2) = \{i_2\}$.

Let us consider a, i_2. Note that JUMP(**if** $a > 0$ **goto** i_2) is non empty and trivial.

The following proposition is true

(80) NIC$(a{:=}f_b, i_1) = \{\text{Next}(i_1)\}$.

Let us consider a, b, f. Observe that JUMP$(a{:=}f_b)$ is empty.

Next we state the proposition

(81) NIC$(f_b{:=}a, i_1) = \{\text{Next}(i_1)\}$.

Let us consider a, b, f. One can check that JUMP$(f_b{:=}a)$ is empty.

The following proposition is true

(82) NIC$(a{:=}\text{len}f, i_1) = \{\text{Next}(i_1)\}$.

Let us consider a, f. Observe that JUMP$(a{:=}\text{len}f)$ is empty.

The following proposition is true

(83) NIC$(f{:=}\langle\underbrace{0, \ldots, 0}_{a}\rangle, i_1) = \{\text{Next}(i_1)\}$.

Let us consider a, f. Note that JUMP$(f{:=}\langle\underbrace{0, \ldots, 0}_{a}\rangle)$ is empty.

The following two propositions are true:

(84) $\mathrm{SUCC}(i_1) = \{i_1, \mathrm{Next}(i_1)\}$.

(85) Let f be a function from $\mathbb{N}$ into the instruction locations of $\mathbf{SCM}_{\mathrm{FSA}}$. Suppose that for every natural number k holds $f(k) = \mathrm{insloc}(k)$. Then

(i) f is bijective, and

(ii) for every natural number k holds $f(k+1) \in \mathrm{SUCC}(f(k))$ and for every natural number j such that $f(j) \in \mathrm{SUCC}(f(k))$ holds $k \leqslant j$.

Let us observe that $\mathbf{SCM}_{\mathrm{FSA}}$ is standard.

The following propositions are true:

(86) $\mathrm{il}_{\mathbf{SCM}_{\mathrm{FSA}}}(k) = \mathrm{insloc}(k)$.

(87) $\mathrm{Next}(\mathrm{il}_{\mathbf{SCM}_{\mathrm{FSA}}}(k)) = \mathrm{il}_{\mathbf{SCM}_{\mathrm{FSA}}}(k+1)$.

(88) $\mathrm{Next}(i_1) = \mathrm{NextLoc}\, i_1$.

Let us mention that $\mathrm{InsCode}(\mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}})$ is jump-only.

Let us mention that $\mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}}$ is jump-only.

Let us consider i_2. One can verify that InsCode(goto i_2) is jump-only.

Let us consider i_2. Observe that goto i_2 is jump-only non sequential and non instruction location free.

Let us consider a, i_2. One can check that InsCode(**if** $a = 0$ **goto** i_2) is jump-only and InsCode(**if** $a > 0$ **goto** i_2) is jump-only.

Let us consider a, i_2. Observe that **if** $a = 0$ **goto** i_2 is jump-only non sequential and non instruction location free and **if** $a > 0$ **goto** i_2 is jump-only non sequential and non instruction location free.

Let us consider a, b. One can verify the following observations:

* InsCode(a:=b) is non jump-only,
* InsCode(AddTo(a, b)) is non jump-only,
* InsCode(SubFrom(a, b)) is non jump-only,
* InsCode(MultBy(a, b)) is non jump-only, and
* InsCode(Divide(a, b)) is non jump-only.

Let us consider a, b. One can verify the following observations:

* a:=b is non jump-only and sequential,
* AddTo(a, b) is non jump-only and sequential,
* SubFrom(a, b) is non jump-only and sequential,
* MultBy(a, b) is non jump-only and sequential, and
* Divide(a, b) is non jump-only and sequential.

Let us consider a, b, f. One can check that InsCode(b:=f_a) is non jump-only and InsCode(f_a:=b) is non jump-only.

Let us consider a, b, f. Observe that b:=f_a is non jump-only and sequential and f_a:=b is non jump-only and sequential.

Let us consider a, f. One can check that InsCode(a:=len$f$) is non jump-only and InsCode(f:=$\langle\underbrace{0,\ldots,0}_{a}\rangle$) is non jump-only.

Let us consider a, f. Note that a:=len$f$ is non jump-only and sequential and f:=$\langle\underbrace{0,\ldots,0}_{a}\rangle$ is non jump-only and sequential.

One can verify that $\mathbf{SCM}_{\mathrm{FSA}}$ is homogeneous and has explicit jumps and no implicit jumps.

Let us note that $\mathbf{SCM}_{\mathrm{FSA}}$ is regular.

The following propositions are true:

(89) $\mathrm{IncAddr}(\mathrm{goto}\ i_2, k) = \mathrm{goto}\ \mathrm{il}_{\mathbf{SCM}_{\mathrm{FSA}}}(\mathrm{locnum}(i_2) + k)$.

(90) $\mathrm{IncAddr}(\mathbf{if}\ a = 0\ \mathbf{goto}\ i_2, k) = \mathbf{if}\ a = 0\ \mathbf{goto}\ \mathrm{il}_{\mathbf{SCM}_{\mathrm{FSA}}}(\mathrm{locnum}(i_2) + k)$.

(91) $\mathrm{IncAddr}(\mathbf{if}\ a > 0\ \mathbf{goto}\ i_2, k) = \mathbf{if}\ a > 0\ \mathbf{goto}\ \mathrm{il}_{\mathbf{SCM}_{\mathrm{FSA}}}(\mathrm{locnum}(i_2) + k)$.

Let us note that $\mathbf{SCM}_{\mathrm{FSA}}$ is IC-good and Exec-preserving.

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Robbins Algebras vs. Boolean Algebras[1]

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Summary. In the early 1930s, Huntington proposed several axiom systems for Boolean algebras. Robbins slightly changed one of them and asked if the resulted system is still a basis for variety of Boolean algebras. The solution (afirmative answer) was given in 1996 by McCune with the help of automated theorem prover EQP/OTTER. Some simplified and restucturized versions of this proof are known. In our version of proof that all Robbins algebras are Boolean we use the results of McCune [5], Huntington [2, 4, 3] and Dahn [1].

MML Identifier: `ROBBINS1`.

The papers [7] and [6] provide the terminology and notation for this paper.

1. Preliminaries

We introduce complemented lattice structures which are extensions of $\sqcup$-semi lattice structure and are systems
$\langle$ a carrier, a join operation, a complement operation $\rangle$,
where the carrier is a set, the join operation is a binary operation on the carrier, and the complement operation is a unary operation on the carrier.

We introduce ortholattice structures which are extensions of complemented lattice structure and lattice structure and are systems
$\langle$ a carrier, a join operation, a meet operation, a complement operation $\rangle$,
where the carrier is a set, the join operation and the meet operation are binary operations on the carrier, and the complement operation is a unary operation on the carrier.

The strict complemented lattice structure TrivComplLat is defined as follows:

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(Def. 1) $\mathrm{TrivComplLat} = \langle \{\emptyset\}, \mathrm{op}_2, \mathrm{op}_1 \rangle$.

The strict ortholattice structure TrivOrtLat is defined by:

(Def. 2) $\mathrm{TrivOrtLat} = \langle \{\emptyset\}, \mathrm{op}_2, \mathrm{op}_2, \mathrm{op}_1 \rangle$.

Let us note that TrivComplLat is non empty and trivial and TrivOrtLat is non empty and trivial.

Let us mention that there exists an ortholattice structure which is strict, non empty, and trivial and there exists a complemented lattice structure which is strict, non empty, and trivial.

Let L be a non empty complemented lattice structure and let x be an element of the carrier of L. The functor x^{c} yielding an element of L is defined as follows:

(Def. 3) $x^{\mathrm{c}} = (\text{the complement operation of } L)(x)$.

Let L be a non empty complemented lattice structure and let x, y be elements of the carrier of L. We introduce $x + y$ as a synonym of $x \sqcup y$.

Let L be a non empty complemented lattice structure and let x, y be elements of the carrier of L. The functor $x * y$ yields an element of L and is defined by:

(Def. 4) $x * y = (x^{\mathrm{c}} \sqcup y^{\mathrm{c}})^{\mathrm{c}}$.

Let L be a non empty complemented lattice structure. We say that L is Robbins if and only if:

(Def. 5) For all elements x, y of the carrier of L holds $((x+y)^{\mathrm{c}} + (x+y^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}} = x$.

We say that L is Huntington if and only if:

(Def. 6) For all elements x, y of the carrier of L holds $(x^{\mathrm{c}} + y^{\mathrm{c}})^{\mathrm{c}} + (x^{\mathrm{c}} + y)^{\mathrm{c}} = x$.

Let G be a non empty $\sqcup$-semi lattice structure. We say that G is join-idempotent if and only if:

(Def. 7) For every element x of the carrier of G holds $x \sqcup x = x$.

Let us observe that TrivComplLat is join-commutative join-associative Robbins Huntington and join-idempotent and TrivOrtLat is join-commutative join-associative Huntington and Robbins.

Let us mention that TrivOrtLat is meet-commutative meet-associative meet-absorbing and join-absorbing.

One can verify that there exists a non empty complemented lattice structure which is strict, join-associative, join-commutative, Robbins, join-idempotent, and Huntington.

Let us observe that there exists a non empty ortholattice structure which is strict, lattice-like, Robbins, and Huntington.

Let L be a join-commutative non empty complemented lattice structure and let x, y be elements of the carrier of L. Let us observe that the functor $x + y$ is commutative.

Next we state several propositions:

(1) Let L be a Huntington join-commutative join-associative non empty complemented lattice structure and a, b be elements of the carrier of L. Then $a * b + a * b^{\mathrm{c}} = a$.

(2) Let L be a Huntington join-commutative join-associative non empty complemented lattice structure and a be an element of the carrier of L. Then $a + a^{\mathrm{c}} = a^{\mathrm{c}} + (a^{\mathrm{c}})^{\mathrm{c}}$.

(3) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and x be an element of the carrier of L. Then $(x^{\mathrm{c}})^{\mathrm{c}} = x$.

(4) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of the carrier of L. Then $a + a^{\mathrm{c}} = b + b^{\mathrm{c}}$.

(5) Let L be a join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure. Then there exists an element c of the carrier of L such that for every element a of the carrier of L holds
$c + a = c$ and $a + a^{\mathrm{c}} = c$.

(6) Every join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure is upper-bounded.

One can verify that every non empty complemented lattice structure which is join-commutative, join-associative, join-idempotent, and Huntington is also upper-bounded.

Let L be a join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure. Then $\top_L$ can be characterized by the condition:

(Def. 8) There exists an element a of the carrier of L such that $\top_L = a + a^{\mathrm{c}}$.

One can prove the following propositions:

(7) Let L be a join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure. Then there exists an element c of the carrier of L such that for every element a of the carrier of L holds
$c * a = c$ and $(a + a^{\mathrm{c}})^{\mathrm{c}} = c$.

(8) Let L be a join-commutative join-associative non empty complemented lattice structure and a, b be elements of the carrier of L. Then $a * b = b * a$.

Let L be a join-commutative join-associative non empty complemented lattice structure and let x, y be elements of the carrier of L. Let us note that the functor $x * y$ is commutative.

Let L be a join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure. The functor $\perp^{\mathrm{C}}_{L}$ yielding an element of L is defined as follows:

(Def. 9) For every element a of the carrier of L holds $\bot_L^{\mathrm{C}} * a = \bot_L^{\mathrm{C}}$.

One can prove the following propositions:

(9) Let L be a join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure and a be an element of the carrier of L. Then $\bot_L^{\mathrm{C}} = (a + a^{\mathrm{c}})^{\mathrm{c}}$.

(10) Let L be a join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure. Then $(\top_L)^{\mathrm{c}} = \bot_L^{\mathrm{C}}$ and $\top_L = (\bot_L^{\mathrm{C}})^{\mathrm{c}}$.

(11) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of the carrier of L. If $a^{\mathrm{c}} = b^{\mathrm{c}}$, then $a = b$.

(12) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of the carrier of L. Then $a + (b + b^{\mathrm{c}})^{\mathrm{c}} = a$.

(13) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a be an element of the carrier of L. Then $a + a = a$.

Let us note that every non empty complemented lattice structure which is join-commutative, join-associative, and Huntington is also join-idempotent.

One can prove the following propositions:

(14) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a be an element of the carrier of L. Then $a + \bot_L^{\mathrm{C}} = a$.

(15) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a be an element of the carrier of L. Then $a * \top_L = a$.

(16) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a be an element of the carrier of L. Then $a * a^{\mathrm{c}} = \bot_L^{\mathrm{C}}$.

(17) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of the carrier of L. Then $a * (b * c) = (a * b) * c$.

(18) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of the carrier of L. Then $a + b = (a^{\mathrm{c}} * b^{\mathrm{c}})^{\mathrm{c}}$.

(19) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a be an element of the carrier of L. Then $a * a = a$.

(20) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a be an element of the carrier of L.

Then $a + \top_L = \top_L$.

(21) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of the carrier of L. Then $a + a * b = a$.

(22) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of the carrier of L. Then $a * (a + b) = a$.

(23) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of the carrier of L. If $a^{\rm c} + b = \top_L$ and $b^{\rm c} + a = \top_L$, then $a = b$.

(24) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of the carrier of L. If $a + b = \top_L$ and $a * b = \bot_L^{\rm C}$, then $a^{\rm c} = b$.

(25) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of the carrier of L. Then $a * b * c + a * b * c^{\rm c} + a * b^{\rm c} * c + a * b^{\rm c} * c^{\rm c} + a^{\rm c} * b * c + a^{\rm c} * b * c^{\rm c} + a^{\rm c} * b^{\rm c} * c + a^{\rm c} * b^{\rm c} * c^{\rm c} = \top_L$.

(26) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of the carrier of L. Then

(i) $a * c * (b * c^{\rm c}) = \bot_L^{\rm C}$,
(ii) $a * b * c * (a^{\rm c} * b * c) = \bot_L^{\rm C}$,
(iii) $a * b^{\rm c} * c * (a^{\rm c} * b * c) = \bot_L^{\rm C}$,
(iv) $a * b * c * (a^{\rm c} * b^{\rm c} * c) = \bot_L^{\rm C}$,
(v) $a * b * c^{\rm c} * (a^{\rm c} * b^{\rm c} * c^{\rm c}) = \bot_L^{\rm C}$, and
(vi) $a * b^{\rm c} * c * (a^{\rm c} * b * c) = \bot_L^{\rm C}$.

(27) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of the carrier of L. Then $a * b + a * c = a * b * c + a * b * c^{\rm c} + a * b^{\rm c} * c$.

(28) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of the carrier of L. Then $(a * (b + c))^{\rm c} = a * b^{\rm c} * c^{\rm c} + a^{\rm c} * b * c + a^{\rm c} * b * c^{\rm c} + a^{\rm c} * b^{\rm c} * c + a^{\rm c} * b^{\rm c} * c^{\rm c}$.

(29) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of the carrier of L. Then $a * b + a * c + (a * (b + c))^{\rm c} = \top_L$.

(30) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of the carrier of L. Then $(a * b + a * c) * (a * (b + c))^{\rm c} = \bot_L^{\rm C}$.

(31) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of the carrier of L.

Then $a * (b + c) = a * b + a * c$.

(32) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of the carrier of L. Then $a + b * c = (a + b) * (a + c)$.

2. Pre-Ortholattices

Let L be a non empty ortholattice structure. We say that L is well-complemented if and only if:

(Def. 10) For every element a of the carrier of L holds a^{c} is a complement of a.

Let us observe that TrivOrtLat is Boolean and well-complemented.

A pre-ortholattice is a lattice-like non empty ortholattice structure.

Let us mention that there exists a pre-ortholattice which is strict, Boolean, and well-complemented.

We now state two propositions:

(33) Let L be a distributive well-complemented pre-ortholattice and x be an element of the carrier of L. Then $(x^{\mathrm{c}})^{\mathrm{c}} = x$.

(34) Let L be a bounded distributive well-complemented pre-ortholattice and x, y be elements of the carrier of L. Then $x \sqcap y = (x^{\mathrm{c}} \sqcup y^{\mathrm{c}})^{\mathrm{c}}$.

3. Correspondence between Boolean Pre-OrthoLattices and Boolean Lattices

Let L be a non empty complemented lattice structure. The functor CLatt L yielding a strict ortholattice structure is defined by the conditions (Def. 11).

(Def. 11)(i) The carrier of CLatt L = the carrier of L,

(ii) the join operation of CLatt L = the join operation of L,

(iii) the complement operation of CLatt L = the complement operation of L, and

(iv) for all elements a, b of the carrier of L holds (the meet operation of CLatt L)$(a, b) = a * b$.

Let L be a non empty complemented lattice structure. One can verify that CLatt L is non empty.

Let L be a join-commutative non empty complemented lattice structure. One can check that CLatt L is join-commutative.

Let L be a join-associative non empty complemented lattice structure. One can check that CLatt L is join-associative.

Let L be a join-commutative join-associative non empty complemented lattice structure. Observe that CLatt L is meet-commutative.

The following proposition is true

(35) Let L be a non empty complemented lattice structure, a, b be elements of the carrier of L, and a', b' be elements of the carrier of CLatt L. If $a = a'$ and $b = b'$, then $a * b = a' \sqcap b'$ and $a + b = a' \sqcup b'$ and $a^{\rm c} = a'^{\rm c}$.

Let L be a join-commutative join-associative Huntington non empty complemented lattice structure. Observe that CLatt L is meet-associative join-absorbing and meet-absorbing.

Let L be a Huntington non empty complemented lattice structure. Note that CLatt L is Huntington.

Let L be a join-commutative join-associative Huntington non empty complemented lattice structure. Note that CLatt L is lower-bounded.

We now state the proposition

(36) For every join-commutative join-associative Huntington non empty complemented lattice structure L holds $\perp_L^{\rm C} = \perp_{{\rm CLatt}\, L}$.

Let L be a join-commutative join-associative Huntington non empty complemented lattice structure. One can check that CLatt L is complemented distributive and bounded.

4. Proofs according to Bernd Ingo Dahn

Let G be a non empty complemented lattice structure and let x be an element of the carrier of G. We introduce $-x$ as a synonym of $x^{\rm c}$.

Let G be a join-commutative non empty complemented lattice structure. Let us observe that G is Huntington if and only if:

(Def. 12) For all elements x, y of the carrier of G holds $-(-x + -y) + -(x + -y) = y$.

Let G be a non empty complemented lattice structure. We say that G has idempotent element if and only if:

(Def. 13) There exists an element x of the carrier of G such that $x + x = x$.

In the sequel G is a Robbins join-associative join-commutative non empty complemented lattice structure and x, y, z are elements of the carrier of G.

Let G be a non empty complemented lattice structure and let x, y be elements of the carrier of G. The functor $\delta(x, y)$ yielding an element of G is defined by:

(Def. 14) $\delta(x, y) = -(-x + y)$.

Let G be a non empty complemented lattice structure and let x, y be elements of the carrier of G. The functor $\mathrm{Expand}(x, y)$ yields an element of G and is defined by:

(Def. 15) $\mathrm{Expand}(x, y) = \delta(x + y, \delta(x, y))$.

Let G be a non empty complemented lattice structure and let x be an element of the carrier of G. The functor x_0 yielding an element of G is defined by:

(Def. 16) $x_0 = -(-x + x)$.

The functor $2x$ yielding an element of G is defined as follows:

(Def. 17) $2x = x + x$.

Let G be a non empty complemented lattice structure and let x be an element of the carrier of G. The functor x_1 yielding an element of G is defined by:

(Def. 18) $x_1 = x_0 + x$.

The functor x_2 yields an element of G and is defined as follows:

(Def. 19) $x_2 = x_0 + 2x$.

The functor x_3 yields an element of G and is defined by:

(Def. 20) $x_3 = x_0 + (2x + x)$.

The functor x_4 yielding an element of G is defined as follows:

(Def. 21) $x_4 = x_0 + (2x + 2x)$.

We now state a number of propositions:

(37) $\delta(x + y, \delta(x, y)) = y$.

(38) $\mathrm{Expand}(x, y) = y$.

(39) $\delta(-x + y, z) = -(\delta(x, y) + z)$.

(40) $\delta(x, x) = x_0$.

(41) $\delta(2x, x_0) = x$.

(42) $\delta(x_2, x) = x_0$.

(43) $x_2 + x = x_3$.

(44) $x_4 + x_0 = x_3 + x_1$.

(45) $x_3 + x_0 = x_2 + x_1$.

(46) $x_3 + x = x_4$.

(47) $\delta(x_3, x_0) = x$.

(48) If $-x = -y$, then $\delta(x, z) = \delta(y, z)$.

(49) $\delta(x, -y) = \delta(y, -x)$.

(50) $\delta(x_3, x) = x_0$.

(51) $\delta(x_1 + x_3, x) = x_0$.

(52) $\delta(x_1 + x_2, x) = x_0$.

(53) $\delta(x_1 + x_3, x_0) = x$.

Let us consider G, x. The functor $\beta(x)$ yielding an element of G is defined as follows:

(Def. 22) $\beta(x) = -(x_1 + x_3) + x + -x_3$.

We now state three propositions:

(54) $\delta(\beta(x), x) = -x_3$.

(55) $\delta(\beta(x), x) = -(x_1 + x_3)$.

(56) There exist y, z such that $-(y + z) = -z$.

5. Proofs according to William McCune

One can prove the following two propositions:

(57) If for every z holds $--z = z$, then G is Huntington.

(58) If G has idempotent element, then G is Huntington.

Let us observe that TrivComplLat has idempotent element.

One can check that every Robbins join-associative join-commutative non empty complemented lattice structure which has idempotent element is Huntington.

One can prove the following two propositions:

(59) If there exist elements c, d of the carrier of G such that $c + d = c$, then G is Huntington.

(60) There exist y, z such that $y + z = z$.

One can verify that every join-associative join-commutative non empty complemented lattice structure which is Robbins is also Huntington.

Let L be a non empty ortholattice structure. We say that L is de Morgan if and only if:

(Def. 23) For all elements x, y of the carrier of L holds $x \sqcap y = (x^{\mathrm{c}} \sqcup y^{\mathrm{c}})^{\mathrm{c}}$.

Let L be a non empty complemented lattice structure. One can verify that CLatt L is de Morgan.

Next we state two propositions:

(61) Let L be a well-complemented join-commutative meet-commutative non empty ortholattice structure and x be an element of the carrier of L. Then $x + x^{\mathrm{c}} = \top_L$ and $x \sqcap x^{\mathrm{c}} = \bot_L$.

(62) For every bounded distributive well-complemented pre-ortholattice L holds $(\top_L)^{\mathrm{c}} = \bot_L$.

Let us observe that TrivOrtLat is de Morgan.

One can verify that there exists a pre-ortholattice which is strict, de Morgan, Boolean, Robbins, and Huntington.

Let us note that every non empty ortholattice structure which is join-associative, join-commutative, and de Morgan is also meet-commutative.

One can prove the following proposition

(63) For every Huntington de Morgan pre-ortholattice L holds $\perp_L^{\mathrm{c}} = \perp_L$.

One can verify that every well-complemented pre-ortholattice which is Boolean is also Huntington.

Let us note that every de Morgan pre-ortholattice which is Huntington is also Boolean.

One can verify that every pre-ortholattice which is Robbins and de Morgan is also Boolean and every well-complemented pre-ortholattice which is Boolean is also Robbins.

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Properties of Fuzzy Relation

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Summary. In this article, we introduce four fuzzy relations and the composition, and some useful properties are shown by them. In section 2, the definition of converse relation R^{-1} of fuzzy relation R and properties concerning it are described. In the next section, we define the composition of the fuzzy relation and show some properties. In the final section we describe the identity relation, the universe relation and the zero relation.

MML Identifier: `FUZZY_4`.

The notation and terminology used here are introduced in the following papers: [5], [6], [2], [9], [4], [3], [8], [7], and [1].

1. Basic Properties of the Membership Function

We follow the rules: x, y, z are sets and C_1, C_2, C_3 are non empty sets.

Let C_1 be a non empty set and let F be a membership function of C_1. One can check that $\operatorname{rng} F$ is non empty.

Next we state four propositions:

(1) Let F be a membership function of C_1. Then $\operatorname{rng} F$ is bounded and for every x such that $x \in \operatorname{dom} F$ holds $F(x) \leqslant \sup \operatorname{rng} F$ and for every x such that $x \in \operatorname{dom} F$ holds $F(x) \geqslant \inf \operatorname{rng} F$.

(2) For all membership functions F, G of C_1 such that for every x such that $x \in C_1$ holds $F(x) \leqslant G(x)$ holds $\sup \operatorname{rng} F \leqslant \sup \operatorname{rng} G$.

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(3) For every Membership function f of C_1, C_2 and for every element c of $[: C_1, C_2 :]$ holds $0 \leqslant f(c)$ and $f(c) \leqslant 1$.

(4) For every Membership function f of C_1, C_2 and for all x, y such that $\langle x, y\rangle \in [: C_1, C_2 :]$ holds $0 \leqslant f(\langle x, y\rangle)$ and $f(\langle x, y\rangle) \leqslant 1$.

2. Definition of Converse Fuzzy Relation and some Properties

Let C_1, C_2 be non empty sets and let h be a Membership function of C_2, C_1. The functor converse h yielding a Membership function of C_1, C_2 is defined by:

(Def. 1) For all x, y such that $\langle x, y\rangle \in [: C_1, C_2 :]$ holds $(\text{converse}\, h)(\langle x, y\rangle) = h(\langle y, x\rangle)$.

Let C_1, C_2 be non empty sets, let f be a Membership function of C_2, C_1, and let R be a fuzzy relation of C_2, C_1, f. The functor R^{-1} yields a fuzzy relation of C_1, C_2, converse f and is defined by:

(Def. 2) $R^{-1} = [: [: C_1, C_2 :], (\text{converse}\, f)^\circ [: C_1, C_2 :] :]$.

The following propositions are true:

(5) For every Membership function f of C_1, C_2 holds $\text{converse}\,\text{converse}\, f = f$.

(6) For every Membership function f of C_1, C_2 and for every fuzzy relation R of C_1, C_2, f holds $(R^{-1})^{-1} = R$.

(7) For every Membership function f of C_1, C_2 holds $1\text{-minus}\,\text{converse}\, f = \text{converse}\, 1\text{-minus}\, f$.

(8) For every Membership function f of C_1, C_2 and for every fuzzy relation R of C_1, C_2, f holds $(R^{-1})^{\mathrm{c}} = (R^{\mathrm{c}})^{-1}$.

(9) For all Membership functions f, g of C_1, C_2 holds $\text{converse}\max(f, g) = \max(\text{converse}\, f, \text{converse}\, g)$.

(10) Let f, g be Membership functions of C_1, C_2, R be a fuzzy relation of C_1, C_2, f, and S be a fuzzy relation of C_1, C_2, g. Then $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.

(11) For all Membership functions f, g of C_1, C_2 holds $\text{converse}\min(f, g) = \min(\text{converse}\, f, \text{converse}\, g)$.

(12) Let f, g be Membership functions of C_1, C_2, R be a fuzzy relation of C_1, C_2, f, and S be a fuzzy relation of C_1, C_2, g. Then $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$.

(13) Let f, g be Membership functions of C_1, C_2 and given x, y. If $x \in C_1$ and $y \in C_2$, then if $f(\langle x, y\rangle) \leqslant g(\langle x, y\rangle)$, then $(\text{converse}\, f)(\langle y, x\rangle) \leqslant (\text{converse}\, g)(\langle y, x\rangle)$.

(14) Let f, g be Membership functions of C_1, C_2, R be a fuzzy relation of C_1, C_2, f, and S be a fuzzy relation of C_1, C_2, g. If $R \subseteq S$, then $R^{-1} \subseteq S^{-1}$.

(15) For all Membership functions f, g of C_1, C_2 holds converse $\min(f, \text{1-minus}\, g) = \min(\text{converse}\, f, \text{1-minus converse}\, g)$.

(16) Let f, g be Membership functions of C_1, C_2, R be a fuzzy relation of C_1, C_2, f, and S be a fuzzy relation of C_1, C_2, g. Then $(R \setminus S)^{-1} = R^{-1} \setminus S^{-1}$.

(17) For all Membership functions f, g of C_1, C_2 holds converse $\max(\min(f, \text{1-minus}\, g), \min(\text{1-minus}\, f, g)) = \max(\min(\text{converse}\, f, \text{1-minus converse}\, g), \min(\text{1-minus converse}\, f, \text{converse}\, g))$.

(18) Let f, g be Membership functions of C_1, C_2, R be a fuzzy relation of C_1, C_2, f, and S be a fuzzy relation of C_1, C_2, g. Then $(R \dot{-} S)^{-1} = R^{-1} \dot{-} S^{-1}$.

3. Definition of the Composition and some Properties

Let C_1, C_2, C_3 be non empty sets, let h be a Membership function of C_1, C_2, let g be a Membership function of C_2, C_3, and let x, z be sets. Let us assume that $x \in C_1$ and $z \in C_3$. The functor $\min(h, g, x, z)$ yields a membership function of C_2 and is defined by:

(Def. 3) For every element y of C_2 holds $(\min(h, g, x, z))(y) = \min(h(\langle x, y\rangle), g(\langle y, z\rangle))$.

Let C_1, C_2, C_3 be non empty sets, let h be a Membership function of C_1, C_2, and let g be a Membership function of C_2, C_3. The functor $h\, g$ yields a Membership function of C_1, C_3 and is defined by:

(Def. 4) For all x, z such that $\langle x, z\rangle \in [\!:C_1, C_3:\!]$ holds $(h\, g)(\langle x, z\rangle) = \sup \operatorname{rng} \min(h, g, x, z)$.

Let C_1, C_2, C_3 be non empty sets, let f be a Membership function of C_1, C_2, let g be a Membership function of C_2, C_3, let R be a fuzzy relation of C_1, C_2, f, and let S be a fuzzy relation of C_2, C_3, g. The functor $R\, S$ yields a fuzzy relation of C_1, C_3, $f\, g$ and is defined as follows:

(Def. 5) $R\, S = [\!:[\!:C_1, C_3:\!], (f\, g)^{\circ}[\!:C_1, C_3:\!]:\!]$.

Next we state a number of propositions:

(19) For every Membership function f of C_1, C_2 and for all Membership functions g, h of C_2, C_3 holds $f \max(g, h) = \max(f\, g, f\, h)$.

(20) Let f be a Membership function of C_1, C_2, g, h be Membership functions of C_2, C_3, R be a fuzzy relation of C_1, C_2, f, S be a fuzzy relation of C_2, C_3, g, and T be a fuzzy relation of C_2, C_3, h. Then $R\, (S \cup T) = R\, S \cup R\, T$.

(21) For all Membership functions f, g of C_1, C_2 and for every Membership function h of C_2, C_3 holds $\max(f, g)\, h = \max(f\, h, g\, h)$.

(22) Let f, g be Membership functions of C_1, C_2, h be a Membership function of C_2, C_3, R be a fuzzy relation of C_1, C_2, f, S be a fuzzy relation of C_1, C_2, g, and T be a fuzzy relation of C_2, C_3, h. Then $(R \cup S)\, T = R\, T \cup S\, T$.

(23) Let f be a Membership function of C_1, C_2, g, h be Membership functions of C_2, C_3, and x, z be sets. If $x \in C_1$ and $z \in C_3$, then $(f\ \min(g, h))(\langle x, z\rangle) \leqslant (\min(f\, g, f\, h))(\langle x, z\rangle)$.

(24) Let f be a Membership function of C_1, C_2, g, h be Membership functions of C_2, C_3, R be a fuzzy relation of C_1, C_2, f, S be a fuzzy relation of C_2, C_3, g, and T be a fuzzy relation of C_2, C_3, h. Then $R\,(S \cap T) \subseteq (R\, S) \cap (R\, T)$.

(25) Let f, g be Membership functions of C_1, C_2, h be a Membership function of C_2, C_3, and x, z be sets. If $x \in C_1$ and $z \in C_3$, then $(\min(f, g)\, h)(\langle x, z\rangle) \leqslant (\min(f\, h, g\, h))(\langle x, z\rangle)$.

(26) Let f, g be Membership functions of C_1, C_2, h be a Membership function of C_2, C_3, R be a fuzzy relation of C_1, C_2, f, S be a fuzzy relation of C_1, C_2, g, and T be a fuzzy relation of C_2, C_3, h. Then $(R \cap S)\, T \subseteq (R\, T) \cap (S\, T)$.

(27) For every Membership function f of C_1, C_2 and for every Membership function g of C_2, C_3 holds converse $f\, g$ = converse g converse f.

(28) Let f be a Membership function of C_1, C_2, g be a Membership function of C_2, C_3, R be a fuzzy relation of C_1, C_2, f, and S be a fuzzy relation of C_2, C_3, g. Then $(R\, S)^{-1} = S^{-1}\, R^{-1}$.

(29) Let f, g be Membership functions of C_1, C_2, h, k be Membership functions of C_2, C_3, and x, z be sets. Suppose $x \in C_1$ and $z \in C_3$ and for every set y such that $y \in C_2$ holds $f(\langle x, y\rangle) \leqslant g(\langle x, y\rangle)$ and $h(\langle y, z\rangle) \leqslant k(\langle y, z\rangle)$. Then $(f\, h)(\langle x, z\rangle) \leqslant (g\, k)(\langle x, z\rangle)$.

(30) Let f, g be Membership functions of C_1, C_2, h, k be Membership functions of C_2, C_3, R be a fuzzy relation of C_1, C_2, f, S be a fuzzy relation of C_1, C_2, g, T be a fuzzy relation of C_2, C_3, h, and W be a fuzzy relation of C_2, C_3, k. If $R \subseteq S$ and $T \subseteq W$, then $R\, T \subseteq S\, W$.

4. Definition of Identity Relation and Properties of Universe and Zero Relation

Let C_1, C_2 be non empty sets. The functor $\mathrm{Imf}(C_1, C_2)$ yields a Membership function of C_1, C_2 and is defined as follows:

(Def. 6) For all x, y such that $\langle x, y\rangle \in [\!: C_1, C_2 :\!]$ holds if $x = y$, then $(\mathrm{Imf}(C_1, C_2))(\langle x, y\rangle) = 1$ and if $x \neq y$, then $(\mathrm{Imf}(C_1, C_2))(\langle x, y\rangle) = 0$.

One can prove the following propositions:

(31) For every element c of $[\!: C_1, C_2 :\!]$ holds $(\mathrm{Zmf}(C_1, C_2))(c) = 0$ and $(\mathrm{Umf}(C_1, C_2))(c) = 1$.

(32) For all x, y such that $\langle x, y\rangle \in [: C_1, C_2 :]$ holds $(\mathrm{Zmf}(C_1, C_2))(\langle x, y\rangle) = 0$ and $(\mathrm{Umf}(C_1, C_2))(\langle x, y\rangle) = 1$.

(33) Let f be a Membership function of C_2, C_3, O_1 be a zero relation of C_1, C_2, O_2 be a zero relation of C_1, C_3, and R be a fuzzy relation of C_2, C_3, f. Then $O_1\,R = O_2$.

(34) For every Membership function f of C_1, C_2 holds $f\ \mathrm{Zmf}(C_2, C_3) = \mathrm{Zmf}(C_1, C_3)$.

(35) Let f be a Membership function of C_1, C_2, O_1 be a zero relation of C_2, C_3, O_2 be a zero relation of C_1, C_3, and R be a fuzzy relation of C_1, C_2, f. Then $R\,O_1 = O_2$.

(36) For every Membership function f of C_1, C_1 holds $f\ \mathrm{Zmf}(C_1, C_1) = \mathrm{Zmf}(C_1, C_1)\, f$.

(37) Let f be a Membership function of C_1, C_1, O be a zero relation of C_1, C_1, and R be a fuzzy relation of C_1, C_1, f. Then $R\,O = O\,R$.

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On Outside Fashoda Meet Theorem

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Summary. We have proven the "Fashoda Meet Theorem" in [12]. Here we prove the outside version of it. It says that if Britain and France intended to set the courses for ships to the opposite side of Africa, they must also meet.

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The articles [19], [8], [1], [2], [3], [4], [12], [13], [11], [5], [14], [7], [10], [20], [17], [18], [16], [9], [15], and [6] provide the terminology and notation for this paper.

One can prove the following propositions:

(1) For all real numbers a, b such that $a \neq 0$ and $b \neq 0$ holds $\frac{a}{b} \cdot \frac{b}{a} = 1$.

(2) For every real number a such that $1 \leqslant a$ holds $a \leqslant a^{\mathbf{2}}$.

(3) For every real number a such that $-1 \geqslant a$ holds $-a \leqslant a^{\mathbf{2}}$.

(4) For every real number a such that $-1 > a$ holds $-a < a^{\mathbf{2}}$.

(5) For all real numbers a, b such that $b^{\mathbf{2}} \leqslant a^{\mathbf{2}}$ and $a \geqslant 0$ holds $-a \leqslant b$ and $b \leqslant a$.

(6) For all real numbers a, b such that $b^{\mathbf{2}} < a^{\mathbf{2}}$ and $a \geqslant 0$ holds $-a < b$ and $b < a$.

(7) For all real numbers a, b such that $-a \leqslant b$ and $b \leqslant a$ holds $b^{\mathbf{2}} \leqslant a^{\mathbf{2}}$.

(8) For all real numbers a, b such that $-a < b$ and $b < a$ holds $b^{\mathbf{2}} < a^{\mathbf{2}}$.

In the sequel T, T_1, T_2, S denote non empty topological spaces.

Next we state a number of propositions:

(9) Let f be a map from T_1 into S, g be a map from T_2 into S, and F_1, F_2 be subsets of T. Suppose that T_1 is a subspace of T and T_2 is a subspace of T and $F_1 = \Omega_{(T_1)}$ and $F_2 = \Omega_{(T_2)}$ and $\Omega_{(T_1)} \cup \Omega_{(T_2)} = \Omega_T$ and F_1 is closed and F_2 is closed and f is continuous and g is continuous and for every set p such that $p \in \Omega_{(T_1)} \cap \Omega_{(T_2)}$ holds $f(p) = g(p)$. Then there exists a map h from T into S such that $h = f{+\cdot}g$ and h is continuous.

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(10) Let n be a natural number, q_2 be a point of $\mathcal{E}^n$, q be a point of $\mathcal{E}^n_{\mathrm{T}}$, and r be a real number. If $q = q_2$, then $\mathrm{Ball}(q_2, r) = \{q_3; q_3$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $|q - q_3| < r\}$.

(11) $(0_{\mathcal{E}^2_{\mathrm{T}}})_{\mathbf{1}} = 0$ and $(0_{\mathcal{E}^2_{\mathrm{T}}})_{\mathbf{2}} = 0$.

(12) $1.\mathrm{REAL}\,2 = \langle(1$ **qua** real number$), (1$ **qua** real number$)\rangle$.

(13) $(1.\mathrm{REAL}\,2)_{\mathbf{1}} = 1$ and $(1.\mathrm{REAL}\,2)_{\mathbf{2}} = 1$.

(14) $\mathrm{dom}\,\mathrm{proj1} =$ the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and $\mathrm{dom}\,\mathrm{proj1} = \mathcal{R}^2$.

(15) $\mathrm{dom}\,\mathrm{proj2} =$ the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and $\mathrm{dom}\,\mathrm{proj2} = \mathcal{R}^2$.

(16) proj1 is a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$.

(17) proj2 is a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$.

(18) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $p = [\mathrm{proj1}(p), \mathrm{proj2}(p)]$.

(19) For every subset B of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ such that $B = \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ holds $B^{\mathrm{c}} \neq \emptyset$ and (the carrier of $\mathcal{E}^2_{\mathrm{T}}$) $\setminus B \neq \emptyset$.

(20) Let X, Y be non empty topological spaces and f be a map from X into Y. Then f is continuous if and only if for every point p of X and for every subset V of Y such that $f(p) \in V$ and V is open there exists a subset W of X such that $p \in W$ and W is open and $f^{\circ}W \subseteq V$.

(21) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$ and G be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose G is open and $p \in G$. Then there exists a real number r such that $r > 0$ and $\{q; q$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{1}} - r < q_{\mathbf{1}} \wedge q_{\mathbf{1}} < p_{\mathbf{1}} + r \wedge p_{\mathbf{2}} - r < q_{\mathbf{2}} \wedge q_{\mathbf{2}} < p_{\mathbf{2}} + r\} \subseteq G$.

(22) Let X, Y, Z be non empty topological spaces, B be a subset of Y, C be a subset of Z, f be a map from X into Y, and h be a map from $Y{\restriction}B$ into $Z{\restriction}C$. Suppose f is continuous and h is continuous and $\mathrm{rng}\, f \subseteq B$ and $B \neq \emptyset$ and $C \neq \emptyset$. Then there exists a map g from X into Z such that g is continuous and $g = h \cdot f$.

In the sequel p, q are points of $\mathcal{E}^2_{\mathrm{T}}$.

The function OutInSq from (the carrier of $\mathcal{E}^2_{\mathrm{T}}$) $\setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ into (the carrier of $\mathcal{E}^2_{\mathrm{T}}$) $\setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ is defined by the condition (Def. 1).

(Def. 1) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$. Then

(i) if $p_{\mathbf{2}} \leqslant p_{\mathbf{1}}$ and $-p_{\mathbf{1}} \leqslant p_{\mathbf{2}}$ or $p_{\mathbf{2}} \geqslant p_{\mathbf{1}}$ and $p_{\mathbf{2}} \leqslant -p_{\mathbf{1}}$, then $\mathrm{OutInSq}(p) = [\frac{1}{p_{\mathbf{1}}}, \frac{\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}}}{p_{\mathbf{1}}}]$, and

(ii) if $p_{\mathbf{2}} \not\leqslant p_{\mathbf{1}}$ or $-p_{\mathbf{1}} \not\leqslant p_{\mathbf{2}}$ and if $p_{\mathbf{2}} \not\geqslant p_{\mathbf{1}}$ or $p_{\mathbf{2}} \not\leqslant -p_{\mathbf{1}}$, then $\mathrm{OutInSq}(p) = [\frac{\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}}}{p_{\mathbf{2}}}, \frac{1}{p_{\mathbf{2}}}]$.

Next we state a number of propositions:

(23) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} \not\leqslant p_{\mathbf{1}}$ or $-p_{\mathbf{1}} \not\leqslant p_{\mathbf{2}}$ but $p_{\mathbf{2}} \not\geqslant p_{\mathbf{1}}$ or $p_{\mathbf{2}} \not\leqslant -p_{\mathbf{1}}$. Then $p_{\mathbf{1}} \leqslant p_{\mathbf{2}}$ and $-p_{\mathbf{2}} \leqslant p_{\mathbf{1}}$ or $p_{\mathbf{1}} \geqslant p_{\mathbf{2}}$ and $p_{\mathbf{1}} \leqslant -p_{\mathbf{2}}$.

(24) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$. Then

(i) if $\mathbf{p_1} \leqslant \mathbf{p_2}$ and $-\mathbf{p_2} \leqslant \mathbf{p_1}$ or $\mathbf{p_1} \geqslant \mathbf{p_2}$ and $\mathbf{p_1} \leqslant -\mathbf{p_2}$, then $\mathrm{OutInSq}(p) = [\frac{\frac{\mathbf{p_1}}{\mathbf{p_2}}}{\mathbf{p_2}}, \frac{1}{\mathbf{p_2}}]$, and

(ii) if $\mathbf{p_1} \not\leqslant \mathbf{p_2}$ or $-\mathbf{p_2} \not\leqslant \mathbf{p_1}$ and if $\mathbf{p_1} \not\geqslant \mathbf{p_2}$ or $\mathbf{p_1} \not\leqslant -\mathbf{p_2}$, then $\mathrm{OutInSq}(p) = [\frac{1}{\mathbf{p_1}}, \frac{\frac{\mathbf{p_2}}{\mathbf{p_1}}}{\mathbf{p_1}}]$.

(25) Let D be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and K_0 be a subset of $(\mathcal{E}^2_{\mathrm{T}}) \restriction D$. Suppose $K_0 = \{p : (\mathbf{p_2} \leqslant \mathbf{p_1} \ \wedge \ -\mathbf{p_1} \leqslant \mathbf{p_2} \ \vee \ \mathbf{p_2} \geqslant \mathbf{p_1} \ \wedge \ \mathbf{p_2} \leqslant -\mathbf{p_1}) \ \wedge \ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then $\mathrm{rng}(\mathrm{OutInSq} \restriction K_0) \subseteq$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction D \restriction K_0$.

(26) Let D be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and K_0 be a subset of $(\mathcal{E}^2_{\mathrm{T}}) \restriction D$. Suppose $K_0 = \{p : (\mathbf{p_1} \leqslant \mathbf{p_2} \ \wedge \ -\mathbf{p_2} \leqslant \mathbf{p_1} \ \vee \ \mathbf{p_1} \geqslant \mathbf{p_2} \ \wedge \ \mathbf{p_1} \leqslant -\mathbf{p_2}) \ \wedge \ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then $\mathrm{rng}(\mathrm{OutInSq} \restriction K_0) \subseteq$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction D \restriction K_0$.

(27) Let K_1 be a set and D be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K_1 = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $(\mathbf{p_2} \leqslant \mathbf{p_1} \ \wedge \ -\mathbf{p_1} \leqslant \mathbf{p_2} \ \vee \ \mathbf{p_2} \geqslant \mathbf{p_1} \ \wedge \ \mathbf{p_2} \leqslant -\mathbf{p_1}) \ \wedge \ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $D^{\mathrm{c}} = \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then K_1 is a non empty subset of $(\mathcal{E}^2_{\mathrm{T}}) \restriction D$ and a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$.

(28) Let K_1 be a set and D be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K_1 = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $(\mathbf{p_1} \leqslant \mathbf{p_2} \ \wedge \ -\mathbf{p_2} \leqslant \mathbf{p_1} \ \vee \ \mathbf{p_1} \geqslant \mathbf{p_2} \ \wedge \ \mathbf{p_1} \leqslant -\mathbf{p_2}) \ \wedge \ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $D^{\mathrm{c}} = \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then K_1 is a non empty subset of $(\mathcal{E}^2_{\mathrm{T}}) \restriction D$ and a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$.

(29) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_1 + r_2$ and g is continuous.

(30) Let X be a non empty topological space and a be a real number. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X holds $g(p) = a$ and g is continuous.

(31) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_1 - r_2$ and g is continuous.

(32) Let X be a non empty topological space and f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = r_1 \cdot r_1$ and g is continuous.

(33) Let X be a non empty topological space, f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$, and a be a real number. Suppose f_1 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = a \cdot r_1$ and g is continuous.

(34) Let X be a non empty topological space, f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$, and a be a real number. Suppose f_1 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = r_1 + a$ and g is continuous.

(35) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_1 \cdot r_2$ and g is continuous.

(36) Let X be a non empty topological space and f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and for every point q of X holds $f_1(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = \frac{1}{r_1}$ and g is continuous.

(37) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = \frac{r_1}{r_2}$ and g is continuous.

(38) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that
 (i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = \frac{\frac{r_1}{r_2}}{r_2}$, and
 (ii) g is continuous.

(39) Let K_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_0$ into $\mathbb{R}^{\mathbf{1}}$. If for every point p of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_0$ holds $f(p) = \mathrm{proj1}(p)$, then f is continuous.

(40) Let K_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_0$ into $\mathbb{R}^{\mathbf{1}}$. If for every point p of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_0$ holds $f(p) = \mathrm{proj2}(p)$, then f is continuous.

(41) Let K_2 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_2$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that
 (i) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_2$ holds $f(p) = \frac{1}{p_{\mathbf{1}}}$, and
 (ii) for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_2$ holds $q_{\mathbf{1}} \neq 0$.

 Then f is continuous.

(42) Let K_2 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_2$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that
 (i) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_2$ holds $f(p) = \frac{1}{p_{\mathbf{2}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_2$ holds $q_{\mathbf{2}} \neq 0$.
Then f is continuous.

(43) Let K_2 be a non empty subset of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_2$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_2$ holds $f(p) = \frac{\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}}}{p_{\mathbf{1}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_2$ holds $q_{\mathbf{1}} \neq 0$.
Then f is continuous.

(44) Let K_2 be a non empty subset of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_2$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_2$ holds $f(p) = \frac{\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}}}{p_{\mathbf{2}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_2$ holds $q_{\mathbf{2}} \neq 0$.
Then f is continuous.

(45) Let K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$, f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$, and f_1, f_2 be maps from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) f_1 is continuous,

(ii) f_2 is continuous,

(iii) $K_0 \neq \emptyset$,

(iv) $B_0 \neq \emptyset$, and

(v) for all real numbers x, y, r, s such that $[x, y] \in K_0$ and $r = f_1([x, y])$ and $s = f_2([x, y])$ holds $f([x, y]) = [r, s]$.
Then f is continuous.

(46) Let K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $f = \mathrm{OutInSq}\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : (p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \ \wedge \ -p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \ \vee \ p_{\mathbf{2}} \geqslant p_{\mathbf{1}} \ \wedge \ p_{\mathbf{2}} \leqslant -p_{\mathbf{1}}) \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$.
Then f is continuous.

(47) Let K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $f = \mathrm{OutInSq}\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : (p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \ \wedge \ -p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \ \vee \ p_{\mathbf{1}} \geqslant p_{\mathbf{2}} \ \wedge \ p_{\mathbf{1}} \leqslant -p_{\mathbf{2}}) \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$.
Then f is continuous.

In this article we present several logical schemes. The scheme *TopSubset* concerns a unary predicate $\mathcal{P}$, and states that:

$\{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $\mathcal{P}[p]\}$ is a subset of $\mathcal{E}^2_{\rm T}$

for all values of the parameters.

The scheme *TopCompl* deals with a subset $\mathcal{A}$ of $\mathcal{E}^2_{\rm T}$ and a unary predicate $\mathcal{P}$, and states that:

$-\mathcal{A} = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: not $\mathcal{P}[p]\}$

provided the parameters meet the following requirement:

- $\mathcal{A} = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $\mathcal{P}[p]\}$.

The scheme *ClosedSubset* deals with two unary functors $\mathcal{F}$ and $\mathcal{G}$ yielding real numbers, and states that:

$\{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $\mathcal{F}(p) \leqslant \mathcal{G}(p)\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$

provided the following conditions are met:

- For all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathcal{F}(p-q) = \mathcal{F}(p) - \mathcal{F}(q)$ and $\mathcal{G}(p-q) = \mathcal{G}(p) - \mathcal{G}(q)$, and
- For all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|p-q|^{\mathbf{2}} = |\mathcal{F}(p-q)|^{\mathbf{2}} + |\mathcal{G}(p-q)|^{\mathbf{2}}$.

One can prove the following propositions:

(48) Let B_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$, K_0 be a subset of $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$, and f be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$. Suppose $f = \mathrm{OutInSq}\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\mathrm{T}}) \setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : (p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \wedge -p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \vee p_{\mathbf{2}} \geqslant p_{\mathbf{1}} \wedge p_{\mathbf{2}} \leqslant -p_{\mathbf{1}}) \wedge p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then f is continuous and K_0 is closed.

(49) Let B_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$, K_0 be a subset of $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$, and f be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$. Suppose $f = \mathrm{OutInSq}\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\mathrm{T}}) \setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : (p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \wedge -p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \vee p_{\mathbf{1}} \geqslant p_{\mathbf{2}} \wedge p_{\mathbf{1}} \leqslant -p_{\mathbf{2}}) \wedge p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then f is continuous and K_0 is closed.

(50) Let D be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $D^{\mathrm{c}} = \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then there exists a map h from $(\mathcal{E}^2_{\mathrm{T}})\restriction D$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction D$ such that $h = \mathrm{OutInSq}$ and h is continuous.

(51) Let B, K_0, K_3 be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) $B = \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$,

(ii) $K_0 = \{p : -1 < p_{\mathbf{1}} \wedge p_{\mathbf{1}} < 1 \wedge -1 < p_{\mathbf{2}} \wedge p_{\mathbf{2}} < 1\}$, and

(iii) $K_3 = \{q : -1 = q_{\mathbf{1}} \wedge -1 \leqslant q_{\mathbf{2}} \wedge q_{\mathbf{2}} \leqslant 1 \vee q_{\mathbf{1}} = 1 \wedge -1 \leqslant q_{\mathbf{2}} \wedge q_{\mathbf{2}} \leqslant 1 \vee -1 = q_{\mathbf{2}} \wedge -1 \leqslant q_{\mathbf{1}} \wedge q_{\mathbf{1}} \leqslant 1 \vee 1 = q_{\mathbf{2}} \wedge -1 \leqslant q_{\mathbf{1}} \wedge q_{\mathbf{1}} \leqslant 1\}$.

Then there exists a map f from $(\mathcal{E}^2_{\mathrm{T}})\restriction B^{\mathrm{c}}$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction B^{\mathrm{c}}$ such that

(iv) f is continuous and one-to-one,

(v) for every point t of $\mathcal{E}^2_{\mathrm{T}}$ such that $t \in K_0$ and $t \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$ holds $f(t) \notin K_0 \cup K_3$,

(vi) for every point r of $\mathcal{E}^2_{\mathrm{T}}$ such that $r \notin K_0 \cup K_3$ holds $f(r) \in K_0$, and

(vii) for every point s of $\mathcal{E}^2_{\mathrm{T}}$ such that $s \in K_3$ holds $f(s) = s$.

(52) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, K_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $K_0 = \{p : -1 < p_{\mathbf{1}} \wedge p_{\mathbf{1}} < 1 \wedge -1 < p_{\mathbf{2}} \wedge p_{\mathbf{2}} < 1\}$ and $f(O)_{\mathbf{1}} = -1$ and $f(I)_{\mathbf{1}} = 1$ and $-1 \leqslant f(O)_{\mathbf{2}}$ and $f(O)_{\mathbf{2}} \leqslant 1$ and $-1 \leqslant f(I)_{\mathbf{2}}$ and $f(I)_{\mathbf{2}} \leqslant 1$ and $g(O)_{\mathbf{2}} = -1$ and $g(I)_{\mathbf{2}} = 1$ and $-1 \leqslant g(O)_{\mathbf{1}}$ and $g(O)_{\mathbf{1}} \leqslant 1$ and $-1 \leqslant g(I)_{\mathbf{1}}$ and $g(I)_{\mathbf{1}} \leqslant 1$ and $\mathrm{rng}\, f \cap K_0 = \emptyset$ and $\mathrm{rng}\, g \cap K_0 = \emptyset$. Then $\mathrm{rng}\, f \cap \mathrm{rng}\, g \neq \emptyset$.

(53) Let A, B, C, D be real numbers and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that for every point t of $\mathcal{E}^2_{\mathrm{T}}$ holds $f(t) = [A \cdot t_{\mathbf{1}} + B, C \cdot t_{\mathbf{2}} + D]$. Then f is continuous.

(54) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $f(O)_{\mathbf{1}} = a$ and $f(I)_{\mathbf{1}} = b$ and $c \leqslant f(O)_{\mathbf{2}}$ and $f(O)_{\mathbf{2}} \leqslant d$ and $c \leqslant f(I)_{\mathbf{2}}$ and $f(I)_{\mathbf{2}} \leqslant d$ and $g(O)_{\mathbf{2}} = c$ and $g(I)_{\mathbf{2}} = d$ and $a \leqslant g(O)_{\mathbf{1}}$ and $g(O)_{\mathbf{1}} \leqslant b$ and $a \leqslant g(I)_{\mathbf{1}}$ and $g(I)_{\mathbf{1}} \leqslant b$ and $a < b$ and $c < d$ and it is not true that there exists a point r of $\mathbb{I}$ such that $a < f(r)_{\mathbf{1}}$ and $f(r)_{\mathbf{1}} < b$ and $c < f(r)_{\mathbf{2}}$ and $f(r)_{\mathbf{2}} < d$ and it is not true that there exists a point r of $\mathbb{I}$ such that $a < g(r)_{\mathbf{1}}$ and $g(r)_{\mathbf{1}} < b$ and $c < g(r)_{\mathbf{2}}$ and $g(r)_{\mathbf{2}} < d$. Then $\operatorname{rng} f \cap \operatorname{rng} g \neq \emptyset$.

(55)(i) $\{p_7; p_7$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $(p_7)_{\mathbf{2}} \leqslant (p_7)_{\mathbf{1}}\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$, and

(ii) $\{p_7; p_7$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $(p_7)_{\mathbf{1}} \leqslant (p_7)_{\mathbf{2}}\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$.

(56)(i) $\{p_7; p_7$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $-(p_7)_{\mathbf{1}} \leqslant (p_7)_{\mathbf{2}}\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$, and

(ii) $\{p_7; p_7$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $(p_7)_{\mathbf{2}} \leqslant -(p_7)_{\mathbf{1}}\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$.

(57)(i) $\{p_7; p_7$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $-(p_7)_{\mathbf{2}} \leqslant (p_7)_{\mathbf{1}}\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$, and

(ii) $\{p_7; p_7$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $(p_7)_{\mathbf{1}} \leqslant -(p_7)_{\mathbf{2}}\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$.

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The Set of Primitive Recursive Functions[1]

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Summary. We follow [23] in defining the set of primitive recursive functions. The important helper notion is the homogeneous function from finite sequences of natural numbers into natural numbers where homogeneous means that all the sequences in the domain are of the same length. The set of all such functions is then used to define the notion of a set closed under composition of functions and under primitive recursion. We call a set primitively recursively closed iff it contains the initial functions (nullary constant function returning 0, unary successor and projection functions for all arities) and is closed under composition and primitive recursion. The set of primitive recursive functions is then defined as the smallest set of functions which is primitive recursively closed. We show that this set can be obtained by primitive recursive approximation. We finish with showing that some simple and well known functions are primitive recursive.

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The articles [17], [22], [3], [4], [6], [20], [18], [7], [8], [2], [5], [11], [1], [15], [9], [16], [24], [25], [14], [12], [21], [19], [13], and [10] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we adopt the following rules: i, j, k, c, m, n are natural numbers, a, x, y, z, X, Y are sets, D, E are non empty sets, R is a binary relation, f, g are functions, and p, q are finite sequences.

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Let X be a non empty set, let n be a natural number, let p be an element of X^n, let i be a natural number, and let x be an element of X. Then $p +\cdot (i, x)$ is an element of X^n.

Let n be a natural number, let t be an element of $\mathbb{N}^n$, and let i be a natural number. Then $t(i)$ is an element of $\mathbb{N}$.

The following propositions are true:

(3)[2] $\langle x, y\rangle +\cdot (1, z) = \langle z, y\rangle$ and $\langle x, y\rangle +\cdot (2, z) = \langle x, z\rangle$.

(5)[3] If $f +\cdot (a, x) = g +\cdot (a, y)$, then $f +\cdot (a, z) = g +\cdot (a, z)$.

(6) $(p +\cdot (i, x))_{\restriction i} = p_{\restriction i}$.

(7) If $p +\cdot (i, a) = q +\cdot (i, a)$, then $p_{\restriction i} = q_{\restriction i}$.

(8) $X^0 = \{\emptyset\}$.

(9) If $n \neq 0$, then $\emptyset^n = \emptyset$.

(10) If $\emptyset \in \operatorname{rng} f$, then $\prod^* f = \emptyset$.

(11) If $\operatorname{rng} f = D$, then $\operatorname{rng} \prod^* \langle f\rangle = D^1$.

(12) If $1 \leqslant i$ and $i \leqslant n + 1$, then for every element p of D^{n+1} holds $p_{\restriction i} \in D^n$.

(13) For every set X and for every set Y of finite sequences of X holds $Y \subseteq X^*$.

2. Sets of Compatible Functions

Let X be a set. We say that X is compatible if and only if:

(Def. 1) For all functions f, g such that $f \in X$ and $g \in X$ holds $f \approx g$.

Let us observe that there exists a set which is non empty, functional, and compatible.

Let X be a functional compatible set. One can verify that $\bigcup X$ is function-like and relation-like.

The following proposition is true

(14) X is functional and compatible iff $\bigcup X$ is a function.

Let X, Y be sets. One can verify that there exists a non empty set of partial functions from X to Y which is non empty and compatible.

The following propositions are true:

(15) For every non empty functional compatible set X holds $\operatorname{dom} \bigcup X = \bigcup\{\operatorname{dom} f : f$ ranges over elements of $X\}$.

(16) Let X be a functional compatible set and f be a function. If $f \in X$, then $\operatorname{dom} f \subseteq \operatorname{dom} \bigcup X$ and for every set x such that $x \in \operatorname{dom} f$ holds $(\bigcup X)(x) = f(x)$.

[2]The propositions (1) and (2) have been removed.

[3]The proposition (4) has been removed.

(17) For every non empty functional compatible set X holds $\operatorname{rng} \bigcup X = \bigcup \{\operatorname{rng} f : f \text{ ranges over elements of } X\}$.

Let us consider X, Y. Observe that every non empty set of partial functions from X to Y is functional.

We now state the proposition

(18) Let P be a compatible non empty set of partial functions from X to Y. Then $\bigcup P$ is a partial function from X to Y.

3. Homogeneous Relations

Let f be a binary relation. We introduce f is into $\mathbb{N}$ as a synonym of f is natural-yielding.

Let f be a binary relation. We say that f is from tuples on $\mathbb{N}$ if and only if:

(Def. 2) $\operatorname{dom} f \subseteq \mathbb{N}^*$.

One can check that there exists a function which is from tuples on $\mathbb{N}$ and into $\mathbb{N}$.

Let f be a binary relation from tuples on $\mathbb{N}$. We say that f is length total if and only if:

(Def. 3) For all finite sequences x, y of elements of $\mathbb{N}$ such that $\operatorname{len} x = \operatorname{len} y$ and $x \in \operatorname{dom} f$ holds $y \in \operatorname{dom} f$.

Let f be a binary relation. We say that f is homogeneous if and only if:

(Def. 4) For all finite sequences x, y such that $x \in \operatorname{dom} f$ and $y \in \operatorname{dom} f$ holds $\operatorname{len} x = \operatorname{len} y$.

One can prove the following proposition

(19) If $\operatorname{dom} R \subseteq D^n$, then R is homogeneous.

Let us observe that $\emptyset$ is homogeneous.

Let p be a finite sequence and let x be a set. Observe that $\{p\} \longmapsto x$ is non empty and homogeneous.

Let us note that there exists a function which is non empty and homogeneous.

Let f be a homogeneous function and let g be a function. Observe that $g \cdot f$ is homogeneous.

Let X, Y be sets. Note that there exists a partial function from X^* to Y which is homogeneous.

Let X, Y be non empty sets. Observe that there exists a partial function from X^* to Y which is non empty and homogeneous.

Let X be a non empty set. Observe that there exists a partial function from X^* to X which is non empty, homogeneous, and quasi total.

One can check that there exists a function from tuples on $\mathbb{N}$ which is non empty, homogeneous, into $\mathbb{N}$, and length total.

One can check that every partial function from $\mathbb{N}^*$ to $\mathbb{N}$ is into $\mathbb{N}$ and from tuples on $\mathbb{N}$.

Let us observe that every partial function from $\mathbb{N}^*$ to $\mathbb{N}$ which is quasi total is also length total.

The following proposition is true

(20) Every length total function from tuples on $\mathbb{N}$ into $\mathbb{N}$ is a quasi total partial function from $\mathbb{N}^*$ to $\mathbb{N}$.

Let f be a homogeneous binary relation. The functor arity f yielding a natural number is defined by:

(Def. 5)(i) For every finite sequence x such that $x \in \operatorname{dom} f$ holds arity $f = \operatorname{len} x$ if there exists a finite sequence x such that $x \in \operatorname{dom} f$,

(ii) arity $f = 0$, otherwise.

The following propositions are true:

(21) arity $\emptyset = 0$.

(22) For every homogeneous binary relation f such that $\operatorname{dom} f = \{\emptyset\}$ holds arity $f = 0$.

(23) For every homogeneous partial function f from X^* to Y holds $\operatorname{dom} f \subseteq X^{\operatorname{arity} f}$.

(24) For every homogeneous function f from tuples on $\mathbb{N}$ holds $\operatorname{dom} f \subseteq \mathbb{N}^{\operatorname{arity} f}$.

(25) Let f be a homogeneous partial function from X^* to X. Then f is quasi total and non empty if and only if $\operatorname{dom} f = X^{\operatorname{arity} f}$.

(26) Let f be a homogeneous function into $\mathbb{N}$ and from tuples on $\mathbb{N}$. Then f is length total and non empty if and only if $\operatorname{dom} f = \mathbb{N}^{\operatorname{arity} f}$.

(27) For every non empty homogeneous partial function f from D^* to D and for every n such that $\operatorname{dom} f \subseteq D^n$ holds arity $f = n$.

(28) For every homogeneous partial function f from D^* to D and for every n such that $\operatorname{dom} f = D^n$ holds arity $f = n$.

Let R be a binary relation. We say that R has the same arity if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let f, g be functions such that $f \in \operatorname{rng} R$ and $g \in \operatorname{rng} R$. Then

(i) if f is empty, then g is empty or $\operatorname{dom} g = \{\emptyset\}$, and

(ii) if f is non empty and g is non empty, then there exists a natural number n and there exists a non empty set X such that $\operatorname{dom} f \subseteq X^n$ and $\operatorname{dom} g \subseteq X^n$.

Let us note that $\emptyset$ has the same arity.

One can check that there exists a finite sequence which has the same arity. Let X be a set. One can verify that there exists a finite sequence of elements of X which has the same arity and there exists an element of X^* which has the same arity.

Let F be a binary relation with the same arity. The functor arity F yielding a natural number is defined as follows:

(Def. 7)(i) For every homogeneous function f such that $f \in \operatorname{rng} F$ holds arity F = arity f if there exists a homogeneous function f such that $f \in \operatorname{rng} F$,

(ii) arity $F = 0$, otherwise.

Next we state the proposition

(29) For every finite sequence F with the same arity such that $\operatorname{len} F = 0$ holds arity $F = 0$.

Let X be a set. The functor HFuncs X yielding a non empty set of partial functions from X^* to X is defined by:

(Def. 8) HFuncs $X = \{f; f$ ranges over elements of $X^* \dot{\rightarrow} X$: f is homogeneous$\}$.

Next we state the proposition

(30) $\emptyset \in$ HFuncs X.

Let X be a non empty set. Note that there exists an element of HFuncs X which is non empty, homogeneous, and quasi total.

Let X be a set. Observe that every element of HFuncs X is homogeneous.

Let X be a non empty set and let S be a non empty subset of HFuncs X. Note that every element of S is homogeneous.

The following propositions are true:

(31) Every homogeneous function into $\mathbb{N}$ and from tuples on $\mathbb{N}$ is an element of HFuncs $\mathbb{N}$.

(32) Every length total homogeneous function from tuples on $\mathbb{N}$ into $\mathbb{N}$ is a quasi total element of HFuncs $\mathbb{N}$.

(33) Let X be a non empty set and F be a binary relation such that $\operatorname{rng} F \subseteq$ HFuncs X and for all homogeneous functions f, g such that $f \in \operatorname{rng} F$ and $g \in \operatorname{rng} F$ holds arity f = arity g. Then F has the same arity.

Let n, m be natural numbers. The functor $\mathrm{const}_n(m)$ yields a homogeneous function into $\mathbb{N}$ and from tuples on $\mathbb{N}$ and is defined by:

(Def. 9) $\mathrm{const}_n(m) = \mathbb{N}^n \longmapsto m$.

We now state the proposition

(34) $\mathrm{const}_n(m) \in$ HFuncs $\mathbb{N}$.

Let n, m be natural numbers. One can check that $\mathrm{const}_n(m)$ is length total and non empty.

We now state two propositions:

(35) arity $\mathrm{const}_n(m) = n$.

(36) For every element t of $\mathbb{N}^n$ holds $(\mathrm{const}_n(m))(t) = m$.

Let n, i be natural numbers. The functor $\mathrm{succ}_n(i)$ yields a homogeneous function into $\mathbb{N}$ and from tuples on $\mathbb{N}$ and is defined by:

(Def. 10) $\operatorname{dom} \operatorname{succ}_n(i) = \mathbb{N}^n$ and for every element p of $\mathbb{N}^n$ holds $(\operatorname{succ}_n(i))(p) = p_i + 1$.

We now state the proposition

(37) $\operatorname{succ}_n(i) \in \operatorname{HFuncs} \mathbb{N}$.

Let n, i be natural numbers. One can check that $\operatorname{succ}_n(i)$ is length total and non empty.

Next we state the proposition

(38) $\operatorname{arity} \operatorname{succ}_n(i) = n$.

Let n, i be natural numbers. The functor $\operatorname{proj}_n(i)$ yielding a homogeneous function into $\mathbb{N}$ and from tuples on $\mathbb{N}$ is defined by:

(Def. 11) $\operatorname{proj}_n(i) = \operatorname{proj}(n \mapsto \mathbb{N}, i)$.

The following two propositions are true:

(39) $\operatorname{proj}_n(i) \in \operatorname{HFuncs} \mathbb{N}$.

(40) $\operatorname{dom} \operatorname{proj}_n(i) = \mathbb{N}^n$ and if $1 \leqslant i$ and $i \leqslant n$, then $\operatorname{rng} \operatorname{proj}_n(i) = \mathbb{N}$.

Let n, i be natural numbers. One can verify that $\operatorname{proj}_n(i)$ is length total and non empty.

We now state two propositions:

(41) $\operatorname{arity} \operatorname{proj}_n(i) = n$.

(42) For every element t of $\mathbb{N}^n$ holds $(\operatorname{proj}_n(i))(t) = t(i)$.

Let X be a set. Observe that $\operatorname{HFuncs} X$ is functional.

We now state three propositions:

(43) Let F be a function from D into $\operatorname{HFuncs} E$. Suppose $\operatorname{rng} F$ is compatible and for every element x of D holds $\operatorname{dom} F(x) \subseteq E^n$. Then there exists an element f of $\operatorname{HFuncs} E$ such that $f = \bigcup F$ and $\operatorname{dom} f \subseteq E^n$.

(44) For every function F from $\mathbb{N}$ into $\operatorname{HFuncs} D$ such that for every i holds $F(i) \subseteq F(i+1)$ holds $\bigcup F \in \operatorname{HFuncs} D$.

(45) For every finite sequence F of elements of $\operatorname{HFuncs} D$ with the same arity holds $\operatorname{dom} \prod^* F \subseteq D^{\operatorname{arity} F}$.

Let X be a non empty set and let F be a finite sequence of elements of $\operatorname{HFuncs} X$ with the same arity. Observe that $\prod^* F$ is homogeneous.

The following proposition is true

(46) Let f be an element of $\operatorname{HFuncs} D$ and F be a finite sequence of elements of $\operatorname{HFuncs} D$ with the same arity. Then $\operatorname{dom}(f \cdot \prod^* F) \subseteq D^{\operatorname{arity} F}$ and $\operatorname{rng}(f \cdot \prod^* F) \subseteq D$ and $f \cdot \prod^* F \in \operatorname{HFuncs} D$.

Let X, Y be non empty sets, let P be a non empty set of partial functions from X to Y, and let S be a non empty subset of P. We see that the element of S is an element of P.

Let f be a homogeneous function from tuples on $\mathbb{N}$. One can check that $\langle f \rangle$ has the same arity.

Next we state several propositions:

(47) For every homogeneous function f into $\mathbb{N}$ and from tuples on $\mathbb{N}$ holds $\operatorname{arity}\langle f\rangle = \operatorname{arity} f$.

(48) Let f, g be non empty elements of HFuncs $\mathbb{N}$ and F be a finite sequence of elements of HFuncs $\mathbb{N}$ with the same arity. If $g = f \cdot \prod^* F$, then $\operatorname{arity} g = \operatorname{arity} F$.

(49) Let f be a non empty quasi total element of HFuncs D and F be a finite sequence of elements of HFuncs D with the same arity. Suppose $\operatorname{arity} f = \operatorname{len} F$ and F is non empty and for every element h of HFuncs D such that $h \in \operatorname{rng} F$ holds h is quasi total and non empty. Then $f \cdot \prod^* F$ is a non empty quasi total element of HFuncs D and $\operatorname{dom}(f \cdot \prod^* F) = D^{\operatorname{arity} F}$.

(50) Let f be a quasi total element of HFuncs D and F be a finite sequence of elements of HFuncs D with the same arity. Suppose $\operatorname{arity} f = \operatorname{len} F$ and for every element h of HFuncs D such that $h \in \operatorname{rng} F$ holds h is quasi total. Then $f \cdot \prod^* F$ is a quasi total element of HFuncs D.

(51) For all non empty quasi total elements f, g of HFuncs D such that $\operatorname{arity} f = 0$ and $\operatorname{arity} g = 0$ and $f(\emptyset) = g(\emptyset)$ holds $f = g$.

(52) Let f, g be non empty length total homogeneous functions from tuples on $\mathbb{N}$ into $\mathbb{N}$. If $\operatorname{arity} f = 0$ and $\operatorname{arity} g = 0$ and $f(\emptyset) = g(\emptyset)$, then $f = g$.

4. Primitive Recursiveness

We adopt the following convention: f_1, f_2 are non empty homogeneous functions into $\mathbb{N}$ and from tuples on $\mathbb{N}$, e_1, e_2 are homogeneous functions into $\mathbb{N}$ and from tuples on $\mathbb{N}$, and p is an element of $\mathbb{N}^{\operatorname{arity} f_1+1}$.

Let g, f_1, f_2 be homogeneous functions into $\mathbb{N}$ and from tuples on $\mathbb{N}$ and let i be a natural number. We say that g is primitive recursively expressed by f_1, f_2 and i if and only if the condition (Def. 12) is satisfied.

(Def. 12) There exists a natural number n such that

(i) $\operatorname{dom} g \subseteq \mathbb{N}^n$,

(ii) $i \geqslant 1$,

(iii) $i \leqslant n$,

(iv) $\operatorname{arity} f_1 + 1 = n$,

(v) $n + 1 = \operatorname{arity} f_2$, and

(vi) for every finite sequence p of elements of $\mathbb{N}$ such that $\operatorname{len} p = n$ holds $p +\cdot (i, 0) \in \operatorname{dom} g$ iff $p_{\restriction i} \in \operatorname{dom} f_1$ and if $p +\cdot (i, 0) \in \operatorname{dom} g$, then $g(p +\cdot (i, 0)) = f_1(p_{\restriction i})$ and for every natural number n holds $p +\cdot (i, n+1) \in \operatorname{dom} g$ iff $p +\cdot (i, n) \in \operatorname{dom} g$ and $(p +\cdot (i, n)) \frown \langle g(p +\cdot (i, n))\rangle \in \operatorname{dom} f_2$ and if $p +\cdot (i, n+1) \in \operatorname{dom} g$, then $g(p +\cdot (i, n+1)) = f_2((p +\cdot (i, n)) \frown \langle g(p +\cdot (i, n))\rangle)$.

Let f_1, f_2 be homogeneous functions into $\mathbb{N}$ and from tuples on $\mathbb{N}$, let i be a natural number, and let p be a finite sequence of elements of $\mathbb{N}$. The functor $\mathrm{primrec}(f_1, f_2, i, p)$ yielding an element of HFuncs $\mathbb{N}$ is defined by the condition (Def. 13).

(Def. 13) There exists a function F from $\mathbb{N}$ into HFuncs $\mathbb{N}$ such that

(i) $\mathrm{primrec}(f_1, f_2, i, p) = F(p_i)$,

(ii) if $i \in \mathrm{dom}\, p$ and $p_{\restriction i} \in \mathrm{dom}\, f_1$, then $F(0) = \{p +\cdot (i, 0)\} \longmapsto f_1(p_{\restriction i})$,

(iii) if $i \notin \mathrm{dom}\, p$ or $p_{\restriction i} \notin \mathrm{dom}\, f_1$, then $F(0) = \emptyset$, and

(iv) for every natural number m holds if $i \in \mathrm{dom}\, p$ and $p +\cdot (i, m) \in \mathrm{dom}\, F(m)$ and $(p +\cdot (i, m)) ^\frown \langle F(m)(p +\cdot (i, m))\rangle \in \mathrm{dom}\, f_2$, then $F(m+1) = F(m) +\cdot (\{p +\cdot (i, m+1)\} \longmapsto f_2((p +\cdot (i, m)) ^\frown \langle F(m)(p +\cdot (i, m))\rangle))$ and if $i \notin \mathrm{dom}\, p$ or $p +\cdot (i, m) \notin \mathrm{dom}\, F(m)$ or $(p +\cdot (i, m)) ^\frown \langle F(m)(p +\cdot (i, m))\rangle \notin \mathrm{dom}\, f_2$, then $F(m + 1) = F(m)$.

We now state several propositions:

(53) For all finite sequences p, q of elements of $\mathbb{N}$ such that $q \in \mathrm{dom}\,\mathrm{primrec}(e_1, e_2, i, p)$ there exists k such that $q = p +\cdot (i, k)$.

(54) For every finite sequence p of elements of $\mathbb{N}$ such that $i \notin \mathrm{dom}\, p$ holds $\mathrm{primrec}(e_1, e_2, i, p) = \emptyset$.

(55) For all finite sequences p, q of elements of $\mathbb{N}$ holds $\mathrm{primrec}(e_1, e_2, i, p) \approx \mathrm{primrec}(e_1, e_2, i, q)$.

(56) For every finite sequence p of elements of $\mathbb{N}$ holds $\mathrm{dom}\,\mathrm{primrec}(e_1, e_2, i, p) \subseteq \mathbb{N}^{1+\mathrm{arity}\, e_1}$.

(57) For every finite sequence p of elements of $\mathbb{N}$ such that e_1 is empty holds $\mathrm{primrec}(e_1, e_2, i, p)$ is empty.

(58) If f_1 is length total and f_2 is length total and $\mathrm{arity}\, f_1 + 2 = \mathrm{arity}\, f_2$ and $1 \leqslant i$ and $i \leqslant 1 + \mathrm{arity}\, f_1$, then $p \in \mathrm{dom}\,\mathrm{primrec}(f_1, f_2, i, p)$.

Let f_1, f_2 be homogeneous functions into $\mathbb{N}$ and from tuples on $\mathbb{N}$ and let i be a natural number. The functor $\mathrm{primrec}(f_1, f_2, i)$ yielding an element of HFuncs $\mathbb{N}$ is defined as follows:

(Def. 14) There exists a function G from $\mathbb{N}^{\mathrm{arity}\, f_1+1}$ into HFuncs $\mathbb{N}$ such that $\mathrm{primrec}(f_1, f_2, i) = \bigcup G$ and for every element p of $\mathbb{N}^{\mathrm{arity}\, f_1+1}$ holds $G(p) = \mathrm{primrec}(f_1, f_2, i, p)$.

One can prove the following propositions:

(59) If e_1 is empty, then $\mathrm{primrec}(e_1, e_2, i)$ is empty.

(60) $\mathrm{dom}\,\mathrm{primrec}(f_1, f_2, i) \subseteq \mathbb{N}^{\mathrm{arity}\, f_1+1}$.

(61) If f_1 is length total and f_2 is length total and $\mathrm{arity}\, f_1 + 2 = \mathrm{arity}\, f_2$ and $1 \leqslant i$ and $i \leqslant 1 + \mathrm{arity}\, f_1$, then $\mathrm{dom}\,\mathrm{primrec}(f_1, f_2, i) = \mathbb{N}^{\mathrm{arity}\, f_1+1}$ and $\mathrm{arity}\,\mathrm{primrec}(f_1, f_2, i) = \mathrm{arity}\, f_1 + 1$.

(62) If $i \in \mathrm{dom}\, p$, then $p +\cdot (i, 0) \in \mathrm{dom}\,\mathrm{primrec}(f_1, f_2, i)$ iff $p_{\restriction i} \in \mathrm{dom}\, f_1$.

(63) If $i \in \operatorname{dom} p$ and $p +\cdot (i, 0) \in \operatorname{dom} \operatorname{primrec}(f_1, f_2, i)$, then $(\operatorname{primrec}(f_1, f_2, i))(p +\cdot (i, 0)) = f_1(p_{\restriction i})$.

(64) If $i \in \operatorname{dom} p$ and f_1 is length total, then $(\operatorname{primrec}(f_1, f_2, i))(p +\cdot (i, 0)) = f_1(p_{\restriction i})$.

(65) If $i \in \operatorname{dom} p$, then $p +\cdot (i, m+1) \in \operatorname{dom} \operatorname{primrec}(f_1, f_2, i)$ iff $p +\cdot (i, m) \in \operatorname{dom} \operatorname{primrec}(f_1, f_2, i)$ and $(p +\cdot (i, m)) \frown \langle (\operatorname{primrec}(f_1, f_2, i))(p +\cdot (i, m)) \rangle \in \operatorname{dom} f_2$.

(66) If $i \in \operatorname{dom} p$ and $p +\cdot (i, m + 1) \in \operatorname{dom} \operatorname{primrec}(f_1, f_2, i)$, then $(\operatorname{primrec}(f_1, f_2, i))(p +\cdot (i, m + 1)) = f_2((p +\cdot (i, m)) \frown \langle (\operatorname{primrec}(f_1, f_2, i))(p +\cdot (i, m)) \rangle)$.

(67) Suppose f_1 is length total and f_2 is length total and $\operatorname{arity} f_1 + 2 = \operatorname{arity} f_2$ and $1 \leqslant i$ and $i \leqslant 1 + \operatorname{arity} f_1$. Then $(\operatorname{primrec}(f_1, f_2, i))(p +\cdot (i, m+1)) = f_2((p +\cdot (i, m)) \frown \langle (\operatorname{primrec}(f_1, f_2, i))(p +\cdot (i, m)) \rangle)$.

(68) If $\operatorname{arity} f_1 + 2 = \operatorname{arity} f_2$ and $1 \leqslant i$ and $i \leqslant \operatorname{arity} f_1 + 1$, then $\operatorname{primrec}(f_1, f_2, i)$ is primitive recursively expressed by f_1, f_2 and i.

(69) Suppose $1 \leqslant i$ and $i \leqslant \operatorname{arity} f_1 + 1$. Let g be an element of HFuncs $\mathbb{N}$. If g is primitive recursively expressed by f_1, f_2 and i, then $g = \operatorname{primrec}(f_1, f_2, i)$.

5. The Set of Primitive Recursive Functions

Let X be a set. We say that X is composition closed if and only if the condition (Def. 15) is satisfied.

(Def. 15) Let f be an element of HFuncs $\mathbb{N}$ and F be a finite sequence of elements of HFuncs $\mathbb{N}$ with the same arity. If $f \in X$ and $\operatorname{arity} f = \operatorname{len} F$ and $\operatorname{rng} F \subseteq X$, then $f \cdot \prod^* F \in X$.

We say that X is primitive recursion closed if and only if the condition (Def. 16) is satisfied.

(Def. 16) Let g, f_1, f_2 be elements of HFuncs $\mathbb{N}$ and i be a natural number. Suppose g is primitive recursively expressed by f_1, f_2 and i and $f_1 \in X$ and $f_2 \in X$. Then $g \in X$.

Let X be a set. We say that X is primitive recursively closed if and only if the conditions (Def. 17) are satisfied.

(Def. 17)(i) $\operatorname{const}_0(0) \in X$,

(ii) $\operatorname{succ}_1(1) \in X$,

(iii) for all natural numbers n, i such that $1 \leqslant i$ and $i \leqslant n$ holds $\operatorname{proj}_n(i) \in X$, and

(iv) X is composition closed and primitive recursion closed.

We now state the proposition

(70) HFuncs $\mathbb{N}$ is primitive recursively closed.

One can check that there exists a subset of HFuncs $\mathbb{N}$ which is primitive recursively closed and non empty.

In the sequel P is a primitive recursively closed non empty subset of HFuncs $\mathbb{N}$.

We now state several propositions:

(71) For every element g of HFuncs $\mathbb{N}$ such that $e_1 = \emptyset$ and g is primitive recursively expressed by e_1, e_2 and i holds $g = \emptyset$.

(72) Let g be an element of HFuncs $\mathbb{N}$, f_1, f_2 be quasi total elements of HFuncs $\mathbb{N}$, and i be a natural number. Suppose g is primitive recursively expressed by f_1, f_2 and i. Then g is quasi total and if f_1 is non empty, then g is non empty.

(73) $\mathrm{const}_n(c) \in P$.

(74) If $1 \leqslant i$ and $i \leqslant n$, then $\mathrm{succ}_n(i) \in P$.

(75) $\emptyset \in P$.

(76) Let f be an element of P and F be a finite sequence of elements of P with the same arity. If arity $f = \mathrm{len}\, F$, then $f \cdot \prod^* F \in P$.

(77) Let f_1, f_2 be elements of P. Suppose arity $f_1 + 2 =$ arity f_2. Let i be a natural number. If $1 \leqslant i$ and $i \leqslant$ arity $f_1 + 1$, then $\mathrm{primrec}(f_1, f_2, i) \in P$.

The subset PrimRec of HFuncs $\mathbb{N}$ is defined as follows:

(Def. 18) PrimRec $= \bigcap\{R; R$ ranges over elements of $2^{\mathrm{HFuncs}\,\mathbb{N}}$: R is primitive recursively closed$\}$.

The following proposition is true

(78) For every subset X of HFuncs $\mathbb{N}$ such that X is primitive recursively closed holds PrimRec $\subseteq X$.

Let us observe that PrimRec is non empty and primitive recursively closed.

One can check that every element of PrimRec is homogeneous.

Let x be a set. We say that x is primitive recursive if and only if:

(Def. 19) $x \in$ PrimRec.

Let us note that every set which is primitive recursive is also relation-like and function-like.

Let us observe that every binary relation which is primitive recursive is also homogeneous, into $\mathbb{N}$, and from tuples on $\mathbb{N}$.

Let us observe that every element of PrimRec is primitive recursive.

Let us note that there exists a function which is primitive recursive and there exists an element of HFuncs $\mathbb{N}$ which is primitive recursive.

The initial functions constitute a subset of HFuncs $\mathbb{N}$ defined as follows:

(Def. 20) The initial functions $= \{\mathrm{const}_0(0), \mathrm{succ}_1(1)\} \cup \{\mathrm{proj}_n(i); n$ ranges over natural numbers, i ranges over natural numbers: $1 \leqslant i \ \wedge\ i \leqslant n\}$.

Let Q be a subset of HFuncs $\mathbb{N}$. The primitive recursion closure of Q is a subset of HFuncs $\mathbb{N}$ and is defined by the condition (Def. 21).

(Def. 21) The primitive recursion closure of $Q = Q \cup \{g; g$ ranges over elements of HFuncs $\mathbb{N}$: $\bigvee_{f_1, f_2 : \text{element of HFuncs}\,\mathbb{N}} \bigvee_{i : \text{natural number}} (f_1 \in Q \ \wedge\ f_2 \in Q \ \wedge\ g$ is primitive recursively expressed by f_1, f_2 and $i)\}$.

The composition closure of Q is a subset of HFuncs $\mathbb{N}$ and is defined by the condition (Def. 22).

(Def. 22) The composition closure of $Q = Q \cup \{f \cdot \prod^* F; f$ ranges over elements of HFuncs $\mathbb{N}, F$ ranges over elements of $(\text{HFuncs}\,\mathbb{N})^*$ with the same arity: $f \in Q \ \wedge\ \text{arity}\, f = \text{len}\, F \ \wedge\ \text{rng}\, F \subseteq Q\}$.

The function $\text{PrimRec}^{\approx}$ from $\mathbb{N}$ into $2^{\text{HFuncs}\,\mathbb{N}}$ is defined by the conditions (Def. 23).

(Def. 23)(i) $\text{PrimRec}^{\approx}(0) =$ the initial functions, and

(ii) for every natural number m holds $\text{PrimRec}^{\approx}(m+1) =$ (the primitive recursion closure of $\text{PrimRec}^{\approx}(m)) \cup$ (the composition closure of $\text{PrimRec}^{\approx}(m))$.

One can prove the following propositions:

(79) If $m \leqslant n$, then $\text{PrimRec}^{\approx}(m) \subseteq \text{PrimRec}^{\approx}(n)$.

(80) $\bigcup(\text{PrimRec}^{\approx})$ is primitive recursively closed.

(81) $\text{PrimRec} = \bigcup(\text{PrimRec}^{\approx})$.

(82) For every element f of HFuncs $\mathbb{N}$ such that $f \in \text{PrimRec}^{\approx}(m)$ holds f is quasi total.

Let us note that every element of PrimRec is quasi total and homogeneous.

Let us observe that every element of HFuncs $\mathbb{N}$ which is primitive recursive is also quasi total.

Let us observe that every function from tuples on $\mathbb{N}$ which is primitive recursive is also length total and there exists an element of PrimRec which is non empty.

6. Examples

Let f be a homogeneous binary relation. We say that f is nullary if and only if:

(Def. 24) $\text{arity}\, f = 0$.

We say that f is unary if and only if:

(Def. 25) $\text{arity}\, f = 1$.

We say that f is binary if and only if:

(Def. 26) $\text{arity}\, f = 2$.

We say that f is ternary if and only if:

(Def. 27) arity $f = 3$.

One can check the following observations:

* every homogeneous function which is unary is also non empty,
* every homogeneous function which is binary is also non empty, and
* every homogeneous function which is ternary is also non empty.

One can check the following observations:

* $\mathrm{proj}_1(1)$ is primitive recursive,
* $\mathrm{proj}_2(1)$ is primitive recursive,
* $\mathrm{proj}_2(2)$ is primitive recursive,
* $\mathrm{succ}_1(1)$ is primitive recursive, and
* $\mathrm{succ}_3(3)$ is primitive recursive.

Let i be a natural number. One can check the following observations:

* $\mathrm{const}_0(i)$ is nullary,
* $\mathrm{const}_1(i)$ is unary,
* $\mathrm{const}_2(i)$ is binary,
* $\mathrm{const}_3(i)$ is ternary,
* $\mathrm{proj}_1(i)$ is unary,
* $\mathrm{proj}_2(i)$ is binary,
* $\mathrm{proj}_3(i)$ is ternary,
* $\mathrm{succ}_1(i)$ is unary,
* $\mathrm{succ}_2(i)$ is binary, and
* $\mathrm{succ}_3(i)$ is ternary.

Let j be a natural number. One can check that $\mathrm{const}_i(j)$ is primitive recursive.

One can verify the following observations:

* there exists a homogeneous function which is nullary, primitive recursive, and non empty,
* there exists a homogeneous function which is unary and primitive recursive,
* there exists a homogeneous function which is binary and primitive recursive, and
* there exists a homogeneous function which is ternary and primitive recursive.

One can verify the following observations:

* there exists a homogeneous function from tuples on $\mathbb{N}$ which is non empty, nullary, length total, and into $\mathbb{N}$,

* there exists a homogeneous function from tuples on $\mathbb{N}$ which is non empty, unary, length total, and into $\mathbb{N}$,
* there exists a homogeneous function from tuples on $\mathbb{N}$ which is non empty, binary, length total, and into $\mathbb{N}$, and
* there exists a homogeneous function from tuples on $\mathbb{N}$ which is non empty, ternary, length total, and into $\mathbb{N}$.

Let f be a nullary non empty primitive recursive function and let g be a binary primitive recursive function. One can check that $\mathrm{primrec}(f, g, 1)$ is primitive recursive and unary.

Let f be a unary primitive recursive function and let g be a ternary primitive recursive function. One can verify that $\mathrm{primrec}(f, g, 1)$ is primitive recursive and binary and $\mathrm{primrec}(f, g, 2)$ is primitive recursive and binary.

The following propositions are true:

(83) Let f_1 be a unary length total homogeneous function from tuples on $\mathbb{N}$ into $\mathbb{N}$ and f_2 be a non empty homogeneous function into $\mathbb{N}$ and from tuples on $\mathbb{N}$. Then $(\mathrm{primrec}(f_1, f_2, 2))(\langle i, 0\rangle) = f_1(\langle i\rangle)$.

(84) If f_1 is length total and $\mathrm{arity}\, f_1 = 0$, then $(\mathrm{primrec}(f_1, f_2, 1))(\langle 0\rangle) = f_1(\emptyset)$.

(85) Let f_1 be a unary length total homogeneous function from tuples on $\mathbb{N}$ into $\mathbb{N}$ and f_2 be a ternary length total homogeneous function from tuples on $\mathbb{N}$ into $\mathbb{N}$. Then $(\mathrm{primrec}(f_1, f_2, 2))(\langle i, j+1\rangle) = f_2(\langle i, j, (\mathrm{primrec}(f_1, f_2, 2))(\langle i, j\rangle)\rangle)$.

(86) If f_1 is length total and f_2 is length total and $\mathrm{arity}\, f_1 = 0$ and $\mathrm{arity}\, f_2 = 2$, then $(\mathrm{primrec}(f_1, f_2, 1))(\langle i+1\rangle) = f_2(\langle i, (\mathrm{primrec}(f_1, f_2, 1))(\langle i\rangle)\rangle)$.

Let g be a function. The functor ${}^{\langle 1,?,2\rangle}g$ yielding a function is defined by:

(Def. 28) ${}^{\langle 1,?,2\rangle}g = g \cdot \prod^{*}\langle \mathrm{proj}_3(1), \mathrm{proj}_3(3)\rangle$.

Let g be a function into $\mathbb{N}$ and from tuples on $\mathbb{N}$. Observe that ${}^{\langle 1,?,2\rangle}g$ is into $\mathbb{N}$ and from tuples on $\mathbb{N}$.

Let g be a homogeneous function. Note that ${}^{\langle 1,?,2\rangle}g$ is homogeneous.

Let g be a binary length total homogeneous function from tuples on $\mathbb{N}$ into $\mathbb{N}$. Observe that ${}^{\langle 1,?,2\rangle}g$ is non empty ternary and length total.

The following propositions are true:

(87) Let f be a binary length total homogeneous function from tuples on $\mathbb{N}$ into $\mathbb{N}$. Then $({}^{\langle 1,?,2\rangle}f)(\langle i, j, k\rangle) = f(\langle i, k\rangle)$.

(88) For every binary primitive recursive function g holds ${}^{\langle 1,?,2\rangle}g \in \mathrm{PrimRec}$.

Let f be a binary primitive recursive homogeneous function. Observe that ${}^{\langle 1,?,2\rangle}f$ is primitive recursive and ternary.

The binary primitive recursive function $[+]$ is defined by:

(Def. 29) $[+] = \mathrm{primrec}(\mathrm{proj}_1(1), \mathrm{succ}_3(3), 2)$.

We now state the proposition

(89) $[+](\langle i, j\rangle) = i + j$.

The binary primitive recursive function $[*]$ is defined by:

(Def. 30) $[*] = \mathrm{primrec}(\mathrm{const}_1(0), {}^{\langle 1,?,2\rangle}[+], 2)$.

Next we state the proposition

(90) For all natural numbers i, j holds $[*](\langle i, j\rangle) = i \cdot j$.

Let g, h be binary primitive recursive homogeneous functions. Note that $\langle g, h\rangle$ has the same arity.

Let f, g, h be binary primitive recursive functions. Observe that $f \cdot \prod^* \langle g, h\rangle$ is primitive recursive.

Let f, g, h be binary primitive recursive functions. Observe that $f \cdot \prod^* \langle g, h\rangle$ is binary.

Let f be a unary primitive recursive function and let g be a primitive recursive function. Note that $f \cdot \prod^* \langle g\rangle$ is primitive recursive.

Let f be a unary primitive recursive function and let g be a binary primitive recursive function. One can verify that $f \cdot \prod^* \langle g\rangle$ is binary.

The unary primitive recursive function $[!]$ is defined by:

(Def. 31) $[!] = \mathrm{primrec}(\mathrm{const}_0(1), [*] \cdot \prod^* \langle \mathrm{succ}_1(1) \cdot \prod^* \langle \mathrm{proj}_2(1)\rangle, \mathrm{proj}_2(2)\rangle, 1)$.

In this article we present several logical schemes. The scheme *Primrec1* deals with a unary length total homogeneous function $\mathcal{A}$ from tuples on $\mathbb{N}$ into $\mathbb{N}$, a binary length total homogeneous function $\mathcal{B}$ from tuples on $\mathbb{N}$ into $\mathbb{N}$, a unary functor $\mathcal{F}$ yielding a natural number, and a binary functor $\mathcal{G}$ yielding a natural number, and states that:

For all natural numbers i, j holds $(\mathcal{A} \cdot \prod^* \langle \mathcal{B}\rangle)(\langle i, j\rangle) = \mathcal{F}(\mathcal{G}(i, j))$

provided the parameters meet the following requirements:

- For every natural number i holds $\mathcal{A}(\langle i\rangle) = \mathcal{F}(i)$, and
- For all natural numbers i, j holds $\mathcal{B}(\langle i, j\rangle) = \mathcal{G}(i, j)$.

The scheme *Primrec2* deals with binary length total homogeneous functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ from tuples on $\mathbb{N}$ into $\mathbb{N}$ and three binary functors $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ yielding natural numbers, and states that:

For all natural numbers i, j holds $(\mathcal{A} \cdot \prod^* \langle \mathcal{B}, \mathcal{C}\rangle)(\langle i, j\rangle) = \mathcal{F}(\mathcal{G}(i, j), \mathcal{H}(i, j))$

provided the parameters meet the following conditions:

- For all natural numbers i, j holds $\mathcal{A}(\langle i, j\rangle) = \mathcal{F}(i, j)$,
- For all natural numbers i, j holds $\mathcal{B}(\langle i, j\rangle) = \mathcal{G}(i, j)$, and
- For all natural numbers i, j holds $\mathcal{C}(\langle i, j\rangle) = \mathcal{H}(i, j)$.

The following proposition is true

(91) $[!](\langle i\rangle) = i!$.

The binary primitive recursive function $[\wedge]$ is defined by:

(Def. 32) $[\wedge] = \mathrm{primrec}(\mathrm{const}_1(1), {}^{\langle 1,?,2\rangle}[*], 2)$.

One can prove the following proposition

(92) $[\hat{\ }](\langle i, j\rangle) = i^j$.

The unary primitive recursive function [pred] is defined as follows:

(Def. 33) $[\mathrm{pred}] = \mathrm{primrec}(\mathrm{const}_0(0), \mathrm{proj}_2(1), 1)$.

The following proposition is true

(93) $[\mathrm{pred}](\langle 0\rangle) = 0$ and $[\mathrm{pred}](\langle i+1\rangle) = i$.

The binary primitive recursive function $[-]$ is defined as follows:

(Def. 34) $[-] = \mathrm{primrec}(\mathrm{proj}_1(1), {}^{\langle 1,?,2\rangle}([\mathrm{pred}] \cdot \prod^* \langle \mathrm{proj}_2(2)\rangle), 2)$.

The following proposition is true

(94) $[-](\langle i, j\rangle) = i -' j$.

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Introduction to Turing Machines

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Summary. A Turing machine can be viewed as a simple kind of computer, whose operations are constrainted to reading and writing symbols on a tape, or moving along the tape to the left or right. In theory, one has proven that the computability of Turing machines is equivalent to recursive functions. This article defines and verifies the Turing machines of summation and three primitive functions which are successor, zero and project functions. It is difficult to compute sophisticated functions by simple Turing machines. Therefore, we define the combination of two Turing machines.

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The notation and terminology used in this paper are introduced in the following articles: [3], [4], [13], [2], [5], [18], [14], [6], [7], [8], [12], [17], [16], [1], [11], [20], [10], [19], [15], and [9].

1. Preliminaries

In this paper n, i, j, k denote natural numbers.

Let A, B be non empty sets, let f be a function from A into B, and let g be a partial function from A to B. Then $f {+}\!\cdot g$ is a function from A into B.

Let X, Y be non empty sets, let a be an element of X, and let b be an element of Y. Then $a \mapsto\!\!\!\!\dot{\ }\!\!\longrightarrow b$ is a partial function from X to Y.

Let n be a natural number. The functor $\mathrm{Seg}_M\, n$ yielding a subset of $\mathbb{N}$ is defined as follows:

(Def. 1) $\mathrm{Seg}_M\, n = \{k : k \leqslant n\}$.

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Let n be a natural number. One can verify that $\mathrm{Seg}_M\, n$ is finite and non empty.

One can prove the following propositions:

(1) $k \in \mathrm{Seg}_M\, n$ iff $k \leqslant n$.

(2) For every function f and for all sets x, y, z, u, v such that $u \neq x$ holds $(f{+}\!\cdot(\langle x,\, y\rangle \longmapsto z))(\langle u,\, v\rangle) = f(\langle u,\, v\rangle)$.

(3) For every function f and for all sets x, y, z, u, v such that $v \neq y$ holds $(f{+}\!\cdot(\langle x,\, y\rangle \longmapsto z))(\langle u,\, v\rangle) = f(\langle u,\, v\rangle)$.

In the sequel i_1, i_2, i_3, i_4 denote elements of $\mathbb{Z}$.

We now state three propositions:

(4) $\sum\langle i_1, i_2\rangle = i_1 + i_2$.

(5) $\sum\langle i_1, i_2, i_3\rangle = i_1 + i_2 + i_3$.

(6) $\sum\langle i_1, i_2, i_3, i_4\rangle = i_1 + i_2 + i_3 + i_4$.

Let f be a finite sequence of elements of $\mathbb{N}$ and let i be a natural number. The functor $\mathrm{Prefix}(f, i)$ yields a finite sequence of elements of $\mathbb{Z}$ and is defined by:

(Def. 2) $\mathrm{Prefix}(f, i) = f \restriction \mathrm{Seg}\, i$.

Next we state two propositions:

(7) For all natural numbers x_1, x_2 holds $\sum \mathrm{Prefix}(\langle x_1, x_2\rangle, 1) = x_1$ and $\sum \mathrm{Prefix}(\langle x_1, x_2\rangle, 2) = x_1 + x_2$.

(8) For all natural numbers x_1, x_2, x_3 holds $\sum \mathrm{Prefix}(\langle x_1, x_2, x_3\rangle, 1) = x_1$ and $\sum \mathrm{Prefix}(\langle x_1, x_2, x_3\rangle, 2) = x_1 + x_2$ and $\sum \mathrm{Prefix}(\langle x_1, x_2, x_3\rangle, 3) = x_1 + x_2 + x_3$.

2. Definitions and Terminology for Turing Machine

We consider Turing machine structures as systems

$\langle$ symbols, control states, a transition, an initial state, an accepting state $\rangle$,

where the symbols and the control states constitute finite non empty sets, the transition is a function from $[\!:$ the control states, the symbols $:\!]$ into $[\!:$ the control states, the symbols, $\{-1, 0, 1\}$ $:\!]$, and the initial state and the accepting state are elements of the control states.

Let T be a Turing machine structure. A state of T is an element of the control states of T. A tape of T is an element of (the symbols of $T)^{\mathbb{Z}}$. A symbol of T is an element of the symbols of T.

Let T be a Turing machine structure, let t be a tape of T, let h be an integer, and let s be a symbol of T. The functor $\text{Tape-Chg}(t, h, s)$ yields a tape of T and is defined as follows:

(Def. 3) $\text{Tape-Chg}(t, h, s) = t +\cdot (h \dot{\longmapsto} s)$.

Let T be a Turing machine structure. A State of T is an element of $[\!:$ the control states of T, $\mathbb{Z}$, (the symbols of $T)^{\mathbb{Z}}$ $:\!]$. A transition-source of T is an element of $[\!:$ the control states of T, the symbols of T $:\!]$. A transition-target of T is an element of $[\!:$ the control states of T, the symbols of T, $\{-1, 0, 1\}$ $:\!]$.

Let T be a Turing machine structure and let g be a transition-target of T. The functor offset(g) yields an integer and is defined as follows:

(Def. 4) $\text{offset}(g) = g_{\mathbf{3}}$.

Let T be a Turing machine structure and let s be a State of T. The functor Head(s) yielding an integer is defined by:

(Def. 5) $\text{Head}(s) = s_{\mathbf{2}}$.

Let T be a Turing machine structure and let s be a State of T. The functor s-target yielding a transition-target of T is defined by:

(Def. 6) $s\text{-target} = (\text{the transition of } T)(\langle s_{\mathbf{1}}, (s_{\mathbf{3}} \textbf{ qua } \text{tape of } T)(\text{Head}(s))\rangle)$.

Let T be a Turing machine structure and let s be a State of T. The functor Following(s) yields a State of T and is defined as follows:

(Def. 7) $$\text{Following}(s) = \begin{cases} \langle s\text{-target}_{\mathbf{1}}, \text{Head}(s) + \text{offset}(s\text{-target}), \\ \quad \text{Tape-Chg}(s_{\mathbf{3}}, \text{Head}(s), s\text{-target}_{\mathbf{2}})\rangle, \\ \quad \text{if } s_{\mathbf{1}} \neq \text{the accepting state of } T, \\ s, \text{ otherwise.} \end{cases}$$

Let T be a Turing machine structure and let s be a State of T. The functor Computation(s) yielding a function from $\mathbb{N}$ into $[\!:$ the control states of T, $\mathbb{Z}$, (the symbols of $T)^{\mathbb{Z}}$ $:\!]$ is defined as follows:

(Def. 8) $(\text{Computation}(s))(0) = s$ and for every i holds $(\text{Computation}(s))(i{+}1) = \text{Following}((\text{Computation}(s))(i))$.

In the sequel T is a Turing machine structure and s is a State of T.

The following propositions are true:

(9) Let T be a Turing machine structure and s be a State of T. If $s_{\mathbf{1}} =$ the accepting state of T, then $s = \text{Following}(s)$.

(10) $(\text{Computation}(s))(0) = s$.

(11) $(\text{Computation}(s))(k + 1) = \text{Following}((\text{Computation}(s))(k))$.

(12) $(\text{Computation}(s))(1) = \text{Following}(s)$.

(13) $(\text{Computation}(s))(i + k) = (\text{Computation}((\text{Computation}(s))(i)))(k)$.

(14) If $i \leqslant j$ and $\text{Following}((\text{Computation}(s))(i)) = (\text{Computation}(s))(i)$, then $(\text{Computation}(s))(j) = (\text{Computation}(s))(i)$.

(15) If $i \leqslant j$ and $(\text{Computation}(s))(i)_{\mathbf{1}} =$ the accepting state of T, then $(\text{Computation}(s))(j) = (\text{Computation}(s))(i)$.

Let T be a Turing machine structure and let s be a State of T. We say that s is accepting if and only if:

(Def. 9) There exists k such that $(\mathrm{Computation}(s))(k)_{\mathbf{1}} =$ the accepting state of T.

Let T be a Turing machine structure and let s be a State of T. Let us assume that s is accepting. The functor $\mathrm{Result}(s)$ yielding a State of T is defined by:

(Def. 10) There exists k such that $\mathrm{Result}(s) = (\mathrm{Computation}(s))(k)$ and $(\mathrm{Computation}(s))(k)_{\mathbf{1}} =$ the accepting state of T.

We now state the proposition

(16) Let T be a Turing machine structure and s be a State of T. Suppose s is accepting. Then there exists a natural number k such that

(i) $(\mathrm{Computation}(s))(k)_{\mathbf{1}} =$ the accepting state of T,

(ii) $\mathrm{Result}(s) = (\mathrm{Computation}(s))(k)$, and

(iii) for every natural number i such that $i < k$ holds $(\mathrm{Computation}(s))(i)_{\mathbf{1}} \neq$ the accepting state of T.

Let A, B be non empty sets and let y be a set. Let us assume that $y \in B$. The functor $\mathrm{id}(A, B, y)$ yields a function from A into $[\!: A, B :\!]$ and is defined as follows:

(Def. 11) For every element x of A holds $(\mathrm{id}(A, B, y))(x) = \langle x, y \rangle$.

The function SumTran from $[\!: \mathrm{Seg}_M 5, \{0,1\} :\!]$ into $[\!: \mathrm{Seg}_M 5, \{0,1\}, \{-1,0,1\} :\!]$ is defined as follows:

(Def. 12) $\mathrm{SumTran} = \mathrm{id}([\!: \mathrm{Seg}_M 5, \{0,1\} :\!], \{-1,0,1\}, 0) +\cdot (\langle 0, 0\rangle \dot{\longmapsto} \langle 0, 0, 1\rangle) +\cdot (\langle 0, 1\rangle \dot{\longmapsto} \langle 1, 0, 1\rangle) +\cdot (\langle 1, 1\rangle \dot{\longmapsto} \langle 1, 1, 1\rangle) +\cdot (\langle 1, 0\rangle \dot{\longmapsto} \langle 2, 1, 1\rangle) +\cdot (\langle 2, 1\rangle \dot{\longmapsto} \langle 2, 1, 1\rangle) +\cdot (\langle 2, 0\rangle \dot{\longmapsto} \langle 3, 0, -1\rangle) +\cdot (\langle 3, 1\rangle \dot{\longmapsto} \langle 4, 0, -1\rangle) +\cdot (\langle 4, 1\rangle \dot{\longmapsto} \langle 4, 1, -1\rangle) +\cdot (\langle 4, 0\rangle \dot{\longmapsto} \langle 5, 0, 0\rangle)$.

Next we state the proposition

(17) $\mathrm{SumTran}(\langle 0, 0\rangle) = \langle 0, 0, 1\rangle$ and $\mathrm{SumTran}(\langle 0, 1\rangle) = \langle 1, 0, 1\rangle$ and $\mathrm{SumTran}(\langle 1, 1\rangle) = \langle 1, 1, 1\rangle$ and $\mathrm{SumTran}(\langle 1, 0\rangle) = \langle 2, 1, 1\rangle$ and $\mathrm{SumTran}(\langle 2, 1\rangle) = \langle 2, 1, 1\rangle$ and $\mathrm{SumTran}(\langle 2, 0\rangle) = \langle 3, 0, -1\rangle$ and $\mathrm{SumTran}(\langle 3, 1\rangle) = \langle 4, 0, -1\rangle$ and $\mathrm{SumTran}(\langle 4, 1\rangle) = \langle 4, 1, -1\rangle$ and $\mathrm{SumTran}(\langle 4, 0\rangle) = \langle 5, 0, 0\rangle$.

Let T be a Turing machine structure, let t be a tape of T, and let i, j be integers. We say that t is 1 between i, j if and only if:

(Def. 13) $t(i) = 0$ and $t(j) = 0$ and for every integer k such that $i < k$ and $k < j$ holds $t(k) = 1$.

Let f be a finite sequence of elements of $\mathbb{N}$, let T be a Turing machine structure, and let t be a tape of T. We say that t stores data f if and only if:

(Def. 14) For every natural number i such that $1 \leqslant i$ and $i < \mathrm{len}\, f$ holds t is 1 between $\sum \mathrm{Prefix}(f, i) + 2 \cdot (i - 1)$, $\sum \mathrm{Prefix}(f, i + 1) + 2 \cdot i$.

We now state several propositions:

(18) Let T be a Turing machine structure, t be a tape of T, and s, n be natural numbers. If t stores data $\langle s, n\rangle$, then t is 1 between s, $s+n+2$.

(19) Let T be a Turing machine structure, t be a tape of T, and s, n be natural numbers. If t is 1 between s, $s+n+2$, then t stores data $\langle s, n\rangle$.

(20) Let T be a Turing machine structure, t be a tape of T, and s, n be natural numbers. Suppose t stores data $\langle s, n\rangle$. Then $t(s) = 0$ and $t(s+n+2) = 0$ and for every integer i such that $s < i$ and $i < s+n+2$ holds $t(i) = 1$.

(21) Let T be a Turing machine structure, t be a tape of T, and s, n_1, n_2 be natural numbers. Suppose t stores data $\langle s, n_1, n_2\rangle$. Then t is 1 between s, $s+n_1+2$ and 1 between $s+n_1+2$, $s+n_1+n_2+4$.

(22) Let T be a Turing machine structure, t be a tape of T, and s, n_1, n_2 be natural numbers. Suppose t stores data $\langle s, n_1, n_2\rangle$. Then

(i) $t(s) = 0$,

(ii) $t(s+n_1+2) = 0$,

(iii) $t(s+n_1+n_2+4) = 0$,

(iv) for every integer i such that $s < i$ and $i < s+n_1+2$ holds $t(i) = 1$, and

(v) for every integer i such that $s+n_1+2 < i$ and $i < s+n_1+n_2+4$ holds $t(i) = 1$.

(23) Let f be a finite sequence of elements of $\mathbb{N}$ and s be a natural number. If $\operatorname{len} f \geqslant 1$, then $\sum \operatorname{Prefix}(\langle s\rangle \frown f, 1) = s$ and $\sum \operatorname{Prefix}(\langle s\rangle \frown f, 2) = s + f_1$.

(24) Let f be a finite sequence of elements of $\mathbb{N}$ and s be a natural number. Suppose $\operatorname{len} f \geqslant 3$. Then $\sum \operatorname{Prefix}(\langle s\rangle \frown f, 1) = s$ and $\sum \operatorname{Prefix}(\langle s\rangle \frown f, 2) = s + f_1$ and $\sum \operatorname{Prefix}(\langle s\rangle \frown f, 3) = s + f_1 + f_2$ and $\sum \operatorname{Prefix}(\langle s\rangle \frown f, 4) = s + f_1 + f_2 + f_3$.

(25) Let T be a Turing machine structure, t be a tape of T, s be a natural number, and f be a finite sequence of elements of $\mathbb{N}$. If $\operatorname{len} f \geqslant 1$ and t stores data $\langle s\rangle \frown f$, then t is 1 between s, $s+f_1+2$.

(26) Let T be a Turing machine structure, t be a tape of T, s be a natural number, and f be a finite sequence of elements of $\mathbb{N}$. Suppose $\operatorname{len} f \geqslant 3$ and t stores data $\langle s\rangle \frown f$. Then t is 1 between s, $s+f_1+2$, 1 between $s+f_1+2$, $s+f_1+f_2+4$, and 1 between $s+f_1+f_2+4$, $s+f_1+f_2+f_3+6$.

3. Summation of Two Natural Numbers

The strict Turing machine structure SumTuring is defined by the conditions (Def. 15).

(Def. 15)(i) The symbols of SumTuring $= \{0, 1\}$,

(ii) the control states of SumTuring $= \operatorname{Seg}_M 5$,

(iii) the transition of SumTuring = SumTran,
(iv) the initial state of SumTuring = 0, and
(v) the accepting state of SumTuring = 5.

Next we state several propositions:

(27) Let T be a Turing machine structure, s be a State of T, and p, h, t be sets. If $s = \langle p, h, t\rangle$, then $\mathrm{Head}(s) = h$.

(28) Let T be a Turing machine structure, t be a tape of T, h be an integer, and s be a symbol of T. If $t(h) = s$, then $\text{Tape-Chg}(t, h, s) = t$.

(29) Let T be a Turing machine structure, s be a State of T, and p, h, t be sets. Suppose $s = \langle p, h, t\rangle$ and $p \neq$ the accepting state of T. Then $\mathrm{Following}(s) = \langle s\text{-target}_{\mathbf{1}}, \mathrm{Head}(s) + \mathrm{offset}(s\text{-target}), \text{Tape-Chg}(s_{\mathbf{3}}, \mathrm{Head}(s), s\text{-target}_{\mathbf{2}})\rangle$.

(30) Let T be a Turing machine structure, t be a tape of T, h be an integer, s be a symbol of T, and i be a set. Then $(\text{Tape-Chg}(t, h, s))(h) = s$ and if $i \neq h$, then $(\text{Tape-Chg}(t, h, s))(i) = t(i)$.

(31) Let s be a State of SumTuring, t be a tape of SumTuring, and h_1, n_1, n_2 be natural numbers. Suppose $s = \langle 0, h_1, t\rangle$ and t stores data $\langle h_1, n_1, n_2\rangle$. Then s is accepting and $(\mathrm{Result}(s))_{\mathbf{2}} = 1 + h_1$ and $(\mathrm{Result}(s))_{\mathbf{3}}$ stores data $\langle 1 + h_1, n_1 + n_2\rangle$.

Let T be a Turing machine structure and let F be a function. We say that T computes F if and only if the condition (Def. 16) is satisfied.

(Def. 16) Let s be a State of T, t be a tape of T, a be a natural number, and x be a finite sequence of elements of $\mathbb{N}$. Suppose $x \in \operatorname{dom} F$ and $s = \langle$the initial state of T, a, $t\rangle$ and t stores data $\langle a\rangle \frown x$. Then s is accepting and there exist natural numbers b, y such that $(\mathrm{Result}(s))_{\mathbf{2}} = b$ and $y = F(x)$ and $(\mathrm{Result}(s))_{\mathbf{3}}$ stores data $\langle b\rangle \frown \langle y\rangle$.

Next we state two propositions:

(32) $\operatorname{dom}[+] \subseteq \mathbb{N}^2$.

(33) SumTuring computes $[+]$.

4. Computing Successor Function

The function SuccTran from $[\!:\mathrm{Seg}_M 4, \{0,1\}:\!]$ into $[\!:\mathrm{Seg}_M 4, \{0,1\}, \{-1,0,1\}:\!]$ is defined as follows:

(Def. 17) $\mathrm{SuccTran} = \mathrm{id}([\!:\mathrm{Seg}_M 4, \{0,1\}:\!], \{-1,0,1\}, 0) {+\!\cdot} (\langle 0, 0\rangle \mapsto \langle 1, 0, 1\rangle) {+\!\cdot} (\langle 1, 1\rangle \mapsto \langle 1, 1, 1\rangle) {+\!\cdot} (\langle 1, 0\rangle \mapsto \langle 2, 1, 1\rangle) {+\!\cdot} (\langle 2, 0\rangle \mapsto \langle 3, 0, -1\rangle) {+\!\cdot} (\langle 2, 1\rangle \mapsto \langle 3, 0, -1\rangle) {+\!\cdot} (\langle 3, 1\rangle \mapsto \langle 3, 1, -1\rangle) {+\!\cdot} (\langle 3, 0\rangle \mapsto \langle 4, 0, 0\rangle)$.

We now state the proposition

(34) SuccTran($\langle 0, 0\rangle$) = $\langle 1, 0, 1\rangle$ and SuccTran($\langle 1, 1\rangle$) = $\langle 1, 1, 1\rangle$ and SuccTran($\langle 1, 0\rangle$) = $\langle 2, 1, 1\rangle$ and SuccTran($\langle 2, 0\rangle$) = $\langle 3, 0, -1\rangle$ and SuccTran($\langle 2, 1\rangle$) = $\langle 3, 0, -1\rangle$ and SuccTran($\langle 3, 1\rangle$) = $\langle 3, 1, -1\rangle$ and SuccTran($\langle 3, 0\rangle$) = $\langle 4, 0, 0\rangle$.

The strict Turing machine structure SuccTuring is defined by the conditions (Def. 18).

(Def. 18)(i) The symbols of SuccTuring $= \{0, 1\}$,
(ii) the control states of SuccTuring $= \mathrm{Seg}_M\, 4$,
(iii) the transition of SuccTuring = SuccTran,
(iv) the initial state of SuccTuring = 0, and
(v) the accepting state of SuccTuring = 4.

The following propositions are true:

(36)[1] Let s be a State of SuccTuring, t be a tape of SuccTuring, and h_1, n be natural numbers. Suppose $s = \langle 0, h_1, t\rangle$ and t stores data $\langle h_1, n\rangle$. Then s is accepting and $(\mathrm{Result}(s))_{\mathbf{2}} = h_1$ and $(\mathrm{Result}(s))_{\mathbf{3}}$ stores data $\langle h_1, n+1\rangle$.

(37) SuccTuring computes $\mathrm{succ}_1(1)$.

5. Computing Zero Function

The function ZeroTran from $[\!:\mathrm{Seg}_M\, 4,\ \{0,1\}:\!]$ into $[\!:\mathrm{Seg}_M\, 4,\ \{0,1\},\ \{-1,0,1\}:\!]$ is defined as follows:

(Def. 19) ZeroTran $= \mathrm{id}([\!:\mathrm{Seg}_M\, 4,\ \{0,1\}:\!], \{-1,0,1\}, 1) +\cdot (\langle 0, 0\rangle \longmapsto \langle 1, 0, 1\rangle) +\cdot (\langle 1, 1\rangle \longmapsto \langle 2, 1, 1\rangle) +\cdot (\langle 2, 0\rangle \longmapsto \langle 3, 0, -1\rangle) +\cdot (\langle 2, 1\rangle \longmapsto \langle 3, 0, -1\rangle) +\cdot (\langle 3, 1\rangle \longmapsto \langle 4, 1, -1\rangle)$.

Next we state the proposition

(38) ZeroTran($\langle 0, 0\rangle$) = $\langle 1, 0, 1\rangle$ and ZeroTran($\langle 1, 1\rangle$) = $\langle 2, 1, 1\rangle$ and ZeroTran($\langle 2, 0\rangle$) = $\langle 3, 0, -1\rangle$ and ZeroTran($\langle 2, 1\rangle$) = $\langle 3, 0, -1\rangle$ and ZeroTran($\langle 3, 1\rangle$) = $\langle 4, 1, -1\rangle$.

The strict Turing machine structure ZeroTuring is defined by the conditions (Def. 20).

(Def. 20)(i) The symbols of ZeroTuring $= \{0, 1\}$,
(ii) the control states of ZeroTuring $= \mathrm{Seg}_M\, 4$,
(iii) the transition of ZeroTuring = ZeroTran,
(iv) the initial state of ZeroTuring = 0, and
(v) the accepting state of ZeroTuring = 4.

We now state two propositions:

[1] The proposition (35) has been removed.

(39) Let s be a State of ZeroTuring, t be a tape of ZeroTuring, h_1 be a natural number, and f be a finite sequence of elements of $\mathbb{N}$. Suppose $\operatorname{len} f \geqslant 1$ and $s = \langle 0, h_1, t\rangle$ and t stores data $\langle h_1\rangle \frown f$. Then s is accepting and $(\mathrm{Result}(s))_{\mathbf{2}} = h_1$ and $(\mathrm{Result}(s))_{\mathbf{3}}$ stores data $\langle h_1, 0\rangle$.

(40) If $n \geqslant 1$, then ZeroTuring computes $\mathrm{const}_n(0)$.

6. Computing n-ary Project Function

The function n-proj3Tran from $[\!:\mathrm{Seg}_M 3, \{0,1\}:\!]$ into $[\!:\mathrm{Seg}_M 3, \{0,1\}, \{-1,0,1\}:\!]$ is defined by:

(Def. 21) $n\text{-proj3Tran} = \mathrm{id}([\!:\mathrm{Seg}_M 3, \{0,1\}:\!], \{-1,0,1\}, 0) {+\cdot} (\langle 0, 0\rangle \longmapsto \langle 1, 0, 1\rangle) {+\cdot} (\langle 1, 1\rangle \longmapsto \langle 1, 0, 1\rangle) {+\cdot} (\langle 1, 0\rangle \longmapsto \langle 2, 0, 1\rangle) {+\cdot} (\langle 2, 1\rangle \longmapsto \langle 2, 0, 1\rangle) {+\cdot} (\langle 2, 0\rangle \longmapsto \langle 3, 0, 0\rangle)$.

The following proposition is true

(41) n-proj3Tran$(\langle 0, 0\rangle) = \langle 1, 0, 1\rangle$ and n-proj3Tran$(\langle 1, 1\rangle) = \langle 1, 0, 1\rangle$ and n-proj3Tran$(\langle 1, 0\rangle) = \langle 2, 0, 1\rangle$ and n-proj3Tran$(\langle 2, 1\rangle) = \langle 2, 0, 1\rangle$ and n-proj3Tran$(\langle 2, 0\rangle) = \langle 3, 0, 0\rangle$.

The strict Turing machine structure n-proj3Turing is defined by the conditions (Def. 22).

(Def. 22)(i) The symbols of n-proj3Turing $= \{0,1\}$,
(ii) the control states of n-proj3Turing $= \mathrm{Seg}_M 3$,
(iii) the transition of n-proj3Turing $=$ n-proj3Tran,
(iv) the initial state of n-proj3Turing $= 0$, and
(v) the accepting state of n-proj3Turing $= 3$.

Next we state two propositions:

(42) Let s be a State of n-proj3Turing, t be a tape of n-proj3Turing, h_1 be a natural number, and f be a finite sequence of elements of $\mathbb{N}$. Suppose $\operatorname{len} f \geqslant 3$ and $s = \langle 0, h_1, t\rangle$ and t stores data $\langle h_1\rangle \frown f$. Then s is accepting and $(\mathrm{Result}(s))_{\mathbf{2}} = h_1 + f_1 + f_2 + 4$ and $(\mathrm{Result}(s))_{\mathbf{3}}$ stores data $\langle h_1 + f_1 + f_2 + 4, f_3\rangle$.

(43) If $n \geqslant 3$, then n-proj3Turing computes $\mathrm{proj}_n(3)$.

7. Combining Two Turing Machines into One

Let t_1, t_2 be Turing machine structures. The functor SeqStates(t_1, t_2) yielding a finite non empty set is defined by the condition (Def. 23).

(Def. 23) $\mathrm{SeqStates}(t_1, t_2) = [\!:$ the control states of t_1, $\{$the initial state of $t_2\}:\!] \cup [\!:\{$the accepting state of $t_1\}$, the control states of $t_2:\!]$.

One can prove the following four propositions:

(44) Let t_1, t_2 be Turing machine structures. Then
   (i) $\langle$the initial state of t_1, the initial state of $t_2\rangle \in \mathrm{SeqStates}(t_1, t_2)$, and
   (ii) $\langle$the accepting state of t_1, the accepting state of $t_2\rangle \in \mathrm{SeqStates}(t_1, t_2)$.

(45) For all Turing machine structures s, t and for every state x of s holds $\langle x$, the initial state of $t\rangle \in \mathrm{SeqStates}(s, t)$.

(46) For all Turing machine structures s, t and for every state x of t holds $\langle$the accepting state of s, $x\rangle \in \mathrm{SeqStates}(s, t)$.

(47) Let s, t be Turing machine structures and x be an element of $\mathrm{SeqStates}(s, t)$. Then there exists a state x_1 of s and there exists a state x_2 of t such that $x = \langle x_1, x_2\rangle$.

Let s, t be Turing machine structures and let x be a transition-target of s. The functor $1^{\mathrm{st}}\mathrm{SeqTran}(s, t, x)$ yielding an element of $[\!:\mathrm{SeqStates}(s, t)$, (the symbols of s) $\cup$ (the symbols of t), $\{-1, 0, 1\}:\!]$ is defined as follows:

(Def. 24) $1^{\mathrm{st}}\mathrm{SeqTran}(s, t, x) = \langle\langle x_{\mathbf{1}}$, the initial state of $t\rangle$, $x_{\mathbf{2}}$, $x_{\mathbf{3}}\rangle$.

Let s, t be Turing machine structures and let x be a transition-target of t. The functor $2^{\mathrm{nd}}\mathrm{SeqTran}(s, t, x)$ yielding an element of $[\!:\mathrm{SeqStates}(s, t)$, (the symbols of s) $\cup$ (the symbols of t), $\{-1, 0, 1\}:\!]$ is defined as follows:

(Def. 25) $2^{\mathrm{nd}}\mathrm{SeqTran}(s, t, x) = \langle\langle$the accepting state of s, $x_{\mathbf{1}}\rangle$, $x_{\mathbf{2}}$, $x_{\mathbf{3}}\rangle$.

Let s, t be Turing machine structures and let x be an element of $\mathrm{SeqStates}(s, t)$. Then $x_{\mathbf{1}}$ is a state of s. Then $x_{\mathbf{2}}$ is a state of t.

Let s, t be Turing machine structures and let x be an element of $[\!:\mathrm{SeqStates}(s, t)$, (the symbols of s) $\cup$ (the symbols of t) $:\!]$. The functor $1^{\mathrm{st}}\mathrm{SeqState}\,x$ yields a state of s and is defined by:

(Def. 26) $1^{\mathrm{st}}\mathrm{SeqState}\,x = (x_{\mathbf{1}})_{\mathbf{1}}$.

The functor $2^{\mathrm{nd}}\mathrm{SeqState}\,x$ yielding a state of t is defined as follows:

(Def. 27) $2^{\mathrm{nd}}\mathrm{SeqState}\,x = (x_{\mathbf{1}})_{\mathbf{2}}$.

Let X, Y, Z be non empty sets and let x be an element of $[\!: X, Y \cup Z :\!]$. Let us assume that there exist a set u and an element y of Y such that $x = \langle u, y\rangle$. The functor $1^{\mathrm{st}}\mathrm{SeqSymbol}\,x$ yielding an element of Y is defined as follows:

(Def. 28) $1^{\mathrm{st}}\mathrm{SeqSymbol}\,x = x_{\mathbf{2}}$.

Let X, Y, Z be non empty sets and let x be an element of $[\!: X, Y \cup Z :\!]$. Let us assume that there exist a set u and an element z of Z such that $x = \langle u, z\rangle$. The functor $2^{\mathrm{nd}}\mathrm{SeqSymbol}\,x$ yielding an element of Z is defined by:

(Def. 29) $2^{\mathrm{nd}}\mathrm{SeqSymbol}\,x = x_{\mathbf{2}}$.

Let s, t be Turing machine structures and let x be an element of $[\!:\mathrm{SeqStates}(s, t)$, (the symbols of s) $\cup$ (the symbols of t) $:\!]$. The functor $\mathrm{SeqTran}(s, t, x)$ yielding an element of $[\!:\mathrm{SeqStates}(s, t)$, (the symbols of s) $\cup$ (the symbols of t), $\{-1, 0, 1\}:\!]$ is defined by:

(Def. 30)
$$\mathrm{SeqTran}(s,t,x) = \begin{cases} 1^{\mathrm{st}}\mathrm{SeqTran}(s,t,(\text{the transition of } s)(\langle 1^{\mathrm{st}}\mathrm{SeqState}\, x, \\ \quad 1^{\mathrm{st}}\mathrm{SeqSymbol}\, x\rangle)), \text{ if there exists a state } p \text{ of } s \\ \quad \text{and there exists a symbol } y \text{ of } s \text{ such that } x = \\ \quad \langle\langle p, \text{ the initial state of } t\rangle, y\rangle \text{ and } p \neq \text{the accepting} \\ \quad \text{state of } s, \\ 2^{\mathrm{nd}}\mathrm{SeqTran}(s,t,(\text{the transition of } t)(\langle 2^{\mathrm{nd}}\mathrm{SeqState}\, x, \\ \quad 2^{\mathrm{nd}}\mathrm{SeqSymbol}\, x\rangle)), \text{ if there exists a state } q \text{ of } t \\ \quad \text{and there exists a symbol } y \text{ of } t \text{ such that } x = \\ \quad \langle\langle \text{the accepting state of } s,\ q\rangle, y\rangle, \\ \langle x_{\mathbf{1}}, x_{\mathbf{2}}, -1\rangle, \text{ otherwise.} \end{cases}$$

Let s, t be Turing machine structures. The functor $\mathrm{SeqTran}(s,t)$ yielding a function from $[\!:\mathrm{SeqStates}(s,t), (\text{the symbols of } s) \cup (\text{the symbols of } t):\!]$ into $[\!:\mathrm{SeqStates}(s,t), (\text{the symbols of } s) \cup (\text{the symbols of } t), \{-1,0,1\}:\!]$ is defined by:

(Def. 31) For every element x of $[\!:\mathrm{SeqStates}(s,t), (\text{the symbols of } s)\cup(\text{the symbols of } t):\!]$ holds $(\mathrm{SeqTran}(s,t))(x) = \mathrm{SeqTran}(s,t,x)$.

Let T_1, T_2 be Turing machine structures. The functor $T_1;\, T_2$ yielding a strict Turing machine structure is defined by the conditions (Def. 32).

(Def. 32)(i) The symbols of $T_1;\, T_2 = (\text{the symbols of } T_1) \cup (\text{the symbols of } T_2)$,

(ii) the control states of $T_1;\, T_2 = \mathrm{SeqStates}(T_1, T_2)$,

(iii) the transition of $T_1;\, T_2 = \mathrm{SeqTran}(T_1, T_2)$,

(iv) the initial state of $T_1;\, T_2 = \langle$the initial state of T_1, the initial state of $T_2\rangle$, and

(v) the accepting state of $T_1;\, T_2 = \langle$the accepting state of T_1, the accepting state of $T_2\rangle$.

We now state several propositions:

(48) Let T_1, T_2 be Turing machine structures, g be a transition-target of T_1, p be a state of T_1, and y be a symbol of T_1. Suppose $p \neq$ the accepting state of T_1 and $g = (\text{the transition of } T_1)(\langle p, y\rangle)$. Then $(\text{the transition of } T_1;\, T_2)(\langle\langle p, \text{the initial state of } T_2\rangle, y\rangle) = \langle\langle g_{\mathbf{1}}, \text{the initial state of } T_2\rangle, g_{\mathbf{2}}, g_{\mathbf{3}}\rangle$.

(49) Let T_1, T_2 be Turing machine structures, g be a transition-target of T_2, q be a state of T_2, and y be a symbol of T_2. Suppose $g = (\text{the transition of } T_2)(\langle q, y\rangle)$. Then $(\text{the transition of } T_1;\, T_2)(\langle\langle \text{the accepting state of } T_1, q\rangle, y\rangle) = \langle\langle \text{the accepting state of } T_1,\ g_{\mathbf{1}}\rangle, g_{\mathbf{2}}, g_{\mathbf{3}}\rangle$.

(50) Let T_1, T_2 be Turing machine structures, s_1 be a State of T_1, h be a natural number, t be a tape of T_1, s_2 be a State of T_2, and s_3 be a State of $T_1;\, T_2$. Suppose that

(i) s_1 is accepting,

(ii) $s_1 = \langle \text{the initial state of } T_1,\ h,\ t\rangle$,

(iii) s_2 is accepting,

(iv) $s_2 = \langle$the initial state of T_2, $(\mathrm{Result}(s_1))_{\mathbf{2}}$, $(\mathrm{Result}(s_1))_{\mathbf{3}}\rangle$, and

(v) $s_3 = \langle$the initial state of T_1; T_2, h, $t\rangle$.

Then s_3 is accepting and $(\mathrm{Result}(s_3))_{\mathbf{2}} = (\mathrm{Result}(s_2))_{\mathbf{2}}$ and $(\mathrm{Result}(s_3))_{\mathbf{3}} = (\mathrm{Result}(s_2))_{\mathbf{3}}$.

(51) Let t_3, t_4 be Turing machine structures and t be a tape of t_3. If the symbols of t_3 = the symbols of t_4, then t is a tape of t_3; t_4.

(52) Let t_3, t_4 be Turing machine structures and t be a tape of t_3; t_4. Suppose the symbols of t_3 = the symbols of t_4. Then t is a tape of t_3 and a tape of t_4.

(53) Let f be a finite sequence of elements of $\mathbb{N}$, t_3, t_4 be Turing machine structures, t_1 be a tape of t_3, and t_2 be a tape of t_4. If $t_1 = t_2$ and t_1 stores data f, then t_2 stores data f.

(54) Let s be a State of ZeroTuring ; SuccTuring, t be a tape of ZeroTuring, and h_1, n be natural numbers. Suppose $s = \langle\langle 0, 0\rangle, h_1, t\rangle$ and t stores data $\langle h_1, n\rangle$. Then s is accepting and $(\mathrm{Result}(s))_{\mathbf{2}} = h_1$ and $(\mathrm{Result}(s))_{\mathbf{3}}$ stores data $\langle h_1, 1\rangle$.

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On the Characterizations of Compactness

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Summary. In the paper we show equivalence of the convergence of filters on a topological space and the convergence of nets in the space. We also give, five characterizations of compactness. Namely, for any topological space T we proved that following condition are equivalent:

- T is compact,
- every ultrafilter on T is convergent,
- every proper filter on T has cluster point,
- every net in T has cluster point,
- every net in T has convergent subnet,
- every Cauchy net in T is convergent.

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The articles [18], [13], [4], [11], [6], [16], [12], [19], [10], [17], [14], [8], [5], [1], [2], [9], [7], [15], and [3] provide the notation and terminology for this paper.

In this paper X is a set.

The following propositions are true:

(1) The carrier of $2^X_\subseteq = 2^X$.

(2) For every non empty set X and for every proper filter F of $2^X_\subseteq$ and for every set A such that $A \in F$ holds A is not empty.

Let T be a non empty topological space and let x be a point of T. The neighborhood system of x is a subset of $2^{\Omega_T}_\subseteq$ and is defined by:

(Def. 1) The neighborhood system of $x = \{A : A$ ranges over neighbourhoods of $x\}$.

The following proposition is true

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(3) Let T be a non empty topological space, x be a point of T, and A be a set. Then $A \in$ the neighborhood system of x if and only if A is a neighbourhood of x.

Let T be a non empty topological space and let x be a point of T. Observe that the neighborhood system of x is non empty proper upper and filtered.

One can prove the following propositions:

(4) Let T be a non empty topological space, x be a point of T, and F be an upper subset of $2^{\Omega_T}_{\subseteq}$. Then x is a convergence point of F, T if and only if the neighborhood system of $x \subseteq F$.

(5) For every non empty topological space T holds every point x of T is a convergence point of the neighborhood system of x, T.

(6) Let T be a non empty topological space and A be a subset of T. Then A is open if and only if for every point x of T such that $x \in A$ and for every filter F of $2^{\Omega_T}_{\subseteq}$ such that x is a convergence point of F, T holds $A \in F$.

Let S be a non empty 1-sorted structure and let N be a non empty net structure over S. A subset of S is called a subset of S reachable by N if:

(Def. 2) There exists an element i of N such that it $=$ rng (the mapping of $N{\restriction}i$).

The following proposition is true

(7) Let S be a non empty 1-sorted structure, N be a non empty net structure over S, and i be an element of N. Then rng (the mapping of $N{\restriction}i$) is a subset of S reachable by N.

Let S be a non empty 1-sorted structure and let N be a reflexive non empty net structure over S. Note that every subset of S reachable by N is non empty.

We now state three propositions:

(8) Let S be a non empty 1-sorted structure, N be a net in S, i be an element of N, and x be a set. Then $x \in$ rng (the mapping of $N{\restriction}i$) if and only if there exists an element j of N such that $i \leqslant j$ and $x = N(j)$.

(9) Let S be a non empty 1-sorted structure, N be a net in S, and A be a subset of S reachable by N. Then N is eventually in A.

(10) Let S be a non empty 1-sorted structure, N be a net in S, and F be a finite non empty set. Suppose every element of F is a subset of S reachable by N. Then there exists a subset B of S reachable by N such that $B \subseteq \bigcap F$.

Let T be a non empty 1-sorted structure and let N be a non empty net structure over T. The filter of N is a subset of $2^{\Omega_T}_{\subseteq}$ and is defined by:

(Def. 3) The filter of $N = \{A; A$ ranges over subsets of T: N is eventually in $A\}$.

The following proposition is true

(11) Let T be a non empty 1-sorted structure, N be a non empty net structure over T, and A be a set. Then $A \in$ the filter of N if and only if N is eventually in A and A is a subset of T.

Let T be a non empty 1-sorted structure and let N be a non empty net structure over T. Note that the filter of N is non empty and upper.

Let T be a non empty 1-sorted structure and let N be a net in T. One can verify that the filter of N is proper and filtered.

We now state two propositions:

(12) Let T be a non empty topological space, N be a net in T, and x be a point of T. Then x is a cluster point of N if and only if x is a cluster point of the filter of N, T.

(13) Let T be a non empty topological space, N be a net in T, and x be a point of T. Then $x \in \operatorname{Lim} N$ if and only if x is a convergence point of the filter of N, T.

Let L be a non empty 1-sorted structure, let O be a non empty subset of L, and let F be a filter of $2^{O}_{\subseteq}$. The net of F is a strict non empty net structure over L and is defined by the conditions (Def. 4).

(Def. 4)(i) The carrier of the net of $F = \{\langle a, f\rangle; a$ ranges over elements of L, f ranges over elements of F: $a \in f\}$,

(ii) for all elements i, j of the net of F holds $i \leqslant j$ iff $j_{\mathbf{2}} \subseteq i_{\mathbf{2}}$, and

(iii) for every element i of the net of F holds (the net of $F)(i) = i_{\mathbf{1}}$.

Let L be a non empty 1-sorted structure, let O be a non empty subset of L, and let F be a filter of $2^{O}_{\subseteq}$. Note that the net of F is reflexive and transitive.

Let L be a non empty 1-sorted structure, let O be a non empty subset of L, and let F be a proper filter of $2^{O}_{\subseteq}$. One can verify that the net of F is directed.

The following propositions are true:

(14) For every non empty 1-sorted structure T and for every filter F of $2^{\Omega_T}_{\subseteq}$ holds $F \setminus \{\emptyset\} =$ the filter of the net of F.

(15) Let T be a non empty 1-sorted structure and F be a proper filter of $2^{\Omega_T}_{\subseteq}$. Then $F =$ the filter of the net of F.

(16) Let T be a non empty 1-sorted structure, F be a filter of $2^{\Omega_T}_{\subseteq}$, and A be a non empty subset of T. Then $A \in F$ if and only if the net of F is eventually in A.

(17) Let T be a non empty topological space, F be a proper filter of $2^{\Omega_T}_{\subseteq}$, and x be a point of T. Then x is a cluster point of the net of F if and only if x is a cluster point of F, T.

(18) Let T be a non empty topological space, F be a proper filter of $2^{\Omega_T}_{\subseteq}$, and x be a point of T. Then $x \in \operatorname{Lim}$ (the net of F) if and only if x is a convergence point of F, T.

(19) Let T be a non empty topological space, A be a subset of T, and x be a point of T. Then $x \in \overline{A}$ if and only if for every neighbourhood O of x holds O meets A.

(20) Let T be a non empty topological space, x be a point of T, and A be a subset of T. Suppose $x \in \overline{A}$. Let F be a proper filter of $2_{\subseteq}^{\Omega_T}$. If $F =$ the neighborhood system of x, then the net of F is often in A.

(21) Let T be a non empty 1-sorted structure, A be a set, and N be a net in T. If N is eventually in A, then every subnet of N is eventually in A.

(22) Let T be a non empty topological space and F, G, x be sets. Suppose $F \subseteq G$ and x is a convergence point of F, T. Then x is a convergence point of G, T.

(23) Let T be a non empty topological space, A be a subset of T, and x be a point of T. Then $x \in \overline{A}$ if and only if there exists a net N in T such that N is eventually in A and x is a cluster point of N.

(24) Let T be a non empty topological space, A be a subset of T, and x be a point of T. Then $x \in \overline{A}$ if and only if there exists a convergent net N in T such that N is eventually in A and $x \in \operatorname{Lim} N$.

(25) Let T be a non empty topological space and A be a subset of T. Then A is closed if and only if for every net N in T such that N is eventually in A and for every point x of T such that x is a cluster point of N holds $x \in A$.

(26) Let T be a non empty topological space and A be a subset of T. Then A is closed if and only if for every convergent net N in T such that N is eventually in A and for every point x of T such that $x \in \operatorname{Lim} N$ holds $x \in A$.

(27) Let T be a non empty topological space, A be a subset of T, and x be a point of T. Then $x \in \overline{A}$ if and only if there exists a proper filter F of $2_{\subseteq}^{\Omega_T}$ such that $A \in F$ and x is a cluster point of F, T.

(28) Let T be a non empty topological space, A be a subset of T, and x be a point of T. Then $x \in \overline{A}$ if and only if there exists an ultra filter F of $2_{\subseteq}^{\Omega_T}$ such that $A \in F$ and x is a convergence point of F, T.

(29) Let T be a non empty topological space and A be a subset of T. Then A is closed if and only if for every proper filter F of $2_{\subseteq}^{\Omega_T}$ such that $A \in F$ and for every point x of T such that x is a cluster point of F, T holds $x \in A$.

(30) Let T be a non empty topological space and A be a subset of T. Then A is closed if and only if for every ultra filter F of $2_{\subseteq}^{\Omega_T}$ such that $A \in F$ and for every point x of T such that x is a convergence point of F, T holds $x \in A$.

(31) Let T be a non empty topological space, N be a net in T, and s be a point of T. Then s is a cluster point of N if and only if for every subset A of T reachable by N holds $s \in \overline{A}$.

(32) Let T be a non empty topological space and F be a family of subsets of

the carrier of T. If F is closed, then $\mathrm{FinMeetCl}(F)$ is closed.

(33) Let T be a non empty topological space. Then T is compact if and only if for every ultra filter F of $2_{\subseteq}^{\Omega_T}$ holds there exists a point of T which is a convergence point of F, T.

(34) Let T be a non empty topological space. Then T is compact if and only if for every proper filter F of $2_{\subseteq}^{\Omega_T}$ holds there exists a point of T which is a cluster point of F, T.

(35) Let T be a non empty topological space. Then T is compact if and only if for every net N in T holds there exists a point of T which is a cluster point of N.

(36) Let T be a non empty topological space. Then T is compact if and only if for every net N in T such that $N \in \mathrm{NetUniv}(T)$ holds there exists a point of T which is a cluster point of N.

Let L be a non empty 1-sorted structure and let N be a transitive net structure over L. Note that every full structure of a subnet of N is transitive.

Let L be a non empty 1-sorted structure and let N be a non empty directed net structure over L. Note that there exists a structure of a subnet of N which is strict, non empty, directed, and full.

The following proposition is true

(37) For every non empty topological space T holds T is compact iff for every net N in T holds there exists a subnet of N which is convergent.

Let S be a non empty 1-sorted structure and let N be a non empty net structure over S. We say that N is Cauchy if and only if:

(Def. 5) For every subset A of S holds N is eventually in A or eventually in $-A$.

Let S be a non empty 1-sorted structure and let F be an ultra filter of $2_{\subseteq}^{\Omega_S}$. Observe that the net of F is Cauchy.

Next we state the proposition

(38) Let T be a non empty topological space. Then T is compact if and only if for every net N in T such that N is Cauchy holds N is convergent.

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Compactness of Lim-inf Topology

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Summary. Formalization of [10], chapter III, section 3 (3.4–3.6).

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The papers [15], [9], [1], [18], [21], [14], [22], [17], [12], [8], [20], [6], [16], [3], [4], [13], [7], [2], [11], [23], [19], and [5] provide the notation and terminology for this paper.

Let L be a non empty poset, let X be a non empty subset of L, and let F be a filter of $2^X_\subseteq$. The functor $\liminf F$ yielding an element of L is defined by:

(Def. 1) $\liminf F = \bigsqcup_L \{\inf B; B$ ranges over subsets of L: $B \in F\}$.

One can prove the following proposition

(1) Let L_1, L_2 be complete lattices. Suppose the relational structure of $L_1 =$ the relational structure of L_2. Let X_1 be a non empty subset of L_1, X_2 be a non empty subset of L_2, F_1 be a filter of $2^{X_1}_\subseteq$, and F_2 be a filter of $2^{X_2}_\subseteq$. If $F_1 = F_2$, then $\liminf F_1 = \liminf F_2$.

Let L be a non empty FR-structure. We say that L is lim-inf if and only if:

(Def. 2) The topology of $L = \xi(L)$.

Let us note that every non empty FR-structure which is lim-inf is also topological space-like.

One can check that every top-lattice which is trivial is also lim-inf.

One can check that there exists a top-lattice which is lim-inf, continuous, and complete.

We now state several propositions:

(2) Let L_1, L_2 be non empty 1-sorted structures. Suppose the carrier of $L_1 =$ the carrier of L_2. Let N_1 be a net structure over L_1. Then there exists a strict net structure N_2 over L_2 such that

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(i) the relational structure of N_1 = the relational structure of N_2, and
(ii) the mapping of N_1 = the mapping of N_2.

(3) Let L_1, L_2 be non empty 1-sorted structures. Suppose the carrier of L_1 = the carrier of L_2. Let N_1 be a net structure over L_1. Suppose $N_1 \in \mathrm{NetUniv}(L_1)$. Then there exists a strict net N_2 in L_2 such that
(i) $N_2 \in \mathrm{NetUniv}(L_2)$,
(ii) the relational structure of N_1 = the relational structure of N_2, and
(iii) the mapping of N_1 = the mapping of N_2.

(4) Let L_1, L_2 be inf-complete up-complete semilattices. Suppose the relational structure of L_1 = the relational structure of L_2. Let N_1 be a net in L_1 and N_2 be a net in L_2. Suppose that
(i) the relational structure of N_1 = the relational structure of N_2, and
(ii) the mapping of N_1 = the mapping of N_2.
Then $\liminf N_1 = \liminf N_2$.

(5) Let L_1, L_2 be non empty 1-sorted structures. Suppose the carrier of L_1 = the carrier of L_2. Let N_1 be a net in L_1 and N_2 be a net in L_2. Suppose that
(i) the relational structure of N_1 = the relational structure of N_2, and
(ii) the mapping of N_1 = the mapping of N_2.
Let S_1 be a subnet of N_1. Then there exists a strict subnet S_2 of N_2 such that
(iii) the relational structure of S_1 = the relational structure of S_2, and
(iv) the mapping of S_1 = the mapping of S_2.

(6) Let L_1, L_2 be inf-complete up-complete semilattices. Suppose the relational structure of L_1 = the relational structure of L_2. Let N_1 be a net structure over L_1 and a be a set. Suppose $\langle N_1, a\rangle \in$ the lim inf convergence of L_1. Then there exists a strict net N_2 in L_2 such that
(i) $\langle N_2, a\rangle \in$ the lim inf convergence of L_2,
(ii) the relational structure of N_1 = the relational structure of N_2, and
(iii) the mapping of N_1 = the mapping of N_2.

(7) Let L_1, L_2 be non empty 1-sorted structures, N_1 be a non empty net structure over L_1, and N_2 be a non empty net structure over L_2. Suppose that
(i) the relational structure of N_1 = the relational structure of N_2, and
(ii) the mapping of N_1 = the mapping of N_2.
Let X be a set. If N_1 is eventually in X, then N_2 is eventually in X.

(8) Let L_1, L_2 be inf-complete up-complete semilattices. Suppose the relational structure of L_1 = the relational structure of L_2. Then ConvergenceSpace(the lim inf convergence of L_1) = ConvergenceSpace(the lim inf convergence of L_2).

(9) Let L_1, L_2 be inf-complete up-complete semilattices. Suppose the relational structure of L_1 = the relational structure of L_2. Then $\xi(L_1) = \xi(L_2)$.

Let R be an inf-complete non empty reflexive relational structure. Note that every topological augmentation of R is inf-complete.

Let R be a semilattice. One can verify that every topological augmentation of R has g.l.b.'s.

Let L be an inf-complete up-complete semilattice. One can check that there exists a topological augmentation of L which is strict and lim-inf.

The following proposition is true

(10) Let L be an inf-complete up-complete semilattice and X be a lim-inf topological augmentation of L. Then $\xi(L)$ = the topology of X.

Let L be an inf-complete up-complete semilattice. The functor $\Xi(L)$ yielding a strict topological augmentation of L is defined by:

(Def. 3) $\Xi(L)$ is lim-inf.

Let L be an inf-complete up-complete semilattice. One can check that $\Xi(L)$ is lim-inf.

Next we state a number of propositions:

(11) For every complete lattice L and for every net N in L holds $\liminf N = \bigsqcup_L \{\inf(N{\restriction}i) : i$ ranges over elements of $N\}$.

(12) Let L be a complete lattice, F be a proper filter of $2^{\Omega_L}_{\subseteq}$, and f be a subset of L. Suppose $f \in F$. Let i be an element of the net of F. If $i_{\mathbf{2}} = f$, then $\inf f = \inf((\text{the net of } F){\restriction}i)$.

(13) For every complete lattice L and for every proper filter F of $2^{\Omega_L}_{\subseteq}$ holds $\liminf F = \liminf$ (the net of F).

(14) For every complete lattice L and for every proper filter F of $2^{\Omega_L}_{\subseteq}$ holds the net of $F \in \mathrm{NetUniv}(L)$.

(15) Let L be a complete lattice, F be an ultra filter of $2^{\Omega_L}_{\subseteq}$, and p be a greater or equal to id map from the net of F into the net of F. Then $\liminf F \geqslant \inf((\text{the net of } F) \cdot p)$.

(16) Let L be a complete lattice, F be an ultra filter of $2^{\Omega_L}_{\subseteq}$, and M be a subnet of the net of F. Then $\liminf F = \liminf M$.

(17) Let L be a non empty 1-sorted structure, N be a net in L, and A be a set. Suppose N is often in A. Then there exists a strict subnet N' of N such that rng (the mapping of N') $\subseteq A$ and N' is a structure of a subnet of N.

(18) Let L be a complete lim-inf top-lattice and A be a non empty subset of L. Then A is closed if and only if for every ultra filter F of $2^{\Omega_L}_{\subseteq}$ such that $A \in F$ holds $\liminf F \in A$.

(19) For every non empty reflexive relational structure L holds $\sigma(L) \subseteq \xi(L)$.

(20) Let T_1, T_2 be non empty topological spaces and B be a prebasis of T_1. Suppose $B \subseteq$ the topology of T_2 and the carrier of $T_1 \in$ the topology of T_2. Then the topology of $T_1 \subseteq$ the topology of T_2.

(21) For every complete lattice L holds $\omega(L) \subseteq \xi(L)$.

(22) Let T_1, T_2 be topological spaces and T be a non empty topological space. Suppose T is a topological extension of T_1 and a topological extension of T_2. Let R be a refinement of T_1 and T_2. Then T is a topological extension of R.

(23) Let T_1 be a topological space, T_2 be a topological extension of T_1, and A be a subset of T_1. Then

(i) if A is open, then A is an open subset of T_2, and

(ii) if A is closed, then A is a closed subset of T_2.

(24) For every complete lattice L holds $\lambda(L) \subseteq \xi(L)$.

(25) Let L be a complete lattice, T be a lim-inf topological augmentation of L, and S be a Lawson correct topological augmentation of L. Then T is a topological extension of S.

(26) For every complete lim-inf top-lattice L and for every ultra filter F of $2_{\subseteq}^{\Omega_L}$ holds $\liminf F$ is a convergence point of F, L.

(27) Every complete lim-inf top-lattice is compact and T_1.

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Miscellaneous Facts about Functors

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Summary. In the paper we show useful facts concerning reverse and inclusion functors and the restriction of functors. We also introduce a new notation for the intersection of categories and the isomorphism under arbitrary functors.

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The notation and terminology used in this paper have been introduced in the following articles: [11], [12], [15], [13], [7], [2], [3], [4], [9], [14], [5], [10], [16], [17], [8], [1], and [6].

1. Reverse Functors

The following propositions are true:

(1) Let A, B be transitive non empty category structures with units and F be a feasible reflexive functor structure from A to B. Suppose F is coreflexive and bijective. Let a be an object of A and b be an object of B. Then $F(a) = b$ if and only if $F^{-1}(b) = a$.

(2) Let A, B be transitive non empty category structures with units, F be a precovariant feasible functor structure from A to B, and G be a precovariant feasible functor structure from B to A. Suppose F is bijective and $G = F^{-1}$. Let a_1, a_2 be objects of A. Suppose $\langle a_1, a_2 \rangle \neq \emptyset$. Let f be a morphism from a_1 to a_2 and g be a morphism from $F(a_1)$ to $F(a_2)$. Then $F(f) = g$ if and only if $G(g) = f$.

(3) Let A, B be transitive non empty category structures with units, F be a precontravariant feasible functor structure from A to B, and G be

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a precontravariant feasible functor structure from B to A. Suppose F is bijective and $G = F^{-1}$. Let a_1, a_2 be objects of A. Suppose $\langle a_1, a_2 \rangle \neq \emptyset$. Let f be a morphism from a_1 to a_2 and g be a morphism from $F(a_2)$ to $F(a_1)$. Then $F(f) = g$ if and only if $G(g) = f$.

(4) Let A, B be categories and F be a functor from A to B. Suppose F is bijective. Let G be a functor from B to A. If $F \cdot G = \mathrm{id}_B$, then the functor structure of $G = F^{-1}$.

(5) Let A, B be categories and F be a functor from A to B. Suppose F is bijective. Let G be a functor from B to A. If $G \cdot F = \mathrm{id}_A$, then the functor structure of $G = F^{-1}$.

(6) Let A, B be categories and F be a covariant functor from A to B. Suppose F is bijective. Let G be a covariant functor from B to A. Suppose that

(i) for every object b of B holds $F(G(b)) = b$, and

(ii) for all objects a, b of B such that $\langle a, b \rangle \neq \emptyset$ and for every morphism f from a to b holds $F(G(f)) = f$.

Then the functor structure of $G = F^{-1}$.

(7) Let A, B be categories and F be a contravariant functor from A to B. Suppose F is bijective. Let G be a contravariant functor from B to A. Suppose that

(i) for every object b of B holds $F(G(b)) = b$, and

(ii) for all objects a, b of B such that $\langle a, b \rangle \neq \emptyset$ and for every morphism f from a to b holds $F(G(f)) = f$.

Then the functor structure of $G = F^{-1}$.

(8) Let A, B be categories and F be a covariant functor from A to B. Suppose F is bijective. Let G be a covariant functor from B to A. Suppose that

(i) for every object a of A holds $G(F(a)) = a$, and

(ii) for all objects a, b of A such that $\langle a, b \rangle \neq \emptyset$ and for every morphism f from a to b holds $G(F(f)) = f$.

Then the functor structure of $G = F^{-1}$.

(9) Let A, B be categories and F be a contravariant functor from A to B. Suppose F is bijective. Let G be a contravariant functor from B to A. Suppose that

(i) for every object a of A holds $G(F(a)) = a$, and

(ii) for all objects a, b of A such that $\langle a, b \rangle \neq \emptyset$ and for every morphism f from a to b holds $G(F(f)) = f$.

Then the functor structure of $G = F^{-1}$.

2. Intersection of Categories

Let A, B be category structures. We say that A and B have the same composition if and only if:

(Def. 1) For all sets a_1, a_2, a_3 holds (the composition of A)($\langle a_1, a_2, a_3\rangle$) $\approx$ (the composition of B)($\langle a_1, a_2, a_3\rangle$).

Let us note that the predicate A and B have the same composition is symmetric.

Next we state three propositions:

(10) Let A, B be category structures. Then A and B have the same composition if and only if for all sets a_1, a_2, a_3, x such that $x \in \mathrm{dom}$ (the composition of A)($\langle a_1, a_2, a_3\rangle$) and $x \in \mathrm{dom}$ (the composition of B)($\langle a_1, a_2, a_3\rangle$) holds (the composition of A)($\langle a_1, a_2, a_3\rangle$)($x$) = (the composition of B)($\langle a_1, a_2, a_3\rangle$)($x$).

(11) Let A, B be transitive non empty category structures. Then A and B have the same composition if and only if for all objects a_1, a_2, a_3 of A such that $\langle a_1, a_2\rangle \neq \emptyset$ and $\langle a_2, a_3\rangle \neq \emptyset$ and for all objects b_1, b_2, b_3 of B such that $\langle b_1, b_2\rangle \neq \emptyset$ and $\langle b_2, b_3\rangle \neq \emptyset$ and $b_1 = a_1$ and $b_2 = a_2$ and $b_3 = a_3$ and for every morphism f_1 from a_1 to a_2 and for every morphism g_1 from b_1 to b_2 such that $g_1 = f_1$ and for every morphism f_2 from a_2 to a_3 and for every morphism g_2 from b_2 to b_3 such that $g_2 = f_2$ holds $f_2 \cdot f_1 = g_2 \cdot g_1$.

(12) For all para-functional semi-functional categories A, B holds A and B have the same composition.

Let f, g be functions. The functor $\mathrm{Intersect}(f, g)$ yielding a function is defined as follows:

(Def. 2) $\mathrm{dom}\,\mathrm{Intersect}(f, g) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and for every set x such that $x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$ holds $(\mathrm{Intersect}(f, g))(x) = f(x) \cap g(x)$.

Let us notice that the functor $\mathrm{Intersect}(f, g)$ is commutative.

One can prove the following propositions:

(13) For every set I and for all many sorted sets A, B indexed by I holds $\mathrm{Intersect}(A, B) = A \cap B$.

(14) Let I, J be sets, A be a many sorted set indexed by I, and B be a many sorted set indexed by J. Then $\mathrm{Intersect}(A, B)$ is a many sorted set indexed by $I \cap J$.

(15) Let I, J be sets, A be a many sorted set indexed by I, B be a function, and C be a many sorted set indexed by J. If $C = \mathrm{Intersect}(A, B)$, then $C \mathrel{\dot{\subseteq}} A$.

(16) Let A_1, A_2, B_1, B_2 be sets, f be a function from A_1 into A_2, and g be a function from B_1 into B_2. If $f \approx g$, then $f \cap g$ is a function from $A_1 \cap B_1$ into $A_2 \cap B_2$.

(17) Let I_1, I_2 be sets, A_1, B_1 be many sorted sets indexed by I_1, A_2, B_2 be many sorted sets indexed by I_2, and A, B be many sorted sets indexed by $I_1 \cap I_2$. Suppose $A = \mathrm{Intersect}(A_1, A_2)$ and $B = \mathrm{Intersect}(B_1, B_2)$. Let F be a many sorted function from A_1 into B_1 and G be a many sorted function from A_2 into B_2. Suppose that for every set x such that $x \in \mathrm{dom}\, F$ and $x \in \mathrm{dom}\, G$ holds $F(x) \approx G(x)$. Then $\mathrm{Intersect}(F, G)$ is a many sorted function from A into B.

(18) Let I, J be sets, F be a many sorted set indexed by $[\!: I, I :\!]$, and G be a many sorted set indexed by $[\!: J, J :\!]$. Then there exists a many sorted set H indexed by $[\!: I \cap J, I \cap J :\!]$ such that $H = \mathrm{Intersect}(F, G)$ and $\mathrm{Intersect}(\{\!|F|\!\}, \{\!|G|\!\}) = \{\!|H|\!\}$.

(19) Let I, J be sets, F_1, F_2 be many sorted sets indexed by $[\!: I, I :\!]$, and G_1, G_2 be many sorted sets indexed by $[\!: J, J :\!]$. Then there exist many sorted sets H_1, H_2 indexed by $[\!: I \cap J, I \cap J :\!]$ such that $H_1 = \mathrm{Intersect}(F_1, G_1)$ and $H_2 = \mathrm{Intersect}(F_2, G_2)$ and $\mathrm{Intersect}(\{\!|F_1, F_2|\!\}, \{\!|G_1, G_2|\!\}) = \{\!|H_1, H_2|\!\}$.

Let A, B be category structures. Let us assume that A and B have the same composition. The functor $\mathrm{Intersect}(A, B)$ yields a strict category structure and is defined by the conditions (Def. 3).

(Def. 3)(i) The carrier of $\mathrm{Intersect}(A, B) =$ (the carrier of A) $\cap$ (the carrier of B),

(ii) the arrows of $\mathrm{Intersect}(A, B) =$ Intersect(the arrows of A, the arrows of B), and

(iii) the composition of $\mathrm{Intersect}(A, B) =$ Intersect(the composition of A, the composition of B).

The following propositions are true:

(20) For all category structures A, B such that A and B have the same composition holds $\mathrm{Intersect}(A, B) = \mathrm{Intersect}(B, A)$.

(21) Let A, B be category structures. Suppose A and B have the same composition. Then $\mathrm{Intersect}(A, B)$ is a substructure of A.

(22) Let A, B be category structures. Suppose A and B have the same composition. Let a_1, a_2 be objects of A, b_1, b_2 be objects of B, and o_1, o_2 be objects of $\mathrm{Intersect}(A, B)$. If $o_1 = a_1$ and $o_1 = b_1$ and $o_2 = a_2$ and $o_2 = b_2$, then $\langle o_1, o_2 \rangle = (\langle a_1, a_2 \rangle) \cap (\langle b_1, b_2 \rangle)$.

(23) Let A, B be transitive category structures. If A and B have the same composition, then $\mathrm{Intersect}(A, B)$ is transitive.

(24) Let A, B be category structures. Suppose A and B have the same composition. Let a_1, a_2 be objects of A, b_1, b_2 be objects of B, and o_1, o_2 be objects of $\mathrm{Intersect}(A, B)$. Suppose $o_1 = a_1$ and $o_1 = b_1$ and $o_2 = a_2$ and $o_2 = b_2$ and $\langle a_1, a_2 \rangle \neq \emptyset$ and $\langle b_1, b_2 \rangle \neq \emptyset$. Let f be a morphism from a_1 to a_2 and g be a morphism from b_1 to b_2. If $f = g$, then $f \in \langle o_1, o_2 \rangle$.

(25) Let A, B be non empty category structures with units. Suppose A and B have the same composition. Let a be an object of A, b be an object of B, and o be an object of Intersect(A, B). If $o = a$ and $o = b$ and $\mathrm{id}_a = \mathrm{id}_b$, then $\mathrm{id}_a \in \langle o, o \rangle$.

(26) Let A, B be categories. Suppose that
(i) A and B have the same composition,
(ii) Intersect(A, B) is non empty, and
(iii) for every object a of A and for every object b of B such that $a = b$ holds $\mathrm{id}_a = \mathrm{id}_b$.

Then Intersect(A, B) is a subcategory of A.

3. Subcategories

The scheme *SubcategoryUniq* deals with a category $\mathcal{A}$, non empty subcategories $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, a unary predicate $\mathcal{P}$, and a ternary predicate $\mathcal{Q}$, and states that:

The category structure of $\mathcal{B}$ = the category structure of $\mathcal{C}$

provided the following requirements are met:

- For every object a of $\mathcal{A}$ holds a is an object of $\mathcal{B}$ iff $\mathcal{P}[a]$,
- Let a, b be objects of $\mathcal{A}$ and a', b' be objects of $\mathcal{B}$. Suppose $a' = a$ and $b' = b$ and $\langle a, b \rangle \neq \emptyset$. Let f be a morphism from a to b. Then $f \in \langle a', b' \rangle$ if and only if $\mathcal{Q}[a, b, f]$,
- For every object a of $\mathcal{A}$ holds a is an object of $\mathcal{C}$ iff $\mathcal{P}[a]$, and
- Let a, b be objects of $\mathcal{A}$ and a', b' be objects of $\mathcal{C}$. Suppose $a' = a$ and $b' = b$ and $\langle a, b \rangle \neq \emptyset$. Let f be a morphism from a to b. Then $f \in \langle a', b' \rangle$ if and only if $\mathcal{Q}[a, b, f]$.

The following proposition is true

(27) Let A be a non empty category structure and B be a non empty substructure of A. Then B is full if and only if for all objects a_1, a_2 of A and for all objects b_1, b_2 of B such that $b_1 = a_1$ and $b_2 = a_2$ holds $\langle b_1, b_2 \rangle = \langle a_1, a_2 \rangle$.

Now we present two schemes. The scheme *FullSubcategoryEx* deals with a category $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

There exists a strict full non empty subcategory B of $\mathcal{A}$ such that for every object a of $\mathcal{A}$ holds a is an object of B if and only if $\mathcal{P}[a]$

provided the parameters satisfy the following condition:

- There exists an object a of $\mathcal{A}$ such that $\mathcal{P}[a]$.

The scheme *FullSubcategoryUniq* deals with a category $\mathcal{A}$, full non empty subcategories $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

The category structure of $\mathcal{B}$ = the category structure of $\mathcal{C}$

provided the parameters meet the following conditions:

- For every object a of $\mathcal{A}$ holds a is an object of $\mathcal{B}$ iff $\mathcal{P}[a]$, and
- For every object a of $\mathcal{A}$ holds a is an object of $\mathcal{C}$ iff $\mathcal{P}[a]$.

4. Inclusion Functors and Functor Restrictions

Let f be a function yielding function and let x, y be sets. Observe that $f(x, y)$ is relation-like and function-like.

One can prove the following proposition

(28) Let A be a category, C be a non empty subcategory of A, and a, b be objects of C. If $\langle a, b\rangle \neq \emptyset$, then for every morphism f from a to b holds $(\overset{C}{\hookrightarrow})(f) = f$.

Let A be a category and let C be a non empty subcategory of A. Note that $\overset{C}{\hookrightarrow}$ is id-preserving and comp-preserving.

Let A be a category and let C be a non empty subcategory of A. One can verify that $\overset{C}{\hookrightarrow}$ is precovariant.

Let A be a category and let C be a non empty subcategory of A. Then $\overset{C}{\hookrightarrow}$ is a strict covariant functor from C to A.

Let A, B be categories, let C be a non empty subcategory of A, and let F be a covariant functor from A to B. Then $F \restriction C$ is a strict covariant functor from C to B.

Let A, B be categories, let C be a non empty subcategory of A, and let F be a contravariant functor from A to B. Then $F \restriction C$ is a strict contravariant functor from C to B.

Next we state several propositions:

(29) Let A, B be categories, C be a non empty subcategory of A, F be a functor structure from A to B, a be an object of A, and c be an object of C. If $c = a$, then $(F \restriction C)(c) = F(a)$.

(30) Let A, B be categories, C be a non empty subcategory of A, F be a covariant functor from A to B, a, b be objects of A, and c, d be objects of C. Suppose $c = a$ and $d = b$ and $\langle c, d\rangle \neq \emptyset$. Let f be a morphism from a to b and g be a morphism from c to d. If $g = f$, then $(F \restriction C)(g) = F(f)$.

(31) Let A, B be categories, C be a non empty subcategory of A, F be a contravariant functor from A to B, a, b be objects of A, and c, d be objects of C. Suppose $c = a$ and $d = b$ and $\langle c, d\rangle \neq \emptyset$. Let f be a morphism from a to b and g be a morphism from c to d. If $g = f$, then $(F \restriction C)(g) = F(f)$.

(32) Let A, B be non empty graphs and F be a bimap structure from A into B. Suppose F is precovariant and one-to-one. Let a, b be objects of A. If $F(a) = F(b)$, then $a = b$.

(33) Let A, B be non empty reflexive graphs and F be a feasible precovariant functor structure from A to B. Suppose F is faithful. Let a, b be objects of A. Suppose $\langle a, b\rangle \neq \emptyset$. Let f, g be morphisms from a to b. If $F(f) = F(g)$, then $f = g$.

(34) Let A, B be non empty graphs and F be a precovariant functor structure from A to B. Suppose F is surjective. Let a, b be objects of B. Suppose $\langle a, b\rangle \neq \emptyset$. Let f be a morphism from a to b. Then there exist objects c, d of A and there exists a morphism g from c to d such that $a = F(c)$ and $b = F(d)$ and $\langle c, d\rangle \neq \emptyset$ and $f = F(g)$.

(35) Let A, B be non empty graphs and F be a bimap structure from A into B. Suppose F is precontravariant and one-to-one. Let a, b be objects of A. If $F(a) = F(b)$, then $a = b$.

(36) Let A, B be non empty reflexive graphs and F be a feasible precontravariant functor structure from A to B. Suppose F is faithful. Let a, b be objects of A. Suppose $\langle a, b\rangle \neq \emptyset$. Let f, g be morphisms from a to b. If $F(f) = F(g)$, then $f = g$.

(37) Let A, B be non empty graphs and F be a precontravariant functor structure from A to B. Suppose F is surjective. Let a, b be objects of B. Suppose $\langle a, b\rangle \neq \emptyset$. Let f be a morphism from a to b. Then there exist objects c, d of A and there exists a morphism g from c to d such that $b = F(c)$ and $a = F(d)$ and $\langle c, d\rangle \neq \emptyset$ and $f = F(g)$.

5. Isomorphisms under Arbitrary Functor

Let A, B be categories, let F be a functor structure from A to B, and let A', B' be categories. We say that A' and B' are isomorphic under F if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) A' is a subcategory of A,

(ii) B' is a subcategory of B, and

(iii) there exists a covariant functor G from A' to B' such that G is bijective and for every object a' of A' and for every object a of A such that $a' = a$ holds $G(a') = F(a)$ and for all objects b', c' of A' and for all objects b, c of A such that $\langle b', c'\rangle \neq \emptyset$ and $b' = b$ and $c' = c$ and for every morphism f' from b' to c' and for every morphism f from b to c such that $f' = f$ holds $G(f') = (\text{Morph-Map}_F(b, c))(f)$.

We say that A' and B' are anti-isomorphic under F if and only if the conditions (Def. 5) are satisfied.

(Def. 5)(i) A' is a subcategory of A,

(ii) B' is a subcategory of B, and

(iii) there exists a contravariant functor G from A' to B' such that G is bijective and for every object a' of A' and for every object a of A such that $a' = a$ holds $G(a') = F(a)$ and for all objects b', c' of A' and for all objects b, c of A such that $\langle b', c' \rangle \neq \emptyset$ and $b' = b$ and $c' = c$ and for every morphism f' from b' to c' and for every morphism f from b to c such that $f' = f$ holds $G(f') = (\text{Morph-Map}_F(b, c))(f)$.

We now state several propositions:

(38) Let A, B, A_1, B_1 be categories and F be a functor structure from A to B. If A_1 and B_1 are isomorphic under F, then A_1 and B_1 are isomorphic.

(39) Let A, B, A_1, B_1 be categories and F be a functor structure from A to B. Suppose A_1 and B_1 are anti-isomorphic under F. Then A_1, B_1 are anti-isomorphic.

(40) Let A, B be categories and F be a covariant functor from A to B. If A and B are isomorphic under F, then F is bijective.

(41) Let A, B be categories and F be a contravariant functor from A to B. If A and B are anti-isomorphic under F, then F is bijective.

(42) Let A, B be categories and F be a covariant functor from A to B. If F is bijective, then A and B are isomorphic under F.

(43) Let A, B be categories and F be a contravariant functor from A to B. If F is bijective, then A and B are anti-isomorphic under F.

Now we present two schemes. The scheme *CoBijectRestriction* deals with non empty categories $\mathcal{A}$, $\mathcal{B}$, a covariant functor $\mathcal{C}$ from $\mathcal{A}$ to $\mathcal{B}$, a non empty subcategory $\mathcal{D}$ of $\mathcal{A}$, and a non empty subcategory $\mathcal{E}$ of $\mathcal{B}$, and states that:

$\mathcal{D}$ and $\mathcal{E}$ are isomorphic under $\mathcal{C}$

provided the parameters satisfy the following conditions:

- $\mathcal{C}$ is bijective,
- For every object a of $\mathcal{A}$ holds a is an object of $\mathcal{D}$ iff $\mathcal{C}(a)$ is an object of $\mathcal{E}$, and
- Let a, b be objects of $\mathcal{A}$. Suppose $\langle a, b \rangle \neq \emptyset$. Let a_1, b_1 be objects of $\mathcal{D}$. Suppose $a_1 = a$ and $b_1 = b$. Let a_2, b_2 be objects of $\mathcal{E}$. Suppose $a_2 = \mathcal{C}(a)$ and $b_2 = \mathcal{C}(b)$. Let f be a morphism from a to b. Then $f \in \langle a_1, b_1 \rangle$ if and only if $\mathcal{C}(f) \in \langle a_2, b_2 \rangle$.

The scheme *ContraBijectRestriction* deals with non empty categories $\mathcal{A}$, $\mathcal{B}$, a contravariant functor $\mathcal{C}$ from $\mathcal{A}$ to $\mathcal{B}$, a non empty subcategory $\mathcal{D}$ of $\mathcal{A}$, and a non empty subcategory $\mathcal{E}$ of $\mathcal{B}$, and states that:

$\mathcal{D}$ and $\mathcal{E}$ are anti-isomorphic under $\mathcal{C}$

provided the parameters meet the following conditions:

- $\mathcal{C}$ is bijective,
- For every object a of $\mathcal{A}$ holds a is an object of $\mathcal{D}$ iff $\mathcal{C}(a)$ is an object of $\mathcal{E}$, and

- Let a, b be objects of $\mathcal{A}$. Suppose $\langle a, b\rangle \neq \emptyset$. Let a_1, b_1 be objects of $\mathcal{D}$. Suppose $a_1 = a$ and $b_1 = b$. Let a_2, b_2 be objects of $\mathcal{E}$. Suppose $a_2 = \mathcal{C}(a)$ and $b_2 = \mathcal{C}(b)$. Let f be a morphism from a to b. Then $f \in \langle a_1, b_1\rangle$ if and only if $\mathcal{C}(f) \in \langle b_2, a_2\rangle$.

The following propositions are true:

(44) For every category A and for every non empty subcategory B of A holds B and B are isomorphic under id_A.

(45) For all functions f, g such that $f \subseteq g$ holds $\curvearrowleft f \subseteq \curvearrowleft g$.

(46) For all functions f, g such that $\mathrm{dom}\, f$ is a binary relation and $\curvearrowleft f \subseteq \curvearrowleft g$ holds $f \subseteq g$.

(47) Let I, J be sets, A be a many sorted set indexed by $[\!: I, I :\!]$, and B be a many sorted set indexed by $[\!: J, J :\!]$. If $A \mathrel{\dot\subseteq} B$, then $\curvearrowleft A \mathrel{\dot\subseteq} \curvearrowleft B$.

(48) Let A be a transitive non empty category structure and B be a transitive non empty substructure of A. Then B^{op} is a substructure of A^{op}.

(49) For every category A and for every non empty subcategory B of A holds B^{op} is a subcategory of A^{op}.

(50) Let A be a category and B be a non empty subcategory of A. Then B and B^{op} are anti-isomorphic under the dualizing functor from A into A^{op}.

(51) Let A_1, A_2 be categories and F be a covariant functor from A_1 to A_2. Suppose F is bijective. Let B_1 be a non empty subcategory of A_1 and B_2 be a non empty subcategory of A_2. Suppose B_1 and B_2 are isomorphic under F. Then B_2 and B_1 are isomorphic under F^{-1}.

(52) Let A_1, A_2 be categories and F be a contravariant functor from A_1 to A_2. Suppose F is bijective. Let B_1 be a non empty subcategory of A_1 and B_2 be a non empty subcategory of A_2. Suppose B_1 and B_2 are anti-isomorphic under F. Then B_2 and B_1 are anti-isomorphic under F^{-1}.

(53) Let A_1, A_2, A_3 be categories, F be a covariant functor from A_1 to A_2, G be a covariant functor from A_2 to A_3, B_1 be a non empty subcategory of A_1, B_2 be a non empty subcategory of A_2, and B_3 be a non empty subcategory of A_3. Suppose B_1 and B_2 are isomorphic under F and B_2 and B_3 are isomorphic under G. Then B_1 and B_3 are isomorphic under $G \cdot F$.

(54) Let A_1, A_2, A_3 be categories, F be a contravariant functor from A_1 to A_2, G be a covariant functor from A_2 to A_3, B_1 be a non empty subcategory of A_1, B_2 be a non empty subcategory of A_2, and B_3 be a non empty subcategory of A_3. Suppose B_1 and B_2 are anti-isomorphic under F and B_2 and B_3 are isomorphic under G. Then B_1 and B_3 are anti-isomorphic under $G \cdot F$.

(55) Let A_1, A_2, A_3 be categories, F be a covariant functor from A_1 to A_2, G be a contravariant functor from A_2 to A_3, B_1 be a non empty subcategory

of A_1, B_2 be a non empty subcategory of A_2, and B_3 be a non empty subcategory of A_3. Suppose B_1 and B_2 are isomorphic under F and B_2 and B_3 are anti-isomorphic under G. Then B_1 and B_3 are anti-isomorphic under $G \cdot F$.

(56) Let A_1, A_2, A_3 be categories, F be a contravariant functor from A_1 to A_2, G be a contravariant functor from A_2 to A_3, B_1 be a non empty subcategory of A_1, B_2 be a non empty subcategory of A_2, and B_3 be a non empty subcategory of A_3. Suppose B_1 and B_2 are anti-isomorphic under F and B_2 and B_3 are anti-isomorphic under G. Then B_1 and B_3 are isomorphic under $G \cdot F$.

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Categorial Background for Duality Theory

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Summary. In the paper, we develop the notation of lattice-wise categories as concrete categories (see [8]) of lattices. Namely, the categories based on [17] with lattices as objects and at least monotone maps between them as morphisms. As examples, we introduce the categories *UPS*, *CONT*, and *ALG* with complete, continuous, and algebraic lattices, respectively, as objects and directed suprema preserving maps as morphisms. Some useful schemes to construct categories of lattices and functors between them are also presented.

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The terminology and notation used in this paper are introduced in the following papers: [17], [18], [12], [20], [9], [14], [4], [19], [1], [15], [21], [22], [16], [10], [11], [6], [7], [13], [2], [3], [8], and [5].

1. Lattice-wise Categories

In this paper x, y are sets.

Let a be a set. a as 1-sorted is a 1-sorted structure and is defined as follows:

(Def. 1) a as 1-sorted $= \begin{cases} a, & \text{if } a \text{ is a 1-sorted structure,} \\ \langle a \rangle, & \text{otherwise.} \end{cases}$

Let W be a set. The functor POSETS(W) is defined as follows:

(Def. 2) $x \in$ POSETS(W) iff x is a strict poset and the carrier of x as 1-sorted $\in W$.

Let W be a non empty set. One can check that POSETS(W) is non empty.

Let W be a set with non empty elements. Note that POSETS(W) is poset-membered.

Let C be a category. We say that C is carrier-underlaid if and only if:

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(Def. 3) For every object a of C there exists a 1-sorted structure S such that $a = S$ and the carrier of a = the carrier of S.

Let C be a category. We say that C is lattice-wise if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) C is semi-functional and set-id-inheriting,

(ii) every object of C is a lattice, and

(iii) for all objects a, b of C and for all lattices A, B such that $A = a$ and $B = b$ holds $\langle a, b\rangle \subseteq B^{A}_{\leqslant}$.

Let C be a category. We say that C has complete lattices if and only if:

(Def. 5) C is lattice-wise and every object of C is a complete lattice.

One can check that every category which has complete lattices is lattice-wise and every category which is lattice-wise is also concrete and carrier-underlaid.

One can verify that there exists a category which is strict and has complete lattices.

We now state two propositions:

(1) Let C be a carrier-underlaid category and a be an object of C. Then the carrier of a = the carrier of a as 1-sorted.

(2) Let C be a set-id-inheriting carrier-underlaid category and a be an object of C. Then $\mathrm{id}_a = \mathrm{id}_{a \text{ as 1-sorted}}$.

Let C be a lattice-wise category and let a be an object of C. Then a as 1-sorted is a lattice and it can be characterized by the condition:

(Def. 6) a as 1-sorted $= a$.

We introduce $\mathbb{L}_a$ as a synonym of a as 1-sorted.

Let C be a category with complete lattices and let a be an object of C. Then a as 1-sorted is a complete lattice. We introduce $\mathbb{L}_a$ as a synonym of a as 1-sorted.

Let C be a lattice-wise category and let a, b be objects of C. Let us assume that $\langle a, b\rangle \neq \emptyset$. Let f be a morphism from a to b. The functor $^{@}f$ yielding a monotone map from $\mathbb{L}_a$ into $\mathbb{L}_b$ is defined as follows:

(Def. 7) $^{@}f = f$.

The following proposition is true

(3) Let C be a lattice-wise category and a, b, c be objects of C. Suppose $\langle a, b\rangle \neq \emptyset$ and $\langle b, c\rangle \neq \emptyset$. Let f be a morphism from a to b and g be a morphism from b to c. Then $g \cdot f = (^{@}g) \cdot (^{@}f)$.

In this article we present several logical schemes. The scheme *CLCatEx1* deals with a non empty set $\mathcal{A}$ and a ternary predicate $\mathcal{P}$, and states that:

There exists a lattice-wise strict category C such that

(i) the carrier of $C = \mathcal{A}$, and

(ii) for all objects a, b of C and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b\rangle$ iff $\mathcal{P}[\mathbb{L}_a, \mathbb{L}_b, f]$

provided the following conditions are met:

- Every element of $\mathcal{A}$ is a lattice,
- Let a, b, c be lattices. Suppose $a \in \mathcal{A}$ and $b \in \mathcal{A}$ and $c \in \mathcal{A}$. Let f be a map from a into b and g be a map from b into c. If $\mathcal{P}[a, b, f]$ and $\mathcal{P}[b, c, g]$, then $\mathcal{P}[a, c, g \cdot f]$, and
- For every lattice a such that $a \in \mathcal{A}$ holds $\mathcal{P}[a, a, \mathrm{id}_a]$.

The scheme *CLCatEx2* deals with a non empty set $\mathcal{A}$, a unary predicate $\mathcal{P}$, and a ternary predicate $\mathcal{Q}$, and states that:

There exists a lattice-wise strict category C such that
(i) for every lattice x holds x is an object of C iff x is strict and $\mathcal{P}[x]$ and the carrier of $x \in \mathcal{A}$, and
(ii) for all objects a, b of C and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b\rangle$ iff $\mathcal{Q}[\mathbb{L}_a, \mathbb{L}_b, f]$

provided the parameters satisfy the following conditions:

- There exists a strict lattice x such that $\mathcal{P}[x]$ and the carrier of $x \in \mathcal{A}$,
- Let a, b, c be lattices. Suppose $\mathcal{P}[a]$ and $\mathcal{P}[b]$ and $\mathcal{P}[c]$. Let f be a map from a into b and g be a map from b into c. If $\mathcal{Q}[a, b, f]$ and $\mathcal{Q}[b, c, g]$, then $\mathcal{Q}[a, c, g \cdot f]$, and
- For every lattice a such that $\mathcal{P}[a]$ holds $\mathcal{Q}[a, a, \mathrm{id}_a]$.

The scheme *CLCatUniq1* deals with a non empty set $\mathcal{A}$ and a ternary predicate $\mathcal{P}$, and states that:

Let C_1, C_2 be lattice-wise categories. Suppose that
(i) the carrier of $C_1 = \mathcal{A}$,
(ii) for all objects a, b of C_1 and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b\rangle$ iff $\mathcal{P}[a, b, f]$,
(iii) the carrier of $C_2 = \mathcal{A}$, and
(iv) for all objects a, b of C_2 and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b\rangle$ iff $\mathcal{P}[a, b, f]$.

Then the category structure of C_1 = the category structure of C_2

for all values of the parameters.

The scheme *CLCatUniq2* deals with a non empty set $\mathcal{A}$, a unary predicate $\mathcal{P}$, and a ternary predicate $\mathcal{Q}$, and states that:

Let C_1, C_2 be lattice-wise categories. Suppose that
(i) for every lattice x holds x is an object of C_1 iff x is strict and $\mathcal{P}[x]$ and the carrier of $x \in \mathcal{A}$,
(ii) for all objects a, b of C_1 and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b\rangle$ iff $\mathcal{Q}[a, b, f]$,
(iii) for every lattice x holds x is an object of C_2 iff x is strict and $\mathcal{P}[x]$ and the carrier of $x \in \mathcal{A}$, and

(iv) for all objects a, b of C_2 and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b \rangle$ iff $\mathcal{Q}[a, b, f]$.

Then the category structure of C_1 = the category structure of C_2

for all values of the parameters.

The scheme *CLCovariantFunctorEx* deals with lattice-wise categories $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a lattice, a ternary functor $\mathcal{G}$ yielding a function, and two ternary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

There exists a covariant strict functor F from $\mathcal{A}$ to $\mathcal{B}$ such that

(i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(\mathbb{L}_a)$, and

(ii) for all objects a, b of $\mathcal{A}$ such that $\langle a, b \rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{G}(\mathbb{L}_a, \mathbb{L}_b, {}^{@}f)$

provided the parameters meet the following conditions:

- Let a, b be lattices and f be a map from a into b. Then $f \in$ (the arrows of $\mathcal{A}$)(a, b) if and only if $a \in$ the carrier of $\mathcal{A}$ and $b \in$ the carrier of $\mathcal{A}$ and $\mathcal{P}[a, b, f]$,
- Let a, b be lattices and f be a map from a into b. Then $f \in$ (the arrows of $\mathcal{B}$)(a, b) if and only if $a \in$ the carrier of $\mathcal{B}$ and $b \in$ the carrier of $\mathcal{B}$ and $\mathcal{Q}[a, b, f]$,
- For every lattice a such that $a \in$ the carrier of $\mathcal{A}$ holds $\mathcal{F}(a) \in$ the carrier of $\mathcal{B}$,
- Let a, b be lattices and f be a map from a into b. If $\mathcal{P}[a, b, f]$, then $\mathcal{G}(a, b, f)$ is a map from $\mathcal{F}(a)$ into $\mathcal{F}(b)$ and $\mathcal{Q}[\mathcal{F}(a), \mathcal{F}(b), \mathcal{G}(a, b, f)]$,
- For every lattice a such that $a \in$ the carrier of $\mathcal{A}$ holds $\mathcal{G}(a, a, \mathrm{id}_a) = \mathrm{id}_{\mathcal{F}(a)}$, and
- Let a, b, c be lattices, f be a map from a into b, and g be a map from b into c. If $\mathcal{P}[a, b, f]$ and $\mathcal{P}[b, c, g]$, then $\mathcal{G}(a, c, g \cdot f) = \mathcal{G}(b, c, g) \cdot \mathcal{G}(a, b, f)$.

The scheme *CLContravariantFunctorEx* deals with lattice-wise categories $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a lattice, a ternary functor $\mathcal{G}$ yielding a function, and two ternary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

There exists a contravariant strict functor F from $\mathcal{A}$ to $\mathcal{B}$ such that

(i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(\mathbb{L}_a)$, and

(ii) for all objects a, b of $\mathcal{A}$ such that $\langle a, b \rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{G}(\mathbb{L}_a, \mathbb{L}_b, {}^{@}f)$

provided the parameters satisfy the following conditions:

- Let a, b be lattices and f be a map from a into b. Then $f \in$ (the arrows of $\mathcal{A}$)(a, b) if and only if $a \in$ the carrier of $\mathcal{A}$ and $b \in$ the carrier of $\mathcal{A}$ and $\mathcal{P}[a, b, f]$,

- Let a, b be lattices and f be a map from a into b. Then $f \in$ (the arrows of $\mathcal{B}$)(a, b) if and only if $a \in$ the carrier of $\mathcal{B}$ and $b \in$ the carrier of $\mathcal{B}$ and $\mathcal{Q}[a, b, f]$,
- For every lattice a such that $a \in$ the carrier of $\mathcal{A}$ holds $\mathcal{F}(a) \in$ the carrier of $\mathcal{B}$,
- Let a, b be lattices and f be a map from a into b. If $\mathcal{P}[a, b, f]$, then $\mathcal{G}(a, b, f)$ is a map from $\mathcal{F}(b)$ into $\mathcal{F}(a)$ and $\mathcal{Q}[\mathcal{F}(b), \mathcal{F}(a), \mathcal{G}(a, b, f)]$,
- For every lattice a such that $a \in$ the carrier of $\mathcal{A}$ holds $\mathcal{G}(a, a, \mathrm{id}_a) = \mathrm{id}_{\mathcal{F}(a)}$, and
- Let a, b, c be lattices, f be a map from a into b, and g be a map from b into c. If $\mathcal{P}[a, b, f]$ and $\mathcal{P}[b, c, g]$, then $\mathcal{G}(a, c, g \cdot f) = \mathcal{G}(a, b, f) \cdot \mathcal{G}(b, c, g)$.

The scheme *CLCatIsomorphism* deals with lattice-wise categories $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a lattice, a ternary functor $\mathcal{G}$ yielding a function, and two ternary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\mathcal{A}$ and $\mathcal{B}$ are isomorphic

provided the parameters meet the following conditions:

- Let a, b be lattices and f be a map from a into b. Then $f \in$ (the arrows of $\mathcal{A}$)(a, b) if and only if $a \in$ the carrier of $\mathcal{A}$ and $b \in$ the carrier of $\mathcal{A}$ and $\mathcal{P}[a, b, f]$,
- Let a, b be lattices and f be a map from a into b. Then $f \in$ (the arrows of $\mathcal{B}$)(a, b) if and only if $a \in$ the carrier of $\mathcal{B}$ and $b \in$ the carrier of $\mathcal{B}$ and $\mathcal{Q}[a, b, f]$,
- There exists a covariant functor F from $\mathcal{A}$ to $\mathcal{B}$ such that
  (i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(a)$, and
  (ii) for all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{G}(a, b, f)$,
- For all lattices a, b such that $a \in$ the carrier of $\mathcal{A}$ and $b \in$ the carrier of $\mathcal{A}$ holds if $\mathcal{F}(a) = \mathcal{F}(b)$, then $a = b$,
- For all lattices a, b and for all maps f, g from a into b such that $\mathcal{P}[a, b, f]$ and $\mathcal{P}[a, b, g]$ holds if $\mathcal{G}(a, b, f) = \mathcal{G}(a, b, g)$, then $f = g$, and
- Let a, b be lattices and f be a map from a into b. Suppose $\mathcal{Q}[a, b, f]$. Then there exist lattices c, d and there exists a map g from c into d such that $c \in$ the carrier of $\mathcal{A}$ and $d \in$ the carrier of $\mathcal{A}$ and $\mathcal{P}[c, d, g]$ and $a = \mathcal{F}(c)$ and $b = \mathcal{F}(d)$ and $f = \mathcal{G}(c, d, g)$.

The scheme *CLCatAntiIsomorphism* deals with lattice-wise categories $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a lattice, a ternary functor $\mathcal{G}$ yielding a function, and two ternary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\mathcal{A}$, $\mathcal{B}$ are anti-isomorphic

provided the following conditions are met:

- Let a, b be lattices and f be a map from a into b. Then $f \in$ (the arrows of $\mathcal{A}$)(a, b) if and only if $a \in$ the carrier of $\mathcal{A}$ and $b \in$ the carrier of $\mathcal{A}$ and $\mathcal{P}[a, b, f]$,
- Let a, b be lattices and f be a map from a into b. Then $f \in$ (the arrows of $\mathcal{B}$)(a, b) if and only if $a \in$ the carrier of $\mathcal{B}$ and $b \in$ the carrier of $\mathcal{B}$ and $\mathcal{Q}[a, b, f]$,
- There exists a contravariant functor F from $\mathcal{A}$ to $\mathcal{B}$ such that
  (i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(a)$, and
  (ii) for all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{G}(a, b, f)$,
- For all lattices a, b such that $a \in$ the carrier of $\mathcal{A}$ and $b \in$ the carrier of $\mathcal{A}$ holds if $\mathcal{F}(a) = \mathcal{F}(b)$, then $a = b$,
- For all lattices a, b and for all maps f, g from a into b such that $\mathcal{G}(a, b, f) = \mathcal{G}(a, b, g)$ holds $f = g$, and
- Let a, b be lattices and f be a map from a into b. Suppose $\mathcal{Q}[a, b, f]$. Then there exist lattices c, d and there exists a map g from c into d such that $c \in$ the carrier of $\mathcal{A}$ and $d \in$ the carrier of $\mathcal{A}$ and $\mathcal{P}[c, d, g]$ and $b = \mathcal{F}(c)$ and $a = \mathcal{F}(d)$ and $f = \mathcal{G}(c, d, g)$.

2. Equivalence of Lattice-wise Categories

Let C be a lattice-wise category. We say that C has all isomorphisms if and only if:

(Def. 8) For all objects a, b of C and for every map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ such that f is isomorphic holds $f \in \langle a, b\rangle$.

One can verify that there exists a strict lattice-wise category which has all isomorphisms.

The following propositions are true:

(4) Let C be a lattice-wise category with all isomorphisms, a, b be objects of C, and f be a morphism from a to b. If ${}^{@}f$ is isomorphic, then f is iso.

(5) Let C be a lattice-wise category and a, b be objects of C. Suppose $\langle a, b\rangle \neq \emptyset$ and $\langle b, a\rangle \neq \emptyset$. Let f be a morphism from a to b. If f is iso, then ${}^{@}f$ is isomorphic.

The scheme *CLCatEquivalence* deals with lattice-wise categories $\mathcal{A}$, $\mathcal{B}$, two unary functors $\mathcal{F}$ and $\mathcal{G}$ yielding lattices, two ternary functors $\mathcal{H}$ and $\mathcal{I}$ yielding functions, two unary functors $\mathcal{A}$ and $\mathcal{B}$ yielding functions, and two ternary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\mathcal{A}$ and $\mathcal{B}$ are equivalent

provided the parameters satisfy the following conditions:

- For all objects a, b of $\mathcal{A}$ and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b\rangle$ iff $\mathcal{P}[\mathbb{L}_a, \mathbb{L}_b, f]$,
- For all objects a, b of $\mathcal{B}$ and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b\rangle$ iff $\mathcal{Q}[\mathbb{L}_a, \mathbb{L}_b, f]$,
- There exists a covariant functor F from $\mathcal{A}$ to $\mathcal{B}$ such that
  (i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(a)$, and
  (ii) for all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{H}(a, b, f)$,
- There exists a covariant functor G from $\mathcal{B}$ to $\mathcal{A}$ such that
  (i) for every object a of $\mathcal{B}$ holds $G(a) = \mathcal{G}(a)$, and
  (ii) for all objects a, b of $\mathcal{B}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $G(f) = \mathcal{I}(a, b, f)$,
- Let a be a lattice. Suppose $a \in$ the carrier of $\mathcal{A}$. Then there exists a monotone map f from $\mathcal{G}(\mathcal{F}(a))$ into a such that $f = \mathcal{A}(a)$ and f is isomorphic and $\mathcal{P}[\mathcal{G}(\mathcal{F}(a)), a, f]$ and $\mathcal{P}[a, \mathcal{G}(\mathcal{F}(a)), f^{-1}]$,
- Let a be a lattice. Suppose $a \in$ the carrier of $\mathcal{B}$. Then there exists a monotone map f from a into $\mathcal{F}(\mathcal{G}(a))$ such that $f = \mathcal{B}(a)$ and f is isomorphic and $\mathcal{Q}[a, \mathcal{F}(\mathcal{G}(a)), f]$ and $\mathcal{Q}[\mathcal{F}(\mathcal{G}(a)), a, f^{-1}]$,
- For all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $\mathcal{A}(b) \cdot \mathcal{I}(\mathcal{F}(a), \mathcal{F}(b), \mathcal{H}(a, b, f)) = ({}^{@}f) \cdot \mathcal{A}(a)$, and
- For all objects a, b of $\mathcal{B}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $\mathcal{H}(\mathcal{G}(a), \mathcal{G}(b), \mathcal{I}(a, b, f)) \cdot \mathcal{B}(a) = \mathcal{B}(b) \cdot ({}^{@}f)$.

3. *UPS* Category

Let R be a binary relation. We say that R is upper-bounded if and only if:

(Def. 9) There exists x such that for every y such that $y \in \operatorname{field} R$ holds $\langle y, x\rangle \in R$.

Let us note that every binary relation which is well-ordering is also reflexive, transitive, antisymmetric, connected, and well founded.

Let us mention that there exists a binary relation which is well-ordering.

Next we state the proposition

(6) Let f be an one-to-one function and R be a binary relation. Then $\langle x, y\rangle \in f \cdot R \cdot f^{-1}$ if and only if $x \in \operatorname{dom} f$ and $y \in \operatorname{dom} f$ and $\langle f(x), f(y)\rangle \in R$.

Let f be an one-to-one function and let R be a reflexive binary relation. Note that $f \cdot R \cdot f^{-1}$ is reflexive.

Let f be an one-to-one function and let R be an antisymmetric binary relation. Note that $f \cdot R \cdot f^{-1}$ is antisymmetric.

Let f be an one-to-one function and let R be a transitive binary relation. Note that $f \cdot R \cdot f^{-1}$ is transitive.

Next we state the proposition

(7) Let X be a set and A be an ordinal number. If $X \approx A$, then there exists an order R in X such that R well orders X and $\overline{R} = A$.

Let X be a non empty set. Observe that there exists an order in X which is upper-bounded and well-ordering.

Next we state four propositions:

(8) Let P be a reflexive non empty relational structure. Then P is upper-bounded if and only if the internal relation of P is upper-bounded.

(9) Let P be an upper-bounded non empty poset. Suppose the internal relation of P is well-ordering. Then P is connected, complete, and continuous.

(10) Let P be an upper-bounded non empty poset. Suppose the internal relation of P is well-ordering. Let x, y be elements of P. If $y < x$, then there exists an element z of P such that z is compact and $y \leqslant z$ and $z \leqslant x$.

(11) Let P be an upper-bounded non empty poset. If the internal relation of P is well-ordering, then P is algebraic.

Let X be a non empty set and let R be an upper-bounded well-ordering order in X. Observe that $\langle X, R\rangle$ is complete connected continuous and algebraic.

Let us observe that every set which is non trivial has a non-empty element.

Let W be a non empty set. Let us assume that there exists an element w of W such that w is non empty. The functor UPS_W yielding a lattice-wise strict category is defined by the conditions (Def. 10).

(Def. 10)(i) For every lattice x holds x is an object of UPS_W iff x is strict and complete and the carrier of $x \in W$, and

(ii) for all objects a, b of UPS_W and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b\rangle$ iff f is directed-sups-preserving.

Let W be a set with a non-empty element. Observe that UPS_W has complete lattices and all isomorphisms.

One can prove the following four propositions:

(12) For every set W with a non-empty element holds the carrier of $UPS_W \subseteq$ POSETS(W).

(13) Let W be a set with a non-empty element and given x. Then x is an object of UPS_W if and only if x is a complete lattice and $x \in$ POSETS(W).

(14) Let W be a set with a non-empty element and L be a lattice. Suppose the carrier of $L \in W$. Then L is an object of UPS_W if and only if L is strict and complete.

(15) Let W be a set with a non-empty element, a, b be objects of UPS_W, and f be a set. Then $f \in \langle a, b \rangle$ if and only if f is a directed-sups-preserving map from $\mathbb{L}_a$ into $\mathbb{L}_b$.

Let W be a set with a non-empty element and let a, b be objects of UPS_W. Observe that $\langle a, b \rangle$ is non empty.

4. Lattice-wise Subcategories

Next we state the proposition

(16) Let A be a category, B be a non empty subcategory of A, a be an object of A, and b be an object of B. If $b = a$, then the carrier of $b =$ the carrier of a.

Let A be a set-id-inheriting category. Observe that every non empty subcategory of A is set-id-inheriting.

Let A be a para-functional category. One can verify that every non empty subcategory of A is para-functional.

Let A be a semi-functional category. Note that every non empty transitive substructure of A is semi-functional.

Let A be a carrier-underlaid category. Note that every non empty subcategory of A is carrier-underlaid.

Let A be a lattice-wise category. Observe that every non empty subcategory of A is lattice-wise.

Let A be a lattice-wise category with all isomorphisms. Observe that every non empty subcategory of A which is full has all isomorphisms.

Let A be a category with complete lattices. One can check that every non empty subcategory of A has complete lattices.

Let W be a set with a non-empty element. The functor $CONT_W$ yielding a strict full non empty subcategory of UPS_W is defined by:

(Def. 11) For every object a of UPS_W holds a is an object of $CONT_W$ iff $\mathbb{L}_a$ is continuous.

Let W be a set with a non-empty element. The functor ALG_W yielding a strict full non empty subcategory of $CONT_W$ is defined by:

(Def. 12) For every object a of $CONT_W$ holds a is an object of ALG_W iff $\mathbb{L}_a$ is algebraic.

The following four propositions are true:

(17) Let W be a set with a non-empty element and L be a lattice. Suppose the carrier of $L \in W$. Then L is an object of $CONT_W$ if and only if L is strict, complete, and continuous.

(18) Let W be a set with a non-empty element and L be a lattice. Suppose the carrier of $L \in W$. Then L is an object of ALG_W if and only if L is strict, complete, and algebraic.

(19) Let W be a set with a non-empty element, a, b be objects of $CONT_W$, and f be a set. Then $f \in \langle a, b \rangle$ if and only if f is a directed-sups-preserving map from $\mathbb{L}_a$ into $\mathbb{L}_b$.

(20) Let W be a set with a non-empty element, a, b be objects of ALG_W, and f be a set. Then $f \in \langle a, b \rangle$ if and only if f is a directed-sups-preserving map from $\mathbb{L}_a$ into $\mathbb{L}_b$.

Let W be a set with a non-empty element and let a, b be objects of $CONT_W$. One can check that $\langle a, b \rangle$ is non empty.

Let W be a set with a non-empty element and let a, b be objects of ALG_W. One can check that $\langle a, b \rangle$ is non empty.

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Duality Based on the Galois Connection. Part I

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Summary. In the paper, we investigate the duality of categories of complete lattices and maps preserving suprema or infima according to [12, p. 179–183; 1.1–1.12]. The duality is based on the concept of the Galois connection.

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The papers [20], [8], [19], [21], [9], [16], [1], [23], [17], [25], [24], [18], [11], [14], [27], [22], [13], [3], [10], [4], [15], [7], [6], [2], [26], and [5] provide the terminology and notation for this paper.

1. Infs-preserving and Sups-preserving Maps

Let S, T be complete lattices. One can check that there exists a connection between S and T which is Galois.

Next we state the proposition

(1) Let S, T, S', T' be non empty relational structures. Suppose that
  (i) the relational structure of S = the relational structure of S', and
  (ii) the relational structure of T = the relational structure of T'.

  Let c be a connection between S and T and c' be a connection between S' and T'. If $c = c'$, then if c is Galois, then c' is Galois.

Let S, T be lattices and let g be a map from S into T. Let us assume that S is complete and T is complete and g is infs-preserving. The lower adjoint of g is a map from T into S and is defined as follows:

(Def. 1) $\langle g$, the lower adjoint of $g\rangle$ is Galois.

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Let S, T be lattices and let d be a map from T into S. Let us assume that S is complete and T is complete and d is sups-preserving. The upper adjoint of d is a map from S into T and is defined as follows:

(Def. 2) $\langle$the upper adjoint of d, $d\rangle$ is Galois.

Let S, T be complete lattices and let g be an infs-preserving map from S into T. One can verify that the lower adjoint of g is lower adjoint.

Let S, T be complete lattices and let d be a sups-preserving map from T into S. One can check that the upper adjoint of d is upper adjoint.

The following two propositions are true:

(2) Let S, T be complete lattices, g be an infs-preserving map from S into T, and t be an element of T. Then (the lower adjoint of g)$(t) = \inf(g^{-1}(\uparrow t))$.

(3) Let S, T be complete lattices, d be a sups-preserving map from T into S, and s be an element of S. Then (the upper adjoint of d)$(s) = \sup(d^{-1}(\downarrow s))$.

Let S, T be relational structures and let f be a function from the carrier of S into the carrier of T. The functor $f^{\rm op}$ yielding a map from $S^{\rm op}$ into $T^{\rm op}$ is defined as follows:

(Def. 3) $f^{\rm op} = f$.

Let S, T be complete lattices and let g be an infs-preserving map from S into T. One can verify that $g^{\rm op}$ is lower adjoint.

Let S, T be complete lattices and let d be a sups-preserving map from S into T. Observe that $d^{\rm op}$ is upper adjoint.

We now state several propositions:

(4) Let S, T be complete lattices and g be an infs-preserving map from S into T. Then the lower adjoint of $g =$ the upper adjoint of $g^{\rm op}$.

(5) Let S, T be complete lattices and d be a sups-preserving map from S into T. Then the lower adjoint of $d^{\rm op} =$ the upper adjoint of d.

(6) For every non empty relational structure L holds $\langle {\rm id}_L, {\rm id}_L\rangle$ is Galois.

(7) For every complete lattice L holds the lower adjoint of ${\rm id}_L = {\rm id}_L$ and the upper adjoint of ${\rm id}_L = {\rm id}_L$.

(8) Let L_1, L_2, L_3 be complete lattices, g_1 be an infs-preserving map from L_1 into L_2, and g_2 be an infs-preserving map from L_2 into L_3. Then the lower adjoint of $g_2 \cdot g_1 =$ (the lower adjoint of g_1) $\cdot$ (the lower adjoint of g_2).

(9) Let L_1, L_2, L_3 be complete lattices, d_1 be a sups-preserving map from L_1 into L_2, and d_2 be a sups-preserving map from L_2 into L_3. Then the upper adjoint of $d_2 \cdot d_1 =$ (the upper adjoint of d_1) $\cdot$ (the upper adjoint of d_2).

(10) Let S, T be complete lattices and g be an infs-preserving map from S into T. Then the upper adjoint of the lower adjoint of $g = g$.

(11) Let S, T be complete lattices and d be a sups-preserving map from S into T. Then the lower adjoint of the upper adjoint of $d = d$.

(12) Let C be a non empty category structure and a, b, f be sets. Suppose $f \in$ (the arrows of C)(a, b). Then there exist objects o_1, o_2 of C such that $o_1 = a$ and $o_2 = b$ and $f \in \langle o_1, o_2 \rangle$ and f is a morphism from o_1 to o_2.

Let W be a non empty set. Let us assume that there exists an element w of W such that w is non empty. The functor INF_W yields a lattice-wise strict category and is defined by the conditions (Def. 4).

(Def. 4)(i) For every lattice x holds x is an object of INF_W iff x is strict and complete and the carrier of $x \in W$, and

(ii) for all objects a, b of INF_W and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b \rangle$ iff f is infs-preserving.

Let W be a non empty set. Let us assume that there exists an element w of W such that w is non empty. The functor SUP_W yields a lattice-wise strict category and is defined by the conditions (Def. 5).

(Def. 5)(i) For every lattice x holds x is an object of SUP_W iff x is strict and complete and the carrier of $x \in W$, and

(ii) for all objects a, b of SUP_W and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b \rangle$ iff f is sups-preserving.

Let W be a set with a non-empty element. Observe that INF_W has complete lattices and SUP_W has complete lattices.

One can prove the following propositions:

(13) Let W be a set with a non-empty element and L be a lattice. Then L is an object of INF_W if and only if L is strict and complete and the carrier of $L \in W$.

(14) Let W be a set with a non-empty element, a, b be objects of INF_W, and f be a set. Then $f \in \langle a, b \rangle$ if and only if f is an infs-preserving map from $\mathbb{L}_a$ into $\mathbb{L}_b$.

(15) Let W be a set with a non-empty element and L be a lattice. Then L is an object of SUP_W if and only if L is strict and complete and the carrier of $L \in W$.

(16) Let W be a set with a non-empty element, a, b be objects of SUP_W, and f be a set. Then $f \in \langle a, b \rangle$ if and only if f is a sups-preserving map from $\mathbb{L}_a$ into $\mathbb{L}_b$.

(17) For every set W with a non-empty element holds the carrier of $INF_W =$ the carrier of SUP_W.

Let W be a set with a non-empty element. The functor $\mathrm{LowerAdj}_W$ yields a contravariant strict functor from INF_W to SUP_W and is defined by the conditions (Def. 6).

(Def. 6)(i) For every object a of INF_W holds $\mathrm{LowerAdj}_W(a) = \mathbb{L}_a$, and

(ii) for all objects a, b of INF_W such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $\mathrm{LowerAdj}_W(f) =$ the lower adjoint of ${}^{@}f$.

The functor $\mathrm{UpperAdj}_W$ yields a contravariant strict functor from SUP_W to INF_W and is defined by the conditions (Def. 7).

(Def. 7)(i) For every object a of SUP_W holds $\mathrm{UpperAdj}_W(a) = \mathbb{L}_a$, and

(ii) for all objects a, b of SUP_W such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $\mathrm{UpperAdj}_W(f) =$ the upper adjoint of ${}^{@}f$.

Let W be a set with a non-empty element. Observe that $\mathrm{LowerAdj}_W$ is bijective and $\mathrm{UpperAdj}_W$ is bijective.

We now state several propositions:

(18) For every set W with a non-empty element holds $(\mathrm{LowerAdj}_W)^{-1} = \mathrm{UpperAdj}_W$ and $(\mathrm{UpperAdj}_W)^{-1} = \mathrm{LowerAdj}_W$.

(19) For every set W with a non-empty element holds $\mathrm{LowerAdj}_W \cdot \mathrm{UpperAdj}_W = \mathrm{id}_{\mathit{SUP}_W}$ and $\mathrm{UpperAdj}_W \cdot \mathrm{LowerAdj}_W = \mathrm{id}_{\mathit{INF}_W}$.

(20) For every set W with a non-empty element holds INF_W, SUP_W are anti-isomorphic.

(21) For every set W with a non-empty element holds INF_W and SUP_W are anti-isomorphic under $\mathrm{LowerAdj}_W$.

(22) For every set W with a non-empty element holds SUP_W and INF_W are anti-isomorphic under $\mathrm{UpperAdj}_W$.

2. Scott Continuous Maps and Continuous Lattices

Next we state the proposition

(23) Let S, T be complete lattices and g be an infs-preserving map from S into T. Then g is directed-sups-preserving if and only if for every Scott topological augmentation X of T and for every Scott topological augmentation Y of S and for every open subset V of X holds $\uparrow$((the lower adjoint of $g)^{\circ}V$) is an open subset of Y.

Let S, T be non empty reflexive relational structures and let f be a map from S into T. We say that f is waybelow-preserving if and only if:

(Def. 8) For all elements x, y of S such that $x \ll y$ holds $f(x) \ll f(y)$.

We now state two propositions:

(24) Let S, T be complete lattices and g be an infs-preserving map from S into T. Suppose g is directed-sups-preserving. Then the lower adjoint of g is waybelow-preserving.

(25) Let S be a complete lattice, T be a complete continuous lattice, and g be an infs-preserving map from S into T. Suppose the lower adjoint of g is waybelow-preserving. Then g is directed-sups-preserving.

Let S, T be topological spaces and let f be a map from S into T. We say that f is relatively open if and only if:

(Def. 9) For every open subset V of S holds $f^\circ V$ is an open subset of $T \restriction \operatorname{rng} f$.

One can prove the following propositions:

(26) Let X, Y be non empty topological spaces and d be a map from X into Y. Then d is relatively open if and only if d° is open.

(27) Let S, T be complete lattices, g be an infs-preserving map from S into T, X be a Scott topological augmentation of T, Y be a Scott topological augmentation of S, and V be an open subset of X. Then (the lower adjoint of $g)^\circ V = \operatorname{rng}$ (the lower adjoint of $g) \cap \uparrow$((the lower adjoint of $g)^\circ V$).

(28) Let S, T be complete lattices, g be an infs-preserving map from S into T, X be a Scott topological augmentation of T, and Y be a Scott topological augmentation of S. Suppose that for every open subset V of X holds $\uparrow$((the lower adjoint of $g)^\circ V$) is an open subset of Y. Let d be a map from X into Y. If $d =$ the lower adjoint of g, then d is relatively open.

Let X, Y be complete lattices and let f be a sups-preserving map from X into Y. One can check that $\operatorname{Im} f$ is complete.

Next we state four propositions:

(29) Let S, T be complete lattices, g be an infs-preserving map from S into T, X be a Scott topological augmentation of T, Y be a Scott topological augmentation of S, Z be a Scott topological augmentation of Im (the lower adjoint of g), d be a map from X into Y, and d' be a map from X into Z. Suppose $d =$ the lower adjoint of g and $d' = d$. If d is relatively open, then d' is open.

(30) Let T_1, T_2, S_1, S_2 be topological structures. Suppose that
  (i) the topological structure of $T_1 =$ the topological structure of T_2, and
  (ii) the topological structure of $S_1 =$ the topological structure of S_2.

  If S_1 is a subspace of T_1, then S_2 is a subspace of T_2.

(31) For every topological structure T holds $T \restriction \Omega_T =$ the topological structure of T.

(32) Let S, T be complete lattices and g be an infs-preserving map from S into T. Suppose g is one-to-one. Let X be a Scott topological augmentation of T, Y be a Scott topological augmentation of S, and d be a map from X into Y. Suppose $d =$ the lower adjoint of g. Then g is directed-sups-preserving if and only if d is open.

Let X be a complete lattice and let f be a projection map from X into X. One can verify that $\operatorname{Im} f$ is complete.

We now state a number of propositions:

(33) Let L be a complete lattice and k be a kernel map from L into L. Then
  (i) k° is infs-preserving,

(ii) $k_\circ$ is sups-preserving,

(iii) the lower adjoint of $k^\circ = k_\circ$, and

(iv) the upper adjoint of $k_\circ = k^\circ$.

(34) Let L be a complete lattice and k be a kernel map from L into L. Then k is directed-sups-preserving if and only if k° is directed-sups-preserving.

(35) Let L be a complete lattice and k be a kernel map from L into L. Then k is directed-sups-preserving if and only if for every Scott topological augmentation X of $\operatorname{Im} k$ and for every Scott topological augmentation Y of L and for every subset V of L such that V is an open subset of X holds $\uparrow V$ is an open subset of Y.

(36) Let L be a complete lattice, S be a sups-inheriting non empty full relational substructure of L, x, y be elements of L, and a, b be elements of S. If $a = x$ and $b = y$, then if $x \ll y$, then $a \ll b$.

(37) Let L be a complete lattice and k be a kernel map from L into L. Suppose k is directed-sups-preserving. Let x, y be elements of L and a, b be elements of $\operatorname{Im} k$. If $a = x$ and $b = y$, then $x \ll y$ iff $a \ll b$.

(38) Let L be a complete lattice and k be a kernel map from L into L. Suppose that

(i) $\operatorname{Im} k$ is continuous, and

(ii) for all elements x, y of L and for all elements a, b of $\operatorname{Im} k$ such that $a = x$ and $b = y$ holds $x \ll y$ iff $a \ll b$.

Then k is directed-sups-preserving.

(39) Let L be a complete lattice and c be a closure map from L into L. Then

(i) c° is sups-preserving,

(ii) $c_\circ$ is infs-preserving,

(iii) the upper adjoint of $c^\circ = c_\circ$, and

(iv) the lower adjoint of $c_\circ = c^\circ$.

(40) Let L be a complete lattice and c be a closure map from L into L. Then $\operatorname{Im} c$ is directed-sups-inheriting if and only if $c_\circ$ is directed-sups-preserving.

(41) Let L be a complete lattice and c be a closure map from L into L. Then $\operatorname{Im} c$ is directed-sups-inheriting if and only if for every Scott topological augmentation X of $\operatorname{Im} c$ and for every Scott topological augmentation Y of L and for every map f from Y into X such that $f = c$ holds f is open.

(42) Let L be a complete lattice and c be a closure map from L into L. If $\operatorname{Im} c$ is directed-sups-inheriting, then c° is waybelow-preserving.

(43) Let L be a continuous complete lattice and c be a closure map from L into L. If c° is waybelow-preserving, then $\operatorname{Im} c$ is directed-sups-inheriting.

3. Duality of Subcategories of INF and SUP

Let W be a non empty set. The functor $INF^{\uparrow}_W$ yielding a strict non empty subcategory of INF_W is defined by the conditions (Def. 10).

(Def. 10)(i) Every object of INF_W is an object of $INF^{\uparrow}_W$, and

(ii) for all objects a, b of INF_W and for all objects a', b' of $INF^{\uparrow}_W$ such that $a' = a$ and $b' = b$ and $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $f \in \langle a', b'\rangle$ iff ${}^{@}f$ is directed-sups-preserving.

Let W be a set with a non-empty element. The functor SUP^{0}_W yields a strict non empty subcategory of SUP_W and is defined by the conditions (Def. 11).

(Def. 11)(i) Every object of SUP_W is an object of SUP^{0}_W, and

(ii) for all objects a, b of SUP_W and for all objects a', b' of SUP^{0}_W such that $a' = a$ and $b' = b$ and $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $f \in \langle a', b'\rangle$ iff the upper adjoint of ${}^{@}f$ is directed-sups-preserving.

The following propositions are true:

(44) Let S be a non empty relational structure, T be a non empty reflexive antisymmetric relational structure, t be an element of T, and X be a non empty subset of S. Then $S \longmapsto t$ preserves sup of X and $S \longmapsto t$ preserves inf of X.

(45) Let S be a non empty relational structure and T be a lower-bounded non empty reflexive antisymmetric relational structure. Then $S \longmapsto \bot_T$ is sups-preserving.

(46) Let S be a non empty relational structure and T be an upper-bounded non empty reflexive antisymmetric relational structure. Then $S \longmapsto \top_T$ is infs-preserving.

Let S be a non empty relational structure and let T be an upper-bounded non empty reflexive antisymmetric relational structure. Observe that $S \longmapsto \top_T$ is directed-sups-preserving and infs-preserving.

Let S be a non empty relational structure and let T be a lower-bounded non empty reflexive antisymmetric relational structure. Observe that $S \longmapsto \bot_T$ is filtered-infs-preserving and sups-preserving.

Let S be a non empty relational structure and let T be an upper-bounded non empty reflexive antisymmetric relational structure. Note that there exists a map from S into T which is directed-sups-preserving and infs-preserving.

Let S be a non empty relational structure and let T be a lower-bounded non empty reflexive antisymmetric relational structure. One can check that there exists a map from S into T which is filtered-infs-preserving and sups-preserving.

Next we state several propositions:

(47) Let W be a set with a non-empty element and L be a lattice. Then L is an object of $INF^{\uparrow}_{W}$ if and only if L is strict and complete and the carrier of $L \in W$.

(48) Let W be a set with a non-empty element, a, b be objects of $INF^{\uparrow}_{W}$, and f be a set. Then $f \in \langle a, b \rangle$ if and only if f is a directed-sups-preserving infs-preserving map from $\mathbb{L}_a$ into $\mathbb{L}_b$.

(49) Let W be a set with a non-empty element and L be a lattice. Then L is an object of SUP^{0}_{W} if and only if L is strict and complete and the carrier of $L \in W$.

(50) Let W be a set with a non-empty element, a, b be objects of SUP^{0}_{W}, and f be a set. Then $f \in \langle a, b \rangle$ if and only if there exists a sups-preserving map g from $\mathbb{L}_a$ into $\mathbb{L}_b$ such that $g = f$ and the upper adjoint of g is directed-sups-preserving.

(51) For every set W with a non-empty element holds $INF^{\uparrow}_{W} = \mathrm{Intersect}(INF_W, UPS_W)$.

Let W be a set with a non-empty element. The functor CL_W yielding a strict full non empty subcategory of $INF^{\uparrow}_{W}$ is defined as follows:

(Def. 12) For every object a of $INF^{\uparrow}_{W}$ holds a is an object of CL_W iff $\mathbb{L}_a$ is continuous.

Let W be a set with a non-empty element. Observe that CL_W has complete lattices.

One can prove the following two propositions:

(52) Let W be a set with a non-empty element and L be a lattice. Suppose the carrier of $L \in W$. Then L is an object of CL_W if and only if L is strict, complete, and continuous.

(53) Let W be a set with a non-empty element, a, b be objects of CL_W, and f be a set. Then $f \in \langle a, b \rangle$ if and only if f is an infs-preserving directed-sups-preserving map from $\mathbb{L}_a$ into $\mathbb{L}_b$.

Let W be a set with a non-empty element. The functor CL^{op}_{W} yields a strict full non empty subcategory of SUP^{0}_{W} and is defined by:

(Def. 13) For every object a of SUP^{0}_{W} holds a is an object of CL^{op}_{W} iff $\mathbb{L}_a$ is continuous.

Next we state several propositions:

(54) Let W be a set with a non-empty element and L be a lattice. Suppose the carrier of $L \in W$. Then L is an object of CL^{op}_{W} if and only if L is strict, complete, and continuous.

(55) Let W be a set with a non-empty element, a, b be objects of CL^{op}_{W}, and f be a set. Then $f \in \langle a, b \rangle$ if and only if there exists a sups-preserving map g from $\mathbb{L}_a$ into $\mathbb{L}_b$ such that $g = f$ and the upper adjoint of g is directed-sups-preserving.

(56) For every set W with a non-empty element holds $INF^{\uparrow}_{W}$ and SUP^{0}_{W} are anti-isomorphic under $\mathrm{LowerAdj}_W$.

(57) For every set W with a non-empty element holds SUP^{0}_{W} and $INF^{\uparrow}_{W}$ are anti-isomorphic under $\mathrm{UpperAdj}_W$.

(58) For every set W with a non-empty element holds CL_W and CL^{op}_W are anti-isomorphic under $\mathrm{LowerAdj}_W$.

(59) For every set W with a non-empty element holds CL^{op}_W and CL_W are anti-isomorphic under $\mathrm{UpperAdj}_W$.

4. Compact Preserving Maps and Sup-semilattices Morphisms

Let S, T be non empty reflexive relational structures and let f be a map from S into T. We say that f is compact-preserving if and only if:

(Def. 14) For every element s of S such that s is compact holds $f(s)$ is compact.

One can prove the following propositions:

(60) Let S, T be complete lattices and d be a sups-preserving map from T into S. If d is waybelow-preserving, then d is compact-preserving.

(61) Let S, T be complete lattices and d be a sups-preserving map from T into S. Suppose T is algebraic and d is compact-preserving. Then d is waybelow-preserving.

(62) Let R, S, T be non empty relational structures, X be a subset of R, f be a map from R into S, and g be a map from S into T. Suppose f preserves sup of X and g preserves sup of $f^{\circ}X$. Then $g \cdot f$ preserves sup of X.

Let S, T be non empty relational structures and let f be a map from S into T. We say that f is finite-sups-preserving if and only if:

(Def. 15) For every finite subset X of S holds f preserves sup of X.

We say that f is bottom-preserving if and only if:

(Def. 16) f preserves sup of $\emptyset_S$.

Next we state the proposition

(63) Let R, S, T be non empty relational structures, f be a map from R into S, and g be a map from S into T. Suppose f is finite-sups-preserving and g is finite-sups-preserving. Then $g \cdot f$ is finite-sups-preserving.

Let S, T be non empty antisymmetric lower-bounded relational structures and let f be a map from S into T. Let us observe that f is bottom-preserving if and only if:

(Def. 17) $f(\perp_S) = \perp_T$.

Let L be a non empty relational structure and let S be a relational substructure of L. We say that S is finite-sups-inheriting if and only if:

(Def. 18) For every finite subset X of S such that $\sup X$ exists in L holds $\bigsqcup_L X \in$ the carrier of S.

We say that S is bottom-inheriting if and only if:

(Def. 19) $\perp_L \in$ the carrier of S.

Let S, T be non empty relational structures. Observe that every map from S into T which is sups-preserving is also bottom-preserving.

Let L be a lower-bounded antisymmetric non empty relational structure. Note that every relational substructure of L which is finite-sups-inheriting is also bottom-inheriting and join-inheriting.

Let L be a non empty relational structure. One can check that every relational substructure of L which is sups-inheriting is also finite-sups-inheriting.

Let S, T be lower-bounded non empty posets. One can verify that there exists a map from S into T which is sups-preserving.

Let L be a lower-bounded antisymmetric non empty relational structure. Observe that every full relational substructure of L which is bottom-inheriting is also non empty and lower-bounded.

Let L be a lower-bounded antisymmetric non empty relational structure. Note that there exists a relational substructure of L which is non empty, sups-inheriting, finite-sups-inheriting, bottom-inheriting, and full.

Next we state the proposition

(64) Let L be a lower-bounded antisymmetric non empty relational structure and S be a non empty bottom-inheriting full relational substructure of L. Then $\perp_S = \perp_L$.

Let L be a lower-bounded non empty poset with l.u.b.'s. Note that every full relational substructure of L which is bottom-inheriting and join-inheriting is also finite-sups-inheriting.

Next we state two propositions:

(65) Let S, T be non empty relational structures and f be a map from S into T. Suppose f is finite-sups-preserving. Then f is join-preserving and bottom-preserving.

(66) Let S, T be lower-bounded posets with l.u.b.'s and f be a map from S into T. Suppose f is join-preserving and bottom-preserving. Then f is finite-sups-preserving.

Let S, T be non empty relational structures. One can check that every map from S into T which is sups-preserving is also finite-sups-preserving and every map from S into T which is finite-sups-preserving is also join-preserving and bottom-preserving.

Let S be a non empty relational structure and let T be a lower-bounded non empty reflexive antisymmetric relational structure. Observe that there exists a map from S into T which is sups-preserving and finite-sups-preserving.

Let L be a lower-bounded non empty poset. One can check that CompactSublatt(L) is lower-bounded.

One can prove the following propositions:

(67) Let S be a relational structure, T be a non empty relational structure, f be a map from S into T, S' be a relational substructure of S, and T' be a relational substructure of T. Suppose f°(the carrier of S') $\subseteq$ the carrier of T'. Then $f{\restriction}$the carrier of S' is a map from S' into T'.

(68) Let S, T be lattices, f be a join-preserving map from S into T, S' be a non empty join-inheriting full relational substructure of S, T' be a non empty join-inheriting full relational substructure of T, and g be a map from S' into T'. If $g = f{\restriction}$the carrier of S', then g is join-preserving.

(69) Let S, T be lower-bounded lattices, f be a finite-sups-preserving map from S into T, S' be a non empty finite-sups-inheriting full relational substructure of S, T' be a non empty finite-sups-inheriting full relational substructure of T, and g be a map from S' into T'. If $g = f{\restriction}$the carrier of S', then g is finite-sups-preserving.

Let L be a complete lattice. One can verify that CompactSublatt(L) is finite-sups-inheriting.

Next we state two propositions:

(70) Let S, T be complete lattices and d be a sups-preserving map from T into S. Then d is compact-preserving if and only if $d{\restriction}$the carrier of CompactSublatt(T) is a finite-sups-preserving map from CompactSublatt(T) into CompactSublatt(S).

(71) Let S, T be complete lattices. Suppose T is algebraic. Let g be an infs-preserving map from S into T. Then g is directed-sups-preserving if and only if (the lower adjoint of g)${\restriction}$the carrier of CompactSublatt(T) is a finite-sups-preserving map from CompactSublatt(T) into CompactSublatt(S).

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Yet Another Construction of Free Algebra

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The notation and terminology used here are introduced in the following papers: [27], [21], [10], [15], [14], [9], [12], [8], [13], [23], [20], [6], [25], [11], [16], [7], [24], [17], [18], [19], [28], [29], [26], [22], [1], [3], [4], [5], and [2].

In this paper X, x, z are sets.

Let S be a non empty non void many sorted signature and let A be a non empty algebra over S. Observe that $\bigcup$(the sorts of A) is non empty.

Let S be a non empty non void many sorted signature and let A be a non empty algebra over S.

(Def. 1) An element of $\bigcup$(the sorts of A) is said to be an element of A.

We now state two propositions:

(1) For every function f such that $X \subseteq \operatorname{dom} f$ and f is one-to-one holds $f^{-1}(f^{\circ}X) = X$.

(2) Let I be a set, A be a many sorted set indexed by I, and F be a many sorted function indexed by I. If F is "1-1" and $A \subseteq \operatorname{dom}_{\kappa} F(\kappa)$, then $F^{-1}(F \circ A) = A$.

Let S be a non void signature and let X be a many sorted set indexed by the carrier of S. The functor $\mathrm{Free}_S(X)$ yields a strict algebra over S and is defined by:

(Def. 2) There exists a subset A of $\mathrm{Free}(X \cup ((\text{the carrier of } S) \longmapsto \{0\}))$ such that $\mathrm{Free}_S(X) = \mathrm{Gen}(A)$ and $A = (\mathrm{Reverse}(X \cup ((\text{the carrier of } S) \longmapsto \{0\})))^{-1}(X)$.

We now state four propositions:

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(3) Let S be a non void signature, X be a non-empty many sorted set indexed by the carrier of S, and s be a sort symbol of S. Then $\langle x, s\rangle \in$ the carrier of DTConMSA(X) if and only if $x \in X(s)$.

(4) Let S be a non void signature, Y be a non-empty many sorted set indexed by the carrier of S, X be a many sorted set indexed by the carrier of S, and s be a sort symbol of S. Then $x \in X(s)$ and $x \in Y(s)$ if and only if the root tree of $\langle x, s\rangle \in ((\mathrm{Reverse}(Y))^{-1}(X))(s)$.

(5) Let S be a non void signature, X be a many sorted set indexed by the carrier of S, and s be a sort symbol of S. If $x \in X(s)$, then the root tree of $\langle x, s\rangle \in$ (the sorts of $\mathrm{Free}_S(X))(s)$.

(6) Let S be a non void signature, X be a many sorted set indexed by the carrier of S, and o be an operation symbol of S. Suppose $\mathrm{Arity}(o) = \emptyset$. Then the root tree of $\langle o$, the carrier of $S\rangle \in$ (the sorts of $\mathrm{Free}_S(X)$)(the result sort of o).

Let S be a non void signature and let X be a non empty yielding many sorted set indexed by the carrier of S. Observe that $\mathrm{Free}_S(X)$ is non empty.

One can prove the following three propositions:

(7) Let S be a non void signature and X be a non-empty many sorted set indexed by the carrier of S. Then x is an element of $\mathrm{Free}(X)$ if and only if x is a term of S over X.

(8) Let S be a non void signature, X be a non-empty many sorted set indexed by the carrier of S, s be a sort symbol of S, and x be a term of S over X. Then $x \in$ (the sorts of $\mathrm{Free}(X))(s)$ if and only if the sort of $x = s$.

(9) Let S be a non void signature and X be a non empty yielding many sorted set indexed by the carrier of S. Then every element of $\mathrm{Free}_S(X)$ is a term of S over $X \cup ((\text{the carrier of } S) \longmapsto \{0\})$.

Let S be a non empty non void many sorted signature and let X be a non empty yielding many sorted set indexed by the carrier of S. Note that every element of $\mathrm{Free}_S(X)$ is relation-like and function-like.

Let S be a non empty non void many sorted signature and let X be a non empty yielding many sorted set indexed by the carrier of S. Note that every element of $\mathrm{Free}_S(X)$ is finite and decorated tree-like.

Let S be a non empty non void many sorted signature and let X be a non empty yielding many sorted set indexed by the carrier of S. Observe that every element of $\mathrm{Free}_S(X)$ is finite-branching.

One can check that every decorated tree is non empty.

Let S be a many sorted signature and let t be a non empty binary relation. The functor $\mathrm{Var}_S\, t$ yields a many sorted set indexed by the carrier of S and is defined as follows:

(Def. 3) For every set s such that $s \in$ the carrier of S holds $(\mathrm{Var}_S\, t)(s) = \{a_{\mathbf{1}}; a$

ranges over elements of $\operatorname{rng} t : a_{\mathbf{2}} = s\}$.

Let S be a many sorted signature, let X be a many sorted set indexed by the carrier of S, and let t be a non empty binary relation. The functor $\mathrm{Var}_X\, t$ yielding a many sorted subset indexed by X is defined by:

(Def. 4) $\mathrm{Var}_X\, t = X \cap \mathrm{Var}_S\, t$.

We now state several propositions:

(10) Let S be a many sorted signature, X be a many sorted set indexed by the carrier of S, t be a non empty binary relation, and V be a many sorted subset indexed by X. Then $V = \mathrm{Var}_X\, t$ if and only if for every set s such that $s \in$ the carrier of S holds $V(s) = X(s) \cap \{a_{\mathbf{1}}; a$ ranges over elements of $\operatorname{rng} t : a_{\mathbf{2}} = s\}$.

(11) Let S be a many sorted signature and s, x be sets. Then

(i) if $s \in$ the carrier of S, then $(\mathrm{Var}_S\,(\text{the root tree of } \langle x, s\rangle))(s) = \{x\}$, and

(ii) for every set s' such that $s' \neq s$ or $s \notin$ the carrier of S holds $(\mathrm{Var}_S\,(\text{the root tree of } \langle x, s\rangle))(s') = \emptyset$.

(12) Let S be a many sorted signature and s be a set. Suppose $s \in$ the carrier of S. Let p be a decorated tree yielding finite sequence. Then $x \in (\mathrm{Var}_S(\langle z,$ the carrier of $S\rangle\text{-tree}(p)))(s)$ if and only if there exists a decorated tree t such that $t \in \operatorname{rng} p$ and $x \in (\mathrm{Var}_S\, t)(s)$.

(13) Let S be a many sorted signature, X be a many sorted set indexed by the carrier of S, and s, x be sets. Then

(i) if $x \in X(s)$, then $(\mathrm{Var}_X\,(\text{the root tree of } \langle x, s\rangle))(s) = \{x\}$, and

(ii) for every set s' such that $s' \neq s$ or $x \notin X(s)$ holds $(\mathrm{Var}_X\,(\text{the root tree of } \langle x, s\rangle))(s') = \emptyset$.

(14) Let S be a many sorted signature, X be a many sorted set indexed by the carrier of S, and s be a set. Suppose $s \in$ the carrier of S. Let p be a decorated tree yielding finite sequence. Then $x \in (\mathrm{Var}_X(\langle z,$ the carrier of $S\rangle\text{-tree}(p)))(s)$ if and only if there exists a decorated tree t such that $t \in \operatorname{rng} p$ and $x \in (\mathrm{Var}_X\, t)(s)$.

(15) Let S be a non void signature, X be a non-empty many sorted set indexed by the carrier of S, and t be a term of S over X. Then $\mathrm{Var}_S\, t \subseteq X$.

Let S be a non void signature, let X be a non-empty many sorted set indexed by the carrier of S, and let t be a term of S over X. The functor Var_t yielding a many sorted subset indexed by X is defined by:

(Def. 5) $\mathrm{Var}_t = \mathrm{Var}_S\, t$.

The following proposition is true

(16) Let S be a non void signature, X be a non-empty many sorted set indexed by the carrier of S, and t be a term of S over X. Then $\mathrm{Var}_t = \mathrm{Var}_X\, t$.

Let S be a non void signature, let Y be a non-empty many sorted set indexed by the carrier of S, and let X be a many sorted set indexed by the carrier of S. The functor S-$\mathrm{Terms}^Y(X)$ yielding a subset of $\mathrm{Free}(Y)$ is defined as follows:

(Def. 6) For every sort symbol s of S holds $(S\text{-Terms}^Y(X))(s) = \{t; t$ ranges over terms of S over Y: the sort of $t = s \ \wedge\ \mathrm{Var}_t \subseteq X\}$.

One can prove the following propositions:

(17) Let S be a non void signature, Y be a non-empty many sorted set indexed by the carrier of S, X be a many sorted set indexed by the carrier of S, and s be a sort symbol of S. If $x \in (S\text{-Terms}^Y(X))(s)$, then x is a term of S over Y.

(18) Let S be a non void signature, Y be a non-empty many sorted set indexed by the carrier of S, X be a many sorted set indexed by the carrier of S, t be a term of S over Y, and s be a sort symbol of S. If $t \in (S\text{-Terms}^Y(X))(s)$, then the sort of $t = s$ and $\mathrm{Var}_t \subseteq X$.

(19) Let S be a non void signature, Y be a non-empty many sorted set indexed by the carrier of S, X be a many sorted set indexed by the carrier of S, and s be a sort symbol of S. Then the root tree of $\langle x,\ s\rangle \in (S\text{-Terms}^Y(X))(s)$ if and only if $x \in X(s)$ and $x \in Y(s)$.

(20) Let S be a non void signature, Y be a non-empty many sorted set indexed by the carrier of S, X be a many sorted set indexed by the carrier of S, o be an operation symbol of S, and p be an argument sequence of $\mathrm{Sym}(o, Y)$. Then $\mathrm{Sym}(o, Y)\text{-tree}(p) \in (S\text{-Terms}^Y(X))$(the result sort of o) if and only if $\mathrm{rng}\, p \subseteq \bigcup(S\text{-Terms}^Y(X))$.

(21) Let S be a non void signature, X be a non-empty many sorted set indexed by the carrier of S, and A be a subset of $\mathrm{Free}(X)$. Then A is operations closed if and only if for every operation symbol o of S and for every argument sequence p of $\mathrm{Sym}(o, X)$ such that $\mathrm{rng}\, p \subseteq \bigcup A$ holds $\mathrm{Sym}(o, X)\text{-tree}(p) \in A$(the result sort of o).

(22) Let S be a non void signature, Y be a non-empty many sorted set indexed by the carrier of S, and X be a many sorted set indexed by the carrier of S. Then S-$\mathrm{Terms}^Y(X)$ is operations closed.

(23) Let S be a non void signature, Y be a non-empty many sorted set indexed by the carrier of S, and X be a many sorted set indexed by the carrier of S. Then $(\mathrm{Reverse}(Y))^{-1}(X) \subseteq S\text{-Terms}^Y(X)$.

(24) Let S be a non void signature, X be a many sorted set indexed by the carrier of S, t be a term of S over $X \cup (($the carrier of $S) \longmapsto \{0\})$, and s be a sort symbol of S. If $t \in (S\text{-Terms}^{X\cup((\text{the carrier of } S)\longmapsto\{0\})}(X))(s)$, then $t \in$ (the sorts of $\mathrm{Free}_S(X))(s)$.

(25) Let S be a non void signature and X be a many sorted set indexed by the carrier of S. Then the sorts of $\mathrm{Free}_S(X) =$

S-Terms$^{X \cup ((\text{the carrier of } S) \longmapsto \{0\})}(X)$.

(26) Let S be a non void signature and X be a many sorted set indexed by the carrier of S. Then $\mathrm{Free}(X \cup ((\text{the carrier of } S) \longmapsto \{0\}))\restriction(S\text{-Terms}^{X \cup ((\text{the carrier of } S) \longmapsto \{0\})}(X)) = \mathrm{Free}_S(X)$.

(27) Let S be a non void signature, X, Y be non-empty many sorted sets indexed by the carrier of S, A be a subalgebra of $\mathrm{Free}(X)$, and B be a subalgebra of $\mathrm{Free}(Y)$. Suppose the sorts of $A =$ the sorts of B. Then the algebra of $A =$ the algebra of B.

(28) Let S be a non void signature, X be a non empty yielding many sorted set indexed by the carrier of S, Y be a many sorted set indexed by the carrier of S, and t be an element of $\mathrm{Free}_S(X)$. Then $\mathrm{Var}_S\, t \subseteq X$.

(29) Let S be a non void signature, X be a non-empty many sorted set indexed by the carrier of S, and t be a term of S over X. Then $\mathrm{Var}_t \subseteq X$.

(30) Let S be a non void signature, X, Y be non-empty many sorted sets indexed by the carrier of S, t_1 be a term of S over X, and t_2 be a term of S over Y. If $t_1 = t_2$, then the sort of $t_1 =$ the sort of t_2.

(31) Let S be a non void signature, X, Y be non-empty many sorted sets indexed by the carrier of S, and t be a term of S over Y. If $\mathrm{Var}_t \subseteq X$, then t is a term of S over X.

(32) Let S be a non void signature and X be a non-empty many sorted set indexed by the carrier of S. Then $\mathrm{Free}_S(X) = \mathrm{Free}(X)$.

(33) Let S be a non void signature, Y be a non-empty many sorted set indexed by the carrier of S, t be a term of S over Y, and p be an element of $\mathrm{dom}\, t$. Then $\mathrm{Var}_{t\restriction p} \subseteq \mathrm{Var}_t$.

(34) Let S be a non void signature, X be a non empty yielding many sorted set indexed by the carrier of S, t be an element of $\mathrm{Free}_S(X)$, and p be an element of $\mathrm{dom}\, t$. Then $t\restriction p$ is an element of $\mathrm{Free}_S(X)$.

(35) Let S be a non void signature, X be a non-empty many sorted set indexed by the carrier of S, t be a term of S over X, and a be an element of $\mathrm{rng}\, t$. Then $a = \langle a_{\mathbf{1}}, a_{\mathbf{2}} \rangle$.

(36) Let S be a non void signature, X be a non empty yielding many sorted set indexed by the carrier of S, t be an element of $\mathrm{Free}_S(X)$, and s be a sort symbol of S. Then
  (i) if $x \in (\mathrm{Var}_S\, t)(s)$, then $\langle x, s \rangle \in \mathrm{rng}\, t$, and
  (ii) if $\langle x, s \rangle \in \mathrm{rng}\, t$, then $x \in (\mathrm{Var}_S\, t)(s)$ and $x \in X(s)$.

(37) Let S be a non void signature and X be a many sorted set indexed by the carrier of S. Suppose that for every sort symbol s of S such that $X(s) = \emptyset$ there exists an operation symbol o of S such that the result sort of $o = s$ and $\mathrm{Arity}(o) = \emptyset$. Then $\mathrm{Free}_S(X)$ is non-empty.

(38) Let S be a non void signature, A be an algebra over S, B be a subalgebra

of A, and o be an operation symbol of S. Then $\mathrm{Args}(o, B) \subseteq \mathrm{Args}(o, A)$.

(39) For every non void signature S and for every feasible algebra A over S holds every subalgebra of A is feasible.

The following proposition is true

(40) Let S be a non void signature and X be a many sorted set indexed by the carrier of S. Then $\mathrm{Free}_S(X)$ is feasible and free.

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Upper and Lower Sequence of a Cage[1]

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The notation and terminology used in this paper are introduced in the following papers: [21], [7], [15], [8], [2], [19], [4], [17], [3], [14], [13], [6], [1], [5], [11], [22], [12], [18], [20], [16], [9], and [10].

1. Preliminaries

In this paper n is a natural number.

One can prove the following propositions:

(1) For every non empty subset X of $\mathcal{E}^2_{\rm T}$ and for every compact subset Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ holds $\text{N-bound}\,X \leqslant \text{N-bound}\,Y$.

(2) For every non empty subset X of $\mathcal{E}^2_{\rm T}$ and for every compact subset Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ holds $\text{E-bound}\,X \leqslant \text{E-bound}\,Y$.

(3) For every non empty subset X of $\mathcal{E}^2_{\rm T}$ and for every compact subset Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ holds $\text{S-bound}\,X \geqslant \text{S-bound}\,Y$.

(4) For every non empty subset X of $\mathcal{E}^2_{\rm T}$ and for every compact subset Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ holds $\text{W-bound}\,X \geqslant \text{W-bound}\,Y$.

(5) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is in the area of g. Let p be an element of the carrier of $\mathcal{E}^2_{\rm T}$. If $p \in \operatorname{rng} f$, then $f -\!\!: p$ is in the area of g.

(6) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is in the area of g. Let p be an element of the carrier of $\mathcal{E}^2_{\rm T}$. If $p \in \operatorname{rng} f$, then $f :\!\!- p$ is in the area of g.

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(7) For every non empty finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \widetilde{\mathcal{L}}(f)$ holds $\downharpoonleft p, f \neq \emptyset$.

(8) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $\operatorname{len} \downharpoonleft f, p \geqslant 2$, then $f(1) \in \widetilde{\mathcal{L}}(\downharpoonleft f, p)$.

(9) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$, then $f(1) \notin \widetilde{\mathcal{L}}(\operatorname{mid}(f, \operatorname{Index}(p, f) + 1, \operatorname{len} f))$.

(10) For all natural numbers i, j, m, n such that $i + j = m + n$ and $i \leqslant m$ and $j \leqslant n$ holds $i = m$.

(11) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $f(1) \in \widetilde{\mathcal{L}}(\downharpoonleft p, f)$, then $f(1) = p$.

2. About Upper and Lower Sequence of a Cage

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. The functor $\operatorname{UpperSeq}(C, n)$ yielding a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 1) $\operatorname{UpperSeq}(C, n) = ((\operatorname{Cage}(C, n))^{\text{W-min}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))}_{\circlearrowleft}){-\!:}\text{E-max}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$.

The following proposition is true

(12) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number n holds $\operatorname{len} \operatorname{UpperSeq}(C, n) = (\text{E-max}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))) \looparrowleft ((\operatorname{Cage}(C, n))^{\text{W-min}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))}_{\circlearrowleft})$.

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. The functor $\operatorname{LowerSeq}(C, n)$ yields a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 2) $\operatorname{LowerSeq}(C, n) = ((\operatorname{Cage}(C, n))^{\text{W-min}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))}_{\circlearrowleft}){:\!-}\text{E-max}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$.

Next we state the proposition

(13) Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Then $\operatorname{len} \operatorname{LowerSeq}(C, n) = (\operatorname{len}((\operatorname{Cage}(C, n))^{\text{W-min}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))}_{\circlearrowleft}) - (\text{E-max}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))) \looparrowleft ((\operatorname{Cage}(C, n))^{\text{W-min}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))}_{\circlearrowleft})) + 1$.

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. Note that $\operatorname{UpperSeq}(C, n)$ is non empty and $\operatorname{LowerSeq}(C, n)$ is non empty.

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. Observe that $\operatorname{UpperSeq}(C, n)$ is one-to-one special unfolded and s.n.c. and $\operatorname{LowerSeq}(C, n)$ is one-to-one special unfolded and s.n.c..

The following propositions are true:

(14) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number n holds $\mathrm{len}\,\mathrm{UpperSeq}(C,n) + \mathrm{len}\,\mathrm{LowerSeq}(C,n) = \mathrm{len}\,\mathrm{Cage}(C,n) + 1$.

(15) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number n holds $(\mathrm{Cage}(C,n))_{\circlearrowleft}^{\text{W-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))} = \mathrm{UpperSeq}(C,n) \frown \mathrm{LowerSeq}(C,n)$.

(16) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number n holds $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n) \frown \mathrm{LowerSeq}(C,n))$.

(17) For every compact non vertical non horizontal non empty subset C of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number n holds $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) \cup \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n))$.

(18) For every simple closed curve P holds $\text{W-min}\,P \neq \text{E-min}\,P$.

(19) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number n holds $\mathrm{len}\,\mathrm{UpperSeq}(C,n) \geqslant 3$ and $\mathrm{len}\,\mathrm{LowerSeq}(C,n) \geqslant 3$.

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. Observe that $\mathrm{UpperSeq}(C,n)$ is special sequence and $\mathrm{LowerSeq}(C,n)$ is special sequence.

Next we state several propositions:

(20) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number n holds $\widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n)) = \{\text{W-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)), \text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))\}$.

(21) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{UpperSeq}(C,n)$ is in the area of $\mathrm{Cage}(C,n)$.

(22) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{LowerSeq}(C,n)$ is in the area of $\mathrm{Cage}(C,n)$.

(23) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $((\mathrm{Cage}(C,n))_2)_{\mathbf{2}} = \text{N-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$.

(24) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and k be a natural number. If $1 \leqslant k$ and $k+1 \leqslant \mathrm{len}\,\mathrm{Cage}(C,n)$ and $(\mathrm{Cage}(C,n))_k = \text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$, then $((\mathrm{Cage}(C,n))_{k+1})_{\mathbf{1}} = \text{E-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$.

(25) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and k be a natural number. If $1 \leqslant k$ and $k+1 \leqslant \mathrm{len}\,\mathrm{Cage}(C,n)$ and $(\mathrm{Cage}(C,n))_k = \text{S-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$, then $((\mathrm{Cage}(C,n))_{k+1})_{\mathbf{2}} = \text{S-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$.

(26) Let C be a compact connected non vertical non horizontal subset of

$\mathcal{E}^2_{\mathrm{T}}$ and k be a natural number. If $1 \leqslant k$ and $k+1 \leqslant \operatorname{len}\operatorname{Cage}(C,n)$ and $(\operatorname{Cage}(C,n))_k = \text{W-min}\,\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$, then $((\operatorname{Cage}(C,n))_{k+1})_{\mathbf{1}} = \text{W-bound}\,\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$.

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On Polynomials with Coefficients in a Ring of Polynomials

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Summary. The main result of the paper is, that the ring of polynomials with o_1 variables and coefficients in the ring of polynomials with o_2 variables and coefficient in a ring L is isomorphic with the ring with $o_1 + o_2$ variables, and coefficients in L.

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The papers [18], [4], [3], [6], [15], [14], [9], [1], [2], [13], [12], [10], [5], [16], [7], [17], [8], and [11] provide the notation and terminology for this paper.

1. Preliminaries

In this paper o_1, o_2 are ordinal numbers.

Let L_1, L_2 be non empty double loop structures. Let us note that the predicate L_1 is ring isomorphic to L_2 is reflexive. We introduce L_1 and L_2 are isomorphic as a synonym of L_1 is ring isomorphic to L_2.

We now state the proposition

(1) Let B be a set. Suppose that for every set x holds $x \in B$ iff there exists an ordinal number o such that $x = o_1 + o$ and $o \in o_2$. Then $o_1 + o_2 = o_1 \cup B$.

Let o_1 be an ordinal number and let o_2 be a non empty ordinal number. Note that $o_1 + o_2$ is non empty and $o_2 + o_1$ is non empty.

One can prove the following proposition

(2) Let n be an ordinal number and a, b be bags of n. Suppose $a < b$. Then there exists an ordinal number o such that $o \in n$ and $a(o) < b(o)$ and for every ordinal number l such that $l \in o$ holds $a(l) = b(l)$.

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2. About Bags

Let o_1, o_2 be ordinal numbers, let a be an element of $\mathrm{Bags}\, o_1$, and let b be an element of $\mathrm{Bags}\, o_2$. The functor $a+b$ yielding an element of $\mathrm{Bags}(o_1+o_2)$ is defined as follows:

(Def. 1) For every ordinal number o holds if $o \in o_1$, then $(a+b)(o) = a(o)$ and if $o \in (o_1+o_2) \setminus o_1$, then $(a+b)(o) = b(o - o_1)$.

One can prove the following propositions:

(3) For every element a of $\mathrm{Bags}\, o_1$ and for every element b of $\mathrm{Bags}\, o_2$ such that $o_2 = \emptyset$ holds $a+b = a$.

(4) For every element a of $\mathrm{Bags}\, o_1$ and for every element b of $\mathrm{Bags}\, o_2$ such that $o_1 = \emptyset$ holds $a+b = b$.

(5) For every element b_1 of $\mathrm{Bags}\, o_1$ and for every element b_2 of $\mathrm{Bags}\, o_2$ holds $b_1+b_2 = \mathrm{EmptyBag}(o_1+o_2)$ iff $b_1 = \mathrm{EmptyBag}\, o_1$ and $b_2 = \mathrm{EmptyBag}\, o_2$.

(6) For every element c of $\mathrm{Bags}(o_1+o_2)$ there exists an element c_1 of $\mathrm{Bags}\, o_1$ and there exists an element c_2 of $\mathrm{Bags}\, o_2$ such that $c = c_1 + c_2$.

(7) For all elements b_1, c_1 of $\mathrm{Bags}\, o_1$ and for all elements b_2, c_2 of $\mathrm{Bags}\, o_2$ such that $b_1 + b_2 = c_1 + c_2$ holds $b_1 = c_1$ and $b_2 = c_2$.

(8) Let n be an ordinal number, L be an Abelian add-associative right zeroed right complementable distributive associative non empty double loop structure, and p, q, r be serieses of n, L. Then $(p+q)*r = p*r+q*r$.

3. Main Results

Let n be an ordinal number and let L be a right zeroed Abelian add-associative right complementable unital distributive associative non trivial non empty double loop structure. Observe that $\mathrm{Polynom\text{-}Ring}(n, L)$ is non trivial and distributive.

Let o_1, o_2 be non empty ordinal numbers, let L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, and let P be a polynomial of o_1, $\mathrm{Polynom\text{-}Ring}(o_2, L)$. The functor $\mathrm{Compress}\, P$ yields a polynomial of $o_1 + o_2$, L and is defined by the condition (Def. 2).

(Def. 2) Let b be an element of $\mathrm{Bags}(o_1 + o_2)$. Then there exists an element b_1 of $\mathrm{Bags}\, o_1$ and there exists an element b_2 of $\mathrm{Bags}\, o_2$ and there exists a polynomial Q_1 of o_2, L such that $Q_1 = P(b_1)$ and $b = b_1 + b_2$ and $(\mathrm{Compress}\, P)(b) = Q_1(b_2)$.

Next we state several propositions:

(9) For all elements b_1, c_1 of $\mathrm{Bags}\, o_1$ and for all elements b_2, c_2 of $\mathrm{Bags}\, o_2$ such that $b_1 \mid c_1$ and $b_2 \mid c_2$ holds $b_1 + b_2 \mid c_1 + c_2$.

(10) Let b be a bag of $o_1 + o_2$, b_1 be an element of $\mathrm{Bags}\, o_1$, and b_2 be an element of $\mathrm{Bags}\, o_2$. Suppose $b \mid b_1 + b_2$. Then there exists an element c_1 of $\mathrm{Bags}\, o_1$ and there exists an element c_2 of $\mathrm{Bags}\, o_2$ such that $c_1 \mid b_1$ and $c_2 \mid b_2$ and $b = c_1 + c_2$.

(11) For all elements a_1, b_1 of $\mathrm{Bags}\, o_1$ and for all elements a_2, b_2 of $\mathrm{Bags}\, o_2$ holds $a_1 + a_2 < b_1 + b_2$ iff $a_1 < b_1$ or $a_1 = b_1$ and $a_2 < b_2$.

(12) Let b_1 be an element of $\mathrm{Bags}\, o_1$, b_2 be an element of $\mathrm{Bags}\, o_2$, and G be a finite sequence of elements of $(\mathrm{Bags}(o_1 + o_2))^*$. Suppose that
  (i) $\mathrm{dom}\, G = \mathrm{Seg}\, \mathrm{len}\, \mathrm{divisors}\, b_1$, and
  (ii) for every natural number i such that $i \in \mathrm{Seg}\, \mathrm{len}\, \mathrm{divisors}\, b_1$ there exists an element a'_1 of $\mathrm{Bags}\, o_1$ and there exists a finite sequence F_1 of elements of $\mathrm{Bags}(o_1 + o_2)$ such that $F_1 = G_i$ and $\pi_i\, \mathrm{divisors}\, b_1 = a'_1$ and $\mathrm{len}\, F_1 = \mathrm{len}\, \mathrm{divisors}\, b_2$ and for every natural number m such that $m \in \mathrm{dom}\, F_1$ there exists an element a''_1 of $\mathrm{Bags}\, o_2$ such that $\pi_m\, \mathrm{divisors}\, b_2 = a''_1$ and $\pi_m F_1 = a'_1 + a''_1$.

  Then $\mathrm{divisors}(b_1 + b_2) = \mathrm{Flat}(G)$.

(13) For all elements a_1, b_1, c_1 of $\mathrm{Bags}\, o_1$ and for all elements a_2, b_2, c_2 of $\mathrm{Bags}\, o_2$ such that $c_1 = b_1 -' a_1$ and $c_2 = b_2 -' a_2$ holds $(b_1 + b_2) -' (a_1 + a_2) = c_1 + c_2$.

(14) Let b_1 be an element of $\mathrm{Bags}\, o_1$, b_2 be an element of $\mathrm{Bags}\, o_2$, and G be a finite sequence of elements of $((\mathrm{Bags}(o_1 + o_2))^2)^*$. Suppose that
  (i) $\mathrm{dom}\, G = \mathrm{Seg}\, \mathrm{len}\, \mathrm{decomp}\, b_1$, and
  (ii) for every natural number i such that $i \in \mathrm{Seg}\, \mathrm{len}\, \mathrm{decomp}\, b_1$ there exist elements a'_1, b'_1 of $\mathrm{Bags}\, o_1$ and there exists a finite sequence F_1 of elements of $(\mathrm{Bags}(o_1 + o_2))^2$ such that $F_1 = G_i$ and $\pi_i\, \mathrm{decomp}\, b_1 = \langle a'_1, b'_1 \rangle$ and $\mathrm{len}\, F_1 = \mathrm{len}\, \mathrm{decomp}\, b_2$ and for every natural number m such that $m \in \mathrm{dom}\, F_1$ there exist elements a''_1, b''_1 of $\mathrm{Bags}\, o_2$ such that $\pi_m\, \mathrm{decomp}\, b_2 = \langle a''_1, b''_1 \rangle$ and $\pi_m F_1 = \langle a'_1 + a''_1, b'_1 + b''_1 \rangle$.

  Then $\mathrm{decomp}(b_1 + b_2) = \mathrm{Flat}(G)$.

(15) Let o_1, o_2 be non empty ordinal numbers and L be an Abelian right zeroed add-associative right complementable unital distributive associative well unital non trivial non empty double loop structure. Then $\text{Polynom-Ring}(o_1, \text{Polynom-Ring}(o_2, L))$ and $\text{Polynom-Ring}(o_1 + o_2, L)$ are isomorphic.

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On Cosets in Segre's Product of Partial Linear Spaces

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Summary. This paper is a continuation of [12]. We prove that the family of cosets in the Segre's product of partial linear spaces remains invariant under automorphisms.

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The terminology and notation used in this paper are introduced in the following articles: [13], [20], [1], [3], [4], [7], [6], [2], [18], [12], [15], [11], [14], [5], [10], [21], [16], [19], [17], [9], and [8].

1. Preliminaries on Finite Sequences

Let D be a set, let p be a finite sequence of elements of D, and let i, j be natural numbers. The functor $\mathrm{Del}(p,i,j)$ yields a finite sequence of elements of D and is defined by:

(Def. 1) $\mathrm{Del}(p,i,j) = (p \restriction (i -' 1)) ^\frown (p_{\downharpoonright j})$.

We now state several propositions:

(1) For every set D and for every finite sequence p of elements of D and for all natural numbers i, j holds $\mathrm{rng}\,\mathrm{Del}(p,i,j) \subseteq \mathrm{rng}\, p$.

(2) Let D be a set, p be a finite sequence of elements of D, and i, j be natural numbers. If $i \in \mathrm{dom}\, p$ and $j \in \mathrm{dom}\, p$, then $\mathrm{len}\,\mathrm{Del}(p,i,j) = ((\mathrm{len}\, p - j) + i) - 1$.

(3) Let D be a set, p be a finite sequence of elements of D, and i, j be natural numbers. If $i \in \mathrm{dom}\, p$ and $j \in \mathrm{dom}\, p$, then if $\mathrm{len}\,\mathrm{Del}(p,i,j) = 0$, then $i = 1$ and $j = \mathrm{len}\, p$.

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(4) Let D be a set, p be a finite sequence of elements of D, and i, j, k be natural numbers. If $i \in \operatorname{dom} p$ and $1 \leqslant k$ and $k \leqslant i-1$, then $(\operatorname{Del}(p,i,j))(k) = p(k)$.

(5) For all finite sequences p, q and for every natural number k such that $\operatorname{len} p + 1 \leqslant k$ holds $(p \frown q)(k) = q(k - \operatorname{len} p)$.

(6) Let D be a set, p be a finite sequence of elements of D, and i, j, k be natural numbers. Suppose $i \in \operatorname{dom} p$ and $j \in \operatorname{dom} p$ and $i \leqslant j$ and $i \leqslant k$ and $k \leqslant ((\operatorname{len} p - j) + i) - 1$. Then $(\operatorname{Del}(p,i,j))(k) = p((j -' i) + k + 1)$.

The scheme *FinSeqOneToOne* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, a finite sequence $\mathcal{D}$ of elements of $\mathcal{C}$, and a binary predicate $\mathcal{P}$, and states that:

> There exists an one-to-one finite sequence g of elements of $\mathcal{C}$ such that $\mathcal{A} = g(1)$ and $\mathcal{B} = g(\operatorname{len} g)$ and $\operatorname{rng} g \subseteq \operatorname{rng} \mathcal{D}$ and for every natural number j such that $1 \leqslant j$ and $j < \operatorname{len} g$ holds $\mathcal{P}[g(j), g(j+1)]$

provided the following requirements are met:

- $\mathcal{A} = \mathcal{D}(1)$ and $\mathcal{B} = \mathcal{D}(\operatorname{len} \mathcal{D})$, and
- For every natural number i and for all sets d_1, d_2 such that $1 \leqslant i$ and $i < \operatorname{len} \mathcal{D}$ and $d_1 = \mathcal{D}(i)$ and $d_2 = \mathcal{D}(i+1)$ holds $\mathcal{P}[d_1, d_2]$.

2. Segre Cosets

Next we state the proposition

(7) Let I be a non empty set, A be a 1-sorted yielding many sorted set indexed by I, L be a many sorted subset indexed by the support of A, i be an element of I, and S be a subset of the carrier of $A(i)$. Then $L +\cdot (i, S)$ is a many sorted subset indexed by the support of A.

Let I be a non empty set and let A be a non-Trivial-yielding TopStruct-yielding many sorted set indexed by I. A subset of Segre_Product A is called a Segre-Coset of A if it satisfies the condition (Def. 2).

(Def. 2) There exists a Segre-like non trivial-yielding many sorted subset L indexed by the support of A such that it $= \prod L$ and $L(\operatorname{index}(L)) = \Omega_{A(\operatorname{index}(L))}$.

The following proposition is true

(8) Let I be a non empty set, A be a non-Trivial-yielding TopStruct-yielding many sorted set indexed by I, and B_1, B_2 be Segre-Cosets of A. If $2 \subseteq \overline{\overline{B_1 \cap B_2}}$, then $B_1 = B_2$.

Let S be a topological structure and let X, Y be subsets of the carrier of S. We say that X and Y are joinable if and only if the condition (Def. 3) is satisfied.

(Def. 3) There exists a finite sequence f of elements of $2^{\text{the carrier of } S}$ such that

(i) $X = f(1)$,

(ii) $Y = f(\operatorname{len} f)$,

(iii) for every subset W of the carrier of S such that $W \in \operatorname{rng} f$ holds W is closed under lines and strong, and

(iv) for every natural number i such that $1 \leqslant i$ and $i < \operatorname{len} f$ holds $2 \subseteq \overline{\overline{f(i) \cap f(i+1)}}$.

One can prove the following three propositions:

(9) Let S be a topological structure and X, Y be subsets of the carrier of S. Suppose X and Y are joinable. Then there exists an one-to-one finite sequence f of elements of $2^{\text{the carrier of } S}$ such that

(i) $X = f(1)$,

(ii) $Y = f(\operatorname{len} f)$,

(iii) for every subset W of the carrier of S such that $W \in \operatorname{rng} f$ holds W is closed under lines and strong, and

(iv) for every natural number i such that $1 \leqslant i$ and $i < \operatorname{len} f$ holds $2 \subseteq \overline{\overline{f(i) \cap f(i+1)}}$.

(10) Let S be a topological structure and X be a subset of the carrier of S. If X is closed under lines and strong, then X and X are joinable.

(11) Let I be a non empty set, A be a PLS-yielding many sorted set indexed by I, and X, Y be subsets of the carrier of Segre_Product A. Suppose that

(i) X is non trivial, closed under lines, and strong,

(ii) Y is non trivial, closed under lines, and strong, and

(iii) X and Y are joinable.

Let X_1, Y_1 be Segre-like non trivial-yielding many sorted subsets indexed by the support of A. Suppose $X = \prod X_1$ and $Y = \prod Y_1$. Then $\operatorname{index}(X_1) = \operatorname{index}(Y_1)$ and for every set i such that $i \neq \operatorname{index}(X_1)$ holds $X_1(i) = Y_1(i)$.

3. Collineations of Segre Product

One can prove the following proposition

(12) Let S be a 1-sorted structure, T be a non empty 1-sorted structure, and f be a map from S into T. If f is bijective, then f^{-1} is bijective.

Let S, T be topological structures and let f be a map from S into T. We say that f is isomorphic if and only if:

(Def. 4) f is bijective and open and f^{-1} is bijective and open.

Let S be a non empty topological structure. Observe that there exists a map from S into S which is isomorphic.

Let S be a non empty topological structure. A collineation of S is an isomorphic map from S into S.

Let S be a non empty non void topological structure, let f be a collineation of S, and let l be a block of S. Then $f^\circ l$ is a block of S.

Let S be a non empty non void topological structure, let f be a collineation of S, and let l be a block of S. Then $f^{-1}(l)$ is a block of S.

Next we state a number of propositions:

(13) For every non empty topological structure S and for every collineation f of S holds f^{-1} is a collineation of S.

(14) Let S be a non empty topological structure, f be a collineation of S, and X be a subset of the carrier of S. If X is non trivial, then $f^\circ X$ is non trivial.

(15) Let S be a non empty topological structure, f be a collineation of S, and X be a subset of the carrier of S. If X is non trivial, then $f^{-1}(X)$ is non trivial.

(16) Let S be a non empty non void topological structure, f be a collineation of S, and X be a subset of the carrier of S. If X is strong, then $f^\circ X$ is strong.

(17) Let S be a non empty non void topological structure, f be a collineation of S, and X be a subset of the carrier of S. If X is strong, then $f^{-1}(X)$ is strong.

(18) Let S be a non empty non void topological structure, f be a collineation of S, and X be a subset of the carrier of S. If X is closed under lines, then $f^\circ X$ is closed under lines.

(19) Let S be a non empty non void topological structure, f be a collineation of S, and X be a subset of the carrier of S. If X is closed under lines, then $f^{-1}(X)$ is closed under lines.

(20) Let S be a non empty non void topological structure, f be a collineation of S, and X, Y be subsets of the carrier of S. Suppose X is non trivial and Y is non trivial and X and Y are joinable. Then $f^\circ X$ and $f^\circ Y$ are joinable.

(21) Let S be a non empty non void topological structure, f be a collineation of S, and X, Y be subsets of the carrier of S. Suppose X is non trivial and Y is non trivial and X and Y are joinable. Then $f^{-1}(X)$ and $f^{-1}(Y)$ are joinable.

(22) Let I be a non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is strongly connected. Let W be a subset of the carrier of Segre_Product A. Suppose W is non trivial, strong, and closed under lines. Then $\bigcup\{Y; Y$ ranges over subsets of the carrier of Segre_Product A : Y is non trivial, strong, and

closed under lines $\wedge$ W and Y are joinable} is a Segre-Coset of A.

(23) Let I be a non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is strongly connected. Let B be a set. Then B is a Segre-Coset of A if and only if there exists a subset W of the carrier of Segre_Product A such that W is non trivial, strong, and closed under lines and $B = \bigcup\{Y; Y$ ranges over subsets of the carrier of Segre_Product A : Y is non trivial, strong, and closed under lines $\wedge$ W and Y are joinable$\}$.

(24) Let I be a non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is strongly connected. Let B be a Segre-Coset of A and f be a collineation of Segre_Product A. Then $f^{\circ}B$ is a Segre-Coset of A.

(25) Let I be a non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is strongly connected. Let B be a Segre-Coset of A and f be a collineation of Segre_Product A. Then $f^{-1}(B)$ is a Segre-Coset of A.

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On the Simple Closed Curve Property of the Circle and the Fashoda Meet Theorem

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Summary. First, we prove the fact that the circle is the simple closed curve, which was defined as a curve homeomorphic to the square. For this proof, we introduce a mapping which is a homeomorphism from 2-dimensional plane to itself. This mapping maps the square to the circle. Secondly, we prove the Fashoda meet theorem for the circle using this homeomorphism.

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The terminology and notation used in this paper have been introduced in the following articles: [17], [5], [7], [1], [2], [11], [3], [12], [4], [13], [10], [18], [15], [16], [14], [8], [9], and [6].

1. Preliminaries

In this paper x, y, z, u, a are real numbers.

We now state a number of propositions:

(1) If $x^{\mathbf{2}} = y^{\mathbf{2}}$, then $x = y$ or $x = -y$.

(2) If $x^{\mathbf{2}} = 1$, then $x = 1$ or $x = -1$.

(3) If $0 \leqslant x$ and $x \leqslant 1$, then $x^{\mathbf{2}} \leqslant x$.

(4) If $a \geqslant 0$ and $(x - a) \cdot (x + a) \leqslant 0$, then $-a \leqslant x$ and $x \leqslant a$.

(5) If $x^{\mathbf{2}} - 1 \leqslant 0$, then $-1 \leqslant x$ and $x \leqslant 1$.

(6) $x < y$ and $x < z$ iff $x < \min(y, z)$.

(7) If $0 < x$, then $\frac{x}{3} < x$ and $\frac{x}{4} < x$.

(8) If $x \geqslant 1$, then $\sqrt{x} \geqslant 1$ and if $x > 1$, then $\sqrt{x} > 1$.

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(9) If $x \leqslant y$ and $z \leqslant u$, then $[y, z] \subseteq [x, u]$.

(10) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $|p| = \sqrt{(p_{\mathbf{1}})^{\mathbf{2}} + (p_{\mathbf{2}})^{\mathbf{2}}}$ and $|p|^{\mathbf{2}} = (p_{\mathbf{1}})^{\mathbf{2}} + (p_{\mathbf{2}})^{\mathbf{2}}$.

(11) For every function f and for all sets B, C holds $(f{\restriction}B)^{\circ}C = f^{\circ}(C \cap B)$.

(12) Let X be a topological structure, Y be a non empty topological structure, f be a map from X into Y, and P be a subset of X. Then $f{\restriction}P$ is a map from $X{\restriction}P$ into Y.

(13) Let X, Y be non empty topological spaces, p_0 be a point of X, D be a non empty subset of X, E be a non empty subset of Y, and f be a map from X into Y. Suppose that $D^{\mathrm{c}} = \{p_0\}$ and $E^{\mathrm{c}} = \{f(p_0)\}$ and X is a T_2 space and Y is a T_2 space and for every point p of $X{\restriction}D$ holds $f(p) \neq f(p_0)$ and there exists a map h from $X{\restriction}D$ into $Y{\restriction}E$ such that $h = f{\restriction}D$ and h is continuous and for every subset V of Y such that $f(p_0) \in V$ and V is open there exists a subset W of X such that $p_0 \in W$ and W is open and $f^{\circ}W \subseteq V$. Then f is continuous.

2. The Circle is a Simple Closed Curve

In the sequel p, q denote points of $\mathcal{E}^2_{\mathrm{T}}$.

The function SqCirc from the carrier of $\mathcal{E}^2_{\mathrm{T}}$ into the carrier of $\mathcal{E}^2_{\mathrm{T}}$ is defined by the condition (Def. 1).

(Def. 1) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Then

(i) if $p = 0_{\mathcal{E}^2_{\mathrm{T}}}$, then $\mathrm{SqCirc}(p) = p$,

(ii) if $p_{\mathbf{2}} \leqslant p_{\mathbf{1}}$ and $-p_{\mathbf{1}} \leqslant p_{\mathbf{2}}$ or $p_{\mathbf{2}} \geqslant p_{\mathbf{1}}$ and $p_{\mathbf{2}} \leqslant -p_{\mathbf{1}}$ and if $p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$, then $\mathrm{SqCirc}(p) = [\frac{p_{\mathbf{1}}}{\sqrt{1+(\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}}, \frac{p_{\mathbf{2}}}{\sqrt{1+(\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}}]$, and

(iii) if $p_{\mathbf{2}} \not\leqslant p_{\mathbf{1}}$ or $-p_{\mathbf{1}} \not\leqslant p_{\mathbf{2}}$ but $p_{\mathbf{2}} \not\geqslant p_{\mathbf{1}}$ or $p_{\mathbf{2}} \not\leqslant -p_{\mathbf{1}}$ and $p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$, then $\mathrm{SqCirc}(p) = [\frac{p_{\mathbf{1}}}{\sqrt{1+(\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}}, \frac{p_{\mathbf{2}}}{\sqrt{1+(\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}}]$.

We now state a number of propositions:

(14) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$. Then

(i) if $p_{\mathbf{1}} \leqslant p_{\mathbf{2}}$ and $-p_{\mathbf{2}} \leqslant p_{\mathbf{1}}$ or $p_{\mathbf{1}} \geqslant p_{\mathbf{2}}$ and $p_{\mathbf{1}} \leqslant -p_{\mathbf{2}}$, then $\mathrm{SqCirc}(p) = [\frac{p_{\mathbf{1}}}{\sqrt{1+(\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}}, \frac{p_{\mathbf{2}}}{\sqrt{1+(\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}}]$, and

(ii) if $p_{\mathbf{1}} \not\leqslant p_{\mathbf{2}}$ or $-p_{\mathbf{2}} \not\leqslant p_{\mathbf{1}}$ and if $p_{\mathbf{1}} \not\geqslant p_{\mathbf{2}}$ or $p_{\mathbf{1}} \not\leqslant -p_{\mathbf{2}}$, then $\mathrm{SqCirc}(p) = [\frac{p_{\mathbf{1}}}{\sqrt{1+(\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}}, \frac{p_{\mathbf{2}}}{\sqrt{1+(\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}}]$.

(15) Let X be a non empty topological space and f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and for every point q of X there exists a real number r such that $f_1(q) = r$ and $r \geqslant 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = \sqrt{r_1}$ and g is continuous.

(16) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that
 (i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = (\frac{r_1}{r_2})^{\mathbf{2}}$, and
 (ii) g is continuous.

(17) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that
 (i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = 1 + (\frac{r_1}{r_2})^{\mathbf{2}}$, and
 (ii) g is continuous.

(18) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that
 (i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = \sqrt{1 + (\frac{r_1}{r_2})^{\mathbf{2}}}$, and
 (ii) g is continuous.

(19) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that
 (i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = \frac{r_1}{\sqrt{1+(\frac{r_1}{r_2})^{\mathbf{2}}}}$, and
 (ii) g is continuous.

(20) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that
 (i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = \frac{r_2}{\sqrt{1+(\frac{r_1}{r_2})^{\mathbf{2}}}}$, and
 (ii) g is continuous.

(21) Let K_1 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that
 (i) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction K_1$ holds $f(p) = \frac{p_{\mathbf{1}}}{\sqrt{1+(\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}}$, and
 (ii) for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction K_1$ holds $q_{\mathbf{1}} \neq 0$.

 Then f is continuous.

(22) Let K_1 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $f(p) = \frac{p_{\mathbf{2}}}{\sqrt{1+(\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $q_{\mathbf{1}} \neq 0$.

Then f is continuous.

(23) Let K_1 be a non empty subset of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $f(p) = \frac{p_{\mathbf{2}}}{\sqrt{1+(\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $q_{\mathbf{2}} \neq 0$.

Then f is continuous.

(24) Let K_1 be a non empty subset of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $f(p) = \frac{p_{\mathbf{1}}}{\sqrt{1+(\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $q_{\mathbf{2}} \neq 0$.

Then f is continuous.

(25) Let K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $f = \mathrm{SqCirc}\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : (p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \ \wedge\ -p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \ \vee\ p_{\mathbf{2}} \geqslant p_{\mathbf{1}} \ \wedge\ p_{\mathbf{2}} \leqslant -p_{\mathbf{1}}) \ \wedge\ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(26) Let K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $f = \mathrm{SqCirc}\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : (p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \ \wedge\ -p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \ \vee\ p_{\mathbf{1}} \geqslant p_{\mathbf{2}} \ \wedge\ p_{\mathbf{1}} \leqslant -p_{\mathbf{2}}) \ \wedge\ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

In this article we present several logical schemes. The scheme *TopIncl* concerns a unary predicate $\mathcal{P}$, and states that:

$\{p : \mathcal{P}[p] \ \wedge\ p \neq 0_{\mathcal{E}^2_{\rm T}}\} \subseteq$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$

for all values of the parameters.

The scheme *TopInter* concerns a unary predicate $\mathcal{P}$, and states that:

$\{p : \mathcal{P}[p] \ \wedge\ p \neq 0_{\mathcal{E}^2_{\rm T}}\} = \{p_7; p_7$ ranges over points of $\mathcal{E}^2_{\rm T}$: $\mathcal{P}[p_7]\} \cap$ ((the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$)

for all values of the parameters.

Next we state several propositions:

(27) Let B_0 be a subset of $\mathcal{E}^2_{\rm T}$, K_0 be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $f = \mathrm{SqCirc}\restriction K_0$ and $B_0 =$ (the

carrier of $\mathcal{E}^2_{\mathrm{T}}) \setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : (p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \ \wedge \ -p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \ \vee \ p_{\mathbf{2}} \geqslant p_{\mathbf{1}} \ \wedge \ p_{\mathbf{2}} \leqslant -p_{\mathbf{1}}) \ \wedge \ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then f is continuous and K_0 is closed.

(28) Let B_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$, K_0 be a subset of $(\mathcal{E}^2_{\mathrm{T}}) \restriction B_0$, and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction B_0 \restriction K_0$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction B_0$. Suppose $f = \mathrm{SqCirc} \restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\mathrm{T}}) \setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : (p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \ \wedge \ -p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \ \vee \ p_{\mathbf{1}} \geqslant p_{\mathbf{2}} \ \wedge \ p_{\mathbf{1}} \leqslant -p_{\mathbf{2}}) \ \wedge \ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then f is continuous and K_0 is closed.

(29) Let D be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $D^{\mathrm{c}} = \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then there exists a map h from $(\mathcal{E}^2_{\mathrm{T}}) \restriction D$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction D$ such that $h = \mathrm{SqCirc} \restriction D$ and h is continuous.

(30) For every non empty subset D of $\mathcal{E}^2_{\mathrm{T}}$ such that $D =$ (the carrier of $\mathcal{E}^2_{\mathrm{T}}) \setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ holds $D^{\mathrm{c}} = \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$.

(31) There exists a map h from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ such that $h = \mathrm{SqCirc}$ and h is continuous.

(32) SqCirc is one-to-one.

Let us observe that SqCirc is one-to-one.

One can prove the following propositions:

(33) Let K_2, C_1 be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) $K_2 = \{q : -1 = q_{\mathbf{1}} \ \wedge \ -1 \leqslant q_{\mathbf{2}} \ \wedge \ q_{\mathbf{2}} \leqslant 1 \ \vee \ q_{\mathbf{1}} = 1 \ \wedge \ -1 \leqslant q_{\mathbf{2}} \ \wedge \ q_{\mathbf{2}} \leqslant 1 \ \vee \ -1 = q_{\mathbf{2}} \ \wedge \ -1 \leqslant q_{\mathbf{1}} \ \wedge \ q_{\mathbf{1}} \leqslant 1 \ \vee \ 1 = q_{\mathbf{2}} \ \wedge \ -1 \leqslant q_{\mathbf{1}} \ \wedge \ q_{\mathbf{1}} \leqslant 1\}$, and

(ii) $C_1 = \{p_2; p_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p_2| = 1\}$.

Then $\mathrm{SqCirc}^{\circ} K_2 = C_1$.

(34) Let P, K_2 be subsets of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_2$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction P$. Suppose that

(i) $K_2 = \{q : -1 = q_{\mathbf{1}} \ \wedge \ -1 \leqslant q_{\mathbf{2}} \ \wedge \ q_{\mathbf{2}} \leqslant 1 \ \vee \ q_{\mathbf{1}} = 1 \ \wedge \ -1 \leqslant q_{\mathbf{2}} \ \wedge \ q_{\mathbf{2}} \leqslant 1 \ \vee \ -1 = q_{\mathbf{2}} \ \wedge \ -1 \leqslant q_{\mathbf{1}} \ \wedge \ q_{\mathbf{1}} \leqslant 1 \ \vee \ 1 = q_{\mathbf{2}} \ \wedge \ -1 \leqslant q_{\mathbf{1}} \ \wedge \ q_{\mathbf{1}} \leqslant 1\}$, and

(ii) f is a homeomorphism.

Then P is a simple closed curve.

(35) Let K_2 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K_2 = \{q : -1 = q_{\mathbf{1}} \wedge -1 \leqslant q_{\mathbf{2}} \wedge q_{\mathbf{2}} \leqslant 1 \vee q_{\mathbf{1}} = 1 \wedge -1 \leqslant q_{\mathbf{2}} \wedge q_{\mathbf{2}} \leqslant 1 \vee -1 = q_{\mathbf{2}} \wedge -1 \leqslant q_{\mathbf{1}} \wedge q_{\mathbf{1}} \leqslant 1 \vee 1 = q_{\mathbf{2}} \wedge -1 \leqslant q_{\mathbf{1}} \wedge q_{\mathbf{1}} \leqslant 1\}$. Then K_2 is a simple closed curve and compact.

(36) For every subset C_1 of $\mathcal{E}^2_{\mathrm{T}}$ such that $C_1 = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ holds C_1 is a simple closed curve.

3. The Fashoda Meet Theorem for the Circle

Next we state a number of propositions:

(37) Let K_0, C_0 be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K_0 = \{p : -1 \leqslant p_{\mathbf{1}} \ \wedge \ p_{\mathbf{1}} \leqslant 1 \ \wedge \ -1 \leqslant p_{\mathbf{2}} \ \wedge \ p_{\mathbf{2}} \leqslant 1\}$ and $C_0 = \{p_1; p_1$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p_1| \leqslant 1\}$. Then $\mathrm{SqCirc}^{-1}(C_0) \subseteq K_0$.

(38) Let given p. Then

(i) if $p = 0_{\mathcal{E}^2_{\mathrm{T}}}$, then $\mathrm{SqCirc}^{-1}(p) = 0_{\mathcal{E}^2_{\mathrm{T}}}$,

(ii) if $p_{\mathbf{2}} \leqslant p_{\mathbf{1}}$ and $-p_{\mathbf{1}} \leqslant p_{\mathbf{2}}$ or $p_{\mathbf{2}} \geqslant p_{\mathbf{1}}$ and $p_{\mathbf{2}} \leqslant -p_{\mathbf{1}}$ and if $p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$, then $\mathrm{SqCirc}^{-1}(p) = [p_{\mathbf{1}} \cdot \sqrt{1 + (\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}, p_{\mathbf{2}} \cdot \sqrt{1 + (\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}]$, and

(iii) if $p_{\mathbf{2}} \nleqslant p_{\mathbf{1}}$ or $-p_{\mathbf{1}} \nleqslant p_{\mathbf{2}}$ but $p_{\mathbf{2}} \ngeqslant p_{\mathbf{1}}$ or $p_{\mathbf{2}} \nleqslant -p_{\mathbf{1}}$ and $p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$, then $\mathrm{SqCirc}^{-1}(p) = [p_{\mathbf{1}} \cdot \sqrt{1 + (\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}, p_{\mathbf{2}} \cdot \sqrt{1 + (\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}]$.

(39) SqCirc^{-1} is a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$.

(40) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$. Then

(i) if $p_{\mathbf{1}} \leqslant p_{\mathbf{2}}$ and $-p_{\mathbf{2}} \leqslant p_{\mathbf{1}}$ or $p_{\mathbf{1}} \geqslant p_{\mathbf{2}}$ and $p_{\mathbf{1}} \leqslant -p_{\mathbf{2}}$, then $\mathrm{SqCirc}^{-1}(p) = [p_{\mathbf{1}} \cdot \sqrt{1 + (\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}, p_{\mathbf{2}} \cdot \sqrt{1 + (\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}]$, and

(ii) if $p_{\mathbf{1}} \nleqslant p_{\mathbf{2}}$ or $-p_{\mathbf{2}} \nleqslant p_{\mathbf{1}}$ and if $p_{\mathbf{1}} \ngeqslant p_{\mathbf{2}}$ or $p_{\mathbf{1}} \nleqslant -p_{\mathbf{2}}$, then $\mathrm{SqCirc}^{-1}(p) = [p_{\mathbf{1}} \cdot \sqrt{1 + (\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}, p_{\mathbf{2}} \cdot \sqrt{1 + (\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}]$.

(41) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that

(i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_1 \cdot \sqrt{1 + (\frac{r_1}{r_2})^{\mathbf{2}}}$, and

(ii) g is continuous.

(42) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that

(i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_2 \cdot \sqrt{1 + (\frac{r_1}{r_2})^{\mathbf{2}}}$, and

(ii) g is continuous.

(43) Let K_1 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $f(p) = p_{\mathbf{1}} \cdot \sqrt{1 + (\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $q_{\mathbf{1}} \neq 0$.

Then f is continuous.

(44) Let K_1 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $f(p) = p_{\mathbf{2}} \cdot \sqrt{1 + (\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $q_{\mathbf{1}} \neq 0$.

Then f is continuous.

(45) Let K_1 be a non empty subset of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_1$ into $\mathbb{R}^{\bf 1}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $f(p) = p_{\bf 2} \cdot \sqrt{1 + (\frac{p_{\bf 1}}{p_{\bf 2}})^{\bf 2}}$, and

(ii) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $q_{\bf 2} \neq 0$.

Then f is continuous.

(46) Let K_1 be a non empty subset of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_1$ into $\mathbb{R}^{\bf 1}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $f(p) = p_{\bf 1} \cdot \sqrt{1 + (\frac{p_{\bf 1}}{p_{\bf 2}})^{\bf 2}}$, and

(ii) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $q_{\bf 2} \neq 0$.

Then f is continuous.

(47) Let K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $f = \mathrm{SqCirc}^{-1}\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : (p_{\bf 2} \leqslant p_{\bf 1} \ \wedge \ -p_{\bf 1} \leqslant p_{\bf 2} \ \vee \ p_{\bf 2} \geqslant p_{\bf 1} \ \wedge \ p_{\bf 2} \leqslant -p_{\bf 1}) \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(48) Let K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $f = \mathrm{SqCirc}^{-1}\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : (p_{\bf 1} \leqslant p_{\bf 2} \ \wedge \ -p_{\bf 2} \leqslant p_{\bf 1} \ \vee \ p_{\bf 1} \geqslant p_{\bf 2} \ \wedge \ p_{\bf 1} \leqslant -p_{\bf 2}) \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(49) Let B_0 be a subset of $\mathcal{E}^2_{\rm T}$, K_0 be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $f = \mathrm{SqCirc}^{-1}\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : (p_{\bf 2} \leqslant p_{\bf 1} \ \wedge \ -p_{\bf 1} \leqslant p_{\bf 2} \ \vee \ p_{\bf 2} \geqslant p_{\bf 1} \ \wedge \ p_{\bf 2} \leqslant -p_{\bf 1}) \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous and K_0 is closed.

(50) Let B_0 be a subset of $\mathcal{E}^2_{\rm T}$, K_0 be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $f = \mathrm{SqCirc}^{-1}\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : (p_{\bf 1} \leqslant p_{\bf 2} \ \wedge \ -p_{\bf 2} \leqslant p_{\bf 1} \ \vee \ p_{\bf 1} \geqslant p_{\bf 2} \ \wedge \ p_{\bf 1} \leqslant -p_{\bf 2}) \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous and K_0 is closed.

(51) Let D be a non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $D^{\rm c} = \{0_{\mathcal{E}^2_{\rm T}}\}$. Then there exists a map h from $(\mathcal{E}^2_{\rm T})\restriction D$ into $(\mathcal{E}^2_{\rm T})\restriction D$ such that $h = \mathrm{SqCirc}^{-1}\restriction D$ and h is continuous.

(52) There exists a map h from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ such that $h = \mathrm{SqCirc}^{-1}$ and h is continuous.

(54)[1](i) SqCirc is a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$,

(ii) $\mathrm{rng}\,\mathrm{SqCirc} =$ the carrier of $\mathcal{E}^2_{\rm T}$, and

[1]The proposition (53) has been removed.

(iii) for every map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ such that $f = \mathrm{SqCirc}$ holds f is a homeomorphism.

(55) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, C_0, K_3, K_4, K_5, K_6 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \leqslant 1\}$ and $K_3 = \{q_1; q_1$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_1| = 1 \wedge (q_1)_{\mathbf{2}} \leqslant (q_1)_{\mathbf{1}} \wedge (q_1)_{\mathbf{2}} \geqslant -(q_1)_{\mathbf{1}}\}$ and $K_4 = \{q_2; q_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_2| = 1 \wedge (q_2)_{\mathbf{2}} \geqslant (q_2)_{\mathbf{1}} \wedge (q_2)_{\mathbf{2}} \leqslant -(q_2)_{\mathbf{1}}\}$ and $K_5 = \{q_3; q_3$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_3| = 1 \wedge (q_3)_{\mathbf{2}} \geqslant (q_3)_{\mathbf{1}} \wedge (q_3)_{\mathbf{2}} \geqslant -(q_3)_{\mathbf{1}}\}$ and $K_6 = \{q_4; q_4$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_4| = 1 \wedge (q_4)_{\mathbf{2}} \leqslant (q_4)_{\mathbf{1}} \wedge (q_4)_{\mathbf{2}} \leqslant -(q_4)_{\mathbf{1}}\}$ and $f(O) \in K_4$ and $f(I) \in K_3$ and $g(O) \in K_6$ and $g(I) \in K_5$ and $\mathrm{rng}\, f \subseteq C_0$ and $\mathrm{rng}\, g \subseteq C_0$. Then $\mathrm{rng}\, f \cap \mathrm{rng}\, g \neq \emptyset$.

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Pythagorean Triples

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Summary. A Pythagorean triple is a set of positive integers $\{a, b, c\}$ with $a^2 + b^2 = c^2$. We prove that every Pythagorean triple is of the form

$$a = n^2 - m^2 \qquad b = 2mn \qquad c = n^2 + m^2$$

or is a multiple of such a triple. Using this characterization we show that for every $n > 2$ there exists a Pythagorean triple X with $n \in X$. Also we show that even the set of *simplified* Pythagorean triples is infinite.

MML Identifier: PYTHTRIP.

The articles [6], [7], [2], [8], [5], [1], [3], [4], and [9] provide the terminology and notation for this paper.

1. Relative Primeness

We follow the rules: a, b, c, k, m, n are natural numbers and i is an integer.

Let us consider m, n. Let us observe that m and n are relative prime if and only if:

(Def. 1) For every k such that $k \mid m$ and $k \mid n$ holds $k = 1$.

Let us consider m, n. Let us observe that m and n are relative prime if and only if:

(Def. 2) For every prime natural number p holds $p \nmid m$ or $p \nmid n$.

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2. Squares

Let n be a number. We say that n is square if and only if:

(Def. 3) There exists m such that $n = m^{\mathbf{2}}$.

Let us observe that every number which is square is also natural.

Let n be a natural number. Observe that $n^{\mathbf{2}}$ is square.

Let us observe that there exists a natural number which is even and square.

One can check that there exists a natural number which is odd and square.

One can check that there exists a number which is even and square.

One can check that there exists a number which is odd and square.

Let m, n be square numbers. Observe that $m \cdot n$ is square.

We now state the proposition

(1) If $m \cdot n$ is square and m and n are relative prime, then m is square and n is square.

Let i be an even integer. Observe that $i^{\mathbf{2}}$ is even.

Let i be an odd integer. Observe that $i^{\mathbf{2}}$ is odd.

Next we state three propositions:

(2) i is even iff $i^{\mathbf{2}}$ is even.

(3) If i is even, then $i^{\mathbf{2}} \bmod 4 = 0$.

(4) If i is odd, then $i^{\mathbf{2}} \bmod 4 = 1$.

Let m, n be odd square numbers. Note that $m + n$ is non square.

One can prove the following two propositions:

(5) If $m^{\mathbf{2}} = n^{\mathbf{2}}$, then $m = n$.

(6) $m \mid n$ iff $m^{\mathbf{2}} \mid n^{\mathbf{2}}$.

3. Distributive Law for HCF

We now state two propositions:

(7) $m \mid n$ or $k = 0$ iff $k \cdot m \mid k \cdot n$.

(8) $\gcd(k \cdot m, k \cdot n) = k \cdot \gcd(m, n)$.

4. Unbounded Sets are Infinite

We now state the proposition

(9) For every set X such that for every m there exists n such that $n \geqslant m$ and $n \in X$ holds X is infinite.

5. Pythagorean Triples

We now state three propositions:

(10) If a and b are relative prime, then a is odd or b is odd.

(11) Suppose $a^{\mathbf{2}} + b^{\mathbf{2}} = c^{\mathbf{2}}$ and a and b are relative prime and a is odd. Then there exist m, n such that $m \leqslant n$ and $a = n^{\mathbf{2}} - m^{\mathbf{2}}$ and $b = 2 \cdot m \cdot n$ and $c = n^{\mathbf{2}} + m^{\mathbf{2}}$.

(12) If $a = n^{\mathbf{2}} - m^{\mathbf{2}}$ and $b = 2 \cdot m \cdot n$ and $c = n^{\mathbf{2}} + m^{\mathbf{2}}$, then $a^{\mathbf{2}} + b^{\mathbf{2}} = c^{\mathbf{2}}$.

A subset of $\mathbb{N}$ is called a Pythagorean triple if:

(Def. 4) There exist a, b, c such that $a^{\mathbf{2}} + b^{\mathbf{2}} = c^{\mathbf{2}}$ and it $= \{a, b, c\}$.

In the sequel X is a Pythagorean triple.

Let us note that every Pythagorean triple is finite.

Let us note that the Pythagorean triple can be characterized by the following (equivalent) condition:

(Def. 5) There exist k, m, n such that $m \leqslant n$ and it $= \{k \cdot (n^{\mathbf{2}} - m^{\mathbf{2}}), k \cdot (2 \cdot m \cdot n), k \cdot (n^{\mathbf{2}} + m^{\mathbf{2}})\}$.

Let us consider X. We say that X is degenerate if and only if:

(Def. 6) $0 \in X$.

We now state the proposition

(13) If $n > 2$, then there exists X such that X is non degenerate and $n \in X$.

Let us consider X. We say that X is simplified if and only if:

(Def. 7) For every k such that for every n such that $n \in X$ holds $k \mid n$ holds $k = 1$.

Let us consider X. Let us observe that X is simplified if and only if:

(Def. 8) There exist m, n such that $m \in X$ and $n \in X$ and m and n are relative prime.

One can prove the following proposition

(14) If $n > 0$, then there exists X such that X is non degenerate and simplified and $4 \cdot n \in X$.

Let us note that there exists a Pythagorean triple which is non degenerate and simplified.

The following propositions are true:

(15) $\{3, 4, 5\}$ is a non degenerate simplified Pythagorean triple.

(16) $\{X : X \text{ is non degenerate} \wedge X \text{ is simplified}\}$ is infinite.

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Some Remarks on Finite Sequences on Go-boards[1]

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Summary. This paper shows some properties of finite sequences on Go-boards. It also provides the partial correspondence between two ways of decomposition of curves induced by cages.

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The articles [20], [24], [8], [19], [9], [2], [3], [22], [4], [15], [14], [16], [18], [5], [7], [13], [1], [6], [12], [17], [23], [21], [10], and [11] provide the terminology and notation for this paper.

We follow the rules: i, j, k, n denote natural numbers, f denotes a finite sequence of elements of the carrier of $\mathcal{E}^2_{\rm T}$, and G denotes a Go-board.

We now state several propositions:

(1) Suppose that
- (i) f is a sequence which elements belong to G,
- (ii) $\mathcal{L}(G \circ (i,j), G \circ (i,k))$ meets $\widetilde{\mathcal{L}}(f)$,
- (iii) $\langle i,\, j\rangle \in$ the indices of G,
- (iv) $\langle i,\, k\rangle \in$ the indices of G, and
- (v) $j \leqslant k$.

Then there exists n such that $j \leqslant n$ and $n \leqslant k$ and $(G \circ (i,n))_{\mathbf{2}} = \inf(\mathrm{proj2}^\circ(\mathcal{L}(G \circ (i,j), G \circ (i,k)) \cap \widetilde{\mathcal{L}}(f)))$.

(2) Suppose that
- (i) f is a sequence which elements belong to G,
- (ii) $\mathcal{L}(G \circ (i,j), G \circ (i,k))$ meets $\widetilde{\mathcal{L}}(f)$,
- (iii) $\langle i,\, j\rangle \in$ the indices of G,
- (iv) $\langle i,\, k\rangle \in$ the indices of G, and

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(v) $j \leqslant k$.

Then there exists n such that $j \leqslant n$ and $n \leqslant k$ and $(G \circ (i, n))_{\mathbf{2}} = \sup(\mathrm{proj2}^{\circ}(\mathcal{L}(G \circ (i, j), G \circ (i, k)) \cap \widetilde{\mathcal{L}}(f)))$.

(3) Suppose that
(i) f is a sequence which elements belong to G,
(ii) $\mathcal{L}(G \circ (j, i), G \circ (k, i))$ meets $\widetilde{\mathcal{L}}(f)$,
(iii) $\langle j,\, i\rangle \in$ the indices of G,
(iv) $\langle k,\, i\rangle \in$ the indices of G, and
(v) $j \leqslant k$.

Then there exists n such that $j \leqslant n$ and $n \leqslant k$ and $(G \circ (n, i))_{\mathbf{1}} = \inf(\mathrm{proj1}^{\circ}(\mathcal{L}(G \circ (j, i), G \circ (k, i)) \cap \widetilde{\mathcal{L}}(f)))$.

(4) Suppose that
(i) f is a sequence which elements belong to G,
(ii) $\mathcal{L}(G \circ (j, i), G \circ (k, i))$ meets $\widetilde{\mathcal{L}}(f)$,
(iii) $\langle j,\, i\rangle \in$ the indices of G,
(iv) $\langle k,\, i\rangle \in$ the indices of G, and
(v) $j \leqslant k$.

Then there exists n such that $j \leqslant n$ and $n \leqslant k$ and $(G \circ (n, i))_{\mathbf{1}} = \sup(\mathrm{proj1}^{\circ}(\mathcal{L}(G \circ (j, i), G \circ (k, i)) \cap \widetilde{\mathcal{L}}(f)))$.

(5) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number n holds $(\mathrm{UpperSeq}(C, n))_1 = \text{W-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$.

(6) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number n holds $(\mathrm{LowerSeq}(C, n))_1 = \text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$.

(7) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number n holds $(\mathrm{UpperSeq}(C, n))_{\mathrm{len}\,\mathrm{UpperSeq}(C,n)} = \text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$.

(8) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number n holds $(\mathrm{LowerSeq}(C, n))_{\mathrm{len}\,\mathrm{LowerSeq}(C,n)} = \text{W-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$.

(9) Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Then $\widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C, n)) = \mathrm{UpperArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$ and $\widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C, n)) = \mathrm{LowerArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$ or $\widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C, n)) = \mathrm{LowerArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$ and $\widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C, n)) = \mathrm{UpperArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$.

We adopt the following convention: C is a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ satisfying conditions of simple closed curve, p is a point of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, j_1, i_2, j_2 are natural numbers.

Next we state four propositions:

(10) Let C be a connected compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Then $\mathrm{UpperSeq}(C, n)$ is a sequence which elements belong to $\mathrm{Gauge}(C, n)$.

(11) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) f is a sequence which elements belong to G,

(ii) there exist i, j such that $\langle i,\ j\rangle \in$ the indices of G and $p = G \circ (i, j)$, and

(iii) for all i_1, j_1, i_2, j_2 such that $\langle i_1,\ j_1\rangle \in$ the indices of G and $\langle i_2,\ j_2\rangle \in$ the indices of G and $p = G \circ (i_1, j_1)$ and $f_1 = G \circ (i_2, j_2)$ holds $|i_2 - i_1| + |j_2 - j_1| = 1$.

Then $\langle p\rangle \frown f$ is a sequence which elements belong to G.

(12) Let C be a connected compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Then $\mathrm{LowerSeq}(C, n)$ is a sequence which elements belong to $\mathrm{Gauge}(C, n)$.

(13) Suppose $p_{\mathbf{1}} = \frac{\text{W-bound}\,C + \text{E-bound}\,C}{2}$ and $p_{\mathbf{2}} = \inf(\mathrm{proj2}^{\circ}(\mathcal{L}(\mathrm{Gauge}(C, 1) \circ (\mathrm{Center}\,\mathrm{Gauge}(C, 1), 1), \mathrm{Gauge}(C, 1) \circ (\mathrm{Center}\,\mathrm{Gauge}(C, 1), \mathrm{width}\,\mathrm{Gauge}(C, 1))) \cap \mathrm{UpperArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C, i+1))))$. Then there exists j such that $1 \leqslant j$ and $j \leqslant \mathrm{width}\,\mathrm{Gauge}(C, i+1)$ and $p = \mathrm{Gauge}(C, i+1) \circ (\mathrm{Center}\,\mathrm{Gauge}(C, i+1), j)$.

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Upper and Lower Sequence on the Cage. Part II[1]

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The terminology and notation used here are introduced in the following articles: [29], [9], [22], [10], [1], [3], [27], [5], [25], [4], [16], [20], [15], [17], [19], [12], [21], [6], [8], [14], [23], [7], [2], [13], [30], [18], [26], [28], [24], and [11].

In this paper n is a natural number.

Let us note that there exists a finite sequence which is trivial.

The following proposition is true

(1) For every trivial finite sequence f holds f is empty or there exists a set x such that $f = \langle x \rangle$.

Let p be a non trivial finite sequence. Observe that $\mathrm{Rev}(p)$ is non trivial.

We now state four propositions:

(2) Let D be a non empty set, f be a finite sequence of elements of D, G be a matrix over D, and p be a set. Suppose f is a sequence which elements belong to G. Then $f -: p$ is a sequence which elements belong to G.

(3) Let D be a non empty set, f be a finite sequence of elements of D, G be a matrix over D, and p be an element of D. Suppose $p \in \mathrm{rng}\, f$. Suppose f is a sequence which elements belong to G. Then $f :- p$ is a sequence which elements belong to G.

(4) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$. Then $\mathrm{UpperSeq}(C, n)$ is a sequence which elements belong to $\mathrm{Gauge}(C, n)$.

(5) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$. Then $\mathrm{LowerSeq}(C, n)$ is a sequence which elements belong to $\mathrm{Gauge}(C, n)$.

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Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. Note that UpperSeq(C, n) is standard and LowerSeq(C, n) is standard.

One can prove the following propositions:

(6) Let G be a column **Y**-constant line **Y**-increasing matrix over $\mathcal{E}^2_{\mathrm{T}}$ and i_1, i_2, j_1, j_2 be natural numbers. Suppose $\langle i_1, j_1\rangle \in$ the indices of G and $\langle i_2, j_2\rangle \in$ the indices of G. If $(G \circ (i_1, j_1))_{\mathbf{2}} = (G \circ (i_2, j_2))_{\mathbf{2}}$, then $j_1 = j_2$.

(7) Let G be a line **X**-constant column **X**-increasing matrix over $\mathcal{E}^2_{\mathrm{T}}$ and i_1, i_2, j_1, j_2 be natural numbers. Suppose $\langle i_1, j_1\rangle \in$ the indices of G and $\langle i_2, j_2\rangle \in$ the indices of G. If $(G \circ (i_1, j_1))_{\mathbf{1}} = (G \circ (i_2, j_2))_{\mathbf{1}}$, then $i_1 = i_2$.

(8) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds N-min $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(9) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds N-max $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(10) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds E-min $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(11) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds E-max $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(12) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds S-min $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(13) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds S-max $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(14) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds W-min $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(15) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds W-max $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(16) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $f_1 \neq$ N-min $\widetilde{\mathcal{L}}(f)$ and $f_{\operatorname{len} f} \neq$ N-min $\widetilde{\mathcal{L}}(f)$ or $f_1 \neq$ N-max $\widetilde{\mathcal{L}}(f)$ and $f_{\operatorname{len} f} \neq$ N-max $\widetilde{\mathcal{L}}(f)$, then (N-min $\widetilde{\mathcal{L}}(f))_{\mathbf{1}} <$ (N-max $\widetilde{\mathcal{L}}(f))_{\mathbf{1}}$.

(17) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $f_1 \neq$ N-min $\widetilde{\mathcal{L}}(f)$ and $f_{\operatorname{len} f} \neq$ N-min $\widetilde{\mathcal{L}}(f)$ or $f_1 \neq$ N-max $\widetilde{\mathcal{L}}(f)$ and $f_{\operatorname{len} f} \neq$ N-max $\widetilde{\mathcal{L}}(f)$, then N-min $\widetilde{\mathcal{L}}(f) \neq$ N-max $\widetilde{\mathcal{L}}(f)$.

(18) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $f_1 \neq$ S-min $\widetilde{\mathcal{L}}(f)$ and $f_{\operatorname{len} f} \neq$ S-min $\widetilde{\mathcal{L}}(f)$ or $f_1 \neq$ S-max $\widetilde{\mathcal{L}}(f)$ and $f_{\operatorname{len} f} \neq$ S-max $\widetilde{\mathcal{L}}(f)$, then (S-min $\widetilde{\mathcal{L}}(f))_{\mathbf{1}} <$ (S-max $\widetilde{\mathcal{L}}(f))_{\mathbf{1}}$.

(19) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $f_1 \neq$ S-min $\widetilde{\mathcal{L}}(f)$ and $f_{\operatorname{len} f} \neq$ S-min $\widetilde{\mathcal{L}}(f)$ or $f_1 \neq$

S-max $\widetilde{\mathcal{L}}(f)$ and $f_{\mathrm{len}\, f} \neq$ S-max $\widetilde{\mathcal{L}}(f)$, then S-min $\widetilde{\mathcal{L}}(f) \neq$ S-max $\widetilde{\mathcal{L}}(f)$.

(20) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $f_1 \neq$ W-min $\widetilde{\mathcal{L}}(f)$ and $f_{\mathrm{len}\, f} \neq$ W-min $\widetilde{\mathcal{L}}(f)$ or $f_1 \neq$ W-max $\widetilde{\mathcal{L}}(f)$ and $f_{\mathrm{len}\, f} \neq$ W-max $\widetilde{\mathcal{L}}(f)$, then $(\text{W-min}\, \widetilde{\mathcal{L}}(f))_{\mathbf{2}} < (\text{W-max}\, \widetilde{\mathcal{L}}(f))_{\mathbf{2}}$.

(21) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $f_1 \neq$ W-min $\widetilde{\mathcal{L}}(f)$ and $f_{\mathrm{len}\, f} \neq$ W-min $\widetilde{\mathcal{L}}(f)$ or $f_1 \neq$ W-max $\widetilde{\mathcal{L}}(f)$ and $f_{\mathrm{len}\, f} \neq$ W-max $\widetilde{\mathcal{L}}(f)$, then W-min $\widetilde{\mathcal{L}}(f) \neq$ W-max $\widetilde{\mathcal{L}}(f)$.

(22) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $f_1 \neq$ E-min $\widetilde{\mathcal{L}}(f)$ and $f_{\mathrm{len}\, f} \neq$ E-min $\widetilde{\mathcal{L}}(f)$ or $f_1 \neq$ E-max $\widetilde{\mathcal{L}}(f)$ and $f_{\mathrm{len}\, f} \neq$ E-max $\widetilde{\mathcal{L}}(f)$, then $(\text{E-min}\, \widetilde{\mathcal{L}}(f))_{\mathbf{2}} < (\text{E-max}\, \widetilde{\mathcal{L}}(f))_{\mathbf{2}}$.

(23) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $f_1 \neq$ E-min $\widetilde{\mathcal{L}}(f)$ and $f_{\mathrm{len}\, f} \neq$ E-min $\widetilde{\mathcal{L}}(f)$ or $f_1 \neq$ E-max $\widetilde{\mathcal{L}}(f)$ and $f_{\mathrm{len}\, f} \neq$ E-max $\widetilde{\mathcal{L}}(f)$, then E-min $\widetilde{\mathcal{L}}(f) \neq$ E-max $\widetilde{\mathcal{L}}(f)$.

(24) Let D be a non empty set, f be a finite sequence of elements of D, and p, q be elements of D. If $p \in \mathrm{rng}\, f$ and $q \in \mathrm{rng}\, f$ and $q \leftarrow\!\!\!\!\!\!\!\! \hookleftarrow f \leqslant p \hookleftarrow f$, then $(f -: p) :- q = (f :- q) -: p$.

(25) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Then $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n) -: \text{W-min}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n) :- \text{W-min}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) = \{\text{N-min}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)), \text{W-min}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))\}$.

(26) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{LowerSeq}(C,n) = ((\mathrm{Cage}(C,n))^{\text{E-max}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}_{\circlearrowleft}) -: \text{W-min}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$.

(27) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\text{W-min}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \hookleftarrow \mathrm{UpperSeq}(C,n) = 1$.

(28) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\text{W-min}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \hookleftarrow \mathrm{UpperSeq}(C,n) < (\text{W-max}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \hookleftarrow \mathrm{UpperSeq}(C,n)$.

(29) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\text{W-max}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \hookleftarrow \mathrm{UpperSeq}(C,n) \leqslant (\text{N-min}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \hookleftarrow \mathrm{UpperSeq}(C,n)$.

(30) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\text{N-min}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \hookleftarrow \mathrm{UpperSeq}(C,n) < (\text{N-max}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \hookleftarrow \mathrm{UpperSeq}(C,n)$.

(31) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\text{N-max}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \hookleftarrow \mathrm{UpperSeq}(C,n) \leqslant (\text{E-max}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \hookleftarrow \mathrm{UpperSeq}(C,n)$.

(32) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \looparrowleft \mathrm{UpperSeq}(C,n) = \mathrm{len}\,\mathrm{UpperSeq}(C,n)$.

(33) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \looparrowleft \mathrm{LowerSeq}(C,n) = 1$.

(34) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \looparrowleft \mathrm{LowerSeq}(C,n) < (\text{E-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \looparrowleft \mathrm{LowerSeq}(C,n)$.

(35) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\text{E-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \looparrowleft \mathrm{LowerSeq}(C,n) \leqslant (\text{S-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \looparrowleft \mathrm{LowerSeq}(C,n)$.

(36) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\text{S-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \looparrowleft \mathrm{LowerSeq}(C,n) < (\text{S-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \looparrowleft \mathrm{LowerSeq}(C,n)$.

(37) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\text{S-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \looparrowleft \mathrm{LowerSeq}(C,n) \leqslant (\text{W-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \looparrowleft \mathrm{LowerSeq}(C,n)$.

(38) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\text{W-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \looparrowleft \mathrm{LowerSeq}(C,n) = \mathrm{len}\,\mathrm{LowerSeq}(C,n)$.

(39) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $((\mathrm{UpperSeq}(C,n))_2)_{\mathbf{1}} = \text{W-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$.

(40) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $((\mathrm{LowerSeq}(C,n))_2)_{\mathbf{1}} = \text{E-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$.

(41) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $\text{W-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) + \text{E-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \text{W-bound}\,C + \text{E-bound}\,C$.

(42) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $\text{S-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) + \text{N-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \text{S-bound}\,C + \text{N-bound}\,C$.

(43) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n, i be natural numbers. If $1 \leqslant i$ and $i \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $n > 0$, then $(\mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n), i))_{\mathbf{1}} = \frac{\text{W-bound}\,C + \text{E-bound}\,C}{2}$.

(44) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n, i be natural numbers. If $1 \leqslant i$ and $i \leqslant \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $n > 0$, then $(\mathrm{Gauge}(C,n) \circ (i, \mathrm{Center}\,\mathrm{Gauge}(C,n)))_{\mathbf{2}} = \frac{\text{S-bound}\,C + \text{N-bound}\,C}{2}$.

(45) Let f be a S-sequence in $\mathbb{R}^2$ and k_1, k_2 be natural numbers. If $1 \leqslant k_1$ and $k_1 \leqslant \mathrm{len}\,f$ and $1 \leqslant k_2$ and $k_2 \leqslant \mathrm{len}\,f$ and $f_1 \in \widetilde{\mathcal{L}}(\mathrm{mid}(f, k_1, k_2))$, then $k_1 = 1$ or $k_2 = 1$.

(46) Let f be a S-sequence in $\mathbb{R}^2$ and k_1, k_2 be natural numbers. If $1 \leqslant k_1$ and $k_1 \leqslant \mathrm{len}\,f$ and $1 \leqslant k_2$ and $k_2 \leqslant \mathrm{len}\,f$ and $f_{\mathrm{len}\,f} \in \widetilde{\mathcal{L}}(\mathrm{mid}(f, k_1, k_2))$, then $k_1 = \mathrm{len}\,f$ or $k_2 = \mathrm{len}\,f$.

(47) Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Then $\operatorname{rng} \operatorname{UpperSeq}(C,n) \subseteq \operatorname{rng} \operatorname{Cage}(C,n)$ and $\operatorname{rng} \operatorname{LowerSeq}(C,n) \subseteq \operatorname{rng} \operatorname{Cage}(C,n)$.

(48) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $\operatorname{UpperSeq}(C,n)$ is a h.c. for $\operatorname{Cage}(C,n)$.

(49) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $\operatorname{Rev}(\operatorname{LowerSeq}(C,n))$ is a h.c. for $\operatorname{Cage}(C,n)$.

(50) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i be a natural number. If $1 < i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(C,n)$, then $\operatorname{Gauge}(C,n) \circ (i,1) \notin \operatorname{rng} \operatorname{UpperSeq}(C,n)$.

(51) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i be a natural number. If $1 \leqslant i$ and $i < \operatorname{len} \operatorname{Gauge}(C,n)$, then $\operatorname{Gauge}(C,n) \circ (i, \operatorname{width} \operatorname{Gauge}(C,n)) \notin \operatorname{rng} \operatorname{LowerSeq}(C,n)$.

(52) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i be a natural number. If $1 < i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(C,n)$, then $\operatorname{Gauge}(C,n) \circ (i,1) \notin \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C,n))$.

(53) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i be a natural number. If $1 \leqslant i$ and $i < \operatorname{len} \operatorname{Gauge}(C,n)$, then $\operatorname{Gauge}(C,n) \circ (i, \operatorname{width} \operatorname{Gauge}(C,n)) \notin \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C,n))$.

(54) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant \operatorname{width} \operatorname{Gauge}(C,n)$ and $\operatorname{Gauge}(C,n) \circ (i,j) \in \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$. Then $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i,1), \operatorname{Gauge}(C,n) \circ (i,j))$ meets $\widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C,n))$.

(55) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. If $n > 0$, then $\operatorname{FPoint}(\widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C,n)), \text{W-min}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)), \text{E-max}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)),$ $\operatorname{VerticalLine} \frac{\text{W-bound}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)) + \text{E-bound}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))}{2}) \in \operatorname{rng} \operatorname{UpperSeq}(C,n)$.

(56) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. If $n > 0$, then $\operatorname{LPoint}(\widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C,n)), \text{E-max}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)), \text{W-min}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)),$ $\operatorname{VerticalLine} \frac{\text{W-bound}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)) + \text{E-bound}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))}{2}) \in \operatorname{rng} \operatorname{LowerSeq}(C,n)$.

(57) For every S-sequence f in $\mathbb{R}^2$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \operatorname{rng} f$ holds $\downharpoonleft f, p = \operatorname{mid}(f, 1, p \hookleftarrow f)$.

(58) Let f be a S-sequence in $\mathbb{R}^2$ and Q be a closed subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $\widetilde{\mathcal{L}}(f)$ meets Q and $f_1 \notin Q$ and $\operatorname{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\operatorname{len} f}, Q) \in \operatorname{rng} f$. Then $\widetilde{\mathcal{L}}(\operatorname{mid}(f, 1, (\operatorname{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\operatorname{len} f}, Q)) \hookleftarrow f)) \cap Q = \{\operatorname{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\operatorname{len} f}, Q)\}$.

(59) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Suppose $n > 0$.

Let k be a natural number. Suppose $1 \leqslant k$ and $k < $ (FPoint($\widetilde{\mathcal{L}}$(UpperSeq(C,n)), W-min $\widetilde{\mathcal{L}}$(Cage(C,n)), E-max $\widetilde{\mathcal{L}}$(Cage(C,n)), VerticalLine $\frac{\text{W-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))+\text{E-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))}{2}$)) $\hookleftarrow$ UpperSeq(C,n). Then $((\text{UpperSeq}(C,n))_k)_{\mathbf{1}} < \frac{\text{W-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))+\text{E-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))}{2}$.

(60) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Suppose $n > 0$. Let k be a natural number. Suppose $1 \leqslant k$ and $k < $ (FPoint($\widetilde{\mathcal{L}}$(Rev(LowerSeq(C,n))), W-min $\widetilde{\mathcal{L}}$(Cage(C,n)), E-max $\widetilde{\mathcal{L}}$(Cage (C,n)), VerticalLine $\frac{\text{W-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))+\text{E-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))}{2}$)) $\hookleftarrow$ Rev(LowerSeq(C,n)). Then $((\text{Rev}(\text{LowerSeq}(C,n)))_k)_{\mathbf{1}} < \frac{\text{W-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))+\text{E-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))}{2}$.

(61) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Suppose $n > 0$. Let q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $q \in$ rng mid(UpperSeq(C,n), 2, (FPoint($\widetilde{\mathcal{L}}$(UpperSeq(C,n)), W-min $\widetilde{\mathcal{L}}$(Cage (C,n)), E-max $\widetilde{\mathcal{L}}$(Cage(C,n)), VerticalLine $\frac{\text{W-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))+\text{E-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))}{2}$)) $\hookleftarrow$ UpperSeq(C,n))). Then $q_{\mathbf{1}} \leqslant \frac{\text{W-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))+\text{E-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))}{2}$.

(62) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Suppose $n > 0$. Then (FPoint($\widetilde{\mathcal{L}}$(UpperSeq(C,n)), W-min $\widetilde{\mathcal{L}}$(Cage(C,n)), E-max $\widetilde{\mathcal{L}}$(Cage(C,n)), VerticalLine $\frac{\text{W-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))+\text{E-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))}{2}$))$_{\mathbf{2}} >$ (LPoint($\widetilde{\mathcal{L}}$ (LowerSeq(C,n)), E-max $\widetilde{\mathcal{L}}$(Cage(C,n)), W-min $\widetilde{\mathcal{L}}$(Cage(C,n)), VerticalLine $\frac{\text{W-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))+\text{E-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))}{2}$))$_{\mathbf{2}}$.

(63) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. If $n > 0$, then $\widetilde{\mathcal{L}}$(UpperSeq(C,n)) = UpperArc $\widetilde{\mathcal{L}}$(Cage(C,n)).

(64) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. If $n > 0$, then $\widetilde{\mathcal{L}}$(LowerSeq(C,n)) = LowerArc $\widetilde{\mathcal{L}}$(Cage(C,n)).

(65) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Suppose $n > 0$. Let i, j be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant$ len Gauge(C,n) and $1 \leqslant j$ and $j \leqslant$ width Gauge(C,n) and Gauge$(C,n) \circ (i,j) \in \widetilde{\mathcal{L}}$(Cage$(C,n)$). Then $\mathcal{L}$(Gauge$(C,n) \circ (i,1)$, Gauge$(C,n) \circ (i,j)$) meets LowerArc $\widetilde{\mathcal{L}}$(Cage(C,n)).

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Zero-Based Finite Sequences

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The terminology and notation used in this paper are introduced in the following papers: [11], [4], [7], [6], [5], [1], [3], [2], [8], [12], [13], [10], and [9].

We follow the rules: k, n are natural numbers, x, y, z, y_1, y_2, X are sets, and f is a function.

One can prove the following propositions:

(1) $n \in n+1$.

(2) If $k \leqslant n$, then $k = k \cap n$.

(3) If $k = k \cap n$, then $k \leqslant n$.

(4) $n \cup \{n\} = n+1$.

(5) $\mathrm{Seg}\, n \subseteq n+1$.

(6) $n+1 = \{0\} \cup \mathrm{Seg}\, n$.

(7) For every function r holds r is finite and transfinite sequence-like iff there exists n such that $\mathrm{dom}\, r = n$.

Let us mention that there exists a function which is finite and transfinite sequence-like.

A finite 0-sequence is a finite transfinite sequence.

In the sequel p, q, r denote finite 0-sequences.

Observe that every set which is natural is also finite. Let us consider p. One can verify that $\mathrm{dom}\, p$ is natural.

Let us consider p. Then $\overline{\overline{p}}$ is a natural number and it can be characterized by the condition:

(Def. 1) $\overline{\overline{p}} = \mathrm{dom}\, p$.

We introduce $\mathrm{len}\, p$ as a synonym of $\overline{\overline{p}}$.

Let us consider p. Then $\mathrm{dom}\, p$ is a subset of $\mathbb{N}$.

Next we state the proposition

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(8) If there exists k such that $\mathrm{dom}\, f \subseteq k$, then there exists p such that $f \subseteq p$.

In this article we present several logical schemes. The scheme *XSeqEx* deals with a natural number $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists p such that $\mathrm{dom}\, p = \mathcal{A}$ and for every k such that $k \in \mathcal{A}$ holds $\mathcal{P}[k, p(k)]$

provided the following conditions are satisfied:

- For all k, y_1, y_2 such that $k \in \mathcal{A}$ and $\mathcal{P}[k, y_1]$ and $\mathcal{P}[k, y_2]$ holds $y_1 = y_2$, and
- For every k such that $k \in \mathcal{A}$ there exists x such that $\mathcal{P}[k, x]$.

The scheme *SeqLambda* deals with a natural number $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a finite 0-sequence p such that $\mathrm{len}\, p = \mathcal{A}$ and for every k such that $k \in \mathcal{A}$ holds $p(k) = \mathcal{F}(k)$

for all values of the parameters.

Next we state several propositions:

(9) If $z \in p$, then there exists k such that $k \in \mathrm{dom}\, p$ and $z = \langle k, p(k) \rangle$.

(10) If $\mathrm{dom}\, p = \mathrm{dom}\, q$ and for every k such that $k \in \mathrm{dom}\, p$ holds $p(k) = q(k)$, then $p = q$.

(11) If $\mathrm{len}\, p = \mathrm{len}\, q$ and for every k such that $k < \mathrm{len}\, p$ holds $p(k) = q(k)$, then $p = q$.

(12) $p \restriction n$ is a finite 0-sequence.

(13) If $\mathrm{rng}\, p \subseteq \mathrm{dom}\, f$, then $f \cdot p$ is a finite 0-sequence.

(14) If $k < \mathrm{len}\, p$ and $q = p \restriction k$, then $\mathrm{len}\, q = k$ and $\mathrm{dom}\, q = k$.

Let D be a set. Observe that there exists a transfinite sequence of elements of D which is finite.

Let D be a set. A finite 0-sequence of D is a finite transfinite sequence of elements of D.

We now state the proposition

(15) For every set D holds every finite 0-sequence of D is a partial function from $\mathbb{N}$ to D.

One can verify that $\emptyset$ is transfinite sequence-like.

Let D be a set. Observe that there exists a partial function from $\mathbb{N}$ to D which is finite and transfinite sequence-like.

In the sequel D is a set.

Next we state two propositions:

(16) For every finite 0-sequence p of D holds $p \restriction k$ is a finite 0-sequence of D.

(17) For every non empty set D there exists a finite 0-sequence p of D such that $\mathrm{len}\, p = k$.

One can verify that there exists a finite 0-sequence which is empty.

One can prove the following propositions:

(18) $\operatorname{len} p = 0$ iff $p = \emptyset$.

(19) For every set D holds $\emptyset$ is a finite 0-sequence of D.

Let D be a set. One can verify that there exists a finite 0-sequence of D which is empty.

Let us consider x. The functor $\langle_0 x\rangle$ yielding a set is defined as follows:

(Def. 2) $\langle_0 x\rangle = \{\langle 0, x\rangle\}$.

Let D be a set. The functor $\langle\rangle_D$ yields an empty finite 0-sequence of D and is defined by:

(Def. 3) $\langle\rangle_D = \emptyset$.

Let us consider p, q. Observe that $p \frown q$ is finite. Then $p \frown q$ can be characterized by the condition:

(Def. 4) $\operatorname{dom}(p \frown q) = \operatorname{len} p + \operatorname{len} q$ and for every k such that $k \in \operatorname{dom} p$ holds $(p \frown q)(k) = p(k)$ and for every k such that $k \in \operatorname{dom} q$ holds $(p \frown q)(\operatorname{len} p + k) = q(k)$.

The following propositions are true:

(20) $\operatorname{len}(p \frown q) = \operatorname{len} p + \operatorname{len} q$.

(21) If $\operatorname{len} p \leqslant k$ and $k < \operatorname{len} p + \operatorname{len} q$, then $(p \frown q)(k) = q(k - \operatorname{len} p)$.

(22) If $\operatorname{len} p \leqslant k$ and $k < \operatorname{len}(p \frown q)$, then $(p \frown q)(k) = q(k - \operatorname{len} p)$.

(23) If $k \in \operatorname{dom}(p \frown q)$, then $k \in \operatorname{dom} p$ or there exists n such that $n \in \operatorname{dom} q$ and $k = \operatorname{len} p + n$.

(24) For all transfinite sequences p, q holds $\operatorname{dom} p \subseteq \operatorname{dom}(p \frown q)$.

(25) If $x \in \operatorname{dom} q$, then there exists k such that $k = x$ and $\operatorname{len} p + k \in \operatorname{dom}(p \frown q)$.

(26) If $k \in \operatorname{dom} q$, then $\operatorname{len} p + k \in \operatorname{dom}(p \frown q)$.

(27) $\operatorname{rng} p \subseteq \operatorname{rng}(p \frown q)$.

(28) $\operatorname{rng} q \subseteq \operatorname{rng}(p \frown q)$.

(29) $\operatorname{rng}(p \frown q) = \operatorname{rng} p \cup \operatorname{rng} q$.

(30) $(p \frown q) \frown r = p \frown (q \frown r)$.

(31) If $p \frown r = q \frown r$ or $r \frown p = r \frown q$, then $p = q$.

(32) $p \frown \emptyset = p$ and $\emptyset \frown p = p$.

(33) If $p \frown q = \emptyset$, then $p = \emptyset$ and $q = \emptyset$.

Let D be a set and let p, q be finite 0-sequences of D. Then $p \frown q$ is a transfinite sequence of elements of D.

Let us consider x. Then $\langle_0 x\rangle$ is a function and it can be characterized by the condition:

(Def. 5) $\operatorname{dom}\langle_0 x\rangle = 1$ and $\langle_0 x\rangle(0) = x$.

Let us consider x. One can verify that $\langle_0 x\rangle$ is function-like and relation-like.

Let us consider x. One can check that $\langle_0 x\rangle$ is finite and transfinite sequence-like.

One can prove the following proposition

(34) Suppose $p \frown q$ is a finite 0-sequence of D. Then p is a finite 0-sequence of D and q is a finite 0-sequence of D.

Let us consider x, y. The functor $\langle_0 x, y\rangle$ yielding a set is defined by:

(Def. 6) $\langle_0 x, y\rangle = \langle_0 x\rangle \frown \langle_0 y\rangle$.

Let us consider z. The functor $\langle_0 x, y, z\rangle$ yields a set and is defined by:

(Def. 7) $\langle_0 x, y, z\rangle = \langle_0 x\rangle \frown \langle_0 y\rangle \frown \langle_0 z\rangle$.

Let us consider x, y. One can check that $\langle_0 x, y\rangle$ is function-like and relation-like. Let us consider z. One can verify that $\langle_0 x, y, z\rangle$ is function-like and relation-like.

Let us consider x, y. One can check that $\langle_0 x, y\rangle$ is finite and transfinite sequence-like. Let us consider z. Observe that $\langle_0 x, y, z\rangle$ is finite and transfinite sequence-like.

One can prove the following propositions:

(35) $\langle_0 x\rangle = \{\langle 0, x\rangle\}$.

(36) $p = \langle_0 x\rangle$ iff $\operatorname{dom} p = 1$ and $\operatorname{rng} p = \{x\}$.

(37) $p = \langle_0 x\rangle$ iff $\operatorname{len} p = 1$ and $\operatorname{rng} p = \{x\}$.

(38) $p = \langle_0 x\rangle$ iff $\operatorname{len} p = 1$ and $p(0) = x$.

(39) $(\langle_0 x\rangle \frown p)(0) = x$.

(40) $(p \frown \langle_0 x\rangle)(\operatorname{len} p) = x$.

(41) $\langle_0 x, y, z\rangle = \langle_0 x\rangle \frown \langle_0 y, z\rangle$ and $\langle_0 x, y, z\rangle = \langle_0 x, y\rangle \frown \langle_0 z\rangle$.

(42) $p = \langle_0 x, y\rangle$ iff $\operatorname{len} p = 2$ and $p(0) = x$ and $p(1) = y$.

(43) $p = \langle_0 x, y, z\rangle$ iff $\operatorname{len} p = 3$ and $p(0) = x$ and $p(1) = y$ and $p(2) = z$.

(44) If $p \neq \emptyset$, then there exist q, x such that $p = q \frown \langle_0 x\rangle$.

Let D be a non empty set and let x be an element of D. Then $\langle_0 x\rangle$ is a finite 0-sequence of D.

The scheme *IndXSeq* concerns a unary predicate $\mathcal{P}$, and states that:

For every p holds $\mathcal{P}[p]$

provided the following conditions are met:

- $\mathcal{P}[\emptyset]$, and
- For all p, x such that $\mathcal{P}[p]$ holds $\mathcal{P}[p \frown \langle_0 x\rangle]$.

We now state the proposition

(45) For all finite 0-sequences p, q, r, s such that $p \frown q = r \frown s$ and $\operatorname{len} p \leqslant \operatorname{len} r$ there exists a finite 0-sequence t such that $p \frown t = r$.

Let D be a set. The functor D^ω yields a set and is defined as follows:

(Def. 8) $x \in D^\omega$ iff x is a finite 0-sequence of D.

Let D be a set. One can check that D^ω is non empty.

One can prove the following propositions:

(46) $x \in D^\omega$ iff x is a finite 0-sequence of D.

(47) $\emptyset \in D^\omega$.

The scheme *SepSeq* deals with a non empty set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

There exists X such that for every x holds $x \in X$ iff there exists p such that $p \in \mathcal{A}^\omega$ and $\mathcal{P}[p]$ and $x = p$

for all values of the parameters.

Let p be a finite 0-sequence and let i, x be sets. Note that $p +\cdot (i, x)$ is finite and transfinite sequence-like. We introduce $\mathrm{Replace}(p, i, x)$ as a synonym of $p +\cdot (i, x)$.

One can prove the following proposition

(48) Let p be a finite 0-sequence, i be a natural number, and x be a set. Then $\mathrm{len}\,\mathrm{Replace}(p, i, x) = \mathrm{len}\,p$ and if $i < \mathrm{len}\,p$, then $(\mathrm{Replace}(p, i, x))(i) = x$ and for every natural number j such that $j \neq i$ holds $(\mathrm{Replace}(p, i, x))(j) = p(j)$.

Let D be a non empty set, let p be a finite 0-sequence of D, let i be a natural number, and let a be an element of D. Then $\mathrm{Replace}(p, i, a)$ is a finite 0-sequence of D.

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More on the External Approximation of a Continuum[1]

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Summary. The main goal was to prove two facts:

- the gauge is the Go-board of a corresponding cage,
- the left components of the complement of the curve determined by a cage are monotonic w.r.t. the index of the approximation.

Some auxiliary facts are proved, too. At the end new notions needed for internal approximation are defined and some useful lemmas are proved.

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The terminology and notation used in this paper have been introduced in the following articles: [28], [40], [1], [3], [12], [29], [14], [4], [5], [37], [33], [13], [6], [20], [21], [26], [32], [9], [35], [24], [18], [27], [25], [8], [11], [17], [2], [36], [38], [30], [10], [16], [41], [43], [42], [19], [23], [34], [39], [31], [15], [44], [22], and [7].

1. Preliminaries

For simplicity, we follow the rules: m, k, j, j_1, i, i_1, i_2, n are natural numbers, r, s, r_1, t are real numbers, C, D are compact non vertical non horizontal non empty subsets of $\mathcal{E}^2_{\rm T}$, f is a finite sequence of elements of the carrier of $\mathcal{E}^2_{\rm T}$, G is a Go-board, and p is a point of $\mathcal{E}^2_{\rm T}$.

We now state three propositions:

(1) For all sets A, x, y such that A meets $\{x, y\}$ holds $x \in A$ or $y \in A$.

(2) If $r < 0$ and $r_1 \leqslant r$ and $0 \leqslant t$, then $\frac{t}{r} \leqslant \frac{t}{r_1}$.

(3) For every set X and for every binary relation R such that R is reflexive in X holds $X \subseteq \operatorname{field} R$.

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Let us observe that there exists a set which has a non-empty element.

Let D be a non empty set with a non-empty element. Observe that there exists a finite sequence of elements of D^* which is non empty and non-empty.

Let D be a non empty set with non empty elements. One can check that there exists a finite sequence of elements of D^* which is non empty and non-empty.

Let F be a non-empty function yielding function. Note that $\mathrm{rng}_\kappa F(\kappa)$ is non-empty.

Let us note that every finite sequence of elements of $\mathbb{R}$ which is increasing is also one-to-one.

One can prove the following propositions:

(4) For all points p, q of $\mathcal{E}^2_\mathrm{T}$ holds $\mathcal{L}(p,q) \setminus \{p,q\}$ is convex.

(5) For all points p, q of $\mathcal{E}^2_\mathrm{T}$ holds $\mathcal{L}(p,q) \setminus \{p,q\}$ is connected.

(6) For all points p, q of $\mathcal{E}^2_\mathrm{T}$ such that $p \neq q$ holds $p \in \overline{\mathcal{L}(p,q) \setminus \{p,q\}}$.

(7) For all points p, q of $\mathcal{E}^2_\mathrm{T}$ such that $p \neq q$ holds $\overline{\mathcal{L}(p,q) \setminus \{p,q\}} = \mathcal{L}(p,q)$.

(8) Let S be a subset of the carrier of $\mathcal{E}^2_\mathrm{T}$ and p, q be points of $\mathcal{E}^2_\mathrm{T}$. If $p \neq q$ and $\mathcal{L}(p,q) \setminus \{p,q\} \subseteq S$, then $\mathcal{L}(p,q) \subseteq \overline{S}$.

2. Transforming Finite Sets to Finite Sequences

The binary relation RealOrd on $\mathbb{R}$ is defined by:

(Def. 1) $\mathrm{RealOrd} = \{\langle r,\ s\rangle : r \leqslant s\}$.

Next we state two propositions:

(9) If $\langle r,\ s\rangle \in \mathrm{RealOrd}$, then $r \leqslant s$.

(10) $\mathrm{field}\,\mathrm{RealOrd} = \mathbb{R}$.

Let us note that RealOrd is ordering and linear-order.

The following propositions are true:

(11) RealOrd linearly orders $\mathbb{R}$.

(12) For every finite subset A of $\mathbb{R}$ holds $\mathrm{SgmX}(\mathrm{RealOrd}, A)$ is increasing.

(13) For every finite sequence f of elements of $\mathbb{R}$ and for every finite subset A of $\mathbb{R}$ such that $A = \mathrm{rng}\, f$ holds $\mathrm{SgmX}(\mathrm{RealOrd}, A) = \mathrm{Inc}(f)$.

Let A be a finite subset of $\mathbb{R}$. One can verify that $\mathrm{SgmX}(\mathrm{RealOrd}, A)$ is increasing.

Next we state two propositions:

(14) Let X be a non empty set, A be a finite subset of X, and R be an order in X. If R linearly orders A, then $\mathrm{len}\,\mathrm{SgmX}(R, A) = \mathrm{card}\, A$.

(15) For every non empty set X and for every finite subset A of X and for every linear-order order R in X holds $\mathrm{len}\,\mathrm{SgmX}(R, A) = \mathrm{card}\, A$.

3. On the Construction of Go-boards

Next we state two propositions:

(16) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathbf{X}$-coordinate$(f) = $ proj1 $\cdot f$.

(17) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathbf{Y}$-coordinate$(f) = $ proj2 $\cdot f$.

Let D be a non empty set and let M be a finite sequence of elements of D^*. Then Values M is a subset of D.

Let D be a non empty set with non empty elements and let M be a non empty non-empty finite sequence of elements of D^*. One can verify that Values M is non empty.

The following propositions are true:

(18) For every non empty set D and for every matrix M over D and for every i such that $i \in \mathrm{Seg}\,\mathrm{width}\,M$ holds $\mathrm{rng}(M_{\square,i}) \subseteq \mathrm{Values}\,M$.

(19) For every non empty set D and for every matrix M over D and for every i such that $i \in \mathrm{dom}\,M$ holds $\mathrm{rng}\,\mathrm{Line}(M,i) \subseteq \mathrm{Values}\,M$.

(20) For every column $\mathbf{X}$-increasing non empty yielding matrix G over $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{len}\,G \leqslant \mathrm{card}(\mathrm{proj1}^\circ\,\mathrm{Values}\,G)$.

(21) For every line $\mathbf{X}$-constant matrix G over $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{card}(\mathrm{proj1}^\circ\,\mathrm{Values}\,G) \leqslant \mathrm{len}\,G$.

(22) For every line $\mathbf{X}$-constant column $\mathbf{X}$-increasing non empty yielding matrix G over $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{len}\,G = \mathrm{card}(\mathrm{proj1}^\circ\,\mathrm{Values}\,G)$.

(23) For every line $\mathbf{Y}$-increasing non empty yielding matrix G over $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{width}\,G \leqslant \mathrm{card}(\mathrm{proj2}^\circ\,\mathrm{Values}\,G)$.

(24) For every column $\mathbf{Y}$-constant non empty yielding matrix G over $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{card}(\mathrm{proj2}^\circ\,\mathrm{Values}\,G) \leqslant \mathrm{width}\,G$.

(25) For every column $\mathbf{Y}$-constant line $\mathbf{Y}$-increasing non empty yielding matrix G over $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{width}\,G = \mathrm{card}(\mathrm{proj2}^\circ\,\mathrm{Values}\,G)$.

4. More about Go-boards

Next we state several propositions:

(26) For every standard special circular sequence f such that $1 \leqslant k$ and $k+1 \leqslant \mathrm{len}\,f$ holds $\mathcal{L}(f,k) \subseteq \mathrm{leftcell}(f,k)$.

(27) For every standard special circular sequence f such that $1 \leqslant k$ and $k+1 \leqslant \mathrm{len}\,f$ holds $\mathrm{left\_cell}(f,k,\text{the Go-board of } f) = \mathrm{leftcell}(f,k)$.

(28) For every standard special circular sequence f such that $1 \leqslant k$ and $k+1 \leqslant \mathrm{len}\,f$ holds $\mathcal{L}(f,k) \subseteq \mathrm{rightcell}(f,k)$.

(29) For every standard special circular sequence f such that $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$ holds right_cell$(f, k,$the Go-board of $f) = \operatorname{rightcell}(f, k)$.

(30) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a non constant standard special circular sequence. If P is a component of $(\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$, then $P = \operatorname{RightComp}(f)$ or $P = \operatorname{LeftComp}(f)$.

(31) Let f be a non constant standard special circular sequence. Suppose f is a sequence which elements belong to G. Let given k. If $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$, then Int right_cell$(f, k, G) \subseteq \operatorname{RightComp}(f)$ and Int left_cell$(f, k, G) \subseteq \operatorname{LeftComp}(f)$.

(32) Let i_1, j_1, i_2, j_2 be natural numbers and G be a Go-board. Suppose $\langle i_1, j_1\rangle \in$ the indices of G and $\langle i_2, j_2\rangle \in$ the indices of G and $G \circ (i_1, j_1) = G \circ (i_2, j_2)$. Then $i_1 = i_2$ and $j_1 = j_2$.

(33) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and M be a Go-board. Suppose f is a sequence which elements belong to M. Then $\operatorname{mid}(f, i_1, i_2)$ is a sequence which elements belong to M.

Let us mention that every Go-board is non empty and non-empty.

The following propositions are true:

(34) For every Go-board G such that $1 \leqslant i$ and $i \leqslant \operatorname{len} G$ holds $(\operatorname{SgmX}(\operatorname{RealOrd}, \operatorname{proj1}^\circ \operatorname{Values} G))(i) = (G \circ (i, 1))_{\mathbf{1}}$.

(35) For every Go-board G such that $1 \leqslant j$ and $j \leqslant \operatorname{width} G$ holds $(\operatorname{SgmX}(\operatorname{RealOrd}, \operatorname{proj2}^\circ \operatorname{Values} G))(j) = (G \circ (1, j))_{\mathbf{2}}$.

(36) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and G be a Go-board. Suppose that

(i) f is a sequence which elements belong to G,

(ii) there exists i such that $\langle 1, i\rangle \in$ the indices of G and $G \circ (1, i) \in \operatorname{rng} f$, and

(iii) there exists i such that $\langle \operatorname{len} G, i\rangle \in$ the indices of G and $G \circ (\operatorname{len} G, i) \in \operatorname{rng} f$.

Then $\operatorname{proj1}^\circ \operatorname{rng} f = \operatorname{proj1}^\circ \operatorname{Values} G$.

(37) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and G be a Go-board. Suppose that

(i) f is a sequence which elements belong to G,

(ii) there exists i such that $\langle i, 1\rangle \in$ the indices of G and $G \circ (i, 1) \in \operatorname{rng} f$, and

(iii) there exists i such that $\langle i, \operatorname{width} G\rangle \in$ the indices of G and $G \circ (i, \operatorname{width} G) \in \operatorname{rng} f$.

Then $\operatorname{proj2}^\circ \operatorname{rng} f = \operatorname{proj2}^\circ \operatorname{Values} G$.

Let G be a Go-board. Observe that $\operatorname{Values} G$ is non empty.

One can prove the following three propositions:

(38) For every Go-board G holds $G =$ the Go-board of SgmX(RealOrd,

proj1° Values G), SgmX(RealOrd, proj2° Values G).

(39) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and G be a Go-board. If proj1° rng f = proj1° Values G and proj2° rng f = proj2° Values G, then $G =$ the Go-board of f.

(40) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and G be a Go-board. Suppose that

(i) f is a sequence which elements belong to G,

(ii) there exists i such that $\langle 1, i\rangle \in$ the indices of G and $G \circ (1, i) \in \operatorname{rng} f$,

(iii) there exists i such that $\langle i, 1\rangle \in$ the indices of G and $G \circ (i, 1) \in \operatorname{rng} f$,

(iv) there exists i such that $\langle \operatorname{len} G, i\rangle \in$ the indices of G and $G \circ (\operatorname{len} G, i) \in \operatorname{rng} f$, and

(v) there exists i such that $\langle i, \operatorname{width} G\rangle \in$ the indices of G and $G \circ (i, \operatorname{width} G) \in \operatorname{rng} f$.

Then $G =$ the Go-board of f.

5. More about Gauges

The following propositions are true:

(41) If $m \leqslant n$ and $1 \leqslant i$ and $i + 1 \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$, then $\lfloor \frac{i-2}{2^{n-'m}} + 2 \rfloor$ is a natural number.

(42) If $m \leqslant n$ and $1 \leqslant i$ and $i + 1 \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$, then $1 \leqslant \lfloor \frac{i-2}{2^{n-'m}} + 2 \rfloor$ and $\lfloor \frac{i-2}{2^{n-'m}} + 2 \rfloor + 1 \leqslant \operatorname{len} \operatorname{Gauge}(C, m)$.

(43) Suppose $m \leqslant n$ and $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant j$ and $j + 1 \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$. Then there exist i_1, j_1 such that $i_1 = \lfloor \frac{i-2}{2^{n-'m}} + 2 \rfloor$ and $j_1 = \lfloor \frac{j-2}{2^{n-'m}} + 2 \rfloor$ and $\operatorname{cell}(\operatorname{Gauge}(C, n), i, j) \subseteq \operatorname{cell}(\operatorname{Gauge}(C, m), i_1, j_1)$.

(44) Suppose $m \leqslant n$ and $1 \leqslant i$ and $i + 1 \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant j$ and $j + 1 \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$. Then there exist i_1, j_1 such that $1 \leqslant i_1$ and $i_1 + 1 \leqslant \operatorname{len} \operatorname{Gauge}(C, m)$ and $1 \leqslant j_1$ and $j_1 + 1 \leqslant \operatorname{width} \operatorname{Gauge}(C, m)$ and $\operatorname{cell}(\operatorname{Gauge}(C, n), i, j) \subseteq \operatorname{cell}(\operatorname{Gauge}(C, m), i_1, j_1)$.

(45) If $i \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$, then $\operatorname{cell}(\operatorname{Gauge}(C, n), i, 0) \subseteq \operatorname{UBD} C$.

(46) If $i \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$, then $\operatorname{cell}(\operatorname{Gauge}(C, n), i, \operatorname{width} \operatorname{Gauge}(C, n)) \subseteq \operatorname{UBD} C$.

(47) For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that P is Bounded holds UBD P is not Bounded.

(48) Let f be a non constant standard special circular sequence. If $f^p_{\circlearrowleft}$ is clockwise oriented, then f is clockwise oriented.

(49) For every non constant standard special circular sequence f such that $\operatorname{LeftComp}(f) = \operatorname{UBD} \widetilde{\mathcal{L}}(f)$ holds f is clockwise oriented.

6. More about Cages

The following propositions are true:

(50) $\overline{\mathrm{LeftComp}(\mathrm{Cage}(C,i))}^{\mathrm{c}} = \mathrm{RightComp}(\mathrm{Cage}(C,i))$.

(51) If C is connected, then the Go-board of $\mathrm{Cage}(C,n) = \mathrm{Gauge}(C,n)$.

(52) If C is connected, then $\text{N-min}\, C \in \mathrm{rightcell}(\mathrm{Cage}(C,n),1)$.

(53) If C is connected and $i \leqslant j$, then $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,j)) \subseteq \overline{\mathrm{RightComp}(\mathrm{Cage}(C,i))}$.

(54) If C is connected and $i \leqslant j$, then $\mathrm{LeftComp}(\mathrm{Cage}(C,i)) \subseteq \mathrm{LeftComp}(\mathrm{Cage}(C,j))$.

(55) If C is connected and $i \leqslant j$, then $\mathrm{RightComp}(\mathrm{Cage}(C,j)) \subseteq \mathrm{RightComp}(\mathrm{Cage}(C,i))$.

7. Preparing the Internal Approximation

Let us consider C, n. The functor X-SpanStart(C,n) yielding a natural number is defined as follows:

(Def. 2) $\text{X-SpanStart}(C,n) = 2^{n -' 1} + 2$.

Next we state three propositions:

(56) $\text{X-SpanStart}(C,n) = \mathrm{Center}\,\mathrm{Gauge}(C,n)$.

(57) $2 < \text{X-SpanStart}(C,n)$ and $\text{X-SpanStart}(C,n) < \mathrm{len}\,\mathrm{Gauge}(C,n)$.

(58) $1 \leqslant \text{X-SpanStart}(C,n) -' 1$ and $\text{X-SpanStart}(C,n) -' 1 < \mathrm{len}\,\mathrm{Gauge}(C,n)$.

Let us consider C, n. We say that n is sufficiently large for C if and only if:

(Def. 3) There exists j such that $j < \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $\mathrm{cell}(\mathrm{Gauge}(C,n), \text{X-SpanStart}(C,n) -' 1, j) \subseteq \mathrm{BDD}\, C$.

One can prove the following propositions:

(59) If n is sufficiently large for C, then $n \geqslant 1$.

(60) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_1 be natural numbers. Suppose that

(i) $\mathrm{left\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_1)$,

(iv) $\langle i_1, j_1 + 1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1, j_1 + 1)$.

Then $\langle i_1 -' 1, j_1 + 1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(61) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_1 be natural numbers. Suppose that

- (i) $\mathrm{left\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,
- (ii) $\langle i_1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,
- (iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_1)$,
- (iv) $\langle i_1 + 1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and
- (v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1 + 1, j_1)$.

Then $\langle i_1 + 1, j_1 + 1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(62) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let j_1, i_2 be natural numbers. Suppose that

- (i) $\mathrm{left\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,
- (ii) $\langle i_2 + 1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,
- (iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_2 + 1, j_1)$,
- (iv) $\langle i_2, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and
- (v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_2, j_1)$.

Then $\langle i_2, j_1 -' 1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(63) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_2 be natural numbers. Suppose that

- (i) $\mathrm{left\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,
- (ii) $\langle i_1, j_2 + 1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,
- (iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_2 + 1)$,
- (iv) $\langle i_1, j_2 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and
- (v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1, j_2)$.

Then $\langle i_1 + 1, j_2 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(64) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_1 be natural numbers. Suppose that

- (i) $\mathrm{front\_left\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,
- (ii) $\langle i_1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,
- (iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_1)$,
- (iv) $\langle i_1, j_1 + 1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and
- (v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1, j_1 + 1)$.

Then $\langle i_1, j_1 + 2 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(65) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_1 be natural numbers. Suppose that

(i) $\mathrm{front\_left\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_1,\, j_1\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_1)$,

(iv) $\langle i_1 + 1,\, j_1\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1 + 1, j_1)$.

Then $\langle i_1 + 2,\, j_1\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(66) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let j_1, i_2 be natural numbers. Suppose that

(i) $\mathrm{front\_left\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_2 + 1,\, j_1\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_2 + 1, j_1)$,

(iv) $\langle i_2,\, j_1\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_2, j_1)$.

Then $\langle i_2 -' 1,\, j_1\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(67) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_2 be natural numbers. Suppose that

(i) $\mathrm{front\_left\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_1,\, j_2 + 1\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_2 + 1)$,

(iv) $\langle i_1,\, j_2\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1, j_2)$.

Then $\langle i_1,\, j_2 -' 1\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(68) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_1 be natural numbers. Suppose that

(i) $\mathrm{front\_right\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_1,\, j_1\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_1)$,

(iv) $\langle i_1,\, j_1 + 1\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1, j_1 + 1)$.

Then $\langle i_1 + 1,\, j_1 + 1\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(69) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\rm T}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is a sequence which elements belong to Gauge(C,n) and len $f > 1$. Let i_1, j_1 be natural numbers. Suppose that

(i) front_right_cell$(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_1, j_1\rangle \in$ the indices of Gauge(C,n),

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_1)$,

(iv) $\langle i_1+1, j_1\rangle \in$ the indices of Gauge(C,n), and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1+1, j_1)$.

Then $\langle i_1+1, j_1 -' 1\rangle \in$ the indices of Gauge(C,n).

(70) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\rm T}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is a sequence which elements belong to Gauge(C,n) and len $f > 1$. Let j_1, i_2 be natural numbers. Suppose that

(i) front_right_cell$(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_2+1, j_1\rangle \in$ the indices of Gauge(C,n),

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_2+1, j_1)$,

(iv) $\langle i_2, j_1\rangle \in$ the indices of Gauge(C,n), and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_2, j_1)$.

Then $\langle i_2, j_1+1\rangle \in$ the indices of Gauge(C,n).

(71) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\rm T}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is a sequence which elements belong to Gauge(C,n) and len $f > 1$. Let i_1, j_2 be natural numbers. Suppose that

(i) front_right_cell$(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_1, j_2+1\rangle \in$ the indices of Gauge(C,n),

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_2+1)$,

(iv) $\langle i_1, j_2\rangle \in$ the indices of Gauge(C,n), and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1, j_2)$.

Then $\langle i_1 -' 1, j_2\rangle \in$ the indices of Gauge(C,n).

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More on the Finite Sequences on the Plane[1]

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Summary. We continue proving lemmas needed for the proof of the Jordan curve theorem. The main goal was to prove the last theorem being a mutation of the first theorem in [13].

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The articles [16], [7], [2], [4], [19], [6], [18], [5], [12], [15], [14], [9], [1], [3], [21], [22], [11], [10], [20], [17], and [8] provide the terminology and notation for this paper.

1. Preliminaries

The following proposition is true

(1) For all sets A, x, y such that $A \subseteq \{x, y\}$ and $x \in A$ and $y \notin A$ holds $A = \{x\}$.

Let us note that there exists a function which is trivial.

2. Finite Sequences

We adopt the following convention: G denotes a Go-board and i, j, k, m, n denote natural numbers.

Let us note that there exists a finite sequence which is non constant.

Next we state a number of propositions:

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(2) For every non trivial finite sequence f holds $1 < \operatorname{len} f$.

(3) For every non trivial set D and for every non constant circular finite sequence f of elements of D holds $\operatorname{len} f > 2$.

(4) For every finite sequence f and for every set x holds $x \in \operatorname{rng} f$ or $x \looparrowleft f = 0$.

(5) Let p be a set, D be a non empty set, f be a non empty finite sequence of elements of D, and g be a finite sequence of elements of D. If $p \looparrowleft f = \operatorname{len} f$, then $f \frown g \rightarrow p = g$.

(6) For every non empty set D and for every non empty one-to-one finite sequence f of elements of D holds $f_{\operatorname{len} f} \looparrowleft f = \operatorname{len} f$.

(7) For all finite sequences f, g holds $\operatorname{len} f \leqslant \operatorname{len}(f \frown\frown g)$.

(8) For all finite sequences f, g and for every set x such that $x \in \operatorname{rng} f$ holds $x \looparrowleft f = x \looparrowleft (f \frown\frown g)$.

(9) For every non empty finite sequence f and for every finite sequence g holds $\operatorname{len} g \leqslant \operatorname{len}(f \frown\frown g)$.

(10) For all finite sequences f, g holds $\operatorname{rng} f \subseteq \operatorname{rng}(f \frown\frown g)$.

(11) Let D be a non empty set, f be a non empty finite sequence of elements of D, and g be a non trivial finite sequence of elements of D. If $g_{\operatorname{len} g} = f_1$, then $f \frown\frown g$ is circular.

(12) Let D be a non empty set, M be a matrix over D, f be a finite sequence of elements of D, and g be a non empty finite sequence of elements of D. Suppose $f_{\operatorname{len} f} = g_1$ and f is a sequence which elements belong to M and g is a sequence which elements belong to M. Then $f \frown\frown g$ is a sequence which elements belong to M.

(13) For every set D and for every finite sequence f of elements of D such that $1 \leqslant k$ holds $\langle f(k+1), \ldots, f(\operatorname{len} f)\rangle = f_{\lfloor k}$.

(14) For every set D and for every finite sequence f of elements of D such that $k \leqslant \operatorname{len} f$ holds $\langle f(1), \ldots, f(k)\rangle = f \restriction k$.

(15) Let p be a set, D be a non empty set, f be a non empty finite sequence of elements of D, and g be a finite sequence of elements of D. If $p \looparrowleft f = \operatorname{len} f$, then $f \frown g \leftarrow p = \langle f(1), \ldots, f(\operatorname{len} f -' 1)\rangle$.

(16) Let D be a non empty set and f, g be non empty finite sequences of elements of D. If $g_1 \looparrowleft f = \operatorname{len} f$, then $(f \frown\frown g) :\!-\, g_1 = g$.

(17) Let D be a non empty set and f, g be non empty finite sequences of elements of D. If $g_1 \looparrowleft f = \operatorname{len} f$, then $(f \frown\frown g) -\!:\, g_1 = f$.

(18) Let D be a non trivial set, f be a non empty finite sequence of elements of D, and g be a non trivial finite sequence of elements of D. Suppose $g_1 = f_{\operatorname{len} f}$ and for every i such that $1 \leqslant i$ and $i < \operatorname{len} f$ holds $f_i \neq g_1$. Then $(f \frown\frown g)^{g_1}_{\circlearrowleft} = g \frown\frown f$.

3. On the Plane

We now state several propositions:

(19) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathcal{L}(f,1) = \widetilde{\mathcal{L}}(f{\restriction}2)$.

(20) For every s.c.c. finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ and for every n such that $n < \operatorname{len} f$ holds $f{\restriction}n$ is s.n.c..

(21) For every s.c.c. finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ and for every n such that $1 \leqslant n$ holds $f_{\lfloor n}$ is s.n.c..

(22) Let f be a circular s.c.c. finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and given n. If $n < \operatorname{len} f$ and $\operatorname{len} f > 4$, then $f{\restriction}n$ is one-to-one.

(23) Let f be a circular s.c.c. finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $\operatorname{len} f > 4$. Let i, j be natural numbers. If $1 < i$ and $i < j$ and $j \leqslant \operatorname{len} f$, then $f_i \neq f_j$.

(24) Let f be a circular s.c.c. finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and given n. If $1 \leqslant n$ and $\operatorname{len} f > 4$, then $f_{\lfloor n}$ is one-to-one.

(25) For every special non empty finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\langle f(m), \ldots, f(n) \rangle$ is special.

(26) Let f be a special non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and g be a special non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $f_{\operatorname{len} f} = g_1$, then $f \frown g$ is special.

(27) For every circular unfolded s.c.c. finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that $\operatorname{len} f > 4$ holds $\mathcal{L}(f,1) \cap \widetilde{\mathcal{L}}(f_{\lfloor 1}) = \{f_1, f_2\}$.

Let us note that there exists a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ which is one-to-one, special, unfolded, s.n.c., and non empty.

We now state several propositions:

(28) For all finite sequences f, g of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that $j < \operatorname{len} f$ holds $\mathcal{L}(f \frown g, j) = \mathcal{L}(f, j)$.

(29) For all non empty finite sequences f, g of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that $1 \leqslant j$ and $j + 1 < \operatorname{len} g$ holds $\mathcal{L}(f \frown g, \operatorname{len} f + j) = \mathcal{L}(g, j+1)$.

(30) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and g be a non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $f_{\operatorname{len} f} = g_1$, then $\mathcal{L}(f \frown g, \operatorname{len} f) = \mathcal{L}(g, 1)$.

(31) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and g be a non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $j + 1 < \operatorname{len} g$ and $f_{\operatorname{len} f} = g_1$, then $\mathcal{L}(f \frown g, \operatorname{len} f + j) = \mathcal{L}(g, j+1)$.

(32) Let f be a non empty s.n.c. unfolded finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and given i. If $1 \leqslant i$ and $i < \operatorname{len} f$, then $\mathcal{L}(f, i) \cap \operatorname{rng} f = \{f_i, f_{i+1}\}$.

(33) Let f, g be non trivial s.n.c. one-to-one unfolded finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{f_1, g_1\}$ and $f_1 = g_{\mathrm{len}\, g}$ and $g_1 = f_{\mathrm{len}\, f}$, then $f \frown g$ is s.c.c..

In the sequel f, g are finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$.

The following propositions are true:

(34) If f is unfolded and g is unfolded and $f_{\mathrm{len}\, f} = g_1$ and $\mathcal{L}(f, \mathrm{len}\, f -' 1) \cap \mathcal{L}(g, 1) = \{f_{\mathrm{len}\, f}\}$, then $f \frown g$ is unfolded.

(35) If f is non empty and g is non trivial and $f_{\mathrm{len}\, f} = g_1$, then $\widetilde{\mathcal{L}}(f \frown g) = \widetilde{\mathcal{L}}(f) \cup \widetilde{\mathcal{L}}(g)$.

(36) Suppose that

(i) for every n such that $n \in \mathrm{dom}\, f$ there exist i, j such that $\langle i, j \rangle \in$ the indices of G and $f_n = G \circ (i, j)$,

(ii) f is non constant, circular, unfolded, s.c.c., and special, and

(iii) $\mathrm{len}\, f > 4$.

Then there exists g such that

(iv) g is a sequence which elements belong to G, unfolded, s.c.c., and special,

(v) $\widetilde{\mathcal{L}}(f) = \widetilde{\mathcal{L}}(g)$,

(vi) $f_1 = g_1$,

(vii) $f_{\mathrm{len}\, f} = g_{\mathrm{len}\, g}$, and

(viii) $\mathrm{len}\, f \leqslant \mathrm{len}\, g$.

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More on Multivariate Polynomials: Monomials and Constant Polynomials

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Summary. In this article we give some technical concepts for multivariate polynomials with arbitrary number of variables. Monomials and constant polynomials are introduced and their properties with respect to the eval functor are shown. In addition, the multiplication of polynomials with coefficients is defined and investigated.

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The notation and terminology used here are introduced in the following articles: [6], [10], [15], [13], [1], [12], [7], [9], [8], [2], [14], [3], [11], [4], and [5].

1. Preliminaries

Let us note that there exists a non empty zero structure which is non trivial.

Let us observe that every zero structure which is non trivial is also non empty.

Let us mention that there exists a non trivial double loop structure which is Abelian, left zeroed, right zeroed, add-associative, right complementable, unital, associative, commutative, distributive, and integral domain-like.

Let R be a non empty zero structure and let a be an element of R. We say that a is non-zero if and only if:

(Def. 1) $a \neq 0_R$.

Let R be a non trivial zero structure. Note that there exists an element of R which is non-zero.

Let X be a set, let R be a non empty zero structure, and let p be a series of X, R. We say that p is non-zero if and only if:

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(Def. 2) $p \neq 0\_(X, R)$.

Let X be a set and let R be a non trivial zero structure. One can check that there exists a series of X, R which is non-zero.

Let n be an ordinal number and let R be a non trivial zero structure. Note that there exists a polynomial of n, R which is non-zero.

The following two propositions are true:

(1) Let X be a set, R be a non empty zero structure, and s be a series of X, R. Then $s = 0\_(X, R)$ if and only if $\mathrm{Support}\, s = \emptyset$.

(2) Let X be a set and R be a non empty zero structure. Then R is non trivial if and only if there exists a series s of X, R such that $\mathrm{Support}\, s \neq \emptyset$.

Let X be a set and let b be a bag of X. We say that b is univariate if and only if:

(Def. 3) There exists an element u of X such that $\mathrm{support}\, b = \{u\}$.

Let X be a non empty set. Note that there exists a bag of X which is univariate.

Let X be a non empty set. Note that every bag of X which is univariate is also non empty.

2. Polynomials without Variables

We now state three propositions:

(3) For every bag b of $\emptyset$ holds $b = \mathrm{EmptyBag}\, \emptyset$.

(4) Let L be a right zeroed add-associative right complementable well unital distributive non trivial double loop structure, p be a polynomial of $\emptyset$, L, and x be a function from $\emptyset$ into L. Then $\mathrm{eval}(p, x) = p(\mathrm{EmptyBag}\, \emptyset)$.

(5) Let L be a right zeroed add-associative right complementable well unital distributive non trivial double loop structure. Then $\text{Polynom-Ring}(\emptyset, L)$ is ring isomorphic to L.

3. Monomials

Let X be a set, let L be a non empty zero structure, and let p be a series of X, L. We say that p is monomial-like if and only if:

(Def. 4) There exists a bag b of X such that for every bag b' of X such that $b' \neq b$ holds $p(b') = 0_L$.

Let X be a set and let L be a non empty zero structure. Note that there exists a series of X, L which is monomial-like.

Let X be a set and let L be a non empty zero structure. A monomial of X, L is a monomial-like series of X, L.

Let X be a set and let L be a non empty zero structure. One can check that every series of X, L which is monomial-like is also finite-Support.

The following proposition is true

(6) Let X be a set, L be a non empty zero structure, and p be a series of X, L. Then p is a monomial of X, L if and only if $\mathrm{Support}\, p = \emptyset$ or there exists a bag b of X such that $\mathrm{Support}\, p = \{b\}$.

Let X be a set, let L be a non empty zero structure, let a be an element of L, and let b be a bag of X. The functor $\mathrm{Monom}(a, b)$ yields a monomial of X, L and is defined as follows:

(Def. 5) $\mathrm{Monom}(a, b) = 0\_(X, L) +\cdot (b, a)$.

Let X be a set, let L be a non empty zero structure, and let m be a monomial of X, L. The functor $\mathrm{term}\, m$ yielding a bag of X is defined by:

(Def. 6) $m(\mathrm{term}\, m) \neq 0_L$ or $\mathrm{Support}\, m = \emptyset$ and $\mathrm{term}\, m = \mathrm{EmptyBag}\, X$.

Let X be a set, let L be a non empty zero structure, and let m be a monomial of X, L. The functor coefficient m yields an element of L and is defined by:

(Def. 7) $\mathrm{coefficient}\, m = m(\mathrm{term}\, m)$.

One can prove the following propositions:

(7) For every set X and for every non empty zero structure L and for every monomial m of X, L holds $\mathrm{Support}\, m = \emptyset$ or $\mathrm{Support}\, m = \{\mathrm{term}\, m\}$.

(8) For every set X and for every non empty zero structure L and for every bag b of X holds $\mathrm{coefficient}\, \mathrm{Monom}(0_L, b) = 0_L$ and $\mathrm{term}\, \mathrm{Monom}(0_L, b) = \mathrm{EmptyBag}\, X$.

(9) Let X be a set, L be a non empty zero structure, a be an element of L, and b be a bag of X. Then $\mathrm{coefficient}\, \mathrm{Monom}(a, b) = a$.

(10) Let X be a set, L be a non trivial zero structure, a be a non-zero element of L, and b be a bag of X. Then $\mathrm{term}\, \mathrm{Monom}(a, b) = b$.

(11) For every set X and for every non empty zero structure L and for every monomial m of X, L holds $\mathrm{Monom}(\mathrm{coefficient}\, m, \mathrm{term}\, m) = m$.

(12) Let n be an ordinal number, L be a right zeroed add-associative right complementable unital distributive non trivial double loop structure, m be a monomial of n, L, and x be a function from n into L. Then $\mathrm{eval}(m, x) = \mathrm{coefficient}\, m \cdot \mathrm{eval}(\mathrm{term}\, m, x)$.

(13) Let n be an ordinal number, L be a right zeroed add-associative right complementable unital distributive non trivial double loop structure, a be an element of L, b be a bag of n, and x be a function from n into L. Then $\mathrm{eval}(\mathrm{Monom}(a, b), x) = a \cdot \mathrm{eval}(b, x)$.

4. Constant Polynomials

Let X be a set, let L be a non empty zero structure, and let p be a series of X, L. We say that p is constant if and only if:

(Def. 8) For every bag b of X such that $b \neq \mathrm{EmptyBag}\, X$ holds $p(b) = 0_L$.

Let X be a set and let L be a non empty zero structure. Observe that there exists a series of X, L which is constant.

Let X be a set and let L be a non empty zero structure. A constant polynomial of X, L is a constant series of X, L.

Let X be a set and let L be a non empty zero structure. One can check that every series of X, L which is constant is also monomial-like.

The following proposition is true

(14) Let X be a set, L be a non empty zero structure, and p be a series of X, L. Then p is a constant polynomial of X, L if and only if $p = 0\_(X, L)$ or $\mathrm{Support}\, p = \{\mathrm{EmptyBag}\, X\}$.

Let X be a set and let L be a non empty zero structure. Observe that $0\_(X, L)$ is constant.

Let X be a set and let L be a unital non empty double loop structure. One can check that $1\_(X, L)$ is constant.

The following propositions are true:

(15) Let X be a set, L be a non empty zero structure, and c be a constant polynomial of X, L. Then $\mathrm{Support}\, c = \emptyset$ or $\mathrm{Support}\, c = \{\mathrm{EmptyBag}\, X\}$.

(16) Let X be a set, L be a non empty zero structure, and c be a constant polynomial of X, L. Then $\mathrm{term}\, c = \mathrm{EmptyBag}\, X$ and $\mathrm{coefficient}\, c = c(\mathrm{EmptyBag}\, X)$.

Let X be a set, let L be a non empty zero structure, and let a be an element of L. The functor $a\,\_(X, L)$ yielding a series of X, L is defined by:

(Def. 9) $a\,\_(X, L) = 0\_(X, L) +\cdot (\mathrm{EmptyBag}\, X, a)$.

Let X be a set, let L be a non empty zero structure, and let a be an element of L. Observe that $a\,\_(X, L)$ is constant.

We now state several propositions:

(17) Let X be a set, L be a non empty zero structure, and p be a series of X, L. Then p is a constant polynomial of X, L if and only if there exists an element a of L such that $p = a\,\_(X, L)$.

(18) Let X be a set, L be a non empty multiplicative loop with zero structure, and a be an element of L. Then $(a\,\_(X, L))(\mathrm{EmptyBag}\, X) = a$ and for every bag b of X such that $b \neq \mathrm{EmptyBag}\, X$ holds $(a\,\_(X, L))(b) = 0_L$.

(19) For every set X and for every non empty zero structure L holds $0_L\,\_(X, L) = 0\_(X, L)$.

(20) For every set X and for every unital non empty multiplicative loop with zero structure L holds $1_L\,\_(X, L) = 1\_(X, L)$.

(21) Let X be a set, L be a non empty zero structure, and a, b be elements of L. Then $a\,\_(X, L) = b\,\_(X, L)$ if and only if $a = b$.

(22) For every set X and for every non empty zero structure L and for every element a of L holds $\mathrm{Support}\, a\,\_(X, L) = \emptyset$ or $\mathrm{Support}\, a\,\_(X, L) = \{\mathrm{EmptyBag}\, X\}$.

(23) For every set X and for every non empty zero structure L and for every element a of L holds $\mathrm{term}\, a\,\_(X, L) = \mathrm{EmptyBag}\, X$ and $\mathrm{coefficient}\, a\,\_(X, L) = a$.

(24) Let n be an ordinal number, L be a right zeroed add-associative right complementable unital distributive non trivial double loop structure, c be a constant polynomial of n, L, and x be a function from n into L. Then $\mathrm{eval}(c, x) = \mathrm{coefficient}\, c$.

(25) Let n be an ordinal number, L be a right zeroed add-associative right complementable unital distributive non trivial double loop structure, a be an element of L, and x be a function from n into L. Then $\mathrm{eval}(a\,\_(n, L), x) = a$.

5. Multiplication with Coefficients

Let X be a set, let L be a non empty multiplicative loop with zero structure, let p be a series of X, L, and let a be an element of L. The functor $a \cdot p$ yields a series of X, L and is defined by:

(Def. 10) For every bag b of X holds $(a \cdot p)(b) = a \cdot p(b)$.

The functor $p \cdot a$ yields a series of X, L and is defined by:

(Def. 11) For every bag b of X holds $(p \cdot a)(b) = p(b) \cdot a$.

Let X be a set, let L be a left zeroed right zeroed add-cancelable distributive non empty double loop structure, let p be a finite-Support series of X, L, and let a be an element of L. Note that $a \cdot p$ is finite-Support and $p \cdot a$ is finite-Support.

One can prove the following propositions:

(26) Let X be a set, L be a commutative non empty multiplicative loop with zero structure, p be a series of X, L, and a be an element of L. Then $a \cdot p = p \cdot a$.

(27) Let n be an ordinal number, L be an add-associative right complementable right zeroed left distributive non empty double loop structure, p be a series of n, L, and a be an element of L. Then $a \cdot p = (a\,\_(n, L)) * p$.

(28) Let n be an ordinal number, L be an add-associative right complementable right zeroed right distributive non empty double loop structure, p be a series of n, L, and a be an element of L. Then $p \cdot a = p * (a \_(n, L))$.

(29) Let n be an ordinal number, L be an Abelian left zeroed right zeroed add-associative right complementable unital associative commutative distributive non trivial double loop structure, p be a polynomial of n, L, a be an element of L, and x be a function from n into L. Then $\mathrm{eval}(a \cdot p, x) = a \cdot \mathrm{eval}(p, x)$.

(30) Let n be an ordinal number, L be a left zeroed right zeroed add-left-cancelable add-associative right complementable unital associative integral domain-like distributive non trivial double loop structure, p be a polynomial of n, L, a be an element of L, and x be a function from n into L. Then $\mathrm{eval}(a \cdot p, x) = a \cdot \mathrm{eval}(p, x)$.

(31) Let n be an ordinal number, L be an Abelian left zeroed right zeroed add-associative right complementable unital associative commutative distributive non trivial double loop structure, p be a polynomial of n, L, a be an element of L, and x be a function from n into L. Then $\mathrm{eval}(p \cdot a, x) = \mathrm{eval}(p, x) \cdot a$.

(32) Let n be an ordinal number, L be a left zeroed right zeroed add-left-cancelable add-associative right complementable unital associative commutative distributive integral domain-like non trivial double loop structure, p be a polynomial of n, L, a be an element of L, and x be a function from n into L. Then $\mathrm{eval}(p \cdot a, x) = \mathrm{eval}(p, x) \cdot a$.

(33) Let n be an ordinal number, L be an Abelian left zeroed right zeroed add-associative right complementable unital associative commutative distributive non trivial double loop structure, p be a polynomial of n, L, a be an element of L, and x be a function from n into L. Then $\mathrm{eval}((a \_(n, L)) * p, x) = a \cdot \mathrm{eval}(p, x)$.

(34) Let n be an ordinal number, L be an Abelian left zeroed right zeroed add-associative right complementable unital associative commutative distributive non trivial double loop structure, p be a polynomial of n, L, a be an element of L, and x be a function from n into L. Then $\mathrm{eval}(p * (a \_(n, L)), x) = \mathrm{eval}(p, x) \cdot a$.

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On State Machines of Calculating Type

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Summary. In this article, we show properties of calculating type state machines. In the first section, we have defined calculating type state machines of which the state transition only depends on the first input. We have also proved theorems of the state machines. In the second section, we defined Moore machines with final states. We also introduced the concept of result of the Moore machines. In the last section, we proved the correctness of several calculating type of Moore machines.

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The terminology and notation used in this paper have been introduced in the following articles: [10], [3], [16], [11], [2], [14], [9], [4], [5], [1], [8], [17], [7], [13], [15], [12], and [6].

1. Calculating Type

For simplicity, we use the following convention: m denotes a natural number, x, y denote real numbers, i, j denote non empty natural numbers, I, O denote non empty sets, s, s_1, s_2, s_3 denote elements of I, w, w_1, w_2 denote finite sequences of elements of I, t denotes an element of O, S denotes a non empty FSM over I, and q, q_1 denote states of S.

Let us consider I, S, q, w. We introduce $\mathrm{GEN}(w, q)$ as a synonym of (q, w)-admissible.

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Let us consider I, S, q, w. Note that GEN(w, q) is non empty.

The following propositions are true:

(1) GEN$(\langle s\rangle, q) = \langle q,$ (the transition of S)$(\langle q, s\rangle)\rangle$.

(2) GEN$(\langle s_1, s_2\rangle, q) = \langle q,$ (the transition of S)$(\langle q, s_1\rangle)$, (the transition of S)$(\langle$(the transition of S)$(\langle q, s_1\rangle), s_2\rangle)\rangle$.

(3) GEN$(\langle s_1, s_2, s_3\rangle, q) = \langle q,$ (the transition of S)$(\langle q, s_1\rangle)$, (the transition of S)$(\langle$(the transition of S)$(\langle q, s_1\rangle), s_2\rangle)$, (the transition of S)$(\langle$(the transition of S)$(\langle$(the transition of S)$(\langle q, s_1\rangle), s_2\rangle), s_3\rangle)\rangle$.

Let us consider I, S. We say that S is calculating type if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let given j and given w_1, w_2. Suppose $w_1(1) = w_2(1)$ and $j \leqslant \operatorname{len} w_1 + 1$ and $j \leqslant \operatorname{len} w_2 + 1$. Then (GEN($w_1$, the initial state of S))(j) = (GEN(w_2, the initial state of S))(j).

The following propositions are true:

(4) Suppose S is calculating type. Let given w_1, w_2. Suppose $w_1(1) = w_2(1)$. Then GEN(w_1, the initial state of S) and GEN(w_2, the initial state of S) are c=-comparable.

(5) Suppose that for all w_1, w_2 such that $w_1(1) = w_2(1)$ holds GEN(w_1, the initial state of S) and GEN(w_2, the initial state of S) are c=-comparable. Then S is calculating type.

(6) Suppose S is calculating type. Let given w_1, w_2. Suppose $w_1(1) = w_2(1)$ and $\operatorname{len} w_1 = \operatorname{len} w_2$. Then GEN($w_1$, the initial state of S) = GEN(w_2, the initial state of S).

(7) Suppose that for all w_1, w_2 such that $w_1(1) = w_2(1)$ and $\operatorname{len} w_1 = \operatorname{len} w_2$ holds GEN(w_1, the initial state of S) = GEN(w_2, the initial state of S). Then S is calculating type.

Let us consider I, S, q, s. We say that q is accessible via s if and only if:

(Def. 2) There exists a finite sequence w of elements of I such that the initial state of $S \xrightarrow{\langle s\rangle ^\frown w} q$.

Let us consider I, S, q. We say that q is accessible if and only if:

(Def. 3) There exists a finite sequence w of elements of I such that the initial state of $S \xrightarrow{w} q$.

We now state four propositions:

(8) If q is accessible via s, then q is accessible.

(9) If q is accessible and $q \neq$ the initial state of S, then there exists s such that q is accessible via s.

(10) The initial state of S is accessible.

(11) Suppose S is calculating type and q is accessible via s. Then there exists a non empty natural number m such that for every w if $\operatorname{len} w + 1 \geqslant m$

and $w(1) = s$, then $q = (\mathrm{GEN}(w, \text{the initial state of } S))(m)$ and for every i such that $i < m$ holds $(\mathrm{GEN}(w, \text{the initial state of } S))(i) \neq q$.

Let us consider I, S. We say that S is regular if and only if:

(Def. 4) Every state of S is accessible.

We now state several propositions:

(12) If for all s_1, s_2, q holds (the transition of S)($\langle q, s_1 \rangle$) = (the transition of S)($\langle q, s_2 \rangle$), then S is calculating type.

(13) Let given S. Suppose that

(i) for all s_1, s_2, q such that $q \neq$ the initial state of S holds (the transition of S)($\langle q, s_1 \rangle$) = (the transition of S)($\langle q, s_2 \rangle$), and

(ii) for all s, q_1 holds (the transition of S)($\langle q_1, s \rangle$) $\neq$ the initial state of S.

Then S is calculating type.

(14) Suppose S is regular and calculating type. Let given s_1, s_2, q. If $q \neq$ the initial state of S, then $(\mathrm{GEN}(\langle s_1 \rangle, q))(2) = (\mathrm{GEN}(\langle s_2 \rangle, q))(2)$.

(15) Suppose S is regular and calculating type. Let given s_1, s_2, q. Suppose $q \neq$ the initial state of S. Then (the transition of S)($\langle q, s_1 \rangle$) = (the transition of S)($\langle q, s_2 \rangle$).

(16) Suppose S is regular and calculating type. Let given s, s_1, s_2, q. Suppose (the transition of S)($\langle$the initial state of S, $s_1 \rangle$) $\neq$ (the transition of S)($\langle$the initial state of S, $s_2 \rangle$). Then (the transition of S)($\langle q, s \rangle$) $\neq$ the initial state of S.

2. State Machine with Final States

Let I be a set. We introduce state machines over I with final states which are extensions of FSM over I and are systems

$\langle$ a carrier, a transition, an initial state, final states $\rangle$,

where the carrier is a set, the transition is a function from [: the carrier, I :] into the carrier, the initial state is an element of the carrier, and the final states constitute a subset of the carrier.

Let I be a set. One can check that there exists a state machine over I with final states which is non empty.

Let us consider I, s and let S be a non empty state machine over I with final states. We say that s leads to final state of S if and only if:

(Def. 5) There exists a state q of S such that q is accessible via s and $q \in$ the final states of S.

Let us consider I and let S be a non empty state machine over I with final states. We say that S is halting if and only if:

(Def. 6) s leads to final state of S.

Let I be a set and let O be a non empty set. We consider Moore state machines over I and O with final states as extensions of state machine over I with final states and Moore-FSM over I, O as systems

⟨ a carrier, a transition, an output function, an initial state, final states ⟩,

where the carrier is a set, the transition is a function from [: the carrier, I :] into the carrier, the output function is a function from the carrier into O, the initial state is an element of the carrier, and the final states constitute a subset of the carrier.

Let I be a set and let O be a non empty set. Observe that there exists a Moore state machine over I and O with final states which is non empty and strict.

Let us consider I, O, let i, f be sets, and let o be a function from $\{i, f\}$ into O. The functor I-TwoStatesMooreSM(i, f, o) yielding a non empty strict Moore state machine over I and O with final states is defined by the conditions (Def. 7).

(Def. 7)(i) The carrier of I-TwoStatesMooreSM$(i, f, o) = \{i, f\}$,

(ii) the transition of I-TwoStatesMooreSM$(i, f, o) =$ [: $\{i, f\}$, I :] $\longmapsto f$,

(iii) the output function of I-TwoStatesMooreSM$(i, f, o) = o$,

(iv) the initial state of I-TwoStatesMooreSM$(i, f, o) = i$, and

(v) the final states of I-TwoStatesMooreSM$(i, f, o) = \{f\}$.

One can prove the following proposition

(17) Let i, f be sets, o be a function from $\{i, f\}$ into O, and given j. If $1 < j$ and $j \leqslant \operatorname{len} w + 1$, then (GEN($w$, the initial state of I-TwoStatesMooreSM(i, f, o)))$(j) = f$.

Let us consider I, O, let i, f be sets, and let o be a function from $\{i, f\}$ into O. Observe that I-TwoStatesMooreSM(i, f, o) is calculating type.

Let us consider I, O, let i, f be sets, and let o be a function from $\{i, f\}$ into O. One can check that I-TwoStatesMooreSM(i, f, o) is halting.

In the sequel n, m are non empty natural numbers and M is a non empty Moore state machine over I and O with final states.

Next we state the proposition

(18) Suppose that

(i) M is calculating type,

(ii) s leads to final state of M, and

(iii) the initial state of $M \notin$ the final states of M.

Then there exists a non empty natural number m such that

(iv) for every w such that $\operatorname{len} w + 1 \geqslant m$ and $w(1) = s$ holds (GEN(w, the initial state of M))$(m) \in$ the final states of M, and

(v) for all w, j such that $j \leqslant \operatorname{len} w + 1$ and $w(1) = s$ and $j < m$ holds (GEN(w, the initial state of M))$(j) \notin$ the final states of M.

3. Correctness of a Result of State Machine

Let us consider I, O, M, s and let t be a set. We say that t is a result of s in M if and only if the condition (Def. 8) is satisfied.

(Def. 8) There exists m such that for every w if $w(1) = s$, then if $m \leqslant \operatorname{len} w + 1$, then $t =$ (the output function of M)((GEN(w, the initial state of M))(m)) and (GEN(w, the initial state of M))(m) $\in$ the final states of M and for every n such that $n < m$ and $n \leqslant \operatorname{len} w + 1$ holds (GEN(w, the initial state of M))(n) $\notin$ the final states of M.

We now state several propositions:

(19) Suppose the initial state of $M \in$ the final states of M. Then (the output function of M)(the initial state of M) is a result of s in M.

(20) Suppose that
(i) M is calculating type,
(ii) s leads to final state of M, and
(iii) the initial state of $M \notin$ the final states of M.
Then there exists t which is a result of s in M.

(21) Suppose M is calculating type and s leads to final state of M. Let t_1, t_2 be elements of O. If t_1 is a result of s in M and t_2 is a result of s in M, then $t_1 = t_2$.

(22) Let X be a non empty set, f be a binary operation on X, and M be a non empty Moore state machine over $[: X, X :]$ and $X \cup \{X\}$ with final states. Suppose that
(i) M is calculating type,
(ii) the carrier of $M = X \cup \{X\}$,
(iii) the final states of $M = X$,
(iv) the initial state of $M = X$,
(v) the output function of $M = \mathrm{id}_{\text{the carrier of } M}$, and
(vi) for all elements x, y of X holds (the transition of M)($\langle$the initial state of M, $\langle x, y\rangle\rangle$) $= f(x, y)$.
Then M is halting and for all elements x, y of X holds $f(x, y)$ is a result of $\langle x, y\rangle$ in M.

(23) Let M be a non empty Moore state machine over $[: \mathbb{R}, \mathbb{R} :]$ and $\mathbb{R} \cup \{\mathbb{R}\}$ with final states. Suppose that M is calculating type and the carrier of $M = \mathbb{R} \cup \{\mathbb{R}\}$ and the final states of $M = \mathbb{R}$ and the initial state of $M = \mathbb{R}$ and the output function of $M = \mathrm{id}_{\text{the carrier of } M}$ and for all x, y such that $x \geqslant y$ holds (the transition of M)($\langle$the initial state of M, $\langle x, y\rangle\rangle$) $= x$ and for all x, y such that $x < y$ holds (the transition of M)($\langle$the initial state of M, $\langle x, y\rangle\rangle$) $= y$. Let x, y be elements of $\mathbb{R}$. Then $\max(x, y)$ is a result of $\langle x, y\rangle$ in M.

(24) Let M be a non empty Moore state machine over $[:\mathbb{R}, \mathbb{R}:]$ and $\mathbb{R} \cup \{\mathbb{R}\}$ with final states. Suppose that M is calculating type and the carrier of $M = \mathbb{R} \cup \{\mathbb{R}\}$ and the final states of $M = \mathbb{R}$ and the initial state of $M = \mathbb{R}$ and the output function of $M = \mathrm{id}_{\text{the carrier of } M}$ and for all x, y such that $x < y$ holds (the transition of M)($\langle$the initial state of M, $\langle x, y\rangle\rangle$) $= x$ and for all x, y such that $x \geqslant y$ holds (the transition of M)($\langle$the initial state of M, $\langle x, y\rangle\rangle$) $= y$. Let x, y be elements of $\mathbb{R}$. Then $\min(x, y)$ is a result of $\langle x, y\rangle$ in M.

(25) Let M be a non empty Moore state machine over $[:\mathbb{R}, \mathbb{R}:]$ and $\mathbb{R} \cup \{\mathbb{R}\}$ with final states. Suppose that

(i) M is calculating type,

(ii) the carrier of $M = \mathbb{R} \cup \{\mathbb{R}\}$,

(iii) the final states of $M = \mathbb{R}$,

(iv) the initial state of $M = \mathbb{R}$,

(v) the output function of $M = \mathrm{id}_{\text{the carrier of } M}$, and

(vi) for all x, y holds (the transition of M)($\langle$the initial state of M, $\langle x, y\rangle\rangle$) $= x + y$.

Let x, y be elements of $\mathbb{R}$. Then $x + y$ is a result of $\langle x, y\rangle$ in M.

(26) Let M be a non empty Moore state machine over $[:\mathbb{R}, \mathbb{R}:]$ and $\mathbb{R} \cup \{\mathbb{R}\}$ with final states. Suppose that M is calculating type and the carrier of $M = \mathbb{R} \cup \{\mathbb{R}\}$ and the final states of $M = \mathbb{R}$ and the initial state of $M = \mathbb{R}$ and the output function of $M = \mathrm{id}_{\text{the carrier of } M}$ and for all x, y such that $x > 0$ or $y > 0$ holds (the transition of M)($\langle$the initial state of M, $\langle x, y\rangle\rangle$) $= 1$ and for all x, y such that $x = 0$ or $y = 0$ but $x \leqslant 0$ but $y \leqslant 0$ holds (the transition of M)($\langle$the initial state of M, $\langle x, y\rangle\rangle$) $= 0$ and for all x, y such that $x < 0$ and $y < 0$ holds (the transition of M)($\langle$the initial state of M, $\langle x, y\rangle\rangle$) $= -1$. Let x, y be elements of $\mathbb{R}$. Then $\max(\operatorname{sgn} x, \operatorname{sgn} y)$ is a result of $\langle x, y\rangle$ in M.

Let us consider I, O. Note that there exists a non empty Moore state machine over I and O with final states which is calculating type and halting.

Let us consider I. Observe that there exists a non empty state machine over I with final states which is calculating type and halting.

Let us consider I, O, let M be a calculating type halting non empty Moore state machine over I and O with final states, and let us consider s. The functor $\mathrm{Result}(s, M)$ yields an element of O and is defined as follows:

(Def. 9) $\mathrm{Result}(s, M)$ is a result of s in M.

Next we state several propositions:

(27) For every function f from $\{0, 1\}$ into O holds $\mathrm{Result}(s, I\text{-TwoStatesMooreSM}(0, 1, f)) = f(1)$.

(28) Let M be a calculating type halting non empty Moore state machine over $[:\mathbb{R}, \mathbb{R}:]$ and $\mathbb{R} \cup \{\mathbb{R}\}$ with final states. Suppose that

(i) the carrier of $M = \mathbb{R} \cup \{\mathbb{R}\}$,
(ii) the final states of $M = \mathbb{R}$,
(iii) the initial state of $M = \mathbb{R}$,
(iv) the output function of $M = \mathrm{id}_{\text{the carrier of } M}$,
(v) for all x, y such that $x \geqslant y$ holds (the transition of M)($\langle$the initial state of M, $\langle x, y\rangle\rangle) = x$, and
(vi) for all x, y such that $x < y$ holds (the transition of M)($\langle$the initial state of M, $\langle x, y\rangle\rangle) = y$.

Let x, y be elements of $\mathbb{R}$. Then $\mathrm{Result}(\langle x, y\rangle, M) = \max(x, y)$.

(29) Let M be a calculating type halting non empty Moore state machine over $[:\mathbb{R}, \mathbb{R}:]$ and $\mathbb{R} \cup \{\mathbb{R}\}$ with final states. Suppose that
(i) the carrier of $M = \mathbb{R} \cup \{\mathbb{R}\}$,
(ii) the final states of $M = \mathbb{R}$,
(iii) the initial state of $M = \mathbb{R}$,
(iv) the output function of $M = \mathrm{id}_{\text{the carrier of } M}$,
(v) for all x, y such that $x < y$ holds (the transition of M)($\langle$the initial state of M, $\langle x, y\rangle\rangle) = x$, and
(vi) for all x, y such that $x \geqslant y$ holds (the transition of M)($\langle$the initial state of M, $\langle x, y\rangle\rangle) = y$.

Let x, y be elements of $\mathbb{R}$. Then $\mathrm{Result}(\langle x, y\rangle, M) = \min(x, y)$.

(30) Let M be a calculating type halting non empty Moore state machine over $[:\mathbb{R}, \mathbb{R}:]$ and $\mathbb{R} \cup \{\mathbb{R}\}$ with final states. Suppose that
(i) the carrier of $M = \mathbb{R} \cup \{\mathbb{R}\}$,
(ii) the final states of $M = \mathbb{R}$,
(iii) the initial state of $M = \mathbb{R}$,
(iv) the output function of $M = \mathrm{id}_{\text{the carrier of } M}$, and
(v) for all x, y holds (the transition of M)($\langle$the initial state of M, $\langle x, y\rangle\rangle) = x + y$.

Let x, y be elements of $\mathbb{R}$. Then $\mathrm{Result}(\langle x, y\rangle, M) = x + y$.

(31) Let M be a calculating type halting non empty Moore state machine over $[:\mathbb{R}, \mathbb{R}:]$ and $\mathbb{R} \cup \{\mathbb{R}\}$ with final states. Suppose that the carrier of $M = \mathbb{R} \cup \{\mathbb{R}\}$ and the final states of $M = \mathbb{R}$ and the initial state of $M = \mathbb{R}$ and the output function of $M = \mathrm{id}_{\text{the carrier of } M}$ and for all x, y such that $x > 0$ or $y > 0$ holds (the transition of M)($\langle$the initial state of M, $\langle x, y\rangle\rangle) = 1$ and for all x, y such that $x = 0$ or $y = 0$ but $x \leqslant 0$ but $y \leqslant 0$ holds (the transition of M)($\langle$the initial state of M, $\langle x, y\rangle\rangle) = 0$ and for all x, y such that $x < 0$ and $y < 0$ holds (the transition of M)($\langle$the initial state of M, $\langle x, y\rangle\rangle) = -1$. Let x, y be elements of $\mathbb{R}$. Then $\mathrm{Result}(\langle x, y\rangle, M) = \max(\operatorname{sgn} x, \operatorname{sgn} y)$.

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Hierarchies and Classifications of Sets[1]

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Summary. This article is a continuation of [2] article. Further properties of classification of sets are proved. The notion of hierarchy of a set is introduced. Properties of partitions and hierarchies are shown. The main theorem says that for each hierarchy there exists a classification which the union is equal to the considered hierarchy.

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The terminology and notation used here have been introduced in the following articles: [7], [11], [6], [9], [4], [12], [5], [10], [8], [2], [3], and [1].

1. Tree and Classification of a Set

For simplicity, we follow the rules: A denotes a relational structure, X denotes a non empty set, P_1, P_2, P_3, Y, a, b, c, x denote sets, and S_1 denotes a subset of Y.

Let us consider A. We say that A has superior elements if and only if:

(Def. 1) There exists an element of A which is superior of the internal relation of A.

Let us consider A. We say that A has comparable down elements if and only if:

(Def. 2) For all elements x, y of A such that there exists an element z of A such that $z \leqslant x$ and $z \leqslant y$ holds $x \leqslant y$ or $y \leqslant x$.

The following proposition is true

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(1) For every set a holds $\langle\{\{a\}\},\subseteq\rangle$ is non empty, reflexive, transitive, and antisymmetric and has superior elements and comparable down elements.

Let us observe that there exists a relational structure which is non empty, reflexive, transitive, antisymmetric, and strict and has superior elements and comparable down elements.

A tree is a poset with superior elements and comparable down elements.

Next we state four propositions:

(2) For every equivalence relation E_1 of X and for all sets x, y, z such that $z \in [x]_{(E_1)}$ and $z \in [y]_{(E_1)}$ holds $[x]_{(E_1)} = [y]_{(E_1)}$.

(3) For every partition P of X and for all sets x, y, z such that $x \in P$ and $y \in P$ and $z \in x$ and $z \in y$ holds $x = y$.

(4) For all sets C, x such that C is a classification of X and $x \in \bigcup C$ holds $x \subseteq X$.

(5) For every set C such that C is a strong classification of X holds $\langle\bigcup C,\subseteq\rangle$ is a tree.

2. The Hierarchy of a Set

Let us consider Y. We say that Y is hierarchic if and only if:

(Def. 3) For all sets x, y such that $x \in Y$ and $y \in Y$ holds $x \subseteq y$ or $y \subseteq x$ or x misses y.

One can verify that every set which is trivial is also hierarchic.

Let us note that there exists a set which is non trivial and hierarchic.

The following propositions are true:

(6) $\emptyset$ is hierarchic.

(7) $\{\emptyset\}$ is hierarchic.

Let us consider Y. A family of subsets of Y is said to be a hierarchy of Y if:

(Def. 4) It is hierarchic.

Let us consider Y. We say that Y is mutually-disjoint if and only if:

(Def. 5) For all sets x, y such that $x \in Y$ and $y \in Y$ and $x \neq y$ holds x misses y.

In the sequel H denotes a hierarchy of Y.

Let us consider Y. Observe that there exists a family of subsets of Y which is mutually-disjoint.

Next we state three propositions:

(8) $\emptyset$ is mutually-disjoint.

(9) $\{\emptyset\}$ is mutually-disjoint.

(10) $\{a\}$ is mutually-disjoint.

Let us consider Y and let F be a family of subsets of Y. We say that F is T_3 if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let A be a subset of Y. Suppose $A \in F$. Let x be an element of Y. If $x \notin A$, then there exists a subset B of Y such that $x \in B$ and $B \in F$ and A misses B.

We now state the proposition

(11) For every family F of subsets of Y such that $F = \emptyset$ holds F is T_3.

Let us consider Y. One can verify that there exists a hierarchy of Y which is covering and T_3.

Let us consider Y and let F be a family of subsets of Y. We say that F is lower-bounded if and only if the condition (Def. 7) is satisfied.

(Def. 7) Let B be a set. Suppose $B \neq \emptyset$ and $B \subseteq F$ and for all a, b such that $a \in B$ and $b \in B$ holds $a \subseteq b$ or $b \subseteq a$. Then there exists c such that $c \in F$ and $c \subseteq \bigcap B$.

Next we state the proposition

(12) Let B be a mutually-disjoint family of subsets of Y. Suppose that for every set b such that $b \in B$ holds S_1 misses b and $Y \neq \emptyset$. Then $B \cup \{S_1\}$ is a mutually-disjoint family of subsets of Y and if $S_1 \neq \emptyset$, then $\bigcup(B \cup \{S_1\}) \neq \bigcup B$.

Let us consider Y and let F be a family of subsets of Y. We say that F has maximum elements if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let S be a subset of Y. Suppose $S \in F$. Then there exists a subset T of Y such that $S \subseteq T$ and $T \in F$ and for every subset V of Y such that $T \subseteq V$ and $V \in F$ holds $V = Y$.

3. Some Properties of Partitions, Hierarchies and Classifications of Sets

The following propositions are true:

(13) For every covering hierarchy H of Y such that H has maximum elements there exists a partition P of Y such that $P \subseteq H$.

(14) Let H be a covering hierarchy of Y and B be a mutually-disjoint family of subsets of Y. Suppose $B \subseteq H$ and for every mutually-disjoint family C of subsets of Y such that $C \subseteq H$ and $\bigcup B \subseteq \bigcup C$ holds $B = C$. Then B is a partition of Y.

(15) Let H be a covering T_3 hierarchy of Y. Suppose H is lower-bounded and $\emptyset \notin H$. Let A be a subset of Y and B be a mutually-disjoint family of subsets of Y. Suppose that

(i) $A \in B$,

(ii) $B \subseteq H$, and

(iii) for every mutually-disjoint family C of subsets of Y such that $A \in C$ and $C \subseteq H$ and $\bigcup B \subseteq \bigcup C$ holds $\bigcup B = \bigcup C$.

Then B is a partition of Y.

(16) Let H be a covering T_3 hierarchy of Y. Suppose H is lower-bounded and $\emptyset \notin H$. Let A be a subset of Y and B be a mutually-disjoint family of subsets of Y. Suppose $A \in B$ and $B \subseteq H$ and for every mutually-disjoint family C of subsets of Y such that $A \in C$ and $C \subseteq H$ and $B \subseteq C$ holds $B = C$. Then B is a partition of Y.

(17) Let H be a covering T_3 hierarchy of Y. Suppose H is lower-bounded and $\emptyset \notin H$. Let A be a subset of Y. If $A \in H$, then there exists a partition P of Y such that $A \in P$ and $P \subseteq H$.

(18) Let h be a non empty set, P_4 be a partition of X, and h_1 be a set. Suppose $h_1 \in P_4$ and $h \subseteq h_1$. Let P_6 be a partition of X. Suppose $h \in P_6$ and for every x such that $x \in P_6$ holds $x \subseteq h_1$ or $h_1 \subseteq x$ or h_1 misses x. Let P_5 be a set. Suppose that for every a holds $a \in P_5$ iff $a \in P_6$ and $a \subseteq h_1$. Then $P_5 \cup (P_4 \setminus \{h_1\})$ is a partition of X and $P_5 \cup (P_4 \setminus \{h_1\})$ is finer than P_4.

(19) Let h be a non empty set. Suppose $h \subseteq X$. Let P_8 be a partition of X. Suppose there exists a set h_2 such that $h_2 \in P_8$ and $h_2 \subseteq h$ and for every x such that $x \in P_8$ holds $x \subseteq h$ or $h \subseteq x$ or h misses x. Let P_7 be a set. Suppose that for every x holds $x \in P_7$ iff $x \in P_8$ and x misses h. Then

(i) $P_7 \cup \{h\}$ is a partition of X,

(ii) P_8 is finer than $P_7 \cup \{h\}$, and

(iii) for every partition P_4 of X such that P_8 is finer than P_4 and for every set h_1 such that $h_1 \in P_4$ and $h \subseteq h_1$ holds $P_7 \cup \{h\}$ is finer than P_4.

(20) Let H be a covering T_3 hierarchy of X. Suppose that

(i) H is lower-bounded,

(ii) $\emptyset \notin H$, and

(iii) for every set C_1 such that $C_1 \neq \emptyset$ and $C_1 \subseteq \mathrm{PARTITIONS}(X)$ and for all sets P_9, P_{10} such that $P_9 \in C_1$ and $P_{10} \in C_1$ holds P_9 is finer than P_{10} or P_{10} is finer than P_9 there exist P_1, P_2 such that $P_1 \in C_1$ and $P_2 \in C_1$ and for every P_3 such that $P_3 \in C_1$ holds P_3 is finer than P_2 and P_1 is finer than P_3.

Then there exists a classification C of X such that $\bigcup C = H$.

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