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The Canonical Formulae

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The articles [23], [29], [11], [28], [14], [2], [27], [12], [30], [8], [5], [3], [20], [9], [6], [22], [7], [10], [1], [4], [15], [17], [18], [24], [25], [19], [16], [21], [13], and [26] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(1) For every integer i holds i is even iff $i-1$ is odd.

(2) For every integer i holds i is odd iff $i-1$ is even.

(3) Let X be a trivial set and x be a set. Suppose $x \in X$. Let f be a function from X into X. Then x is a fixpoint of f.

Let A, B, C be sets. Note that every function from A into C^B is function yielding.

One can prove the following three propositions:

(4) For every function yielding function f holds $\mathrm{Sub_f}\,\mathrm{rng}\, f = \mathrm{rng}\, f$.

(5) For all sets A, B, x and for every function f such that $x \in A$ and $f \in B^A$ holds $f(x) \in B$.

(6) For all sets A, B, C such that if $C = \emptyset$, then $B = \emptyset$ or $A = \emptyset$ and for every function f from A into C^B holds $\mathrm{dom}_\kappa\, f(\kappa) = A \longmapsto B$.

Let us note that $\emptyset$ is function yielding.

In the sequel n is a natural number and p, q, r are elements of HP-WFF.

Next we state the proposition

(7) For every set x holds $\emptyset(x) = \emptyset$.

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Let A be a set and let B be a functional set. One can verify that every function from A into B is function yielding.

One can prove the following propositions:

(8) For every set X and for every subset A of X holds $[0 \longmapsto 1, 1 \longmapsto 0] \cdot \chi_{A,X} = \chi_{A^{c},X}$.

(9) For every set X and for every subset A of X holds $[0 \longmapsto 1, 1 \longmapsto 0] \cdot \chi_{A^{c},X} = \chi_{A,X}$.

(10) For all sets a, b, x, y, x', y' such that $a \neq b$ and $[a \longmapsto x, b \longmapsto y] = [a \longmapsto x', b \longmapsto y']$ holds $x = x'$ and $y = y'$.

(11) For all sets a, b, x, y, X, Y such that $a \neq b$ and $x \in X$ and $y \in Y$ holds $[a \longmapsto x, b \longmapsto y] \in \prod[a \longmapsto X, b \longmapsto Y]$.

(12) For every non empty set D and for every function f from 2 into D there exist elements d_1, d_2 of D such that $f = [0 \longmapsto d_1, 1 \longmapsto d_2]$.

(13) For all sets a, b, c, d such that $a \neq b$ holds $[a \longmapsto c, b \longmapsto d] \cdot [a \longmapsto b, b \longmapsto a] = [a \longmapsto d, b \longmapsto c]$.

(14) For all sets a, b, c, d and for every function f such that $a \neq b$ and $c \in \operatorname{dom} f$ and $d \in \operatorname{dom} f$ holds $f \cdot [a \longmapsto c, b \longmapsto d] = [a \longmapsto f(c), b \longmapsto f(d)]$.

2. The Cartesian Product of Functions and the Frege Function

Let f, g be one-to-one functions. Note that $[\!: f, g :\!]$ is one-to-one.

We now state a number of propositions:

(15) Let A, B be non empty sets, C, D be sets, f be a function from C into A, and g be a function from D into B. Then $\pi_1(A \times B) \cdot [\!: f, g :\!] = f \cdot \pi_1(C \times D)$.

(16) Let A, B be non empty sets, C, D be sets, f be a function from C into A, and g be a function from D into B. Then $\pi_2(A \times B) \cdot [\!: f, g :\!] = g \cdot \pi_2(C \times D)$.

(17) For every function g holds $\emptyset \hookleftarrow g = \emptyset$.

(18) For every function yielding function f and for all functions g, h holds $f \hookleftarrow g \cdot h = (f \cdot h) \hookleftarrow (g \cdot h)$.

(19) Let C be a set, A be a non empty set, f be a function from A into $C^{(\emptyset \textbf{ qua } \mathrm{set})}$, and g be a function from A into $\emptyset$. Then $\operatorname{rng}(f \hookleftarrow g) = \{\emptyset\}$.

(20) Let A, B, C be sets such that if $B = \emptyset$, then $A = \emptyset$. Let f be a function from A into C^{B} and g be a function from A into B. Then $\operatorname{rng}(f \hookleftarrow g) \subseteq C$.

(21) For all sets A, B, C such that if $C = \emptyset$, then $B = \emptyset$ or $A = \emptyset$ and for every function f from A into C^{B} holds $\operatorname{dom} \operatorname{Frege}(f) = B^{A}$.

(22) $\operatorname{Frege}(\emptyset) = \{\emptyset\} \longmapsto \emptyset$.

(23) For all sets A, B, C such that if $C = \emptyset$, then $B = \emptyset$ or $A = \emptyset$ and for every function f from A into C^{B} holds $\operatorname{rng} \operatorname{Frege}(f) \subseteq C^{A}$.

(24) Let A, B, C be sets such that if $C = \emptyset$, then $B = \emptyset$ or $A = \emptyset$. Let f be a function from A into C^B. Then $\mathrm{Frege}(f)$ is a function from B^A into C^A.

3. About Permutations

Let A be a set. Observe that every permutation of A is one-to-one.

The following proposition is true

(25) For all sets A, B and for every permutation P of A and for every permutation Q of B holds $[\!: P, Q :\!]$ is permutation-like.

Let A, B be non empty sets, let P be a permutation of A, and let Q be a function from B into B. The functor $P \Rightarrow Q$ yielding a function from B^A into B^A is defined as follows:

(Def. 1) For every function f from A into B holds $(P \Rightarrow Q)(f) = Q \cdot f \cdot P^{-1}$.

Let A, B be non empty sets, let P be a permutation of A, and let Q be a permutation of B. Observe that $P \Rightarrow Q$ is permutation-like.

Next we state three propositions:

(26) Let A, B be non empty sets, P be a permutation of A, Q be a permutation of B, and f be a function from A into B. Then $(P \Rightarrow Q)^{-1}(f) = Q^{-1} \cdot f \cdot P$.

(27) For all non empty sets A, B and for every permutation P of A and for every permutation Q of B holds $(P \Rightarrow Q)^{-1} = P^{-1} \Rightarrow Q^{-1}$.

(28) Let A, B, C be non empty sets, f be a function from A into C^B, g be a function from A into B, P be a permutation of B, and Q be a permutation of C. Then $((P \Rightarrow Q) \cdot f) \looparrowleft (P \cdot g) = Q \cdot f \looparrowleft g$.

4. Set Valuations

A SetValuation is a non-empty many sorted set indexed by $\mathbb{N}$.

In the sequel V denotes a SetValuation.

Let us consider V. The functor $\mathrm{SetVal}\,V$ yielding a many sorted set indexed by HP-WFF is defined by the conditions (Def. 2).

(Def. 2)(i) $(\mathrm{SetVal}\,V)(\mathrm{VERUM}) = 1$,

(ii) for every n holds $(\mathrm{SetVal}\,V)(\mathrm{prop}\,n) = V(n)$, and

(iii) for all p, q holds $(\mathrm{SetVal}\,V)(p \wedge q) = [\!: (\mathrm{SetVal}\,V)(p), (\mathrm{SetVal}\,V)(q) :\!]$ and $(\mathrm{SetVal}\,V)(p \Rightarrow q) = (\mathrm{SetVal}\,V)(q)^{(\mathrm{SetVal}\,V)(p)}$.

Let us consider V, p. The functor $\mathrm{SetVal}(V, p)$ is defined as follows:

(Def. 3) $\mathrm{SetVal}(V, p) = (\mathrm{SetVal}\,V)(p)$.

Let us consider V, p. One can check that $\mathrm{SetVal}(V,p)$ is non empty.

Next we state four propositions:

(29) $\mathrm{SetVal}(V,\mathrm{VERUM}) = 1$.

(30) $\mathrm{SetVal}(V,\mathrm{prop}\,n) = V(n)$.

(31) $\mathrm{SetVal}(V,p \wedge q) = [\!:\mathrm{SetVal}(V,p),\ \mathrm{SetVal}(V,q):\!]$.

(32) $\mathrm{SetVal}(V,p \Rightarrow q) = (\mathrm{SetVal}(V,q))^{\mathrm{SetVal}(V,p)}$.

Let us consider V, p, q. Observe that $\mathrm{SetVal}(V,p \Rightarrow q)$ is functional.

Let us consider V, p, q, r. Note that every element of $\mathrm{SetVal}(V,p \Rightarrow (q \Rightarrow r))$ is function yielding.

Let us consider V, p, q, r. One can check that there exists a function from $\mathrm{SetVal}(V,p \Rightarrow q)$ into $\mathrm{SetVal}(V,p \Rightarrow r)$ which is function yielding and there exists an element of $\mathrm{SetVal}(V,p \Rightarrow (q \Rightarrow r))$ which is function yielding.

5. Permuting Set Valuations

Let us consider V. A function is called a permutation of V if:

(Def. 4) $\mathrm{dom}\,\mathrm{it} = \mathbb{N}$ and for every n holds $\mathrm{it}(n)$ is a permutation of $V(n)$.

In the sequel P is a permutation of V.

Let us consider V, P. The functor $\mathrm{Perm}\,P$ yielding a many sorted function from $\mathrm{SetVal}\,V$ into $\mathrm{SetVal}\,V$ is defined by the conditions (Def. 5).

(Def. 5)(i) $(\mathrm{Perm}\,P)(\mathrm{VERUM}) = \mathrm{id}_1$,

(ii) for every n holds $(\mathrm{Perm}\,P)(\mathrm{prop}\,n) = P(n)$, and

(iii) for all p, q there exists a permutation p' of $\mathrm{SetVal}(V,p)$ and there exists a permutation q' of $\mathrm{SetVal}(V,q)$ such that $p' = (\mathrm{Perm}\,P)(p)$ and $q' = (\mathrm{Perm}\,P)(q)$ and $(\mathrm{Perm}\,P)(p \wedge q) = [\!:p',\ q':\!]$ and $(\mathrm{Perm}\,P)(p \Rightarrow q) = p' \Rightarrow q'$.

Let us consider V, P, p. The functor $\mathrm{Perm}(P,p)$ yields a function from $\mathrm{SetVal}(V,p)$ into $\mathrm{SetVal}(V,p)$ and is defined by:

(Def. 6) $\mathrm{Perm}(P,p) = (\mathrm{Perm}\,P)(p)$.

Next we state four propositions:

(33) $\mathrm{Perm}(P,\mathrm{VERUM}) = \mathrm{id}_{\mathrm{SetVal}(V,\mathrm{VERUM})}$.

(34) $\mathrm{Perm}(P,\mathrm{prop}\,n) = P(n)$.

(35) $\mathrm{Perm}(P,p \wedge q) = [\!:\mathrm{Perm}(P,p),\ \mathrm{Perm}(P,q):\!]$.

(36) For every permutation p' of $\mathrm{SetVal}(V,p)$ and for every permutation q' of $\mathrm{SetVal}(V,q)$ such that $p' = \mathrm{Perm}(P,p)$ and $q' = \mathrm{Perm}(P,q)$ holds $\mathrm{Perm}(P,p \Rightarrow q) = p' \Rightarrow q'$.

Let us consider V, P, p. One can check that $\mathrm{Perm}(P,p)$ is permutation-like.

We now state four propositions:

(37) For every function g from $\mathrm{SetVal}(V,p)$ into $\mathrm{SetVal}(V,q)$ holds $(\mathrm{Perm}(P,p \Rightarrow q))(g) = \mathrm{Perm}(P,q) \cdot g \cdot (\mathrm{Perm}(P,p))^{-1}$.

(38) For every function g from $\mathrm{SetVal}(V,p)$ into $\mathrm{SetVal}(V,q)$ holds $(\mathrm{Perm}(P,p \Rightarrow q))^{-1}(g) = (\mathrm{Perm}(P,q))^{-1} \cdot g \cdot \mathrm{Perm}(P,p)$.

(39) For all functions f, g from $\mathrm{SetVal}(V,p)$ into $\mathrm{SetVal}(V,q)$ such that $f = (\mathrm{Perm}(P,p \Rightarrow q))(g)$ holds $\mathrm{Perm}(P,q) \cdot g = f \cdot \mathrm{Perm}(P,p)$.

(40) Let given V, P be a permutation of V, and x be a set. Suppose x is a fixpoint of $\mathrm{Perm}(P,p)$. Let f be a function. If f is a fixpoint of $\mathrm{Perm}(P,p \Rightarrow q)$, then $f(x)$ is a fixpoint of $\mathrm{Perm}(P,q)$.

6. Canonical Formulae

Let us consider p. We say that p is canonical if and only if:

(Def. 7) For every V there exists a set x such that for every permutation P of V holds x is a fixpoint of $\mathrm{Perm}(P,p)$.

Let us observe that VERUM is canonical.

Next we state several propositions:

(41) $p \Rightarrow (q \Rightarrow p)$ is canonical.

(42) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ is canonical.

(43) $p \wedge q \Rightarrow p$ is canonical.

(44) $p \wedge q \Rightarrow q$ is canonical.

(45) $p \Rightarrow (q \Rightarrow p \wedge q)$ is canonical.

(46) If p is canonical and $p \Rightarrow q$ is canonical, then q is canonical.

(47) If $p \in$ HP_TAUT, then p is canonical.

Let us observe that there exists an element of HP-WFF which is canonical.

7. Pseudo-Canonical Formulae

Let us consider p. We say that p is pseudo-canonical if and only if:

(Def. 8) For every V and for every permutation P of V holds there exists a set which is a fixpoint of $\mathrm{Perm}(P,p)$.

Let us observe that every element of HP-WFF which is canonical is also pseudo-canonical.

One can prove the following propositions:

(48) $p \Rightarrow (q \Rightarrow p)$ is pseudo-canonical.

(49) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ is pseudo-canonical.

(50) $p \wedge q \Rightarrow p$ is pseudo-canonical.

(51) $p \wedge q \Rightarrow q$ is pseudo-canonical.

(52) $p \Rightarrow (q \Rightarrow p \wedge q)$ is pseudo-canonical.

(53) If p is pseudo-canonical and $p \Rightarrow q$ is pseudo-canonical, then q is pseudo-canonical.

(54) Let given p, q, given V, and P be a permutation of V. Suppose there exists a set which is a fixpoint of $\mathrm{Perm}(P, p)$ and there exists no set which is a fixpoint of $\mathrm{Perm}(P, q)$. Then $p \Rightarrow q$ is not pseudo-canonical.

(55) $((\mathrm{prop}\,0 \Rightarrow \mathrm{prop}\,1) \Rightarrow \mathrm{prop}\,0) \Rightarrow \mathrm{prop}\,0$ is not pseudo-canonical.

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The Incompleteness of the Lattice of Substitutions

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Summary. In [11] we proved that the lattice of substitutions, as defined in [9], is a Heyting lattice (i.e. it is pseudo-complemented and it has the zero element). We show that the lattice needs not to be complete. Obviously, the example has to be infinite, namely we can take the set of natural numbers as variables and a singleton as a set of constants. The incompleteness has been shown for lattices of substitutions defined in terms of [22] and relational structures [18].

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The terminology and notation used here are introduced in the following articles: [13], [20], [14], [4], [8], [17], [5], [10], [2], [22], [16], [1], [18], [6], [12], [21], [19], [9], [15], [3], and [7].

1. Preliminaries

The scheme *SSubsetUniq* deals with a relational structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> Let A_1, A_2 be subsets of $\mathcal{A}$. Suppose for every set x holds $x \in A_1$ iff $\mathcal{P}[x]$ and for every set x holds $x \in A_2$ iff $\mathcal{P}[x]$. Then $A_1 = A_2$

for all values of the parameters.

Let A, x be sets. Observe that $[\!:A, \{x\}:\!]$ is function-like.

Next we state a number of propositions:

(1) For every odd natural number n holds $1 \leqslant n$.

(2) For every finite non empty subset X of $\mathbb{N}$ holds $\max X \in X$.

(3) For every finite non empty subset X of $\mathbb{N}$ there exists a natural number n such that $X \subseteq \operatorname{Seg} n \cup \{0\}$.

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(4) For every finite subset X of $\mathbb{N}$ there exists an odd natural number k such that $k \notin X$.

(5) Let k be a natural number and X be a finite non empty subset of $[\!:\mathbb{N}, \{k\}:\!]$. Then there exists a non empty natural number n such that $X \subseteq [\!:\mathrm{Seg}\, n \cup \{0\}, \{k\}:\!]$.

(6) Let m be a natural number and X be a finite non empty subset of $[\!:\mathbb{N}, \{m\}:\!]$. Then there exists a non empty natural number k such that $\langle 2\cdot k+1, m\rangle \notin X$.

(7) Let m be a natural number and X be a finite subset of $[\!:\mathbb{N}, \{m\}:\!]$. Then there exists a natural number k such that for every natural number l such that $l \geqslant k$ holds $\langle l, m\rangle \notin X$.

(8) For every upper-bounded lattice L holds $\top_L = \top_{\mathrm{Poset}(L)}$.

(9) For every lower-bounded lattice L holds $\bot_L = \bot_{\mathrm{Poset}(L)}$.

(10) Let L be a lower-bounded non empty antisymmetric relational structure and a be an element of L. If $\bot_L \geqslant a$, then $a = \bot_L$.

2. Poset of Substitutions

Next we state four propositions:

(11) For every set V and for every finite set C and for all elements A, B of $\mathrm{Fin}(V \dot{\rightarrow} C)$ such that $A = \emptyset$ and $B \neq \emptyset$ holds $B \rightarrowtail A = \emptyset$.

(12) For all sets V, V', C, C' such that $V \subseteq V'$ and $C \subseteq C'$ holds $\mathrm{SubstitutionSet}(V, C) \subseteq \mathrm{SubstitutionSet}(V', C')$.

(13) Let V, V', C, C' be sets, A be an element of $\mathrm{Fin}(V \dot{\rightarrow} C)$, and B be an element of $\mathrm{Fin}(V' \dot{\rightarrow} C')$. If $V \subseteq V'$ and $C \subseteq C'$ and $A = B$, then $\mu A = \mu B$.

(14) Let V, V', C, C' be sets. Suppose $V \subseteq V'$ and $C \subseteq C'$. Then the join operation of $\mathrm{SubstLatt}(V, C) =$ (the join operation of $\mathrm{SubstLatt}(V', C')$)$\restriction[\!:$ the carrier of $\mathrm{SubstLatt}(V, C)$, the carrier of $\mathrm{SubstLatt}(V, C):\!]$.

Let V, C be sets. The functor $\mathrm{SubstPoset}(V, C)$ yields a relational structure and is defined as follows:

(Def. 1) $\mathrm{SubstPoset}(V, C) = \mathrm{Poset}(\mathrm{SubstLatt}(V, C))$.

Let V, C be sets. One can verify that $\mathrm{SubstPoset}(V, C)$ has l.u.b.'s and g.l.b.'s.

Let V, C be sets. One can verify that $\mathrm{SubstPoset}(V, C)$ is reflexive antisymmetric and transitive.

One can prove the following propositions:

(15) Let V, C be sets and a, b be elements of $\mathrm{SubstPoset}(V, C)$. Then $a \leqslant b$ if and only if for every set x such that $x \in a$ there exists a set y such that $y \in b$ and $y \subseteq x$.

(16) For all sets V, V', C, C' such that $V \subseteq V'$ and $C \subseteq C'$ holds SubstPoset(V,C) is a full relational substructure of SubstPoset(V',C').

Let n, k be natural numbers. The functor $\mathrm{PF_A}(n,k)$ yields an element of $\mathbb{N}\dot{\rightarrow}\{k\}$ and is defined as follows:

(Def. 2) For every set x holds $x \in \mathrm{PF_A}(n,k)$ iff there exists an odd natural number m such that $m \leqslant 2 \cdot n$ and $\langle m, k\rangle = x$ or $\langle 2 \cdot n, k\rangle = x$.

Let n, k be natural numbers. One can verify that $\mathrm{PF_A}(n,k)$ is finite.

Let n, k be natural numbers. The functor $\mathrm{PF_C}(n,k)$ yielding an element of $\mathbb{N}\dot{\rightarrow}\{k\}$ is defined by:

(Def. 3) For every set x holds $x \in \mathrm{PF_C}(n,k)$ iff there exists an odd natural number m such that $m \leqslant 2 \cdot n + 1$ and $\langle m, k\rangle = x$.

Let n, k be natural numbers. Note that $\mathrm{PF_C}(n,k)$ is finite.

The following four propositions are true:

(17) For all natural numbers n, k holds $\langle 2 \cdot n + 1, k\rangle \in \mathrm{PF_C}(n,k)$.

(18) For all natural numbers n, k holds $\mathrm{PF_C}(n,k) \cap \{\langle 2 \cdot n + 3, k\rangle\} = \emptyset$.

(19) For all natural numbers n, k holds $\mathrm{PF_C}(n+1,k) = \mathrm{PF_C}(n,k) \cup \{\langle 2 \cdot n+3, k\rangle\}$.

(20) For all natural numbers n, k holds $\mathrm{PF_C}(n,k) \subset \mathrm{PF_C}(n+1,k)$.

Let n, k be natural numbers. One can verify that $\mathrm{PF_A}(n,k)$ is non empty.

Next we state three propositions:

(21) For all natural numbers n, m, k holds $\mathrm{PF_A}(n,m) \not\subseteq \mathrm{PF_C}(k,m)$.

(22) For all natural numbers n, m, k such that $n \leqslant k$ holds $\mathrm{PF_C}(n,m) \subseteq \mathrm{PF_C}(k,m)$.

(23) For every natural number n holds $\mathrm{PF_A}(1,n) = \{\langle 1, n\rangle, \langle 2, n\rangle\}$.

Let n, k be natural numbers. The functor $\mathrm{PF_B}(n,k)$ yields an element of $\mathrm{Fin}(\mathbb{N}\dot{\rightarrow}\{k\})$ and is defined as follows:

(Def. 4) For every set x holds $x \in \mathrm{PF_B}(n,k)$ iff there exists a non empty natural number m such that $m \leqslant n$ and $x = \mathrm{PF_A}(m,k)$ or $x = \mathrm{PF_C}(n,k)$.

The following propositions are true:

(24) For all natural numbers n, k and for every set x such that $x \in \mathrm{PF_B}(n+1,k)$ there exists a set y such that $y \in \mathrm{PF_B}(n,k)$ and $y \subseteq x$.

(25) For all natural numbers n, k holds $\mathrm{PF_C}(n,k) \notin \mathrm{PF_B}(n+1,k)$.

(26) For all natural numbers n, m, k such that $\mathrm{PF_A}(n,m) \subseteq \mathrm{PF_A}(k,m)$ holds $n = k$.

(27) For all natural numbers n, m, k holds $\mathrm{PF_C}(n,m) \subseteq \mathrm{PF_A}(k,m)$ iff $n < k$.

3. The Incompleteness

The following proposition is true

(28) For all natural numbers n, k holds $\mathrm{PF}_{\mathrm{B}}(n,k)$ is an element of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$.

Let k be a natural number. The functor $\mathrm{PF}_{\mathrm{D}}(k)$ yielding a subset of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$ is defined as follows:

(Def. 5) For every set x holds $x \in \mathrm{PF}_{\mathrm{D}}(k)$ iff there exists a non empty natural number n such that $x = \mathrm{PF}_{\mathrm{B}}(n,k)$.

The following propositions are true:

(29) For every natural number k holds $\mathrm{PF}_{\mathrm{B}}(1,k) = \{\mathrm{PF}_{\mathrm{A}}(1,k), \mathrm{PF}_{\mathrm{C}}(1,k)\}$.

(30) For every natural number k holds $\mathrm{PF}_{\mathrm{B}}(1,k) \neq \{\emptyset\}$.

Let k be a natural number. Note that $\mathrm{PF}_{\mathrm{B}}(1,k)$ is non empty.

We now state four propositions:

(31) For all natural numbers n, k holds $\{\mathrm{PF}_{\mathrm{A}}(n,k)\}$ is an element of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$.

(32) Let k be a natural number, V, X be sets, and a be an element of $\mathrm{SubstPoset}(V,\{k\})$. If $X \in a$, then X is a finite subset of $[\!: V, \{k\} :\!]$.

(33) Let m be a natural number and a be an element of $\mathrm{SubstPoset}(\mathbb{N},\{m\})$. Suppose $\mathrm{PF}_{\mathrm{D}}(m) \geqslant a$. Let X be a non empty set. If $X \in a$, then it is not true that for every natural number n such that $\langle n, m\rangle \in X$ holds n is odd.

(34) Let k be a natural number, a, b be elements of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$, and X be a subset of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$. If $a \leqslant X$ and $b \leqslant X$, then $a \sqcup b \leqslant X$.

Let k be a natural number. Note that there exists an element of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$ which is non empty.

One can prove the following propositions:

(35) For every natural number n and for every element a of $\mathrm{SubstPoset}(\mathbb{N},\{n\})$ such that $\emptyset \in a$ holds $a = \{\emptyset\}$.

(36) Let k be a natural number and a be a non empty element of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$. If $a \neq \{\emptyset\}$, then there exists a finite function f such that $f \in a$ and $f \neq \emptyset$.

(37) Let k be a natural number, a be a non empty element of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$, and a' be an element of $\mathrm{Fin}(\mathbb{N} \dot{\rightarrow} \{k\})$. If $a \neq \{\emptyset\}$ and $a = a'$, then $\mathrm{Involved}\, a'$ is a finite non empty subset of $\mathbb{N}$.

(38) Let k be a natural number, a be an element of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$, a' be an element of $\mathrm{Fin}(\mathbb{N} \dot{\rightarrow} \{k\})$, and B be a finite non empty subset of $\mathbb{N}$. Suppose $B = \mathrm{Involved}\, a'$ and $a' = a$. Let X be a set. If $X \in a$, then for every natural number l such that $l > \max B + 1$ holds $\langle l, k\rangle \notin X$.

(39) For every natural number k holds $\top_{\mathrm{SubstPoset}(\mathbb{N},\{k\})} = \{\emptyset\}$.

(40) For every natural number k holds $\bot_{\mathrm{SubstPoset}(\mathbb{N},\{k\})} = \emptyset$.

(41) For every natural number k and for all elements a, b of $\mathrm{SubstPoset}(\mathbb{N}, \{k\})$ such that $a \leqslant b$ and $a = \{\emptyset\}$ holds $b = \{\emptyset\}$.

(42) For every natural number k and for all elements a, b of $\mathrm{SubstPoset}(\mathbb{N}, \{k\})$ such that $a \leqslant b$ and $b = \emptyset$ holds $a = \emptyset$.

(43) For every natural number m and for every element a of $\mathrm{SubstPoset}(\mathbb{N}, \{m\})$ such that $\mathrm{PF}_{\mathrm{D}}(m) \geqslant a$ holds $a \neq \{\emptyset\}$.

Let m be a natural number. One can verify that $\mathrm{SubstPoset}(\mathbb{N}, \{m\})$ is non complete.

Let m be a natural number. One can check that $\mathrm{SubstLatt}(\mathbb{N}, \{m\})$ is non complete.

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Trigonometric Form of Complex Numbers

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The articles [13], [1], [2], [8], [11], [15], [9], [3], [10], [12], [4], [18], [5], [16], [6], [19], [14], [17], and [7] provide the terminology and notation for this paper.

1. Preliminaries

One can prove the following propositions:

(1) Let F be an add-associative right zeroed right complementable left distributive non empty double loop structure and x be an element of the carrier of F. Then $0_F \cdot x = 0_F$.

(2) Let F be an add-associative right zeroed right complementable right distributive non empty double loop structure and x be an element of the carrier of F. Then $x \cdot 0_F = 0_F$.

The scheme *Regr without 0* concerns a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[1]$

provided the parameters meet the following conditions:

- There exists a non empty natural number k such that $\mathcal{P}[k]$, and
- For every non empty natural number k such that $k \neq 1$ and $\mathcal{P}[k]$ there exists a non empty natural number n such that $n < k$ and $\mathcal{P}[n]$.

One can prove the following propositions:

(3) For every element z of $\mathbb{C}$ holds $\Re(z) \geqslant -|z|$.

(4) For every element z of $\mathbb{C}$ holds $\Im(z) \geqslant -|z|$.

(5) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $\Re(z) \geqslant -|z|$.

(6) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $\Im(z) \geqslant -|z|$.

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(7) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $|z|^{\mathbf{2}} = \Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}$.

(8) For all real numbers x_1, x_2, y_1, y_2 such that $x_1 + x_2 i_{\mathbb{C}_{\mathrm{F}}} = y_1 + y_2 i_{\mathbb{C}_{\mathrm{F}}}$ holds $x_1 = y_1$ and $x_2 = y_2$.

(9) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $z = \Re(z) + \Im(z) i_{\mathbb{C}_{\mathrm{F}}}$.

(10) $0_{\mathbb{C}_{\mathrm{F}}} = 0 + 0 i_{\mathbb{C}_{\mathrm{F}}}$.

(11) $0_{\mathbb{C}_{\mathrm{F}}} =$ the zero of $\mathbb{C}_{\mathrm{F}}$.

(12) For every unital non empty groupoid L and for every element x of the carrier of L holds $\mathrm{power}_L(x, 1) = x$.

(13) For every unital non empty groupoid L and for every element x of the carrier of L holds $\mathrm{power}_L(x, 2) = x \cdot x$.

(14) Let L be an add-associative right zeroed right complementable right distributive unital non empty double loop structure and n be a natural number. If $n > 0$, then $\mathrm{power}_L(0_L, n) = 0_L$.

(15) Let L be an associative commutative unital non empty groupoid, x, y be elements of the carrier of L, and n be a natural number. Then $\mathrm{power}_L(x \cdot y, n) = \mathrm{power}_L(x, n) \cdot \mathrm{power}_L(y, n)$.

(16) For every real number x such that $x > 0$ and for every natural number n holds $\mathrm{power}_{\mathbb{C}_{\mathrm{F}}}(x + 0 i_{\mathbb{C}_{\mathrm{F}}}, n) = x^n + 0 i_{\mathbb{C}_{\mathrm{F}}}$.

(17) For every real number x and for every natural number n such that $x \geqslant 0$ and $n \neq 0$ holds $\sqrt[n]{x}^n = x$.

2. Sinus and Cosinus Properties

One can prove the following propositions:

(20)[1] $\pi + \frac{\pi}{2} = \frac{3}{2} \cdot \pi$ and $\frac{3}{2} \cdot \pi + \frac{\pi}{2} = 2 \cdot \pi$ and $\frac{3}{2} \cdot \pi - \pi = \frac{\pi}{2}$.

(21) $0 < \frac{\pi}{2}$ and $\frac{\pi}{2} < \pi$ and $0 < \pi$ and $-\frac{\pi}{2} < \frac{\pi}{2}$ and $\pi < 2 \cdot \pi$ and $\frac{\pi}{2} < \frac{3}{2} \cdot \pi$ and $-\frac{\pi}{2} < 0$ and $0 < 2 \cdot \pi$ and $\pi < \frac{3}{2} \cdot \pi$ and $\frac{3}{2} \cdot \pi < 2 \cdot \pi$ and $0 < \frac{3}{2} \cdot \pi$.

(22) For all real numbers a, b, c, x such that $x \in]a, c[$ holds $x \in]a, b[$ or $x = b$ or $x \in]b, c[$.

(23) For every real number x such that $x \in]0, \pi[$ holds $\sin(x) > 0$.

(24) For every real number x such that $x \in [0, \pi]$ holds $\sin(x) \geqslant 0$.

(25) For every real number x such that $x \in]\pi, 2 \cdot \pi[$ holds $\sin(x) < 0$.

(26) For every real number x such that $x \in [\pi, 2 \cdot \pi]$ holds $\sin(x) \leqslant 0$.

(27) For every real number x such that $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ holds $\cos(x) > 0$.

(28) For every real number x such that $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ holds $\cos(x) \geqslant 0$.

[1] The notation of π has been changed, previously 'Pai'. The propositions (18) and (19) have been removed.

(29) For every real number x such that $x \in]\frac{\pi}{2}, \frac{3}{2} \cdot \pi[$ holds $\cos(x) < 0$.

(30) For every real number x such that $x \in [\frac{\pi}{2}, \frac{3}{2} \cdot \pi]$ holds $\cos(x) \leqslant 0$.

(31) For every real number x such that $x \in]\frac{3}{2} \cdot \pi, 2 \cdot \pi[$ holds $\cos(x) > 0$.

(32) For every real number x such that $x \in [\frac{3}{2} \cdot \pi, 2 \cdot \pi]$ holds $\cos(x) \geqslant 0$.

(33) For every real number x such that $0 \leqslant x$ and $x < 2 \cdot \pi$ and $\sin x = 0$ holds $x = 0$ or $x = \pi$.

(34) For every real number x such that $0 \leqslant x$ and $x < 2 \cdot \pi$ and $\cos x = 0$ holds $x = \frac{\pi}{2}$ or $x = \frac{3}{2} \cdot \pi$.

(35) sin is increasing on $]-\frac{\pi}{2}, \frac{\pi}{2}[$.

(36) sin is decreasing on $]\frac{\pi}{2}, \frac{3}{2} \cdot \pi[$.

(37) cos is decreasing on $]0, \pi[$.

(38) cos is increasing on $]\pi, 2 \cdot \pi[$.

(39) sin is increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(40) sin is decreasing on $[\frac{\pi}{2}, \frac{3}{2} \cdot \pi]$.

(41) cos is decreasing on $[0, \pi]$.

(42) cos is increasing on $[\pi, 2 \cdot \pi]$.

(43) sin is continuous on $\mathbb{R}$ and for all real numbers x, y holds sin is continuous on $[x, y]$ and sin is continuous on $]x, y[$.

(44) cos is continuous on $\mathbb{R}$ and for all real numbers x, y holds cos is continuous on $[x, y]$ and cos is continuous on $]x, y[$.

(45) For every real number x holds $\sin(x) \in [-1, 1]$ and $\cos(x) \in [-1, 1]$.

(46) $\operatorname{rng} \sin = [-1, 1]$.

(47) $\operatorname{rng} \cos = [-1, 1]$.

(48) $\operatorname{rng}(\sin \restriction [-\frac{\pi}{2}, \frac{\pi}{2}]) = [-1, 1]$.

(49) $\operatorname{rng}(\sin \restriction [\frac{\pi}{2}, \frac{3}{2} \cdot \pi]) = [-1, 1]$.

(50) $\operatorname{rng}(\cos \restriction [0, \pi]) = [-1, 1]$.

(51) $\operatorname{rng}(\cos \restriction [\pi, 2 \cdot \pi]) = [-1, 1]$.

3. Argument of Complex Number

Let z be an element of the carrier of $\mathbb{C}_{\mathrm{F}}$. The functor $\operatorname{Arg} z$ yielding a real number is defined as follows:

(Def. 1)(i) $z = |z| \cdot \cos \operatorname{Arg} z + (|z| \cdot \sin \operatorname{Arg} z) i_{\mathbb{C}_{\mathrm{F}}}$ and $0 \leqslant \operatorname{Arg} z$ and $\operatorname{Arg} z < 2 \cdot \pi$ if $z \neq 0_{\mathbb{C}_{\mathrm{F}}}$,

(ii) $\operatorname{Arg} z = 0$, otherwise.

One can prove the following propositions:

(52) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $0 \leqslant \operatorname{Arg} z$ and $\operatorname{Arg} z < 2 \cdot \pi$.

(53) For every real number x such that $x \geqslant 0$ holds $\operatorname{Arg} x + 0 i_{\mathbb{C}_{\mathrm{F}}} = 0$.

(54) For every real number x such that $x < 0$ holds $\operatorname{Arg} x + 0 i_{\mathbb{C}_{\mathrm{F}}} = \pi$.

(55) For every real number x such that $x > 0$ holds $\operatorname{Arg} 0 + x i_{\mathbb{C}_{\mathrm{F}}} = \frac{\pi}{2}$.

(56) For every real number x such that $x < 0$ holds $\operatorname{Arg} 0 + x i_{\mathbb{C}_{\mathrm{F}}} = \frac{3}{2} \cdot \pi$.

(57) $\operatorname{Arg} \mathbf{1}_{\mathbb{C}_{\mathrm{F}}} = 0$.

(58) $\operatorname{Arg} i_{\mathbb{C}_{\mathrm{F}}} = \frac{\pi}{2}$.

(59) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $\operatorname{Arg} z \in]0, \frac{\pi}{2}[$ iff $\Re(z) > 0$ and $\Im(z) > 0$.

(60) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $\operatorname{Arg} z \in]\frac{\pi}{2}, \pi[$ iff $\Re(z) < 0$ and $\Im(z) > 0$.

(61) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $\operatorname{Arg} z \in]\pi, \frac{3}{2} \cdot \pi[$ iff $\Re(z) < 0$ and $\Im(z) < 0$.

(62) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $\operatorname{Arg} z \in]\frac{3}{2} \cdot \pi, 2 \cdot \pi[$ iff $\Re(z) > 0$ and $\Im(z) < 0$.

(63) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $\Im(z) > 0$ holds $\sin \operatorname{Arg} z > 0$.

(64) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $\Im(z) < 0$ holds $\sin \operatorname{Arg} z < 0$.

(65) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $\Im(z) \geqslant 0$ holds $\sin \operatorname{Arg} z \geqslant 0$.

(66) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $\Im(z) \leqslant 0$ holds $\sin \operatorname{Arg} z \leqslant 0$.

(67) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $\Re(z) > 0$ holds $\cos \operatorname{Arg} z > 0$.

(68) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $\Re(z) < 0$ holds $\cos \operatorname{Arg} z < 0$.

(69) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $\Re(z) \geqslant 0$ holds $\cos \operatorname{Arg} z \geqslant 0$.

(70) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $\Re(z) \leqslant 0$ and $z \neq 0_{\mathbb{C}_{\mathrm{F}}}$ holds $\cos \operatorname{Arg} z \leqslant 0$.

(71) For every real number x and for every natural number n holds $\operatorname{power}_{\mathbb{C}_{\mathrm{F}}}(\cos x + \sin x i_{\mathbb{C}_{\mathrm{F}}}, n) = \cos n \cdot x + \sin n \cdot x i_{\mathbb{C}_{\mathrm{F}}}$.

(72) Let z be an element of the carrier of $\mathbb{C}_{\mathrm{F}}$ and n be a natural number. If $z \neq 0_{\mathbb{C}_{\mathrm{F}}}$ or $n \neq 0$, then $\operatorname{power}_{\mathbb{C}_{\mathrm{F}}}(z, n) = |z|^n \cdot \cos n \cdot \operatorname{Arg} z + (|z|^n \cdot \sin n \cdot \operatorname{Arg} z) i_{\mathbb{C}_{\mathrm{F}}}$.

(73) For every real number x and for all natural numbers n, k such that $n \neq 0$ holds $\operatorname{power}_{\mathbb{C}_{\mathrm{F}}}(\cos \frac{x+2 \cdot \pi \cdot k}{n} + \sin \frac{x+2 \cdot \pi \cdot k}{n} i_{\mathbb{C}_{\mathrm{F}}}, n) = \cos x + \sin x i_{\mathbb{C}_{\mathrm{F}}}$.

(74) Let z be an element of the carrier of $\mathbb{C}_{\rm F}$ and n, k be natural numbers. If $n \neq 0$, then $z = \mathrm{power}_{\mathbb{C}_{\rm F}}(\sqrt[n]{|z|} \cdot \cos \frac{\mathrm{Arg}\, z + 2 \cdot \pi \cdot k}{n} + (\sqrt[n]{|z|} \cdot \sin \frac{\mathrm{Arg}\, z + 2 \cdot \pi \cdot k}{n}) i_{\mathbb{C}_{\rm F}}, n)$.

Let x be an element of the carrier of $\mathbb{C}_{\rm F}$ and let n be a non empty natural number. An element of $\mathbb{C}_{\rm F}$ is called a root of n, x if:

(Def. 2) $\mathrm{power}_{\mathbb{C}_{\rm F}}(\mathrm{it}, n) = x$.

We now state four propositions:

(75) Let x be an element of the carrier of $\mathbb{C}_{\rm F}$, n be a non empty natural number, and k be a natural number. Then $\sqrt[n]{|x|} \cdot \cos \frac{\mathrm{Arg}\, x + 2 \cdot \pi \cdot k}{n} + (\sqrt[n]{|x|} \cdot \sin \frac{\mathrm{Arg}\, x + 2 \cdot \pi \cdot k}{n}) i_{\mathbb{C}_{\rm F}}$ is a root of n, x.

(76) For every element x of the carrier of $\mathbb{C}_{\rm F}$ and for every root v of 1, x holds $v = x$.

(77) For every non empty natural number n and for every root v of n, $0_{\mathbb{C}_{\rm F}}$ holds $v = 0_{\mathbb{C}_{\rm F}}$.

(78) Let n be a non empty natural number, x be an element of the carrier of $\mathbb{C}_{\rm F}$, and v be a root of n, x. If $v = 0_{\mathbb{C}_{\rm F}}$, then $x = 0_{\mathbb{C}_{\rm F}}$.

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Fundamental Theorem of Algebra[1]

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The papers [18], [22], [19], [4], [16], [5], [12], [1], [3], [26], [24], [6], [7], [25], [13], [2], [20], [15], [14], [21], [9], [29], [27], [8], [10], [23], [28], [11], and [17] provide the terminology and notation for this paper.

1. Preliminaries

The following propositions are true:

(1) For all natural numbers n, m such that $n \neq 0$ and $m \neq 0$ holds $(n \cdot m - n - m) + 1 \geqslant 0$.

(2) For all real numbers x, y such that $y > 0$ holds $\frac{\min(x,y)}{\max(x,y)} \leqslant 1$.

(3) For all real numbers x, y such that for every real number c such that $c > 0$ and $c < 1$ holds $c \cdot x \geqslant y$ holds $y \leqslant 0$.

(4) Let p be a finite sequence of elements of $\mathbb{R}$. Suppose that for every natural number n such that $n \in \operatorname{dom} p$ holds $p(n) \geqslant 0$. Let i be a natural number. If $i \in \operatorname{dom} p$, then $\sum p \geqslant p(i)$.

(5) For all real numbers x, y holds $-(x + y i_{\mathbb{C}_{\mathrm{F}}}) = -x + (-y) i_{\mathbb{C}_{\mathrm{F}}}$.

(6) For all real numbers x_1, y_1, x_2, y_2 holds $(x_1 + y_1 i_{\mathbb{C}_{\mathrm{F}}}) - (x_2 + y_2 i_{\mathbb{C}_{\mathrm{F}}}) = (x_1 - x_2) + (y_1 - y_2) i_{\mathbb{C}_{\mathrm{F}}}$.

(7) Let L be a commutative associative left unital distributive field-like non empty double loop structure and f, g, h be elements of the carrier of L. If $h \neq 0_L$, then if $h \cdot g = h \cdot f$ or $g \cdot h = f \cdot h$, then $g = f$.

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In this article we present several logical schemes. The scheme *ExDHGrStrSeq* deals with a non empty groupoid $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an element of the carrier of $\mathcal{A}$, and states that:

There exists a sequence S of $\mathcal{A}$ such that for every natural number n holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

The scheme *ExDdoubleLoopStrSeq* deals with a non empty double loop structure $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an element of the carrier of $\mathcal{A}$, and states that:

There exists a sequence S of $\mathcal{A}$ such that for every natural number n holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

Next we state the proposition

(8) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $z \neq 0_{\mathbb{C}_{\mathrm{F}}}$ and for every natural number n holds $|\mathrm{power}_{\mathbb{C}_{\mathrm{F}}}(z, n)| = |z|^n$.

Let p be a finite sequence of elements of the carrier of $\mathbb{C}_{\mathrm{F}}$. The functor $|p|$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def. 1) $\mathrm{len}\,|p| = \mathrm{len}\,p$ and for every natural number n such that $n \in \mathrm{dom}\,p$ holds $|p|_n = |p_n|$.

We now state several propositions:

(9) $|\varepsilon_{(\text{the carrier of } \mathbb{C}_{\mathrm{F}})}| = \varepsilon_{\mathbb{R}}$.

(10) For every element x of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $|\langle x \rangle| = \langle |x| \rangle$.

(11) For all elements x, y of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $|\langle x, y \rangle| = \langle |x|, |y| \rangle$.

(12) For all elements x, y, z of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $|\langle x, y, z \rangle| = \langle |x|, |y|, |z| \rangle$.

(13) For all finite sequences p, q of elements of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $|p \frown q| = |p| \frown |q|$.

(14) Let p be a finite sequence of elements of the carrier of $\mathbb{C}_{\mathrm{F}}$ and x be an element of the carrier of $\mathbb{C}_{\mathrm{F}}$. Then $|p \frown \langle x \rangle| = |p| \frown \langle |x| \rangle$ and $|\langle x \rangle \frown p| = \langle |x| \rangle \frown |p|$.

(15) For every finite sequence p of elements of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $|\sum p| \leqslant \sum |p|$.

2. Operations on Polynomials

Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure, let p be a Polynomial of L, and let n be a natural number. The functor p^n yields a sequence of L and is defined by:

(Def. 2) $p^n = \mathrm{power}_{\text{Polynom-Ring}\,L}(p, n)$.

Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure, let p be a Polynomial of L, and let n be a natural number. One can verify that p^n is finite-Support.

One can prove the following propositions:

(16) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and p be a Polynomial of L. Then $p^0 = \mathbf{1}.\,L$.

(17) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and p be a Polynomial of L. Then $p^1 = p$.

(18) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and p be a Polynomial of L. Then $p^2 = p * p$.

(19) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and p be a Polynomial of L. Then $p^3 = p * p * p$.

(20) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure, p be a Polynomial of L, and n be a natural number. Then $p^{n+1} = p^n * p$.

(21) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and n be a natural number. Then $(\mathbf{0}.\,L)^{n+1} = \mathbf{0}.\,L$.

(22) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and n be a natural number. Then $(\mathbf{1}.\,L)^n = \mathbf{1}.\,L$.

(23) Let L be a field, p be a Polynomial of L, x be an element of the carrier of L, and n be a natural number. Then $\mathrm{eval}(p^n, x) = \mathrm{power}_L(\mathrm{eval}(p, x), n)$.

(24) Let L be a field and p be a Polynomial of L. If $\mathrm{len}\, p \neq 0$, then for every natural number n holds $\mathrm{len}(p^n) = (n \cdot \mathrm{len}\, p - n) + 1$.

Let L be a non empty groupoid, let p be a sequence of L, and let v be an element of the carrier of L. The functor $v \cdot p$ yields a sequence of L and is defined by:

(Def. 3) For every natural number n holds $(v \cdot p)(n) = v \cdot p(n)$.

Let L be an add-associative right zeroed right complementable right distributive non empty double loop structure, let p be a Polynomial of L, and let v be an element of the carrier of L. Observe that $v \cdot p$ is finite-Support.

We now state several propositions:

(25) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure and p be a Polynomial of L. Then $\mathrm{len}(0_L \cdot p) = 0$.

(26) Let L be an add-associative right zeroed right complementable left unital commutative associative distributive field-like non empty double loop structure, p be a Polynomial of L, and v be an element of the carrier of L. If $v \neq 0_L$, then $\mathrm{len}(v \cdot p) = \mathrm{len}\, p$.

(27) Let L be an add-associative right zeroed right complementable left distributive non empty double loop structure and p be a sequence of L. Then $0_L \cdot p = \mathbf{0}.\, L$.

(28) For every left unital non empty multiplicative loop structure L and for every sequence p of L holds $\mathbf{1}_L \cdot p = p$.

(29) Let L be an add-associative right zeroed right complementable right distributive non empty double loop structure and v be an element of the carrier of L. Then $v \cdot \mathbf{0}.\, L = \mathbf{0}.\, L$.

(30) Let L be an add-associative right zeroed right complementable right unital right distributive non empty double loop structure and v be an element of the carrier of L. Then $v \cdot \mathbf{1}.\, L = \langle v \rangle$.

(31) Let L be an add-associative right zeroed right complementable left unital distributive commutative associative field-like non empty double loop structure, p be a Polynomial of L, and v, x be elements of the carrier of L. Then $\mathrm{eval}(v \cdot p, x) = v \cdot \mathrm{eval}(p, x)$.

(32) Let L be an add-associative right zeroed right complementable right distributive unital non empty double loop structure and p be a Polynomial of L. Then $\mathrm{eval}(p, 0_L) = p(0)$.

Let L be a non empty zero structure and let z_0, z_1 be elements of the carrier of L. The functor $\langle z_0, z_1 \rangle$ yields a sequence of L and is defined by:

(Def. 4) $\langle z_0, z_1 \rangle = \mathbf{0}.\, L +\cdot (0, z_0) +\cdot (1, z_1)$.

The following propositions are true:

(33) Let L be a non empty zero structure and z_0 be an element of the carrier of L. Then $\langle z_0 \rangle(0) = z_0$ and for every natural number n such that $n \geqslant 1$ holds $\langle z_0 \rangle(n) = 0_L$.

(34) For every non empty zero structure L and for every element z_0 of the carrier of L such that $z_0 \neq 0_L$ holds $\mathrm{len}\langle z_0 \rangle = 1$.

(35) For every non empty zero structure L holds $\langle 0_L \rangle = \mathbf{0}.\, L$.

(36) Let L be an add-associative right zeroed right complementable distributive commutative associative left unital field-like non empty double loop structure and x, y be elements of the carrier of L. Then $\langle x \rangle * \langle y \rangle = \langle x \cdot y \rangle$.

(37) Let L be an Abelian add-associative right zeroed right complementable right unital associative commutative distributive field-like non empty do-

uble loop structure, x be an element of the carrier of L, and n be a natural number. Then $\langle x\rangle^n = \langle \mathrm{power}_L(x, n)\rangle$.

(38) Let L be an add-associative right zeroed right complementable unital non empty double loop structure and z_0, x be elements of the carrier of L. Then $\mathrm{eval}(\langle z_0\rangle, x) = z_0$.

(39) Let L be a non empty zero structure and z_0, z_1 be elements of the carrier of L. Then $\langle z_0, z_1\rangle(0) = z_0$ and $\langle z_0, z_1\rangle(1) = z_1$ and for every natural number n such that $n \geqslant 2$ holds $\langle z_0, z_1\rangle(n) = 0_L$.

Let L be a non empty zero structure and let z_0, z_1 be elements of the carrier of L. One can verify that $\langle z_0, z_1\rangle$ is finite-Support.

The following propositions are true:

(40) For every non empty zero structure L and for all elements z_0, z_1 of the carrier of L holds $\mathrm{len}\langle z_0, z_1\rangle \leqslant 2$.

(41) For every non empty zero structure L and for all elements z_0, z_1 of the carrier of L such that $z_1 \neq 0_L$ holds $\mathrm{len}\langle z_0, z_1\rangle = 2$.

(42) For every non empty zero structure L and for every element z_0 of the carrier of L such that $z_0 \neq 0_L$ holds $\mathrm{len}\langle z_0, 0_L\rangle = 1$.

(43) For every non empty zero structure L holds $\langle 0_L, 0_L\rangle = \mathbf{0}.\, L$.

(44) For every non empty zero structure L and for every element z_0 of the carrier of L holds $\langle z_0, 0_L\rangle = \langle z_0\rangle$.

(45) Let L be an add-associative right zeroed right complementable left distributive unital non empty double loop structure and z_0, z_1, x be elements of the carrier of L. Then $\mathrm{eval}(\langle z_0, z_1\rangle, x) = z_0 + z_1 \cdot x$.

(46) Let L be an add-associative right zeroed right complementable left distributive unital non empty double loop structure and z_0, z_1, x be elements of the carrier of L. Then $\mathrm{eval}(\langle z_0, 0_L\rangle, x) = z_0$.

(47) Let L be an add-associative right zeroed right complementable left distributive unital non empty double loop structure and z_0, z_1, x be elements of the carrier of L. Then $\mathrm{eval}(\langle 0_L, z_1\rangle, x) = z_1 \cdot x$.

(48) Let L be an add-associative right zeroed right complementable left distributive well unital non empty double loop structure and z_0, z_1, x be elements of the carrier of L. Then $\mathrm{eval}(\langle z_0, \mathbf{1}_L\rangle, x) = z_0 + x$.

(49) Let L be an add-associative right zeroed right complementable left distributive well unital non empty double loop structure and z_0, z_1, x be elements of the carrier of L. Then $\mathrm{eval}(\langle 0_L, \mathbf{1}_L\rangle, x) = x$.

3. Substitution in Polynomials

Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and let p, q be Polynomials of L. The functor $p[q]$ yielding a Polynomial of L is defined by the condition (Def. 5).

(Def. 5) There exists a finite sequence F of elements of the carrier of Polynom-Ring L such that $p[q] = \sum F$ and $\mathrm{len}\, F = \mathrm{len}\, p$ and for every natural number n such that $n \in \mathrm{dom}\, F$ holds $F(n) = p(n -' 1) \cdot q^{n -' 1}$.

One can prove the following propositions:

(50) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and p be a Polynomial of L. Then $(\mathbf{0}.\, L)[p] = \mathbf{0}.\, L$.

(51) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and p be a Polynomial of L. Then $p[\mathbf{0}.\, L] = \langle p(0) \rangle$.

(52) Let L be an Abelian add-associative right zeroed right complementable right unital associative commutative distributive field-like non empty double loop structure, p be a Polynomial of L, and x be an element of the carrier of L. Then $\mathrm{len}(p[\langle x \rangle]) \leqslant 1$.

(53) For every field L and for all Polynomials p, q of L such that $\mathrm{len}\, p \neq 0$ and $\mathrm{len}\, q > 1$ holds $\mathrm{len}(p[q]) = (\mathrm{len}\, p \cdot \mathrm{len}\, q - \mathrm{len}\, p - \mathrm{len}\, q) + 2$.

(54) Let L be a field, p, q be Polynomials of L, and x be an element of the carrier of L. Then $\mathrm{eval}(p[q], x) = \mathrm{eval}(p, \mathrm{eval}(q, x))$.

4. Fundamental Theorem of Algebra

Let L be a unital non empty double loop structure, let p be a Polynomial of L, and let x be an element of the carrier of L. We say that x is a root of p if and only if:

(Def. 6) $\mathrm{eval}(p, x) = 0_L$.

Let L be a unital non empty double loop structure and let p be a Polynomial of L. We say that p has roots if and only if:

(Def. 7) There exists an element x of the carrier of L such that x is a root of p.

The following proposition is true

(55) For every unital non empty double loop structure L holds $\mathbf{0}.\, L$ has roots.

Let L be a unital non empty double loop structure. One can verify that $\mathbf{0}.\,L$ has roots.

The following proposition is true

(56) Let L be a unital non empty double loop structure and x be an element of the carrier of L. Then x is a root of $\mathbf{0}.\,L$.

Let L be a unital non empty double loop structure. One can verify that there exists a Polynomial of L which has roots.

Let L be a unital non empty double loop structure. We say that L is algebraic-closed if and only if:

(Def. 8) For every Polynomial p of L such that $\operatorname{len} p > 1$ holds p has roots.

Let L be a unital non empty double loop structure and let p be a Polynomial of L. The functor $\operatorname{Roots} p$ yields a subset of L and is defined by:

(Def. 9) For every element x of the carrier of L holds $x \in \operatorname{Roots} p$ iff x is a root of p.

Let L be a commutative associative left unital distributive field-like non empty double loop structure and let p be a Polynomial of L. The functor $\operatorname{NormPolynomial} p$ yielding a sequence of L is defined as follows:

(Def. 10) For every natural number n holds $(\operatorname{NormPolynomial} p)(n) = \frac{p(n)}{p(\operatorname{len} p -' 1)}$.

Let L be an add-associative right zeroed right complementable commutative associative left unital distributive field-like non empty double loop structure and let p be a Polynomial of L. Note that $\operatorname{NormPolynomial} p$ is finite-Support.

The following propositions are true:

(57) Let L be a commutative associative left unital distributive field-like non empty double loop structure and p be a Polynomial of L. If $\operatorname{len} p \neq 0$, then $(\operatorname{NormPolynomial} p)(\operatorname{len} p -' 1) = \mathbf{1}_L$.

(58) For every field L and for every Polynomial p of L such that $\operatorname{len} p \neq 0$ holds $\operatorname{len} \operatorname{NormPolynomial} p = \operatorname{len} p$.

(59) Let L be a field and p be a Polynomial of L. Suppose $\operatorname{len} p \neq 0$. Let x be an element of the carrier of L. Then $\operatorname{eval}(\operatorname{NormPolynomial} p, x) = \frac{\operatorname{eval}(p,x)}{p(\operatorname{len} p -' 1)}$.

(60) Let L be a field and p be a Polynomial of L. Suppose $\operatorname{len} p \neq 0$. Let x be an element of the carrier of L. Then x is a root of p if and only if x is a root of $\operatorname{NormPolynomial} p$.

(61) For every field L and for every Polynomial p of L such that $\operatorname{len} p \neq 0$ holds p has roots iff $\operatorname{NormPolynomial} p$ has roots.

(62) For every field L and for every Polynomial p of L such that $\operatorname{len} p \neq 0$ holds $\operatorname{Roots} p = \operatorname{Roots} \operatorname{NormPolynomial} p$.

(63) $\mathrm{id}_{\mathbb{C}}$ is continuous on $\mathbb{C}$.

(64) For every element x of $\mathbb{C}$ holds $\mathbb{C} \longmapsto x$ is continuous on $\mathbb{C}$.

Let L be a unital non empty groupoid, let x be an element of the carrier of L, and let n be a natural number. The functor $\mathrm{FPower}(x, n)$ yields a map from L into L and is defined as follows:

(Def. 11) For every element y of the carrier of L holds $(\mathrm{FPower}(x, n))(y) = x \cdot \mathrm{power}_L(y, n)$.

The following propositions are true:

(65) For every unital non empty groupoid L holds $\mathrm{FPower}(1_L, 1) = \mathrm{id}_{\text{the carrier of } L}$.

(66) $\mathrm{FPower}(\mathbf{1}_{\mathbb{C}_{\mathrm{F}}}, 2) = \mathrm{id}_{\mathbb{C}}\, \mathrm{id}_{\mathbb{C}}$.

(67) For every unital non empty groupoid L and for every element x of the carrier of L holds $\mathrm{FPower}(x, 0) = (\text{the carrier of } L) \longmapsto x$.

(68) For every element x of the carrier of $\mathbb{C}_{\mathrm{F}}$ there exists an element x_1 of $\mathbb{C}$ such that $x = x_1$ and $\mathrm{FPower}(x, 1) = x_1\, \mathrm{id}_{\mathbb{C}}$.

(69) For every element x of the carrier of $\mathbb{C}_{\mathrm{F}}$ there exists an element x_1 of $\mathbb{C}$ such that $x = x_1$ and $\mathrm{FPower}(x, 2) = x_1\, (\mathrm{id}_{\mathbb{C}}\, \mathrm{id}_{\mathbb{C}})$.

(70) Let x be an element of the carrier of $\mathbb{C}_{\mathrm{F}}$ and n be a natural number. Then there exists a function f from $\mathbb{C}$ into $\mathbb{C}$ such that $f = \mathrm{FPower}(x, n)$ and $\mathrm{FPower}(x, n+1) = f\, \mathrm{id}_{\mathbb{C}}$.

(71) Let x be an element of the carrier of $\mathbb{C}_{\mathrm{F}}$ and n be a natural number. Then there exists a function f from $\mathbb{C}$ into $\mathbb{C}$ such that $f = \mathrm{FPower}(x, n)$ and f is continuous on $\mathbb{C}$.

Let L be a unital non empty double loop structure and let p be a Polynomial of L. The functor $\mathrm{Polynomial\text{-}Function}(L, p)$ yields a map from L into L and is defined as follows:

(Def. 12) For every element x of the carrier of L holds $(\mathrm{Polynomial\text{-}Function}(L, p))(x) = \mathrm{eval}(p, x)$.

The following propositions are true:

(72) For every Polynomial p of $\mathbb{C}_{\mathrm{F}}$ there exists a function f from $\mathbb{C}$ into $\mathbb{C}$ such that $f = \mathrm{Polynomial\text{-}Function}(\mathbb{C}_{\mathrm{F}}, p)$ and f is continuous on $\mathbb{C}$.

(73) Let p be a Polynomial of $\mathbb{C}_{\mathrm{F}}$. Suppose $\mathrm{len}\, p > 2$ and $|p(\mathrm{len}\, p -' 1)| = 1$. Let F be a finite sequence of elements of $\mathbb{R}$. Suppose $\mathrm{len}\, F = \mathrm{len}\, p$ and for every natural number n such that $n \in \mathrm{dom}\, F$ holds $F(n) = |p(n -' 1)|$. Let z be an element of the carrier of $\mathbb{C}_{\mathrm{F}}$. If $|z| > \sum F$, then $|\mathrm{eval}(p, z)| > |p(0)| + 1$.

(74) Let p be a Polynomial of $\mathbb{C}_{\mathrm{F}}$. Suppose $\mathrm{len}\, p > 2$. Then there exists an element z_0 of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that for every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $|\mathrm{eval}(p, z)| \geqslant |\mathrm{eval}(p, z_0)|$.

(75) For every Polynomial p of $\mathbb{C}_{\mathrm{F}}$ such that $\mathrm{len}\, p > 1$ holds p has roots.

Let us note that $\mathbb{C}_{\mathrm{F}}$ is algebraic-closed.

Let us mention that there exists a left unital right unital non empty double loop structure which is algebraic-closed, add-associative, right zeroed, right complementable, Abelian, commutative, associative, distributive, field-like, and non degenerated.

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On Replace Function and Swap Function for Finite Sequences

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Summary. In this article, we show the property of the Replace Function and the Swap Function of finite sequences. In the first section, we prepared some useful theorems for finite sequences. In the second section, we defined the Replace function and proved some theorems about the function. This function replaces an element of a sequence by another value. In the third section, we defined the Swap function and proved some theorems about the function. This function swaps two elements of a sequence. In the last section, we show the property of composed functions of the Replace Function and the Swap Function.

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The notation and terminology used here are introduced in the following papers: [7], [11], [2], [9], [3], [1], [5], [12], [6], [10], [8], and [4].

1. Some Basic Theorems

For simplicity, we adopt the following rules: D denotes a non empty set, f, g, h denote finite sequences of elements of D, p, p_1, p_2, p_3, q denote elements of D, and i, j, k, l, n denote natural numbers.

One can prove the following propositions:

(1) If $1 \leqslant i$ and $j \leqslant \operatorname{len} f$ and $i < j$, then $f = (f \restriction (i -' 1)) \frown \langle f(i) \rangle \frown (f_{\lfloor i} \restriction (j -' i -' 1)) \frown \langle f(j) \rangle \frown (f_{\lfloor j})$.

(2) If $\operatorname{len} g = \operatorname{len} h$ and $\operatorname{len} g < i$ and $i \leqslant \operatorname{len}(g \frown f)$, then $(g \frown f)(i) = (h \frown f)(i)$.

(3) If $1 \leqslant i$ and $i \leqslant \operatorname{len} f$, then $f(i) = (g \frown f)(\operatorname{len} g + i)$.

(4) If $i \in \operatorname{dom}(f_{\lfloor n})$, then $f_{\lfloor n}(i) = f(n + i)$.

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2. Definition of Replace Function and its Properties

Let D be a non empty set, let f be a finite sequence of elements of D, let i be a natural number, and let p be an element of D. Then $f +\cdot (i, p)$ is a finite sequence of elements of D and it can be characterized by the condition:

(Def. 1) $$f +\cdot (i,p) = \begin{cases} (f \restriction (i -' 1)) \frown \langle p \rangle \frown (f_{\restriction i}), & \text{if } 1 \leqslant i \text{ and } i \leqslant \operatorname{len} f, \\ f, & \text{otherwise.} \end{cases}$$

We introduce $\operatorname{Replace}(f, i, p)$ as a synonym of $f +\cdot (i, p)$.

The following propositions are true:

(5) $\operatorname{Replace}(f, 0, p) = f$.

(6) If $i > \operatorname{len} f$, then $\operatorname{Replace}(f, i, p) = f$.

(7) $\operatorname{len} \operatorname{Replace}(f, i, p) = \operatorname{len} f$.

(8) $\operatorname{rng} \operatorname{Replace}(f, i, p) \subseteq \operatorname{rng} f \cup \{p\}$.

(9) If $1 \leqslant i$ and $i \leqslant \operatorname{len} f$, then $p \in \operatorname{rng} \operatorname{Replace}(f, i, p)$.

(10) If $1 \leqslant i$ and $i \leqslant \operatorname{len} f$, then $(\operatorname{Replace}(f, i, p))_i = p$.

(11) If $1 \leqslant i$ and $i \leqslant \operatorname{len} f$, then for every k such that $0 < k$ and $k \leqslant \operatorname{len} f - i$ holds $(\operatorname{Replace}(f, i, p))(i + k) = f_{\restriction i}(k)$.

(12) If $1 \leqslant k$ and $k \leqslant \operatorname{len} f$ and $k \neq i$, then $(\operatorname{Replace}(f, i, p))_k = f_k$.

(13) If $1 \leqslant i$ and $i < j$ and $j \leqslant \operatorname{len} f$, then $\operatorname{Replace}(\operatorname{Replace}(f, j, q), i, p) = (f \restriction (i -' 1)) \frown \langle p \rangle \frown (f_{\restriction i} \restriction (j -' i -' 1)) \frown \langle q \rangle \frown (f_{\restriction j})$.

(14) $\operatorname{Replace}(\langle p \rangle, 1, q) = \langle q \rangle$.

(15) $\operatorname{Replace}(\langle p_1, p_2 \rangle, 1, q) = \langle q, p_2 \rangle$.

(16) $\operatorname{Replace}(\langle p_1, p_2 \rangle, 2, q) = \langle p_1, q \rangle$.

(17) $\operatorname{Replace}(\langle p_1, p_2, p_3 \rangle, 1, q) = \langle q, p_2, p_3 \rangle$.

(18) $\operatorname{Replace}(\langle p_1, p_2, p_3 \rangle, 2, q) = \langle p_1, q, p_3 \rangle$.

(19) $\operatorname{Replace}(\langle p_1, p_2, p_3 \rangle, 3, q) = \langle p_1, p_2, q \rangle$.

3. Definition of Swap Function and its Properties

Let D be a non empty set, let f be a finite sequence of elements of D, and let i, j be natural numbers. The functor $\operatorname{Swap}(f, i, j)$ yields a finite sequence of elements of D and is defined as follows:

(Def. 2) $$\operatorname{Swap}(f,i,j) = \begin{cases} \operatorname{Replace}(\operatorname{Replace}(f, i, f_j), j, f_i), & \text{if } 1 \leqslant i \text{ and } i \leqslant \operatorname{len} f \\ \quad \text{and } 1 \leqslant j \text{ and } j \leqslant \operatorname{len} f, & \\ f, & \text{otherwise.} \end{cases}$$

Next we state a number of propositions:

(20) $\operatorname{len} \operatorname{Swap}(f, i, j) = \operatorname{len} f$.

(21) $\mathrm{Swap}(f, i, i) = f$.

(22) $\mathrm{Swap}(\mathrm{Swap}(f, i, j), j, i) = f$.

(23) $\mathrm{Swap}(f, i, j) = \mathrm{Swap}(f, j, i)$.

(24) $\mathrm{rng}\,\mathrm{Swap}(f, i, j) = \mathrm{rng}\, f$.

(25) $\mathrm{Swap}(\langle p_1, p_2 \rangle, 1, 2) = \langle p_2, p_1 \rangle$.

(26) $\mathrm{Swap}(\langle p_1, p_2, p_3 \rangle, 1, 2) = \langle p_2, p_1, p_3 \rangle$.

(27) $\mathrm{Swap}(\langle p_1, p_2, p_3 \rangle, 1, 3) = \langle p_3, p_2, p_1 \rangle$.

(28) $\mathrm{Swap}(\langle p_1, p_2, p_3 \rangle, 2, 3) = \langle p_1, p_3, p_2 \rangle$.

(29) If $1 \leqslant i$ and $i < j$ and $j \leqslant \mathrm{len}\, f$, then $\mathrm{Swap}(f, i, j) = (f \restriction (i -' 1)) ^\frown \langle f_j \rangle ^\frown (f_{\lfloor i} \restriction (j -' i -' 1)) ^\frown \langle f_i \rangle ^\frown (f_{\lfloor j})$.

(30) If $1 < i$ and $i \leqslant \mathrm{len}\, f$, then $\mathrm{Swap}(f, 1, i) = \langle f_i \rangle ^\frown (f_{\lfloor 1} \restriction (i -' 2)) ^\frown \langle f_1 \rangle ^\frown (f_{\lfloor i})$.

(31) If $1 \leqslant i$ and $i < \mathrm{len}\, f$, then $\mathrm{Swap}(f, i, \mathrm{len}\, f) = (f \restriction (i -' 1)) ^\frown \langle f_{\mathrm{len}\, f} \rangle ^\frown (f_{\lfloor i} \restriction (\mathrm{len}\, f -' i -' 1)) ^\frown \langle f_i \rangle$.

(32) If $i \neq k$ and $j \neq k$ and $1 \leqslant k$ and $k \leqslant \mathrm{len}\, f$, then $(\mathrm{Swap}(f, i, j))_k = f_k$.

(33) If $1 \leqslant i$ and $i \leqslant \mathrm{len}\, f$ and $1 \leqslant j$ and $j \leqslant \mathrm{len}\, f$, then $(\mathrm{Swap}(f, i, j))_i = f_j$ and $(\mathrm{Swap}(f, i, j))_j = f_i$.

4. Properties of Combination Function of Replace Function and Swap Function

We now state four propositions:

(34) If $1 \leqslant i$ and $i \leqslant \mathrm{len}\, f$ and $1 \leqslant j$ and $j \leqslant \mathrm{len}\, f$, then $\mathrm{Replace}(\mathrm{Swap}(f, i, j), i, p) = \mathrm{Swap}(\mathrm{Replace}(f, j, p), i, j)$.

(35) If $i \neq k$ and $j \neq k$ and $1 \leqslant i$ and $i \leqslant \mathrm{len}\, f$ and $1 \leqslant j$ and $j \leqslant \mathrm{len}\, f$ and $1 \leqslant k$ and $k \leqslant \mathrm{len}\, f$, then $\mathrm{Swap}(\mathrm{Replace}(f, k, p), i, j) = \mathrm{Replace}(\mathrm{Swap}(f, i, j), k, p)$.

(36) If $i \neq k$ and $j \neq k$ and $1 \leqslant i$ and $i \leqslant \mathrm{len}\, f$ and $1 \leqslant j$ and $j \leqslant \mathrm{len}\, f$ and $1 \leqslant k$ and $k \leqslant \mathrm{len}\, f$, then $\mathrm{Swap}(\mathrm{Swap}(f, i, j), j, k) = \mathrm{Swap}(\mathrm{Swap}(f, i, k), i, j)$.

(37) Suppose $i \neq k$ and $j \neq k$ and $l \neq i$ and $l \neq j$ and $1 \leqslant i$ and $i \leqslant \mathrm{len}\, f$ and $1 \leqslant j$ and $j \leqslant \mathrm{len}\, f$ and $1 \leqslant k$ and $k \leqslant \mathrm{len}\, f$ and $1 \leqslant l$ and $l \leqslant \mathrm{len}\, f$. Then $\mathrm{Swap}(\mathrm{Swap}(f, i, j), k, l) = \mathrm{Swap}(\mathrm{Swap}(f, k, l), i, j)$.

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The Correctness of the High Speed Array Multiplier Circuits

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Summary. This article introduces the verification of the correctness for the operations and the specification of the high speed array multiplier. We formalize the concepts of 2-by-2 and 3-by-3 bit Plain array multiplier, 3-by-3 Wallace tree multiplier circuit, and show that outputs of the array multiplier are equivalent to outputs of normal (sequencial) multiplier.

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The articles [3], [1], and [2] provide the terminology and notation for this paper.

1. Preliminaries

Let x_0, x_1, y_0, y_1 be sets. The functor $\mathrm{MULT}_{210}(x_1, y_1, x_0, y_0)$ yields a set and is defined as follows:

(Def. 1) $\mathrm{MULT}_{210}(x_1, y_1, x_0, y_0) = \mathrm{AND2}(x_0, y_0)$.

The functor $\mathrm{MULT}_{211}(x_1, y_1, x_0, y_0)$ yields a set and is defined by:

(Def. 2) $\mathrm{MULT}_{211}(x_1, y_1, x_0, y_0) = \mathrm{ADD1}(\mathrm{AND2}(x_1, y_0), \mathrm{AND2}(x_0, y_1), \emptyset)$.

The functor $\mathrm{MULT}_{212}(x_1, y_1, x_0, y_0)$ yielding a set is defined as follows:

(Def. 3) $\mathrm{MULT}_{212}(x_1, y_1, x_0, y_0) = \mathrm{ADD2}(\emptyset, \mathrm{AND2}(x_1, y_1), \mathrm{AND2}(x_1, y_0), \mathrm{AND2}(x_0, y_1), \emptyset)$.

The functor $\mathrm{MULT}_{213}(x_1, y_1, x_0, y_0)$ yields a set and is defined as follows:

(Def. 4) $\mathrm{MULT}_{213}(x_1, y_1, x_0, y_0) = \mathrm{CARR2}(\emptyset, \mathrm{AND2}(x_1, y_1), \mathrm{AND2}(x_1, y_0), \mathrm{AND2}(x_0, y_1), \emptyset)$.

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We now state the proposition

(1) Let x_0, x_1, y_0, y_1, z_0, z_1, z_2, z_3, q_0, q_1, c_1, q_{11}, c_{11} be sets such that NE q_0 iff NE AND2(x_0, y_0) and NE q_1 iff NE XOR3(AND2(x_1, y_0), AND2(x_0, y_1), $\emptyset$) and NE c_1 iff NE MAJ3(AND2(x_1, y_0), AND2(x_0, y_1), $\emptyset$) and NE q_{11} iff NE XOR3(AND2(x_1, y_1), $\emptyset$, c_1) and NE c_{11} iff NE MAJ3(AND2(x_1, y_1), $\emptyset$, c_1) and NE z_0 iff NE q_0 and NE z_1 iff NE q_1 and NE z_2 iff NE q_{11} and NE z_3 iff NE c_{11}. Then

(i) NE z_0 iff NE MULT$_{210}(x_1, y_1, x_0, y_0)$,

(ii) NE z_1 iff NE MULT$_{211}(x_1, y_1, x_0, y_0)$,

(iii) NE z_2 iff NE MULT$_{212}(x_1, y_1, x_0, y_0)$, and

(iv) NE z_3 iff NE MULT$_{213}(x_1, y_1, x_0, y_0)$.

Let x_0, x_1, x_2, y_0, y_1 be sets. The functor MULT$_{310}(x_2, x_1, y_1, x_0, y_0)$ yields a set and is defined by:

(Def. 5) MULT$_{310}(x_2, x_1, y_1, x_0, y_0)$ = AND2(x_0, y_0).

The functor MULT$_{311}(x_2, x_1, y_1, x_0, y_0)$ yields a set and is defined as follows:

(Def. 6) MULT$_{311}(x_2, x_1, y_1, x_0, y_0)$ = ADD1(AND2(x_1, y_0), AND2(x_0, y_1), $\emptyset$).

The functor MULT$_{312}(x_2, x_1, y_1, x_0, y_0)$ yields a set and is defined as follows:

(Def. 7) MULT$_{312}(x_2, x_1, y_1, x_0, y_0)$ = ADD2(AND2(x_2, y_0), AND2(x_1, y_1), AND2(x_1, y_0), AND2(x_0, y_1), $\emptyset$).

The functor MULT$_{313}(x_2, x_1, y_1, x_0, y_0)$ yields a set and is defined as follows:

(Def. 8) MULT$_{313}(x_2, x_1, y_1, x_0, y_0)$ = ADD3($\emptyset$, AND2(x_2, y_1), AND2(x_2, y_0), AND2(x_1, y_1), AND2(x_1, y_0), AND2(x_0, y_1), $\emptyset$).

The functor MULT$_{314}(x_2, x_1, y_1, x_0, y_0)$ yielding a set is defined by:

(Def. 9) MULT$_{314}(x_2, x_1, y_1, x_0, y_0)$ = CARR3($\emptyset$, AND2(x_2, y_1), AND2(x_2, y_0), AND2(x_1, y_1), AND2(x_1, y_0), AND2(x_0, y_1), $\emptyset$).

Let x_0, x_1, x_2, y_0, y_1, y_2 be sets. The functor MULT$_{321}(x_2, y_2, x_1, y_1, x_0, y_0)$ yields a set and is defined by:

(Def. 10) MULT$_{321}(x_2, y_2, x_1, y_1, x_0, y_0)$ = ADD1(MULT$_{312}(x_2, x_1, y_1, x_0, y_0)$, AND2$(x_0, y_2)$, $\emptyset$).

The functor MULT$_{322}(x_2, y_2, x_1, y_1, x_0, y_0)$ yields a set and is defined by:

(Def. 11) MULT$_{322}(x_2, y_2, x_1, y_1, x_0, y_0)$ = ADD2(MULT$_{313}(x_2, x_1, y_1, x_0, y_0)$, AND2$(x_1, y_2)$, MULT$_{312}(x_2, x_1, y_1, x_0, y_0)$, AND2$(x_0, y_2)$, $\emptyset$).

The functor MULT$_{323}(x_2, y_2, x_1, y_1, x_0, y_0)$ yields a set and is defined as follows:

(Def. 12) MULT$_{323}(x_2, y_2, x_1, y_1, x_0, y_0)$ = ADD3(MULT$_{314}(x_2, x_1, y_1, x_0, y_0)$, AND2$(x_2, y_2)$, MULT$_{313}(x_2, x_1, y_1, x_0, y_0)$, AND2$(x_1, y_2)$, MULT$_{312}(x_2, x_1, y_1, x_0, y_0)$, AND2$(x_0, y_2)$, $\emptyset$).

The functor MULT$_{324}(x_2, y_2, x_1, y_1, x_0, y_0)$ yielding a set is defined as follows:

(Def. 13) MULT$_{324}(x_2, y_2, x_1, y_1, x_0, y_0)$ = CARR3(MULT$_{314}(x_2, x_1, y_1, x_0, y_0)$, AND2$(x_2, y_2)$, MULT$_{313}(x_2, x_1, y_1, x_0, y_0)$, AND2$(x_1, y_2)$, MULT$_{312}(x_2, x_1,$

$y_1, x_0, y_0), \mathrm{AND2}(x_0, y_2), \emptyset)$.

Next we state the proposition

(2) Let x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2, z_3, z_4, z_5, q_0, q_1, q_2, c_1, c_2, q_{11}, q_{12}, c_{11}, c_{12}, q_{21}, q_{22}, c_{21}, c_{22} be sets such that NE q_0 iff NE $\mathrm{AND2}(x_0, y_0)$ and NE q_1 iff NE $\mathrm{XOR3}(\mathrm{AND2}(x_1, y_0), \mathrm{AND2}(x_0, y_1), \emptyset)$ and NE c_1 iff NE $\mathrm{MAJ3}(\mathrm{AND2}(x_1, y_0), \mathrm{AND2}(x_0, y_1), \emptyset)$ and NE q_2 iff NE $\mathrm{XOR3}(\mathrm{AND2}(x_2, y_0), \mathrm{AND2}(x_1, y_1), \emptyset)$ and NE c_2 iff NE $\mathrm{MAJ3}(\mathrm{AND2}(x_2, y_0), \mathrm{AND2}(x_1, y_1), \emptyset)$ and NE q_{11} iff NE $\mathrm{XOR3}(q_2, \mathrm{AND2}(x_0, y_2), c_1)$ and NE c_{11} iff NE $\mathrm{MAJ3}(q_2, \mathrm{AND2}(x_0, y_2), c_1)$ and NE q_{12} iff NE $\mathrm{XOR3}(\mathrm{AND2}(x_2, y_1), \mathrm{AND2}(x_1, y_2), c_2)$ and NE c_{12} iff NE $\mathrm{MAJ3}(\mathrm{AND2}(x_2, y_1), \mathrm{AND2}(x_1, y_2), c_2)$ and NE q_{21} iff NE $\mathrm{XOR3}(q_{12}, \emptyset, c_{11})$ and NE c_{21} iff NE $\mathrm{MAJ3}(q_{12}, \emptyset, c_{11})$ and NE q_{22} iff NE $\mathrm{XOR3}(\mathrm{AND2}(x_2, y_2), c_{21}, c_{12})$ and NE c_{22} iff NE $\mathrm{MAJ3}(\mathrm{AND2}(x_2, y_2), c_{21}, c_{12})$ and NE z_0 iff NE q_0 and NE z_1 iff NE q_1 and NE z_2 iff NE q_{11} and NE z_3 iff NE q_{21} and NE z_4 iff NE q_{22} and NE z_5 iff NE c_{22}. Then

(i) NE z_0 iff NE $\mathrm{MULT}_{310}(x_2, x_1, y_1, x_0, y_0)$,

(ii) NE z_1 iff NE $\mathrm{MULT}_{311}(x_2, x_1, y_1, x_0, y_0)$,

(iii) NE z_2 iff NE $\mathrm{MULT}_{321}(x_2, y_2, x_1, y_1, x_0, y_0)$,

(iv) NE z_3 iff NE $\mathrm{MULT}_{322}(x_2, y_2, x_1, y_1, x_0, y_0)$,

(v) NE z_4 iff NE $\mathrm{MULT}_{323}(x_2, y_2, x_1, y_1, x_0, y_0)$, and

(vi) NE z_5 iff NE $\mathrm{MULT}_{324}(x_2, y_2, x_1, y_1, x_0, y_0)$.

2. Logical Equivalence of Wallace Tree Multiplier

One can prove the following proposition

(3) Let x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2, z_3, z_4, z_5, q_0, q_1, q_2, q_3, c_1, c_2, c_3, q_{11}, q_{12}, q_{13}, c_{11}, c_{12}, c_{13} be sets such that NE q_0 iff NE $\mathrm{AND2}(x_0, y_0)$ and NE q_1 iff NE $\mathrm{XOR3}(\mathrm{AND2}(x_1, y_0), \mathrm{AND2}(x_0, y_1), \emptyset)$ and NE c_1 iff NE $\mathrm{MAJ3}(\mathrm{AND2}(x_1, y_0), \mathrm{AND2}(x_0, y_1), \emptyset)$ and NE q_2 iff NE $\mathrm{XOR3}(\mathrm{AND2}(x_2, y_0), \mathrm{AND2}(x_1, y_1), \mathrm{AND2}(x_0, y_2))$ and NE c_2 iff NE $\mathrm{MAJ3}(\mathrm{AND2}(x_2, y_0), \mathrm{AND2}(x_1, y_1), \mathrm{AND2}(x_0, y_2))$ and NE q_3 iff NE $\mathrm{XOR3}(\mathrm{AND2}(x_2, y_1), \mathrm{AND2}(x_1, y_2), \emptyset)$ and NE c_3 iff NE $\mathrm{MAJ3}(\mathrm{AND2}(x_2, y_1), \mathrm{AND2}(x_1, y_2), \emptyset)$ and NE q_{11} iff NE $\mathrm{XOR3}(q_2, c_1, \emptyset)$ and NE c_{11} iff NE $\mathrm{MAJ3}(q_2, c_1, \emptyset)$ and NE q_{12} iff NE $\mathrm{XOR3}(q_3, c_2, c_{11})$ and NE c_{12} iff NE $\mathrm{MAJ3}(q_3, c_2, c_{11})$ and NE q_{13} iff NE $\mathrm{XOR3}(\mathrm{AND2}(x_2, y_2), c_3, c_{12})$ and NE c_{13} iff NE $\mathrm{MAJ3}(\mathrm{AND2}(x_2, y_2), c_3, c_{12})$ and NE z_0 iff NE q_0 and NE z_1 iff NE q_1 and NE z_2 iff NE q_{11} and NE z_3 iff NE q_{12} and NE z_4 iff NE q_{13} and NE z_5 iff NE c_{13}. Then

(i) NE z_0 iff NE $\mathrm{MULT}_{310}(x_2, x_1, y_1, x_0, y_0)$,

(ii) NE z_1 iff NE $\mathrm{MULT}_{311}(x_2, x_1, y_1, x_0, y_0)$,

(iii) NE z_2 iff NE $\mathrm{MULT}_{321}(x_2, y_2, x_1, y_1, x_0, y_0)$,
(iv) NE z_3 iff NE $\mathrm{MULT}_{322}(x_2, y_2, x_1, y_1, x_0, y_0)$,
(v) NE z_4 iff NE $\mathrm{MULT}_{323}(x_2, y_2, x_1, y_1, x_0, y_0)$, and
(vi) NE z_5 iff NE $\mathrm{MULT}_{324}(x_2, y_2, x_1, y_1, x_0, y_0)$.

Let a_1, b_1, c be sets. We introduce $\mathrm{CLAADD1}(a_1, b_1, c)$ as a synonym of $\mathrm{XOR3}(a_1, b_1, c)$. We introduce $\mathrm{CLACARR1}(a_1, b_1, c)$ as a synonym of $\mathrm{MAJ3}(a_1, b_1, c)$.

Let a_1, b_1, a_2, b_2, c be sets. The functor $\mathrm{CLAADD2}(a_2, b_2, a_1, b_1, c)$ yields a set and is defined by:

(Def. 16)[1] $\mathrm{CLAADD2}(a_2, b_2, a_1, b_1, c) = \mathrm{XOR3}(a_2, b_2, \mathrm{MAJ3}(a_1, b_1, c))$.

The functor $\mathrm{CLACARR2}(a_2, b_2, a_1, b_1, c)$ yields a set and is defined by:

(Def. 17) $\mathrm{CLACARR2}(a_2, b_2, a_1, b_1, c) = \mathrm{OR2}(\mathrm{AND2}(a_2, b_2), \mathrm{AND2}(\mathrm{OR2}(a_2, b_2), \mathrm{MAJ3}(a_1, b_1, c)))$.

Let $a_1, b_1, a_2, b_2, a_3, b_3, c$ be sets. The functor $\mathrm{CLAADD3}(a_3, b_3, a_2, b_2, a_1, b_1, c)$ yielding a set is defined as follows:

(Def. 18) $\mathrm{CLAADD3}(a_3, b_3, a_2, b_2, a_1, b_1, c) = \mathrm{XOR3}(a_3, b_3, \mathrm{CLACARR2}(a_2, b_2, a_1, b_1, c))$.

The functor $\mathrm{CLACARR3}(a_3, b_3, a_2, b_2, a_1, b_1, c)$ yields a set and is defined by:

(Def. 19) $\mathrm{CLACARR3}(a_3, b_3, a_2, b_2, a_1, b_1, c) = \mathrm{OR3}(\mathrm{AND2}(a_3, b_3), \mathrm{AND2}(\mathrm{OR2}(a_3, b_3), \mathrm{AND2}(a_2, b_2)), \mathrm{AND3}(\mathrm{OR2}(a_3, b_3), \mathrm{OR2}(a_2, b_2), \mathrm{MAJ3}(a_1, b_1, c)))$.

Let $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, c$ be sets. The functor $\mathrm{CLAADD4}(a_4, b_4, a_3, b_3, a_2, b_2, a_1, b_1, c)$ yielding a set is defined by:

(Def. 20) $\mathrm{CLAADD4}(a_4, b_4, a_3, b_3, a_2, b_2, a_1, b_1, c) = \mathrm{XOR3}(a_4, b_4, \mathrm{CLACARR3}(a_3, b_3, a_2, b_2, a_1, b_1, c))$.

The functor $\mathrm{CLACARR4}(a_4, b_4, a_3, b_3, a_2, b_2, a_1, b_1, c)$ yielding a set is defined as follows:

(Def. 21) $\mathrm{CLACARR4}(a_4, b_4, a_3, b_3, a_2, b_2, a_1, b_1, c) = \mathrm{OR4}(\mathrm{AND2}(a_4, b_4), \mathrm{AND2}(\mathrm{OR2}(a_4, b_4), \mathrm{AND2}(a_3, b_3)), \mathrm{AND3}(\mathrm{OR2}(a_4, b_4), \mathrm{OR2}(a_3, b_3), \mathrm{AND2}(a_2, b_2)), \mathrm{AND4}(\mathrm{OR2}(a_4, b_4), \mathrm{OR2}(a_3, b_3), \mathrm{OR2}(a_2, b_2), \mathrm{MAJ3}(a_1, b_1, c)))$.

One can prove the following proposition

(4) Let x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2, z_3, z_4, z_5, q_0, q_1, q_2, q_3, c_1, c_2, c_3 be sets such that NE q_0 iff NE $\mathrm{AND2}(x_0, y_0)$ and NE q_1 iff NE $\mathrm{XOR3}(\mathrm{AND2}(x_1, y_0), \mathrm{AND2}(x_0, y_1), \emptyset)$ and NE c_1 iff NE $\mathrm{MAJ3}(\mathrm{AND2}(x_1, y_0), \mathrm{AND2}(x_0, y_1), \emptyset)$ and NE q_2 iff NE $\mathrm{XOR3}(\mathrm{AND2}(x_2, y_0), \mathrm{AND2}(x_1, y_1), \mathrm{AND2}(x_0, y_2))$ and NE c_2 iff NE $\mathrm{MAJ3}(\mathrm{AND2}(x_2, y_0), \mathrm{AND2}(x_1, y_1), \mathrm{AND2}(x_0, y_2))$ and NE q_3 iff NE $\mathrm{XOR3}(\mathrm{AND2}(x_2, y_1), \mathrm{AND2}(x_1, y_2), \emptyset)$ and NE c_3 iff NE MAJ3(AND2(x_2,

[1] The definitions (Def. 14) and (Def. 15) have been removed.

$y_1), \mathrm{AND2}(x_1, y_2), \emptyset)$ and NE z_0 iff NE q_0 and NE z_1 iff NE q_1 and NE z_2 iff NE CLAADD1$(q_2, c_1, \emptyset)$ and NE z_3 iff NE CLAADD2$(q_3, c_2, q_2, c_1, \emptyset)$ and NE z_4 iff NE CLAADD3(AND2$(x_2, y_2), c_3, q_3, c_2, q_2, c_1, \emptyset)$ and NE z_5 iff NE CLACARR3(AND2$(x_2, y_2), c_3, q_3, c_2, q_2, c_1, \emptyset)$. Then

(i) NE z_0 iff NE $\mathrm{MULT}_{310}(x_2, x_1, y_1, x_0, y_0)$,
(ii) NE z_1 iff NE $\mathrm{MULT}_{311}(x_2, x_1, y_1, x_0, y_0)$,
(iii) NE z_2 iff NE $\mathrm{MULT}_{321}(x_2, y_2, x_1, y_1, x_0, y_0)$,
(iv) NE z_3 iff NE $\mathrm{MULT}_{322}(x_2, y_2, x_1, y_1, x_0, y_0)$,
(v) NE z_4 iff NE $\mathrm{MULT}_{323}(x_2, y_2, x_1, y_1, x_0, y_0)$, and
(vi) NE z_5 iff NE $\mathrm{MULT}_{324}(x_2, y_2, x_1, y_1, x_0, y_0)$.

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Some Lemmas for the Jordan Curve Theorem[1]

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Summary. I present some miscellaneous simple facts that are still missing in the library. The only common feature is that, most of them, were needed as lemmas in the proof of the Jordan curve theorem.

MML Identifier: `JCT_MISC`.

The articles [11], [8], [17], [14], [9], [2], [3], [7], [1], [10], [4], [12], [5], [18], [19], [6], [15], [16], and [13] provide the notation and terminology for this paper.

1. Preliminaries

The scheme *NonEmpty* deals with a non empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

$\{\mathcal{F}(a) : a$ ranges over elements of $\mathcal{A}\}$ is non empty

for all values of the parameters.

One can prove the following propositions:

(1) For all sets A, B, C such that $A \subseteq B$ and A misses C holds $A \subseteq B \setminus C$.

(2) For all sets X, Y such that X meets $\bigcup Y$ there exists a set Z such that $Z \in Y$ and X meets Z.

(3) For all sets A, B and for every function f such that $A \subseteq \operatorname{dom} f$ and $f^\circ A \subseteq B$ holds $A \subseteq f^{-1}(B)$.

(4) For every function f and for all sets A, B such that A misses B holds $f^{-1}(A)$ misses $f^{-1}(B)$.

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(5) Let S, X be sets, f be a function from S into X, and A be a subset of X such that if $X = \emptyset$, then $S = \emptyset$. Then $(f^{-1}(A))^{\rm c} = f^{-1}(A^{\rm c})$.

(6) Let S be a 1-sorted structure, X be a non empty set, f be a function from the carrier of S into X, and A be a subset of X. Then $-f^{-1}(A) = f^{-1}(A^{\rm c})$.

We use the following convention: i, j, m, n denote natural numbers and r, s, r_0, s_0, t denote real numbers.

Next we state several propositions:

(7) If $m \leqslant n$, then $n -' (n -' m) = m$.

(8) For every real number r such that $1 \leqslant r$ and $i \leqslant j$ holds $r^i \leqslant r^j$.

(9) For all real numbers a, b such that $r \in [a, b]$ and $s \in [a, b]$ holds $\frac{r+s}{2} \in [a, b]$.

(10) For every increasing sequence N_1 of naturals and for all i, j such that $i \leqslant j$ holds $N_1(i) \leqslant N_1(j)$.

(11) $||r_0 - s_0| - |r - s|| \leqslant |r_0 - r| + |s_0 - s|$.

(12) If $t \in]r, s[$, then $|t| < \max(|r|, |s|)$.

Let A, B, C be non empty sets and let f be a function from A into $[\!: B, C :\!]$. Then $\mathrm{pr1}(f)$ is a function from A into B and it can be characterized by the condition:

(Def. 1) For every element x of A holds $\mathrm{pr1}(f)(x) = f(x)_{\mathbf{1}}$.

Then $\mathrm{pr2}(f)$ is a function from A into C and it can be characterized by the condition:

(Def. 2) For every element x of A holds $\mathrm{pr2}(f)(x) = f(x)_{\mathbf{2}}$.

The scheme *DoubleChoice* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a ternary predicate $\mathcal{P}$, and states that:

> There exists a function a from $\mathcal{A}$ into $\mathcal{B}$ and there exists a function b from $\mathcal{A}$ into $\mathcal{C}$ such that for every element i of $\mathcal{A}$ holds $\mathcal{P}[i, a(i), b(i)]$

provided the parameters meet the following requirement:

- For every element i of $\mathcal{A}$ there exists an element a_1 of $\mathcal{B}$ and there exists an element b_1 of $\mathcal{C}$ such that $\mathcal{P}[i, a_1, b_1]$.

We now state the proposition

(13) Let S, T be non empty topological spaces and G be a subset of $[\!: S, T :\!]$. Suppose that for every point x of $[\!: S, T :\!]$ such that $x \in G$ there exists a subset G_1 of S and there exists a subset G_2 of T such that G_1 is open and G_2 is open and $x \in [\!: G_1, G_2 :\!]$ and $[\!: G_1, G_2 :\!] \subseteq G$. Then G is open.

2. Topological Properties of Sets of Real Numbers

One can prove the following proposition

(14) For all compact subsets A, B of $\mathbb{R}$ holds $A \cap B$ is compact.

Let A be a subset of $\mathbb{R}$. We say that A is connected if and only if:

(Def. 3) For all real numbers r, s such that $r \in A$ and $s \in A$ holds $[r, s] \subseteq A$.

The following proposition is true

(15) Let T be a non empty topological space, f be a continuous real map of T, and A be a subset of T. If A is connected, then $f^\circ A$ is connected.

Let A, B be subsets of $\mathbb{R}$. The functor $\rho(A, B)$ yielding a real number is defined by:

(Def. 4) There exists a subset X of $\mathbb{R}$ such that $X = \{|r - s|; r$ ranges over elements of $\mathbb{R}$, s ranges over elements of $\mathbb{R}$: $r \in A \wedge s \in B\}$ and $\rho(A, B) = \inf X$.

Let us notice that the functor $\rho(A, B)$ is commutative.

The following propositions are true:

(16) For all subsets A, B of $\mathbb{R}$ and for all r, s such that $r \in A$ and $s \in B$ holds $|r - s| \geqslant \rho(A, B)$.

(17) For all subsets A, B of $\mathbb{R}$ and for all non empty subsets C, D of $\mathbb{R}$ such that $C \subseteq A$ and $D \subseteq B$ holds $\rho(A, B) \leqslant \rho(C, D)$.

(18) For all non empty compact subsets A, B of $\mathbb{R}$ there exist real numbers r, s such that $r \in A$ and $s \in B$ and $\rho(A, B) = |r - s|$.

(19) For all non empty compact subsets A, B of $\mathbb{R}$ holds $\rho(A, B) \geqslant 0$.

(20) For all non empty compact subsets A, B of $\mathbb{R}$ such that A misses B holds $\rho(A, B) > 0$.

(21) Let e, f be real numbers and A, B be compact subsets of $\mathbb{R}$. Suppose A misses B and $A \subseteq [e, f]$ and $B \subseteq [e, f]$. Let S be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$. Suppose that for every natural number i holds $S(i)$ is connected and $S(i)$ meets A and $S(i)$ meets B. Then there exists a real number r such that $r \in [e, f]$ and $r \notin A \cup B$ and for every natural number i there exists a natural number k such that $i \leqslant k$ and $r \in S(k)$.

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Mahlo and Inaccessible Cardinals

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Summary. This article contains basic ordinal topology: closed unbounded and stationary sets and necessary theorems about them, completness of the centered system of Clubs of M, Mahlo and strongly Mahlo cardinals, the proof that (strongly) Mahlo is (strongly) inaccessible, and the proof that Rank of strongly inaccessible is a model of ZF.

MML Identifier: `CARD_LAR`.

The notation and terminology used in this paper are introduced in the following articles: [15], [1], [6], [7], [16], [5], [11], [10], [8], [9], [3], [4], [17], [18], [12], [14], [13], and [2].

1. Clubs and Mahlo Cardinals

Let S be a set, let X be a set, and let Y be a subset of S. Then $X \cap Y$ is a subset of S.

Let us observe that every ordinal number which is cardinal and infinite is also limit.

Let us note that every ordinal number which is non empty and limit is also infinite.

Let us mention that every aleph which is non limit is also non countable.

Let us observe that there exists an aleph which is regular and non countable.

We use the following convention: A, B denote limit infinite ordinal numbers, B_1, B_2, B_3, C denote ordinal numbers, and X denotes a set.

Let us consider A, X. We say that X is unbounded in A if and only if:

(Def. 1) $X \subseteq A$ and $\sup X = A$.

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We say that X is closed in A if and only if:

(Def. 2) $X \subseteq A$ and for every B such that $B \in A$ holds if $\sup(X \cap B) = B$, then $B \in X$.

Let us consider A, X. We say that X is club in A if and only if:

(Def. 3) X is closed in A and X is unbounded in A.

Next we state the proposition

(1) X is club in A iff X is closed in A and X is unbounded in A.

In the sequel X is a subset of A.

Let us consider A, X. We say that X is unbounded if and only if:

(Def. 4) $\sup X = A$.

We introduce X is bounded as an antonym of X is unbounded. We say that X is closed if and only if:

(Def. 5) For every B such that $B \in A$ holds if $\sup(X \cap B) = B$, then $B \in X$.

We now state several propositions:

(2) X is club in A iff X is closed and unbounded.

(3) $X \subseteq \sup X$.

(4) Suppose X is non empty and for every B_1 such that $B_1 \in X$ there exists B_2 such that $B_2 \in X$ and $B_1 \in B_2$. Then $\sup X$ is limit infinite ordinal number.

(5) X is bounded iff there exists B_1 such that $B_1 \in A$ and $X \subseteq B_1$.

(6) If $\sup(X \cap B) \neq B$, then there exists B_1 such that $B_1 \in B$ and $X \cap B \subseteq B_1$.

(7) X is unbounded iff for every B_1 such that $B_1 \in A$ there exists C such that $C \in X$ and $B_1 \subseteq C$.

(8) If X is unbounded, then X is non empty.

(9) If X is unbounded and $B_1 \in A$, then there exists an element B_3 of A such that $B_3 \in \{B_2; B_2$ ranges over elements of A: $B_2 \in X \ \wedge \ B_1 \in B_2\}$.

Let us consider A, X, B_1. Let us assume that X is unbounded. And let us assume that $B_1 \in A$. The functor $\mathrm{LBound}(B_1, X)$ yields an element of X and is defined by:

(Def. 6) $\mathrm{LBound}(B_1, X) = \inf\{B_2; B_2$ ranges over elements of A: $B_2 \in X \wedge B_1 \in B_2\}$.

Next we state two propositions:

(10) If X is unbounded and $B_1 \in A$, then $\mathrm{LBound}(B_1, X) \in X$ and $B_1 \in \mathrm{LBound}(B_1, X)$.

(11) Ω_A is closed and unbounded.

Let A be a set, let X be a subset of A, and let Y be a set. Then $X \setminus Y$ is a subset of A.

Next we state two propositions:

(12) If $B_1 \in A$ and X is closed and unbounded, then $X \setminus B_1$ is closed and unbounded.

(13) If $B_1 \in A$, then $A \setminus B_1$ is closed and unbounded.

Let us consider A, X. We say that X is stationary if and only if:

(Def. 7) For every subset Y of A such that Y is closed and unbounded holds $X \cap Y$ is non empty.

The following proposition is true

(14) For all subsets X, Y of A such that X is stationary and $X \subseteq Y$ holds Y is stationary.

Let us consider A and let X be a set. We say that X is stationary in A if and only if:

(Def. 8) $X \subseteq A$ and for every subset Y of A such that Y is closed and unbounded holds $X \cap Y$ is non empty.

One can prove the following proposition

(15) For all sets X, Y such that X is stationary in A and $X \subseteq Y$ and $Y \subseteq A$ holds Y is stationary in A.

Let X be a set and let S be a family of subsets of X. We see that the element of S is a subset of X.

The following proposition is true

(16) If X is stationary, then X is unbounded.

Let us consider A, X. The functor $\operatorname{limpoints} X$ yields a subset of A and is defined as follows:

(Def. 9) $\operatorname{limpoints} X = \{B_1; B_1$ ranges over elements of A: B_1 is infinite and $\operatorname{limit} \wedge \sup(X \cap B_1) = B_1\}$.

We now state four propositions:

(17) If $X \cap B_3 \subseteq B_1$, then $B_3 \cap \operatorname{limpoints} X \subseteq \operatorname{succ} B_1$.

(18) If $X \subseteq B_1$, then $\operatorname{limpoints} X \subseteq \operatorname{succ} B_1$.

(19) $\operatorname{limpoints} X$ is closed.

(20) Suppose X is unbounded and $\operatorname{limpoints} X$ is bounded. Then there exists B_1 such that $B_1 \in A$ and $\{\operatorname{succ} B_2; B_2$ ranges over elements of A: $B_2 \in X \wedge B_1 \in \operatorname{succ} B_2\}$ is club in A.

In the sequel M is a non countable aleph and X is a subset of M.

Let us consider M. One can verify that there exists an element of M which is cardinal and infinite.

In the sequel N denotes a cardinal infinite element of M.

Next we state several propositions:

(21) For every aleph M and for every subset X of M such that X is unbounded holds $\operatorname{cf} M \leqslant \overline{\overline{X}}$.

(22) For every family S of subsets of M such that every element of S is closed holds $\bigcap S$ is closed.

(23) If $\aleph_0 < \mathrm{cf}\, M$, then for every function f from $\mathbb{N}$ into X holds $\sup \mathrm{rng}\, f \in M$.

(24) Suppose $\aleph_0 < \mathrm{cf}\, M$. Let S be a non empty family of subsets of M. If $\overline{\overline{S}} < \mathrm{cf}\, M$ and every element of S is closed and unbounded, then $\bigcap S$ is closed and unbounded.

(25) If $\aleph_0 < \mathrm{cf}\, M$ and X is unbounded, then for every B_1 such that $B_1 \in M$ there exists B such that $B \in M$ and $B_1 \in B$ and $B \in \mathrm{limpoints}\, X$.

(26) If $\aleph_0 < \mathrm{cf}\, M$ and X is unbounded, then limpoints X is unbounded.

Let us consider M. We say that M is Mahlo if and only if:

(Def. 10) $\{N : N \text{ is regular}\}$ is stationary in M.

We say that M is strongly Mahlo if and only if:

(Def. 11) $\{N : N \text{ is strongly inaccessible}\}$ is stationary in M.

We now state several propositions:

(27) If M is strongly Mahlo, then M is Mahlo.

(28) If M is Mahlo, then M is regular.

(29) If M is Mahlo, then M is limit.

(30) If M is Mahlo, then M is inaccessible.

(31) If M is strongly Mahlo, then M is strong limit.

(32) If M is strongly Mahlo, then M is strongly inaccessible.

2. Proof that Strongly Inaccessible is Model of ZF

We adopt the following convention: A denotes an ordinal number, x, y denote sets, and X, Y denote sets.

The following propositions are true:

(33) Suppose that for every x such that $x \in X$ there exists y such that $y \in X$ and $x \subseteq y$ and y is a cardinal number. Then $\bigcup X$ is a cardinal number.

(34) For every aleph M such that $\overline{\overline{X}} < \mathrm{cf}\, M$ and for every Y such that $Y \in X$ holds $\overline{\overline{Y}} < M$ holds $\overline{\overline{\bigcup X}} \in M$.

(35) If M is strongly inaccessible and $A \in M$, then $\overline{\overline{\mathbf{R}_A}} < M$.

(36) If M is strongly inaccessible, then $\overline{\overline{\mathbf{R}_M}} = M$.

(37) If M is strongly inaccessible, then $\mathbf{R}_M$ is a Tarski class.

(38) For every non empty ordinal number A holds $\mathbf{R}_A$ is non empty.

Let A be a non empty ordinal number. One can check that $\mathbf{R}_A$ is non empty.

Next we state two propositions:

(39) If M is strongly inaccessible, then $\mathbf{R}_M$ is a universal class.

(40) If M is strongly inaccessible, then $\mathbf{R}_M$ is model of ZF.

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Basic Properties of Extended Real Numbers

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Summary. We introduce product, quotient and absolute value, and we prove some basic properties of extended real numbers.

MML Identifier: `EXTREAL1`.

The articles [3], [4], [5], [1], and [2] provide the notation and terminology for this paper.

1. Preliminaries

In this paper x, y, z denote extended real numbers and a denotes a real number.

One can prove the following propositions:

(1) If $x \neq +\infty$ and $x \neq -\infty$, then x is a real number.

(2) $-\infty < +\infty$.

(3) If $x < y$, then $x \neq +\infty$ and $y \neq -\infty$.

(4) $x = +\infty$ iff $-x = -\infty$ and $x = -\infty$ iff $-x = +\infty$.

(5) If $x \neq +\infty$ or $y \neq -\infty$ and if $x \neq -\infty$ or $y \neq +\infty$, then $x - -y = x + y$.

(6) If $x \neq +\infty$ or $y \neq +\infty$ and if $x \neq -\infty$ or $y \neq -\infty$, then $x + -y = x - y$.

(7) If $x \neq -\infty$ and $y \neq +\infty$ and $x \leqslant y$, then $x \neq +\infty$ and $y \neq -\infty$.

(8) Suppose $x \neq +\infty$ or $y \neq -\infty$ but $x \neq -\infty$ or $y \neq +\infty$ and $y \neq +\infty$ or $z \neq -\infty$ but $y \neq -\infty$ or $z \neq +\infty$ and $x \neq +\infty$ or $z \neq -\infty$ but $x \neq -\infty$ or $z \neq +\infty$. Then $(x + y) + z = x + (y + z)$.

(9) If $-\infty < x$ and $x < +\infty$, then $x + -x = 0_{\overline{\mathbb{R}}}$ and $-x + x = 0_{\overline{\mathbb{R}}}$.

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(10) If $x \neq +\infty$ or $y \neq +\infty$ and if $x \neq -\infty$ or $y \neq -\infty$, then $x - y = x + -y$.

(11) Suppose $x \neq +\infty$ or $y \neq -\infty$ but $x \neq -\infty$ or $y \neq +\infty$ and $y \neq +\infty$ or $z \neq +\infty$ but $y \neq -\infty$ or $z \neq -\infty$ and $x + y \neq +\infty$ or $y - z \neq -\infty$ but $x + y \neq -\infty$ or $y - z \neq +\infty$. Then $(x + y) - z = x + (y - z)$.

2. Operations of Multiplication, Quotient and Absolute Value on Extended Real Numbers

Let x, y be extended real numbers. The functor $x \cdot y$ yields an extended real number and is defined by the conditions (Def. 1).

(Def. 1)(i) There exist real numbers a, b such that $x = a$ and $y = b$ and $x \cdot y = a \cdot b$, or

(ii) $0_{\overline{\mathbb{R}}} < x$ and $y = +\infty$ or $0_{\overline{\mathbb{R}}} < y$ and $x = +\infty$ or $x < 0_{\overline{\mathbb{R}}}$ and $y = -\infty$ or $y < 0_{\overline{\mathbb{R}}}$ and $x = -\infty$ but $x \cdot y = +\infty$, or

(iii) $x < 0_{\overline{\mathbb{R}}}$ and $y = +\infty$ or $y < 0_{\overline{\mathbb{R}}}$ and $x = +\infty$ or $0_{\overline{\mathbb{R}}} < x$ and $y = -\infty$ or $0_{\overline{\mathbb{R}}} < y$ and $x = -\infty$ but $x \cdot y = -\infty$, or

(iv) $x = 0_{\overline{\mathbb{R}}}$ or $y = 0_{\overline{\mathbb{R}}}$ but $x \cdot y = 0_{\overline{\mathbb{R}}}$.

The following propositions are true:

(12) Let x, y be extended real numbers. Then

(i) there exist real numbers a, b such that $x = a$ and $y = b$ and $x \cdot y = a \cdot b$, or

(ii) $0_{\overline{\mathbb{R}}} < x$ and $y = +\infty$ or $0_{\overline{\mathbb{R}}} < y$ and $x = +\infty$ or $x < 0_{\overline{\mathbb{R}}}$ and $y = -\infty$ or $y < 0_{\overline{\mathbb{R}}}$ and $x = -\infty$ but $x \cdot y = +\infty$, or

(iii) $x < 0_{\overline{\mathbb{R}}}$ and $y = +\infty$ or $y < 0_{\overline{\mathbb{R}}}$ and $x = +\infty$ or $0_{\overline{\mathbb{R}}} < x$ and $y = -\infty$ or $0_{\overline{\mathbb{R}}} < y$ and $x = -\infty$ but $x \cdot y = -\infty$, or

(iv) $x = 0_{\overline{\mathbb{R}}}$ or $y = 0_{\overline{\mathbb{R}}}$ but $x \cdot y = 0_{\overline{\mathbb{R}}}$.

(13) For all extended real numbers x, y and for all real numbers a, b such that $x = a$ and $y = b$ holds $x \cdot y = a \cdot b$.

(14) For every extended real number x such that $0_{\overline{\mathbb{R}}} < x$ holds $+\infty \cdot x = +\infty$ and $x \cdot +\infty = +\infty$ and $-\infty \cdot x = -\infty$ and $x \cdot -\infty = -\infty$.

(15) For every extended real number x such that $x < 0_{\overline{\mathbb{R}}}$ holds $+\infty \cdot x = -\infty$ and $x \cdot +\infty = -\infty$ and $-\infty \cdot x = +\infty$ and $x \cdot -\infty = +\infty$.

(16) For all extended real numbers x, y such that $x = 0_{\overline{\mathbb{R}}}$ holds $x \cdot y = 0_{\overline{\mathbb{R}}}$ and $y \cdot x = 0_{\overline{\mathbb{R}}}$.

(17) For all extended real numbers x, y holds $x \cdot y = y \cdot x$.

Let x, y be extended real numbers. Let us notice that the functor $x \cdot y$ is commutative.

One can prove the following propositions:

(18) If $x = a$, then $0 < a$ iff $0_{\overline{\mathbb{R}}} < x$.

(19) If $x = a$, then $a < 0$ iff $x < 0_{\overline{\mathbb{R}}}$.

(20) If $0_{\overline{\mathbb{R}}} < x$ and $0_{\overline{\mathbb{R}}} < y$ or $x < 0_{\overline{\mathbb{R}}}$ and $y < 0_{\overline{\mathbb{R}}}$, then $0_{\overline{\mathbb{R}}} < x \cdot y$.

(21) If $0_{\overline{\mathbb{R}}} < x$ and $y < 0_{\overline{\mathbb{R}}}$ or $x < 0_{\overline{\mathbb{R}}}$ and $0_{\overline{\mathbb{R}}} < y$, then $x \cdot y < 0_{\overline{\mathbb{R}}}$.

(22) $x \cdot y = 0_{\overline{\mathbb{R}}}$ iff $x = 0_{\overline{\mathbb{R}}}$ or $y = 0_{\overline{\mathbb{R}}}$.

(23) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(24) $-0_{\overline{\mathbb{R}}} = 0_{\overline{\mathbb{R}}}$.

(25) $0_{\overline{\mathbb{R}}} < x$ iff $-x < 0_{\overline{\mathbb{R}}}$ and $x < 0_{\overline{\mathbb{R}}}$ iff $0_{\overline{\mathbb{R}}} < -x$.

(26) $-x \cdot y = x \cdot -y$ and $-x \cdot y = (-x) \cdot y$.

(27) If $x \neq +\infty$ and $x \neq -\infty$ and $x \cdot y = +\infty$, then $y = +\infty$ or $y = -\infty$.

(28) If $x \neq +\infty$ and $x \neq -\infty$ and $x \cdot y = -\infty$, then $y = +\infty$ or $y = -\infty$.

(29) If $y \neq +\infty$ or $z \neq -\infty$ but $y \neq -\infty$ or $z \neq +\infty$ and $x \neq +\infty$ and $x \neq -\infty$, then $x \cdot (y + z) = x \cdot y + x \cdot z$.

(30) If $y \neq +\infty$ or $z \neq +\infty$ but $y \neq -\infty$ or $z \neq -\infty$ and $x \neq +\infty$ and $x \neq -\infty$, then $x \cdot (y - z) = x \cdot y - x \cdot z$.

Let x, y be extended real numbers. Let us assume that $x = -\infty$ or $x = +\infty$ but $y = -\infty$ or $y = +\infty$ but $y \neq 0_{\overline{\mathbb{R}}}$. The functor $\frac{x}{y}$ yielding an extended real number is defined by the conditions (Def. 2).

(Def. 2)(i) There exist real numbers a, b such that $x = a$ and $y = b$ and $\frac{x}{y} = \frac{a}{b}$, or

(ii) $x = +\infty$ and $0_{\overline{\mathbb{R}}} < y$ or $x = -\infty$ and $y < 0_{\overline{\mathbb{R}}}$ but $\frac{x}{y} = +\infty$, or

(iii) $x = -\infty$ and $0_{\overline{\mathbb{R}}} < y$ or $x = +\infty$ and $y < 0_{\overline{\mathbb{R}}}$ but $\frac{x}{y} = -\infty$, or

(iv) $y = -\infty$ or $y = +\infty$ but $\frac{x}{y} = 0_{\overline{\mathbb{R}}}$.

The following four propositions are true:

(31) Let x, y be extended real numbers. Suppose $x = -\infty$ or $x = +\infty$ but $y = -\infty$ or $y = +\infty$ but $y \neq 0_{\overline{\mathbb{R}}}$. Then

(i) there exist real numbers a, b such that $x = a$ and $y = b$ and $\frac{x}{y} = \frac{a}{b}$, or

(ii) $x = +\infty$ and $0_{\overline{\mathbb{R}}} < y$ or $x = -\infty$ and $y < 0_{\overline{\mathbb{R}}}$ but $\frac{x}{y} = +\infty$, or

(iii) $x = -\infty$ and $0_{\overline{\mathbb{R}}} < y$ or $x = +\infty$ and $y < 0_{\overline{\mathbb{R}}}$ but $\frac{x}{y} = -\infty$, or

(iv) $y = -\infty$ or $y = +\infty$ but $\frac{x}{y} = 0_{\overline{\mathbb{R}}}$.

(32) Let x, y be extended real numbers. Suppose $y \neq 0_{\overline{\mathbb{R}}}$. Let a, b be real numbers. If $x = a$ and $y = b$, then $\frac{x}{y} = \frac{a}{b}$.

(33) For all extended real numbers x, y such that $x \neq -\infty$ but $x \neq +\infty$ but $y = -\infty$ or $y = +\infty$ holds $\frac{x}{y} = 0_{\overline{\mathbb{R}}}$.

(34) For every extended real number x such that $x \neq -\infty$ and $x \neq +\infty$ and $x \neq 0_{\overline{\mathbb{R}}}$ holds $\frac{x}{x} = 1$.

Let x be an extended real number. The functor $|x|$ yielding an extended real number is defined as follows:

(Def. 3) $$|x| = \begin{cases} x, & \text{if } 0_{\overline{\mathbb{R}}} \leqslant x, \\ -x, & \text{otherwise.} \end{cases}$$

One can prove the following propositions:

(35) For every extended real number x such that $0_{\overline{\mathbb{R}}} \leqslant x$ holds $|x| = x$.

(36) For every extended real number x such that $0_{\overline{\mathbb{R}}} < x$ holds $|x| = x$.

(37) For every extended real number x such that $x < 0_{\overline{\mathbb{R}}}$ holds $|x| = -x$.

(38) For all real numbers a, b holds $\overline{\mathbb{R}}(a \cdot b) = \overline{\mathbb{R}}(a) \cdot \overline{\mathbb{R}}(b)$.

(39) For all real numbers a, b such that $b \neq 0$ holds $\overline{\mathbb{R}}(\frac{a}{b}) = \frac{\overline{\mathbb{R}}(a)}{\overline{\mathbb{R}}(b)}$.

(40) For all extended real numbers x, y such that $x \leqslant y$ and $x < +\infty$ and $-\infty < y$ holds $0_{\overline{\mathbb{R}}} \leqslant y - x$.

(41) For all extended real numbers x, y such that $x < y$ and $x < +\infty$ and $-\infty < y$ holds $0_{\overline{\mathbb{R}}} < y - x$.

(42) If $x \leqslant y$ and $0_{\overline{\mathbb{R}}} \leqslant z$, then $x \cdot z \leqslant y \cdot z$.

(43) If $x \leqslant y$ and $z \leqslant 0_{\overline{\mathbb{R}}}$, then $y \cdot z \leqslant x \cdot z$.

(44) If $x < y$ and $0_{\overline{\mathbb{R}}} < z$ and $z \neq +\infty$, then $x \cdot z < y \cdot z$.

(45) If $x < y$ and $z < 0_{\overline{\mathbb{R}}}$ and $z \neq -\infty$, then $y \cdot z < x \cdot z$.

(46) Suppose x is a real number and y is a real number. Then $x < y$ if and only if there exist real numbers p, q such that $p = x$ and $q = y$ and $p < q$.

(47) If $x \neq -\infty$ and $y \neq +\infty$ and $x \leqslant y$ and $0_{\overline{\mathbb{R}}} < z$, then $\frac{x}{z} \leqslant \frac{y}{z}$.

(48) If $x \leqslant y$ and $0_{\overline{\mathbb{R}}} < z$ and $z \neq +\infty$, then $\frac{x}{z} \leqslant \frac{y}{z}$.

(49) If $x \neq -\infty$ and $y \neq +\infty$ and $x \leqslant y$ and $z < 0_{\overline{\mathbb{R}}}$, then $\frac{y}{z} \leqslant \frac{x}{z}$.

(50) If $x \leqslant y$ and $z < 0_{\overline{\mathbb{R}}}$ and $z \neq -\infty$, then $\frac{y}{z} \leqslant \frac{x}{z}$.

(51) If $x < y$ and $0_{\overline{\mathbb{R}}} < z$ and $z \neq +\infty$, then $\frac{x}{z} < \frac{y}{z}$.

(52) If $x < y$ and $z < 0_{\overline{\mathbb{R}}}$ and $z \neq -\infty$, then $\frac{y}{z} < \frac{x}{z}$.

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Definitions and Basic Properties of Measurable Functions

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Summary. In this article we introduce some definitions concerning measurable functions and prove related properties.

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The papers [18], [10], [8], [9], [16], [6], [5], [2], [7], [1], [13], [12], [11], [19], [20], [14], [17], [3], [4], and [15] provide the notation and terminology for this paper.

1. Cardinal Numbers of $\mathbb{Z}$ and $\mathbb{Q}$

In this paper k is a natural number, r is a real number, i is an integer, and q is a rational number.

The subset $\mathbb{Z}_-$ of $\mathbb{R}$ is defined as follows:

(Def. 1) $r \in \mathbb{Z}_-$ iff there exists k such that $r = -k$.

Let us observe that $\mathbb{Z}_-$ is non empty.

Next we state three propositions:

(1) $\mathbb{N} \approx \mathbb{Z}_-$.

(2) $\mathbb{Z} = \mathbb{Z}_- \cup \mathbb{N}$.

(3) $\mathbb{N} \approx \mathbb{Z}$.

$\mathbb{Z}$ is a subset of $\mathbb{R}$.

Let n be a natural number. The functor $\mathbb{Q}(n)$ yields a subset of $\mathbb{Q}$ and is defined as follows:

(Def. 2) $q \in \mathbb{Q}(n)$ iff there exists i such that $q = \frac{i}{n}$.

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Let n be a natural number. Observe that $\mathbb{Q}(n+1)$ is non empty.

We now state two propositions:

(4) For every natural number n holds $\mathbb{Z} \approx \mathbb{Q}(n+1)$.

(5) $\mathbb{N} \approx \mathbb{Q}$.

2. Basic Operations of Extended Real Valued Functions

Let C be a non empty set, let f be a partial function from C to $\overline{\mathbb{R}}$, and let x be a set. Then $f(x)$ is an extended real number.

Let C be a non empty set and let f_1, f_2 be partial functions from C to $\overline{\mathbb{R}}$. The functor $f_1 + f_2$ yielding a partial function from C to $\overline{\mathbb{R}}$ is defined by:

(Def. 3) $\mathrm{dom}(f_1 + f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2 \setminus ({f_1}^{-1}(\{-\infty\}) \cap {f_2}^{-1}(\{+\infty\}) \cup {f_1}^{-1}(\{+\infty\}) \cap {f_2}^{-1}(\{-\infty\}))$ and for every element c of C such that $c \in \mathrm{dom}(f_1 + f_2)$ holds $(f_1 + f_2)(c) = f_1(c) + f_2(c)$.

The functor $f_1 - f_2$ yields a partial function from C to $\overline{\mathbb{R}}$ and is defined by:

(Def. 4) $\mathrm{dom}(f_1 - f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2 \setminus ({f_1}^{-1}(\{+\infty\}) \cap {f_2}^{-1}(\{+\infty\}) \cup {f_1}^{-1}(\{-\infty\}) \cap {f_2}^{-1}(\{-\infty\}))$ and for every element c of C such that $c \in \mathrm{dom}(f_1 - f_2)$ holds $(f_1 - f_2)(c) = f_1(c) - f_2(c)$.

The functor $f_1\, f_2$ yields a partial function from C to $\overline{\mathbb{R}}$ and is defined as follows:

(Def. 5) $\mathrm{dom}(f_1\, f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every element c of C such that $c \in \mathrm{dom}(f_1\, f_2)$ holds $(f_1\, f_2)(c) = f_1(c) \cdot f_2(c)$.

Let C be a non empty set, let f be a partial function from C to $\overline{\mathbb{R}}$, and let r be a real number. The functor $r\, f$ yielding a partial function from C to $\overline{\mathbb{R}}$ is defined as follows:

(Def. 6) $\mathrm{dom}(r\, f) = \mathrm{dom}\, f$ and for every element c of C such that $c \in \mathrm{dom}(r\, f)$ holds $(r\, f)(c) = \overline{\mathbb{R}}(r) \cdot f(c)$.

The following proposition is true

(6) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and r be a real number. Suppose $r \neq 0$. Let c be an element of C. If $c \in \mathrm{dom}(r\, f)$, then $f(c) = \frac{(r\, f)(c)}{\overline{\mathbb{R}}(r)}$.

Let C be a non empty set and let f be a partial function from C to $\overline{\mathbb{R}}$. The functor $-f$ yielding a partial function from C to $\overline{\mathbb{R}}$ is defined by:

(Def. 7) $\mathrm{dom}(-f) = \mathrm{dom}\, f$ and for every element c of C such that $c \in \mathrm{dom}(-f)$ holds $(-f)(c) = -f(c)$.

The extended real number $\overline{1}$ is defined by:

(Def. 8) $\overline{1} = 1$.

Let C be a non empty set, let f be a partial function from C to $\overline{\mathbb{R}}$, and let r be a real number. The functor $\frac{r}{f}$ yielding a partial function from C to $\overline{\mathbb{R}}$ is defined by:

(Def. 9) $\mathrm{dom}(\frac{r}{f}) = \mathrm{dom}\, f \setminus f^{-1}(\{0_{\overline{\mathbb{R}}}\})$ and for every element c of C such that $c \in \mathrm{dom}(\frac{r}{f})$ holds $(\frac{r}{f})(c) = \frac{\overline{\mathbb{R}}(r)}{f(c)}$.

One can prove the following proposition

(7) Let C be a non empty set and f be a partial function from C to $\overline{\mathbb{R}}$. Then $\mathrm{dom}(\frac{1}{f}) = \mathrm{dom}\, f \setminus f^{-1}(\{0_{\overline{\mathbb{R}}}\})$ and for every element c of C such that $c \in \mathrm{dom}(\frac{1}{f})$ holds $(\frac{1}{f})(c) = \frac{\overline{1}}{f(c)}$.

Let C be a non empty set and let f be a partial function from C to $\overline{\mathbb{R}}$. The functor $|f|$ yields a partial function from C to $\overline{\mathbb{R}}$ and is defined as follows:

(Def. 10) $\mathrm{dom}\,|f| = \mathrm{dom}\, f$ and for every element c of C such that $c \in \mathrm{dom}\,|f|$ holds $|f|(c) = |f(c)|$.

We now state three propositions:

(8) For all extended real numbers x, y such that $x \neq +\infty$ or $y \neq -\infty$ but $x \neq -\infty$ or $y \neq +\infty$ holds $x + y = y + x$.

(9) For every non empty set C and for all partial functions f_1, f_2 from C to $\overline{\mathbb{R}}$ holds $f_1 + f_2 = f_2 + f_1$.

(10) For every non empty set C and for all partial functions f_1, f_2 from C to $\overline{\mathbb{R}}$ holds $f_1\, f_2 = f_2\, f_1$.

Let C be a non empty set and let f_1, f_2 be partial functions from C to $\overline{\mathbb{R}}$. Let us note that the functor $f_1 + f_2$ is commutative. Let us observe that the functor $f_1\, f_2$ is commutative.

3. Level Sets

Next we state several propositions:

(11) For every real number r there exists a natural number n such that $r \leqslant n$.

(12) For every real number r there exists a natural number n such that $-n \leqslant r$.

(13) For all real numbers r, s such that $r < s$ there exists a natural number n such that $\frac{1}{n+1} < s - r$.

(14) For all real numbers r, s such that for every natural number n holds $r - \frac{1}{n+1} \leqslant s$ holds $r \leqslant s$.

(15) For every extended real number a such that for every real number r holds $\overline{\mathbb{R}}(r) < a$ holds $a = +\infty$.

(16) For every extended real number a such that for every real number r holds $a < \overline{\mathbb{R}}(r)$ holds $a = -\infty$.

Let X be a set, let S be a σ-field of subsets of X, and let A be a set. We say that A is measurable on S if and only if:

(Def. 11) $A \in S$.

One can prove the following proposition

(17) Let X, A be sets and S be a σ-field of subsets of X. Then A is measurable on S if and only if for every σ-measure M on S holds A is measurable w.r.t. M.

For simplicity, we use the following convention: X is a non empty set, x is an element of X, f, g are partial functions from X to $\overline{\mathbb{R}}$, S is a σ-field of subsets of X, F is a function from $\mathbb{N}$ into S, A is a set, a is an extended real number, r, s are real numbers, and n is a natural number.

Let us consider X, f, a. The functor LE-dom(f, a) yielding a subset of X is defined by:

(Def. 12) $x \in$ LE-dom(f, a) iff $x \in \operatorname{dom} f$ and there exists an extended real number y such that $y = f(x)$ and $y < a$.

The functor LEQ-dom(f, a) yielding a subset of X is defined by:

(Def. 13) $x \in$ LEQ-dom(f, a) iff $x \in \operatorname{dom} f$ and there exists an extended real number y such that $y = f(x)$ and $y \leqslant a$.

The functor GT-dom(f, a) yields a subset of X and is defined as follows:

(Def. 14) $x \in$ GT-dom(f, a) iff $x \in \operatorname{dom} f$ and there exists an extended real number y such that $y = f(x)$ and $a < y$.

The functor GTE-dom(f, a) yields a subset of X and is defined as follows:

(Def. 15) $x \in$ GTE-dom(f, a) iff $x \in \operatorname{dom} f$ and there exists an extended real number y such that $y = f(x)$ and $a \leqslant y$.

The functor EQ-dom(f, a) yielding a subset of X is defined as follows:

(Def. 16) $x \in$ EQ-dom(f, a) iff $x \in \operatorname{dom} f$ and there exists an extended real number y such that $y = f(x)$ and $a = y$.

One can prove the following propositions:

(18) For all X, S, f, A, a such that $A \subseteq \operatorname{dom} f$ holds $A \cap \text{GTE-dom}(f, a) = A \setminus A \cap \text{LE-dom}(f, a)$.

(19) For all X, S, f, A, a such that $A \subseteq \operatorname{dom} f$ holds $A \cap \text{GT-dom}(f, a) = A \setminus A \cap \text{LEQ-dom}(f, a)$.

(20) For all X, S, f, A, a such that $A \subseteq \operatorname{dom} f$ holds $A \cap \text{LEQ-dom}(f, a) = A \setminus A \cap \text{GT-dom}(f, a)$.

(21) For all X, S, f, A, a such that $A \subseteq \operatorname{dom} f$ holds $A \cap \text{LE-dom}(f, a) = A \setminus A \cap \text{GTE-dom}(f, a)$.

(22) For all X, S, f, A, a holds $A \cap \text{EQ-dom}(f, a) = A \cap \text{GTE-dom}(f, a) \cap \text{LEQ-dom}(f, a)$.

(23) For all X, S, F, f, A, r such that for every n holds $F(n) = A \cap \text{GT-dom}(f, \overline{\mathbb{R}}(r - \frac{1}{n+1}))$ holds $A \cap \text{GTE-dom}(f, \overline{\mathbb{R}}(r)) = \bigcap \operatorname{rng} F$.

(24) For all X, S, F, f, A and for every real number r such that for every n holds $F(n) = A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(r + \frac{1}{n+1}))$ holds $A \cap \text{LEQ-dom}(f, \overline{\mathbb{R}}(r)) =$

$\bigcap \operatorname{rng} F$.

(25) For all X, S, F, f, A and for every real number r such that for every n holds $F(n) = A \cap \text{LEQ-dom}(f, \overline{\mathbb{R}}(r - \frac{1}{n+1}))$ holds $A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(r)) = \bigcup \operatorname{rng} F$.

(26) For all X, S, F, f, A, r such that for every n holds $F(n) = A \cap \text{GTE-dom}(f, \overline{\mathbb{R}}(r + \frac{1}{n+1}))$ holds $A \cap \text{GT-dom}(f, \overline{\mathbb{R}}(r)) = \bigcup \operatorname{rng} F$.

(27) For all X, S, F, f, A such that for every n holds $F(n) = A \cap \text{GT-dom}(f, \overline{\mathbb{R}}(n))$ holds $A \cap \text{EQ-dom}(f, +\infty) = \bigcap \operatorname{rng} F$.

(28) For all X, S, F, f, A such that for every n holds $F(n) = A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(n))$ holds $A \cap \text{LE-dom}(f, +\infty) = \bigcup \operatorname{rng} F$.

(29) For all X, S, F, f, A such that for every n holds $F(n) = A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(-n))$ holds $A \cap \text{EQ-dom}(f, -\infty) = \bigcap \operatorname{rng} F$.

(30) For all X, S, F, f, A such that for every n holds $F(n) = A \cap \text{GT-dom}(f, \overline{\mathbb{R}}(-n))$ holds $A \cap \text{GT-dom}(f, -\infty) = \bigcup \operatorname{rng} F$.

4. Measurable Functions

Let X be a non empty set, let S be a σ-field of subsets of X, let f be a partial function from X to $\overline{\mathbb{R}}$, and let A be an element of S. We say that f is measurable on A if and only if:

(Def. 17) For every real number r holds $A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(r))$ is measurable on S.

In the sequel A, B are elements of S.

Next we state a number of propositions:

(31) Let given X, S, f, A. Suppose $A \subseteq \operatorname{dom} f$. Then f is measurable on A if and only if for every real number r holds $A \cap \text{GTE-dom}(f, \overline{\mathbb{R}}(r))$ is measurable on S.

(32) Let given X, S, f, A. Then f is measurable on A if and only if for every real number r holds $A \cap \text{LEQ-dom}(f, \overline{\mathbb{R}}(r))$ is measurable on S.

(33) Let given X, S, f, A. Suppose $A \subseteq \operatorname{dom} f$. Then f is measurable on A if and only if for every real number r holds $A \cap \text{GT-dom}(f, \overline{\mathbb{R}}(r))$ is measurable on S.

(34) For all X, S, f, A, B such that $B \subseteq A$ and f is measurable on A holds f is measurable on B.

(35) For all X, S, f, A, B such that f is measurable on A and f is measurable on B holds f is measurable on $A \cup B$.

(36) For all X, S, f, A, r, s such that f is measurable on A and $A \subseteq \operatorname{dom} f$ holds $A \cap \text{GT-dom}(f, \overline{\mathbb{R}}(r)) \cap \text{LE-dom}(f, \overline{\mathbb{R}}(s))$ is measurable on S.

(37) For all X, S, f, A such that f is measurable on A and $A \subseteq \operatorname{dom} f$ holds $A \cap \text{EQ-dom}(f, +\infty)$ is measurable on S.

(38) For all X, S, f, A such that f is measurable on A holds $A \cap \text{EQ-dom}(f, -\infty)$ is measurable on S.

(39) For all X, S, f, A such that f is measurable on A and $A \subseteq \text{dom}\, f$ holds $A \cap \text{GT-dom}(f, -\infty) \cap \text{LE-dom}(f, +\infty)$ is measurable on S.

(40) Let given X, S, f, g, A, r. Suppose f is measurable on A and g is measurable on A and $A \subseteq \text{dom}\, g$. Then $A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(r)) \cap \text{GT-dom}(g, \overline{\mathbb{R}}(r))$ is measurable on S.

(41) For all X, S, f, A, r such that f is measurable on A and $A \subseteq \text{dom}\, f$ holds $r\, f$ is measurable on A.

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Gauges and Cages. Part I[1]

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The notation and terminology used in this paper have been introduced in the following articles: [28], [24], [32], [9], [25], [10], [2], [3], [30], [29], [4], [5], [18], [21], [23], [22], [6], [8], [14], [1], [19], [26], [7], [27], [13], [33], [17], [16], [20], [31], [11], [12], and [15].

1. Preliminaries

For simplicity, we use the following convention: i, i_1, i_2, j, j_1, j_2, k, m, n, t denote natural numbers, D denotes a non empty subset of $\mathcal{E}^2_{\rm T}$, E denotes a compact non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$, C denotes a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$, G denotes a Go-board, p, q, x denote points of $\mathcal{E}^2_{\rm T}$, and r, s denote real numbers.

The following propositions are true:

(1) For all real numbers s_1, s_3, s_4, l such that $s_1 \leqslant s_3$ and $s_1 \leqslant s_4$ and $0 \leqslant l$ and $l \leqslant 1$ holds $s_1 \leqslant (1-l) \cdot s_3 + l \cdot s_4$.

(2) For all real numbers s_1, s_3, s_4, l such that $s_3 \leqslant s_1$ and $s_4 \leqslant s_1$ and $0 \leqslant l$ and $l \leqslant 1$ holds $(1-l) \cdot s_3 + l \cdot s_4 \leqslant s_1$.

(3) If $n > 0$, then $m^n \bmod m = 0$.

(4) If $j > 0$ and $i \bmod j = 0$, then $i \div j = \frac{i}{j}$.

(5) If $n > 0$, then $i^n \div i = \frac{i^n}{i}$.

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(6) If $0 < n$ and $1 < r$, then $1 < r^n$.

(7) If $r > 1$ and $m > n$, then $r^m > r^n$.

(8) Let T be a non empty topological space, A be a subset of T, and B, C be subsets of the carrier of T. If A is connected and C is a component of B and $A \cap C \neq \emptyset$ and $A \subseteq B$, then $A \subseteq C$.

Let f be a finite sequence. The functor $\mathrm{Center}\, f$ yields a natural number and is defined as follows:

(Def. 1) $\mathrm{Center}\, f = (\mathrm{len}\, f \div 2) + 1$.

The following two propositions are true:

(9) For every finite sequence f such that $\mathrm{len}\, f$ is odd holds $\mathrm{len}\, f = 2 \cdot \mathrm{Center}\, f - 1$.

(10) For every finite sequence f such that $\mathrm{len}\, f$ is even holds $\mathrm{len}\, f = 2 \cdot \mathrm{Center}\, f - 2$.

2. Some Subsets of the Plane

One can check the following observations:

* there exists a subset of $\mathcal{E}^2_{\mathrm{T}}$ which is compact, non vertical, non horizontal, and non empty and satisfies conditions of simple closed curve,
* there exists a subset of $\mathcal{E}^2_{\mathrm{T}}$ which is compact, non empty, and horizontal, and
* there exists a subset of $\mathcal{E}^2_{\mathrm{T}}$ which is compact, non empty, and vertical.

The following propositions are true:

(11) If $p \in \text{N-most}\, D$, then $p_{\mathbf{2}} = \text{N-bound}\, D$.

(12) If $p \in \text{E-most}\, D$, then $p_{\mathbf{1}} = \text{E-bound}\, D$.

(13) If $p \in \text{S-most}\, D$, then $p_{\mathbf{2}} = \text{S-bound}\, D$.

(14) If $p \in \text{W-most}\, D$, then $p_{\mathbf{1}} = \text{W-bound}\, D$.

(15) $\mathrm{BDD}\, D$ misses D.

(16) For every compact non empty subset S of $\mathcal{E}^2_{\mathrm{T}}$ satisfying conditions of simple closed curve holds $\mathrm{LowerArc}\, S \subseteq S$ and $\mathrm{UpperArc}\, S \subseteq S$.

(17) $p \in \mathrm{VerticalLine}\, p_{\mathbf{1}}$.

(18) $[r, s] \in \mathrm{VerticalLine}\, r$.

(19) For every subset A of $\mathcal{E}^2_{\mathrm{T}}$ such that $A \subseteq \mathrm{VerticalLine}\, s$ holds A is vertical.

(20) $(\mathrm{proj2})([r, s]) = s$ and $(\mathrm{proj1})([r, s]) = r$.

(21) If $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $r \in [(\mathrm{proj2})(p), (\mathrm{proj2})(q)]$, then $[p_{\mathbf{1}}, r] \in \mathcal{L}(p, q)$.

(22) If $p_{\mathbf{2}} = q_{\mathbf{2}}$ and $r \in [(\mathrm{proj1})(p), (\mathrm{proj1})(q)]$, then $[r, p_{\mathbf{2}}] \in \mathcal{L}(p, q)$.

(23) If $p \in$ VerticalLine s and $q \in$ VerticalLine s, then $\mathcal{L}(p,q) \subseteq$ VerticalLine s.

Let S be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ satisfying conditions of simple closed curve. Observe that LowerArc S is non empty and compact and UpperArc S is non empty and compact.

We now state several propositions:

(24) For all subsets A, B of $\mathcal{E}^2_{\mathrm{T}}$ such that A meets B holds $(\mathrm{proj2})^\circ A$ meets $(\mathrm{proj2})^\circ B$.

(25) For all subsets A, B of $\mathcal{E}^2_{\mathrm{T}}$ such that A misses B and $A \subseteq$ VerticalLine s and $B \subseteq$ VerticalLine s holds $(\mathrm{proj2})^\circ A$ misses $(\mathrm{proj2})^\circ B$.

(26) For every closed subset S of $\mathcal{E}^2_{\mathrm{T}}$ such that S is Bounded holds $(\mathrm{proj2})^\circ S$ is closed.

(27) For every subset S of $\mathcal{E}^2_{\mathrm{T}}$ such that S is Bounded holds $(\mathrm{proj2})^\circ S$ is bounded.

(28) For every compact subset S of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\mathrm{proj2})^\circ S$ is compact.

In this article we present several logical schemes. The scheme *TRSubsetEx* deals with a natural number $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> There exists a subset A of $\mathcal{E}^{\mathcal{A}}_{\mathrm{T}}$ such that for every point p of $\mathcal{E}^{\mathcal{A}}_{\mathrm{T}}$ holds $p \in A$ iff $\mathcal{P}[p]$

for all values of the parameters.

The scheme *TRSubsetUniq* deals with a natural number $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> Let A, B be subsets of $\mathcal{E}^{\mathcal{A}}_{\mathrm{T}}$. Suppose for every point p of $\mathcal{E}^{\mathcal{A}}_{\mathrm{T}}$ holds $p \in A$ iff $\mathcal{P}[p]$ and for every point p of $\mathcal{E}^{\mathcal{A}}_{\mathrm{T}}$ holds $p \in B$ iff $\mathcal{P}[p]$. Then $A = B$

for all values of the parameters.

Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. The functor NorthHalfline p yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 2) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ holds $x \in$ NorthHalfline p iff $x_{\mathbf{1}} = p_{\mathbf{1}}$ and $x_{\mathbf{2}} \geqslant p_{\mathbf{2}}$.

The functor EastHalfline p yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 3) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ holds $x \in$ EastHalfline p iff $x_{\mathbf{1}} \geqslant p_{\mathbf{1}}$ and $x_{\mathbf{2}} = p_{\mathbf{2}}$.

The functor SouthHalfline p yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 4) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ holds $x \in$ SouthHalfline p iff $x_{\mathbf{1}} = p_{\mathbf{1}}$ and $x_{\mathbf{2}} \leqslant p_{\mathbf{2}}$.

The functor WestHalfline p yields a subset of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 5) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ holds $x \in$ WestHalfline p iff $x_{\mathbf{1}} \leqslant p_{\mathbf{1}}$ and $x_{\mathbf{2}} = p_{\mathbf{2}}$.

The following propositions are true:

(29) NorthHalfline $p = \{q; q$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $q_{\mathbf{1}} = p_{\mathbf{1}} \ \wedge\ q_{\mathbf{2}} \geqslant p_{\mathbf{2}}\}$.

(30) NorthHalfline $p = \{[p_{\mathbf{1}}, r]; r$ ranges over elements of $\mathbb{R}$: $r \geqslant p_{\mathbf{2}}\}$.

(31) EastHalfline $p = \{q; q$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $q_{\mathbf{1}} \geqslant p_{\mathbf{1}} \ \wedge\ q_{\mathbf{2}} = p_{\mathbf{2}}\}$.

(32) EastHalfline $p = \{[r, p_{\mathbf{2}}]; r$ ranges over elements of $\mathbb{R}$: $r \geqslant p_{\mathbf{1}}\}$.

(33) SouthHalfline $p = \{q; q$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $q_{\mathbf{1}} = p_{\mathbf{1}} \ \wedge \ q_{\mathbf{2}} \leqslant p_{\mathbf{2}}\}$.

(34) SouthHalfline $p = \{[p_{\mathbf{1}}, r]; r$ ranges over elements of $\mathbb{R}$: $r \leqslant p_{\mathbf{2}}\}$.

(35) WestHalfline $p = \{q; q$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $q_{\mathbf{1}} \leqslant p_{\mathbf{1}} \ \wedge \ q_{\mathbf{2}} = p_{\mathbf{2}}\}$.

(36) WestHalfline $p = \{[r, p_{\mathbf{2}}]; r$ ranges over elements of $\mathbb{R}$: $r \leqslant p_{\mathbf{1}}\}$.

Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. One can check the following observations:

* NorthHalfline p is non empty and convex,
* EastHalfline p is non empty and convex,
* SouthHalfline p is non empty and convex, and
* WestHalfline p is non empty and convex.

3. Goboards

We now state a number of propositions:

(37) If $1 \leqslant i$ and $i \leqslant \operatorname{len} G$ and $1 \leqslant j$ and $j \leqslant \operatorname{width} G$, then $G_{i,j} \in \mathcal{L}(G_{i,1}, G_{i,\operatorname{width} G})$.

(38) If $1 \leqslant i$ and $i \leqslant \operatorname{len} G$ and $1 \leqslant j$ and $j \leqslant \operatorname{width} G$, then $G_{i,j} \in \mathcal{L}(G_{1,j}, G_{\operatorname{len} G,j})$.

(39) If $1 \leqslant j_1$ and $j_1 \leqslant \operatorname{width} G$ and $1 \leqslant j_2$ and $j_2 \leqslant \operatorname{width} G$ and $1 \leqslant i_1$ and $i_1 \leqslant i_2$ and $i_2 \leqslant \operatorname{len} G$, then $(G_{i_1,j_1})_{\mathbf{1}} \leqslant (G_{i_2,j_2})_{\mathbf{1}}$.

(40) If $1 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} G$ and $1 \leqslant i_2$ and $i_2 \leqslant \operatorname{len} G$ and $1 \leqslant j_1$ and $j_1 \leqslant j_2$ and $j_2 \leqslant \operatorname{width} G$, then $(G_{i_1,j_1})_{\mathbf{2}} \leqslant (G_{i_2,j_2})_{\mathbf{2}}$.

(41) Let f be a non constant standard special circular sequence. Suppose f is a sequence which elements belong to G and $1 \leqslant t$ and $t \leqslant \operatorname{len} G$. Then $(G_{t,\operatorname{width} G})_{\mathbf{2}} \geqslant$ N-bound $\widetilde{\mathcal{L}}(f)$.

(42) Let f be a non constant standard special circular sequence. Suppose f is a sequence which elements belong to G and $1 \leqslant t$ and $t \leqslant \operatorname{width} G$. Then $(G_{1,t})_{\mathbf{1}} \leqslant$ W-bound $\widetilde{\mathcal{L}}(f)$.

(43) Let f be a non constant standard special circular sequence. Suppose f is a sequence which elements belong to G and $1 \leqslant t$ and $t \leqslant \operatorname{len} G$. Then $(G_{t,1})_{\mathbf{2}} \leqslant$ S-bound $\widetilde{\mathcal{L}}(f)$.

(44) Let f be a non constant standard special circular sequence. Suppose f is a sequence which elements belong to G and $1 \leqslant t$ and $t \leqslant \operatorname{width} G$. Then $(G_{\operatorname{len} G,t})_{\mathbf{1}} \geqslant$ E-bound $\widetilde{\mathcal{L}}(f)$.

(45) If $i \leqslant \operatorname{len} G$ and $j \leqslant \operatorname{width} G$, then $\operatorname{cell}(G, i, j)$ is non empty.

(46) If $i \leqslant \operatorname{len} G$ and $j \leqslant \operatorname{width} G$, then $\operatorname{cell}(G, i, j)$ is connected.

(47) If $i \leqslant \operatorname{len} G$, then $\operatorname{cell}(G, i, 0)$ is not Bounded.

(48) If $i \leqslant \operatorname{len} G$, then $\operatorname{cell}(G, i, \operatorname{width} G)$ is not Bounded.

4. Gauges

One can prove the following propositions:

(49) $\operatorname{width} \operatorname{Gauge}(D, n) = 2^n + 3$.

(50) If $i < j$, then $\operatorname{len} \operatorname{Gauge}(D, i) < \operatorname{len} \operatorname{Gauge}(D, j)$.

(51) If $i \leqslant j$, then $\operatorname{len} \operatorname{Gauge}(D, i) \leqslant \operatorname{len} \operatorname{Gauge}(D, j)$.

(52) If $m \leqslant n$ and $1 < i$ and $i < \operatorname{len} \operatorname{Gauge}(D, m)$, then $1 < 2^{n -' m} \cdot (i - 2) + 2$ and $2^{n -' m} \cdot (i - 2) + 2 < \operatorname{len} \operatorname{Gauge}(D, n)$.

(53) If $m \leqslant n$ and $1 < i$ and $i < \operatorname{width} \operatorname{Gauge}(D, m)$, then $1 < 2^{n -' m} \cdot (i - 2) + 2$ and $2^{n -' m} \cdot (i - 2) + 2 < \operatorname{width} \operatorname{Gauge}(D, n)$.

(54) Suppose $m \leqslant n$ and $1 < i$ and $i < \operatorname{len} \operatorname{Gauge}(D, m)$ and $1 < j$ and $j < \operatorname{width} \operatorname{Gauge}(D, m)$. Let i_1, j_1 be natural numbers. If $i_1 = 2^{n -' m} \cdot (i - 2) + 2$ and $j_1 = 2^{n -' m} \cdot (j - 2) + 2$, then $(\operatorname{Gauge}(D, m))_{i,j} = (\operatorname{Gauge}(D, n))_{i_1, j_1}$.

(55) If $m \leqslant n$ and $1 < i$ and $i + 1 < \operatorname{len} \operatorname{Gauge}(D, m)$, then $1 < 2^{n -' m} \cdot (i - 1) + 2$ and $2^{n -' m} \cdot (i - 1) + 2 \leqslant \operatorname{len} \operatorname{Gauge}(D, n)$.

(56) If $m \leqslant n$ and $1 < i$ and $i + 1 < \operatorname{width} \operatorname{Gauge}(D, m)$, then $1 < 2^{n -' m} \cdot (i - 1) + 2$ and $2^{n -' m} \cdot (i - 1) + 2 \leqslant \operatorname{width} \operatorname{Gauge}(D, n)$.

(57) If $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(D, n)$ and $1 \leqslant j$ and $j \leqslant \operatorname{len} \operatorname{Gauge}(D, m)$ and $n > 0$ and $m > 0$ or $n = 0$ and $m = 0$, then $((\operatorname{Gauge}(D, n))_{\operatorname{Center} \operatorname{Gauge}(D, n), i})_{\mathbf{1}} = ((\operatorname{Gauge}(D, m))_{\operatorname{Center} \operatorname{Gauge}(D, m), j})_{\mathbf{1}}$.

(58) If $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(D, n)$ and $1 \leqslant j$ and $j \leqslant \operatorname{len} \operatorname{Gauge}(D, m)$ and $n > 0$ and $m > 0$ or $n = 0$ and $m = 0$, then $((\operatorname{Gauge}(D, n))_{i, \operatorname{Center} \operatorname{Gauge}(D, n)})_{\mathbf{2}} = ((\operatorname{Gauge}(D, m))_{j, \operatorname{Center} \operatorname{Gauge}(D, m)})_{\mathbf{2}}$.

(59) If $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(C, 1)$, then $((\operatorname{Gauge}(C, 1))_{\operatorname{Center} \operatorname{Gauge}(C, 1), i})_{\mathbf{1}} = \frac{\text{W-bound}\, C + \text{E-bound}\, C}{2}$.

(60) If $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(C, 1)$, then $((\operatorname{Gauge}(C, 1))_{i, \operatorname{Center} \operatorname{Gauge}(C, 1)})_{\mathbf{2}} = \frac{\text{S-bound}\, C + \text{N-bound}\, C}{2}$.

(61) If $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(E, n)$ and $1 \leqslant j$ and $j \leqslant \operatorname{len} \operatorname{Gauge}(E, m)$ and $m \leqslant n$, then $((\operatorname{Gauge}(E, n))_{i, \operatorname{len} \operatorname{Gauge}(E, n)})_{\mathbf{2}} \leqslant ((\operatorname{Gauge}(E, m))_{j, \operatorname{len} \operatorname{Gauge}(E, m)})_{\mathbf{2}}$.

(62) If $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(E, n)$ and $1 \leqslant j$ and $j \leqslant \operatorname{len} \operatorname{Gauge}(E, m)$ and $m \leqslant n$, then $((\operatorname{Gauge}(E, n))_{\operatorname{len} \operatorname{Gauge}(E, n), i})_{\mathbf{1}} \leqslant ((\operatorname{Gauge}(E, m))_{\operatorname{len} \operatorname{Gauge}(E, m), j})_{\mathbf{1}}$.

(63) If $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(E, n)$ and $1 \leqslant j$ and $j \leqslant \operatorname{len} \operatorname{Gauge}(E, m)$ and $m \leqslant n$, then $((\operatorname{Gauge}(E, m))_{1, j})_{\mathbf{1}} \leqslant ((\operatorname{Gauge}(E, n))_{1, i})_{\mathbf{1}}$.

(64) If $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(E, n)$ and $1 \leqslant j$ and $j \leqslant \operatorname{len} \operatorname{Gauge}(E, m)$ and $m \leqslant n$, then $((\operatorname{Gauge}(E, m))_{j,1})_{\mathbf{2}} \leqslant ((\operatorname{Gauge}(E, n))_{i,1})_{\mathbf{2}}$.

(65) If $1 \leqslant m$ and $m \leqslant n$, then $\mathcal{L}((\operatorname{Gauge}(E, n))_{\operatorname{Center} \operatorname{Gauge}(E,n),1}, (\operatorname{Gauge}(E, n))_{\operatorname{Center} \operatorname{Gauge}(E,n), \operatorname{len} \operatorname{Gauge}(E,n)}) \subseteq \mathcal{L}((\operatorname{Gauge}(E, m))_{\operatorname{Center} \operatorname{Gauge}(E,m),1}, (\operatorname{Gauge}(E, m))_{\operatorname{Center} \operatorname{Gauge}(E,m), \operatorname{len} \operatorname{Gauge}(E,m)})$.

(66) If $1 \leqslant m$ and $m \leqslant n$ and $1 \leqslant j$ and $j \leqslant \operatorname{width} \operatorname{Gauge}(E, n)$, then $\mathcal{L}((\operatorname{Gauge}(E, n))_{\operatorname{Center} \operatorname{Gauge}(E,n),1}, (\operatorname{Gauge}(E, n))_{\operatorname{Center} \operatorname{Gauge}(E,n),j}) \subseteq \mathcal{L}((\operatorname{Gauge}(E, m))_{\operatorname{Center} \operatorname{Gauge}(E,m),1}, (\operatorname{Gauge}(E, n))_{\operatorname{Center} \operatorname{Gauge}(E,n),j})$.

(67) If $1 \leqslant m$ and $m \leqslant n$ and $1 \leqslant j$ and $j \leqslant \operatorname{width} \operatorname{Gauge}(E, n)$, then $\mathcal{L}((\operatorname{Gauge}(E, m))_{\operatorname{Center} \operatorname{Gauge}(E,m),1}, (\operatorname{Gauge}(E, n))_{\operatorname{Center} \operatorname{Gauge}(E,n),j}) \subseteq \mathcal{L}((\operatorname{Gauge}(E, m))_{\operatorname{Center} \operatorname{Gauge}(E,m),1}, (\operatorname{Gauge}(E, m))_{\operatorname{Center} \operatorname{Gauge}(E,m), \operatorname{len} \operatorname{Gauge}(E,m)})$.

(68) Suppose $m \leqslant n$ and $1 < i$ and $i + 1 < \operatorname{len} \operatorname{Gauge}(E, m)$ and $1 < j$ and $j + 1 < \operatorname{width} \operatorname{Gauge}(E, m)$. Let i_1, j_1 be natural numbers. Suppose $2^{n -' m} \cdot (i - 2) + 2 \leqslant i_1$ and $i_1 < 2^{n -' m} \cdot (i - 1) + 2$ and $2^{n -' m} \cdot (j - 2) + 2 \leqslant j_1$ and $j_1 < 2^{n -' m} \cdot (j - 1) + 2$. Then $\operatorname{cell}(\operatorname{Gauge}(E, n), i_1, j_1) \subseteq \operatorname{cell}(\operatorname{Gauge}(E, m), i, j)$.

(69) Suppose $m \leqslant n$ and $3 \leqslant i$ and $i < \operatorname{len} \operatorname{Gauge}(E, m)$ and $1 < j$ and $j + 1 < \operatorname{width} \operatorname{Gauge}(E, m)$. Let i_1, j_1 be natural numbers. If $i_1 = 2^{n -' m} \cdot (i - 2) + 2$ and $j_1 = 2^{n -' m} \cdot (j - 2) + 2$, then $\operatorname{cell}(\operatorname{Gauge}(E, n), i_1 -' 1, j_1) \subseteq \operatorname{cell}(\operatorname{Gauge}(E, m), i -' 1, j)$.

(70) If $i \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$, then $\operatorname{cell}(\operatorname{Gauge}(C, n), i, 0) \subseteq \operatorname{UBD} C$.

(71) If $i \leqslant \operatorname{len} \operatorname{Gauge}(E, n)$, then $\operatorname{cell}(\operatorname{Gauge}(E, n), i, \operatorname{width} \operatorname{Gauge}(E, n)) \subseteq \operatorname{UBD} E$.

5. Cages

The following propositions are true:

(72) If $p \in C$, then NorthHalfline p meets $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$.

(73) If $p \in C$, then EastHalfline p meets $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$.

(74) If $p \in C$, then SouthHalfline p meets $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$.

(75) If $p \in C$, then WestHalfline p meets $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$.

(76) There exist k, t such that $1 \leqslant k$ and $k < \operatorname{len} \operatorname{Cage}(C, n)$ and $1 \leqslant t$ and $t \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $(\operatorname{Cage}(C, n))_k = (\operatorname{Gauge}(C, n))_{1,t}$.

(77) There exist k, t such that $1 \leqslant k$ and $k < \operatorname{len} \operatorname{Cage}(C, n)$ and $1 \leqslant t$ and $t \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $(\operatorname{Cage}(C, n))_k = (\operatorname{Gauge}(C, n))_{t,1}$.

(78) There exist k, t such that $1 \leqslant k$ and $k < \operatorname{len} \operatorname{Cage}(C, n)$ and $1 \leqslant t$ and $t \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $(\operatorname{Cage}(C, n))_k = (\operatorname{Gauge}(C, n))_{\operatorname{len} \operatorname{Gauge}(C,n),t}$.

(79) If $1 \leqslant k$ and $k \leqslant \operatorname{len}\operatorname{Cage}(C,n)$ and $1 \leqslant t$ and $t \leqslant \operatorname{len}\operatorname{Gauge}(C,n)$ and $(\operatorname{Cage}(C,n))_k = (\operatorname{Gauge}(C,n))_{t,\operatorname{width}\operatorname{Gauge}(C,n)}$, then $(\operatorname{Cage}(C,n))_k \in$ N-most $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$.

(80) If $1 \leqslant k$ and $k \leqslant \operatorname{len}\operatorname{Cage}(C,n)$ and $1 \leqslant t$ and $t \leqslant \operatorname{width}\operatorname{Gauge}(C,n)$ and $(\operatorname{Cage}(C,n))_k = (\operatorname{Gauge}(C,n))_{1,t}$, then $(\operatorname{Cage}(C,n))_k \in$ W-most $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$.

(81) If $1 \leqslant k$ and $k \leqslant \operatorname{len}\operatorname{Cage}(C,n)$ and $1 \leqslant t$ and $t \leqslant \operatorname{len}\operatorname{Gauge}(C,n)$ and $(\operatorname{Cage}(C,n))_k = (\operatorname{Gauge}(C,n))_{t,1}$, then $(\operatorname{Cage}(C,n))_k \in$ S-most $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$.

(82) If $1 \leqslant k$ and $k \leqslant \operatorname{len}\operatorname{Cage}(C,n)$ and $1 \leqslant t$ and $t \leqslant \operatorname{width}\operatorname{Gauge}(C,n)$ and $(\operatorname{Cage}(C,n))_k = (\operatorname{Gauge}(C,n))_{\operatorname{len}\operatorname{Gauge}(C,n),t}$, then $(\operatorname{Cage}(C,n))_k \in$ E-most $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$.

(83) W-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)) =$ W-bound $C - \frac{\text{E-bound}\,C - \text{W-bound}\,C}{2^n}$.

(84) S-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)) =$ S-bound $C - \frac{\text{N-bound}\,C - \text{S-bound}\,C}{2^n}$.

(85) E-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)) =$ E-bound $C + \frac{\text{E-bound}\,C - \text{W-bound}\,C}{2^n}$.

(86) N-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)) +$ S-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)) =$ N-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,m)) +$ S-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,m))$.

(87) E-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)) +$ W-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)) =$ E-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,m)) +$ W-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,m))$.

(88) If $i < j$, then E-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,j)) <$ E-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,i))$.

(89) If $i < j$, then W-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,i)) <$ W-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,j))$.

(90) If $i < j$, then S-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,i)) <$ S-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,j))$.

(91) If $1 \leqslant i$ and $i \leqslant \operatorname{len}\operatorname{Gauge}(C,n)$, then N-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)) = ((\operatorname{Gauge}(C,n))_{i,\operatorname{len}\operatorname{Gauge}(C,n)})_{\mathbf{2}}$.

(92) If $1 \leqslant i$ and $i \leqslant \operatorname{len}\operatorname{Gauge}(C,n)$, then E-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)) = ((\operatorname{Gauge}(C,n))_{\operatorname{len}\operatorname{Gauge}(C,n),i})_{\mathbf{1}}$.

(93) If $1 \leqslant i$ and $i \leqslant \operatorname{len}\operatorname{Gauge}(C,n)$, then S-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)) = ((\operatorname{Gauge}(C,n))_{i,1})_{\mathbf{2}}$.

(94) If $1 \leqslant i$ and $i \leqslant \operatorname{len}\operatorname{Gauge}(C,n)$, then W-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)) = ((\operatorname{Gauge}(C,n))_{1,i})_{\mathbf{1}}$.

(95) If $x \in C$ and $p \in \operatorname{NorthHalfline} x \cap \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$, then $p_{\mathbf{2}} > x_{\mathbf{2}}$.

(96) If $x \in C$ and $p \in \operatorname{EastHalfline} x \cap \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$, then $p_{\mathbf{1}} > x_{\mathbf{1}}$.

(97) If $x \in C$ and $p \in \operatorname{SouthHalfline} x \cap \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$, then $p_{\mathbf{2}} < x_{\mathbf{2}}$.

(98) If $x \in C$ and $p \in \operatorname{WestHalfline} x \cap \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$, then $p_{\mathbf{1}} < x_{\mathbf{1}}$.

(99) If $x \in$ N-most C and $p \in \operatorname{NorthHalfline} x$ and $1 \leqslant i$ and $i < \operatorname{len}\operatorname{Cage}(C,n)$ and $p \in \mathcal{L}(\operatorname{Cage}(C,n),i)$, then $\mathcal{L}(\operatorname{Cage}(C,n),i)$ is horizontal.

(100) If $x \in \text{E-most}\, C$ and $p \in \text{EastHalfline}\, x$ and $1 \leqslant i$ and $i < \text{len Cage}(C,n)$ and $p \in \mathcal{L}(\text{Cage}(C,n),i)$, then $\mathcal{L}(\text{Cage}(C,n),i)$ is vertical.

(101) If $x \in \text{S-most}\, C$ and $p \in \text{SouthHalfline}\, x$ and $1 \leqslant i$ and $i < \text{len Cage}(C,n)$ and $p \in \mathcal{L}(\text{Cage}(C,n),i)$, then $\mathcal{L}(\text{Cage}(C,n),i)$ is horizontal.

(102) If $x \in \text{W-most}\, C$ and $p \in \text{WestHalfline}\, x$ and $1 \leqslant i$ and $i < \text{len Cage}(C,n)$ and $p \in \mathcal{L}(\text{Cage}(C,n),i)$, then $\mathcal{L}(\text{Cage}(C,n),i)$ is vertical.

(103) If $x \in \text{N-most}\, C$ and $p \in \text{NorthHalfline}\, x \cap \widetilde{\mathcal{L}}(\text{Cage}(C,n))$, then $p_{\mathbf{2}} = \text{N-bound}\, \widetilde{\mathcal{L}}(\text{Cage}(C,n))$.

(104) If $x \in \text{E-most}\, C$ and $p \in \text{EastHalfline}\, x \cap \widetilde{\mathcal{L}}(\text{Cage}(C,n))$, then $p_{\mathbf{1}} = \text{E-bound}\, \widetilde{\mathcal{L}}(\text{Cage}(C,n))$.

(105) If $x \in \text{S-most}\, C$ and $p \in \text{SouthHalfline}\, x \cap \widetilde{\mathcal{L}}(\text{Cage}(C,n))$, then $p_{\mathbf{2}} = \text{S-bound}\, \widetilde{\mathcal{L}}(\text{Cage}(C,n))$.

(106) If $x \in \text{W-most}\, C$ and $p \in \text{WestHalfline}\, x \cap \widetilde{\mathcal{L}}(\text{Cage}(C,n))$, then $p_{\mathbf{1}} = \text{W-bound}\, \widetilde{\mathcal{L}}(\text{Cage}(C,n))$.

(107) If $x \in \text{N-most}\, C$, then there exists a point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $\text{NorthHalfline}\, x \cap \widetilde{\mathcal{L}}(\text{Cage}(C,n)) = \{p\}$.

(108) If $x \in \text{E-most}\, C$, then there exists a point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $\text{EastHalfline}\, x \cap \widetilde{\mathcal{L}}(\text{Cage}(C,n)) = \{p\}$.

(109) If $x \in \text{S-most}\, C$, then there exists a point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $\text{SouthHalfline}\, x \cap \widetilde{\mathcal{L}}(\text{Cage}(C,n)) = \{p\}$.

(110) If $x \in \text{W-most}\, C$, then there exists a point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $\text{WestHalfline}\, x \cap \widetilde{\mathcal{L}}(\text{Cage}(C,n)) = \{p\}$.

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Some Properties of Extended Real Numbers Operations: abs, min and max

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Summary. In this article, we extend some properties concerning real numbers to extended real numbers. Almost all properties included in this article are extended properties of other articles: [9], [6], [8], [10] and [7].

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The terminology and notation used in this paper are introduced in the following papers: [8], [4], [3], [5], [10], [11], [1], and [2].

1. Preliminaries

We follow the rules: x, y, w, z denote extended real numbers and a, b denote real numbers.

The following propositions are true:

(1) If $x \neq +\infty$ or $y \neq -\infty$ and if $x \neq -\infty$ or $y \neq +\infty$, then $x + y = y + x$.

(2) If $x \neq +\infty$ and $x \neq -\infty$, then there exists y such that $x + y = 0_{\overline{\mathbb{R}}}$.

(3) If $x \neq +\infty$ and $x \neq -\infty$ and $x \neq 0_{\overline{\mathbb{R}}}$, then there exists y such that $x \cdot y = \overline{1}$.

(4) $\overline{1} \cdot x = x$ and $x \cdot \overline{1} = x$ and $\overline{\mathbb{R}}(1) \cdot x = x$ and $x \cdot \overline{\mathbb{R}}(1) = x$.

(5) $0_{\overline{\mathbb{R}}} - x = -x$.

(6) $x \neq 0_{\overline{\mathbb{R}}}$ iff $-x \neq 0_{\overline{\mathbb{R}}}$.

(7) If $0_{\overline{\mathbb{R}}} \leqslant x$ and $0_{\overline{\mathbb{R}}} \leqslant y$, then $0_{\overline{\mathbb{R}}} \leqslant x + y$.

(8) If $0_{\overline{\mathbb{R}}} \leqslant x$ and $0_{\overline{\mathbb{R}}} < y$ or $0_{\overline{\mathbb{R}}} < x$ and $0_{\overline{\mathbb{R}}} \leqslant y$, then $0_{\overline{\mathbb{R}}} < x + y$.

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(9) If $x \leqslant 0_{\overline{\mathbb{R}}}$ and $y \leqslant 0_{\overline{\mathbb{R}}}$, then $x + y \leqslant 0_{\overline{\mathbb{R}}}$.

(10) If $x \leqslant 0_{\overline{\mathbb{R}}}$ and $y < 0_{\overline{\mathbb{R}}}$ or $x < 0_{\overline{\mathbb{R}}}$ and $y \leqslant 0_{\overline{\mathbb{R}}}$, then $x + y < 0_{\overline{\mathbb{R}}}$.

(11) If $z \neq +\infty$ and $z \neq -\infty$ and $x + z = y$, then $x = y - z$.

(12) If $x \neq +\infty$ and $x \neq -\infty$ and $x \neq 0_{\overline{\mathbb{R}}}$, then $x \cdot \frac{\overline{1}}{x} = \overline{1}$ and $\frac{\overline{1}}{x} \cdot x = \overline{1}$.

(13) If $x \neq +\infty$ and $x \neq -\infty$, then $x - x = 0_{\overline{\mathbb{R}}}$.

(14) If $x \neq +\infty$ or $y \neq -\infty$ and if $x \neq -\infty$ or $y \neq +\infty$, then $-(x + y) = -x + -y$ and $-(x + y) = -y - x$ and $-(x + y) = -x - y$.

(15) If $x \neq +\infty$ or $y \neq +\infty$ and if $x \neq -\infty$ or $y \neq -\infty$, then $-(x - y) = -x + y$ and $-(x - y) = y - x$.

(16) If $x \neq +\infty$ or $y \neq +\infty$ and if $x \neq -\infty$ or $y \neq -\infty$, then $-(-x + y) = x - y$ and $-(-x + y) = x + -y$.

(17) If $x = +\infty$ and $0_{\overline{\mathbb{R}}} < y$ and $y < +\infty$ or $x = -\infty$ and $y < 0_{\overline{\mathbb{R}}}$ and $-\infty < y$, then $\frac{x}{y} = +\infty$.

(18) If $x = +\infty$ and $y < 0_{\overline{\mathbb{R}}}$ and $-\infty < y$ or $x = -\infty$ and $0_{\overline{\mathbb{R}}} < y$ and $y < +\infty$, then $\frac{x}{y} = -\infty$.

(19) If $-\infty < y$ and $y < +\infty$ and $y \neq 0_{\overline{\mathbb{R}}}$, then $\frac{x \cdot y}{y} = x$ and $x \cdot \frac{y}{y} = x$.

(20) $\overline{1} < +\infty$ and $-\infty < \overline{1}$.

(21) If $x = +\infty$ or $x = -\infty$, then for every y such that $y \in \mathbb{R}$ holds $x + y \neq 0_{\overline{\mathbb{R}}}$.

(22) If $x = +\infty$ or $x = -\infty$, then for every y holds $x \cdot y \neq \overline{1}$.

(23) If $x \neq +\infty$ or $y \neq -\infty$ but $x \neq -\infty$ or $y \neq +\infty$ and $x + y < +\infty$, then $x \neq +\infty$ and $y \neq +\infty$.

(24) If $x \neq +\infty$ or $y \neq -\infty$ but $x \neq -\infty$ or $y \neq +\infty$ and $-\infty < x + y$, then $x \neq -\infty$ and $y \neq -\infty$.

(25) If $x \neq +\infty$ or $y \neq +\infty$ but $x \neq -\infty$ or $y \neq -\infty$ and $x - y < +\infty$, then $x \neq +\infty$ and $y \neq -\infty$.

(26) If $x \neq +\infty$ or $y \neq +\infty$ but $x \neq -\infty$ or $y \neq -\infty$ and $-\infty < x - y$, then $x \neq -\infty$ and $y \neq +\infty$.

(27) If $x \neq +\infty$ or $y \neq -\infty$ but $x \neq -\infty$ or $y \neq +\infty$ and $x + y < z$, then $x \neq +\infty$ and $y \neq +\infty$ and $z \neq -\infty$ and $x < z - y$.

(28) If $z \neq +\infty$ or $y \neq +\infty$ but $z \neq -\infty$ or $y \neq -\infty$ and $x < z - y$, then $x \neq +\infty$ and $y \neq +\infty$ and $z \neq -\infty$ and $x + y < z$.

(29) If $x \neq +\infty$ or $y \neq +\infty$ but $x \neq -\infty$ or $y \neq -\infty$ and $x - y < z$, then $x \neq +\infty$ and $y \neq -\infty$ and $z \neq -\infty$ and $x < z + y$.

(30) If $z \neq +\infty$ or $y \neq -\infty$ but $z \neq -\infty$ or $y \neq +\infty$ and $x < z + y$, then $x \neq +\infty$ and $y \neq -\infty$ and $z \neq -\infty$ and $x - y < z$.

(31) If $x \neq +\infty$ or $y \neq -\infty$ and $x \neq -\infty$ or $y \neq +\infty$ and $y \neq +\infty$ or $z \neq +\infty$ and $y \neq -\infty$ or $z \neq -\infty$ and $x + y \leqslant z$, then $y \neq +\infty$ and $x \leqslant z - y$.

(32) If $x = +\infty$ and $y = -\infty$ and $x = -\infty$ and $y = +\infty$ and $y = +\infty$ and $z = +\infty$ and $y = -\infty$ and $z = -\infty$ and $x \leqslant z - y$, then $y \neq +\infty$ and $x + y \leqslant z$.

(33) If $x \neq +\infty$ or $y \neq +\infty$ and $x \neq -\infty$ or $y \neq -\infty$ and $y \neq +\infty$ or $z \neq -\infty$ and $y \neq -\infty$ or $z \neq +\infty$ and $x - y \leqslant z$, then $y \neq -\infty$ and $x \leqslant z + y$.

(34) If $x = +\infty$ and $y = +\infty$ and $x = -\infty$ and $y = -\infty$ and $y = -\infty$ and $z = +\infty$ and $x \leqslant z + y$, then $y \neq -\infty$ and $x - y \leqslant z$.

(35) If $x \neq +\infty$ and $y \neq +\infty$, then $x + y \neq +\infty$.

(36) If $x \neq -\infty$ and $y \neq -\infty$, then $x + y \neq -\infty$.

(37) If $x \neq +\infty$ and $y \neq -\infty$, then $x - y \neq +\infty$.

(38) If $x \neq -\infty$ and $y \neq +\infty$, then $x - y \neq -\infty$.

(39) Suppose $x = +\infty$ and $y = -\infty$ and $x = -\infty$ and $y = +\infty$ and $y = +\infty$ and $z = +\infty$ and $y = -\infty$ and $z = -\infty$ and $x = +\infty$ and $z = +\infty$ and $x = -\infty$ and $z = -\infty$. Then $(x + y) - z = x + (y - z)$.

(40) Suppose $x = +\infty$ and $y = +\infty$ and $x = -\infty$ and $y = -\infty$ and $y = +\infty$ and $z = -\infty$ and $y = -\infty$ and $z = +\infty$ and $x = +\infty$ and $z = +\infty$ and $x = -\infty$ and $z = -\infty$. Then $x - y - z = x - (y + z)$.

(41) Suppose $x = +\infty$ and $y = +\infty$ and $x = -\infty$ and $y = -\infty$ and $y = +\infty$ and $z = +\infty$ and $y = -\infty$ and $z = -\infty$ and $x = +\infty$ and $z = -\infty$ and $x = -\infty$ and $z = +\infty$. Then $(x - y) + z = x - (y - z)$.

(42) If $x \cdot y \neq +\infty$ and $x \cdot y \neq -\infty$, then x is a real number or y is a real number.

(43) $0_{\overline{\mathbb{R}}} < x$ and $0_{\overline{\mathbb{R}}} < y$ or $x < 0_{\overline{\mathbb{R}}}$ and $y < 0_{\overline{\mathbb{R}}}$ iff $0_{\overline{\mathbb{R}}} < x \cdot y$.

(44) $0_{\overline{\mathbb{R}}} < x$ and $y < 0_{\overline{\mathbb{R}}}$ or $x < 0_{\overline{\mathbb{R}}}$ and $0_{\overline{\mathbb{R}}} < y$ iff $x \cdot y < 0_{\overline{\mathbb{R}}}$.

(45) $0_{\overline{\mathbb{R}}} \leqslant x$ or $0_{\overline{\mathbb{R}}} < x$ but $0_{\overline{\mathbb{R}}} \leqslant y$ or $0_{\overline{\mathbb{R}}} < y$ or $x \leqslant 0_{\overline{\mathbb{R}}}$ or $x < 0_{\overline{\mathbb{R}}}$ but $y \leqslant 0_{\overline{\mathbb{R}}}$ or $y < 0_{\overline{\mathbb{R}}}$ iff $0_{\overline{\mathbb{R}}} \leqslant x \cdot y$.

(46) $x \leqslant 0_{\overline{\mathbb{R}}}$ or $x < 0_{\overline{\mathbb{R}}}$ but $0_{\overline{\mathbb{R}}} \leqslant y$ or $0_{\overline{\mathbb{R}}} < y$ or $0_{\overline{\mathbb{R}}} \leqslant x$ or $0_{\overline{\mathbb{R}}} < x$ but $y \leqslant 0_{\overline{\mathbb{R}}}$ or $y < 0_{\overline{\mathbb{R}}}$ iff $x \cdot y \leqslant 0_{\overline{\mathbb{R}}}$.

(47) $x \leqslant -y$ iff $y \leqslant -x$ and $-x \leqslant y$ iff $-y \leqslant x$.

(48) $x < -y$ iff $y < -x$ and $-x < y$ iff $-y < x$.

2. Basic Properties of abs for Extended Real Numbers

One can prove the following propositions:

(49) If $x = a$, then $|x| = |a|$.

(50) $|x| = x$ or $|x| = -x$.

(51) $0_{\overline{\mathbb{R}}} \leqslant |x|$.

(52) If $x \neq 0_{\overline{\mathbb{R}}}$, then $0_{\overline{\mathbb{R}}} < |x|$.
(53) $x = 0_{\overline{\mathbb{R}}}$ iff $|x| = 0_{\overline{\mathbb{R}}}$.
(54) If $|x| = -x$ and $x \neq 0_{\overline{\mathbb{R}}}$, then $x < 0_{\overline{\mathbb{R}}}$.
(55) If $x \leqslant 0_{\overline{\mathbb{R}}}$, then $|x| = -x$.
(56) $|x \cdot y| = |x| \cdot |y|$.
(57) $-|x| \leqslant x$ and $x \leqslant |x|$.
(58) If $|x| < y$, then $-y < x$ and $x < y$.
(59) If $-y < x$ and $x < y$, then $0_{\overline{\mathbb{R}}} < y$ and $|x| < y$.
(60) $-y \leqslant x$ and $x \leqslant y$ iff $|x| \leqslant y$.
(61) If $x \neq +\infty$ or $y \neq -\infty$ and if $x \neq -\infty$ or $y \neq +\infty$, then $|x+y| \leqslant |x|+|y|$.
(62) If $-\infty < x$ and $x < +\infty$ and $x \neq 0_{\overline{\mathbb{R}}}$, then $|x| \cdot |\frac{\overline{1}}{x}| = \overline{1}$.
(63) If $x = +\infty$ or $x = -\infty$, then $|x| \cdot |\frac{\overline{1}}{x}| = 0_{\overline{\mathbb{R}}}$.
(64) If $x \neq 0_{\overline{\mathbb{R}}}$, then $|\frac{\overline{1}}{x}| = \frac{\overline{1}}{|x|}$.
(65) If $x = -\infty$ or $x = +\infty$ and if $y = -\infty$ or $y = +\infty$ and if $y \neq 0_{\overline{\mathbb{R}}}$, then $|\frac{x}{y}| = \frac{|x|}{|y|}$.
(66) $|x| = |-x|$.
(67) If $x = +\infty$ or $x = -\infty$, then $|x| = +\infty$.
(68) If x is a real number or y is a real number, then $|x| - |y| \leqslant |x - y|$.
(69) If $x \neq +\infty$ or $y \neq +\infty$ and if $x \neq -\infty$ or $y \neq -\infty$, then $|x-y| \leqslant |x|+|y|$.
(70) $||x|| = |x|$.
(71) If $x \neq +\infty$ or $y \neq -\infty$ but $x \neq -\infty$ or $y \neq +\infty$ and $|x| \leqslant z$ and $|y| \leqslant w$, then $|x + y| \leqslant z + w$.
(72) If x is a real number or y is a real number, then $||x| - |y|| \leqslant |x - y|$.
(73) If $0_{\overline{\mathbb{R}}} \leqslant x \cdot y$, then $|x + y| = |x| + |y|$.

3. Definitions of min, max for Extended Real Numbers and their Basic Properties

Next we state the proposition

(74) If $x = a$ and $y = b$, then $b < a$ iff $y < x$ and $b \leqslant a$ iff $y \leqslant x$.

Let us consider x, y. The functor $\min(x, y)$ yielding an extended real number is defined by:

(Def. 1) $$\min(x, y) = \begin{cases} x, & \text{if } x \leqslant y, \\ y, & \text{otherwise.} \end{cases}$$

The functor $\max(x, y)$ yielding an extended real number is defined as follows:

(Def. 2) $\max(x, y) = \begin{cases} x, & \text{if } y \leqslant x, \\ y, & \text{otherwise.} \end{cases}$

One can prove the following propositions:

(75) If $x = -\infty$ or $y = -\infty$, then $\min(x, y) = -\infty$.

(76) If $x = +\infty$ or $y = +\infty$, then $\max(x, y) = +\infty$.

(77) Let x, y be extended real numbers and a, b be real numbers. If $x = a$ and $y = b$, then $\min(x, y) = \min(a, b)$ and $\max(x, y) = \max(a, b)$.

(78) If $y \leqslant x$, then $\min(x, y) = y$.

(79) If $y \nleqslant x$, then $\min(x, y) = x$.

(80) If $x \neq +\infty$ and $y \neq +\infty$ and $x \neq +\infty$ or $y \neq +\infty$ but $x \neq -\infty$ or $y \neq -\infty$, then $\min(x, y) = \frac{(x+y)-|x-y|}{\overline{\mathbb{R}}(2)}$.

(81) $\min(x, y) \leqslant x$ and $\min(y, x) \leqslant x$.

(82) $\min(x, x) = x$.

(83) $\min(x, y) = \min(y, x)$.

(84) $\min(x, y) = x$ or $\min(x, y) = y$.

(85) $x \leqslant y$ and $x \leqslant z$ iff $x \leqslant \min(y, z)$.

(86) If $\min(x, y) = x$, then $x \leqslant y$.

(87) If $\min(x, y) = y$, then $y \leqslant x$.

(88) $\min(x, \min(y, z)) = \min(\min(x, y), z)$.

(89) If $x \leqslant y$, then $\max(x, y) = y$.

(90) If $x \nleqslant y$, then $\max(x, y) = x$.

(91) If $x \neq -\infty$ and $y \neq -\infty$ and $x \neq +\infty$ or $y \neq +\infty$ but $x \neq -\infty$ or $y \neq -\infty$, then $\max(x, y) = \frac{x+y+|x-y|}{\overline{\mathbb{R}}(2)}$.

(92) $x \leqslant \max(x, y)$ and $x \leqslant \max(y, x)$.

(93) $\max(x, x) = x$.

(94) $\max(x, y) = \max(y, x)$.

(95) $\max(x, y) = x$ or $\max(x, y) = y$.

(96) $y \leqslant x$ and $z \leqslant x$ iff $\max(y, z) \leqslant x$.

(97) If $\max(x, y) = x$, then $y \leqslant x$.

(98) If $\max(x, y) = y$, then $x \leqslant y$.

(99) $\max(x, \max(y, z)) = \max(\max(x, y), z)$.

(100) If $x \neq +\infty$ or $y \neq -\infty$ and if $x \neq -\infty$ or $y \neq +\infty$, then $\min(x, y) + \max(x, y) = x + y$.

(101) $\max(x, \min(x, y)) = x$ and $\max(\min(x, y), x) = x$ and $\max(\min(y, x), x) = x$ and $\max(x, \min(y, x)) = x$.

(102) $\min(x, \max(x, y)) = x$ and $\min(\max(x, y), x) = x$ and $\min(\max(y, x), x) = x$ and $\min(x, \max(y, x)) = x$.

(103) $\min(x, \max(y, z)) = \max(\min(x, y), \min(x, z))$ and $\min(\max(y, z), x) = \max(\min(y, x), \min(z, x))$.

(104) $\max(x, \min(y, z)) = \min(\max(x, y), \max(x, z))$ and $\max(\min(y, z), x) = \min(\max(y, x), \max(z, x))$.

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The Concept of Fuzzy Relation and Basic Properties of its Operation

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Summary. This article introduces the fuzzy relation. This is the expansion of usual relation, and the value is given at the fuzzy value. At first, the definition of the fuzzy relation characterized by membership function is described. Next, the definitions of the zero relation and universe relation and basic operations of these relations are shown.

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The papers [8], [1], [5], [9], [3], [4], [6], [7], and [2] provide the terminology and notation for this paper.

1. Definition of Fuzzy Relation

In this paper C_1, C_2 are non empty sets.

Let us consider C_1, C_2. A partial function from $[: C_1, C_2 :]$ to $\mathbb{R}$ is said to be a Membership function of C_1, C_2 if:

(Def. 1) $\operatorname{dom} \mathrm{it} = [: C_1, C_2 :]$ and $\operatorname{rng} \mathrm{it} \subseteq [0, 1]$.

The following proposition is true

(1) $\chi_{[: C_1, C_2 :], [: C_1, C_2 :]}$ is a Membership function of C_1, C_2.

Let C_1, C_2 be non empty sets and let h be a Membership function of C_1, C_2. A set is called a fuzzy relation of C_1, C_2, h if:

(Def. 2) $\mathrm{It} = [: [: C_1, C_2 :], h^{\circ}[: C_1, C_2 :] :]$.

Let C_1, C_2 be non empty sets, let h, g be Membership functions of C_1, C_2, let A be a fuzzy relation of C_1, C_2, h, and let B be a fuzzy relation of C_1, C_2, g. The predicate $A = B$ is defined by:

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(Def. 3) For every element c of $[: C_1, C_2 :]$ holds $h(c) = g(c)$.

Let C_1, C_2 be non empty sets, let h, g be Membership functions of C_1, C_2, let A be a fuzzy relation of C_1, C_2, h, and let B be a fuzzy relation of C_1, C_2, g. The predicate $A \subseteq B$ is defined by:

(Def. 4) For every element c of $[: C_1, C_2 :]$ holds $h(c) \leqslant g(c)$.

For simplicity, we adopt the following rules: f, g, h, h_1 denote Membership functions of C_1, C_2, A denotes a fuzzy relation of C_1, C_2, f, B denotes a fuzzy relation of C_1, C_2, g, D denotes a fuzzy relation of C_1, C_2, h, and D_1 denotes a fuzzy relation of C_1, C_2, h_1.

The following three propositions are true:

(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

(3) $A \subseteq A$.

(4) If $A \subseteq B$ and $B \subseteq D$, then $A \subseteq D$.

2. Intersection, Union and Complement

Let C_1, C_2 be non empty sets and let h, g be Membership functions of C_1, C_2. The functor $\min(h, g)$ yielding a Membership function of C_1, C_2 is defined as follows:

(Def. 5) For every element c of $[: C_1, C_2 :]$ holds $(\min(h, g))(c) = \min(h(c), g(c))$.

Let C_1, C_2 be non empty sets and let h, g be Membership functions of C_1, C_2. The functor $\max(h, g)$ yields a Membership function of C_1, C_2 and is defined by:

(Def. 6) For every element c of $[: C_1, C_2 :]$ holds $(\max(h, g))(c) = \max(h(c), g(c))$.

Let C_1, C_2 be non empty sets and let h be a Membership function of C_1, C_2. The functor 1-minus h yields a Membership function of C_1, C_2 and is defined as follows:

(Def. 7) For every element c of $[: C_1, C_2 :]$ holds $(\text{1-minus}\, h)(c) = 1 - h(c)$.

Let C_1, C_2 be non empty sets, let h, g be Membership functions of C_1, C_2, let A be a fuzzy relation of C_1, C_2, h, and let B be a fuzzy relation of C_1, C_2, g. The functor $A \cap B$ yields a fuzzy relation of C_1, C_2, $\min(h, g)$ and is defined as follows:

(Def. 8) $A \cap B = [: [: C_1, C_2 :], (\min(h, g))^{\circ}[: C_1, C_2 :] :]$.

Let C_1, C_2 be non empty sets, let h, g be Membership functions of C_1, C_2, let A be a fuzzy relation of C_1, C_2, h, and let B be a fuzzy relation of C_1, C_2, g. The functor $A \cup B$ yielding a fuzzy relation of C_1, C_2, $\max(h, g)$ is defined by:

(Def. 9) $A \cup B = [: [: C_1, C_2 :], (\max(h, g))^{\circ}[: C_1, C_2 :] :]$.

Let C_1, C_2 be non empty sets, let h be a Membership function of C_1, C_2, and let A be a fuzzy relation of C_1, C_2, h. The functor A^{c} yielding a fuzzy relation of C_1, C_2, 1-minus h is defined as follows:

(Def. 10) $A^{\mathrm{c}} = [\!:[\!: C_1, C_2 :\!], (\text{1-minus}\, h)^{\circ}[\!: C_1, C_2 :\!]:\!]$.

The following propositions are true:

(5) For every element x of $[\!: C_1, C_2 :\!]$ holds $\min(h(x), g(x)) = (\min(h, g))(x)$ and $\max(h(x), g(x)) = (\max(h, g))(x)$.

(6) $\max(h, h) = h$ and $\min(h, h) = h$ and $\max(h, h) = \min(h, h)$ and $\min(f, g) = \min(g, f)$ and $\max(f, g) = \max(g, f)$.

(7) $f = g$ iff $A = B$.

(8) $A \cap A = A$ and $A \cup A = A$.

(9) $A \cap B = B \cap A$ and $A \cup B = B \cup A$.

(10) $\max(\max(f, g), h) = \max(f, \max(g, h))$ and $\min(\min(f, g), h) = \min(f, \min(g, h))$.

(11) $(A \cup B) \cup D = A \cup (B \cup D)$.

(12) $(A \cap B) \cap D = A \cap (B \cap D)$.

(13) $\max(f, \min(f, g)) = f$ and $\min(f, \max(f, g)) = f$.

(14) $A \cup A \cap B = A$ and $A \cap (A \cup B) = A$.

(15) $\min(f, \max(g, h)) = \max(\min(f, g), \min(f, h))$ and $\max(f, \min(g, h)) = \min(\max(f, g), \max(f, h))$.

(16) $A \cup B \cap D = (A \cup B) \cap (A \cup D)$ and $A \cap (B \cup D) = A \cap B \cup A \cap D$.

(17) 1-minus 1-minus $h = h$.

(18) $(A^{\mathrm{c}})^{\mathrm{c}} = A$.

(19) 1-minus $\max(f, g) = \min(\text{1-minus}\, f, \text{1-minus}\, g)$ and 1-minus $\min(f, g) = \max(\text{1-minus}\, f, \text{1-minus}\, g)$.

(20) $(A \cup B)^{\mathrm{c}} = A^{\mathrm{c}} \cap B^{\mathrm{c}}$ and $(A \cap B)^{\mathrm{c}} = A^{\mathrm{c}} \cup B^{\mathrm{c}}$.

(21) $A \subseteq A \cup B$.

(22) If $A \subseteq D$ and $B \subseteq D$, then $A \cup B \subseteq D$.

(23) If $A \subseteq B$, then $A \cup D \subseteq B \cup D$.

(24) If $A \subseteq B$ and $D \subseteq D_1$, then $A \cup D \subseteq B \cup D_1$.

(25) If $A \subseteq B$, then $A \cup B = B$.

(26) $A \cap B \subseteq A$.

(27) $A \cap B \subseteq A \cup B$.

(28) If $D \subseteq A$ and $D \subseteq B$, then $D \subseteq A \cap B$.

(29) For all elements a, b, c, d of $\mathbb{R}$ such that $a \leqslant b$ and $c \leqslant d$ holds $\min(a, c) \leqslant \min(b, d)$.

(30) For all elements a, b, c of $\mathbb{R}$ such that $a \leqslant b$ holds $\min(a, c) \leqslant \min(b, c)$.

(31) If $A \subseteq B$, then $A \cap D \subseteq B \cap D$.

(32) If $A \subseteq B$ and $D \subseteq D_1$, then $A \cap D \subseteq B \cap D_1$.
(33) If $A \subseteq B$, then $A \cap B = A$.
(34) If $A \cap B \cup A \cap D = A$, then $A \subseteq B \cup D$.
(35) $A = B \cup D$ iff $B \subseteq A$ and $D \subseteq A$ and for all h_1, D_1 such that $B \subseteq D_1$ and $D \subseteq D_1$ holds $A \subseteq D_1$.
(36) $A = B \cap D$ iff $A \subseteq B$ and $A \subseteq D$ and for all h_1, D_1 such that $D_1 \subseteq B$ and $D_1 \subseteq D$ holds $D_1 \subseteq A$.
(37) $A \subseteq B$ iff $B^{\rm c} \subseteq A^{\rm c}$.
(38) If $A \subseteq B^{\rm c}$, then $B \subseteq A^{\rm c}$.
(39) If $A^{\rm c} \subseteq B$, then $B^{\rm c} \subseteq A$.
(40) $(A \cup B)^{\rm c} \subseteq A^{\rm c}$ and $(A \cup B)^{\rm c} \subseteq B^{\rm c}$.
(41) $A^{\rm c} \subseteq (A \cap B)^{\rm c}$ and $B^{\rm c} \subseteq (A \cap B)^{\rm c}$.

3. Exclusive Sum

Let C_1, C_2 be non empty sets, let h, g be Membership functions of C_1, C_2, let A be a fuzzy relation of C_1, C_2, h, and let B be a fuzzy relation of C_1, C_2, g. The functor $A \dot{-} B$ yields a fuzzy relation of C_1, C_2, $\max(\min(h, \text{1-minus}\, g), \min(\text{1-minus}\, h, g))$ and is defined by:

(Def. 11) $A \dot{-} B = [\![[\![C_1, C_2]\!], (\max(\min(h, \text{1-minus}\, g), \min(\text{1-minus}\, h, g)))^{\circ}[\![C_1, C_2]\!]]\!]$.

The following propositions are true:

(42) $A \dot{-} B = A \cap B^{\rm c} \cup A^{\rm c} \cap B$.
(43) $A \dot{-} B = B \dot{-} A$.

4. Zero Relation and Universe Relation

Let C_1, C_2 be non empty sets. A set is called a zero relation of C_1, C_2 if:

(Def. 12) It $= [\![[\![C_1, C_2]\!], (\chi_{\emptyset, [\![C_1, C_2]\!]})^{\circ}[\![C_1, C_2]\!]]\!]$.

Let C_1, C_2 be non empty sets. A set is called a universe relation of C_1, C_2 if:

(Def. 13) It $= [\![[\![C_1, C_2]\!], (\chi_{[\![C_1, C_2]\!], [\![C_1, C_2]\!]})^{\circ}[\![C_1, C_2]\!]]\!]$.

In the sequel X is a universe relation of C_1, C_2 and O is a zero relation of C_1, C_2.

The following proposition is true

(44) $\chi_{\emptyset, [\![C_1, C_2]\!]}$ is a Membership function of C_1, C_2.

Let C_1, C_2 be non empty sets. The functor $\mathrm{Zmf}(C_1, C_2)$ yielding a Membership function of C_1, C_2 is defined as follows:

(Def. 14) $\mathrm{Zmf}(C_1, C_2) = \chi_{\emptyset, [: C_1, C_2 :]}$.

Let C_1, C_2 be non empty sets. The functor $\mathrm{Umf}(C_1, C_2)$ yields a Membership function of C_1, C_2 and is defined as follows:

(Def. 15) $\mathrm{Umf}(C_1, C_2) = \chi_{[: C_1, C_2 :], [: C_1, C_2 :]}$.

Next we state four propositions:

(45) Let h be a Membership function of C_1, C_2. If $h = \chi_{[: C_1, C_2 :], [: C_1, C_2 :]}$, then $[: [: C_1, C_2 :], (\chi_{[: C_1, C_2 :], [: C_1, C_2 :]})^{\circ}[: C_1, C_2 :] :]$ is a fuzzy relation of C_1, C_2, h.

(46) For every Membership function h of C_1, C_2 such that $h = \chi_{\emptyset, [: C_1, C_2 :]}$ holds $[: [: C_1, C_2 :], (\chi_{\emptyset, [: C_1, C_2 :]})^{\circ}[: C_1, C_2 :] :]$ is a fuzzy relation of C_1, C_2, h.

(47) O is a fuzzy relation of C_1, C_2, $\mathrm{Zmf}(C_1, C_2)$.

(48) X is a fuzzy relation of C_1, C_2, $\mathrm{Umf}(C_1, C_2)$.

Let C_1, C_2 be non empty sets. We see that the zero relation of C_1, C_2 is a fuzzy relation of C_1, C_2, $\mathrm{Zmf}(C_1, C_2)$.

Let C_1, C_2 be non empty sets. We see that the universe relation of C_1, C_2 is a fuzzy relation of C_1, C_2, $\mathrm{Umf}(C_1, C_2)$.

In the sequel X denotes a universe relation of C_1, C_2 and O denotes a zero relation of C_1, C_2.

Next we state a number of propositions:

(49) Let a, b be elements of $\mathbb{R}$ and f be a partial function from $[: C_1, C_2 :]$ to $\mathbb{R}$. Suppose $\mathrm{rng}\, f \subseteq [a, b]$ and $\mathrm{dom}\, f \neq \emptyset$ and $a \leqslant b$. Let x be an element of $[: C_1, C_2 :]$. If $x \in \mathrm{dom}\, f$, then $a \leqslant f(x)$ and $f(x) \leqslant b$.

(50) $O \subseteq A$.

(51) $A \subseteq X$.

(52) For every element x of $[: C_1, C_2 :]$ and for every Membership function h of C_1, C_2 holds $(\mathrm{Zmf}(C_1, C_2))(x) \leqslant h(x)$ and $h(x) \leqslant (\mathrm{Umf}(C_1, C_2))(x)$.

(53) $\max(f, \mathrm{Umf}(C_1, C_2)) = \mathrm{Umf}(C_1, C_2)$ and $\min(f, \mathrm{Umf}(C_1, C_2)) = f$ and $\max(f, \mathrm{Zmf}(C_1, C_2)) = f$ and $\min(f, \mathrm{Zmf}(C_1, C_2)) = \mathrm{Zmf}(C_1, C_2)$.

(54) $A \cup X = X$ and $A \cap X = A$.

(55) $A \cup O = A$ and $A \cap O = O$.

(56) If $A \subseteq B$ and $A \subseteq D$ and $B \cap D = O$, then $A = O$.

(57) If $A \subseteq B$ and $B \cap D = O$, then $A \cap D = O$.

(58) If $A \subseteq O$, then $A = O$.

(59) $A \cup B = O$ iff $A = O$ and $B = O$.

(60) If $A \subseteq B \cup D$ and $A \cap D = O$, then $A \subseteq B$.

(61) $\text{1-minus}\, \mathrm{Zmf}(C_1, C_2) = \mathrm{Umf}(C_1, C_2)$ and $\text{1-minus}\, \mathrm{Umf}(C_1, C_2) = \mathrm{Zmf}(C_1, C_2)$.

(62) $O^{\rm c} = X$ and $X^{\rm c} = O$.

(63) $A \dot{-} O = A$ and $O \dot{-} A = A$.

(64) $A \dot{-} X = A^{\rm c}$ and $X \dot{-} A = A^{\rm c}$.

(65) For every element c of $[\!: C_1, C_2 :\!]$ such that $f(c) \leqslant h(c)$ holds $(\max(f, \min(g, h)))(c) = (\min(\max(f, g), h))(c)$.

(66) If $A \subseteq D$, then $A \cup B \cap D = (A \cup B) \cap D$.

Let C_1, C_2 be non empty sets, let f, g be Membership functions of C_1, C_2, let A be a fuzzy relation of C_1, C_2, f, and let B be a fuzzy relation of C_1, C_2, g. The functor $A \setminus B$ yielding a fuzzy relation of C_1, C_2, $\min(f, \text{1-minus}\, g)$ is defined by:

(Def. 16) $A \setminus B = [\!: [\!: C_1, C_2 :\!], (\min(f, \text{1-minus}\, g))^\circ [\!: C_1, C_2 :\!] :\!]$.

One can prove the following propositions:

(67) $A \setminus B = A \cap B^{\rm c}$.

(68) $\text{1-minus} \min(f, \text{1-minus}\, g) = \max(\text{1-minus}\, f, g)$.

(69) $(A \setminus B)^{\rm c} = A^{\rm c} \cup B$.

(70) For every element c of $[\!: C_1, C_2 :\!]$ such that $f(c) \leqslant g(c)$ holds $(\min(f, \text{1-minus}\, h))(c) \leqslant (\min(g, \text{1-minus}\, h))(c)$.

(71) If $A \subseteq B$, then $A \setminus D \subseteq B \setminus D$.

(72) For every element c of $[\!: C_1, C_2 :\!]$ such that $f(c) \leqslant g(c)$ holds $(\min(h, \text{1-minus}\, g))(c) \leqslant (\min(h, \text{1-minus}\, f))(c)$.

(73) If $A \subseteq B$, then $D \setminus B \subseteq D \setminus A$.

(74) For every element c of $[\!: C_1, C_2 :\!]$ such that $f(c) \leqslant g(c)$ and $h(c) \leqslant h_1(c)$ holds $(\min(f, \text{1-minus}\, h_1))(c) \leqslant (\min(g, \text{1-minus}\, h))(c)$.

(75) If $A \subseteq B$ and $D \subseteq D_1$, then $A \setminus D_1 \subseteq B \setminus D$.

(76) For every element c of $[\!: C_1, C_2 :\!]$ holds $(\min(f, \text{1-minus}\, g))(c) \leqslant f(c)$.

(77) $A \setminus B \subseteq A$.

(78) For every element c of $[\!: C_1, C_2 :\!]$ holds $(\min(f, \text{1-minus}\, g))(c) \leqslant (\max(\min(f, \text{1-minus}\, g), \min(\text{1-minus}\, f, g)))(c)$.

(79) $A \setminus B \subseteq A \dot{-} B$.

(80) $A \setminus O = A$.

(81) $O \setminus A = O$.

(82) For every element c of $[\!: C_1, C_2 :\!]$ holds $(\min(f, \text{1-minus}\, g))(c) \leqslant (\min(f, \text{1-minus} \min(f, g)))(c)$.

(83) $A \setminus B \subseteq A \setminus A \cap B$.

(84) For every element c of $[\!: C_1, C_2 :\!]$ holds $(\max(\min(f, g), \min(f, \text{1-minus}\, g)))(c) \leqslant f(c)$.

(85) For every element c of $[\!: C_1, C_2 :\!]$ holds $(\max(f, \min(g, \text{1-minus}\, f)))(c) \leqslant (\max(f, g))(c)$.

(86) $A \cup (B \setminus A) \subseteq A \cup B$.

(87) $A \cap B \cup (A \setminus B) \subseteq A$.

(88) $\min(f, 1\text{-minus} \min(g, 1\text{-minus}\, h)) = \max(\min(f, 1\text{-minus}\, g), \min(f, h))$.

(89) $A \setminus (B \setminus D) = (A \setminus B) \cup A \cap D$.

(90) For every element c of $[\!: C_1, C_2 :\!]$ holds $(\min(f, g))(c) \leqslant (\min(f, 1\text{-minus} \min(f, 1\text{-minus}\, g)))(c)$.

(91) $A \cap B \subseteq A \setminus (A \setminus B)$.

(92) For every element c of $[\!: C_1, C_2 :\!]$ holds $(\min(f, 1\text{-minus}\, g))(c) \leqslant (\min(\max(f, g), 1\text{-minus}\, g))(c)$.

(93) $A \setminus B \subseteq (A \cup B) \setminus B$.

(94) $\min(f, 1\text{-minus} \max(g, h)) = \min(\min(f, 1\text{-minus}\, g), \min(f, 1\text{-minus}\, h))$.

(95) $A \setminus (B \cup D) = (A \setminus B) \cap (A \setminus D)$.

(96) $\min(f, 1\text{-minus} \min(g, h)) = \max(\min(f, 1\text{-minus}\, g), \min(f, 1\text{-minus}\, h))$.

(97) $A \setminus B \cap D = (A \setminus B) \cup (A \setminus D)$.

(98) $\min(\min(f, 1\text{-minus}\, g), 1\text{-minus}\, h) = \min(f, 1\text{-minus} \max(g, h))$.

(99) $A \setminus B \setminus D = A \setminus (B \cup D)$.

(100) For every element c of $[\!: C_1, C_2 :\!]$ holds $(\min(\max(f, g), 1\text{-minus} \min(f, g)))(c) \geqslant (\max(\min(f, 1\text{-minus}\, g), \min(g, 1\text{-minus}\, f)))(c)$.

(101) $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus A \cap B$.

(102) $\min(\max(f, g), 1\text{-minus}\, h) = \max(\min(f, 1\text{-minus}\, h), \min(g, 1\text{-minus}\, h))$.

(103) $(A \cup B) \setminus D = (A \setminus D) \cup (B \setminus D)$.

(104) For every element c of $[\!: C_1, C_2 :\!]$ such that $(\min(f, 1\text{-minus}\, g))(c) \leqslant h(c)$ and $(\min(g, 1\text{-minus}\, f))(c) \leqslant h(c)$ holds $(\max(\min(f, 1\text{-minus}\, g), \min(1\text{-minus}\, f, g)))(c) \leqslant h(c)$.

(105) If $A \setminus B \subseteq D$ and $B \setminus A \subseteq D$, then $A \dot{-} B \subseteq D$.

(106) $A \cap (B \setminus D) = A \cap B \setminus D$.

(107) For every element c of $[\!: C_1, C_2 :\!]$ holds $(\min(f, \min(g, 1\text{-minus}\, h)))(c) \leqslant (\min(\min(f, g), 1\text{-minus} \min(f, h)))(c)$.

(108) $A \cap (B \setminus D) \subseteq A \cap B \setminus A \cap D$.

(109) For every element c of $[\!: C_1, C_2 :\!]$ holds $(\min(\max(f, g), 1\text{-minus} \min(f, g)))(c) \geqslant (\max(\min(f, 1\text{-minus}\, g), \min(1\text{-minus}\, f, g)))(c)$.

(110) $A \dot{-} B \subseteq (A \cup B) \setminus A \cap B$.

(111) For every element c of $[\!: C_1, C_2 :\!]$ holds $(\max(\min(f, g), 1\text{-minus} \max(f, g)))(c) \leqslant (1\text{-minus} \max(\min(f, 1\text{-minus}\, g), \min(1\text{-minus}\, f, g)))(c)$.

(112) $A \cap B \cup (A \cup B)^{\mathrm{c}} \subseteq (A \dot{-} B)^{\mathrm{c}}$.

(113) $\min(\max(\min(f, 1\text{-minus}\, g), \min(1\text{-minus}\, f, g)), 1\text{-minus}\, h) = \max(\min(f, 1\text{-minus} \max(g, h)), \min(g, 1\text{-minus} \max(f, h)))$.

(114) $(A \dot{-} B) \setminus D = (A \setminus (B \cup D)) \cup (B \setminus (A \cup D))$.

(115) For every element c of $[: C_1, C_2 :]$ holds $(\min(f, 1\text{-minus}\max(\min(g, 1\text{-minus}\, h), \min(1\text{-minus}\, g, h))))(c) \geqslant (\max(\min(f, 1\text{-minus}\max(g, h)), \min(\min(f, g), h)))(c)$.

(116) $(A \setminus (B \cup D)) \cup A \cap B \cap D \subseteq A \setminus (B \dot{-} D)$.

(117) For every element c of $[: C_1, C_2 :]$ such that $f(c) \leqslant g(c)$ holds $g(c) \geqslant (\max(f, \min(g, 1\text{-minus}\, f)))(c)$.

(118) If $A \subseteq B$, then $A \cup (B \setminus A) \subseteq B$.

(119) For every element c of $[: C_1, C_2 :]$ holds $(\max(f, g))(c) \geqslant (\max(\max(\min(f, 1\text{-minus}\, g), \min(1\text{-minus}\, f, g)), \min(f, g)))(c)$.

(120) $(A \dot{-} B) \cup A \cap B \subseteq A \cup B$.

(121) If $\min(f, 1\text{-minus}\, g) = \mathrm{Zmf}(C_1, C_2)$, then for every element c of $[: C_1, C_2 :]$ holds $f(c) \leqslant g(c)$.

(122) If $A \setminus B = O$, then $A \subseteq B$.

(123) If $\min(f, g) = \mathrm{Zmf}(C_1, C_2)$, then $\min(f, 1\text{-minus}\, g) = f$.

(124) If $A \cap B = O$, then $A \setminus B = A$.

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The Measurability of Extended Real Valued Functions

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Summary. In this article we prove the measurablility of some extended real valued functions which are $f+g$, $f-g$ and so on. Moreover, we will define the simple function which are defined on the sigma field. It will play an important role for the Lebesgue integral theory.

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The notation and terminology used here are introduced in the following papers: [21], [2], [10], [11], [9], [7], [6], [3], [8], [13], [12], [17], [16], [15], [14], [22], [23], [18], [20], [4], [5], [19], and [1].

1. Finite Valued Function

For simplicity, we adopt the following rules: X is a non empty set, x is an element of X, f, g are partial functions from X to $\overline{\mathbb{R}}$, S is a σ-field of subsets of X, F is a function from $\mathbb{Q}$ into S, p is a rational number, r is a real number, n, m are natural numbers, and A, B are elements of S.

Let us consider X and let us consider f. We say that f is finite if and only if:

(Def. 1) For every x such that $x \in \operatorname{dom} f$ holds $|f(x)| < +\infty$.

Next we state three propositions:

(1) $f = 1\,f$.

(2) For all f, g, A such that f is finite or g is finite holds $\operatorname{dom}(f+g) = \operatorname{dom} f \cap \operatorname{dom} g$ and $\operatorname{dom}(f-g) = \operatorname{dom} f \cap \operatorname{dom} g$.

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(3) Let given f, g, F, r, A. Suppose f is finite and g is finite and for every p holds $F(p) = A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(p)) \cap (A \cap \text{LE-dom}(g, \overline{\mathbb{R}}(r - p)))$. Then $A \cap \text{LE-dom}(f + g, \overline{\mathbb{R}}(r)) = \bigcup \operatorname{rng} F$.

2. Measurability of $f+g$ and $f-g$

The following propositions are true:

(4) There exists a function F from $\mathbb{N}$ into $\mathbb{Q}$ such that F is one-to-one and $\operatorname{dom} F = \mathbb{N}$ and $\operatorname{rng} F = \mathbb{Q}$.

(5) Let X, Y, Z be non empty sets and F be a function from X into Z. If $X \approx Y$, then there exists a function G from Y into Z such that $\operatorname{rng} F = \operatorname{rng} G$.

(6) Let given S, f, g, A. Suppose f is measurable on A and g is measurable on A. Then there exists a function F from $\mathbb{Q}$ into S such that for every rational number p holds $F(p) = A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(p)) \cap (A \cap \text{LE-dom}(g, \overline{\mathbb{R}}(r - p)))$.

(7) Let given f, g, A. Suppose f is finite and g is finite and f is measurable on A and g is measurable on A. Then $f + g$ is measurable on A.

(8) For all sets E, F, G and for every partial function f from E to F holds $f^{-1}(G) \subseteq E$.

(9) For every non empty set C and for all partial functions f_1, f_2 from C to $\overline{\mathbb{R}}$ holds $f_1 - f_2 = f_1 + -f_2$.

(10) For every real number r holds $\overline{\mathbb{R}}(-r) = -\overline{\mathbb{R}}(r)$.

(11) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $-f = (-1)\,f$.

(12) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and r be a real number. If f is finite, then $r\,f$ is finite.

(13) Let given f, g, A. Suppose f is finite and g is finite and f is measurable on A and g is measurable on A and $A \subseteq \operatorname{dom} g$. Then $f - g$ is measurable on A.

3. Definitions of Extended Real Valued Functions $\max_+(f)$ and $\max_-(f)$ and their Basic Properties

Let C be a non empty set and let f be a partial function from C to $\overline{\mathbb{R}}$. The functor $\max_+(f)$ yields a partial function from C to $\overline{\mathbb{R}}$ and is defined as follows:

(Def. 2) $\operatorname{dom} \max_+(f) = \operatorname{dom} f$ and for every element x of C such that $x \in \operatorname{dom} \max_+(f)$ holds $(\max_+(f))(x) = \max(f(x), 0_{\overline{\mathbb{R}}})$.

The functor $\max_-(f)$ yielding a partial function from C to $\overline{\mathbb{R}}$ is defined by:

(Def. 3) $\operatorname{dom}\max_-(f) = \operatorname{dom} f$ and for every element x of C such that $x \in \operatorname{dom}\max_-(f)$ holds $(\max_-(f))(x) = \max(-f(x), 0_{\overline{\mathbb{R}}})$.

The following propositions are true:

(14) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$, then $0_{\overline{\mathbb{R}}} \leqslant (\max_+(f))(x)$.

(15) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$, then $0_{\overline{\mathbb{R}}} \leqslant (\max_-(f))(x)$.

(16) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $\max_-(f) = \max_+(-f)$.

(17) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $0_{\overline{\mathbb{R}}} < (\max_+(f))(x)$, then $(\max_-(f))(x) = 0_{\overline{\mathbb{R}}}$.

(18) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $0_{\overline{\mathbb{R}}} < (\max_-(f))(x)$, then $(\max_+(f))(x) = 0_{\overline{\mathbb{R}}}$.

(19) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $\operatorname{dom} f = \operatorname{dom}(\max_+(f) - \max_-(f))$ and $\operatorname{dom} f = \operatorname{dom}(\max_+(f) + \max_-(f))$.

(20) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$, then $(\max_+(f))(x) = f(x)$ or $(\max_+(f))(x) = 0_{\overline{\mathbb{R}}}$ but $(\max_-(f))(x) = -f(x)$ or $(\max_-(f))(x) = 0_{\overline{\mathbb{R}}}$.

(21) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $(\max_+(f))(x) = f(x)$, then $(\max_-(f))(x) = 0_{\overline{\mathbb{R}}}$.

(22) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $(\max_+(f))(x) = 0_{\overline{\mathbb{R}}}$, then $(\max_-(f))(x) = -f(x)$.

(23) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $(\max_-(f))(x) = -f(x)$, then $(\max_+(f))(x) = 0_{\overline{\mathbb{R}}}$.

(24) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $(\max_-(f))(x) = 0_{\overline{\mathbb{R}}}$, then $(\max_+(f))(x) = f(x)$.

(25) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $f = \max_+(f) - \max_-(f)$.

(26) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $|f| = \max_+(f) + \max_-(f)$.

4. Measurability of $\max_+(f)$, $\max_-(f)$ and $|f|$

Next we state three propositions:

(27) If f is measurable on A, then $\max_+(f)$ is measurable on A.

(28) If f is measurable on A and $A \subseteq \operatorname{dom} f$, then $\max_-(f)$ is measurable on A.

(29) For all f, A such that f is measurable on A and $A \subseteq \operatorname{dom} f$ holds $|f|$ is measurable on A.

5. Definition and Measurability of Characteristic Function

One can prove the following proposition

(30) For all sets A, X holds $\operatorname{rng}(\chi_{A,X}) \subseteq \{0_{\overline{\mathbb{R}}}, \overline{1}\}$.

Let A, X be sets. Then $\chi_{A,X}$ is a partial function from X to $\overline{\mathbb{R}}$.

Next we state two propositions:

(31) $\chi_{A,X}$ is finite.

(32) $\chi_{A,X}$ is measurable on B.

6. Definition and Measurability of Simple Function

Let X be a set and let S be a σ-field of subsets of X. One can check that there exists a finite sequence of elements of S which is disjoint valued.

Let X be a set and let S be a σ-field of subsets of X. A finite sequence of separated subsets of S is a disjoint valued finite sequence of elements of S.

The following propositions are true:

(33) Suppose F is a finite sequence of separated subsets of S. Then there exists a sequence G of separated subsets of S such that $\bigcup \operatorname{rng} F = \bigcup \operatorname{rng} G$ and for every n such that $n \in \operatorname{dom} F$ holds $F(n) = G(n)$ and for every m such that $m \notin \operatorname{dom} F$ holds $G(m) = \emptyset$.

(34) If F is a finite sequence of separated subsets of S, then $\bigcup \operatorname{rng} F \in S$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let f be a partial function from X to $\overline{\mathbb{R}}$. We say that f is simple function in S if and only if the conditions (Def. 5) are satisfied.

(Def. 5)[1] (i) f is finite, and

(ii) there exists a finite sequence F of separated subsets of S such that $\operatorname{dom} f = \bigcup \operatorname{rng} F$ and for every natural number n and for all elements x, y of X such that $n \in \operatorname{dom} F$ and $x \in F(n)$ and $y \in F(n)$ holds $f(x) = f(y)$.

[1] The definition (Def. 4) has been removed.

One can prove the following propositions:

(35) If f is finite, then $\operatorname{rng} f$ is a subset of $\mathbb{R}$.

(36) Suppose F is a finite sequence of separated subsets of S. Let given n. Then $F \restriction \operatorname{Seg} n$ is a finite sequence of separated subsets of S.

(37) If f is simple function in S, then f is measurable on A.

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Some Properties of Cells and Arcs[1]

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The notation and terminology used in this paper are introduced in the following papers: [25], [2], [11], [26], [21], [12], [3], [5], [30], [7], [28], [6], [18], [22], [17], [24], [20], [23], [8], [10], [16], [1], [27], [9], [4], [15], [32], [19], [29], [31], [13], and [14].

For simplicity, we adopt the following convention: E denotes a compact non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$, C denotes a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$, G denotes a Go-board, i, j, m, n denote natural numbers, and p denotes a point of $\mathcal{E}^2_{\rm T}$.

Let us observe that every simple closed curve is non vertical and non horizontal.

Let T be a non empty topological space. Note that there exists a union of components of T which is non empty.

The following propositions are true:

(1) Let T be a non empty topological space and A be a non empty union of components of T. If A is connected, then A is a component of T.

(2) For every finite sequence f holds f is empty iff $\mathrm{Rev}(f)$ is empty.

(3) Let D be a non empty set, f be a finite sequence of elements of D, and given i, j. If $1 \leqslant i$ and $i \leqslant \mathrm{len}\, f$ and $1 \leqslant j$ and $j \leqslant \mathrm{len}\, f$, then $\mathrm{mid}(f, i, j)$ is non empty.

(4) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If $1 \leqslant \mathrm{len}\, f$ and $p \in \widetilde{\mathcal{L}}(f)$, then $(\downharpoonleft f, p)(1) = f(1)$.

(5) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$, then $(\downharpoonleft p, f)(\mathrm{len} \downharpoonleft p, f) = f(\mathrm{len}\, f)$.

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(6) For every simple closed curve P holds W-max $P \neq$ E-max P.

(7) Let D be a non empty set and f be a finite sequence of elements of D. If $1 \leqslant i$ and $i < \operatorname{len} f$, then $(\operatorname{mid}(f, i, \operatorname{len} f -' 1)) \frown \langle f_{\operatorname{len} f} \rangle = \operatorname{mid}(f, i, \operatorname{len} f)$.

(8) For all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \neq q$ and $\mathcal{L}(p, q)$ is vertical holds $\langle p, q \rangle$ is a special sequence.

(9) For all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \neq q$ and $\mathcal{L}(p, q)$ is horizontal holds $\langle p, q \rangle$ is a special sequence.

(10) Let p, q be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$ and v be a point of $\mathcal{E}^2_{\mathrm{T}}$. If p is in the area of q, then $p^{v}_{\circlearrowleft}$ is in the area of q.

(11) For every non trivial finite sequence p of elements of $\mathcal{E}^2_{\mathrm{T}}$ and for every point v of $\mathcal{E}^2_{\mathrm{T}}$ holds $p^{v}_{\circlearrowleft}$ is in the area of p.

(12) For every finite sequence f holds Center $f \geqslant 1$.

(13) For every finite sequence f such that $\operatorname{len} f \geqslant 1$ holds Center $f \leqslant \operatorname{len} f$.

(14) Center $G \leqslant \operatorname{len} G$.

(15) For every finite sequence f such that $\operatorname{len} f \geqslant 2$ holds Center $f > 1$.

(16) For every finite sequence f such that $\operatorname{len} f \geqslant 3$ holds Center $f < \operatorname{len} f$.

(17) Center Gauge$(E, n) = 2^{n -' 1} + 2$.

(18) $E \subseteq \operatorname{cell}(\operatorname{Gauge}(E, 0), 2, 2)$.

(19) $\operatorname{cell}(\operatorname{Gauge}(E, 0), 2, 2) \not\subseteq$ BDD E.

(20) $(\operatorname{Gauge}(C, 1))_{\text{Center Gauge}(C,1),1} = [\frac{\text{W-bound}\, C + \text{E-bound}\, C}{2}, \text{S-bound}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, 1))]$.

(21) $(\operatorname{Gauge}(C, 1))_{\text{Center Gauge}(C,1), \operatorname{len} \operatorname{Gauge}(C,1)} = [\frac{\text{W-bound}\, C + \text{E-bound}\, C}{2}, \text{N-bound}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, 1))]$.

(22) If $1 \leqslant j$ and $j <$ width G and $1 \leqslant m$ and $m \leqslant \operatorname{len} G$ and $1 \leqslant n$ and $n \leqslant$ width G and $p \in \operatorname{cell}(G, \operatorname{len} G, j)$ and $p_{\mathbf{1}} = (G_{m,n})_{\mathbf{1}}$, then $\operatorname{len} G = m$.

(23) Suppose $1 \leqslant i$ and $i \leqslant \operatorname{len} G$ and $1 \leqslant j$ and $j <$ width G and $1 \leqslant m$ and $m \leqslant \operatorname{len} G$ and $1 \leqslant n$ and $n \leqslant$ width G and $p \in \operatorname{cell}(G, i, j)$ and $p_{\mathbf{1}} = (G_{m,n})_{\mathbf{1}}$. Then $i = m$ or $i = m -' 1$.

(24) If $1 \leqslant i$ and $i < \operatorname{len} G$ and $1 \leqslant m$ and $m \leqslant \operatorname{len} G$ and $1 \leqslant n$ and $n \leqslant$ width G and $p \in \operatorname{cell}(G, i, \text{width}\, G)$ and $p_{\mathbf{2}} = (G_{m,n})_{\mathbf{2}}$, then width $G = n$.

(25) Suppose $1 \leqslant i$ and $i < \operatorname{len} G$ and $1 \leqslant j$ and $j \leqslant$ width G and $1 \leqslant m$ and $m \leqslant \operatorname{len} G$ and $1 \leqslant n$ and $n \leqslant$ width G and $p \in \operatorname{cell}(G, i, j)$ and $p_{\mathbf{2}} = (G_{m,n})_{\mathbf{2}}$. Then $j = n$ or $j = n -' 1$.

(26) For every simple closed curve C and for every real number r such that W-bound $C \leqslant r$ and $r \leqslant$ E-bound C holds $\mathcal{L}([r, \text{S-bound}\, C], [r, \text{N-bound}\, C])$ meets UpperArc C.

(27) For every simple closed curve C and for every real number r such that W-bound $C \leqslant r$ and $r \leqslant$ E-bound C holds $\mathcal{L}([r, \text{S-bound}\, C], [r,$

N-bound C]) meets LowerArc C.

(28) Let C be a simple closed curve and i be a natural number. If $1 < i$ and $i < \mathrm{len}\,\mathrm{Gauge}(C,n)$, then $\mathcal{L}((\mathrm{Gauge}(C,n))_{i,1}, (\mathrm{Gauge}(C,n))_{i,\mathrm{len}\,\mathrm{Gauge}(C,n)})$ meets UpperArc C.

(29) Let C be a simple closed curve and i be a natural number. If $1 < i$ and $i < \mathrm{len}\,\mathrm{Gauge}(C,n)$, then $\mathcal{L}((\mathrm{Gauge}(C,n))_{i,1}, (\mathrm{Gauge}(C,n))_{i,\mathrm{len}\,\mathrm{Gauge}(C,n)})$ meets LowerArc C.

(30) For every simple closed curve C holds $\mathcal{L}((\mathrm{Gauge}(C,n))_{\mathrm{Center}\,\mathrm{Gauge}(C,n),1}, (\mathrm{Gauge}(C,n))_{\mathrm{Center}\,\mathrm{Gauge}(C,n),\mathrm{len}\,\mathrm{Gauge}(C,n)})$ meets UpperArc C.

(31) For every simple closed curve C holds $\mathcal{L}((\mathrm{Gauge}(C,n))_{\mathrm{Center}\,\mathrm{Gauge}(C,n),1}, (\mathrm{Gauge}(C,n))_{\mathrm{Center}\,\mathrm{Gauge}(C,n),\mathrm{len}\,\mathrm{Gauge}(C,n)})$ meets LowerArc C.

(32) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i be a natural number. If $1 \leqslant i$ and $i \leqslant \mathrm{len}\,\mathrm{Gauge}(C,n)$, then $\mathcal{L}((\mathrm{Gauge}(C,n))_{i,1}, (\mathrm{Gauge}(C,n))_{i,\mathrm{len}\,\mathrm{Gauge}(C,n)})$ meets UpperArc $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$.

(33) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i be a natural number. If $1 \leqslant i$ and $i \leqslant \mathrm{len}\,\mathrm{Gauge}(C,n)$, then $\mathcal{L}((\mathrm{Gauge}(C,n))_{i,1}, (\mathrm{Gauge}(C,n))_{i,\mathrm{len}\,\mathrm{Gauge}(C,n)})$ meets LowerArc $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$.

(34) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathcal{L}((\mathrm{Gauge}(C,n))_{\mathrm{Center}\,\mathrm{Gauge}(C,n),1}, (\mathrm{Gauge}(C,n))_{\mathrm{Center}\,\mathrm{Gauge}(C,n),\mathrm{len}\,\mathrm{Gauge}(C,n)})$ meets UpperArc $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$.

(35) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathcal{L}((\mathrm{Gauge}(C,n))_{\mathrm{Center}\,\mathrm{Gauge}(C,n),1}, (\mathrm{Gauge}(C,n))_{\mathrm{Center}\,\mathrm{Gauge}(C,n),\mathrm{len}\,\mathrm{Gauge}(C,n)})$ meets LowerArc $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$.

(36) If $j \leqslant \mathrm{width}\,G$, then $\mathrm{cell}(G,0,j)$ is not Bounded.

(37) If $i \leqslant \mathrm{width}\,G$, then $\mathrm{cell}(G,\mathrm{len}\,G,i)$ is not Bounded.

(38) If $j \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$, then $\mathrm{cell}(\mathrm{Gauge}(C,n),0,j) \subseteq \mathrm{UBD}\,C$.

(39) If $j \leqslant \mathrm{len}\,\mathrm{Gauge}(E,n)$, then $\mathrm{cell}(\mathrm{Gauge}(E,n),\mathrm{len}\,\mathrm{Gauge}(E,n),j) \subseteq \mathrm{UBD}\,E$.

(40) If $i \leqslant \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $j \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $\mathrm{cell}(\mathrm{Gauge}(C,n),i,j) \subseteq \mathrm{BDD}\,C$, then $j > 1$.

(41) If $i \leqslant \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $j \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $\mathrm{cell}(\mathrm{Gauge}(C,n),i,j) \subseteq \mathrm{BDD}\,C$, then $i > 1$.

(42) If $i \leqslant \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $j \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $\mathrm{cell}(\mathrm{Gauge}(C,n),i,j) \subseteq \mathrm{BDD}\,C$, then $j+1 < \mathrm{width}\,\mathrm{Gauge}(C,n)$.

(43) If $i \leqslant \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $j \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $\mathrm{cell}(\mathrm{Gauge}(C,n),i,j) \subseteq \mathrm{BDD}\,C$, then $i+1 < \mathrm{len}\,\mathrm{Gauge}(C,n)$.

(44) If there exist i, j such that $i \leqslant \operatorname{len}\operatorname{Gauge}(C,n)$ and $j \leqslant \operatorname{width}\operatorname{Gauge}(C,n)$ and $\operatorname{cell}(\operatorname{Gauge}(C,n),i,j) \subseteq \operatorname{BDD} C$, then $n \geqslant 1$.

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Formal Topological Spaces

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Summary. This article is divided into two parts. In the first part, we prove some useful theorems on finite topological spaces. In the second part, in order to consider a family of neighborhoods to a point in a space, we extend the notion of finite topological space and define a new topological space, which we call formal topological space. We show the relation between formal topological space struct (`FMT_Space_Str`) and the `TopStruct` by giving a function (`NeighSp`). And the following notions are introduced in formal topological spaces: boundary, closure, interior, isolated point, connected point, open set and close set, then some basic facts concerning them are proved. We will discuss the relation between the formal topological space and the finite topological space in future work.

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The papers [5], [3], [2], [1], [6], and [4] provide the notation and terminology for this paper.

1. Some Useful Theorems on Finite Topological Spaces

In this paper F_1 denotes a non empty finite topology space and A denotes a subset of the carrier of F_1.

The following propositions are true:

(1) Let F_1 be a non empty finite topology space and A, B be subsets of the carrier of F_1. If $A \subseteq B$, then $A^i \subseteq B^i$.

(2) $A^\delta = A^b \cap (A^i)^c$.

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(3) $A^\delta = A^b \setminus A^i$.

(4) If A^c is open, then A is closed.

(5) If A^c is closed, then A is open.

Let F_1 be a non empty finite topology space, let x be an element of the carrier of F_1, let y be an element of the carrier of F_1, and let A be a subset of the carrier of F_1. The functor $\mathrm{P}_1(x, y, A)$ yields an element of *Boolean* and is defined by:

(Def. 1) $\mathrm{P}_1(x, y, A) = \begin{cases} \mathit{true}, & \text{if } y \in U(x) \text{ and } y \in A, \\ \mathit{false}, & \text{otherwise.} \end{cases}$

Let F_1 be a non empty finite topology space, let x be an element of the carrier of F_1, let y be an element of the carrier of F_1, and let A be a subset of the carrier of F_1. The functor $\mathrm{P}_2(x, y, A)$ yielding an element of *Boolean* is defined as follows:

(Def. 2) $\mathrm{P}_2(x, y, A) = \begin{cases} \mathit{true}, & \text{if } y \in U(x) \text{ and } y \in A^c, \\ \mathit{false}, & \text{otherwise.} \end{cases}$

We now state three propositions:

(6) Let x, y be elements of the carrier of F_1 and A be a subset of the carrier of F_1. Then $\mathrm{P}_1(x, y, A) = \mathit{true}$ if and only if $y \in U(x)$ and $y \in A$.

(7) Let x, y be elements of the carrier of F_1 and A be a subset of the carrier of F_1. Then $\mathrm{P}_2(x, y, A) = \mathit{true}$ if and only if $y \in U(x)$ and $y \in A^c$.

(8) Let x be an element of the carrier of F_1 and A be a subset of the carrier of F_1. Then $x \in A^\delta$ if and only if there exist elements y_1, y_2 of the carrier of F_1 such that $\mathrm{P}_1(x, y_1, A) = \mathit{true}$ and $\mathrm{P}_2(x, y_2, A) = \mathit{true}$.

Let F_1 be a non empty finite topology space, let x be an element of the carrier of F_1, and let y be an element of the carrier of F_1. The functor $\mathrm{P}_0(x, y)$ yielding an element of *Boolean* is defined as follows:

(Def. 3) $\mathrm{P}_0(x, y) = \begin{cases} \mathit{true}, & \text{if } y \in U(x), \\ \mathit{false}, & \text{otherwise.} \end{cases}$

We now state three propositions:

(9) For all elements x, y of the carrier of F_1 holds $\mathrm{P}_0(x, y) = \mathit{true}$ iff $y \in U(x)$.

(10) Let x be an element of the carrier of F_1 and A be a subset of the carrier of F_1. Then $x \in A^i$ if and only if for every element y of the carrier of F_1 such that $\mathrm{P}_0(x, y) = \mathit{true}$ holds $\mathrm{P}_1(x, y, A) = \mathit{true}$.

(11) Let x be an element of the carrier of F_1 and A be a subset of the carrier of F_1. Then $x \in A^b$ if and only if there exists an element y_1 of the carrier of F_1 such that $\mathrm{P}_1(x, y_1, A) = \mathit{true}$.

Let F_1 be a non empty finite topology space, let x be an element of the carrier of F_1, and let A be a subset of the carrier of F_1. The functor $\mathrm{P_A}(x, A)$ yielding an element of *Boolean* is defined as follows:

(Def. 4) $\mathrm{P}_{\mathrm{A}}(x, A) = \begin{cases} \mathit{true}, & \text{if } x \in A, \\ \mathit{false}, & \text{otherwise.} \end{cases}$

One can prove the following three propositions:

(12) Let x be an element of the carrier of F_1 and A be a subset of the carrier of F_1. Then $\mathrm{P}_{\mathrm{A}}(x, A) = \mathit{true}$ if and only if $x \in A$.

(13) Let x be an element of the carrier of F_1 and A be a subset of the carrier of F_1. Then $x \in A^{\delta_i}$ if and only if the following conditions are satisfied:

(i) there exist elements y_1, y_2 of the carrier of F_1 such that $\mathrm{P}_1(x, y_1, A) = \mathit{true}$ and $\mathrm{P}_2(x, y_2, A) = \mathit{true}$, and

(ii) $\mathrm{P}_{\mathrm{A}}(x, A) = \mathit{true}$.

(14) Let x be an element of the carrier of F_1 and A be a subset of the carrier of F_1. Then $x \in A^{\delta_o}$ if and only if the following conditions are satisfied:

(i) there exist elements y_1, y_2 of the carrier of F_1 such that $\mathrm{P}_1(x, y_1, A) = \mathit{true}$ and $\mathrm{P}_2(x, y_2, A) = \mathit{true}$, and

(ii) $\mathrm{P}_{\mathrm{A}}(x, A) = \mathit{false}$.

Let F_1 be a non empty finite topology space, let x be an element of the carrier of F_1, and let y be an element of the carrier of F_1. The functor $\mathrm{P}_{\mathrm{e}}(x, y)$ yielding an element of *Boolean* is defined by:

(Def. 5) $\mathrm{P}_{\mathrm{e}}(x, y) = \begin{cases} \mathit{true}, & \text{if } x = y, \\ \mathit{false}, & \text{otherwise.} \end{cases}$

The following four propositions are true:

(15) For all elements x, y of the carrier of F_1 holds $\mathrm{P}_{\mathrm{e}}(x, y) = \mathit{true}$ iff $x = y$.

(16) Let x be an element of the carrier of F_1 and A be a subset of the carrier of F_1. Then $x \in A^{s}$ if and only if the following conditions are satisfied:

(i) $\mathrm{P}_{\mathrm{A}}(x, A) = \mathit{true}$, and

(ii) it is not true that there exists an element y of the carrier of F_1 such that $\mathrm{P}_1(x, y, A) = \mathit{true}$ and $\mathrm{P}_{\mathrm{e}}(x, y) = \mathit{false}$.

(17) Let x be an element of the carrier of F_1 and A be a subset of the carrier of F_1. Then $x \in A^{n}$ if and only if the following conditions are satisfied:

(i) $\mathrm{P}_{\mathrm{A}}(x, A) = \mathit{true}$, and

(ii) there exists an element y of the carrier of F_1 such that $\mathrm{P}_1(x, y, A) = \mathit{true}$ and $\mathrm{P}_{\mathrm{e}}(x, y) = \mathit{false}$.

(18) Let x be an element of the carrier of F_1 and A be a subset of the carrier of F_1. Then $x \in A^{f}$ if and only if there exists an element y of the carrier of F_1 such that $\mathrm{P}_{\mathrm{A}}(y, A) = \mathit{true}$ and $\mathrm{P}_0(y, x) = \mathit{true}$.

2. Formal Topological Spaces

We introduce formal topological spaces which are extensions of 1-sorted structure and are systems

⟨ a carrier, a Neighbour-map ⟩,

where the carrier is a set and the Neighbour-map is a function from the carrier into $2^{2^{\text{the carrier}}}$.

Let us observe that there exists a formal topological space which is non empty and strict.

In the sequel T is a non empty topological structure, F_2 is a non empty formal topological space, and x is an element of the carrier of F_2.

Let us consider F_2 and let x be an element of the carrier of F_2. The functor $U_F(x)$ yielding a subset of $2^{\text{the carrier of } F_2}$ is defined as follows:

(Def. 6) $U_F(x) = (\text{the Neighbour-map of } F_2)(x)$.

Next we state the proposition

(19) Let F_2 be a non empty formal topological space and x be an element of the carrier of F_2. Then $U_F(x) = (\text{the Neighbour-map of } F_2)(x)$.

Let us consider T. The functor $\mathrm{NeighSp}\,T$ yielding a non empty strict formal topological space is defined by the conditions (Def. 7).

(Def. 7)(i) The carrier of $\mathrm{NeighSp}\,T$ = the carrier of T, and

(ii) for every point x of $\mathrm{NeighSp}\,T$ holds $U_F(x) = \{V; V$ ranges over subsets of T: $V \in$ the topology of $T \ \wedge\ x \in V\}$.

In the sequel A, B, W, V denote subsets of the carrier of F_2.

Let I_1 be a non empty formal topological space. We say that I_1 is filled if and only if:

(Def. 8) For every element x of the carrier of I_1 and for every subset D of the carrier of I_1 such that $D \in U_F(x)$ holds $x \in D$.

Let us consider F_2 and let us consider A. The functor $A^{F\delta}$ yielding a subset of the carrier of F_2 is defined as follows:

(Def. 9) $A^{F\delta} = \{x : \bigwedge_W (W \in U_F(x) \ \Rightarrow\ W \cap A \neq \emptyset \ \wedge\ W \cap A^{\mathrm{c}} \neq \emptyset)\}$.

The following proposition is true

(20) $x \in A^{F\delta}$ iff for every W such that $W \in U_F(x)$ holds $W \cap A \neq \emptyset$ and $W \cap A^{\mathrm{c}} \neq \emptyset$.

Let us consider F_2 and let us consider A. The functor A^{F_b} yielding a subset of the carrier of F_2 is defined as follows:

(Def. 10) $A^{F_b} = \{x : \bigwedge_W (W \in U_F(x) \ \Rightarrow\ W \cap A \neq \emptyset)\}$.

One can prove the following proposition

(21) $x \in A^{F_b}$ iff for every W such that $W \in U_F(x)$ holds $W \cap A \neq \emptyset$.

Let us consider F_2 and let us consider A. The functor A^{F_i} yielding a subset of the carrier of F_2 is defined as follows:

(Def. 11) $A^{F_i} = \{x : \bigvee_V (V \in U_F(x) \wedge V \subseteq A)\}$.

Next we state the proposition

(22) $x \in A^{F_i}$ iff there exists V such that $V \in U_F(x)$ and $V \subseteq A$.

Let us consider F_2 and let us consider A. The functor A^{F_s} yields a subset of the carrier of F_2 and is defined by:

(Def. 12) $A^{F_s} = \{x : x \in A \wedge \bigvee_V (V \in U_F(x) \wedge (V \setminus \{x\}) \cap A = \emptyset)\}$.

One can prove the following proposition

(23) $x \in A^{F_s}$ iff $x \in A$ and there exists V such that $V \in U_F(x)$ and $(V \setminus \{x\}) \cap A = \emptyset$.

Let us consider F_2 and let us consider A. The functor $A^{F_{on}}$ yields a subset of the carrier of F_2 and is defined by:

(Def. 13) $A^{F_{on}} = A \setminus A^{F_s}$.

We now state a number of propositions:

(24) $x \in A^{F_{on}}$ iff $x \in A$ and for every V such that $V \in U_F(x)$ holds $(V \setminus \{x\}) \cap A \neq \emptyset$.

(25) Let F_2 be a non empty formal topological space and A, B be subsets of the carrier of F_2. If $A \subseteq B$, then $A^{F_b} \subseteq B^{F_b}$.

(26) Let F_2 be a non empty formal topological space and A, B be subsets of the carrier of F_2. If $A \subseteq B$, then $A^{F_i} \subseteq B^{F_i}$.

(27) $A^{F_b} \cup B^{F_b} \subseteq A \cup B^{F_b}$.

(28) $A \cap B^{F_b} \subseteq A^{F_b} \cap B^{F_b}$.

(29) $A^{F_i} \cup B^{F_i} \subseteq A \cup B^{F_i}$.

(30) $A \cap B^{F_i} \subseteq A^{F_i} \cap B^{F_i}$.

(31) Let F_2 be a non empty formal topological space. Then the following statements are equivalent

(i) for every element x of the carrier of F_2 and for all subsets V_1, V_2 of the carrier of F_2 such that $V_1 \in U_F(x)$ and $V_2 \in U_F(x)$ there exists a subset W of the carrier of F_2 such that $W \in U_F(x)$ and $W \subseteq V_1 \cap V_2$,

(ii) for all subsets A, B of the carrier of F_2 holds $A \cup B^{F_b} = A^{F_b} \cup B^{F_b}$.

(32) Let F_2 be a non empty formal topological space. Then the following statements are equivalent

(i) for every element x of the carrier of F_2 and for all subsets V_1, V_2 of the carrier of F_2 such that $V_1 \in U_F(x)$ and $V_2 \in U_F(x)$ there exists a subset W of the carrier of F_2 such that $W \in U_F(x)$ and $W \subseteq V_1 \cap V_2$,

(ii) for all subsets A, B of the carrier of F_2 holds $A^{F_i} \cap B^{F_i} = A \cap B^{F_i}$.

(33) Let F_2 be a non empty formal topological space and A, B be subsets of the carrier of F_2. Suppose that for every element x of the carrier of

F_2 and for all subsets V_1, V_2 of the carrier of F_2 such that $V_1 \in U_F(x)$ and $V_2 \in U_F(x)$ there exists a subset W of the carrier of F_2 such that $W \in U_F(x)$ and $W \subseteq V_1 \cap V_2$. Then $A \cup B^{F\delta} \subseteq A^{F\delta} \cup B^{F\delta}$.

(34) Let F_2 be a non empty formal topological space. Suppose F_2 is filled. Suppose that for all subsets A, B of the carrier of F_2 holds $A \cup B^{F\delta} = A^{F\delta} \cup B^{F\delta}$. Let x be an element of the carrier of F_2 and V_1, V_2 be subsets of the carrier of F_2. Suppose $V_1 \in U_F(x)$ and $V_2 \in U_F(x)$. Then there exists a subset W of the carrier of F_2 such that $W \in U_F(x)$ and $W \subseteq V_1 \cap V_2$.

(35) Let x be an element of the carrier of F_2 and A be a subset of the carrier of F_2. Then $x \in A^{F_s}$ if and only if the following conditions are satisfied:

(i) $x \in A$, and

(ii) $x \notin A \setminus \{x\}^{F_b}$.

(36) Let F_2 be a non empty formal topological space. Then F_2 is filled if and only if for every subset A of the carrier of F_2 holds $A \subseteq A^{F_b}$.

(37) Let F_2 be a non empty formal topological space. Then F_2 is filled if and only if for every subset A of the carrier of F_2 holds $A^{F_i} \subseteq A$.

(38) $(A^{\mathrm{c}F_b})^{\mathrm{c}} = A^{F_i}$.

(39) $(A^{\mathrm{c}F_i})^{\mathrm{c}} = A^{F_b}$.

(40) $A^{F\delta} = A^{F_b} \cap A^{\mathrm{c}F_b}$.

(41) $A^{F\delta} = A^{F_b} \cap (A^{F_i})^{\mathrm{c}}$.

(42) $A^{F\delta} = A^{\mathrm{c}F\delta}$.

(43) $A^{F\delta} = A^{F_b} \setminus A^{F_i}$.

Let us consider F_2 and let us consider A. The functor $A^{F\delta_i}$ yields a subset of the carrier of F_2 and is defined by:

(Def. 14) $A^{F\delta_i} = A \cap A^{F\delta}$.

The functor $A^{F\delta_o}$ yields a subset of the carrier of F_2 and is defined by:

(Def. 15) $A^{F\delta_o} = A^{\mathrm{c}} \cap A^{F\delta}$.

The following proposition is true

(44) $A^{F\delta} = A^{F\delta_i} \cup A^{F\delta_o}$.

Let us consider F_2 and let G be a subset of the carrier of F_2. We say that G is open if and only if:

(Def. 16) $G = G^{F_i}$.

We say that G is closed if and only if:

(Def. 17) $G = G^{F_b}$.

Next we state four propositions:

(45) If A^{c} is open, then A is closed.

(46) If A^{c} is closed, then A is open.

(47) Let F_2 be a non empty formal topological space and A, B be subsets of the carrier of F_2. Suppose F_2 is filled. Suppose that for every element x of the carrier of F_2 holds $\{x\} \in U_F(x)$. Then $A \cap B^{F_b} = A^{F_b} \cap B^{F_b}$.

(48) Let F_2 be a non empty formal topological space and A, B be subsets of the carrier of F_2. Suppose F_2 is filled. Suppose that for every element x of the carrier of F_2 holds $\{x\} \in U_F(x)$. Then $A^{F_i} \cup B^{F_i} = A \cup B^{F_i}$.

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Some Properties of Cells and Gauges[1]

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The terminology and notation used in this paper are introduced in the following articles: [20], [25], [2], [7], [18], [21], [8], [3], [4], [16], [13], [23], [14], [17], [5], [11], [12], [1], [19], [6], [10], [15], [22], [24], and [9].

We adopt the following convention: C denotes a simple closed curve, i, j, n denote natural numbers, and p denotes a point of $\mathcal{E}^2_{\mathrm{T}}$.

The following propositions are true:

(1) $\mathrm{BDD}\,C$ is Bounded.

(2) If $\langle i, j\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$ and $\langle i+1, j\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, then $\rho((\mathrm{Gauge}(C,n))_{1,1}, (\mathrm{Gauge}(C,n))_{2,1}) = |((\mathrm{Gauge}(C,n))_{i+1,j})_{\mathbf{1}} - ((\mathrm{Gauge}(C,n))_{i,j})_{\mathbf{1}}|$.

(3) If $\langle i, j\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$ and $\langle i, j+1\rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, then $\rho((\mathrm{Gauge}(C,n))_{1,1}, (\mathrm{Gauge}(C,n))_{1,2}) = |((\mathrm{Gauge}(C,n))_{i,j+1})_{\mathbf{2}} - ((\mathrm{Gauge}(C,n))_{i,j})_{\mathbf{2}}|$.

(4) For every subset S of $\mathcal{E}^2_{\mathrm{T}}$ such that S is Bounded holds $(\mathrm{proj1})^\circ S$ is bounded.

(5) Let C_1 be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$ and C_2, S be non empty subsets of $\mathcal{E}^2_{\mathrm{T}}$. If $S = C_1 \cup C_2$ and $(\mathrm{proj1})^\circ C_2$ is non empty and lower bounded, then $\mathrm{W\text{-}bound}\,S = \min(\mathrm{W\text{-}bound}\,C_1, \mathrm{W\text{-}bound}\,C_2)$.

(6) For every subset X of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in X$ and X is Bounded holds $\mathrm{W\text{-}bound}\,X \leqslant p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leqslant \mathrm{E\text{-}bound}\,X$ and $\mathrm{S\text{-}bound}\,X \leqslant p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leqslant \mathrm{N\text{-}bound}\,X$.

(7) $p \in \mathrm{WestHalfline}\,p$ and $p \in \mathrm{EastHalfline}\,p$.

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(8) WestHalfline p is non Bounded.

(9) EastHalfline p is non Bounded.

(10) NorthHalfline p is non Bounded.

(11) SouthHalfline p is non Bounded.

(12) If $\mathrm{UBD}\, C \neq \emptyset$, then $\mathrm{UBD}\, C$ is a component of C^{c}.

(13) For every connected subset W_1 of $\mathcal{E}^2_{\mathrm{T}}$ such that W_1 is non Bounded and $W_1 \cap C = \emptyset$ holds $W_1 \subseteq \mathrm{UBD}\, C$.

(14) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{WestHalfline}\, p \cap C = \emptyset$ holds $\mathrm{WestHalfline}\, p \subseteq \mathrm{UBD}\, C$.

(15) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{EastHalfline}\, p \cap C = \emptyset$ holds $\mathrm{EastHalfline}\, p \subseteq \mathrm{UBD}\, C$.

(16) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{SouthHalfline}\, p \cap C = \emptyset$ holds $\mathrm{SouthHalfline}\, p \subseteq \mathrm{UBD}\, C$.

(17) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{NorthHalfline}\, p \cap C = \emptyset$ holds $\mathrm{NorthHalfline}\, p \subseteq \mathrm{UBD}\, C$.

(18) If $\mathrm{BDD}\, C \neq \emptyset$, then $\text{W-bound}\, C \leqslant \text{W-bound}\, \mathrm{BDD}\, C$.

(19) If $\mathrm{BDD}\, C \neq \emptyset$, then $\text{E-bound}\, C \geqslant \text{E-bound}\, \mathrm{BDD}\, C$.

(20) If $\mathrm{BDD}\, C \neq \emptyset$, then $\text{S-bound}\, C \leqslant \text{S-bound}\, \mathrm{BDD}\, C$.

(21) If $\mathrm{BDD}\, C \neq \emptyset$, then $\text{N-bound}\, C \geqslant \text{N-bound}\, \mathrm{BDD}\, C$.

(22) For every integer I such that $p \in \mathrm{BDD}\, C$ and $I = \lfloor \frac{p_{\mathbf{1}} - \text{W-bound}\, C}{\text{E-bound}\, C - \text{W-bound}\, C} \cdot 2^n + 2 \rfloor$ holds $1 < I$.

(23) For every integer I such that $p \in \mathrm{BDD}\, C$ and $I = \lfloor \frac{p_{\mathbf{1}} - \text{W-bound}\, C}{\text{E-bound}\, C - \text{W-bound}\, C} \cdot 2^n + 2 \rfloor$ holds $I + 1 \leqslant \mathrm{len}\, \mathrm{Gauge}(C, n)$.

(24) For every integer J such that $p \in \mathrm{BDD}\, C$ and $J = \lfloor \frac{p_{\mathbf{2}} - \text{S-bound}\, C}{\text{N-bound}\, C - \text{S-bound}\, C} \cdot 2^n + 2 \rfloor$ holds $1 < J$ and $J + 1 \leqslant \mathrm{width}\, \mathrm{Gauge}(C, n)$.

(25) For every integer I such that $I = \lfloor \frac{p_{\mathbf{1}} - \text{W-bound}\, C}{\text{E-bound}\, C - \text{W-bound}\, C} \cdot 2^n + 2 \rfloor$ holds $\text{W-bound}\, C + \frac{\text{E-bound}\, C - \text{W-bound}\, C}{2^n} \cdot (I - 2) \leqslant p_{\mathbf{1}}$.

(26) For every integer I such that $I = \lfloor \frac{p_{\mathbf{1}} - \text{W-bound}\, C}{\text{E-bound}\, C - \text{W-bound}\, C} \cdot 2^n + 2 \rfloor$ holds $p_{\mathbf{1}} < \text{W-bound}\, C + \frac{\text{E-bound}\, C - \text{W-bound}\, C}{2^n} \cdot (I - 1)$.

(27) For every integer J such that $J = \lfloor \frac{p_{\mathbf{2}} - \text{S-bound}\, C}{\text{N-bound}\, C - \text{S-bound}\, C} \cdot 2^n + 2 \rfloor$ holds $\text{S-bound}\, C + \frac{\text{N-bound}\, C - \text{S-bound}\, C}{2^n} \cdot (J - 2) \leqslant p_{\mathbf{2}}$.

(28) For every integer J such that $J = \lfloor \frac{p_{\mathbf{2}} - \text{S-bound}\, C}{\text{N-bound}\, C - \text{S-bound}\, C} \cdot 2^n + 2 \rfloor$ holds $p_{\mathbf{2}} < \text{S-bound}\, C + \frac{\text{N-bound}\, C - \text{S-bound}\, C}{2^n} \cdot (J - 1)$.

(29) Let C be a closed subset of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2$. If $p \in \mathrm{BDD}\, C$, then there exists a real number r such that $r > 0$ and $\mathrm{Ball}(p, r) \subseteq \mathrm{BDD}\, C$.

(30) Let p, q be points of $\mathcal{E}^2_{\mathrm{T}}$ and r be a real number. Suppose $\rho((\mathrm{Gauge}(C, n))_{1,1}, (\mathrm{Gauge}(C, n))_{1,2}) < r$ and $\rho((\mathrm{Gauge}(C, n))_{1,1},$

$(\mathrm{Gauge}(C,n))_{2,1}) < r$ and $p \in \mathrm{cell}(\mathrm{Gauge}(C,n),i,j)$ and $q \in \mathrm{cell}(\mathrm{Gauge}(C,n),i,j)$ and $1 \leqslant i$ and $i+1 \leqslant \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 \leqslant j$ and $j+1 \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$. Then $\rho(p,q) < 2 \cdot r$.

(31) If $p \in \mathrm{BDD}\,C$, then $p_{\mathbf{2}} \neq \text{N-bound}\,\mathrm{BDD}\,C$.

(32) If $p \in \mathrm{BDD}\,C$, then $p_{\mathbf{1}} \neq \text{E-bound}\,\mathrm{BDD}\,C$.

(33) If $p \in \mathrm{BDD}\,C$, then $p_{\mathbf{2}} \neq \text{S-bound}\,\mathrm{BDD}\,C$.

(34) If $p \in \mathrm{BDD}\,C$, then $p_{\mathbf{1}} \neq \text{W-bound}\,\mathrm{BDD}\,C$.

(35) Suppose $p \in \mathrm{BDD}\,C$. Then there exist natural numbers n, i, j such that $1 < i$ and $i < \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 < j$ and $j < \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $p_{\mathbf{1}} \neq ((\mathrm{Gauge}(C,n))_{i,j})_{\mathbf{1}}$ and $p \in \mathrm{cell}(\mathrm{Gauge}(C,n),i,j)$ and $\mathrm{cell}(\mathrm{Gauge}(C,n),i,j) \subseteq \mathrm{BDD}\,C$.

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Again on the Order on a Special Polygon[1]

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The terminology and notation used in this paper have been introduced in the following articles: [6], [2], [14], [4], [12], [3], [11], [1], [5], [8], [16], [10], [9], [13], [15], and [7].

1. Preliminaries

For simplicity, we use the following convention: D denotes a non empty set, f denotes a finite sequence of elements of D, g denotes a circular finite sequence of elements of D, and p, p_1, p_2, p_3, q denote elements of D.

We now state several propositions:

(1) If $q \in \mathrm{rng}(f{\restriction}p \looparrowleft f)$, then $q \looparrowleft f \leqslant p \looparrowleft f$.

(2) If $p \in \mathrm{rng}\, f$ and $q \in \mathrm{rng}\, f$ and $p \looparrowleft f \leqslant q \looparrowleft f$, then $q \looparrowleft (f :- p) = (q \looparrowleft f - p \looparrowleft f) + 1$.

(3) If $p \in \mathrm{rng}\, f$ and $q \in \mathrm{rng}\, f$ and $p \looparrowleft f < q \looparrowleft f$, then $p \looparrowleft (f -: q) = p \looparrowleft f$.

(4) If $p \in \mathrm{rng}\, f$ and $q \in \mathrm{rng}\, f$ and $p \looparrowleft f \leqslant q \looparrowleft f$, then $q \looparrowleft (f^{p}_{\circlearrowleft}) = (q \looparrowleft f - p \looparrowleft f) + 1$.

(5) If $p_1 \in \mathrm{rng}\, f$ and $p_2 \in \mathrm{rng}\, f$ and $p_3 \in \mathrm{rng}\, f$ and $p_1 \looparrowleft f \leqslant p_2 \looparrowleft f$ and $p_2 \looparrowleft f < p_3 \looparrowleft f$, then $p_2 \looparrowleft (f^{p_1}_{\circlearrowleft}) < p_3 \looparrowleft (f^{p_1}_{\circlearrowleft})$.

(6) If $p_1 \in \mathrm{rng}\, f$ and $p_2 \in \mathrm{rng}\, f$ and $p_3 \in \mathrm{rng}\, f$ and $p_1 \looparrowleft f < p_2 \looparrowleft f$ and $p_2 \looparrowleft f \leqslant p_3 \looparrowleft f$, then $p_2 \looparrowleft (f^{p_1}_{\circlearrowleft}) \leqslant p_3 \looparrowleft (f^{p_1}_{\circlearrowleft})$.

(7) If $p \in \mathrm{rng}\, g$ and $\mathrm{len}\, g > 1$, then $p \looparrowleft g < \mathrm{len}\, g$.

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2. Ordering of Special Points on a Standard Special Sequence

We adopt the following rules: f denotes a non constant standard special circular sequence and p, p_1, p_2, p_3, q denote points of $\mathcal{E}^2_{\mathrm{T}}$.

The following propositions are true:

(8) $f_{\restriction 1}$ is one-to-one.

(9) If $1 < q \leftarrow\!\!\!\!\hookleftarrow f$ and $q \in \operatorname{rng} f$, then $(\pi_1 f) \leftarrow\!\!\!\!\hookleftarrow (f^{q}_{\circlearrowleft}) = (\operatorname{len} f + 1) - q \leftarrow\!\!\!\!\hookleftarrow f$.

(10) If $p \in \operatorname{rng} f$ and $q \in \operatorname{rng} f$ and $p \leftarrow\!\!\!\!\hookleftarrow f < q \leftarrow\!\!\!\!\hookleftarrow f$, then $p \leftarrow\!\!\!\!\hookleftarrow (f^{q}_{\circlearrowleft}) = (\operatorname{len} f + p \leftarrow\!\!\!\!\hookleftarrow f) - q \leftarrow\!\!\!\!\hookleftarrow f$.

(11) If $p_1 \in \operatorname{rng} f$ and $p_2 \in \operatorname{rng} f$ and $p_3 \in \operatorname{rng} f$ and $p_1 \leftarrow\!\!\!\!\hookleftarrow f < p_2 \leftarrow\!\!\!\!\hookleftarrow f$ and $p_2 \leftarrow\!\!\!\!\hookleftarrow f < p_3 \leftarrow\!\!\!\!\hookleftarrow f$, then $p_3 \leftarrow\!\!\!\!\hookleftarrow (f^{p_2}_{\circlearrowleft}) < p_1 \leftarrow\!\!\!\!\hookleftarrow (f^{p_2}_{\circlearrowleft})$.

(12) If $p_1 \in \operatorname{rng} f$ and $p_2 \in \operatorname{rng} f$ and $p_3 \in \operatorname{rng} f$ and $p_1 \leftarrow\!\!\!\!\hookleftarrow f < p_2 \leftarrow\!\!\!\!\hookleftarrow f$ and $p_2 \leftarrow\!\!\!\!\hookleftarrow f < p_3 \leftarrow\!\!\!\!\hookleftarrow f$, then $p_1 \leftarrow\!\!\!\!\hookleftarrow (f^{p_3}_{\circlearrowleft}) < p_2 \leftarrow\!\!\!\!\hookleftarrow (f^{p_3}_{\circlearrowleft})$.

(13) If $p_1 \in \operatorname{rng} f$ and $p_2 \in \operatorname{rng} f$ and $p_3 \in \operatorname{rng} f$ and $p_1 \leftarrow\!\!\!\!\hookleftarrow f \leqslant p_2 \leftarrow\!\!\!\!\hookleftarrow f$ and $p_2 \leftarrow\!\!\!\!\hookleftarrow f < p_3 \leftarrow\!\!\!\!\hookleftarrow f$, then $p_1 \leftarrow\!\!\!\!\hookleftarrow (f^{p_3}_{\circlearrowleft}) \leqslant p_2 \leftarrow\!\!\!\!\hookleftarrow (f^{p_3}_{\circlearrowleft})$.

(14) $(\text{S-min}\,\widetilde{\mathcal{L}}(f)) \leftarrow\!\!\!\!\hookleftarrow f < \operatorname{len} f$.

(15) $(\text{S-max}\,\widetilde{\mathcal{L}}(f)) \leftarrow\!\!\!\!\hookleftarrow f < \operatorname{len} f$.

(16) $(\text{E-min}\,\widetilde{\mathcal{L}}(f)) \leftarrow\!\!\!\!\hookleftarrow f < \operatorname{len} f$.

(17) $(\text{E-max}\,\widetilde{\mathcal{L}}(f)) \leftarrow\!\!\!\!\hookleftarrow f < \operatorname{len} f$.

(18) $(\text{N-min}\,\widetilde{\mathcal{L}}(f)) \leftarrow\!\!\!\!\hookleftarrow f < \operatorname{len} f$.

(19) $(\text{N-max}\,\widetilde{\mathcal{L}}(f)) \leftarrow\!\!\!\!\hookleftarrow f < \operatorname{len} f$.

(20) $(\text{W-max}\,\widetilde{\mathcal{L}}(f)) \leftarrow\!\!\!\!\hookleftarrow f < \operatorname{len} f$.

(21) $(\text{W-min}\,\widetilde{\mathcal{L}}(f)) \leftarrow\!\!\!\!\hookleftarrow f < \operatorname{len} f$.

3. Ordering of Special Points on a Clockwise Oriented Sequence

In the sequel z is a clockwise oriented non constant standard special circular sequence.

Next we state a number of propositions:

(22) If $\pi_1 f = \text{W-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{W-min}\,\widetilde{\mathcal{L}}(f)) \leftarrow\!\!\!\!\hookleftarrow f < (\text{W-max}\,\widetilde{\mathcal{L}}(f)) \leftarrow\!\!\!\!\hookleftarrow f$.

(23) If $\pi_1 f = \text{W-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{W-max}\,\widetilde{\mathcal{L}}(f)) \leftarrow\!\!\!\!\hookleftarrow f > 1$.

(24) If $\pi_1 z = \text{W-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{W-max}\,\widetilde{\mathcal{L}}(z) \neq \text{N-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\hookleftarrow z < (\text{N-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\hookleftarrow z$.

(25) If $\pi_1 z = \text{W-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\hookleftarrow z < (\text{N-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\hookleftarrow z$.

(26) If $\pi_1 z = \text{W-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{N-max}\,\widetilde{\mathcal{L}}(z) \neq \text{E-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\hookleftarrow z < (\text{E-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\hookleftarrow z$.

(27) If $\pi_1 z = \text{W-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{E-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(28) If $\pi_1 z = \text{W-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{E-min}\,\widetilde{\mathcal{L}}(z) \neq \text{S-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{S-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(29) If $\pi_1 z = \text{W-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{S-min}\,\widetilde{\mathcal{L}}(z) \neq \text{W-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{S-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(30) If $\pi_1 f = \text{S-max}\,\widetilde{\mathcal{L}}(f)$, then $(\text{S-max}\,\widetilde{\mathcal{L}}(f)) \looparrowleft f < (\text{S-min}\,\widetilde{\mathcal{L}}(f)) \looparrowleft f$.

(31) If $\pi_1 f = \text{S-max}\,\widetilde{\mathcal{L}}(f)$, then $(\text{S-min}\,\widetilde{\mathcal{L}}(f)) \looparrowleft f > 1$.

(32) If $\pi_1 z = \text{S-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{S-min}\,\widetilde{\mathcal{L}}(z) \neq \text{W-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{W-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(33) If $\pi_1 z = \text{S-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{W-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(34) If $\pi_1 z = \text{S-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{W-max}\,\widetilde{\mathcal{L}}(z) \neq \text{N-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{N-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(35) If $\pi_1 z = \text{S-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{N-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(36) If $\pi_1 z = \text{S-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{N-max}\,\widetilde{\mathcal{L}}(z) \neq \text{E-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{E-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(37) If $\pi_1 z = \text{S-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{E-min}\,\widetilde{\mathcal{L}}(z) \neq \text{S-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{E-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(38) If $\pi_1 f = \text{E-max}\,\widetilde{\mathcal{L}}(f)$, then $(\text{E-max}\,\widetilde{\mathcal{L}}(f)) \looparrowleft f < (\text{E-min}\,\widetilde{\mathcal{L}}(f)) \looparrowleft f$.

(39) If $\pi_1 f = \text{E-max}\,\widetilde{\mathcal{L}}(f)$, then $(\text{E-min}\,\widetilde{\mathcal{L}}(f)) \looparrowleft f > 1$.

(40) If $\pi_1 z = \text{E-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{S-max}\,\widetilde{\mathcal{L}}(z) \neq \text{E-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{S-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(41) If $\pi_1 z = \text{E-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{S-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(42) If $\pi_1 z = \text{E-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{S-min}\,\widetilde{\mathcal{L}}(z) \neq \text{W-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{W-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(43) If $\pi_1 z = \text{E-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{W-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(44) If $\pi_1 z = \text{E-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{W-max}\,\widetilde{\mathcal{L}}(z) \neq \text{N-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{N-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(45) If $\pi_1 z = \text{E-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{N-max}\,\widetilde{\mathcal{L}}(z) \neq \text{E-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{N-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(46) If $\pi_1 f = \text{N-max}\,\widetilde{\mathcal{L}}(f)$ and $\text{N-max}\,\widetilde{\mathcal{L}}(f) \neq \text{E-max}\,\widetilde{\mathcal{L}}(f)$, then $(\text{N-max}\,\widetilde{\mathcal{L}}(f)) \looparrowleft f < (\text{E-max}\,\widetilde{\mathcal{L}}(f)) \looparrowleft f$.

(47) If $\pi_1 z = \text{N-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{E-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(48) If $\pi_1 z = \text{N-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{E-min}\,\widetilde{\mathcal{L}}(z) \neq \text{S-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{S-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(49) If $\pi_1 z = \text{N-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{S-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(50) If $\pi_1 z = \text{N-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{S-min}\,\widetilde{\mathcal{L}}(z) \neq \text{W-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{W-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(51) If $\pi_1 z = \text{N-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{W-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(52) If $\pi_1 z = \text{N-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{N-min}\,\widetilde{\mathcal{L}}(z) \neq \text{W-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z < (\text{N-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z$.

(53) If $\pi_1 f = \text{E-min}\,\widetilde{\mathcal{L}}(f)$ and $\text{E-min}\,\widetilde{\mathcal{L}}(f) \neq \text{S-max}\,\widetilde{\mathcal{L}}(f)$, then $(\text{E-min}\,\widetilde{\mathcal{L}}(f)) \leftarrow\!\!\!\!\!\mapsto f < (\text{S-max}\,\widetilde{\mathcal{L}}(f)) \leftarrow\!\!\!\!\!\mapsto f$.

(54) If $\pi_1 z = \text{E-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z < (\text{S-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z$.

(55) If $\pi_1 z = \text{E-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{S-min}\,\widetilde{\mathcal{L}}(z) \neq \text{W-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z < (\text{W-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z$.

(56) If $\pi_1 z = \text{E-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z < (\text{W-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z$.

(57) If $\pi_1 z = \text{E-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{W-max}\,\widetilde{\mathcal{L}}(z) \neq \text{N-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z < (\text{N-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z$.

(58) If $\pi_1 z = \text{E-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z < (\text{N-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z$.

(59) If $\pi_1 z = \text{E-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{E-max}\,\widetilde{\mathcal{L}}(z) \neq \text{N-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z < (\text{E-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z$.

(60) If $\pi_1 f = \text{S-min}\,\widetilde{\mathcal{L}}(f)$ and $\text{S-min}\,\widetilde{\mathcal{L}}(f) \neq \text{W-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{S-min}\,\widetilde{\mathcal{L}}(f)) \leftarrow\!\!\!\!\!\mapsto f < (\text{W-min}\,\widetilde{\mathcal{L}}(f)) \leftarrow\!\!\!\!\!\mapsto f$.

(61) If $\pi_1 z = \text{S-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z < (\text{W-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z$.

(62) If $\pi_1 z = \text{S-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{W-max}\,\widetilde{\mathcal{L}}(z) \neq \text{N-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z < (\text{N-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z$.

(63) If $\pi_1 z = \text{S-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z < (\text{N-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z$.

(64) If $\pi_1 z = \text{S-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{N-max}\,\widetilde{\mathcal{L}}(z) \neq \text{E-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z < (\text{E-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z$.

(65) If $\pi_1 z = \text{S-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z < (\text{E-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z$.

(66) If $\pi_1 z = \text{S-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{S-max}\,\widetilde{\mathcal{L}}(z) \neq \text{E-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z < (\text{S-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z$.

(67) If $\pi_1 f = \text{W-max}\,\widetilde{\mathcal{L}}(f)$ and $\text{W-max}\,\widetilde{\mathcal{L}}(f) \neq \text{N-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{W-max}\,\widetilde{\mathcal{L}}(f)) \leftarrow\!\!\!\!\!\mapsto f < (\text{N-min}\,\widetilde{\mathcal{L}}(f)) \leftarrow\!\!\!\!\!\mapsto f$.

(68) If $\pi_1 z = \text{W-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z < (\text{N-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z$.

(69) If $\pi_1 z = \text{W-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{N-max}\,\widetilde{\mathcal{L}}(z) \neq \text{E-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z < (\text{E-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z$.

(70) If $\pi_1 z = \text{W-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z < (\text{E-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z$.

(71) If $\pi_1 z = \text{W-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{E-min}\,\widetilde{\mathcal{L}}(z) \neq \text{S-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z < (\text{S-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z$.

(72) If $\pi_1 z = \text{W-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-max}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z < (\text{S-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z$.

(73) If $\pi_1 z = \text{W-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{W-min}\,\widetilde{\mathcal{L}}(z) \neq \text{S-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z < (\text{W-min}\,\widetilde{\mathcal{L}}(z)) \leftarrow\!\!\!\!\!\mapsto z$.

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Gauges and Cages. Part II[1]

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The articles [16], [7], [17], [8], [2], [15], [4], [19], [3], [6], [11], [1], [13], [5], [10], [21], [14], [20], [18], [9], and [12] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we use the following convention: a, b, i, k, m, n are natural numbers, r, s are real numbers, D is a non empty subset of $\mathcal{E}^2_{\rm T}$, and C is a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$.

Next we state the proposition

(1) For all sets A, B such that for every set x such that $x \in A$ there exists a set K such that $K \subseteq B$ and $x \subseteq \bigcup K$ holds $\bigcup A \subseteq \bigcup B$.

Let m be an even integer. Note that $m+2$ is even.

Let m be an odd integer. Observe that $m+2$ is odd.

Let m be a non empty natural number. Observe that 2^m is even.

Let n be an even natural number and let m be a non empty natural number. Note that n^m is even.

We now state several propositions:

(2) If $r \neq 0$, then $\frac{1}{r} \cdot r^{i+1} = r^i$.

(3) If $\frac{r}{s}$ is an integer and $s \neq 0$, then $-\lfloor \frac{r}{s} \rfloor = \lfloor \frac{-r}{s} \rfloor + 1$.

(4) If $\frac{r}{s}$ is an integer, then $-\lfloor \frac{r}{s} \rfloor = \lfloor \frac{-r}{s} \rfloor$.

(5) If $n > 0$ and $k \bmod n \neq 0$, then $-(k \div n) = (-k \div n) + 1$.

(6) If $n > 0$ and $k \bmod n = 0$, then $-(k \div n) = -k \div n$.

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2. Gauges and Cages

We now state a number of propositions:

(7) If $2 \leqslant m$ and $m < \operatorname{len} \operatorname{Gauge}(D, i)$ and $1 \leqslant a$ and $a \leqslant \operatorname{len} \operatorname{Gauge}(D, i)$ and $1 \leqslant b$ and $b \leqslant \operatorname{len} \operatorname{Gauge}(D, i+1)$, then $((\operatorname{Gauge}(D, i))_{m,a})_{\mathbf{1}} = ((\operatorname{Gauge}(D, i+1))_{2 \cdot m -' 2, b})_{\mathbf{1}}$.

(8) If $2 \leqslant n$ and $n < \operatorname{len} \operatorname{Gauge}(D, i)$ and $1 \leqslant a$ and $a \leqslant \operatorname{len} \operatorname{Gauge}(D, i)$ and $1 \leqslant b$ and $b \leqslant \operatorname{len} \operatorname{Gauge}(D, i+1)$, then $((\operatorname{Gauge}(D, i))_{a,n})_{\mathbf{2}} = ((\operatorname{Gauge}(D, i+1))_{b, 2 \cdot n -' 2})_{\mathbf{2}}$.

(9) Let D be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $2 \leqslant m$ and $m+1 < \operatorname{len} \operatorname{Gauge}(D, i)$ and $2 \leqslant n$ and $n+1 < \operatorname{len} \operatorname{Gauge}(D, i)$. Then $\operatorname{cell}(\operatorname{Gauge}(D, i), m, n) = \operatorname{cell}(\operatorname{Gauge}(D, i+1), 2 \cdot m -' 2, 2 \cdot n -' 2) \cup \operatorname{cell}(\operatorname{Gauge}(D, i+1), 2 \cdot m -' 1, 2 \cdot n -' 2) \cup \operatorname{cell}(\operatorname{Gauge}(D, i+1), 2 \cdot m -' 2, 2 \cdot n -' 1) \cup \operatorname{cell}(\operatorname{Gauge}(D, i+1), 2 \cdot m -' 1, 2 \cdot n -' 1)$.

(10) Let D be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and k be a natural number. Suppose $2 \leqslant m$ and $m+1 < \operatorname{len} \operatorname{Gauge}(D, i)$ and $2 \leqslant n$ and $n+1 < \operatorname{len} \operatorname{Gauge}(D, i)$. Then $\operatorname{cell}(\operatorname{Gauge}(D, i), m, n) = \bigcup\{\operatorname{cell}(\operatorname{Gauge}(D, i+k), a, b); a$ ranges over natural numbers, b ranges over natural numbers: $(2^k \cdot m - 2^{k+1}) + 2 \leqslant a \ \wedge \ a \leqslant (2^k \cdot m - 2^k) + 1 \ \wedge \ (2^k \cdot n - 2^{k+1}) + 2 \leqslant b \ \wedge \ b \leqslant (2^k \cdot n - 2^k) + 1\}$.

(11) There exists a natural number i such that $1 \leqslant i$ and $i < \operatorname{len} \operatorname{Cage}(C, n)$ and $\operatorname{N-max} C \in \operatorname{right\_cell}(\operatorname{Cage}(C, n), i, \operatorname{Gauge}(C, n))$.

(12) There exists a natural number i such that $1 \leqslant i$ and $i < \operatorname{len} \operatorname{Cage}(C, n)$ and $\operatorname{N-max} C \in \operatorname{rightcell}(\operatorname{Cage}(C, n), i)$.

(13) There exists a natural number i such that $1 \leqslant i$ and $i < \operatorname{len} \operatorname{Cage}(C, n)$ and $\operatorname{E-min} C \in \operatorname{right\_cell}(\operatorname{Cage}(C, n), i, \operatorname{Gauge}(C, n))$.

(14) There exists a natural number i such that $1 \leqslant i$ and $i < \operatorname{len} \operatorname{Cage}(C, n)$ and $\operatorname{E-min} C \in \operatorname{rightcell}(\operatorname{Cage}(C, n), i)$.

(15) There exists a natural number i such that $1 \leqslant i$ and $i < \operatorname{len} \operatorname{Cage}(C, n)$ and $\operatorname{E-max} C \in \operatorname{right\_cell}(\operatorname{Cage}(C, n), i, \operatorname{Gauge}(C, n))$.

(16) There exists a natural number i such that $1 \leqslant i$ and $i < \operatorname{len} \operatorname{Cage}(C, n)$ and $\operatorname{E-max} C \in \operatorname{rightcell}(\operatorname{Cage}(C, n), i)$.

(17) There exists a natural number i such that $1 \leqslant i$ and $i < \operatorname{len} \operatorname{Cage}(C, n)$ and $\operatorname{S-min} C \in \operatorname{right\_cell}(\operatorname{Cage}(C, n), i, \operatorname{Gauge}(C, n))$.

(18) There exists a natural number i such that $1 \leqslant i$ and $i < \operatorname{len} \operatorname{Cage}(C, n)$ and $\operatorname{S-min} C \in \operatorname{rightcell}(\operatorname{Cage}(C, n), i)$.

(19) There exists a natural number i such that $1 \leqslant i$ and $i < \operatorname{len} \operatorname{Cage}(C, n)$ and $\operatorname{S-max} C \in \operatorname{right\_cell}(\operatorname{Cage}(C, n), i, \operatorname{Gauge}(C, n))$.

(20) There exists a natural number i such that $1 \leqslant i$ and $i < \operatorname{len} \operatorname{Cage}(C, n)$ and $\text{S-max}\, C \in \operatorname{rightcell}(\operatorname{Cage}(C, n), i)$.

(21) There exists a natural number i such that $1 \leqslant i$ and $i < \operatorname{len} \operatorname{Cage}(C, n)$ and $\text{W-min}\, C \in \operatorname{right\_cell}(\operatorname{Cage}(C, n), i, \operatorname{Gauge}(C, n))$.

(22) There exists a natural number i such that $1 \leqslant i$ and $i < \operatorname{len} \operatorname{Cage}(C, n)$ and $\text{W-min}\, C \in \operatorname{rightcell}(\operatorname{Cage}(C, n), i)$.

(23) There exists a natural number i such that $1 \leqslant i$ and $i < \operatorname{len} \operatorname{Cage}(C, n)$ and $\text{W-max}\, C \in \operatorname{right\_cell}(\operatorname{Cage}(C, n), i, \operatorname{Gauge}(C, n))$.

(24) There exists a natural number i such that $1 \leqslant i$ and $i < \operatorname{len} \operatorname{Cage}(C, n)$ and $\text{W-max}\, C \in \operatorname{rightcell}(\operatorname{Cage}(C, n), i)$.

(25) There exists a natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $\text{N-min}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) = (\operatorname{Gauge}(C, n))_{i, \operatorname{width} \operatorname{Gauge}(C,n)}$.

(26) There exists a natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $\text{N-max}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) = (\operatorname{Gauge}(C, n))_{i, \operatorname{width} \operatorname{Gauge}(C,n)}$.

(27) There exists a natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $(\operatorname{Gauge}(C, n))_{i, \operatorname{width} \operatorname{Gauge}(C,n)} \in \operatorname{rng} \operatorname{Cage}(C, n)$.

(28) There exists a natural number j such that $1 \leqslant j$ and $j \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\text{E-min}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) = (\operatorname{Gauge}(C, n))_{\operatorname{len} \operatorname{Gauge}(C,n), j}$.

(29) There exists a natural number j such that $1 \leqslant j$ and $j \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\text{E-max}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) = (\operatorname{Gauge}(C, n))_{\operatorname{len} \operatorname{Gauge}(C,n), j}$.

(30) There exists a natural number j such that $1 \leqslant j$ and $j \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $(\operatorname{Gauge}(C, n))_{\operatorname{len} \operatorname{Gauge}(C,n), j} \in \operatorname{rng} \operatorname{Cage}(C, n)$.

(31) There exists a natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $\text{S-min}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) = (\operatorname{Gauge}(C, n))_{i,1}$.

(32) There exists a natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $\text{S-max}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) = (\operatorname{Gauge}(C, n))_{i,1}$.

(33) There exists a natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $(\operatorname{Gauge}(C, n))_{i,1} \in \operatorname{rng} \operatorname{Cage}(C, n)$.

(34) There exists a natural number j such that $1 \leqslant j$ and $j \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\text{W-min}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) = (\operatorname{Gauge}(C, n))_{1,j}$.

(35) There exists a natural number j such that $1 \leqslant j$ and $j \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\text{W-max}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) = (\operatorname{Gauge}(C, n))_{1,j}$.

(36) There exists a natural number j such that $1 \leqslant j$ and $j \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $(\operatorname{Gauge}(C, n))_{1,j} \in \operatorname{rng} \operatorname{Cage}(C, n)$.

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The Binomial Theorem for Algebraic Structures[1]

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Summary. In this paper we prove the well-known binomial theorem for algebraic structures. In doing so we tried to be as modest as possible concerning the algebraic properties of the underlying structure. Consequently, we proved the binomial theorem for "commutative rings" in which the existence of an inverse with respect to addition is replaced by a weaker property of cancellation.

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The articles [5], [7], [2], [3], [8], [1], [6], [11], [9], [10], and [4] provide the terminology and notation for this paper.

1. Preliminaries

Let L be a non empty loop structure. We say that L is add-left-cancelable if and only if:

(Def. 1) For all elements a, b, c of L such that $a+b=a+c$ holds $b=c$.

We say that L is add-right-cancelable if and only if:

(Def. 2) For all elements a, b, c of L such that $b+a=c+a$ holds $b=c$.

We say that L is add-cancelable if and only if:

(Def. 3) For all elements a, b, c of L holds if $a+b=a+c$, then $b=c$ and if $b+a=c+a$, then $b=c$.

One can check the following observations:

* there exists a non empty loop structure which is add-left-cancelable,

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* there exists a non empty loop structure which is add-right-cancelable, and
* there exists a non empty loop structure which is add-cancelable.

Let us note that every non empty loop structure which is add-left-cancelable and add-right-cancelable is also add-cancelable and every non empty loop structure which is add-cancelable is also add-left-cancelable and add-right-cancelable.

One can verify that every non empty loop structure which is Abelian and add-right-cancelable is also add-left-cancelable and every non empty loop structure which is Abelian and add-left-cancelable is also add-right-cancelable.

Let us observe that every non empty loop structure which is right zeroed, right complementable, and add-associative is also add-right-cancelable.

Let us observe that there exists a non empty double loop structure which is Abelian, add-associative, left zeroed, right zeroed, commutative, associative, add-cancelable, distributive, and unital.

We now state two propositions:

(1) Let R be a right zeroed add-left-cancelable left distributive non empty double loop structure and a be an element of R. Then $0_R \cdot a = 0_R$.

(2) Let R be a left zeroed add-right-cancelable right distributive non empty double loop structure and a be an element of R. Then $a \cdot 0_R = 0_R$.

In this article we present several logical schemes. The scheme *Ind2* deals with a natural number $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every natural number i such that $\mathcal{A} \leqslant i$ holds $\mathcal{P}[i]$

provided the following conditions are satisfied:

- $\mathcal{P}[\mathcal{A}]$, and
- For every natural number j such that $\mathcal{A} \leqslant j$ holds if $\mathcal{P}[j]$, then $\mathcal{P}[j+1]$.

The scheme *RecDef1* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and a binary operation $\mathcal{C}$ on $\mathcal{A}$, and states that:

There exists a function g from $[\!:\mathbb{N}, \mathcal{A}:\!]$ into $\mathcal{A}$ such that for every element a of $\mathcal{A}$ holds

$g(0, a) = \mathcal{B}$ and for every natural number n holds $g(n+1, a) = \mathcal{C}(a, g(n, a))$

for all values of the parameters.

The scheme *RecDef2* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and a binary operation $\mathcal{C}$ on $\mathcal{A}$, and states that:

There exists a function g from $[\!:\mathcal{A}, \mathbb{N}:\!]$ into $\mathcal{A}$ such that for every element a of $\mathcal{A}$ holds

$g(a, 0) = \mathcal{B}$ and for every natural number n holds $g(a, n+1) = \mathcal{C}(g(a, n), a)$

for all values of the parameters.

2. On Finite Sequences

One can prove the following propositions:

(3) For every left zeroed non empty loop structure L and for every element a of L holds $\sum\langle a\rangle = a$.

(4) Let R be a left zeroed add-right-cancelable right distributive non empty double loop structure, a be an element of R, and p be a finite sequence of elements of the carrier of R. Then $\sum(a \cdot p) = a \cdot \sum p$.

(5) Let R be a right zeroed add-left-cancelable left distributive non empty double loop structure, a be an element of R, and p be a finite sequence of elements of the carrier of R. Then $\sum(p \cdot a) = \sum p \cdot a$.

(6) Let R be a commutative non empty double loop structure, a be an element of R, and p be a finite sequence of elements of the carrier of R. Then $\sum(p \cdot a) = \sum(a \cdot p)$.

Let R be a non empty loop structure and let p, q be finite sequences of elements of the carrier of R. Let us assume that $\operatorname{dom} p = \operatorname{dom} q$. The functor $p + q$ yields a finite sequence of elements of the carrier of R and is defined by:

(Def. 4) $\operatorname{dom}(p+q) = \operatorname{dom} p$ and for every natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len}(p+q)$ holds $(p+q)_i = p_i + q_i$.

The following proposition is true

(7) Let R be an Abelian right zeroed add-associative non empty loop structure and p, q be finite sequences of elements of the carrier of R. If $\operatorname{dom} p = \operatorname{dom} q$, then $\sum(p+q) = \sum p + \sum q$.

3. On Powers in Rings

Let R be a unital non empty groupoid, let a be an element of R, and let n be a natural number. The functor a^n yielding an element of R is defined as follows:

(Def. 5) $a^n = \operatorname{power}_R(a, n)$.

We now state several propositions:

(8) For every unital non empty groupoid R and for every element a of R holds $a^0 = 1_R$ and $a^1 = a$.

(9) For every unital non empty groupoid R and for every element a of R and for every natural number n holds $a^{n+1} = a^n \cdot a$.

(10) Let R be a unital associative commutative non empty groupoid, a, b be elements of R, and n be a natural number. Then $(a \cdot b)^n = a^n \cdot b^n$.

(11) Let R be a unital associative non empty groupoid, a be an element of R, and n, m be natural numbers. Then $a^{n+m} = a^n \cdot a^m$.

(12) Let R be a unital associative non empty groupoid, a be an element of R, and n, m be natural numbers. Then $(a^n)^m = a^{n \cdot m}$.

4. On Natural Products in Rings

Let R be a non empty loop structure. The functor Nat-mult-left R yielding a function from $[\!:\mathbb{N}, \text{the carrier of } R:\!]$ into the carrier of R is defined by:

(Def. 6) For every element a of R holds $(\text{Nat-mult-left}\, R)(0, a) = 0_R$ and for every natural number n holds $(\text{Nat-mult-left}\, R)(n + 1, a) = a + (\text{Nat-mult-left}\, R)(n, a)$.

The functor Nat-mult-right R yields a function from $[\!:\text{the carrier of } R, \mathbb{N}:\!]$ into the carrier of R and is defined by:

(Def. 7) For every element a of R holds $(\text{Nat-mult-right}\, R)(a, 0) = 0_R$ and for every natural number n holds $(\text{Nat-mult-right}\, R)(a, n + 1) = (\text{Nat-mult-right}\, R)(a, n) + a$.

Let R be a non empty loop structure, let a be an element of R, and let n be a natural number. The functor $n \cdot a$ yields an element of R and is defined by:

(Def. 8) $n \cdot a = (\text{Nat-mult-left}\, R)(n, a)$.

The functor $a \cdot n$ yields an element of R and is defined as follows:

(Def. 9) $a \cdot n = (\text{Nat-mult-right}\, R)(a, n)$.

One can prove the following propositions:

(13) For every non empty loop structure R and for every element a of R holds $0 \cdot a = 0_R$ and $a \cdot 0 = 0_R$.

(14) For every right zeroed non empty loop structure R and for every element a of R holds $1 \cdot a = a$.

(15) For every left zeroed non empty loop structure R and for every element a of R holds $a \cdot 1 = a$.

(16) Let R be a left zeroed add-associative non empty loop structure, a be an element of R, and n, m be natural numbers. Then $(n+m) \cdot a = n \cdot a + m \cdot a$.

(17) Let R be a right zeroed add-associative non empty loop structure, a be an element of R, and n, m be natural numbers. Then $a \cdot (n+m) = a \cdot n + a \cdot m$.

(18) Let R be a left zeroed right zeroed add-associative non empty loop structure, a be an element of R, and n be a natural number. Then $n \cdot a = a \cdot n$.

(19) Let R be an Abelian non empty loop structure, a be an element of R, and n be a natural number. Then $n \cdot a = a \cdot n$.

(20) Let R be a left zeroed right zeroed add-left-cancelable add-associative left distributive non empty double loop structure, a, b be elements of R, and n be a natural number. Then $(n \cdot a) \cdot b = n \cdot (a \cdot b)$.

(21) Let R be a left zeroed right zeroed add-right-cancelable add-associative distributive non empty double loop structure, a, b be elements of R, and n be a natural number. Then $b \cdot (n \cdot a) = (b \cdot a) \cdot n$.

(22) Let R be a left zeroed right zeroed add-associative add-cancelable distributive non empty double loop structure, a, b be elements of R, and n be a natural number. Then $(a \cdot n) \cdot b = a \cdot (n \cdot b)$.

5. The Binomial Theorem

Let k, n be natural numbers. Then $\binom{n}{k}$ is a natural number.

Let R be a unital non empty double loop structure, let a, b be elements of R, and let n be a natural number. The functor $\langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle$ yields a finite sequence of elements of the carrier of R and is defined by the conditions (Def. 10).

(Def. 10)(i) $\mathrm{len} \langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle = n + 1$, and

(ii) for all natural numbers i, l, m such that $i \in \mathrm{dom} \langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle$ and $m = i - 1$ and $l = n - m$ holds $\langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle_i = \binom{n}{m} \cdot a^l \cdot b^m$.

The following four propositions are true:

(23) For every right zeroed unital non empty double loop structure R and for all elements a, b of R holds $\langle \binom{0}{0} a^0 b^0, \ldots, \binom{0}{0} a^0 b^0 \rangle = \langle 1_R \rangle$.

(24) Let R be a right zeroed unital non empty double loop structure, a, b be elements of R, and n be a natural number. Then $\langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle(1) = a^n$.

(25) Let R be a right zeroed unital non empty double loop structure, a, b be elements of R, and n be a natural number. Then $\langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle(n+1) = b^n$.

(26) Let R be an Abelian add-associative left zeroed right zeroed commutative associative add-cancelable distributive unital non empty double loop structure, a, b be elements of R, and n be a natural number. Then $(a + b)^n = \sum \langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle$.

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Ring Ideals

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Summary. We introduce the basic notions of ideal theory in rings. This includes left and right ideals, (finitely) generated ideals and some operations on ideals such as the addition of ideals and the radical of an ideal. In addition we introduce linear combinations to formalize the well-known characterization of generated ideals. Principal ideal domains and Noetherian rings are defined. The latter development follows [3], pages 144–145.

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The terminology and notation used here are introduced in the following articles: [11], [18], [17], [20], [2], [23], [8], [4], [5], [15], [22], [19], [16], [21], [1], [13], [6], [14], [12], [26], [24], [25], [9], [10], and [7].

1. Preliminaries

Let us note that there exists a non empty loop structure which is add-associative, left zeroed, and right zeroed.

Let us observe that there exists a non empty double loop structure which is Abelian, left zeroed, right zeroed, add-cancelable, well unital, add-associative, associative, commutative, distributive, and non trivial.

One can prove the following proposition

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(1) Let V be an add-associative left zeroed right zeroed non empty loop structure and v, u be elements of V. Then $\sum\langle v, u\rangle = v + u$.

2. Ideals

Let L be a non empty loop structure and let F be a subset of L. We say that F is add closed if and only if:

(Def. 1) For all elements x, y of the carrier of L such that $x \in F$ and $y \in F$ holds $x + y \in F$.

Let L be a non empty groupoid and let F be a subset of L. We say that F is left ideal if and only if:

(Def. 2) For all elements p, x of the carrier of L such that $x \in F$ holds $p \cdot x \in F$.

We say that F is right ideal if and only if:

(Def. 3) For all elements p, x of the carrier of L such that $x \in F$ holds $x \cdot p \in F$.

Let L be a non empty loop structure. Observe that there exists a non empty subset of L which is add closed.

Let L be a non empty groupoid. One can verify that there exists a non empty subset of L which is left ideal and there exists a non empty subset of L which is right ideal.

Let L be a non empty double loop structure. One can verify the following observations:

* there exists a non empty subset of L which is add closed, left ideal, and right ideal,
* there exists a non empty subset of L which is add closed and right ideal, and
* there exists a non empty subset of L which is add closed and left ideal.

Let R be a commutative non empty groupoid. Observe that every non empty subset of R which is left ideal is also right ideal and every non empty subset of R which is right ideal is also left ideal.

Let L be a non empty double loop structure. An ideal of L is an add closed left ideal right ideal non empty subset of L.

Let L be a non empty double loop structure. A right ideal of L is an add closed right ideal non empty subset of L.

Let L be a non empty double loop structure. A left ideal of L is an add closed left ideal non empty subset of L.

The following propositions are true:

(2) Let R be a right zeroed add-left-cancelable left distributive non empty double loop structure and I be a left ideal non empty subset of R. Then $0_R \in I$.

(3) Let R be a left zeroed add-right-cancelable right distributive non empty double loop structure and I be a right ideal non empty subset of R. Then $0_R \in I$.

(4) For every right zeroed non empty double loop structure L holds $\{0_L\}$ is add closed.

(5) Let L be a left zeroed add-right-cancelable right distributive non empty double loop structure. Then $\{0_L\}$ is left ideal.

(6) Let L be a right zeroed add-left-cancelable left distributive non empty double loop structure. Then $\{0_L\}$ is right ideal.

(7) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure. Then $\{0_L\}$ is an ideal of L.

(8) Let L be an add-associative right zeroed right complementable right distributive non empty double loop structure. Then $\{0_L\}$ is a left ideal of L.

(9) Let L be an add-associative right zeroed right complementable left distributive non empty double loop structure. Then $\{0_L\}$ is a right ideal of L.

(10) For every non empty double loop structure L holds the carrier of L is an ideal of L.

(11) For every non empty double loop structure L holds the carrier of L is a left ideal of L.

(12) For every non empty double loop structure L holds the carrier of L is a right ideal of L.

Let R be a left zeroed right zeroed add-cancelable distributive non empty double loop structure and let I be an ideal of R. Let us observe that I is trivial if and only if:

(Def. 4) $I = \{0_R\}$.

Let S be a 1-sorted structure and let T be a subset of S. We say that T is proper if and only if:

(Def. 5) $T \neq$ the carrier of S.

Let S be a non empty 1-sorted structure. Note that there exists a subset of S which is proper.

Let R be a non trivial left zeroed right zeroed add-cancelable distributive non empty double loop structure. One can check that there exists an ideal of R which is proper.

The following propositions are true:

(13) Let L be an add-associative right zeroed right complementable left distributive left unital non empty double loop structure, I be a left ideal non empty subset of L, and x be an element of the carrier of L. If $x \in I$, then $-x \in I$.

(14) Let L be an add-associative right zeroed right complementable right distributive right unital non empty double loop structure, I be a right ideal non empty subset of L, and x be an element of the carrier of L. If $x \in I$, then $-x \in I$.

(15) Let L be an add-associative right zeroed right complementable left distributive left unital non empty double loop structure, I be a left ideal of L, and x, y be elements of the carrier of L. If $x \in I$ and $y \in I$, then $x - y \in I$.

(16) Let L be an add-associative right zeroed right complementable right distributive right unital non empty double loop structure, I be a right ideal of L, and x, y be elements of the carrier of L. If $x \in I$ and $y \in I$, then $x - y \in I$.

(17) Let R be a left zeroed right zeroed add-cancelable add-associative distributive non empty double loop structure, I be an add closed right ideal non empty subset of R, a be an element of I, and n be a natural number. Then $n \cdot a \in I$.

(18) Let R be a unital left zeroed right zeroed add-cancelable associative distributive non empty double loop structure, I be a right ideal non empty subset of R, a be an element of I, and n be a natural number. If $n \neq 0$, then $a^n \in I$.

Let R be a non empty loop structure and let I be an add closed non empty subset of R. The functor $\mathrm{add}|(I, R)$ yielding a binary operation on I is defined as follows:

(Def. 6) $\mathrm{add}|(I, R) = (\text{the addition of } R){\restriction}[\!:I, I:\!]$.

Let R be a non empty groupoid and let I be a right ideal non empty subset of R. The functor $\mathrm{mult}|(I, R)$ yielding a binary operation on I is defined as follows:

(Def. 7) $\mathrm{mult}|(I, R) = (\text{the multiplication of } R){\restriction}[\!:I, I:\!]$.

Let R be a non empty loop structure and let I be an add closed non empty subset of R. The functor $\mathrm{Gr}(I, R)$ yields a non empty loop structure and is defined by:

(Def. 8) $\mathrm{Gr}(I, R) = \langle I, \mathrm{add}|(I, R), 0_R (\in I)\rangle$.

Let R be a left zeroed right zeroed add-cancelable add-associative distributive non empty double loop structure and let I be an add closed non empty subset of R. Note that $\mathrm{Gr}(I, R)$ is add-associative.

Let R be a left zeroed right zeroed add-cancelable add-associative distributive non empty double loop structure and let I be an add closed right ideal non empty subset of R. Observe that $\mathrm{Gr}(I, R)$ is right zeroed.

Let R be an Abelian non empty double loop structure and let I be an add closed non empty subset of R. Observe that $\mathrm{Gr}(I, R)$ is Abelian.

Let R be an Abelian right unital left zeroed right zeroed right complementable add-associative distributive non empty double loop structure and let I be an add closed right ideal non empty subset of R. Note that $\mathrm{Gr}(I, R)$ is right complementable.

We now state four propositions:

(19) Let R be a right unital non empty double loop structure and I be a left ideal non empty subset of R. Then I is proper if and only if $\mathbf{1}_R \notin I$.

(20) Let R be a left unital right unital non empty double loop structure and I be a right ideal non empty subset of R. Then I is proper if and only if for every element u of R such that u is unital holds $u \notin I$.

(21) Let R be a right unital non empty double loop structure and I be a left ideal right ideal non empty subset of R. Then I is proper if and only if for every element u of R such that u is unital holds $u \notin I$.

(22) Let R be a non degenerated commutative ring. Then R is a field if and only if for every ideal I of R holds $I = \{0_R\}$ or $I =$ the carrier of R.

3. Linear Combinations

Let R be a non empty multiplicative loop structure and let A be a non empty subset of the carrier of R. A finite sequence of elements of the carrier of R is said to be a linear combination of A if:

(Def. 9) For every set i such that $i \in \mathrm{dom}\,\mathrm{it}$ there exist elements u, v of R and there exists an element a of A such that $\mathrm{it}_i = u \cdot a \cdot v$.

A finite sequence of elements of the carrier of R is said to be a left linear combination of A if:

(Def. 10) For every set i such that $i \in \mathrm{dom}\,\mathrm{it}$ there exists an element u of R and there exists an element a of A such that $\mathrm{it}_i = u \cdot a$.

A finite sequence of elements of the carrier of R is said to be a right linear combination of A if:

(Def. 11) For every set i such that $i \in \mathrm{dom}\,\mathrm{it}$ there exists an element u of R and there exists an element a of A such that $\mathrm{it}_i = a \cdot u$.

Let R be a non empty multiplicative loop structure and let A be a non empty subset of the carrier of R. One can verify the following observations:

* there exists a linear combination of A which is non empty,
* there exists a left linear combination of A which is non empty, and
* there exists a right linear combination of A which is non empty.

Let R be a non empty multiplicative loop structure, let A, B be non empty subsets of the carrier of R, let F be a linear combination of A, and let G be a linear combination of B. Then $F \frown G$ is a linear combination of $A \cup B$.

One can prove the following three propositions:

(23) Let R be an associative non empty multiplicative loop structure, A be a non empty subset of the carrier of R, a be an element of the carrier of R, and F be a linear combination of A. Then $a \cdot F$ is a linear combination of A.

(24) Let R be an associative non empty multiplicative loop structure, A be a non empty subset of the carrier of R, a be an element of the carrier of R, and F be a linear combination of A. Then $F \cdot a$ is a linear combination of A.

(25) Let R be a right unital non empty multiplicative loop structure and A be a non empty subset of the carrier of R. Then every left linear combination of A is a linear combination of A.

Let R be a non empty multiplicative loop structure, let A, B be non empty subsets of the carrier of R, let F be a left linear combination of A, and let G be a left linear combination of B. Then $F \frown G$ is a left linear combination of $A \cup B$.

One can prove the following three propositions:

(26) Let R be an associative non empty multiplicative loop structure, A be a non empty subset of the carrier of R, a be an element of the carrier of R, and F be a left linear combination of A. Then $a \cdot F$ is a left linear combination of A.

(27) Let R be a non empty multiplicative loop structure, A be a non empty subset of the carrier of R, a be an element of the carrier of R, and F be a left linear combination of A. Then $F \cdot a$ is a linear combination of A.

(28) Let R be a left unital non empty multiplicative loop structure and A be a non empty subset of the carrier of R. Then every right linear combination of A is a linear combination of A.

Let R be a non empty multiplicative loop structure, let A, B be non empty subsets of the carrier of R, let F be a right linear combination of A, and let G be a right linear combination of B. Then $F \frown G$ is a right linear combination of $A \cup B$.

Next we state several propositions:

(29) Let R be an associative non empty multiplicative loop structure, A be a non empty subset of the carrier of R, a be an element of the carrier of R, and F be a right linear combination of A. Then $F \cdot a$ is a right linear combination of A.

(30) Let R be an associative non empty multiplicative loop structure, A be a non empty subset of the carrier of R, a be an element of the carrier of R, and F be a right linear combination of A. Then $a \cdot F$ is a linear combination of A.

(31) Let R be a commutative associative non empty multiplicative loop struc-

ture and A be a non empty subset of the carrier of R. Then every linear combination of A is a left linear combination of A and a right linear combination of A.

(32) Let S be a non empty double loop structure, F be a non empty subset of the carrier of S, and l_1 be a non empty linear combination of F. Then there exists a linear combination p of F and there exists an element e of the carrier of S such that $l_1 = p \frown \langle e \rangle$ and $\langle e \rangle$ is a linear combination of F.

(33) Let S be a non empty double loop structure, F be a non empty subset of the carrier of S, and l_1 be a non empty left linear combination of F. Then there exists a left linear combination p of F and there exists an element e of the carrier of S such that $l_1 = p \frown \langle e \rangle$ and $\langle e \rangle$ is a left linear combination of F.

(34) Let S be a non empty double loop structure, F be a non empty subset of the carrier of S, and l_1 be a non empty right linear combination of F. Then there exists a right linear combination p of F and there exists an element e of the carrier of S such that $l_1 = p \frown \langle e \rangle$ and $\langle e \rangle$ is a right linear combination of F.

Let R be a non empty multiplicative loop structure, let A be a non empty subset of the carrier of R, let L be a linear combination of A, and let E be a finite sequence of elements of $[:$the carrier of R, the carrier of R, the carrier of $R:]$. We say that E represents L if and only if:

(Def. 12) $\operatorname{len} E = \operatorname{len} L$ and for every set i such that $i \in \operatorname{dom} L$ holds $L(i) = (E_i)_{\mathbf{1}} \cdot (E_i)_{\mathbf{2}} \cdot (E_i)_{\mathbf{3}}$ and $(E_i)_{\mathbf{2}} \in A$.

The following propositions are true:

(35) Let R be a non empty multiplicative loop structure, A be a non empty subset of the carrier of R, and L be a linear combination of A. Then there exists a finite sequence E of elements of $[:$the carrier of R, the carrier of R, the carrier of $R:]$ such that E represents L.

(36) Let R, S be non empty multiplicative loop structures, F be a non empty subset of the carrier of R, l_1 be a linear combination of F, G be a non empty subset of the carrier of S, P be a function from the carrier of R into the carrier of S, and E be a finite sequence of elements of $[:$the carrier of R, the carrier of R, the carrier of $R:]$. Suppose $P^{\circ}F \subseteq G$ and E represents l_1. Then there exists a linear combination L_1 of G such that $\operatorname{len} l_1 = \operatorname{len} L_1$ and for every set i such that $i \in \operatorname{dom} L_1$ holds $L_1(i) = P((E_i)_{\mathbf{1}}) \cdot P((E_i)_{\mathbf{2}}) \cdot P((E_i)_{\mathbf{3}})$.

Let R be a non empty multiplicative loop structure, let A be a non empty subset of the carrier of R, let L be a left linear combination of A, and let E be a finite sequence of elements of $[:$the carrier of R, the carrier of $R:]$. We say that

E represents L if and only if:

(Def. 13) $\operatorname{len} E = \operatorname{len} L$ and for every set i such that $i \in \operatorname{dom} L$ holds $L(i) = (E_i)_{\mathbf{1}} \cdot (E_i)_{\mathbf{2}}$ and $(E_i)_{\mathbf{2}} \in A$.

One can prove the following two propositions:

(37) Let R be a non empty multiplicative loop structure, A be a non empty subset of the carrier of R, and L be a left linear combination of A. Then there exists a finite sequence E of elements of $[\!:$ the carrier of $R,$ the carrier of $R:\!]$ such that E represents L.

(38) Let R, S be non empty multiplicative loop structures, F be a non empty subset of the carrier of R, l_1 be a left linear combination of F, G be a non empty subset of the carrier of S, P be a function from the carrier of R into the carrier of S, and E be a finite sequence of elements of $[\!:$ the carrier of $R,$ the carrier of $R:\!]$. Suppose $P^\circ F \subseteq G$ and E represents l_1. Then there exists a left linear combination L_1 of G such that $\operatorname{len} l_1 = \operatorname{len} L_1$ and for every set i such that $i \in \operatorname{dom} L_1$ holds $L_1(i) = P((E_i)_{\mathbf{1}}) \cdot P((E_i)_{\mathbf{2}})$.

Let R be a non empty multiplicative loop structure, let A be a non empty subset of the carrier of R, let L be a right linear combination of A, and let E be a finite sequence of elements of $[\!:$ the carrier of $R,$ the carrier of $R:\!]$. We say that E represents L if and only if:

(Def. 14) $\operatorname{len} E = \operatorname{len} L$ and for every set i such that $i \in \operatorname{dom} L$ holds $L(i) = (E_i)_{\mathbf{1}} \cdot (E_i)_{\mathbf{2}}$ and $(E_i)_{\mathbf{1}} \in A$.

One can prove the following propositions:

(39) Let R be a non empty multiplicative loop structure, A be a non empty subset of the carrier of R, and L be a right linear combination of A. Then there exists a finite sequence E of elements of $[\!:$ the carrier of $R,$ the carrier of $R:\!]$ such that E represents L.

(40) Let R, S be non empty multiplicative loop structures, F be a non empty subset of the carrier of R, l_1 be a right linear combination of F, G be a non empty subset of the carrier of S, P be a function from the carrier of R into the carrier of S, and E be a finite sequence of elements of $[\!:$ the carrier of $R,$ the carrier of $R:\!]$. Suppose $P^\circ F \subseteq G$ and E represents l_1. Then there exists a right linear combination L_1 of G such that $\operatorname{len} l_1 = \operatorname{len} L_1$ and for every set i such that $i \in \operatorname{dom} L_1$ holds $L_1(i) = P((E_i)_{\mathbf{1}}) \cdot P((E_i)_{\mathbf{2}})$.

(41) Let R be a non empty multiplicative loop structure, A be a non empty subset of the carrier of R, l be a linear combination of A, and n be a natural number. Then $l \restriction \operatorname{Seg} n$ is a linear combination of A.

(42) Let R be a non empty multiplicative loop structure, A be a non empty subset of the carrier of R, l be a left linear combination of A, and n be a natural number. Then $l \restriction \operatorname{Seg} n$ is a left linear combination of A.

(43) Let R be a non empty multiplicative loop structure, A be a non empty

subset of the carrier of R, l be a right linear combination of A, and n be a natural number. Then $l \restriction \mathrm{Seg}\, n$ is a right linear combination of A.

4. Generated Ideals

Let L be a non empty double loop structure and let F be a subset of the carrier of L. Let us assume that F is non empty. The functor F–ideal yielding an ideal of L is defined by:

(Def. 15) $F \subseteq F$–ideal and for every ideal I of L such that $F \subseteq I$ holds F–ideal $\subseteq I$.

The functor F–left-ideal yields a left ideal of L and is defined by:

(Def. 16) $F \subseteq F$–left-ideal and for every left ideal I of L such that $F \subseteq I$ holds F–left-ideal $\subseteq I$.

The functor F–right-ideal yields a right ideal of L and is defined as follows:

(Def. 17) $F \subseteq F$–right-ideal and for every right ideal I of L such that $F \subseteq I$ holds F–right-ideal $\subseteq I$.

One can prove the following three propositions:

(44) For every non empty double loop structure L and for every ideal I of L holds I–ideal $= I$.

(45) For every non empty double loop structure L and for every left ideal I of L holds I–left-ideal $= I$.

(46) For every non empty double loop structure L and for every right ideal I of L holds I–right-ideal $= I$.

Let L be a non empty double loop structure and let I be an ideal of L. A non empty subset of L is said to be a basis of I if:

(Def. 18) It–ideal $= I$.

We now state a number of propositions:

(47) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure. Then $\{0_L\}$–ideal $= \{0_L\}$.

(48) For every left zeroed right zeroed add-cancelable distributive non empty double loop structure L holds $\{0_L\}$–ideal $= \{0_L\}$.

(49) Let L be a left zeroed right zeroed add-right-cancelable right distributive non empty double loop structure. Then $\{0_L\}$–left-ideal $= \{0_L\}$.

(50) For every right zeroed add-left-cancelable left distributive non empty double loop structure L holds $\{0_L\}$–right-ideal $= \{0_L\}$.

(51) For every well unital non empty double loop structure L holds $\{\mathbf{1}_L\}$–ideal = the carrier of L.

(52) For every right unital non empty double loop structure L holds $\{\mathbf{1}_L\}$–left-ideal $=$ the carrier of L.

(53) For every left unital non empty double loop structure L holds $\{\mathbf{1}_L\}$–right-ideal $=$ the carrier of L.

(54) For every non empty double loop structure L holds Ω_L–ideal $=$ the carrier of L.

(55) For every non empty double loop structure L holds Ω_L–left-ideal $=$ the carrier of L.

(56) For every non empty double loop structure L holds Ω_L–right-ideal $=$ the carrier of L.

(57) Let L be a non empty double loop structure and A, B be non empty subsets of the carrier of L. If $A \subseteq B$, then A–ideal $\subseteq B$–ideal.

(58) Let L be a non empty double loop structure and A, B be non empty subsets of the carrier of L. If $A \subseteq B$, then A–left-ideal $\subseteq B$–left-ideal.

(59) Let L be a non empty double loop structure and A, B be non empty subsets of the carrier of L. If $A \subseteq B$, then A–right-ideal $\subseteq B$–right-ideal.

(60) Let L be an add-associative left zeroed right zeroed add-cancelable associative distributive well unital non empty double loop structure, F be a non empty subset of the carrier of L, and x be a set. Then $x \in F$–ideal if and only if there exists a linear combination f of F such that $x = \sum f$.

(61) Let L be an add-associative left zeroed right zeroed add-cancelable associative distributive well unital non empty double loop structure, F be a non empty subset of the carrier of L, and x be a set. Then $x \in F$–left-ideal if and only if there exists a left linear combination f of F such that $x = \sum f$.

(62) Let L be an add-associative left zeroed right zeroed add-cancelable associative distributive well unital non empty double loop structure, F be a non empty subset of the carrier of L, and x be a set. Then $x \in F$–right-ideal if and only if there exists a right linear combination f of F such that $x = \sum f$.

(63) Let R be an add-associative left zeroed right zeroed add-cancelable well unital associative commutative distributive non empty double loop structure and F be a non empty subset of the carrier of R. Then F–ideal $=$ F–left-ideal and F–ideal $=$ F–right-ideal.

(64) Let R be an add-associative left zeroed right zeroed add-cancelable well unital associative commutative distributive non empty double loop structure and a be an element of R. Then $\{a\}$–ideal $= \{a \cdot r : r$ ranges over elements of $R\}$.

(65) Let R be an Abelian left zeroed right zeroed add-cancelable well unital add-associative associative commutative distributive non empty double loop structure and a, b be elements of R. Then $\{a, b\}$–ideal $= \{a \cdot r + b \cdot s : r$

ranges over elements of R, s ranges over elements of $R\}$.

(66) For every non empty double loop structure R and for every element a of R holds $a \in \{a\}$–ideal.

(67) Let R be an Abelian left zeroed right zeroed right complementable add-associative associative commutative distributive well unital non empty double loop structure, A be a non empty subset of R, and a be an element of R. If $a \in A$–ideal, then $\{a\}$–ideal $\subseteq A$–ideal.

(68) For every non empty double loop structure R and for all elements a, b of R holds $a \in \{a, b\}$–ideal and $b \in \{a, b\}$–ideal.

(69) For every non empty double loop structure R and for all elements a, b of R holds $\{a\}$–ideal $\subseteq \{a, b\}$–ideal and $\{b\}$–ideal $\subseteq \{a, b\}$–ideal.

5. Some Operations on Ideals

Let R be a non empty groupoid, let I be a subset of R, and let a be an element of R. The functor $a \cdot I$ yielding a subset of R is defined by:

(Def. 19) $a \cdot I = \{a \cdot i; i$ ranges over elements of R: $i \in I\}$.

Let R be a non empty multiplicative loop structure, let I be a non empty subset of R, and let a be an element of R. Observe that $a \cdot I$ is non empty.

Let R be a distributive non empty double loop structure, let I be an add closed subset of R, and let a be an element of R. Observe that $a \cdot I$ is add closed.

Let R be an associative non empty double loop structure, let I be a right ideal subset of R, and let a be an element of R. One can check that $a \cdot I$ is right ideal.

One can prove the following propositions:

(70) Let R be a right zeroed add-left-cancelable left distributive non empty double loop structure and I be a non empty subset of R. Then $0_R \cdot I = \{0_R\}$.

(71) For every left unital non empty double loop structure R and for every subset I of R holds $\mathbf{1}_R \cdot I = I$.

Let R be a non empty loop structure and let I, J be subsets of R. The functor $I + J$ yielding a subset of R is defined by:

(Def. 20) $I + J = \{a + b; a$ ranges over elements of R, b ranges over elements of R: $a \in I \ \wedge \ b \in J\}$.

Let R be a non empty loop structure and let I, J be non empty subsets of R. One can check that $I + J$ is non empty.

Let R be an Abelian non empty loop structure and let I, J be subsets of R. Let us observe that the functor $I + J$ is commutative.

Let R be an Abelian add-associative non empty loop structure and let I, J be add closed subsets of R. Note that $I+J$ is add closed.

Let R be a left distributive non empty double loop structure and let I, J be right ideal subsets of R. Observe that $I+J$ is right ideal.

Let R be a right distributive non empty double loop structure and let I, J be left ideal subsets of R. One can check that $I+J$ is left ideal.

One can prove the following propositions:

(72) For every add-associative non empty loop structure R and for all subsets I, J, K of R holds $I+(J+K)=(I+J)+K$.

(73) Let R be a left zeroed right zeroed add-right-cancelable right distributive non empty double loop structure and I, J be right ideal non empty subsets of R. Then $I \subseteq I+J$.

(74) Let R be a left zeroed add-right-cancelable right distributive non empty double loop structure and I, J be right ideal non empty subsets of R. Then $J \subseteq I+J$.

(75) Let R be a non empty loop structure, I, J be subsets of R, and K be an add closed subset of R. If $I \subseteq K$ and $J \subseteq K$, then $I+J \subseteq K$.

(76) Let R be an Abelian left zeroed right zeroed add-cancelable well unital add-associative associative commutative distributive non empty double loop structure and a, b be elements of R. Then $\{a,b\}$–ideal $=\{a\}$–ideal $+$ $\{b\}$–ideal.

Let R be a non empty 1-sorted structure and let I, J be subsets of R. The functor $I \cap J$ yielding a subset of R is defined as follows:

(Def. 21) $I \cap J=\{x; x$ ranges over elements of R: $x \in I \ \wedge \ x \in J\}$.

Let R be a right zeroed add-left-cancelable left distributive non empty double loop structure and let I, J be left ideal non empty subsets of R. Note that $I \cap J$ is non empty.

Let R be a non empty loop structure and let I, J be add closed subsets of R. Note that $I \cap J$ is add closed.

Let R be a non empty multiplicative loop structure and let I, J be left ideal subsets of R. Observe that $I \cap J$ is left ideal.

Let R be a non empty multiplicative loop structure and let I, J be right ideal subsets of R. Note that $I \cap J$ is right ideal.

One can prove the following four propositions:

(77) For every non empty 1-sorted structure R and for all subsets I, J of R holds $I \cap J \subseteq I$ and $I \cap J \subseteq J$.

(78) For every non empty 1-sorted structure R and for all subsets I, J, K of R holds $I \cap (J \cap K)=(I \cap J) \cap K$.

(79) For every non empty 1-sorted structure R and for all subsets I, J, K of R such that $K \subseteq I$ and $K \subseteq J$ holds $K \subseteq I \cap J$.

(80) Let R be an Abelian left zeroed right zeroed right complementable left unital add-associative left distributive non empty double loop structure, I be an add closed left ideal non empty subset of R, J be a subset of R, and K be a non empty subset of R. If $J \subseteq I$, then $I \cap (J + K) = J + I \cap K$.

Let R be a non empty double loop structure and let I, J be subsets of R. The functor $I * J$ yields a subset of R and is defined by the condition (Def. 22).

(Def. 22) $I * J = \{\sum s; s$ ranges over finite sequences of elements of the carrier of R: $\bigwedge_{i\,:\,\text{natural number}} (1 \leqslant i \ \wedge \ i \leqslant \operatorname{len} s \ \Rightarrow \ \bigvee_{a,b\,:\,\text{element of } R} (s(i) = a \cdot b \ \wedge \ a \in I \ \wedge \ b \in J))\}$.

Let R be a non empty double loop structure and let I, J be subsets of R. Note that $I * J$ is non empty.

Let R be a commutative non empty double loop structure and let I, J be subsets of R. Let us observe that the functor $I * J$ is commutative.

Let R be a right zeroed add-associative non empty double loop structure and let I, J be subsets of R. Note that $I * J$ is add closed.

Let R be a right zeroed add-left-cancelable associative left distributive non empty double loop structure and let I, J be right ideal subsets of R. One can check that $I * J$ is right ideal.

Let R be a left zeroed add-right-cancelable associative right distributive non empty double loop structure and let I, J be left ideal subsets of R. Note that $I * J$ is left ideal.

We now state several propositions:

(81) Let R be a left zeroed right zeroed add-left-cancelable left distributive non empty double loop structure and I be a non empty subset of R. Then $\{0_R\} * I = \{0_R\}$.

(82) Let R be a left zeroed right zeroed add-cancelable distributive non empty double loop structure, I be an add closed right ideal non empty subset of R, and J be an add closed left ideal non empty subset of R. Then $I * J \subseteq I \cap J$.

(83) Let R be an Abelian left zeroed right zeroed add-cancelable add-associative associative distributive non empty double loop structure and I, J, K be right ideal non empty subsets of R. Then $I*(J+K) = I*J+I*K$.

(84) Let R be an Abelian left zeroed right zeroed add-cancelable add-associative commutative associative distributive non empty double loop structure and I, J be right ideal non empty subsets of R. Then $(I + J) * (I \cap J) \subseteq I * J$.

(85) Let R be a right zeroed add-left-cancelable left distributive non empty double loop structure and I, J be add closed left ideal non empty subsets of R. Then $(I + J) * (I \cap J) \subseteq I \cap J$.

Let R be a non empty loop structure and let I, J be subsets of R. We say that I, J are co-prime if and only if:

(Def. 23) $I + J =$ the carrier of R.

We now state two propositions:

(86) Let R be a left zeroed left unital non empty double loop structure and I, J be non empty subsets of R. If I, J are co-prime, then $I \cap J \subseteq (I + J) * (I \cap J)$.

(87) Let R be an Abelian left zeroed right zeroed add-cancelable add-associative left unital commutative associative distributive non empty double loop structure, I be an add closed left ideal right ideal non empty subset of R, and J be an add closed left ideal non empty subset of R. If I, J are co-prime, then $I * J = I \cap J$.

Let R be a non empty groupoid and let I, J be subsets of R. The functor $I \,\%\, J$ yields a subset of R and is defined by:

(Def. 24) $I \,\%\, J = \{a; a$ ranges over elements of R: $a \cdot J \subseteq I\}$.

Let R be a right zeroed add-left-cancelable left distributive non empty double loop structure and let I, J be left ideal non empty subsets of R. One can check that $I \,\%\, J$ is non empty.

Let R be a right zeroed add-left-cancelable left distributive non empty double loop structure and let I, J be add closed left ideal non empty subsets of R. One can check that $I \,\%\, J$ is add closed.

Let R be a right zeroed add-left-cancelable left distributive associative commutative non empty double loop structure and let I, J be left ideal non empty subsets of R. Note that $I \,\%\, J$ is left ideal and $I \,\%\, J$ is right ideal.

We now state several propositions:

(88) Let R be a non empty multiplicative loop structure, I be a right ideal non empty subset of R, and J be a subset of R. Then $I \subseteq I \,\%\, J$.

(89) Let R be a right zeroed add-left-cancelable left distributive non empty double loop structure, I be an add closed left ideal non empty subset of R, and J be a subset of R. Then $(I \,\%\, J) * J \subseteq I$.

(90) Let R be a left zeroed add-right-cancelable right distributive non empty double loop structure, I be an add closed right ideal non empty subset of R, and J be a subset of R. Then $(I \,\%\, J) * J \subseteq I$.

(91) Let R be a left zeroed add-right-cancelable right distributive commutative associative non empty double loop structure, I be an add closed right ideal non empty subset of R, and J, K be subsets of R. Then $(I \,\%\, J) \,\%\, K = I \,\%\, (J * K)$.

(92) For every non empty multiplicative loop structure R and for all subsets I, J, K of R holds $(J \cap K) \,\%\, I = (J \,\%\, I) \cap (K \,\%\, I)$.

(93) Let R be a left zeroed right zeroed add-right-cancelable right distributive non empty double loop structure, I be an add closed subset of R, and J, K

be right ideal non empty subsets of R. Then $I\%(J+K) = (I\%J) \cap (I\%K)$.

Let R be a unital non empty double loop structure and let I be a subset of R. The functor $\sqrt{I}$ yielding a subset of R is defined as follows:

(Def. 25) $\sqrt{I} = \{a; a$ ranges over elements of R: $\bigvee_{n\,:\,\text{natural number}} \ a^n \in I\}$.

Let R be a unital non empty double loop structure and let I be a non empty subset of R. One can verify that $\sqrt{I}$ is non empty.

Let R be an Abelian add-associative left zeroed right zeroed commutative associative add-cancelable distributive unital non empty double loop structure and let I be an add closed right ideal non empty subset of R. Observe that $\sqrt{I}$ is add closed.

Let R be a unital commutative associative non empty double loop structure and let I be a left ideal non empty subset of R. Observe that $\sqrt{I}$ is left ideal and $\sqrt{I}$ is right ideal.

One can prove the following propositions:

(94) Let R be a unital associative non empty double loop structure, I be a non empty subset of R, and a be an element of R. Then $a \in \sqrt{I}$ if and only if there exists a natural number n such that $a^n \in \sqrt{I}$.

(95) Let R be a left zeroed right zeroed add-cancelable distributive unital associative non empty double loop structure, I be an add closed right ideal non empty subset of R, and J be an add closed left ideal non empty subset of R. Then $\sqrt{I * J} = \sqrt{I \cap J}$.

6. Noetherian Rings and PIDs

Let L be a non empty double loop structure and let I be an ideal of L. We say that I is finitely generated if and only if:

(Def. 26) There exists a non empty finite subset F of the carrier of L such that $I = F$–ideal.

Let L be a non empty double loop structure. Observe that there exists an ideal of L which is finitely generated.

Let L be a non empty double loop structure and let F be a non empty finite subset of L. Note that F–ideal is finitely generated.

Let L be a non empty double loop structure. We say that L is Noetherian if and only if:

(Def. 27) Every ideal of L is finitely generated.

Let us observe that there exists a non empty double loop structure which is Euclidian, Abelian, add-associative, right zeroed, right complementable, well unital, distributive, commutative, associative, and non degenerated.

Let L be a non empty double loop structure and let I be an ideal of L. We say that I is principal if and only if:

(Def. 28) There exists an element e of the carrier of L such that $I = \{e\}$–ideal.

Let L be a non empty double loop structure. We say that L is PID if and only if:

(Def. 29) Every ideal of L is principal.

One can prove the following three propositions:

(96) Let L be a non empty double loop structure and F be a non empty subset of the carrier of L. Suppose $F \neq \{0_L\}$. Then there exists an element x of the carrier of L such that $x \neq 0_L$ and $x \in F$.

(97) Every add-associative left zeroed right zeroed right complementable distributive left unital Euclidian non empty double loop structure is PID.

(98) For every non empty double loop structure L such that L is PID holds L is Noetherian.

Let us note that INT.Ring is Noetherian.

Let us observe that there exists a non empty double loop structure which is Noetherian, Abelian, add-associative, right zeroed, right complementable, well unital, distributive, commutative, associative, and non degenerated.

Next we state two propositions:

(99) Let R be a Noetherian add-associative left zeroed right zeroed add-cancelable associative distributive well unital non empty double loop structure and B be a non empty subset of the carrier of R. Then there exists a non empty finite subset C of the carrier of R such that $C \subseteq B$ and C–ideal $= B$–ideal.

(100) Let R be a non empty double loop structure. Suppose that for every non empty subset B of the carrier of R there exists a non empty finite subset C of the carrier of R such that $C \subseteq B$ and C–ideal $= B$–ideal. Let a be a sequence of R. Then there exists a natural number m such that $a(m+1) \in (\mathrm{rng}(a \restriction \mathbb{Z}_{m+1}))$–ideal.

Let X, Y be non empty sets, let f be a function from X into Y, and let A be a non empty subset of X. One can check that $f \restriction A$ is non empty.

The following two propositions are true:

(101) Let R be a non empty double loop structure. Suppose that for every sequence a of R there exists a natural number m such that $a(m+1) \in (\mathrm{rng}(a \restriction \mathbb{Z}_{m+1}))$–ideal. Then there does not exist a function F from $\mathbb{N}$ into $2^{\text{the carrier of } R}$ such that

(i) for every natural number i holds $F(i)$ is an ideal of R, and

(ii) for all natural numbers j, k such that $j < k$ holds $F(j) \subseteq F(k)$ and $F(j) \neq F(k)$.

(102) Let R be a non empty double loop structure. Suppose that it is not true that there exists a function F from $\mathbb{N}$ into $2^{\text{the carrier of } R}$ such that for every natural number i holds $F(i)$ is an ideal of R and for all natural numbers j, k such that $j < k$ holds $F(j) \subseteq F(k)$ and $F(j) \neq F(k)$. Then R is Noetherian.

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Hilbert Basis Theorem[1]

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Summary. We prove the Hilbert basis theorem following [5], page 145. First we prove the theorem for the univariate case and then for the multivariate case. Our proof for the latter is slightly different than in [5]. As a base case we take the ring of polynomilas with no variables. We also prove that a polynomial ring with infinite number of variables is not Noetherian.

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The terminology and notation used in this paper are introduced in the following papers: [18], [19], [31], [13], [7], [4], [28], [12], [8], [9], [27], [1], [25], [2], [21], [3], [26], [22], [24], [16], [20], [23], [6], [32], [33], [29], [14], [30], [11], [15], [17], and [10].

1. Preliminaries

One can prove the following propositions:

(1) Let A, B be finite sequences and f be a function. Suppose $\operatorname{rng} A \cup \operatorname{rng} B \subseteq \operatorname{dom} f$. Then there exist finite sequences f_1, f_2 such that $f_1 = f \cdot A$ and $f_2 = f \cdot B$ and $f \cdot (A \frown B) = f_1 \frown f_2$.

(2) For every bag b of 0 holds $\operatorname{decomp} b = \langle\langle\emptyset, \emptyset\rangle\rangle$.

(3) For all natural numbers i, j and for every bag b of j such that $i \leqslant j$ holds $b \restriction i$ is an element of $\mathrm{Bags}\, i$.

(4) Let i, j be sets, b_1, b_2 be bags of j, and b'_1, b'_2 be bags of i. If $b'_1 = b_1 \restriction i$ and $b'_2 = b_2 \restriction i$ and b_1 divides b_2, then b'_1 divides b'_2.

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(5) Let i, j be sets, b_1, b_2 be bags of j, and b'_1, b'_2 be bags of i. If $b'_1 = b_1 \restriction i$ and $b'_2 = b_2 \restriction i$, then $(b_1 -' b_2) \restriction i = b'_1 -' b'_2$ and $(b_1 + b_2) \restriction i = b'_1 + b'_2$.

Let n, k be natural numbers and let b be a bag of n. The functor b extended by k yields an element of Bags $n+1$ and is defined as follows:

(Def. 1) $(b$ extended by $k) \restriction n = b$ and $(b$ extended by $k)(n) = k$.

We now state two propositions:

(6) For every natural number n holds EmptyBag $n+1 =$ EmptyBag n extended by 0.

(7) For every ordinal number n and for all bags b, b_1 of n holds $b_1 \in$ rng divisors b iff b_1 divides b.

Let X be a set and let x be an element of X. The functor UnitBag x yields an element of Bags X and is defined as follows:

(Def. 2) UnitBag $x =$ EmptyBag $X + \cdot\, (x, 1)$.

Next we state four propositions:

(8) For every non empty set X and for every element x of X holds support UnitBag $x = \{x\}$.

(9) Let X be a non empty set and x be an element of X. Then $(\mathrm{UnitBag}\, x)(x) = 1$ and for every element y of X such that $x \neq y$ holds $(\mathrm{UnitBag}\, x)(y) = 0$.

(10) For every non empty set X and for all elements x_1, x_2 of X such that UnitBag $x_1 =$ UnitBag x_2 holds $x_1 = x_2$.

(11) Let X be a non empty ordinal number, x be an element of X, L be a unital non trivial non empty double loop structure, and e be a function from X into L. Then $\mathrm{eval}(\mathrm{UnitBag}\, x, e) = e(x)$.

Let X be a set, let x be an element of X, and let L be a unital non empty multiplicative loop with zero structure. The functor $1\_1(x, L)$ yielding a Series of X, L is defined by:

(Def. 3) $1\_1(x, L) = 0\_(X, L) + \cdot\, (\mathrm{UnitBag}\, x, 1_L)$.

One can prove the following propositions:

(12) Let X be a set, L be a unital non trivial non empty double loop structure, and x be an element of X. Then $(1\_1(x, L))(\mathrm{UnitBag}\, x) = 1_L$ and for every bag b of X such that $b \neq \mathrm{UnitBag}\, x$ holds $(1\_1(x, L))(b) = 0_L$.

(13) Let X be a set, x be an element of X, and L be an add-associative right zeroed right complementable unital right distributive non trivial non empty double loop structure. Then Support $1\_1(x, L) = \{\mathrm{UnitBag}\, x\}$.

Let X be an ordinal number, let x be an element of X, and let L be an add-associative right zeroed right complementable unital right distributive non trivial non empty double loop structure. Observe that $1\_1(x, L)$ is finite-Support.

One can prove the following three propositions:

(14) Let L be an add-associative right zeroed right complementable unital right distributive non trivial non empty double loop structure, X be a non empty set, and x_1, x_2 be elements of X. If $1\_1(x_1, L) = 1\_1(x_2, L)$, then $x_1 = x_2$.

(15) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure, x be an element of the carrier of Polynom-Ring L, and p be a sequence of L. If $x = p$, then $-x = -p$.

(16) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure, x, y be elements of the carrier of Polynom-Ring L, and p, q be sequences of L. If $x = p$ and $y = q$, then $x - y = p - q$.

Let L be a right zeroed add-associative right complementable unital distributive non empty double loop structure and let I be a non empty subset of the carrier of Polynom-Ring L. The functor minlen I yields a non empty subset of I and is defined by:

(Def. 4) $\operatorname{minlen} I = \{x; x$ ranges over elements of I: $\bigwedge_{x', y' : \text{Polynomial of } L} (x' = x \ \wedge \ y' \in I \ \Rightarrow \ \operatorname{len} x' \leqslant \operatorname{len} y')\}$.

We now state the proposition

(17) Let L be a right zeroed add-associative right complementable unital distributive non empty double loop structure, I be a non empty subset of the carrier of Polynom-Ring L, and i_1, i_2 be Polynomials of L. If $i_1 \in \operatorname{minlen} I$ and $i_2 \in I$, then $i_1 \in I$ and $\operatorname{len} i_1 \leqslant \operatorname{len} i_2$.

Let L be a right zeroed add-associative right complementable unital distributive non empty double loop structure, let n be a natural number, and let a be an element of the carrier of L. The functor monomial(a, n) yields a Polynomial of L and is defined as follows:

(Def. 5) For every natural number x holds if $x = n$, then $(\text{monomial}(a, n))(x) = a$ and if $x \neq n$, then $(\text{monomial}(a, n))(x) = 0_L$.

The following four propositions are true:

(18) Let L be a right zeroed add-associative right complementable unital distributive non empty double loop structure, n be a natural number, and a be an element of the carrier of L. Then if $a \neq 0_L$, then $\operatorname{len} \text{monomial}(a, n) = n + 1$ and if $a = 0_L$, then $\operatorname{len} \text{monomial}(a, n) = 0$ and $\operatorname{len} \text{monomial}(a, n) \leqslant n + 1$.

(19) Let L be a right zeroed add-associative right complementable unital distributive non empty double loop structure, n, x be natural numbers, a be an element of the carrier of L, and p be a Polynomial of L. Then $(\text{monomial}(a, n) * p)(x + n) = a \cdot p(x)$.

(20) Let L be a right zeroed add-associative right complementable unital distributive non empty double loop structure, n, x be natural numbers,

a be an element of the carrier of L, and p be a Polynomial of L. Then $(p * \mathrm{monomial}(a, n))(x + n) = p(x) \cdot a$.

(21) Let L be a right zeroed add-associative right complementable unital distributive non empty double loop structure and p, q be Polynomials of L. Then $\mathrm{len}(p * q) \leqslant (\mathrm{len}\, p + \mathrm{len}\, q) -' 1$.

2. On Ring Isomorphism

The following propositions are true:

(22) Let R, S be non empty double loop structures, I be an ideal of R, and P be a map from R into S. If P is a ring isomorphism, then $P^\circ I$ is an ideal of S.

(23) Let R, S be add-associative right zeroed right complementable non empty double loop structures and f be a map from R into S. If f is a ring homomorphism, then $f(0_R) = 0_S$.

(24) Let R, S be add-associative right zeroed right complementable non empty double loop structures, F be a non empty subset of the carrier of R, G be a non empty subset of the carrier of S, P be a map from R into S, l_1 be a linear combination of F, L_1 be a linear combination of G, and E be a finite sequence of elements of $[\!:$ the carrier of R, the carrier of R, the carrier of $R\, :\!]$. Suppose that

(i) P is a ring homomorphism,

(ii) $\mathrm{len}\, l_1 = \mathrm{len}\, L_1$,

(iii) E represents l_1, and

(iv) for every set i such that $i \in \mathrm{dom}\, L_1$ holds $L_1(i) = P((E_i)_{\mathbf{1}}) \cdot P((E_i)_{\mathbf{2}}) \cdot P((E_i)_{\mathbf{3}})$.

Then $P(\sum l_1) = \sum L_1$.

(25) Let R, S be non empty double loop structures and P be a map from R into S. Suppose P is a ring isomorphism. Then there exists a map P_1 from S into R such that P_1 is a ring isomorphism and $P_1 = P^{-1}$.

(26) Let R, S be Abelian add-associative right zeroed right complementable associative distributive well unital non empty double loop structures, F be a non empty subset of the carrier of R, and P be a map from R into S. If P is a ring isomorphism, then $P^\circ F$–ideal $= (P^\circ F)$–ideal.

(27) Let R, S be Abelian add-associative right zeroed right complementable associative distributive well unital non empty double loop structures and P be a map from R into S. If P is a ring isomorphism and R is Noetherian, then S is Noetherian.

(28) Let R be an add-associative right zeroed right complementable associative distributive well unital non trivial non empty double loop structure. Then there exists a map from R into Polynom-Ring$(0, R)$ which is a ring isomorphism.

(29) Let R be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, n be a natural number, b be a bag of n, p_1 be a Polynomial of n, R, and F be a finite sequence of elements of the carrier of Polynom-Ring(n, R). Suppose $p_1 = \sum F$. Then there exists a function g from the carrier of Polynom-Ring(n, R) into the carrier of R such that for every Polynomial p of n, R holds $g(p) = p(b)$ and $p_1(b) = \sum(g \cdot F)$.

Let R be an Abelian add-associative right zeroed right complementable associative distributive well unital commutative non trivial non empty double loop structure and let n be a natural number. The functor upm(n, R) yielding a map from Polynom-Ring Polynom-Ring(n, R) into Polynom-Ring$(n+1, R)$ is defined by the condition (Def. 6).

(Def. 6) Let p_1 be a Polynomial of Polynom-Ring(n, R), p_2 be a Polynomial of n, R, p_3 be a Polynomial of $n+1$, R, and b be a bag of $n+1$. If $p_3 = (\mathrm{upm}(n, R))(p_1)$ and $p_2 = p_1(b(n))$, then $p_3(b) = p_2(b \restriction n)$.

Let R be an Abelian add-associative right zeroed right complementable associative distributive well unital commutative non trivial non empty double loop structure and let n be a natural number. One can verify the following observations:

* upm(n, R) is additive,
* upm(n, R) is multiplicative,
* upm(n, R) is unity-preserving, and
* upm(n, R) is one-to-one.

Let R be an Abelian add-associative right zeroed right complementable associative distributive well unital commutative non trivial non empty double loop structure and let n be a natural number. The functor mpu(n, R) yields a map from Polynom-Ring$(n+1, R)$ into Polynom-Ring Polynom-Ring(n, R) and is defined by the condition (Def. 7).

(Def. 7) Let p_1 be a Polynomial of $n+1$, R, p_2 be a Polynomial of n, R, p_3 be a Polynomial of Polynom-Ring(n, R), i be a natural number, and b be a bag of n. If $p_3 = (\mathrm{mpu}(n, R))(p_1)$ and $p_2 = p_3(i)$, then $p_2(b) = p_1(b$ extended by $i)$.

Next we state two propositions:

(30) Let R be an Abelian add-associative right zeroed right complementable associative distributive well unital commutative non trivial non empty double loop structure, n be a natural number, and p be an element of the

carrier of Polynom-Ring$(n+1, R)$. Then $(\mathrm{upm}(n, R))((\mathrm{mpu}(n, R))(p)) = p$.

(31) Let R be an Abelian add-associative right zeroed right complementable associative distributive well unital commutative non trivial non empty double loop structure and n be a natural number. Then there exists a map from Polynom-Ring Polynom-Ring(n, R) into Polynom-Ring$(n+1, R)$ which is a ring isomorphism.

3. Hilbert Basis Theorem

Let R be a Noetherian Abelian add-associative right zeroed right complementable associative distributive well unital commutative non empty double loop structure. Observe that Polynom-Ring R is Noetherian.

One can prove the following propositions:

(32) Let R be a Noetherian Abelian add-associative right zeroed right complementable associative distributive well unital commutative non empty double loop structure. Then Polynom-Ring R is Noetherian.

(33) Let R be an Abelian add-associative right zeroed right complementable associative distributive well unital non trivial commutative non empty double loop structure. Suppose R is Noetherian. Let n be a natural number. Then Polynom-Ring(n, R) is Noetherian.

(34) Every field is Noetherian.

(35) For every field F and for every natural number n holds Polynom-Ring(n, F) is Noetherian.

(36) Let R be an Abelian right zeroed add-associative right complementable well unital distributive associative commutative non trivial non empty double loop structure and X be an infinite ordinal number. Then Polynom-Ring(X, R) is non Noetherian.

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Dynkin's Lemma in Measure Theory

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Summary. This article formalizes the proof of Dynkin's lemma in measure theory. Dynkin's lemma is a useful tool in measure theory and probability theory: it helps frequently to generalize a statement about all elements of a intersection-stable set system to all elements of the sigma-field generated by that system.

MML Identifier: `DYNKIN`.

The terminology and notation used in this paper have been introduced in the following articles: [5], [11], [1], [4], [2], [3], [7], [6], [12], [13], [8], [10], and [9].

1. Preliminaries

For simplicity, we adopt the following rules: O_1 denotes a non empty set, f denotes a sequence of subsets of O_1, X, A, B denote subsets of O_1, D denotes a non empty subset of 2^{O_1}, n, m denote natural numbers, F denotes a non empty set, and x, Y denote sets.

Next we state two propositions:

(1) For every sequence f of subsets of O_1 and for every x holds $x \in \operatorname{rng} f$ iff there exists n such that $f(n) = x$.

(2) For every n holds $\operatorname{PSeg} n$ is finite.

Let us consider n. One can verify that $\operatorname{PSeg} n$ is finite.

Next we state the proposition

(3) For all sets x, y, z such that $x \subseteq y$ holds x misses $z \setminus y$.

Let a, b, c be sets. The functor a, b followed by c is defined as follows:

(Def. 1) a, b followed by $c = (\mathbb{N} \longmapsto c) {+\cdot} [0 \longmapsto a, 1 \longmapsto b]$.

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Let a, b, c be sets. Observe that a, b followed by c is function-like and relation-like.

Let X be a non empty set and let a, b, c be elements of X. Then a, b followed by c is a function from $\mathbb{N}$ into X.

Next we state the proposition

(4) For every non empty set X and for all elements a, b, c of X holds a, b followed by c is a function from $\mathbb{N}$ into X.

Let O_1 be a non empty set and let a, b, c be subsets of O_1. Then a, b followed by c is a sequence of subsets of O_1.

One can prove the following propositions:

(5) For all sets a, b, c holds $(a, b \text{ followed by } c)(0) = a$ and $(a, b \text{ followed by } c)(1) = b$ and for every n such that $n \neq 0$ and $n \neq 1$ holds $(a, b \text{ followed by } c)(n) = c$.

(6) For all subsets a, b of O_1 holds $\bigcup \mathrm{rng}(a, b \text{ followed by } \emptyset) = a \cup b$.

Let O_1 be a non empty set, let f be a sequence of subsets of O_1, and let X be a subset of O_1. The functor seqIntersection(X, f) yields a sequence of subsets of O_1 and is defined by:

(Def. 2) For every n holds $(\mathrm{seqIntersection}(X, f))(n) = X \cap f(n)$.

2. Disjoint-valued Functions and Intersection

Let us consider O_1 and let us consider f. Let us observe that f is disjoint valued if and only if:

(Def. 3) If $n < m$, then $f(n)$ misses $f(m)$.

We now state the proposition

(7) For every non empty set Y and for every x holds $x \subseteq \bigcap Y$ iff for every element y of Y holds $x \subseteq y$.

Let x be a set. We introduce x is intersection stable as a synonym of x is multiplicative.

Let O_1 be a non empty set, let f be a sequence of subsets of O_1, and let n be an element of $\mathbb{N}$. The functor disjointify(f, n) yielding an element of 2^{O_1} is defined by:

(Def. 5)[1] $\mathrm{disjointify}(f, n) = f(n) \setminus \bigcup \mathrm{rng}(f \restriction \mathrm{PSeg}\, n)$.

Let O_1 be a non empty set and let g be a sequence of subsets of O_1. The functor disjointify g yielding a sequence of subsets of O_1 is defined as follows:

(Def. 6) For every n holds $(\mathrm{disjointify}\, g)(n) = \mathrm{disjointify}(g, n)$.

The following propositions are true:

[1]The definition (Def. 4) has been removed.

(8) For every n holds $(\mathrm{disjointify}\, f)(n) = f(n) \setminus \bigcup \mathrm{rng}(f \restriction \mathrm{PSeg}\, n)$.

(9) For every sequence f of subsets of O_1 holds disjointify f is disjoint valued.

(10) For every sequence f of subsets of O_1 holds $\bigcup \mathrm{rng}\, \mathrm{disjointify}\, f = \bigcup \mathrm{rng}\, f$.

(11) For all subsets x, y of O_1 such that x misses y holds x, y followed by $\emptyset_{(O_1)}$ is disjoint valued.

(12) Let f be a sequence of subsets of O_1. Suppose f is disjoint valued. Let X be a subset of O_1. Then $\mathrm{seqIntersection}(X, f)$ is disjoint valued.

(13) For every sequence f of subsets of O_1 and for every subset X of O_1 holds $X \cap \mathrm{Union}\, f = \mathrm{Union}\, \mathrm{seqIntersection}(X, f)$.

3. Dynkin Systems: Definition and Closure Properties

Let us consider O_1. A subset of 2^{O_1} is called a Dynkin system of O_1 if:

(Def. 7) For every f such that $\mathrm{rng}\, f \subseteq$ it and f is disjoint valued holds $\mathrm{Union}\, f \in$ it and for every X such that $X \in$ it holds $X^{\mathrm{c}} \in$ it and $\emptyset \in$ it.

Let us consider O_1. One can check that every Dynkin system of O_1 is non empty.

The following propositions are true:

(14) 2^{O_1} is a Dynkin system of O_1.

(15) If for every Y such that $Y \in F$ holds Y is a Dynkin system of O_1, then $\bigcap F$ is a Dynkin system of O_1.

(16) If D is a Dynkin system of O_1 and intersection stable, then if $A \in D$ and $B \in D$, then $A \setminus B \in D$.

(17) If D is a Dynkin system of O_1 and intersection stable, then if $A \in D$ and $B \in D$, then $A \cup B \in D$.

(18) Suppose D is a Dynkin system of O_1 and intersection stable. Let x be a finite set. If $x \subseteq D$, then $\bigcup x \in D$.

(19) Suppose D is a Dynkin system of O_1 and intersection stable. Let f be a sequence of subsets of O_1. If $\mathrm{rng}\, f \subseteq D$, then $\mathrm{rng}\, \mathrm{disjointify}\, f \subseteq D$.

(20) Suppose D is a Dynkin system of O_1 and intersection stable. Let f be a sequence of subsets of O_1. If $\mathrm{rng}\, f \subseteq D$, then $\bigcup \mathrm{rng}\, f \in D$.

(21) For every Dynkin system D of O_1 and for all elements x, y of D such that x misses y holds $x \cup y \in D$.

(22) For every Dynkin system D of O_1 and for all elements x, y of D such that $x \subseteq y$ holds $y \setminus x \in D$.

4. Main Steps for Dynkin's Lemma

One can prove the following proposition

(23) If D is a Dynkin system of O_1 and intersection stable, then D is a σ-field of subsets of O_1.

Let O_1 be a non empty set and let E be a subset of 2^{O_1}. The functor $\mathrm{GenDynSys}\,E$ yielding a Dynkin system of O_1 is defined by:

(Def. 8) $E \subseteq \mathrm{GenDynSys}\,E$ and for every Dynkin system D of O_1 such that $E \subseteq D$ holds $\mathrm{GenDynSys}\,E \subseteq D$.

Let O_1 be a non empty set, let G be a set, and let X be a subset of O_1. The functor $\mathrm{DynSys}(X, G)$ yields a subset of 2^{O_1} and is defined as follows:

(Def. 9) For every subset A of O_1 holds $A \in \mathrm{DynSys}(X, G)$ iff $A \cap X \in G$.

Let O_1 be a non empty set, let G be a Dynkin system of O_1, and let X be an element of G. Then $\mathrm{DynSys}(X, G)$ is a Dynkin system of O_1.

Next we state four propositions:

(24) Let E be a subset of 2^{O_1} and X, Y be subsets of O_1. If $X \in E$ and $Y \in \mathrm{GenDynSys}\,E$ and E is intersection stable, then $X \cap Y \in \mathrm{GenDynSys}\,E$.

(25) Let E be a subset of 2^{O_1} and X, Y be subsets of O_1. If $X \in \mathrm{GenDynSys}\,E$ and $Y \in \mathrm{GenDynSys}\,E$ and E is intersection stable, then $X \cap Y \in \mathrm{GenDynSys}\,E$.

(26) For every subset E of 2^{O_1} such that E is intersection stable holds $\mathrm{GenDynSys}\,E$ is intersection stable.

(27) Let E be a subset of 2^{O_1}. Suppose E is intersection stable. Let D be a Dynkin system of O_1. If $E \subseteq D$, then $\sigma(E) \subseteq D$.

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Lower Tolerance. Preliminaries to Wroclaw Taxonomy[1]

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Summary. The paper introduces some preliminary notions concerning the Wroclaw taxonomy according to [16]. The classifications and tolerances are defined and considered w.r.t. sets and metric spaces. We prove theorems showing various classifications based on tolerances.

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The articles [14], [15], [20], [4], [9], [5], [6], [8], [12], [1], [13], [17], [19], [2], [23], [25], [24], [3], [18], [22], [21], [10], [11], and [7] provide the terminology and notation for this paper.

1. Preliminaries

In this paper A, X are non empty sets, f is a partial function from $[: X, X :]$ to $\mathbb{R}$, and a is a real number.

Let us note that there exists a real number which is non negative.

We now state a number of propositions:

(1) For every finite sequence p and for every natural number k such that $k+1 \in \operatorname{dom} p$ and $k \notin \operatorname{dom} p$ holds $k = 0$.

(2) Let p be a finite sequence and i, j be natural numbers. Suppose $i \in \operatorname{dom} p$ and $j \in \operatorname{dom} p$ and for every natural number k such that $k \in \operatorname{dom} p$ and $k+1 \in \operatorname{dom} p$ holds $p(k) = p(k+1)$. Then $p(i) = p(j)$.

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(3) For every set X and for every binary relation R on X such that R is reflexive in X holds $\operatorname{dom} R = X$.

(4) For every set X and for every binary relation R on X such that R is reflexive in X holds $\operatorname{rng} R = X$.

(5) For every set X and for every binary relation R on X such that R is reflexive in X holds R^* is reflexive in X.

(6) Let X, x, y be sets and R be a binary relation on X. Suppose R is reflexive in X. If R reduces x to y and $x \in X$, then $\langle x, y\rangle \in R^*$.

(7) Let X be a set and R be a binary relation on X. If R is reflexive in X and symmetric in X, then R^* is symmetric in X.

(8) For every set X and for every binary relation R on X such that R is reflexive in X holds R^* is transitive in X.

(9) Let X be a non empty set and R be a binary relation on X. Suppose R is reflexive in X and symmetric in X. Then R^* is an equivalence relation of X.

(10) For all binary relations R_1, R_2 on X such that $R_1 \subseteq R_2$ holds ${R_1}^* \subseteq {R_2}^*$.

(11) SmallestPartition(A) is finer than $\{A\}$.

2. The Notion of Classification

Let A be a non empty set. A subset of PARTITIONS(A) is called a classification of A if:

(Def. 1) For all partitions X, Y of A such that $X \in$ it and $Y \in$ it holds X is finer than Y or Y is finer than X.

One can prove the following propositions:

(12) $\{\{A\}\}$ is a classification of A.

(13) $\{$SmallestPartition(A)$\}$ is a classification of A.

(14) For every subset S of PARTITIONS(A) such that $S = \{\{A\},$ SmallestPartition(A)$\}$ holds S is a classification of A.

Let A be a non empty set. A subset of PARTITIONS(A) is called a strong classification of A if:

(Def. 2) It is a classification of A and $\{A\} \in$ it and SmallestPartition(A) $\in$ it.

Next we state the proposition

(15) For every subset S of PARTITIONS(A) such that $S = \{\{A\},$ SmallestPartition(A)$\}$ holds S is a strong classification of A.

3. The Tolerance on a Non Empty Set

Let X be a non empty set, let f be a partial function from $[: X, X :]$ to $\mathbb{R}$, and let a be a real number. The functor $\mathrm{T_l}(f, a)$ yields a binary relation on X and is defined as follows:

(Def. 3) For all elements x, y of X holds $\langle x, y\rangle \in \mathrm{T_l}(f, a)$ iff $f(x, y) \leqslant a$.

The following four propositions are true:

(16) If f is Reflexive and $a \geqslant 0$, then $\mathrm{T_l}(f, a)$ is reflexive in X.

(17) If f is symmetric, then $\mathrm{T_l}(f, a)$ is symmetric in X.

(18) If $a \geqslant 0$ and f is Reflexive and symmetric, then $\mathrm{T_l}(f, a)$ is a tolerance of X.

(19) Let X be a non empty set, f be a partial function from $[: X, X :]$ to $\mathbb{R}$, and a_1, a_2 be real numbers. If $a_1 \leqslant a_2$, then $\mathrm{T_l}(f, a_1) \subseteq \mathrm{T_l}(f, a_2)$.

Let X be a set and let f be a partial function from $[: X, X :]$ to $\mathbb{R}$. We say that f is non-negative if and only if:

(Def. 4) For all elements x, y of X holds $f(x, y) \geqslant 0$.

We now state three propositions:

(20) Let X be a non empty set, f be a partial function from $[: X, X :]$ to $\mathbb{R}$, and x, y be sets. Suppose f is non-negative, Reflexive, and discernible. If $\langle x, y\rangle \in \mathrm{T_l}(f, 0)$, then $x = y$.

(21) Let X be a non empty set, f be a partial function from $[: X, X :]$ to $\mathbb{R}$, and x be an element of X. If f is Reflexive and discernible, then $\langle x, x\rangle \in \mathrm{T_l}(f, 0)$.

(22) Let X be a non empty set, f be a partial function from $[: X, X :]$ to $\mathbb{R}$, and a be a real number. Suppose $\mathrm{T_l}(f, a)$ is reflexive in X and f is symmetric. Then $(\mathrm{T_l}(f, a))^*$ is an equivalence relation of X.

4. The Partitions Defined by Lower Tolerance

Next we state several propositions:

(23) Let X be a non empty set and f be a partial function from $[: X, X :]$ to $\mathbb{R}$. Suppose f is non-negative, Reflexive, and discernible. Then $(\mathrm{T_l}(f, 0))^* = \mathrm{T_l}(f, 0)$.

(24) Let X be a non empty set, f be a partial function from $[: X, X :]$ to $\mathbb{R}$, and R be an equivalence relation of X. Suppose $R = (\mathrm{T_l}(f, 0))^*$ and f is non-negative, Reflexive, and discernible. Then $R = \triangle_X$.

(25) Let X be a non empty set, f be a partial function from $[\!: X, X :\!]$ to $\mathbb{R}$, and R be an equivalence relation of X. Suppose $R = (\mathrm{T}_1(f, 0))^*$ and f is non-negative, Reflexive, and discernible. Then $\mathrm{Classes}\, R = \mathrm{SmallestPartition}(X)$.

(26) Let X be a finite non empty subset of $\mathbb{R}$, f be a function from $[\!: X, X :\!]$ into $\mathbb{R}$, z be a finite non empty subset of $\mathbb{R}$, and A be a real number. If $z = \mathrm{rng}\, f$ and $A \geqslant \max z$, then for all elements x, y of X holds $f(x, y) \leqslant A$.

(27) Let X be a finite non empty subset of $\mathbb{R}$, f be a function from $[\!: X, X :\!]$ into $\mathbb{R}$, z be a finite non empty subset of $\mathbb{R}$, and A be a real number. Suppose $z = \mathrm{rng}\, f$ and $A \geqslant \max z$. Let R be an equivalence relation of X. If $R = (\mathrm{T}_1(f, A))^*$, then $\mathrm{Classes}\, R = \{X\}$.

(28) Let X be a finite non empty subset of $\mathbb{R}$, f be a function from $[\!: X, X :\!]$ into $\mathbb{R}$, z be a finite non empty subset of $\mathbb{R}$, and A be a real number. If $z = \mathrm{rng}\, f$ and $A \geqslant \max z$, then $(\mathrm{T}_1(f, A))^* = \mathrm{T}_1(f, A)$.

5. The Classification on a Non Empty Set

Let X be a non empty set and let f be a partial function from $[\!: X, X :\!]$ to $\mathbb{R}$. The functor $\mathrm{FamClass}\, f$ yielding a subset of $\mathrm{PARTITIONS}(X)$ is defined by the condition (Def. 5).

(Def. 5) Let x be a set. Then $x \in \mathrm{FamClass}\, f$ if and only if there exists a non negative real number a and there exists an equivalence relation R of X such that $R = (\mathrm{T}_1(f, a))^*$ and $\mathrm{Classes}\, R = x$.

We now state four propositions:

(29) Let X be a non empty set, f be a partial function from $[\!: X, X :\!]$ to $\mathbb{R}$, and a be a non negative real number. If $\mathrm{T}_1(f, a)$ is reflexive in X and f is symmetric, then $\mathrm{FamClass}\, f$ is a non empty set.

(30) Let X be a finite non empty subset of $\mathbb{R}$ and f be a function from $[\!: X, X :\!]$ into $\mathbb{R}$. If f is symmetric and non-negative, then $\{X\} \in \mathrm{FamClass}\, f$.

(31) For every non empty set X and for every partial function f from $[\!: X, X :\!]$ to $\mathbb{R}$ holds $\mathrm{FamClass}\, f$ is a classification of X.

(32) Let X be a finite non empty subset of $\mathbb{R}$ and f be a function from $[\!: X, X :\!]$ into $\mathbb{R}$. Suppose $\mathrm{SmallestPartition}(X) \in \mathrm{FamClass}\, f$ and f is symmetric and non-negative. Then $\mathrm{FamClass}\, f$ is a strong classification of X.

6. The Classification on a Metric Space

Let M be a metric structure, let a be a real number, and let x, y be elements of the carrier of M. We say that x, y are in tolerance w.r.t. a if and only if:

(Def. 6) $\rho(x, y) \leqslant a$.

Let M be a non empty metric structure and let a be a real number. The functor $\mathrm{T_m}(M, a)$ yielding a binary relation on M is defined by:

(Def. 7) For all elements x, y of the carrier of M holds $\langle x, y\rangle \in \mathrm{T_m}(M, a)$ iff x, y are in tolerance w.r.t. a.

Next we state two propositions:

(33) For every non empty metric structure M and for every real number a holds $\mathrm{T_m}(M, a) = \mathrm{T_l}$(the distance of M, a).

(34) Let M be a non empty Reflexive symmetric metric structure, a be a real number, and T be a relation between the carrier of M and the carrier of M. If $T = \mathrm{T_m}(M, a)$ and $a \geqslant 0$, then T is a tolerance of the carrier of M.

Let M be a Reflexive symmetric non empty metric structure. The functor MetricFamClass M yielding a subset of PARTITIONS(the carrier of M) is defined by the condition (Def. 8).

(Def. 8) Let x be a set. Then $x \in$ MetricFamClass M if and only if there exists a non negative real number a and there exists an equivalence relation R of M such that $R = (\mathrm{T_m}(M, a))^*$ and Classes $R = x$.

The following propositions are true:

(35) For every Reflexive symmetric non empty metric structure M holds MetricFamClass M = FamClass the distance of M.

(36) Let M be a non empty metric space and R be an equivalence relation of M. If $R = (\mathrm{T_m}(M, 0))^*$, then Classes R = SmallestPartition(the carrier of M).

(37) For every Reflexive symmetric bounded non empty metric structure M such that $a \geqslant \emptyset(\Omega_M)$ holds $\mathrm{T_m}(M, a) = \nabla_{\text{the carrier of } M}$.

(38) For every Reflexive symmetric bounded non empty metric structure M such that $a \geqslant \emptyset(\Omega_M)$ holds $\mathrm{T_m}(M, a) = (\mathrm{T_m}(M, a))^*$.

(39) For every Reflexive symmetric bounded non empty metric structure M such that $a \geqslant \emptyset(\Omega_M)$ holds $(\mathrm{T_m}(M, a))^* = \nabla_{\text{the carrier of } M}$.

(40) Let M be a Reflexive symmetric bounded non empty metric structure, R be an equivalence relation of M, and a be a non negative real number. If $a \geqslant \emptyset(\Omega_M)$ and $R = (\mathrm{T_m}(M, a))^*$, then Classes R = {the carrier of M}.

Let M be a Reflexive symmetric triangle non empty metric structure and let C be a non empty bounded subset of M. Observe that $\emptyset C$ is non negative.

We now state three propositions:

(41) For every bounded non empty metric space M holds {the carrier of M} $\in$ MetricFamClass M.

(42) For every Reflexive symmetric non empty metric structure M holds MetricFamClass M is a classification of the carrier of M.

(43) For every bounded non empty metric space M holds MetricFamClass M is a strong classification of the carrier of M.

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Concrete Categories

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Summary. In the paper, we develop the notation of duality and equivalence of categories and concrete categories based on [9]. The development was motivated by the duality theory for continuous lattices (see [5, p. 189]), where we need to cope with concrete categories of lattices and maps preserving their properties. For example, the category *UPS* of complete lattices and directed suprema preserving maps; or the category *INF* of complete lattices and infima preserving maps. As the main result of this paper it is shown that every category is isomorphic to its concretization (the concrete category with the same objects). Some useful schemes to construct categories and functors are also presented.

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The notation and terminology used here are introduced in the following articles: [9], [10], [7], [2], [13], [11], [6], [3], [4], [1], [14], [15], [12], and [8].

1. Definability of Categories and Functors

In this article we present several logical schemes. The scheme *AltCatStrLambda* deals with a non empty set $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a set, and a 5-ary functor $\mathcal{G}$ yielding a set, and states that:

> There exists a strict non empty transitive category structure C such that
> (i) the carrier of $C = \mathcal{A}$,
> (ii) for all objects a, b of C holds $\langle a, b\rangle = \mathcal{F}(a, b)$, and
> (iii) for all objects a, b, c of C such that $\langle a, b\rangle \neq \emptyset$ and $\langle b, c\rangle \neq \emptyset$ and for every morphism f from a to b and for every morphism g from b to c holds $g \cdot f = \mathcal{G}(a, b, c, f, g)$

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provided the following requirement is met:

- For all elements a, b, c of $\mathcal{A}$ and for all sets f, g such that $f \in \mathcal{F}(a,b)$ and $g \in \mathcal{F}(b,c)$ holds $\mathcal{G}(a,b,c,f,g) \in \mathcal{F}(a,c)$.

The scheme *CatAssocSch* deals with a non empty transitive category structure $\mathcal{A}$ and a 5-ary functor $\mathcal{F}$ yielding a set, and states that:

$\mathcal{A}$ is associative

provided the parameters meet the following requirements:

- Let a, b, c be objects of $\mathcal{A}$. Suppose $\langle a,b\rangle \neq \emptyset$ and $\langle b,c\rangle \neq \emptyset$. Let f be a morphism from a to b and g be a morphism from b to c. Then $g \cdot f = \mathcal{F}(a,b,c,f,g)$, and
- Let a, b, c, d be objects of $\mathcal{A}$ and f, g, h be sets. If $f \in \langle a,b\rangle$ and $g \in \langle b,c\rangle$ and $h \in \langle c,d\rangle$, then $\mathcal{F}(a,c,d,\mathcal{F}(a,b,c,f,g),h) = \mathcal{F}(a,b,d,f,\mathcal{F}(b,c,d,g,h))$.

The scheme *CatUnitsSch* deals with a non empty transitive category structure $\mathcal{A}$ and a 5-ary functor $\mathcal{F}$ yielding a set, and states that:

$\mathcal{A}$ has units

provided the parameters satisfy the following conditions:

- Let a, b, c be objects of $\mathcal{A}$. Suppose $\langle a,b\rangle \neq \emptyset$ and $\langle b,c\rangle \neq \emptyset$. Let f be a morphism from a to b and g be a morphism from b to c. Then $g \cdot f = \mathcal{F}(a,b,c,f,g)$,
- Let a be an object of $\mathcal{A}$. Then there exists a set f such that $f \in \langle a,a\rangle$ and for every object b of $\mathcal{A}$ and for every set g such that $g \in \langle a,b\rangle$ holds $\mathcal{F}(a,a,b,f,g) = g$, and
- Let a be an object of $\mathcal{A}$. Then there exists a set f such that $f \in \langle a,a\rangle$ and for every object b of $\mathcal{A}$ and for every set g such that $g \in \langle b,a\rangle$ holds $\mathcal{F}(b,a,a,g,f) = g$.

The scheme *CategoryLambda* deals with a non empty set $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a set, and a 5-ary functor $\mathcal{G}$ yielding a set, and states that:

There exists a strict category C such that

(i) the carrier of $C = \mathcal{A}$,

(ii) for all objects a, b of C holds $\langle a,b\rangle = \mathcal{F}(a,b)$, and

(iii) for all objects a, b, c of C such that $\langle a,b\rangle \neq \emptyset$ and $\langle b,c\rangle \neq \emptyset$ and for every morphism f from a to b and for every morphism g from b to c holds $g \cdot f = \mathcal{G}(a,b,c,f,g)$

provided the parameters satisfy the following conditions:

- For all elements a, b, c of $\mathcal{A}$ and for all sets f, g such that $f \in \mathcal{F}(a,b)$ and $g \in \mathcal{F}(b,c)$ holds $\mathcal{G}(a,b,c,f,g) \in \mathcal{F}(a,c)$,
- Let a, b, c, d be elements of $\mathcal{A}$ and f, g, h be sets. If $f \in \mathcal{F}(a,b)$ and $g \in \mathcal{F}(b,c)$ and $h \in \mathcal{F}(c,d)$, then $\mathcal{G}(a,c,d,\mathcal{G}(a,b,c,f,g),h) = \mathcal{G}(a,b,d,f,\mathcal{G}(b,c,d,g,h))$,
- Let a be an element of $\mathcal{A}$. Then there exists a set f such that $f \in \mathcal{F}(a,a)$ and for every element b of $\mathcal{A}$ and for every set g such

that $g \in \mathcal{F}(a,b)$ holds $\mathcal{G}(a,a,b,f,g) = g$, and
- Let a be an element of $\mathcal{A}$. Then there exists a set f such that $f \in \mathcal{F}(a,a)$ and for every element b of $\mathcal{A}$ and for every set g such that $g \in \mathcal{F}(b,a)$ holds $\mathcal{G}(b,a,a,g,f) = g$.

The scheme *CategoryLambdaUniq* deals with a non empty set $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a set, and a 5-ary functor $\mathcal{G}$ yielding a set, and states that:

Let C_1, C_2 be non empty transitive category structures. Suppose that

(i) the carrier of $C_1 = \mathcal{A}$,
(ii) for all objects a, b of C_1 holds $\langle a,b \rangle = \mathcal{F}(a,b)$,
(iii) for all objects a, b, c of C_1 such that $\langle a,b \rangle \neq \emptyset$ and $\langle b,c \rangle \neq \emptyset$ and for every morphism f from a to b and for every morphism g from b to c holds $g \cdot f = \mathcal{G}(a,b,c,f,g)$,
(iv) the carrier of $C_2 = \mathcal{A}$,
(v) for all objects a, b of C_2 holds $\langle a,b \rangle = \mathcal{F}(a,b)$, and
(vi) for all objects a, b, c of C_2 such that $\langle a,b \rangle \neq \emptyset$ and $\langle b,c \rangle \neq \emptyset$ and for every morphism f from a to b and for every morphism g from b to c holds $g \cdot f = \mathcal{G}(a,b,c,f,g)$.

Then the category structure of C_1 = the category structure of C_2

for all values of the parameters.

The scheme *CategoryQuasiLambda* deals with a non empty set $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a set, a 5-ary functor $\mathcal{G}$ yielding a set, and a ternary predicate $\mathcal{P}$, and states that:

There exists a strict category C such that

(i) the carrier of $C = \mathcal{A}$,
(ii) for all objects a, b of C and for every set f holds $f \in \langle a,b \rangle$ iff $f \in \mathcal{F}(a,b)$ and $\mathcal{P}[a,b,f]$, and
(iii) for all objects a, b, c of C such that $\langle a,b \rangle \neq \emptyset$ and $\langle b,c \rangle \neq \emptyset$ and for every morphism f from a to b and for every morphism g from b to c holds $g \cdot f = \mathcal{G}(a,b,c,f,g)$

provided the following requirements are met:

- Let a, b, c be elements of $\mathcal{A}$ and f, g be sets. Suppose $f \in \mathcal{F}(a,b)$ and $\mathcal{P}[a,b,f]$ and $g \in \mathcal{F}(b,c)$ and $\mathcal{P}[b,c,g]$. Then $\mathcal{G}(a,b,c,f,g) \in \mathcal{F}(a,c)$ and $\mathcal{P}[a,c,\mathcal{G}(a,b,c,f,g)]$,
- Let a, b, c, d be elements of $\mathcal{A}$ and f, g, h be sets. Suppose $f \in \mathcal{F}(a,b)$ and $\mathcal{P}[a,b,f]$ and $g \in \mathcal{F}(b,c)$ and $\mathcal{P}[b,c,g]$ and $h \in \mathcal{F}(c,d)$ and $\mathcal{P}[c,d,h]$. Then $\mathcal{G}(a,c,d,\mathcal{G}(a,b,c,f,g),h) = \mathcal{G}(a,b,d,f,\mathcal{G}(b,c,d,g,h))$,
- Let a be an element of $\mathcal{A}$. Then there exists a set f such that $f \in \mathcal{F}(a,a)$ and $\mathcal{P}[a,a,f]$ and for every element b of $\mathcal{A}$ and for every

set g such that $g \in \mathcal{F}(a, b)$ and $\mathcal{P}[a, b, g]$ holds $\mathcal{G}(a, a, b, f, g) = g$, and

- Let a be an element of $\mathcal{A}$. Then there exists a set f such that $f \in \mathcal{F}(a, a)$ and $\mathcal{P}[a, a, f]$ and for every element b of $\mathcal{A}$ and for every set g such that $g \in \mathcal{F}(b, a)$ and $\mathcal{P}[b, a, g]$ holds $\mathcal{G}(b, a, a, g, f) = g$.

Let f be a function yielding function and let a, b, c be sets. Note that $f(a, b, c)$ is relation-like and function-like.

Now we present two schemes. The scheme *SubcategoryEx* deals with a category $\mathcal{A}$, a unary predicate $\mathcal{P}$, and a ternary predicate $\mathcal{Q}$, and states that:

There exists a subcategory B of $\mathcal{A}$ such that

(i) for every object a of $\mathcal{A}$ holds a is an object of B iff $\mathcal{P}[a]$, and

(ii) for all objects a, b of $\mathcal{A}$ and for all objects a', b' of B such that $a' = a$ and $b' = b$ and $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $f \in \langle a', b'\rangle$ iff $\mathcal{Q}[a, b, f]$

provided the parameters meet the following requirements:

- There exists an object a of $\mathcal{A}$ such that $\mathcal{P}[a]$,
- Let a, b, c be objects of $\mathcal{A}$. Suppose $\mathcal{P}[a]$ and $\mathcal{P}[b]$ and $\mathcal{P}[c]$ and $\langle a, b\rangle \neq \emptyset$ and $\langle b, c\rangle \neq \emptyset$. Let f be a morphism from a to b and g be a morphism from b to c. If $\mathcal{Q}[a, b, f]$ and $\mathcal{Q}[b, c, g]$, then $\mathcal{Q}[a, c, g \cdot f]$, and
- For every object a of $\mathcal{A}$ such that $\mathcal{P}[a]$ holds $\mathcal{Q}[a, a, \mathrm{id}_a]$.

The scheme *CovariantFunctorLambda* deals with categories $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a ternary functor $\mathcal{G}$ yielding a set, and states that:

There exists a covariant strict functor F from $\mathcal{A}$ to $\mathcal{B}$ such that

(i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(a)$, and

(ii) for all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{G}(a, b, f)$

provided the parameters have the following properties:

- For every object a of $\mathcal{A}$ holds $\mathcal{F}(a)$ is an object of $\mathcal{B}$,
- Let a, b be objects of $\mathcal{A}$. Suppose $\langle a, b\rangle \neq \emptyset$. Let f be a morphism from a to b. Then $\mathcal{G}(a, b, f) \in$ (the arrows of $\mathcal{B}$)($\mathcal{F}(a)$, $\mathcal{F}(b)$),
- Let a, b, c be objects of $\mathcal{A}$. Suppose $\langle a, b\rangle \neq \emptyset$ and $\langle b, c\rangle \neq \emptyset$. Let f be a morphism from a to b, g be a morphism from b to c, and a', b', c' be objects of $\mathcal{B}$. Suppose $a' = \mathcal{F}(a)$ and $b' = \mathcal{F}(b)$ and $c' = \mathcal{F}(c)$. Let f' be a morphism from a' to b' and g' be a morphism from b' to c'. If $f' = \mathcal{G}(a, b, f)$ and $g' = \mathcal{G}(b, c, g)$, then $\mathcal{G}(a, c, g \cdot f) = g' \cdot f'$, and
- For every object a of $\mathcal{A}$ and for every object a' of $\mathcal{B}$ such that $a' = \mathcal{F}(a)$ holds $\mathcal{G}(a, a, \mathrm{id}_a) = \mathrm{id}_{a'}$.

The following proposition is true

(1) Let A, B be categories and F, G be covariant functors from A to B. Suppose that

(i) for every object a of A holds $F(a) = G(a)$, and

(ii) for all objects a, b of A such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = G(f)$.

Then the functor structure of $F =$ the functor structure of G.

The scheme *ContravariantFunctorLambda* deals with categories $\mathcal{A}, \mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a ternary functor $\mathcal{G}$ yielding a set, and states that:

There exists a contravariant strict functor F from $\mathcal{A}$ to $\mathcal{B}$ such that

(i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(a)$, and

(ii) for all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{G}(a, b, f)$

provided the parameters meet the following requirements:

- For every object a of $\mathcal{A}$ holds $\mathcal{F}(a)$ is an object of $\mathcal{B}$,
- Let a, b be objects of $\mathcal{A}$. Suppose $\langle a, b\rangle \neq \emptyset$. Let f be a morphism from a to b. Then $\mathcal{G}(a, b, f) \in$ (the arrows of $\mathcal{B}$)($\mathcal{F}(b)$, $\mathcal{F}(a)$),
- Let a, b, c be objects of $\mathcal{A}$. Suppose $\langle a, b\rangle \neq \emptyset$ and $\langle b, c\rangle \neq \emptyset$. Let f be a morphism from a to b, g be a morphism from b to c, and a', b', c' be objects of $\mathcal{B}$. Suppose $a' = \mathcal{F}(a)$ and $b' = \mathcal{F}(b)$ and $c' = \mathcal{F}(c)$. Let f' be a morphism from b' to a' and g' be a morphism from c' to b'. If $f' = \mathcal{G}(a, b, f)$ and $g' = \mathcal{G}(b, c, g)$, then $\mathcal{G}(a, c, g \cdot f) = f' \cdot g'$, and
- For every object a of $\mathcal{A}$ and for every object a' of $\mathcal{B}$ such that $a' = \mathcal{F}(a)$ holds $\mathcal{G}(a, a, \mathrm{id}_a) = \mathrm{id}_{a'}$.

One can prove the following proposition

(2) Let A, B be categories and F, G be contravariant functors from A to B. Suppose that

(i) for every object a of A holds $F(a) = G(a)$, and

(ii) for all objects a, b of A such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = G(f)$.

Then the functor structure of $F =$ the functor structure of G.

2. Isomorphism and Equivalence of Categories

Let A, B, C be non empty sets and let f be a function from $[\!: A, B :\!]$ into C. Let us observe that f is one-to-one if and only if:

(Def. 1) For all elements a_1, a_2 of A and for all elements b_1, b_2 of B such that $f(a_1, b_1) = f(a_2, b_2)$ holds $a_1 = a_2$ and $b_1 = b_2$.

Now we present four schemes. The scheme *CoBijectiveSch* deals with categories $\mathcal{A}$, $\mathcal{B}$, a covariant functor $\mathcal{C}$ from $\mathcal{A}$ to $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a ternary functor $\mathcal{C}$ yielding a set, and states that:

$\mathcal{C}$ is bijective

provided the parameters meet the following requirements:

- For every object a of $\mathcal{A}$ holds $\mathcal{C}(a) = \mathcal{F}(a)$,
- For all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $\mathcal{C}(f) = \mathcal{C}(a, b, f)$,
- For all objects a, b of $\mathcal{A}$ such that $\mathcal{F}(a) = \mathcal{F}(b)$ holds $a = b$,
- For all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for all morphisms f, g from a to b such that $\mathcal{C}(a, b, f) = \mathcal{C}(a, b, g)$ holds $f = g$, and
- Let a, b be objects of $\mathcal{B}$. Suppose $\langle a, b\rangle \neq \emptyset$. Let f be a morphism from a to b. Then there exist objects c, d of $\mathcal{A}$ and there exists a morphism g from c to d such that $a = \mathcal{F}(c)$ and $b = \mathcal{F}(d)$ and $\langle c, d\rangle \neq \emptyset$ and $f = \mathcal{C}(c, d, g)$.

The scheme *CatIsomorphism* deals with categories $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a ternary functor $\mathcal{G}$ yielding a set, and states that:

$\mathcal{A}$ and $\mathcal{B}$ are isomorphic

provided the parameters meet the following requirements:

- There exists a covariant functor F from $\mathcal{A}$ to $\mathcal{B}$ such that
  (i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(a)$, and
  (ii) for all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{G}(a, b, f)$,
- For all objects a, b of $\mathcal{A}$ such that $\mathcal{F}(a) = \mathcal{F}(b)$ holds $a = b$,
- For all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for all morphisms f, g from a to b such that $\mathcal{G}(a, b, f) = \mathcal{G}(a, b, g)$ holds $f = g$, and
- Let a, b be objects of $\mathcal{B}$. Suppose $\langle a, b\rangle \neq \emptyset$. Let f be a morphism from a to b. Then there exist objects c, d of $\mathcal{A}$ and there exists a morphism g from c to d such that $a = \mathcal{F}(c)$ and $b = \mathcal{F}(d)$ and $\langle c, d\rangle \neq \emptyset$ and $f = \mathcal{G}(c, d, g)$.

The scheme *ContraBijectiveSch* deals with categories $\mathcal{A}$, $\mathcal{B}$, a contravariant functor $\mathcal{C}$ from $\mathcal{A}$ to $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a ternary functor $\mathcal{C}$ yielding a set, and states that:

$\mathcal{C}$ is bijective

provided the following conditions are met:

- For every object a of $\mathcal{A}$ holds $\mathcal{C}(a) = \mathcal{F}(a)$,
- For all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $\mathcal{C}(f) = \mathcal{C}(a, b, f)$,
- For all objects a, b of $\mathcal{A}$ such that $\mathcal{F}(a) = \mathcal{F}(b)$ holds $a = b$,
- For all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for all morphisms f, g from a to b such that $\mathcal{C}(a, b, f) = \mathcal{C}(a, b, g)$ holds $f = g$, and

- Let a, b be objects of $\mathcal{B}$. Suppose $\langle a, b\rangle \neq \emptyset$. Let f be a morphism from a to b. Then there exist objects c, d of $\mathcal{A}$ and there exists a morphism g from c to d such that $b = \mathcal{F}(c)$ and $a = \mathcal{F}(d)$ and $\langle c, d\rangle \neq \emptyset$ and $f = \mathcal{C}(c, d, g)$.

The scheme *CatAntiIsomorphism* deals with categories $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a ternary functor $\mathcal{G}$ yielding a set, and states that:

$\mathcal{A}$, $\mathcal{B}$ are anti-isomorphic

provided the parameters meet the following conditions:

- There exists a contravariant functor F from $\mathcal{A}$ to $\mathcal{B}$ such that
  (i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(a)$, and
  (ii) for all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{G}(a, b, f)$,
- For all objects a, b of $\mathcal{A}$ such that $\mathcal{F}(a) = \mathcal{F}(b)$ holds $a = b$,
- For all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for all morphisms f, g from a to b such that $\mathcal{G}(a, b, f) = \mathcal{G}(a, b, g)$ holds $f = g$, and
- Let a, b be objects of $\mathcal{B}$. Suppose $\langle a, b\rangle \neq \emptyset$. Let f be a morphism from a to b. Then there exist objects c, d of $\mathcal{A}$ and there exists a morphism g from c to d such that $b = \mathcal{F}(c)$ and $a = \mathcal{F}(d)$ and $\langle c, d\rangle \neq \emptyset$ and $f = \mathcal{G}(c, d, g)$.

Let A, B be categories. We say that A and B are equivalent if and only if the condition (Def. 2) is satisfied.

(Def. 2) There exists a covariant functor F from A to B and there exists a covariant functor G from B to A such that $G \cdot F$ and id_A are naturally equivalent and $F \cdot G$ and id_B are naturally equivalent.

Let us notice that the predicate A and B are equivalent is reflexive and symmetric.

The following propositions are true:

(3) Let A, B, C be non empty reflexive graphs, F_1, F_2 be feasible functor structures from A to B, and G_1, G_2 be functor structures from B to C. Suppose that

(i) the functor structure of $F_1 =$ the functor structure of F_2, and

(ii) the functor structure of $G_1 =$ the functor structure of G_2.

Then $G_1 \cdot F_1 = G_2 \cdot F_2$.

(4) Let A, B, C be categories. Suppose A and B are equivalent and B and C are equivalent. Then A and C are equivalent.

(5) For all categories A, B such that A and B are isomorphic holds A and B are equivalent.

Now we present two schemes. The scheme *NatTransLambda* deals with categories $\mathcal{A}$, $\mathcal{B}$, covariant functors $\mathcal{C}$, $\mathcal{D}$ from $\mathcal{A}$ to $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

> There exists a natural transformation t from $\mathcal{C}$ to $\mathcal{D}$ such that $\mathcal{C}$ is naturally transformable to $\mathcal{D}$ and for every object a of $\mathcal{A}$ holds $t[a] = \mathcal{F}(a)$

provided the parameters have the following properties:

- For every object a of $\mathcal{A}$ holds $\mathcal{F}(a) \in \langle \mathcal{C}(a), \mathcal{D}(a) \rangle$, and
- Let a, b be objects of $\mathcal{A}$. Suppose $\langle a, b \rangle \neq \emptyset$. Let f be a morphism from a to b and g_1 be a morphism from $\mathcal{C}(a)$ to $\mathcal{D}(a)$. Suppose $g_1 = \mathcal{F}(a)$. Let g_2 be a morphism from $\mathcal{C}(b)$ to $\mathcal{D}(b)$. If $g_2 = \mathcal{F}(b)$, then $g_2 \cdot \mathcal{C}(f) = \mathcal{D}(f) \cdot g_1$.

The scheme *NatEquivalenceLambda* deals with categories $\mathcal{A}$, $\mathcal{B}$, covariant functors $\mathcal{C}$, $\mathcal{D}$ from $\mathcal{A}$ to $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

> There exists a natural equivalence t of $\mathcal{C}$ and $\mathcal{D}$ such that $\mathcal{C}$ and $\mathcal{D}$ are naturally equivalent and for every object a of $\mathcal{A}$ holds $t[a] = \mathcal{F}(a)$

provided the following conditions are satisfied:

- Let a be an object of $\mathcal{A}$. Then $\mathcal{F}(a) \in \langle \mathcal{C}(a), \mathcal{D}(a) \rangle$ and $\langle \mathcal{D}(a), \mathcal{C}(a) \rangle \neq \emptyset$ and for every morphism f from $\mathcal{C}(a)$ to $\mathcal{D}(a)$ such that $f = \mathcal{F}(a)$ holds f is iso, and
- Let a, b be objects of $\mathcal{A}$. Suppose $\langle a, b \rangle \neq \emptyset$. Let f be a morphism from a to b and g_1 be a morphism from $\mathcal{C}(a)$ to $\mathcal{D}(a)$. Suppose $g_1 = \mathcal{F}(a)$. Let g_2 be a morphism from $\mathcal{C}(b)$ to $\mathcal{D}(b)$. If $g_2 = \mathcal{F}(b)$, then $g_2 \cdot \mathcal{C}(f) = \mathcal{D}(f) \cdot g_1$.

3. Dual Categories

Let C_1, C_2 be non empty category structures. We say that C_1 and C_2 are opposite if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) The carrier of C_2 = the carrier of C_1,

(ii) the arrows of $C_2 = \curvearrowleft$(the arrows of C_1), and

(iii) for all objects a, b, c of C_1 and for all objects a', b', c' of C_2 such that $a' = a$ and $b' = b$ and $c' = c$ holds (the composition of C_2)(a', b', c') $= \curvearrowleft$(the composition of C_1)(c, b, a).

Let us note that the predicate C_1 and C_2 are opposite is symmetric.

Next we state several propositions:

(6) For all non empty category structures A, B such that A and B are opposite holds every object of A is an object of B.

(7) Let A, B be non empty category structures. Suppose A and B are opposite. Let a, b be objects of A and a', b' be objects of B. If $a' = a$ and $b' = b$, then $\langle a, b \rangle = \langle b', a' \rangle$.

(8) Let A, B be non empty category structures. Suppose A and B are opposite. Let a, b be objects of A and a', b' be objects of B. If $a' = a$ and $b' = b$, then every morphism from a to b is a morphism from b' to a'.

(9) Let C_1, C_2 be non empty transitive category structures. Then C_1 and C_2 are opposite if and only if the following conditions are satisfied:

(i) the carrier of $C_2 =$ the carrier of C_1, and

(ii) for all objects a, b, c of C_1 and for all objects a', b', c' of C_2 such that $a' = a$ and $b' = b$ and $c' = c$ holds $\langle a, b\rangle = \langle b', a'\rangle$ and if $\langle a, b\rangle \neq \emptyset$ and $\langle b, c\rangle \neq \emptyset$, then for every morphism f from a to b and for every morphism g from b to c and for every morphism f' from b' to a' and for every morphism g' from c' to b' such that $f' = f$ and $g' = g$ holds $f' \cdot g' = g \cdot f$.

(10) Let A, B be categories. Suppose A and B are opposite. Let a be an object of A and b be an object of B. If $a = b$, then $\mathrm{id}_a = \mathrm{id}_b$.

(11) Let C be a transitive non empty category structure. Then there exists a strict transitive non empty category structure C' such that C and C' are opposite.

(12) Let A, B be transitive non empty category structures. Suppose A and B are opposite. If A is associative, then B is associative.

(13) For all transitive non empty category structures A, B such that A and B are opposite holds if A has units, then B has units.

(14) Let C, C_1, C_2 be non empty category structures. Suppose C and C_1 are opposite. Then C and C_2 are opposite if and only if the category structure of $C_1 =$ the category structure of C_2.

Let C be a transitive non empty category structure. The functor C^{op} yields a strict transitive non empty category structure and is defined as follows:

(Def. 4) C and C^{op} are opposite.

Let C be an associative transitive non empty category structure. One can check that C^{op} is associative.

Let C be a transitive non empty category structure with units. One can verify that C^{op} has units.

Let A, B be categories. Let us assume that A and B are opposite. The dualizing functor from A into B is a contravariant strict functor from A to B and is defined by the conditions (Def. 5).

(Def. 5)(i) For every object a of A holds (the dualizing functor from A into B)$(a) = a$, and

(ii) for all objects a, b of A such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds (the dualizing functor from A into B)$(f) = f$.

Next we state two propositions:

(15) Let A, B be categories. Suppose A and B are opposite. Then (the dualizing functor from A into B) $\cdot$ (the dualizing functor from B into A) $= \mathrm{id}_B$.

(16) Let A, B be categories. Suppose A and B are opposite. Then the dualizing functor from A into B is bijective.

Let A be a category. One can verify that the dualizing functor from A into A^{op} is bijective and the dualizing functor from A^{op} into A is bijective.

Next we state a number of propositions:

(17) For all categories A, B such that A and B are opposite holds A, B are anti-isomorphic.

(18) Let A, B, C be categories. Suppose A and B are opposite. Then A and C are isomorphic if and only if B, C are anti-isomorphic.

(19) Let A, B, C, D be categories. Suppose A and B are opposite and C and D are opposite. If A and C are isomorphic, then B and D are isomorphic.

(20) Let A, B, C, D be categories. Suppose A and B are opposite and C and D are opposite. If A, C are anti-isomorphic, then B, D are anti-isomorphic.

(21) Let A, B be categories. Suppose A and B are opposite. Let a, b be objects of A. Suppose $\langle a, b\rangle \neq \emptyset$ and $\langle b, a\rangle \neq \emptyset$. Let a', b' be objects of B. Suppose $a' = a$ and $b' = b$. Let f be a morphism from a to b and f' be a morphism from b' to a'. If $f' = f$, then f is retraction iff f' is coretraction.

(22) Let A, B be categories. Suppose A and B are opposite. Let a, b be objects of A. Suppose $\langle a, b\rangle \neq \emptyset$ and $\langle b, a\rangle \neq \emptyset$. Let a', b' be objects of B. Suppose $a' = a$ and $b' = b$. Let f be a morphism from a to b and f' be a morphism from b' to a'. If $f' = f$, then f is coretraction iff f' is retraction.

(23) Let A, B be categories. Suppose A and B are opposite. Let a, b be objects of A. Suppose $\langle a, b\rangle \neq \emptyset$ and $\langle b, a\rangle \neq \emptyset$. Let a', b' be objects of B. Suppose $a' = a$ and $b' = b$. Let f be a morphism from a to b and f' be a morphism from b' to a'. If $f' = f$ and f is retraction and coretraction, then $f'^{-1} = f^{-1}$.

(24) Let A, B be categories. Suppose A and B are opposite. Let a, b be objects of A. Suppose $\langle a, b\rangle \neq \emptyset$ and $\langle b, a\rangle \neq \emptyset$. Let a', b' be objects of B. Suppose $a' = a$ and $b' = b$. Let f be a morphism from a to b and f' be a morphism from b' to a'. If $f' = f$, then f is iso iff f' is iso.

(25) Let A, B, C, D be categories. Suppose A and B are opposite and C and D are opposite. Let F, G be covariant functors from B to C. Suppose F and G are naturally equivalent. Then (the dualizing functor from C into D) $\cdot\, G \,\cdot$ the dualizing functor from A into B and (the dualizing functor from C into D) $\cdot\, F \,\cdot$ the dualizing functor from A into B are naturally equivalent.

(26) Let A, B, C, D be categories. Suppose A and B are opposite and C and D are opposite. If A and C are equivalent, then B and D are equivalent.

Let A, B be categories. We say that A and B are dual if and only if:

(Def. 6) A and B^{op} are equivalent.

Let us note that the predicate A and B are dual is symmetric.

We now state four propositions:

(27) For all categories A, B such that A, B are anti-isomorphic holds A and B are dual.

(28) Let A, B, C be categories. Suppose A and B are opposite. Then A and C are equivalent if and only if B and C are dual.

(29) For all categories A, B, C such that A and B are dual and B and C are equivalent holds A and C are dual.

(30) For all categories A, B, C such that A and B are dual and B and C are dual holds A and C are equivalent.

4. Concrete Categories

The following proposition is true

(31) For all sets X, Y, x holds $x \in Y^X$ iff x is a function and $\pi_1(x) = X$ and $\pi_2(x) \subseteq Y$.

Let C be a 1-sorted structure. A many sorted set indexed by C is a many sorted set indexed by the carrier of C.

Let C be a category. We say that C is para-functional if and only if:

(Def. 7) There exists a many sorted set F indexed by C such that for all objects a_1, a_2 of C holds $\langle a_1, a_2 \rangle \subseteq F(a_2)^{F(a_1)}$.

Let us note that every category which is quasi-functional is also para-functional.

Let C be a category and let a be a set. C-carrier of a is defined as follows:

(Def. 8)(i) There exists an object b of C such that $b = a$ and C-carrier of $a = \pi_1(\mathrm{id}_b)$ if a is an object of C,

(ii) C-carrier of $a = \emptyset$, otherwise.

Let C be a category and let a be an object of C. Then C-carrier of a can be characterized by the condition:

(Def. 9) C-carrier of $a = \pi_1(\mathrm{id}_a)$.

We introduce the carrier of a as a synonym of C-carrier of a.

We now state two propositions:

(32) For every non empty set A and for every object a of Ens_A holds the identity morphism of $a =$ the identity function on a.

(33) For every non empty set A and for every object a of Ens_A holds the carrier of $a = a$.

Let C be a category. We say that C is set-id-inheriting if and only if:

(Def. 10) For every object a of C holds $\mathrm{id}_a = \mathrm{id}_{\text{the carrier of } a}$.

Let A be a non empty set. Observe that Ens_A is set-id-inheriting.

Let C be a category. We say that C is concrete if and only if:

(Def. 11) C is para-functional, semi-functional, and set-id-inheriting.

One can verify that every category which is concrete is also para-functional, semi-functional, and set-id-inheriting and every category which is para-functional, semi-functional, and set-id-inheriting is also concrete.

Let us mention that there exists a category which is concrete, quasi-functional, and strict.

The following propositions are true:

(34) Let C be a category. Then C is para-functional if and only if for all objects a_1, a_2 of C holds $\langle a_1, a_2\rangle \subseteq (\text{the carrier of } a_2)^{\text{the carrier of } a_1}$.

(35) Let C be a para-functional category and a, b be objects of C. Suppose $\langle a, b\rangle \neq \emptyset$. Then every morphism from a to b is a function from the carrier of a into the carrier of b.

Let A be a para-functional category and let a, b be objects of A. One can verify that every morphism from a to b is function-like and relation-like.

We now state four propositions:

(36) Let C be a para-functional category and a, b be objects of C. Suppose $\langle a, b\rangle \neq \emptyset$. Let f be a morphism from a to b. Then $\operatorname{dom} f =$ the carrier of a and $\operatorname{rng} f \subseteq$ the carrier of b.

(37) For every para-functional semi-functional category C and for every object a of C holds the carrier of $a = \operatorname{dom}(\mathrm{id}_a)$.

(38) Let C be a para-functional semi-functional category and a, b, c be objects of C. Suppose $\langle a, b\rangle \neq \emptyset$ and $\langle b, c\rangle \neq \emptyset$. Let f be a morphism from a to b and g be a morphism from b to c. Then $g \cdot f = (g$ **qua** function$) \cdot (f$ **qua** function$)$.

(39) Let C be a para-functional semi-functional category and a be an object of C. If $\mathrm{id}_{\text{the carrier of } a} \in \langle a, a\rangle$, then $\mathrm{id}_a = \mathrm{id}_{\text{the carrier of } a}$.

Now we present several schemes. The scheme *ConcreteCategoryLambda* deals with a non empty set $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a set, and a unary functor $\mathcal{G}$ yielding a set, and states that:

There exists a concrete strict category C such that

(i) the carrier of $C = \mathcal{A}$,

(ii) for every object a of C holds the carrier of $a = \mathcal{G}(a)$, and

(iii) for all objects a, b of C holds $\langle a, b\rangle = \mathcal{F}(a, b)$

provided the following requirements are met:

- For all elements a, b, c of $\mathcal{A}$ and for all functions f, g such that $f \in \mathcal{F}(a, b)$ and $g \in \mathcal{F}(b, c)$ holds $g \cdot f \in \mathcal{F}(a, c)$,
- For all elements a, b of $\mathcal{A}$ holds $\mathcal{F}(a, b) \subseteq \mathcal{G}(b)^{\mathcal{G}(a)}$, and
- For every element a of $\mathcal{A}$ holds $\mathrm{id}_{\mathcal{G}(a)} \in \mathcal{F}(a, a)$.

The scheme *ConcreteCategoryQuasiLambda* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and a ternary predicate $\mathcal{P}$, and states that:

There exists a concrete strict category C such that

(i) the carrier of $C = \mathcal{A}$,

(ii) for every object a of C holds the carrier of $a = \mathcal{F}(a)$, and

(iii) for all elements a, b of $\mathcal{A}$ and for every function f holds $f \in$ (the arrows of C)(a, b) iff $f \in \mathcal{F}(b)^{\mathcal{F}(a)}$ and $\mathcal{P}[a, b, f]$

provided the parameters satisfy the following conditions:

- For all elements a, b, c of $\mathcal{A}$ and for all functions f, g such that $\mathcal{P}[a, b, f]$ and $\mathcal{P}[b, c, g]$ holds $\mathcal{P}[a, c, g \cdot f]$, and
- For every element a of $\mathcal{A}$ holds $\mathcal{P}[a, a, \mathrm{id}_{\mathcal{F}(a)}]$.

The scheme *ConcreteCategoryEx* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, a binary predicate $\mathcal{P}$, and a ternary predicate $\mathcal{Q}$, and states that:

There exists a concrete strict category C such that

(i) the carrier of $C = \mathcal{A}$,

(ii) for every object a of C and for every set x holds $x \in$ the carrier of a iff $x \in \mathcal{F}(a)$ and $\mathcal{P}[a, x]$, and

(iii) for all elements a, b of $\mathcal{A}$ and for every function f holds $f \in$ (the arrows of C)(a, b) iff $f \in$ (C-carrier of b)$^{C\text{-carrier of } a}$ and $\mathcal{Q}[a, b, f]$

provided the following requirements are met:

- For all elements a, b, c of $\mathcal{A}$ and for all functions f, g such that $\mathcal{Q}[a, b, f]$ and $\mathcal{Q}[b, c, g]$ holds $\mathcal{Q}[a, c, g \cdot f]$, and
- Let a be an element of $\mathcal{A}$ and X be a set. If for every set x holds $x \in X$ iff $x \in \mathcal{F}(a)$ and $\mathcal{P}[a, x]$, then $\mathcal{Q}[a, a, \mathrm{id}_X]$.

The scheme *ConcreteCategoryUniq1* deals with a non empty set $\mathcal{A}$ and a binary functor $\mathcal{F}$ yielding a set, and states that:

Let C_1, C_2 be para-functional semi-functional categories. Suppose that

(i) the carrier of $C_1 = \mathcal{A}$,

(ii) for all objects a, b of C_1 holds $\langle a, b \rangle = \mathcal{F}(a, b)$,

(iii) the carrier of $C_2 = \mathcal{A}$, and

(iv) for all objects a, b of C_2 holds $\langle a, b \rangle = \mathcal{F}(a, b)$.

Then the category structure of C_1 = the category structure of C_2

for all values of the parameters.

The scheme *ConcreteCategoryUniq2* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and a ternary predicate $\mathcal{P}$, and states that:

Let C_1, C_2 be para-functional semi-functional categories. Suppose that

(i) the carrier of $C_1 = \mathcal{A}$,

(ii) for all elements a, b of $\mathcal{A}$ and for every function f holds $f \in$ (the arrows of C_1)(a, b) iff $f \in \mathcal{F}(b)^{\mathcal{F}(a)}$ and $\mathcal{P}[a, b, f]$,

(iii) the carrier of $C_2 = \mathcal{A}$, and

(iv) for all elements a, b of $\mathcal{A}$ and for every function f holds $f \in$ (the arrows of C_2)(a, b) iff $f \in \mathcal{F}(b)^{\mathcal{F}(a)}$ and $\mathcal{P}[a, b, f]$.

Then the category structure of $C_1 =$ the category structure of C_2

for all values of the parameters.

The scheme *ConcreteCategoryUniq3* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, a binary predicate $\mathcal{P}$, and a ternary predicate $\mathcal{Q}$, and states that:

Let C_1, C_2 be para-functional semi-functional categories. Suppose that

(i) the carrier of $C_1 = \mathcal{A}$,

(ii) for every object a of C_1 and for every set x holds $x \in$ the carrier of a iff $x \in \mathcal{F}(a)$ and $\mathcal{P}[a, x]$,

(iii) for all elements a, b of $\mathcal{A}$ and for every function f holds $f \in$ (the arrows of C_1)(a, b) iff $f \in (C_1\text{-carrier of } b)^{C_1\text{-carrier of } a}$ and $\mathcal{Q}[a, b, f]$,

(iv) the carrier of $C_2 = \mathcal{A}$,

(v) for every object a of C_2 and for every set x holds $x \in$ the carrier of a iff $x \in \mathcal{F}(a)$ and $\mathcal{P}[a, x]$, and

(vi) for all elements a, b of $\mathcal{A}$ and for every function f holds $f \in$ (the arrows of C_2)(a, b) iff $f \in (C_2\text{-carrier of } b)^{C_2\text{-carrier of } a}$ and $\mathcal{Q}[a, b, f]$.

Then the category structure of $C_1 =$ the category structure of C_2

for all values of the parameters.

5. Equivalence Between Concrete Categories

One can prove the following propositions:

(40) Let C be a concrete category and a, b be objects of C. Suppose $\langle a, b\rangle \neq \emptyset$ and $\langle b, a\rangle \neq \emptyset$. Let f be a morphism from a to b. If f is retraction, then $\operatorname{rng} f =$ the carrier of b.

(41) Let C be a concrete category and a, b be objects of C. Suppose $\langle a, b\rangle \neq \emptyset$ and $\langle b, a\rangle \neq \emptyset$. Let f be a morphism from a to b. If f is coretraction, then f is one-to-one.

(42) Let C be a concrete category and a, b be objects of C. Suppose $\langle a, b\rangle \neq \emptyset$ and $\langle b, a\rangle \neq \emptyset$. Let f be a morphism from a to b. If f is iso, then f is one-to-one and $\operatorname{rng} f =$ the carrier of b.

(43) Let C be a para-functional semi-functional category and a, b be objects of C. Suppose $\langle a, b\rangle \neq \emptyset$. Let f be a morphism from a to b. If f is one-to-one and (f **qua** function) $^{-1} \in \langle b, a\rangle$, then f is iso.

(44) Let C be a concrete category and a, b be objects of C. Suppose $\langle a, b\rangle \neq \emptyset$ and $\langle b, a\rangle \neq \emptyset$. Let f be a morphism from a to b. If f is iso, then $f^{-1} =$ (f **qua** function) $^{-1}$.

The scheme *ConcreteCatEquivalence* deals with para-functional semi-functional categories $\mathcal{A}$, $\mathcal{B}$, two unary functors $\mathcal{F}$ and $\mathcal{G}$ yielding sets, two ternary functors $\mathcal{H}$ and $\mathcal{I}$ yielding functions, and two unary functors $\mathcal{A}$ and $\mathcal{B}$ yielding functions, and states that:

$\mathcal{A}$ and $\mathcal{B}$ are equivalent

provided the following conditions are met:

- There exists a covariant functor F from $\mathcal{A}$ to $\mathcal{B}$ such that
  (i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(a)$, and
  (ii) for all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{H}(a, b, f)$,
- There exists a covariant functor G from $\mathcal{B}$ to $\mathcal{A}$ such that
  (i) for every object a of $\mathcal{B}$ holds $G(a) = \mathcal{G}(a)$, and
  (ii) for all objects a, b of $\mathcal{B}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $G(f) = \mathcal{I}(a, b, f)$,
- For all objects a, b of $\mathcal{A}$ such that $a = \mathcal{G}(\mathcal{F}(b))$ holds $\mathcal{A}(b) \in \langle a, b\rangle$ and $\mathcal{A}(b)^{-1} \in \langle b, a\rangle$ and $\mathcal{A}(b)$ is one-to-one,
- For all objects a, b of $\mathcal{B}$ such that $b = \mathcal{F}(\mathcal{G}(a))$ holds $\mathcal{B}(a) \in \langle a, b\rangle$ and $\mathcal{B}(a)^{-1} \in \langle b, a\rangle$ and $\mathcal{B}(a)$ is one-to-one,
- For all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $\mathcal{A}(b) \cdot \mathcal{I}(\mathcal{F}(a), \mathcal{F}(b), \mathcal{H}(a, b, f)) = f \cdot \mathcal{A}(a)$, and
- For all objects a, b of $\mathcal{B}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $\mathcal{H}(\mathcal{G}(a), \mathcal{G}(b), \mathcal{I}(a, b, f)) \cdot \mathcal{B}(a) = \mathcal{B}(b) \cdot f$.

6. Concretization of Categories

Let C be a category. The concretized C is a concrete strict category and is defined by the conditions (Def. 12).

(Def. 12)(i) The carrier of the concretized $C =$ the carrier of C,

(ii) for every object a of the concretized C and for every set x holds $x \in$ the carrier of a iff $x \in$ Union disjoint (the arrows of C) and $a = x_{\mathbf{2},\mathbf{2}}$, and

(iii) for all elements a, b of the carrier of C and for every function f holds $f \in$ (the arrows of the concretized C)(a, b) iff $f \in$ ((the concretized C)-carrier of b)$^{\text{(the concretized } C\text{)-carrier of } a}$ and there exist objects f_1, f_2 of C and there exists a morphism g from f_1 to f_2 such that $f_1 = a$ and $f_2 = b$ and $\langle f_1, f_2 \rangle \neq \emptyset$ and for every object o of C such that $\langle o, f_1 \rangle \neq \emptyset$ and for every morphism h from o to f_1 holds $f(\langle\!\langle h, \langle\!\langle o, f_1 \rangle\!\rangle \rangle\!\rangle) = \langle\!\langle g \cdot h, \langle\!\langle o, f_2 \rangle\!\rangle \rangle\!\rangle$.

One can prove the following proposition

(45) Let A be a category, a be an object of A, and x be a set. Then $x \in$ (the concretized A)-carrier of a if and only if there exists an object b of A and there exists a morphism f from b to a such that $\langle b, a \rangle \neq \emptyset$ and $x = \langle\!\langle f, \langle\!\langle b, a \rangle\!\rangle \rangle\!\rangle$.

Let A be a category and let a be an object of A. Observe that (the concretized A)-carrier of a is non empty.

One can prove the following two propositions:

(46) Let A be a category and a, b be objects of A. Suppose $\langle a, b \rangle \neq \emptyset$. Let f be a morphism from a to b. Then there exists a function F from (the concretized A)-carrier of a into (the concretized A)-carrier of b such that

(i) $F \in$ (the arrows of the concretized A)(a, b), and

(ii) for every object c of A and for every morphism g from c to a such that $\langle c, a \rangle \neq \emptyset$ holds $F(\langle\!\langle g, \langle\!\langle c, a \rangle\!\rangle \rangle\!\rangle) = \langle\!\langle f \cdot g, \langle\!\langle c, b \rangle\!\rangle \rangle\!\rangle$.

(47) Let A be a category and a, b be objects of A. Suppose $\langle a, b \rangle \neq \emptyset$. Let F_1, F_2 be functions. Suppose that

(i) $F_1 \in$ (the arrows of the concretized A)(a, b),

(ii) $F_2 \in$ (the arrows of the concretized A)(a, b), and

(iii) $F_1(\langle\!\langle \mathrm{id}_a, \langle\!\langle a, a \rangle\!\rangle \rangle\!\rangle) = F_2(\langle\!\langle \mathrm{id}_a, \langle\!\langle a, a \rangle\!\rangle \rangle\!\rangle)$.

Then $F_1 = F_2$.

The scheme *NonUniqMSFunctionEx* deals with a set $\mathcal{A}$, many sorted sets $\mathcal{B}$, $\mathcal{C}$ indexed by $\mathcal{A}$, and a ternary predicate $\mathcal{P}$, and states that:

> There exists a many sorted function F from $\mathcal{B}$ into $\mathcal{C}$ such that for all sets i, x if $i \in \mathcal{A}$ and $x \in \mathcal{B}(i)$, then $F(i)(x) \in \mathcal{C}(i)$ and $\mathcal{P}[i, x, F(i)(x)]$

provided the following condition is met:

- For all sets i, x such that $i \in \mathcal{A}$ and $x \in \mathcal{B}(i)$ there exists a set y such that $y \in \mathcal{C}(i)$ and $\mathcal{P}[i, x, y]$.

Let A be a category. The concretization of A is a covariant strict functor from A to the concretized A and is defined by the conditions (Def. 13).

(Def. 13)(i) For every object a of A holds (the concretization of A)$(a) = a$, and

(ii) for all objects a, b of A such that $\langle a, b \rangle \neq \emptyset$ and for every morphism f from a to b holds (the concretization of A)$(f)(\langle\!\langle \mathrm{id}_a, \langle\!\langle a, a \rangle\!\rangle \rangle\!\rangle) = \langle\!\langle f, \langle\!\langle a, b \rangle\!\rangle \rangle\!\rangle$.

Let A be a category. One can check that the concretization of A is bijective.

The following proposition is true

(48) For every category A holds A and the concretized A are isomorphic.

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Classes of Independent Partitions

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Summary. The paper includes proofs of few theorems proved earlier by Shunichi Kobayashi in many different settings.

MML Identifier: `PARTIT_2`.

The terminology and notation used in this paper have been introduced in the following articles: [1], [3], [4], [5], [9], [2], [10], [12], [11], [7], [6], and [8].

1. Preliminaries

Let X, Y be sets and let R, S be relations between X and Y. Let us observe that $R \subseteq S$ if and only if:

(Def. 1) For every element x of X and for every element y of Y such that $\langle x, y\rangle \in R$ holds $\langle x, y\rangle \in S$.

For simplicity, we adopt the following rules: Y is a non empty set, a is an element of $Boolean^Y$, G is a subset of PARTITIONS(Y), and P, Q are partitions of Y.

Let Y be a non empty set and let G be a non empty subset of PARTITIONS(Y). We see that the element of G is a partition of Y.

One can prove the following propositions:

(1) $\bigwedge \emptyset_{\mathrm{PARTITIONS}(Y)} = \mathcal{O}(Y)$.

(2) For all equivalence relations R, S of Y holds $R \cup S \subseteq R \cdot S$.

(3) For every binary relation R on Y holds $R \subseteq \nabla_Y$.

(4) For every equivalence relation R of Y holds $\nabla_Y \cdot R = \nabla_Y$ and $R \cdot \nabla_Y = \nabla_Y$.

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(5) For every partition P of Y and for all elements x, y of Y holds $\langle x, y\rangle \in \equiv_P$ iff $x \in \mathrm{EqClass}(y, P)$.

(6) Let P, Q, R be partitions of Y. Suppose $\equiv_R = \equiv_P \cdot \equiv_Q$. Let x, y be elements of Y. Then $x \in \mathrm{EqClass}(y, R)$ if and only if there exists an element z of Y such that $x \in \mathrm{EqClass}(z, P)$ and $z \in \mathrm{EqClass}(y, Q)$.

(7) Let R, S be binary relations and Y be a set. If R is reflexive in Y and S is reflexive in Y, then $R \cdot S$ is reflexive in Y.

(8) For every binary relation R and for every set Y such that R is reflexive in Y holds $Y \subseteq \mathrm{field}\, R$.

(9) For every set Y and for every binary relation R on Y such that R is reflexive in Y holds $Y = \mathrm{field}\, R$.

(10) For all equivalence relations R, S of Y such that $R \cdot S = S \cdot R$ holds $R \cdot S$ is an equivalence relation of Y.

2. Boolean-Valued Functions

The following propositions are true:

(11) For all elements a, b of $\mathit{Boolean}^Y$ such that $a \Subset b$ holds $\neg b \Subset \neg a$.

(12) For every element a of $\mathit{Boolean}^Y$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for every partition A of Y holds $\forall_{a,A} G \Subset a$.

(13) Let a, b be elements of $\mathit{Boolean}^Y$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and P be a partition of Y. If $a \Subset b$, then $\forall_{a,P} G \Subset \forall_{b,P} G$.

(14) For every element a of $\mathit{Boolean}^Y$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for every partition A of Y holds $a \Subset \exists_{a,A} G$.

(15) Let a, b be elements of $\mathit{Boolean}^Y$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and P be a partition of Y. If $a \Subset b$, then $\exists_{a,P} G \Subset \exists_{b,P} G$.

3. Independent Classes of Partitions

One can prove the following four propositions:

(16) If G is independent, then for all subsets P, Q of $\mathrm{PARTITIONS}(Y)$ such that $P \subseteq G$ and $Q \subseteq G$ holds $\equiv_{\bigwedge P} \cdot \equiv_{\bigwedge Q} = \equiv_{\bigwedge Q} \cdot \equiv_{\bigwedge P}$.

(17) If G is independent, then $\forall_{\forall_{a,P} G, Q} G = \forall_{\forall_{a,Q} G, P} G$.

(18) If G is independent, then $\exists_{\exists_{a,P} G, Q} G = \exists_{\exists_{a,Q} G, P} G$.

(19) Let a be an element of $\mathit{Boolean}^Y$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and P, Q be partitions of Y. If G is independent, then $\exists_{\forall_{a,P} G, Q} G \Subset \forall_{\exists_{a,Q} G, P} G$.

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Some Properties of Dyadic Numbers and Intervals

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Summary. The article is the second part of a paper proving the fundamental Urysohn Theorem concerning the existence of a real valued continuous function on a normal topological space. The paper is divided into two parts. In the first part, we introduce some definitions and theorems concerning properties of intervals; in the second we prove some of properties of dyadic numbers used in proving Urysohn Lemma.

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The terminology and notation used here have been introduced in the following articles: [9], [10], [11], [3], [4], [8], [7], [6], [12], [1], [2], and [5].

The following proposition is true

(1) For every interval A such that $A \neq \emptyset$ holds if $\inf A < \sup A$, then $\mathrm{vol}(A) = \sup A - \inf A$ and if $\sup A = \inf A$, then $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$.

Let A be a subset of $\mathbb{R}$ and let x be a real number. The functor $x \cdot A$ yielding a subset of $\mathbb{R}$ is defined as follows:

(Def. 1) For every real number y holds $y \in x \cdot A$ iff there exists a real number z such that $z \in A$ and $y = x \cdot z$.

Next we state a number of propositions:

(2) For every subset A of $\mathbb{R}$ and for every real number x such that $x \neq 0$ holds $x^{-1} \cdot (x \cdot A) = A$.

(3) For every real number x such that $x \neq 0$ and for every subset A of $\mathbb{R}$ such that $A = \mathbb{R}$ holds $x \cdot A = A$.

(4) For every subset A of $\mathbb{R}$ such that $A \neq \emptyset$ holds $0 \cdot A = \{0\}$.

(5) For every subset A of $\mathbb{R}$ such that $A \neq \emptyset$ holds $0 \cdot A = \{0\}$.

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(6) For every real number x holds $x \cdot \emptyset = \emptyset$.

(7) For every real number y holds $y < 0$ or $y = 0$ or $0 < y$.

(8) Let a, b be extended real numbers. Suppose $a \leqslant b$. Then $a = -\infty$ and $b = -\infty$ or $a = -\infty$ and $b \in \mathbb{R}$ or $a = -\infty$ and $b = +\infty$ or $a \in \mathbb{R}$ and $b \in \mathbb{R}$ or $a \in \mathbb{R}$ and $b = +\infty$ or $a = +\infty$ and $b = +\infty$.

(9) For every extended real number x holds $[x, x]$ is an interval.

(10) For every interval A holds $0 \cdot A$ is an interval.

(11) For all real numbers q, x such that $x \neq 0$ holds $q = x \cdot \frac{q}{x}$.

(12) For all real numbers p, q, x such that $0 < x$ and $x \cdot p < x \cdot q$ holds $p < q$.

(13) For all real numbers p, q, x such that $x < 0$ and $x \cdot p < x \cdot q$ holds $q < p$.

(14) For all real numbers p, q, x such that $0 < x$ and $x \cdot p \leqslant x \cdot q$ holds $p \leqslant q$.

(15) For all real numbers p, q, x such that $x < 0$ and $x \cdot p \leqslant x \cdot q$ holds $q \leqslant p$.

(16) Let A be an interval and x be a real number. If $x \neq 0$, then if A is open interval, then $x \cdot A$ is open interval.

(17) Let A be an interval and x be a real number. If $x \neq 0$, then if A is closed interval, then $x \cdot A$ is closed interval.

(18) Let A be an interval and x be a real number. Suppose $0 < x$. If A is right open interval, then $x \cdot A$ is right open interval.

(19) Let A be an interval and x be a real number. Suppose $x < 0$. If A is right open interval, then $x \cdot A$ is left open interval.

(20) Let A be an interval and x be a real number. Suppose $0 < x$. If A is left open interval, then $x \cdot A$ is left open interval.

(21) Let A be an interval and x be a real number. Suppose $x < 0$. If A is left open interval, then $x \cdot A$ is right open interval.

(22) Let A be an interval. Suppose $A \neq \emptyset$. Let x be a real number. Suppose $0 < x$. Let B be an interval. Suppose $B = x \cdot A$. Suppose $A = [\inf A, \sup A]$. Then $B = [\inf B, \sup B]$ and for all real numbers s, t such that $s = \inf A$ and $t = \sup A$ holds $\inf B = x \cdot s$ and $\sup B = x \cdot t$.

(23) Let A be an interval. Suppose $A \neq \emptyset$. Let x be a real number. Suppose $0 < x$. Let B be an interval. Suppose $B = x \cdot A$. Suppose $A =]\inf A, \sup A]$. Then $B =]\inf B, \sup B]$ and for all real numbers s, t such that $s = \inf A$ and $t = \sup A$ holds $\inf B = x \cdot s$ and $\sup B = x \cdot t$.

(24) Let A be an interval. Suppose $A \neq \emptyset$. Let x be a real number. Suppose $0 < x$. Let B be an interval. Suppose $B = x \cdot A$. Suppose $A =]\inf A, \sup A[$. Then $B =]\inf B, \sup B[$ and for all real numbers s, t such that $s = \inf A$ and $t = \sup A$ holds $\inf B = x \cdot s$ and $\sup B = x \cdot t$.

(25) Let A be an interval. Suppose $A \neq \emptyset$. Let x be a real number. Suppose $0 < x$. Let B be an interval. Suppose $B = x \cdot A$. Suppose $A = [\inf A, \sup A[$.

Then $B = [\inf B, \sup B[$ and for all real numbers s, t such that $s = \inf A$ and $t = \sup A$ holds $\inf B = x \cdot s$ and $\sup B = x \cdot t$.

(26) For every interval A and for every real number x holds $x \cdot A$ is an interval.

Let A be an interval and let x be a real number. Observe that $x \cdot A$ is interval.

The following propositions are true:

(27) Let A be an interval and x be a real number. If $0 \leqslant x$, then for every real number y such that $y = \mathrm{vol}(A)$ holds $x \cdot y = \mathrm{vol}(x \cdot A)$.

(28) For all real numbers x, y, z such that $x < y$ and $y \leqslant z$ or $x \leqslant y$ and $y < z$ holds $x < z$.

(29) For every natural number n holds $n < 2^n$.

(30) For every integer n such that $0 \leqslant n$ holds n is a natural number.

(31) For all natural numbers n, m such that $n < m$ holds $2^n < 2^m$.

(32) For every real number e_1 such that $0 < e_1$ there exists a natural number n such that $1 < 2^n \cdot e_1$.

(33) For all real numbers a, b such that $0 \leqslant a$ and $1 < b - a$ there exists a natural number n such that $a < n$ and $n < b$.

(34) For every integer n such that $0 < n$ holds n is a natural number.

(35) For every rational number n such that $0 \leqslant n$ holds $0 \leqslant \mathrm{num}\, n$.

(36) For every rational number n such that $0 < n$ holds $0 < \mathrm{num}\, n$.

(37) For all real numbers a, b, c, d such that $0 < b$ and $0 < d$ or $b < 0$ and $d < 0$ holds if $\frac{a}{b} < \frac{c}{d}$, then $a \cdot d < c \cdot b$.

(38) For every natural number n holds $\mathrm{dyadic}(n) \subseteq \mathrm{DYADIC}$.

(39) For all real numbers a, b such that $a < b$ and $0 \leqslant a$ and $b \leqslant 1$ there exists a real number c such that $c \in \mathrm{DYADIC}$ and $a < c$ and $c < b$.

(40) For all real numbers a, b such that $a < b$ there exists a real number c such that $c \in \mathrm{DOM}$ and $a < c$ and $c < b$.

(41) For every non empty subset A of $\overline{\mathbb{R}}$ and for all extended real numbers a, b such that $A \subseteq [a, b]$ holds $a \leqslant \inf A$ and $\sup A \leqslant b$.

(42) $0 \in \mathrm{DYADIC}$ and $1 \in \mathrm{DYADIC}$.

(43) For all extended real numbers a, b such that $a = 0$ and $b = 1$ holds $\mathrm{DYADIC} \subseteq [a, b]$.

(44) For all natural numbers n, k such that $n \leqslant k$ holds $\mathrm{dyadic}(n) \subseteq \mathrm{dyadic}(k)$.

(45) For all real numbers a, b, c, d such that $a < c$ and $c < b$ and $a < d$ and $d < b$ holds $|d - c| < b - a$.

(46) Let e_1 be a real number. Suppose $0 < e_1$. Let d be a real number. Suppose $0 < d$ and $d \leqslant 1$. Then there exist real numbers r_1, r_2 such that $r_1 \in \mathrm{DYADIC} \cup \mathbb{R}_{>1}$ and $r_2 \in \mathrm{DYADIC} \cup \mathbb{R}_{>1}$ and $0 < r_1$ and $r_1 < d$ and $d < r_2$ and $r_2 - r_1 < e_1$.

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The Urysohn Lemma

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Summary. This article is the third part of a paper proving the fundamental Urysohn Theorem concerning the existence of a real valued continuous function on a normal topological space. The paper is divided into two parts. In the first part, we describe the construction of the function solving thesis of the Urysohn Lemma. The second part contains the proof of the Urysohn Lemma in normal space and the proof of the same theorem for compact space.

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The notation and terminology used here have been introduced in the following papers: [15], [10], [7], [8], [4], [1], [9], [6], [12], [16], [17], [13], [14], [2], [3], [11], and [5].

Let D be a non empty subset of $\mathbb{R}$. One can check that every element of D is real.

One can prove the following proposition

(1) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let n be a natural number. Then there exists a function G from $\mathrm{dyadic}(n)$ into $2^{\text{the carrier of } T}$ such that for all elements r_1, r_2 of $\mathrm{dyadic}(n)$ if $r_1 < r_2$, then $G(r_1)$ is open and $G(r_2)$ is open and $\overline{G(r_1)} \subseteq G(r_2)$ and $A \subseteq G(0)$ and $B = \Omega_T \setminus G(1)$.

Let T be a non empty topological space, let A, B be subsets of T, and let n be a natural number. Let us assume that T is a T_4 space and $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. A function from $\mathrm{dyadic}(n)$ into $2^{\text{the carrier of } T}$ is said to be a drizzle of A, B, n if it satisfies the condition (Def. 1).

(Def. 1) Let r_1, r_2 be elements of $\mathrm{dyadic}(n)$. Suppose $r_1 < r_2$. Then $\mathrm{it}(r_1)$ is open and $\mathrm{it}(r_2)$ is open and $\overline{\mathrm{it}(r_1)} \subseteq \mathrm{it}(r_2)$ and $A \subseteq \mathrm{it}(0)$ and $B = \Omega_T \setminus \mathrm{it}(1)$.

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One can prove the following propositions:

(2) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let n be a natural number and D be a drizzle of A, B, n. Then $A \subseteq D(0)$ and $B = \Omega_T \setminus D(1)$.

(3) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let n be a natural number and G be a drizzle of A, B, n. Then there exists a drizzle F of A, B, $n+1$ such that for every element r of $\mathrm{dyadic}(n+1)$ if $r \in \mathrm{dyadic}(n)$, then $F(r) = G(r)$.

Let A, B be non empty sets, let F be a function from $\mathbb{N}$ into $A \dot{\rightarrow} B$, and let n be a natural number. Then $F(n)$ is a partial function from A to B.

Next we state the proposition

(4) Let T be a non empty topological space, A, B be subsets of T, and n be a natural number. Then every drizzle of A, B, n is an element of $\mathrm{DYADIC} \dot{\rightarrow} 2^{\text{the carrier of } T}$.

Let A, B be non empty sets, let F be a function from $\mathbb{N}$ into $A \dot{\rightarrow} B$, and let n be a natural number. Then $F(n)$ is an element of $A \dot{\rightarrow} B$.

One can prove the following proposition

(5) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Then there exists a sequence F of partial functions from DYADIC into $2^{\text{the carrier of } T}$ such that for every natural number n holds $F(n)$ is a drizzle of A, B, n and for every element r of $\operatorname{dom} F(n)$ holds $F(n)(r) = F(n+1)(r)$.

Let T be a non empty topological space and let A, B be subsets of T. Let us assume that T is a T_4 space and $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. A sequence of partial functions from DYADIC into $2^{\text{the carrier of } T}$ is said to be a rain of A, B if it satisfies the condition (Def. 2).

(Def. 2) Let n be a natural number. Then $\mathrm{it}(n)$ is a drizzle of A, B, n and for every element r of $\operatorname{dom} \mathrm{it}(n)$ holds $\mathrm{it}(n)(r) = \mathrm{it}(n+1)(r)$.

Let x be a real number. Let us assume that $x \in \mathrm{DYADIC}$. The functor $\mathrm{InfDyadic}\, x$ yields a natural number and is defined by:

(Def. 3) $x \in \mathrm{dyadic}(0)$ iff $\mathrm{InfDyadic}\, x = 0$ and for every natural number n such that $x \in \mathrm{dyadic}(n+1)$ and $x \notin \mathrm{dyadic}(n)$ holds $\mathrm{InfDyadic}\, x = n+1$.

The following propositions are true:

(6) For every real number x such that $x \in \mathrm{DYADIC}$ holds $x \in \mathrm{dyadic}(\mathrm{InfDyadic}\, x)$.

(7) For every real number x such that $x \in \mathrm{DYADIC}$ and for every natural number n such that $\mathrm{InfDyadic}\, x \leqslant n$ holds $x \in \mathrm{dyadic}(n)$.

(8) For every real number x such that $x \in$ DYADIC and for every natural number n such that $x \in \mathrm{dyadic}(n)$ holds InfDyadic $x \leqslant n$.

(9) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let G be a rain of A, B and x be a real number. If $x \in$ DYADIC, then for every natural number n holds $G(\mathrm{InfDyadic}\,x)(x) = G(\mathrm{InfDyadic}\,x + n)(x)$.

(10) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let G be a rain of A, B and x be a real number. Suppose $x \in$ DYADIC . Then there exists an element y of $2^{\text{the carrier of } T}$ such that for every natural number n if $x \in \mathrm{dyadic}(n)$, then $y = G(n)(x)$.

(11) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let G be a rain of A, B. Then there exists a function F from DOM into $2^{\text{the carrier of } T}$ such that for every real number x holds

(i) if $x \in \mathbb{R}_{<0}$, then $F(x) = \emptyset$,

(ii) if $x \in \mathbb{R}_{>1}$, then $F(x) =$ the carrier of T, and

(iii) if $x \in$ DYADIC, then for every natural number n such that $x \in \mathrm{dyadic}(n)$ holds $F(x) = G(n)(x)$.

Let T be a non empty topological space and let A, B be subsets of T. Let us assume that T is a T_4 space and $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let R be a rain of A, B. The functor Tempest R yielding a function from DOM into $2^{\text{the carrier of } T}$ is defined by the condition (Def. 4).

(Def. 4) Let x be a real number such that $x \in$ DOM . Then

(i) if $x \in \mathbb{R}_{<0}$, then $(\mathrm{Tempest}\,R)(x) = \emptyset$,

(ii) if $x \in \mathbb{R}_{>1}$, then $(\mathrm{Tempest}\,R)(x) =$ the carrier of T, and

(iii) if $x \in$ DYADIC, then for every natural number n such that $x \in \mathrm{dyadic}(n)$ holds $(\mathrm{Tempest}\,R)(x) = R(n)(x)$.

Let X be a non empty set, let T be a topological space, let F be a function from X into $2^{\text{the carrier of } T}$, and let x be an element of X. Then $F(x)$ is a subset of T.

One can prove the following three propositions:

(12) Let T be a non empty topological space and A, B be subsets of T. Suppose T is a T_4 space and $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let G be a rain of A, B and r be a real number. If $r \in$ DOM, then for every subset C of T such that $C = (\mathrm{Tempest}\,G)(r)$ holds C is open.

(13) Let T be a non empty topological space and A, B be subsets of T. Suppose T is a T_4 space and $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let G be a rain of A, B and r_1, r_2 be real numbers. Suppose

$r_1 \in$ DOM and $r_2 \in$ DOM and $r_1 < r_2$. Let C be a subset of T. If $C = (\text{Tempest}\, G)(r_1)$, then $\overline{C} \subseteq (\text{Tempest}\, G)(r_2)$.

(14) Let T be a non empty topological space, A, B be subsets of T, G be a rain of A, B, and p be a point of T. Then there exists a subset R of $\overline{\mathbb{R}}$ such that for every set x holds $x \in R$ if and only if the following conditions are satisfied:

(i) $x \in$ DYADIC, and

(ii) for every real number s such that $s = x$ holds $p \notin (\text{Tempest}\, G)(s)$.

Let T be a non empty topological space, let A, B be subsets of T, let R be a rain of A, B, and let p be a point of T. The functor Rainbow(p, R) yielding a subset of $\overline{\mathbb{R}}$ is defined by:

(Def. 5) For every set x holds $x \in \text{Rainbow}(p, R)$ iff $x \in$ DYADIC and for every real number s such that $s = x$ holds $p \notin (\text{Tempest}\, R)(s)$.

Let T, S be non empty topological spaces, let F be a function from the carrier of T into the carrier of S, and let p be a point of T. Then $F(p)$ is a point of S.

One can prove the following propositions:

(15) Let T be a non empty topological space, A, B be subsets of T, G be a rain of A, B, and p be a point of T. Then Rainbow$(p, G) \subseteq$ DYADIC.

(16) Let T be a non empty topological space, A, B be subsets of T, and R be a rain of A, B. Then there exists a map F from T into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of T holds
if Rainbow$(p, R) = \emptyset$, then $F(p) = 0$ and for every non empty subset S of $\overline{\mathbb{R}}$ such that $S = \text{Rainbow}(p, R)$ holds $F(p) = \sup S$.

Let T be a non empty topological space, let A, B be subsets of T, and let R be a rain of A, B. The functor Thunder R yielding a map from T into $\mathbb{R}^{\mathbf{1}}$ is defined by the condition (Def. 6).

(Def. 6) Let p be a point of T. Then if Rainbow$(p, R) = \emptyset$, then $(\text{Thunder}\, R)(p) = 0$ and for every non empty subset S of $\overline{\mathbb{R}}$ such that $S = \text{Rainbow}(p, R)$ holds $(\text{Thunder}\, R)(p) = \sup S$.

Let T be a non empty topological space, let F be a map from T into $\mathbb{R}^{\mathbf{1}}$, and let p be a point of T. Then $F(p)$ is a real number.

One can prove the following propositions:

(17) Let T be a non empty topological space, A, B be subsets of T, G be a rain of A, B, p be a point of T, and S be a non empty subset of $\overline{\mathbb{R}}$. Suppose $S = \text{Rainbow}(p, G)$. Let ℓ_1 be an extended real number. If $\ell_1 = 1$, then $0_{\overline{\mathbb{R}}} \leqslant \sup S$ and $\sup S \leqslant \ell_1$.

(18) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let G be a rain of A, B, r be an element of DOM, and p be a

point of T. If $(\mathrm{Thunder}\,G)(p) < r$, then $p \in (\mathrm{Tempest}\,G)(r)$.

(19) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let G be a rain of A, B and r be a real number. Suppose $r \in \mathrm{DYADIC} \cup \mathbb{R}_{>1}$ and $0 < r$. Let p be a point of T. If $p \in (\mathrm{Tempest}\,G)(r)$, then $(\mathrm{Thunder}\,G)(p) \leqslant r$.

(20) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let G be a rain of A, B, n be a natural number, and r_1 be an element of DOM. If $0 < r_1$, then for every point p of T such that $r_1 < (\mathrm{Thunder}\,G)(p)$ holds $p \notin (\mathrm{Tempest}\,G)(r_1)$.

(21) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let G be a rain of A, B. Then

(i) $\mathrm{Thunder}\,G$ is continuous, and

(ii) for every point x of T holds $0 \leqslant (\mathrm{Thunder}\,G)(x)$ and $(\mathrm{Thunder}\,G)(x) \leqslant 1$ and if $x \in A$, then $(\mathrm{Thunder}\,G)(x) = 0$ and if $x \in B$, then $(\mathrm{Thunder}\,G)(x) = 1$.

(22) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Then there exists a map F from T into $\mathbb{R}^{\mathbf{1}}$ such that

(i) F is continuous, and

(ii) for every point x of T holds $0 \leqslant F(x)$ and $F(x) \leqslant 1$ and if $x \in A$, then $F(x) = 0$ and if $x \in B$, then $F(x) = 1$.

(23) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose A is closed and B is closed and $A \cap B = \emptyset$. Then there exists a map F from T into $\mathbb{R}^{\mathbf{1}}$ such that

(i) F is continuous, and

(ii) for every point x of T holds $0 \leqslant F(x)$ and $F(x) \leqslant 1$ and if $x \in A$, then $F(x) = 0$ and if $x \in B$, then $F(x) = 1$.

(24) Let T be a non empty topological space. Suppose T is a T_2 space and compact. Let A, B be subsets of T. Suppose A is closed and B is closed and $A \cap B = \emptyset$. Then there exists a map F from T into $\mathbb{R}^{\mathbf{1}}$ such that

(i) F is continuous, and

(ii) for every point x of T holds $0 \leqslant F(x)$ and $F(x) \leqslant 1$ and if $x \in A$, then $F(x) = 0$ and if $x \in B$, then $F(x) = 1$.

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The Algebra of Polynomials[1]

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Summary. In this paper we define the algebra of formal power series and the algebra of polynomials over an arbitrary field and prove some properties of these structures. We also formulate and prove theorems showing some general properties of sequences. These preliminaries will be used for defining and considering linear functionals on the algebra of polynomials.

MML Identifier: `POLYALG1`.

The notation and terminology used here are introduced in the following papers: [9], [13], [1], [2], [3], [12], [8], [7], [11], [16], [5], [14], [10], [15], [6], and [4].

1. Preliminaries

Let F be a 1-sorted structure. We introduce algebra structures over F which are extensions of double loop structure and vector space structure over F and are systems

$\langle$ a carrier, an addition, a multiplication, a reverse-map, a zero, a unity, a left multiplication $\rangle$,

where the carrier is a set, the addition and the multiplication are binary operations on the carrier, the reverse-map is a unary operation on the carrier, the zero and the unity are elements of the carrier, and the left multiplication is a function from $[\!:$ the carrier of F, the carrier $:\!]$ into the carrier.

Let L be a non empty double loop structure. Note that there exists an algebra structure over L which is strict and non empty.

Let L be a non empty double loop structure and let A be a non empty algebra structure over L. We say that A is mix-associative if and only if:

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(Def. 1) For every element a of L and for all elements x, y of A holds $a \cdot (x \cdot y) = (a \cdot x) \cdot y$.

Let L be a non empty double loop structure. Note that there exists a non empty algebra structure over L which is well unital, distributive, vector space-like, and mix-associative.

Let L be a non empty double loop structure. An algebra of L is a well unital distributive vector space-like mix-associative non empty algebra structure over L.

Next we state two propositions:

(1) For all sets X, Y and for every function f from $[\!:X, Y:\!]$ into X holds $\operatorname{dom} f = [\!:X, Y:\!]$.

(2) For all sets X, Y and for every function f from $[\!:X, Y:\!]$ into Y holds $\operatorname{dom} f = [\!:X, Y:\!]$.

2. The Algebra of Formal Power Series

Let L be a non empty double loop structure. The functor Formal-Series L yields a strict non empty algebra structure over L and is defined by the conditions (Def. 2).

(Def. 2) For every set x holds $x \in$ the carrier of Formal-Series L iff x is a sequence of L and for all elements x, y of the carrier of Formal-Series L and for all sequences p, q of L such that $x = p$ and $y = q$ holds $x + y = p + q$ and for all elements x, y of the carrier of Formal-Series L and for all sequences p, q of L such that $x = p$ and $y = q$ holds $x \cdot y = p * q$ and for every element x of the carrier of Formal-Series L and for every sequence p of L such that $x = p$ holds $-x = -p$ and for every element a of L and for every element x of the carrier of Formal-Series L and for every sequence p of L such that $x = p$ holds $a \cdot x = a \cdot p$ and $0_{\text{Formal-Series}\,L} = \mathbf{0}.\,L$ and $\mathbf{1}_{\text{Formal-Series}\,L} = \mathbf{1}.\,L$.

Let L be an Abelian non empty double loop structure. Note that Formal-Series L is Abelian.

Let L be an add-associative non empty double loop structure. Note that Formal-Series L is add-associative.

Let L be a right zeroed non empty double loop structure. Note that Formal-Series L is right zeroed.

Let L be an add-associative right zeroed right complementable non empty double loop structure. Note that Formal-Series L is right complementable.

Let L be an Abelian add-associative right zeroed commutative non empty double loop structure. Observe that Formal-Series L is commutative.

Let L be an Abelian add-associative right zeroed right complementable unital associative distributive non empty double loop structure. Note that Formal-Series L is associative.

Let L be an add-associative right zeroed right complementable right unital right distributive non empty double loop structure. Note that Formal-Series L is right unital.

One can verify that there exists a non empty double loop structure which is add-associative, associative, right zeroed, left zeroed, right unital, left unital, right complementable, and distributive.

We now state three propositions:

(3) For every non empty set D and for every non empty finite sequence f of elements of D holds $f_{\lfloor 1} = f_{\restriction 1}$.

(4) For every non empty set D and for every non empty finite sequence f of elements of D holds $f = \langle f(1) \rangle \frown (f_{\restriction 1})$.

(5) Let L be an add-associative right zeroed left unital right complementable left distributive non empty double loop structure and p be a sequence of L. Then $\mathbf{1}. L * p = p$.

Let L be an add-associative right zeroed right complementable left unital left distributive non empty double loop structure. One can verify that Formal-Series L is left unital.

Let L be an Abelian add-associative right zeroed right complementable distributive non empty double loop structure. One can check that Formal-Series L is right distributive and Formal-Series L is left distributive.

We now state four propositions:

(6) Let L be an Abelian add-associative right zeroed right complementable distributive non empty double loop structure, a be an element of L, and p, q be sequences of L. Then $a \cdot (p + q) = a \cdot p + a \cdot q$.

(7) Let L be an Abelian add-associative right zeroed right complementable distributive non empty double loop structure, a, b be elements of L, and p be a sequence of L. Then $(a + b) \cdot p = a \cdot p + b \cdot p$.

(8) Let L be an associative non empty double loop structure, a, b be elements of L, and p be a sequence of L. Then $(a \cdot b) \cdot p = a \cdot (b \cdot p)$.

(9) Let L be an associative left unital non empty double loop structure and p be a sequence of L. Then (the unity of L) $\cdot\, p = p$.

Let L be an Abelian add-associative associative right zeroed right complementable left unital distributive non empty double loop structure. One can check that Formal-Series L is vector space-like.

We now state the proposition

(10) Let L be an Abelian left zeroed add-associative associative right zeroed right complementable distributive non empty double loop structure, a be

an element of L, and p, q be sequences of L. Then $a \cdot (p * q) = (a \cdot p) * q$.

Let L be an Abelian left zeroed add-associative associative right zeroed right complementable distributive non empty double loop structure. One can verify that Formal-Series L is mix-associative.

Let L be a left zeroed right zeroed add-associative left unital right unital right complementable distributive non empty double loop structure. Observe that Formal-Series L is well unital.

Let L be a 1-sorted structure and let A be an algebra structure over L. An algebra structure over L is said to be a subalgebra of A if it satisfies the conditions (Def. 3).

(Def. 3) The carrier of it $\subseteq$ the carrier of A and $\mathbf{1}_{\rm it} = \mathbf{1}_A$ and $0_{\rm it} = 0_A$ and the addition of it $=$ (the addition of A)$\restriction[\!:$ the carrier of it, the carrier of it $:\!]$ and the multiplication of it $=$ (the multiplication of A)$\restriction[\!:$ the carrier of it, the carrier of it $:\!]$ and the reverse-map of it $=$ (the reverse-map of A)$\restriction$(the carrier of it) and the left multiplication of it $=$ (the left multiplication of A)$\restriction[\!:$ the carrier of L, the carrier of it $:\!]$.

We now state four propositions:

(11) For every 1-sorted structure L holds every algebra structure A over L is a subalgebra of A.

(12) Let L be a 1-sorted structure and A, B, C be algebra structures over L. Suppose A is a subalgebra of B and B is a subalgebra of C. Then A is a subalgebra of C.

(13) Let L be a 1-sorted structure and A, B be algebra structures over L. Suppose A is a subalgebra of B and B is a subalgebra of A. Then the algebra structure of $A =$ the algebra structure of B.

(14) Let L be a 1-sorted structure and A, B be algebra structures over L. Suppose the algebra structure of $A =$ the algebra structure of B. Then A is a subalgebra of B and B is a subalgebra of A.

Let L be a non empty 1-sorted structure. Observe that there exists an algebra structure over L which is non empty and strict.

Let L be a 1-sorted structure and let B be an algebra structure over L. Observe that there exists a subalgebra of B which is strict.

Let L be a non empty 1-sorted structure and let B be a non empty algebra structure over L. Note that there exists a subalgebra of B which is strict and non empty.

Let L be a non empty groupoid, let B be a non empty algebra structure over L, and let A be a subset of B. We say that A is operations closed if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) A is linearly closed,

(ii) for all elements x, y of B such that $x \in A$ and $y \in A$ holds $x \cdot y \in A$,

(iii) for every element x of B such that $x \in A$ holds $-x \in A$,
(iv) $\mathbf{1}_B \in A$, and
(v) $0_B \in A$.

The following propositions are true:

(15) Let L be a non empty groupoid, B be a non empty algebra structure over L, A be a non empty subalgebra of B, x, y be elements of the carrier of B, and x', y' be elements of the carrier of A. If $x = x'$ and $y = y'$, then $x + y = x' + y'$.

(16) Let L be a non empty groupoid, B be a non empty algebra structure over L, A be a non empty subalgebra of B, x, y be elements of the carrier of B, and x', y' be elements of the carrier of A. If $x = x'$ and $y = y'$, then $x \cdot y = x' \cdot y'$.

(17) Let L be a non empty groupoid, B be a non empty algebra structure over L, A be a non empty subalgebra of B, a be an element of the carrier of L, x be an element of the carrier of B, and x' be an element of the carrier of A. If $x = x'$, then $a \cdot x = a \cdot x'$.

(18) Let L be a non empty groupoid, B be a non empty algebra structure over L, A be a non empty subalgebra of B, x be an element of the carrier of B, and x' be an element of the carrier of A. If $x = x'$, then $-x = -x'$.

(19) Let L be a non empty groupoid, B be a non empty algebra structure over L, and A be a non empty subalgebra of B. Then there exists a subset C of B such that the carrier of $A = C$ and C is operations closed.

(20) Let L be a non empty groupoid, B be a non empty algebra structure over L, and A be a subset of B. Suppose A is operations closed. Then there exists a strict subalgebra C of B such that the carrier of $C = A$.

(21) Let L be a non empty groupoid, B be a non empty algebra structure over L, A be a non empty subset of B, and X be a family of subsets of the carrier of B. Suppose that for every set Y holds $Y \in X$ iff $Y \in 2^{\text{the carrier of } B}$ and there exists a subalgebra C of B such that $Y =$ the carrier of C and $A \subseteq Y$. Then $\bigcap X$ is operations closed.

Let L be a non empty groupoid, let B be a non empty algebra structure over L, and let A be a non empty subset of B. The functor $\operatorname{GenAlg} A$ yielding a strict non empty subalgebra of B is defined by the conditions (Def. 5).

(Def. 5)(i) $A \subseteq$ the carrier of $\operatorname{GenAlg} A$, and
(ii) for every subalgebra C of B such that $A \subseteq$ the carrier of C holds the carrier of $\operatorname{GenAlg} A \subseteq$ the carrier of C.

We now state the proposition

(22) Let L be a non empty groupoid, B be a non empty algebra structure over L, and A be a non empty subset of B. If A is operations closed, then the carrier of $\operatorname{GenAlg} A = A$.

3. The Algebra of Polynomials

Let L be an add-associative right zeroed right complementable distributive non empty double loop structure. The functor Polynom-Algebra L yields a strict non empty algebra structure over L and is defined as follows:

(Def. 6) There exists a non empty subset A of Formal-Series L such that $A =$ the carrier of Polynom-Ring L and Polynom-Algebra $L =$ GenAlg A.

Let L be an add-associative right zeroed right complementable distributive non empty double loop structure. One can verify that Polynom-Ring L is looplike.

The following propositions are true:

(23) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure and A be a non empty subset of Formal-Series L. If $A =$ the carrier of Polynom-Ring L, then A is operations closed.

(24) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure. Then the double loop structure of Polynom-Algebra $L =$ Polynom-Ring L.

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Circuit Generated by Terms and Circuit Calculating Terms

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Summary. In the paper we investigate the dependence between the structure of circuits and sets of terms. Circuits in our terminology (see [19]) are treated as locally-finite many sorted algebras over special signatures. Such approach enables to formalize every real circuit. The goal of this investigation is to specify circuits by terms and, enentualy, to have methods of formal verification of real circuits. The following notation is introduced in this paper:

- structural equivalence of circuits, i.e. equivalence of many sorted signatures,
- embedding of a circuit into another one,
- similarity of circuits (a concept narrower than isomorphism of many sorted algebras over equivalent signatures),
- calculation of terms by a circuit according to an algebra,
- specification of circuits by terms and an algebra.

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The articles [27], [3], [18], [19], [20], [11], [10], [17], [12], [13], [14], [22], [21], [9], [25], [1], [15], [24], [7], [28], [26], [23], [2], [5], [6], [8], [16], and [4] provide the terminology and notation for this paper.

1. Circuit Structure Generated by Terms

One can prove the following proposition

(1) Let S be a non empty non void many sorted signature, A be a non-empty algebra over S, V be a variables family of A, t be a term of S over V, and T be a term of A over V. If $T = t$, then the sort of T = the sort of t.

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Let D be a non empty set and let X be a subset of D. Then id_X is a function from X into D.

Let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, and let X be a non empty subset of S-Terms(V). The functor X-CircuitStr yields a non empty strict many sorted signature and is defined by the condition (Def. 1).

(Def. 1) X-CircuitStr $= \langle$Subtrees(X), [: the operation symbols of S, {the carrier of S} :]-Subtrees(X), [: the operation symbols of S, {the carrier of S} :]-ImmediateSubtrees(X), $\mathrm{id}_{[:\text{ the operation symbols of } S,\ \{\text{the carrier of } S\} :]\text{-Subtrees}(X)}\rangle$.

Let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, and let X be a non empty subset of S-Terms(V). Observe that X-CircuitStr is unsplit.

In the sequel S denotes a non empty non void many sorted signature, V denotes a non-empty many sorted set indexed by the carrier of S, A denotes a non-empty algebra over S, and X denotes a non empty subset of S-Terms(V).

The following propositions are true:

(2) X-CircuitStr is void if and only if for every element t of X holds t is root and $t(\emptyset) \notin$ [: the operation symbols of S, {the carrier of S} :].

(3) X is a set with a compound term of S over V iff X-CircuitStr is non void.

Let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, and let X be a set with a compound term of S over V. One can check that X-CircuitStr is non void.

The following three propositions are true:

(4)(i) Every vertex of X-CircuitStr is a term of S over V, and

(ii) for every set s such that $s \in$ the operation symbols of X-CircuitStr holds s is a compound term of S over V.

(5) Let t be a vertex of X-CircuitStr. Then $t \in$ the operation symbols of X-CircuitStr if and only if t is a compound term of S over V.

(6) Let X be a set with a compound term of S over V and g be a gate of X-CircuitStr. Then (the result sort of X-CircuitStr)(g) $= g$ and the result sort of $g = g$.

Let us consider S, V, let X be a set with a compound term of S over V, and let g be a gate of X-CircuitStr. Note that Arity(g) is decorated tree yielding.

Let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, and let X be a non empty subset of S-Terms(V). Note that every vertex of X-CircuitStr is finite, function-like, and relation-like.

Let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, and let X be a non empty subset of S-Terms(V). One can verify that every vertex of X-CircuitStr is decorated tree-like.

Let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, and let X be a set with a compound term of S over V. One can check that every gate of X-CircuitStr is finite, function-like, and relation-like.

Let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, and let X be a set with a compound term of S over V. Note that every gate of X-CircuitStr is decorated tree-like.

Next we state the proposition

(7) Let X_1, X_2 be non empty subsets of S-Terms(V). Then the arity of X_1-CircuitStr $\approx$ the arity of X_2-CircuitStr and the result sort of X_1-CircuitStr $\approx$ the result sort of X_2-CircuitStr.

Let X, Y be constituted of decorated trees sets. Note that $X \cup Y$ is constituted of decorated trees.

One can prove the following propositions:

(8) For all constituted of decorated trees non empty sets X_1, X_2 holds $\mathrm{Subtrees}(X_1 \cup X_2) = \mathrm{Subtrees}(X_1) \cup \mathrm{Subtrees}(X_2)$.

(9) For all constituted of decorated trees non empty sets X_1, X_2 and for every set C holds C-Subtrees$(X_1 \cup X_2) = (C\text{-Subtrees}(X_1)) \cup (C\text{-Subtrees}(X_2))$.

(10) Let X_1, X_2 be constituted of decorated trees non empty sets. If every element of X_1 is finite and every element of X_2 is finite, then for every set C holds C-ImmediateSubtrees$(X_1 \cup X_2) = (C\text{-ImmediateSubtrees}(X_1)) +\cdot (C\text{-ImmediateSubtrees}(X_2))$.

(11) For all non empty subsets X_1, X_2 of S-Terms(V) holds $(X_1 \cup X_2)$-CircuitStr $= (X_1\text{-CircuitStr}) +\cdot (X_2\text{-CircuitStr})$.

(12) Let x be a set. Then $x \in$ InputVertices(X-CircuitStr) if and only if the following conditions are satisfied:

- (i) $x \in \mathrm{Subtrees}(X)$, and
- (ii) there exists a sort symbol s of S and there exists an element v of $V(s)$ such that $x =$ the root tree of $\langle v, s \rangle$.

(13) For every set X with a compound term of S over V and for every gate g of X-CircuitStr holds $g = g(\emptyset)$-tree(Arity(g)).

2. Circuit Generated by Terms

Let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, let X be a non empty subset of

S-Terms(V), let v be a vertex of X-CircuitStr, and let A be an algebra over S. The sort of v w.r.t. A is defined as follows:

(Def. 2) For every term u of S over V such that $u = v$ holds the sort of v w.r.t. $A =$ (the sorts of A)(the sort of u).

Let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, let X be a non empty subset of S-Terms(V), let v be a vertex of X-CircuitStr, and let A be a non-empty algebra over S. Note that the sort of v w.r.t. A is non empty.

Let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, and let X be a non empty subset of S-Terms(V). Let us assume that X is a set with a compound term of S over V. Let o be a gate of X-CircuitStr and let A be an algebra over S. The action of o w.r.t A is a function and is defined by the condition (Def. 3).

(Def. 3) Let X' be a set with a compound term of S over V. Suppose $X' = X$. Let o' be a gate of X'-CircuitStr. Suppose $o' = o$. Then the action of o w.r.t $A =$ (the characteristics of A)($o'(\emptyset)_{\mathbf{1}}$).

The scheme *MSFuncEx* deals with a non empty set $\mathcal{A}$, non-empty many sorted sets $\mathcal{B}$, $\mathcal{C}$ indexed by $\mathcal{A}$, and a ternary predicate $\mathcal{P}$, and states that:

There exists a many sorted function f from $\mathcal{B}$ into $\mathcal{C}$ such that for every element i of $\mathcal{A}$ and for every element a of $\mathcal{B}(i)$ holds $\mathcal{P}[i, a, f(i)(a)]$

provided the following requirement is met:

- For every element i of $\mathcal{A}$ and for every element a of $\mathcal{B}(i)$ there exists an element b of $\mathcal{C}(i)$ such that $\mathcal{P}[i, a, b]$.

Let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, let X be a non empty subset of S-Terms(V), and let A be an algebra over S. The functor X-CircuitSorts(A) yielding a many sorted set indexed by the carrier of X-CircuitStr is defined by:

(Def. 4) For every vertex v of X-CircuitStr holds (X-CircuitSorts(A))(v) = the sort of v w.r.t. A.

Let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, let X be a non empty subset of S-Terms(V), and let A be a non-empty algebra over S. Note that X-CircuitSorts(A) is non-empty.

We now state the proposition

(14) Let X be a set with a compound term of S over V, g be a gate of X-CircuitStr, and o be an operation symbol of S. If $g(\emptyset) = \langle o$, the carrier of $S\rangle$, then (X-CircuitSorts(A)) $\cdot$ Arity(g) = (the sorts of A) $\cdot$ Arity(o).

Let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, let X be a non empty subset of S-Terms(V), and let A be a non-empty algebra over

S. The functor X-CircuitCharact(A) yields a many sorted function from (X-CircuitSorts(A))$^{\#}$ $\cdot$ the arity of X-CircuitStr into (X-CircuitSorts(A)) $\cdot$ the result sort of X-CircuitStr and is defined by:

(Def. 5) For every gate g of X-CircuitStr such that $g \in$ the operation symbols of X-CircuitStr holds (X-CircuitCharact(A))(g) = the action of g w.r.t A.

Let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, let X be a non empty subset of S-Terms(V), and let A be a non-empty algebra over S. The functor X-Circuit(A) yielding a non-empty strict algebra over X-CircuitStr is defined by:

(Def. 6) X-Circuit(A) = $\langle X$-CircuitSorts(A), X-CircuitCharact(A)$\rangle$.

Next we state four propositions:

(15) For every vertex v of X-CircuitStr holds (the sorts of X-Circuit(A))(v) = the sort of v w.r.t. A.

(16) Let A be a locally-finite non-empty algebra over S, X be a set with a compound term of S over V, and g be an operation symbol of X-CircuitStr. Then Den(g, X-Circuit(A)) = the action of g w.r.t A.

(17) Let A be a locally-finite non-empty algebra over S, X be a set with a compound term of S over V, g be an operation symbol of X-CircuitStr, and o be an operation symbol of S. If $g(\emptyset) = \langle o$, the carrier of $S\rangle$, then Den(g, X-Circuit(A)) = Den(o, A).

(18) Let A be a locally-finite non-empty algebra over S and X be a non empty subset of S-Terms(V). Then X-Circuit(A) is locally-finite.

Let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, let X be a set with a compound term of S over V, and let A be a locally-finite non-empty algebra over S. Note that X-Circuit(A) is locally-finite.

The following two propositions are true:

(19) Let S be a non empty non void many sorted signature, V be a non-empty many sorted set indexed by the carrier of S, X_1, X_2 be sets with compound terms of S over V, and A be a non-empty algebra over S. Then X_1-Circuit(A) $\approx$ X_2-Circuit(A).

(20) Let S be a non empty non void many sorted signature, V be a non-empty many sorted set indexed by the carrier of S, X_1, X_2 be sets with compound terms of S over V, and A be a non-empty algebra over S. Then ($X_1 \cup X_2$)-Circuit(A) = (X_1-Circuit(A))$+\cdot$(X_2-Circuit(A)).

3. Correctness of a Term Circuit

In the sequel S is a non empty non void many sorted signature, A is a non-empty locally-finite algebra over S, V is a variables family of A, and X is a set with a compound term of S over V.

Let S be a non empty non void many sorted signature, let A be a non-empty algebra over S, let V be a variables family of A, and let t be a decorated tree. Let us assume that t is a term of S over V. Let f be a many sorted function from V into the sorts of A. The functor $[\![t]\!]_A(f)$ is defined by:

(Def. 7) There exists a term t' of A over V such that $t' = t$ and $[\![t]\!]_A(f) = t' \mathbin{@} f$.

Let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, let X be a set with a compound term of S over V, let A be a non-empty locally-finite algebra over S, and let s be a state of X-Circuit(A). A many sorted function from V into the sorts of A is said to be a valuation compatible with s if it satisfies the condition (Def. 8).

(Def. 8) Let x be a vertex of S and v be an element of $V(x)$. If the root tree of $\langle v, x\rangle \in$ Subtrees(X), then it$(x)(v) = s$(the root tree of $\langle v, x\rangle$).

Next we state the proposition

(21) Let s be a state of X-Circuit(A), f be a valuation compatible with s, and n be a natural number. Then f is a valuation compatible with Following(s, n).

Let x be a set, let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, and let p be a finite sequence of elements of S-Terms(V). One can verify that x-tree(p) is finite.

The following propositions are true:

(22) Let s be a state of X-Circuit(A), f be a valuation compatible with s, and t be a term of S over V. If $t \in$ Subtrees(X), then Following$(s, 1 + \operatorname{height} \operatorname{dom} t)$ is stable at t and $(\text{Following}(s, 1 + \operatorname{height} \operatorname{dom} t))(t) = [\![t]\!]_A(f)$.

(23) Suppose that it is not true that there exists a term t of S over V and there exists an operation symbol o of S such that $t \in$ Subtrees(X) and $t(\emptyset) = \langle o, \text{the carrier of } S\rangle$ and Arity$(o) = \emptyset$. Let s be a state of X-Circuit(A), f be a valuation compatible with s, and t be a term of S over V. If $t \in$ Subtrees(X), then Following$(s, \operatorname{height} \operatorname{dom} t)$ is stable at t and $(\text{Following}(s, \operatorname{height} \operatorname{dom} t))(t) = [\![t]\!]_A(f)$.

4. Circuit Similarity

Let X be a set. One can verify that id_X is one-to-one.

Let f be an one-to-one function. One can verify that f^{-1} is one-to-one. Let g be an one-to-one function. Note that $g \cdot f$ is one-to-one.

Let S_1, S_2 be non empty many sorted signatures and let f, g be functions. We say that S_1 and S_2 are equivalent w.r.t. f and g if and only if the conditions (Def. 9) are satisfied.

(Def. 9)(i) f is one-to-one,

(ii) g is one-to-one,

(iii) f and g form morphism between S_1 and S_2, and

(iv) f^{-1} and g^{-1} form morphism between S_2 and S_1.

One can prove the following propositions:

(24) Let S_1, S_2 be non empty many sorted signatures and f, g be functions. Suppose S_1 and S_2 are equivalent w.r.t. f and g. Then the carrier of $S_2 = f^\circ$(the carrier of S_1) and the operation symbols of $S_2 = g^\circ$(the operation symbols of S_1).

(25) Let S_1, S_2 be non empty many sorted signatures and f, g be functions. Suppose S_1 and S_2 are equivalent w.r.t. f and g. Then $\mathrm{rng}\, f =$ the carrier of S_2 and $\mathrm{rng}\, g =$ the operation symbols of S_2.

(26) Let S be a non empty many sorted signature. Then S and S are equivalent w.r.t. $\mathrm{id}_{\text{the carrier of } S}$ and $\mathrm{id}_{\text{the operation symbols of } S}$.

(27) Let S_1, S_2 be non empty many sorted signatures and f, g be functions. Suppose S_1 and S_2 are equivalent w.r.t. f and g. Then S_2 and S_1 are equivalent w.r.t. f^{-1} and g^{-1}.

(28) Let S_1, S_2, S_3 be non empty many sorted signatures and f_1, g_1, f_2, g_2 be functions. Suppose S_1 and S_2 are equivalent w.r.t. f_1 and g_1 and S_2 and S_3 are equivalent w.r.t. f_2 and g_2. Then S_1 and S_3 are equivalent w.r.t. $f_2 \cdot f_1$ and $g_2 \cdot g_1$.

(29) Let S_1, S_2 be non empty many sorted signatures and f, g be functions. Suppose S_1 and S_2 are equivalent w.r.t. f and g. Then $f^\circ\, \mathrm{InputVertices}(S_1) = \mathrm{InputVertices}(S_2)$ and $f^\circ\, \mathrm{InnerVertices}(S_1) = \mathrm{InnerVertices}(S_2)$.

Let S_1, S_2 be non empty many sorted signatures. We say that S_1 and S_2 are equivalent if and only if:

(Def. 10) There exist one-to-one functions f, g such that S_1 and S_2 are equivalent w.r.t. f and g.

Let us notice that the predicate S_1 and S_2 are equivalent is reflexive and symmetric.

One can prove the following proposition

(30) Let S_1, S_2, S_3 be non empty many sorted signatures. Suppose S_1 and S_2 are equivalent and S_2 and S_3 are equivalent. Then S_1 and S_3 are equivalent.

Let S_1, S_2 be non empty many sorted signatures and let f be a function. We say that f preserves inputs of S_1 in S_2 if and only if:

(Def. 11) $f^\circ\,\mathrm{InputVertices}(S_1) \subseteq \mathrm{InputVertices}(S_2)$.

Next we state four propositions:

(31) Let S_1, S_2 be non empty many sorted signatures and f, g be functions. Suppose f and g form morphism between S_1 and S_2. Let v be a vertex of S_1. Then $f(v)$ is a vertex of S_2.

(32) Let S_1, S_2 be non empty non void many sorted signatures and f, g be functions. Suppose f and g form morphism between S_1 and S_2. Let v be a gate of S_1. Then $g(v)$ is a gate of S_2.

(33) Let S_1, S_2 be non empty many sorted signatures and f, g be functions. If f and g form morphism between S_1 and S_2, then $f^\circ\,\mathrm{InnerVertices}(S_1) \subseteq \mathrm{InnerVertices}(S_2)$.

(34) Let S_1, S_2 be circuit-like non void non empty many sorted signatures and f, g be functions. Suppose f and g form morphism between S_1 and S_2. Let v_1 be a vertex of S_1. Suppose $v_1 \in \mathrm{InnerVertices}(S_1)$. Let v_2 be a vertex of S_2. If $v_2 = f(v_1)$, then the action at $v_2 = g$(the action at v_1).

Let S_1, S_2 be non empty many sorted signatures, let f, g be functions, let C_1 be a non-empty algebra over S_1, and let C_2 be a non-empty algebra over S_2. We say that f and g form embedding of C_1 into C_2 if and only if the conditions (Def. 12) are satisfied.

(Def. 12)(i) f is one-to-one,

(ii) g is one-to-one,

(iii) f and g form morphism between S_1 and S_2,

(iv) the sorts of $C_1 =$ (the sorts of C_2) $\cdot f$, and

(v) the characteristics of $C_1 =$ (the characteristics of C_2) $\cdot g$.

The following propositions are true:

(35) Let S be a non empty many sorted signature and C be a non-empty algebra over S. Then $\mathrm{id}_{\text{the carrier of } S}$ and $\mathrm{id}_{\text{the operation symbols of } S}$ form embedding of C into C.

(36) Let S_1, S_2, S_3 be non empty many sorted signatures, f_1, g_1, f_2, g_2 be functions, C_1 be a non-empty algebra over S_1, C_2 be a non-empty algebra over S_2, and C_3 be a non-empty algebra over S_3. Suppose f_1 and g_1 form embedding of C_1 into C_2 and f_2 and g_2 form embedding of C_2 into C_3. Then $f_2 \cdot f_1$ and $g_2 \cdot g_1$ form embedding of C_1 into C_3.

Let S_1, S_2 be non empty many sorted signatures, let f, g be functions, let C_1 be a non-empty algebra over S_1, and let C_2 be a non-empty algebra over S_2.

We say that C_1 and C_2 are similar w.r.t. f and g if and only if:

(Def. 13) f and g form embedding of C_1 into C_2 and f^{-1} and g^{-1} form embedding of C_2 into C_1.

The following propositions are true:

(37) Let S_1, S_2 be non empty many sorted signatures, f, g be functions, C_1 be a non-empty algebra over S_1, and C_2 be a non-empty algebra over S_2. Suppose C_1 and C_2 are similar w.r.t. f and g. Then S_1 and S_2 are equivalent w.r.t. f and g.

(38) Let S_1, S_2 be non empty many sorted signatures, f, g be functions, C_1 be a non-empty algebra over S_1, and C_2 be a non-empty algebra over S_2. Then C_1 and C_2 are similar w.r.t. f and g if and only if the following conditions are satisfied:

(i) S_1 and S_2 are equivalent w.r.t. f and g,

(ii) the sorts of $C_1 =$ (the sorts of C_2) $\cdot f$, and

(iii) the characteristics of $C_1 =$ (the characteristics of C_2) $\cdot g$.

(39) Let S be a non empty many sorted signature and C be a non-empty algebra over S. Then C and C are similar w.r.t. $\mathrm{id}_{\text{the carrier of } S}$ and $\mathrm{id}_{\text{the operation symbols of } S}$.

(40) Let S_1, S_2 be non empty many sorted signatures, f, g be functions, C_1 be a non-empty algebra over S_1, and C_2 be a non-empty algebra over S_2. Suppose C_1 and C_2 are similar w.r.t. f and g. Then C_2 and C_1 are similar w.r.t. f^{-1} and g^{-1}.

(41) Let S_1, S_2, S_3 be non empty many sorted signatures, f_1, g_1, f_2, g_2 be functions, C_1 be a non-empty algebra over S_1, C_2 be a non-empty algebra over S_2, and C_3 be a non-empty algebra over S_3. Suppose C_1 and C_2 are similar w.r.t. f_1 and g_1 and C_2 and C_3 are similar w.r.t. f_2 and g_2. Then C_1 and C_3 are similar w.r.t. $f_2 \cdot f_1$ and $g_2 \cdot g_1$.

Let S_1, S_2 be non empty many sorted signatures, let C_1 be a non-empty algebra over S_1, and let C_2 be a non-empty algebra over S_2. We say that C_1 and C_2 are similar if and only if:

(Def. 14) There exist functions f, g such that C_1 and C_2 are similar w.r.t. f and g.

For simplicity, we use the following convention: G_1, G_2 denote circuit-like non void non empty many sorted signatures, f, g denote functions, C_1 denotes a non-empty circuit of G_1, and C_2 denotes a non-empty circuit of G_2.

Next we state a number of propositions:

(42) Suppose f and g form embedding of C_1 into C_2. Then

(i) $\mathrm{dom}\, f =$ the carrier of G_1,

(ii) $\mathrm{rng}\, f \subseteq$ the carrier of G_2,

(iii) $\mathrm{dom}\, g =$ the operation symbols of G_1, and

(iv) $\operatorname{rng} g \subseteq$ the operation symbols of G_2.

(43) Suppose f and g form embedding of C_1 into C_2. Let o_1 be a gate of G_1 and o_2 be a gate of G_2. If $o_2 = g(o_1)$, then $\mathrm{Den}(o_2, C_2) = \mathrm{Den}(o_1, C_1)$.

(44) Suppose f and g form embedding of C_1 into C_2. Let o_1 be a gate of G_1 and o_2 be a gate of G_2. Suppose $o_2 = g(o_1)$. Let s_1 be a state of C_1 and s_2 be a state of C_2. If $s_1 = s_2 \cdot f$, then o_2 depends-on-in $s_2 =$ o_1 depends-on-in s_1.

(45) If f and g form embedding of C_1 into C_2, then for every state s of C_2 holds $s \cdot f$ is a state of C_1.

(46) Suppose f and g form embedding of C_1 into C_2. Let s_2 be a state of C_2 and s_1 be a state of C_1. Suppose $s_1 = s_2 \cdot f$ and for every vertex v of G_1 such that $v \in \mathrm{InputVertices}(G_1)$ holds s_2 is stable at $f(v)$. Then $\mathrm{Following}(s_1) = \mathrm{Following}(s_2) \cdot f$.

(47) Suppose f and g form embedding of C_1 into C_2 and f preserves inputs of G_1 in G_2. Let s_2 be a state of C_2 and s_1 be a state of C_1. If $s_1 = s_2 \cdot f$, then $\mathrm{Following}(s_1) = \mathrm{Following}(s_2) \cdot f$.

(48) Suppose f and g form embedding of C_1 into C_2 and f preserves inputs of G_1 in G_2. Let s_2 be a state of C_2 and s_1 be a state of C_1. If $s_1 = s_2 \cdot f$, then for every natural number n holds $\mathrm{Following}(s_1, n) = \mathrm{Following}(s_2, n) \cdot f$.

(49) Suppose f and g form embedding of C_1 into C_2 and f preserves inputs of G_1 in G_2. Let s_2 be a state of C_2 and s_1 be a state of C_1. If $s_1 = s_2 \cdot f$, then if s_2 is stable, then s_1 is stable.

(50) Suppose f and g form embedding of C_1 into C_2 and f preserves inputs of G_1 in G_2. Let s_2 be a state of C_2 and s_1 be a state of C_1. Suppose $s_1 = s_2 \cdot f$. Let v_1 be a vertex of G_1. Then s_1 is stable at v_1 if and only if s_2 is stable at $f(v_1)$.

(51) If C_1 and C_2 are similar w.r.t. f and g, then for every state s of C_2 holds $s \cdot f$ is a state of C_1.

(52) Suppose C_1 and C_2 are similar w.r.t. f and g. Let s_1 be a state of C_1 and s_2 be a state of C_2. Then $s_1 = s_2 \cdot f$ if and only if $s_2 = s_1 \cdot f^{-1}$.

(53) If C_1 and C_2 are similar w.r.t. f and g, then $f^\circ\,\mathrm{InputVertices}(G_1) = \mathrm{InputVertices}(G_2)$ and $f^\circ\,\mathrm{InnerVertices}(G_1) = \mathrm{InnerVertices}(G_2)$.

(54) If C_1 and C_2 are similar w.r.t. f and g, then f preserves inputs of G_1 in G_2.

(55) Suppose C_1 and C_2 are similar w.r.t. f and g. Let s_1 be a state of C_1 and s_2 be a state of C_2. If $s_1 = s_2 \cdot f$, then $\mathrm{Following}(s_1) = \mathrm{Following}(s_2) \cdot f$.

(56) Suppose C_1 and C_2 are similar w.r.t. f and g. Let s_1 be a state of C_1 and s_2 be a state of C_2. If $s_1 = s_2 \cdot f$, then for every natural number n holds $\mathrm{Following}(s_1, n) = \mathrm{Following}(s_2, n) \cdot f$.

(57) Suppose C_1 and C_2 are similar w.r.t. f and g. Let s_1 be a state of C_1 and s_2 be a state of C_2. If $s_1 = s_2 \cdot f$, then s_1 is stable iff s_2 is stable.

(58) Suppose C_1 and C_2 are similar w.r.t. f and g. Let s_1 be a state of C_1 and s_2 be a state of C_2. Suppose $s_1 = s_2 \cdot f$. Let v_1 be a vertex of G_1. Then s_1 is stable at v_1 if and only if s_2 is stable at $f(v_1)$.

5. Term Specification

Let S be a non empty non void many sorted signature, let A be a non-empty algebra over S, let V be a non-empty many sorted set indexed by the carrier of S, let X be a non empty subset of S-Terms(V), let G be a circuit-like non void non empty many sorted signature, and let C be a non-empty circuit of G. We say that C calculates X in A if and only if:

(Def. 15) There exist f, g such that f and g form embedding of X-Circuit(A) into C and f preserves inputs of X-CircuitStr in G.

We say that X and A specify C if and only if:

(Def. 16) C and X-Circuit(A) are similar.

Let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, let A be a non-empty algebra over S, let X be a non empty subset of S-Terms(V), let G be a circuit-like non void non empty many sorted signature, and let C be a non-empty circuit of G. Let us assume that C calculates X in A. An one-to-one function is said to be a sort map from X and A into C if:

(Def. 17) It preserves inputs of X-CircuitStr in G and there exists g such that it and g form embedding of X-Circuit(A) into C.

Let S be a non empty non void many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, let A be a non-empty algebra over S, let X be a non empty subset of S-Terms(V), let G be a circuit-like non void non empty many sorted signature, and let C be a non-empty circuit of G. Let us assume that C calculates X in A. Let f be a sort map from X and A into C. An one-to-one function is said to be an operation map from X and A into C obeying f if:

(Def. 18) f and it form embedding of X-Circuit(A) into C.

The following propositions are true:

(59) Let G be a circuit-like non void non empty many sorted signature and C be a non-empty circuit of G. If X and A specify C, then C calculates X in A.

(60) Let G be a circuit-like non void non empty many sorted signature and C be a non-empty circuit of G. Suppose C calculates X in A. Let f be

a sort map from X and A into C and t be a term of S over V. Suppose $t \in \mathrm{Subtrees}(X)$. Let s be a state of C. Then

(i) Following$(s, 1 + \mathrm{height}\,\mathrm{dom}\, t)$ is stable at $f(t)$, and

(ii) for every state s' of X-Circuit(A) such that $s' = s \cdot f$ and for every valuation h compatible with s' holds (Following$(s, 1+\mathrm{height}\,\mathrm{dom}\, t))(f(t)) = [\![t]\!]_A(h)$.

(61) Let G be a circuit-like non void non empty many sorted signature and C be a non-empty circuit of G. Suppose C calculates X in A. Let t be a term of S over V. Suppose $t \in \mathrm{Subtrees}(X)$. Then there exists a vertex v of G such that for every state s of C holds

(i) Following$(s, 1 + \mathrm{height}\,\mathrm{dom}\, t)$ is stable at v, and

(ii) there exists a sort map f from X and A into C such that for every state s' of X-Circuit(A) such that $s' = s \cdot f$ and for every valuation h compatible with s' holds (Following$(s, 1 + \mathrm{height}\,\mathrm{dom}\, t))(v) = [\![t]\!]_A(h)$.

(62) Let G be a circuit-like non void non empty many sorted signature and C be a non-empty circuit of G. Suppose X and A specify C. Let f be a sort map from X and A into C, s be a state of C, and t be a term of S over V. Suppose $t \in \mathrm{Subtrees}(X)$. Then

(i) Following$(s, 1 + \mathrm{height}\,\mathrm{dom}\, t)$ is stable at $f(t)$, and

(ii) for every state s' of X-Circuit(A) such that $s' = s \cdot f$ and for every valuation h compatible with s' holds (Following$(s, 1+\mathrm{height}\,\mathrm{dom}\, t))(f(t)) = [\![t]\!]_A(h)$.

(63) Let G be a circuit-like non void non empty many sorted signature and C be a non-empty circuit of G. Suppose X and A specify C. Let t be a term of S over V. Suppose $t \in \mathrm{Subtrees}(X)$. Then there exists a vertex v of G such that for every state s of C holds

(i) Following$(s, 1 + \mathrm{height}\,\mathrm{dom}\, t)$ is stable at v, and

(ii) there exists a sort map f from X and A into C such that for every state s' of X-Circuit(A) such that $s' = s \cdot f$ and for every valuation h compatible with s' holds (Following$(s, 1 + \mathrm{height}\,\mathrm{dom}\, t))(v) = [\![t]\!]_A(h)$.

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