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Bounding Boxes for Special Sequences in $\mathcal{E}^2$

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Summary. This is the continuation of the proof of the Jordan Theorem according to [18].

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The articles [16], [8], [6], [2], [21], [20], [5], [3], [12], [13], [15], [9], [1], [14], [17], [4], [23], [11], [10], [22], [19], and [7] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we use the following convention: p, q denote points of $\mathcal{E}^2_{\rm T}$, s, r denote real numbers, h denotes a non constant standard special circular sequence, g denotes a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, f denotes a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$, and I, i_1, i, j, k denote natural numbers.

We now state a number of propositions:

(1) Let B be a subset of $\mathbb{R}$. Suppose there exists a real number r_1 such that $r_1 \in B$ and B is lower bounded and for every r such that $r \in B$ holds $s \leqslant r$. Then $s \leqslant \inf B$.

(2) Let B be a subset of $\mathbb{R}$. Suppose there exists a real number r_1 such that $r_1 \in B$ and B is upper bounded and for every r such that $r \in B$ holds $s \geqslant r$. Then $s \geqslant \sup B$.

(3) $\pi_{\operatorname{len} h} h \in \mathcal{L}(h, \operatorname{len} h -' 1)$.

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(4) If $3 \leqslant i$, then $i \bmod (i -' 1) = 1$.

(5) If $p \in \operatorname{rng} h$, then there exists a natural number i such that $1 \leqslant i$ and $i + 1 \leqslant \operatorname{len} h$ and $h(i) = p$.

(6) For every finite sequence g of elements of $\mathbb{R}$ such that $r \in \operatorname{rng} g$ holds $(\operatorname{Inc}(g))(1) \leqslant r$ and $r \leqslant (\operatorname{Inc}(g))(\operatorname{len} \operatorname{Inc}(g))$.

(7) Suppose $1 \leqslant i$ and $i \leqslant \operatorname{len} h$ and $1 \leqslant I$ and $I \leqslant$ width the Go-board of h. Then $((\text{the Go-board of } h)_{1,I})_{\mathbf{1}} \leqslant (\pi_i h)_{\mathbf{1}}$ and $(\pi_i h)_{\mathbf{1}} \leqslant ((\text{the Go-board of } h)_{\text{len the Go-board of } h,\ I})_{\mathbf{1}}$.

(8) Suppose $1 \leqslant i$ and $i \leqslant \operatorname{len} h$ and $1 \leqslant I$ and $I \leqslant$ len the Go-board of h. Then $((\text{the Go-board of } h)_{I,1})_{\mathbf{2}} \leqslant (\pi_i h)_{\mathbf{2}}$ and $(\pi_i h)_{\mathbf{2}} \leqslant ((\text{the Go-board of } h)_{I,\text{width the Go-board of } h})_{\mathbf{2}}$.

(9) Suppose $1 \leqslant i$ and $i \leqslant$ len the Go-board of f. Then there exist k, j such that $k \in \operatorname{dom} f$ and $\langle i, j \rangle \in$ the indices of the Go-board of f and $\pi_k f = (\text{the Go-board of } f)_{i,j}$.

(10) Suppose $1 \leqslant j$ and $j \leqslant$ width the Go-board of f. Then there exist k, i such that $k \in \operatorname{dom} f$ and $\langle i, j \rangle \in$ the indices of the Go-board of f and $\pi_k f = (\text{the Go-board of } f)_{i,j}$.

(11) Suppose $1 \leqslant i$ and $i \leqslant$ len the Go-board of f and $1 \leqslant j$ and $j \leqslant$ width the Go-board of f. Then there exists k such that $k \in \operatorname{dom} f$ and $\langle i, j \rangle \in$ the indices of the Go-board of f and $(\pi_k f)_{\mathbf{1}} = ((\text{the Go-board of } f)_{i,j})_{\mathbf{1}}$.

(12) Suppose $1 \leqslant i$ and $i \leqslant$ len the Go-board of f and $1 \leqslant j$ and $j \leqslant$ width the Go-board of f. Then there exists k such that $k \in \operatorname{dom} f$ and $\langle i, j \rangle \in$ the indices of the Go-board of f and $(\pi_k f)_{\mathbf{2}} = ((\text{the Go-board of } f)_{i,j})_{\mathbf{2}}$.

2. Extrema of Projections

One can prove the following propositions:

(13) If $1 \leqslant i$ and $i \leqslant \operatorname{len} h$, then S-bound $\widetilde{\mathcal{L}}(h) \leqslant (\pi_i h)_{\mathbf{2}}$ and $(\pi_i h)_{\mathbf{2}} \leqslant$ N-bound $\widetilde{\mathcal{L}}(h)$.

(14) If $1 \leqslant i$ and $i \leqslant \operatorname{len} h$, then W-bound $\widetilde{\mathcal{L}}(h) \leqslant (\pi_i h)_{\mathbf{1}}$ and $(\pi_i h)_{\mathbf{1}} \leqslant$ E-bound $\widetilde{\mathcal{L}}(h)$.

(15) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{2}} : q_{\mathbf{1}} = \text{W-bound}\,\widetilde{\mathcal{L}}(h) \wedge q \in \widetilde{\mathcal{L}}(h)\}$ holds $X = (\text{proj2} \restriction \text{W-most}\,\widetilde{\mathcal{L}}(h))^{\circ}$(the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction \text{W-most}\,\widetilde{\mathcal{L}}(h)$).

(16) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{2}} : q_{\mathbf{1}} = \text{E-bound}\,\widetilde{\mathcal{L}}(h) \wedge q \in \widetilde{\mathcal{L}}(h)\}$ holds $X = (\text{proj2} \restriction \text{E-most}\,\widetilde{\mathcal{L}}(h))^{\circ}$(the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction \text{E-most}\,\widetilde{\mathcal{L}}(h)$).

(17) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{1}} : q_{\mathbf{2}} = \text{N-bound}\,\widetilde{\mathcal{L}}(h) \wedge q \in \widetilde{\mathcal{L}}(h)\}$ holds $X = (\text{proj1} \restriction \text{N-most}\,\widetilde{\mathcal{L}}(h))^{\circ}$(the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction \text{N-most}\,\widetilde{\mathcal{L}}(h)$).

(18) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{1}} : q_{\mathbf{2}} = \text{S-bound}\,\widetilde{\mathcal{L}}(h) \wedge q \in \widetilde{\mathcal{L}}(h)\}$ holds $X = (\text{proj1} \restriction \text{S-most}\,\widetilde{\mathcal{L}}(h))^{\circ}(\text{the carrier of } (\mathcal{E}^2_{\mathrm{T}}) \restriction \text{S-most}\,\widetilde{\mathcal{L}}(h))$.

(19) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{1}} : q \in \widetilde{\mathcal{L}}(g)\}$ holds $X = (\text{proj1} \restriction \widetilde{\mathcal{L}}(g))^{\circ}(\text{the carrier of } (\mathcal{E}^2_{\mathrm{T}}) \restriction \widetilde{\mathcal{L}}(g))$.

(20) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{2}} : q \in \widetilde{\mathcal{L}}(g)\}$ holds $X = (\text{proj2} \restriction \widetilde{\mathcal{L}}(g))^{\circ}(\text{the carrier of } (\mathcal{E}^2_{\mathrm{T}}) \restriction \widetilde{\mathcal{L}}(g))$.

(21) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{2}} : q_{\mathbf{1}} = \text{W-bound}\,\widetilde{\mathcal{L}}(h) \wedge q \in \widetilde{\mathcal{L}}(h)\}$ holds $\inf X = \inf(\text{proj2} \restriction \text{W-most}\,\widetilde{\mathcal{L}}(h))$.

(22) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{2}} : q_{\mathbf{1}} = \text{W-bound}\,\widetilde{\mathcal{L}}(h) \wedge q \in \widetilde{\mathcal{L}}(h)\}$ holds $\sup X = \sup(\text{proj2} \restriction \text{W-most}\,\widetilde{\mathcal{L}}(h))$.

(23) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{2}} : q_{\mathbf{1}} = \text{E-bound}\,\widetilde{\mathcal{L}}(h) \wedge q \in \widetilde{\mathcal{L}}(h)\}$ holds $\inf X = \inf(\text{proj2} \restriction \text{E-most}\,\widetilde{\mathcal{L}}(h))$.

(24) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{2}} : q_{\mathbf{1}} = \text{E-bound}\,\widetilde{\mathcal{L}}(h) \wedge q \in \widetilde{\mathcal{L}}(h)\}$ holds $\sup X = \sup(\text{proj2} \restriction \text{E-most}\,\widetilde{\mathcal{L}}(h))$.

(25) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{1}} : q \in \widetilde{\mathcal{L}}(g)\}$ holds $\inf X = \inf(\text{proj1} \restriction \widetilde{\mathcal{L}}(g))$.

(26) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{1}} : q_{\mathbf{2}} = \text{S-bound}\,\widetilde{\mathcal{L}}(h) \wedge q \in \widetilde{\mathcal{L}}(h)\}$ holds $\inf X = \inf(\text{proj1} \restriction \text{S-most}\,\widetilde{\mathcal{L}}(h))$.

(27) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{1}} : q_{\mathbf{2}} = \text{S-bound}\,\widetilde{\mathcal{L}}(h) \wedge q \in \widetilde{\mathcal{L}}(h)\}$ holds $\sup X = \sup(\text{proj1} \restriction \text{S-most}\,\widetilde{\mathcal{L}}(h))$.

(28) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{1}} : q_{\mathbf{2}} = \text{N-bound}\,\widetilde{\mathcal{L}}(h) \wedge q \in \widetilde{\mathcal{L}}(h)\}$ holds $\inf X = \inf(\text{proj1} \restriction \text{N-most}\,\widetilde{\mathcal{L}}(h))$.

(29) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{1}} : q_{\mathbf{2}} = \text{N-bound}\,\widetilde{\mathcal{L}}(h) \wedge q \in \widetilde{\mathcal{L}}(h)\}$ holds $\sup X = \sup(\text{proj1} \restriction \text{N-most}\,\widetilde{\mathcal{L}}(h))$.

(30) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{2}} : q \in \widetilde{\mathcal{L}}(g)\}$ holds $\inf X = \inf(\text{proj2} \restriction \widetilde{\mathcal{L}}(g))$.

(31) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{1}} : q \in \widetilde{\mathcal{L}}(g)\}$ holds $\sup X = \sup(\text{proj1} \restriction \widetilde{\mathcal{L}}(g))$.

(32) For every subset X of $\mathbb{R}$ such that $X = \{q_{\mathbf{2}} : q \in \widetilde{\mathcal{L}}(g)\}$ holds $\sup X = \sup(\text{proj2} \restriction \widetilde{\mathcal{L}}(g))$.

(33) If $p \in \widetilde{\mathcal{L}}(h)$ and $1 \leqslant I$ and $I \leqslant \text{width}\,\text{the Go-board of } h$, then $((\text{the Go-board of } h)_{1,I})_{\mathbf{1}} \leqslant p_{\mathbf{1}}$.

(34) If $p \in \widetilde{\mathcal{L}}(h)$ and $1 \leqslant I$ and $I \leqslant \text{width}\,\text{the Go-board of } h$, then $p_{\mathbf{1}} \leqslant ((\text{the Go-board of } h)_{\text{len}\,\text{the Go-board of } h,\, I})_{\mathbf{1}}$.

(35) If $p \in \widetilde{\mathcal{L}}(h)$ and $1 \leqslant I$ and $I \leqslant \text{len}\,\text{the Go-board of } h$, then $((\text{the Go-board of } h)_{I,1})_{\mathbf{2}} \leqslant p_{\mathbf{2}}$.

(36) If $p \in \widetilde{\mathcal{L}}(h)$ and $1 \leqslant I$ and $I \leqslant \text{len}\,\text{the Go-board of } h$, then $p_{\mathbf{2}} \leqslant ((\text{the Go-board of } h)_{I,\text{width}\,\text{the Go-board of } h})_{\mathbf{2}}$.

(37) Suppose $1 \leqslant i$ and $i \leqslant \operatorname{len}$ the Go-board of h and $1 \leqslant j$ and $j \leqslant \operatorname{width}$ the Go-board of h. Then there exists q such that $q_{\mathbf{1}} = ((\text{the Go-board of } h)_{i,j})_{\mathbf{1}}$ and $q \in \widetilde{\mathcal{L}}(h)$.

(38) Suppose $1 \leqslant i$ and $i \leqslant \operatorname{len}$ the Go-board of h and $1 \leqslant j$ and $j \leqslant \operatorname{width}$ the Go-board of h. Then there exists q such that $q_{\mathbf{2}} = ((\text{the Go-board of } h)_{i,j})_{\mathbf{2}}$ and $q \in \widetilde{\mathcal{L}}(h)$.

(39) W-bound $\widetilde{\mathcal{L}}(h) = ((\text{the Go-board of } h)_{1,1})_{\mathbf{1}}$.

(40) S-bound $\widetilde{\mathcal{L}}(h) = ((\text{the Go-board of } h)_{1,1})_{\mathbf{2}}$.

(41) E-bound $\widetilde{\mathcal{L}}(h) = ((\text{the Go-board of } h)_{\operatorname{len} \text{the Go-board of } h,\, 1})_{\mathbf{1}}$.

(42) N-bound $\widetilde{\mathcal{L}}(h) = ((\text{the Go-board of } h)_{1, \operatorname{width} \text{the Go-board of } h})_{\mathbf{2}}$.

(43) Let Y be a non empty finite subset of $\mathbb{N}$. Suppose that

(i) $1 \leqslant i$,

(ii) $i \leqslant \operatorname{len} f$,

(iii) $1 \leqslant I$,

(iv) $I \leqslant \operatorname{len}$ the Go-board of f,

(v) $Y = \{j : \langle I, j\rangle \in$ the indices of the Go-board of $f \ \wedge\ \bigvee_k (k \in \operatorname{dom} f \ \wedge\ \pi_k f = (\text{the Go-board of } f)_{I,j})\}$,

(vi) $(\pi_i f)_{\mathbf{1}} = ((\text{the Go-board of } f)_{I,1})_{\mathbf{1}}$, and

(vii) $i_1 = \min Y$.

Then $((\text{the Go-board of } f)_{I,i_1})_{\mathbf{2}} \leqslant (\pi_i f)_{\mathbf{2}}$.

(44) Let Y be a non empty finite subset of $\mathbb{N}$. Suppose that

(i) $1 \leqslant i$,

(ii) $i \leqslant \operatorname{len} h$,

(iii) $1 \leqslant I$,

(iv) $I \leqslant \operatorname{width}$ the Go-board of h,

(v) $Y = \{j : \langle j, I\rangle \in$ the indices of the Go-board of $h \ \wedge\ \bigvee_k (k \in \operatorname{dom} h \ \wedge\ \pi_k h = (\text{the Go-board of } h)_{j,I})\}$,

(vi) $(\pi_i h)_{\mathbf{2}} = ((\text{the Go-board of } h)_{1,I})_{\mathbf{2}}$, and

(vii) $i_1 = \min Y$.

Then $((\text{the Go-board of } h)_{i_1,I})_{\mathbf{1}} \leqslant (\pi_i h)_{\mathbf{1}}$.

(45) Let Y be a non empty finite subset of $\mathbb{N}$. Suppose that

(i) $1 \leqslant i$,

(ii) $i \leqslant \operatorname{len} h$,

(iii) $1 \leqslant I$,

(iv) $I \leqslant \operatorname{width}$ the Go-board of h,

(v) $Y = \{j : \langle j, I\rangle \in$ the indices of the Go-board of $h \ \wedge\ \bigvee_k (k \in \operatorname{dom} h \ \wedge\ \pi_k h = (\text{the Go-board of } h)_{j,I})\}$,

(vi) $(\pi_i h)_{\mathbf{2}} = ((\text{the Go-board of } h)_{1,I})_{\mathbf{2}}$, and

(vii) $i_1 = \max Y$.

Then $((\text{the Go-board of } h)_{i_1,I})_{\mathbf{1}} \geqslant (\pi_i h)_{\mathbf{1}}$.

(46) Let Y be a non empty finite subset of $\mathbb{N}$. Suppose that

(i) $1 \leqslant i$,

(ii) $i \leqslant \operatorname{len} f$,

(iii) $1 \leqslant I$,

(iv) $I \leqslant \operatorname{len}$ the Go-board of f,

(v) $Y = \{j : \langle I, j\rangle \in$ the indices of the Go-board of $f \wedge \bigvee_k (k \in \operatorname{dom} f \wedge \pi_k f = (\text{the Go-board of } f)_{I,j})\}$,

(vi) $(\pi_i f)_{\mathbf{1}} = ((\text{the Go-board of } f)_{I,1})_{\mathbf{1}}$, and

(vii) $i_1 = \max Y$.

Then $((\text{the Go-board of } f)_{I,i_1})_{\mathbf{2}} \geqslant (\pi_i f)_{\mathbf{2}}$.

3. Coordinates of the Special Circular Sequences Bounding Boxes

Let g be a non constant standard special circular sequence. The functor $\mathrm{i}_{\mathrm{SW}}\, g$ yields a natural number and is defined as follows:

(Def. 1) $\langle 1, \mathrm{i}_{\mathrm{SW}}\, g\rangle \in$ the indices of the Go-board of g and $(\text{the Go-board of } g)_{1,\mathrm{i}_{\mathrm{SW}}\, g} = \text{W-min}\, \widetilde{\mathcal{L}}(g)$.

The functor $\mathrm{i}_{\mathrm{NW}}\, g$ yields a natural number and is defined by:

(Def. 2) $\langle 1, \mathrm{i}_{\mathrm{NW}}\, g\rangle \in$ the indices of the Go-board of g and $(\text{the Go-board of } g)_{1,\mathrm{i}_{\mathrm{NW}}\, g} = \text{W-max}\, \widetilde{\mathcal{L}}(g)$.

The functor $\mathrm{i}_{\mathrm{SE}}\, g$ yielding a natural number is defined by the conditions (Def. 3).

(Def. 3)(i) $\langle \operatorname{len}$ the Go-board of g, $\mathrm{i}_{\mathrm{SE}}\, g\rangle \in$ the indices of the Go-board of g, and

(ii) $(\text{the Go-board of } g)_{\operatorname{len}\text{the Go-board of } g,\ \mathrm{i}_{\mathrm{SE}}\, g} = \text{E-min}\, \widetilde{\mathcal{L}}(g)$.

The functor $\mathrm{i}_{\mathrm{NE}}\, g$ yielding a natural number is defined by the conditions (Def. 4).

(Def. 4)(i) $\langle \operatorname{len}$ the Go-board of g, $\mathrm{i}_{\mathrm{NE}}\, g\rangle \in$ the indices of the Go-board of g, and

(ii) $(\text{the Go-board of } g)_{\operatorname{len}\text{the Go-board of } g,\ \mathrm{i}_{\mathrm{NE}}\, g} = \text{E-max}\, \widetilde{\mathcal{L}}(g)$.

The functor $\mathrm{i}_{\mathrm{WS}}\, g$ yields a natural number and is defined by:

(Def. 5) $\langle \mathrm{i}_{\mathrm{WS}}\, g, 1\rangle \in$ the indices of the Go-board of g and $(\text{the Go-board of } g)_{\mathrm{i}_{\mathrm{WS}}\, g,1} = \text{S-min}\, \widetilde{\mathcal{L}}(g)$.

The functor $\mathrm{i}_{\mathrm{ES}}\, g$ yields a natural number and is defined by:

(Def. 6) $\langle \mathrm{i}_{\mathrm{ES}}\, g, 1\rangle \in$ the indices of the Go-board of g and $(\text{the Go-board of } g)_{\mathrm{i}_{\mathrm{ES}}\, g,1} = \text{S-max}\, \widetilde{\mathcal{L}}(g)$.

The functor $\mathrm{i}_{\mathrm{WN}}\, g$ yields a natural number and is defined by the conditions (Def. 7).

(Def. 7)(i) $\langle \mathrm{i}_{\mathrm{WN}}\, g$, width the Go-board of $g\rangle \in$ the indices of the Go-board of g, and

(ii) $(\text{the Go-board of } g)_{\mathrm{i}_{\mathrm{WN}}\, g,\text{width the Go-board of } g} = \text{N-min}\, \widetilde{\mathcal{L}}(g)$.

The functor $\mathrm{i_{EN}}\, g$ yields a natural number and is defined by the conditions (Def. 8).

(Def. 8)(i) $\langle \mathrm{i_{EN}}\, g, \mathrm{width}\,\text{the Go-board of } g\rangle \in$ the indices of the Go-board of g, and

(ii) $(\text{the Go-board of } g)_{\mathrm{i_{EN}}\, g, \mathrm{width}\,\text{the Go-board of } g} = \text{N-max}\, \widetilde{\mathcal{L}}(g)$.

Next we state two propositions:

(47)(i) $1 \leqslant \mathrm{i_{WN}}\, h$,

(ii) $\mathrm{i_{WN}}\, h \leqslant \mathrm{len}\,\text{the Go-board of } h$,

(iii) $1 \leqslant \mathrm{i_{EN}}\, h$,

(iv) $\mathrm{i_{EN}}\, h \leqslant \mathrm{len}\,\text{the Go-board of } h$,

(v) $1 \leqslant \mathrm{i_{WS}}\, h$,

(vi) $\mathrm{i_{WS}}\, h \leqslant \mathrm{len}\,\text{the Go-board of } h$,

(vii) $1 \leqslant \mathrm{i_{ES}}\, h$, and

(viii) $\mathrm{i_{ES}}\, h \leqslant \mathrm{len}\,\text{the Go-board of } h$.

(48)(i) $1 \leqslant \mathrm{i_{NE}}\, h$,

(ii) $\mathrm{i_{NE}}\, h \leqslant \mathrm{width}\,\text{the Go-board of } h$,

(iii) $1 \leqslant \mathrm{i_{SE}}\, h$,

(iv) $\mathrm{i_{SE}}\, h \leqslant \mathrm{width}\,\text{the Go-board of } h$,

(v) $1 \leqslant \mathrm{i_{NW}}\, h$,

(vi) $\mathrm{i_{NW}}\, h \leqslant \mathrm{width}\,\text{the Go-board of } h$,

(vii) $1 \leqslant \mathrm{i_{SW}}\, h$, and

(viii) $\mathrm{i_{SW}}\, h \leqslant \mathrm{width}\,\text{the Go-board of } h$.

Let g be a non constant standard special circular sequence. The functor $\mathrm{n_{SW}}\, g$ yields a natural number and is defined as follows:

(Def. 9) $1 \leqslant \mathrm{n_{SW}}\, g$ and $\mathrm{n_{SW}}\, g + 1 \leqslant \mathrm{len}\, g$ and $g(\mathrm{n_{SW}}\, g) = \text{W-min}\, \widetilde{\mathcal{L}}(g)$.

The functor $\mathrm{n_{NW}}\, g$ yielding a natural number is defined as follows:

(Def. 10) $1 \leqslant \mathrm{n_{NW}}\, g$ and $\mathrm{n_{NW}}\, g + 1 \leqslant \mathrm{len}\, g$ and $g(\mathrm{n_{NW}}\, g) = \text{W-max}\, \widetilde{\mathcal{L}}(g)$.

The functor $\mathrm{n_{SE}}\, g$ yielding a natural number is defined by:

(Def. 11) $1 \leqslant \mathrm{n_{SE}}\, g$ and $\mathrm{n_{SE}}\, g + 1 \leqslant \mathrm{len}\, g$ and $g(\mathrm{n_{SE}}\, g) = \text{E-min}\, \widetilde{\mathcal{L}}(g)$.

The functor $\mathrm{n_{NE}}\, g$ yielding a natural number is defined by:

(Def. 12) $1 \leqslant \mathrm{n_{NE}}\, g$ and $\mathrm{n_{NE}}\, g + 1 \leqslant \mathrm{len}\, g$ and $g(\mathrm{n_{NE}}\, g) = \text{E-max}\, \widetilde{\mathcal{L}}(g)$.

The functor $\mathrm{n_{WS}}\, g$ yielding a natural number is defined by:

(Def. 13) $1 \leqslant \mathrm{n_{WS}}\, g$ and $\mathrm{n_{WS}}\, g + 1 \leqslant \mathrm{len}\, g$ and $g(\mathrm{n_{WS}}\, g) = \text{S-min}\, \widetilde{\mathcal{L}}(g)$.

The functor $\mathrm{n_{ES}}\, g$ yields a natural number and is defined as follows:

(Def. 14) $1 \leqslant \mathrm{n_{ES}}\, g$ and $\mathrm{n_{ES}}\, g + 1 \leqslant \mathrm{len}\, g$ and $g(\mathrm{n_{ES}}\, g) = \text{S-max}\, \widetilde{\mathcal{L}}(g)$.

The functor $\mathrm{n_{WN}}\, g$ yielding a natural number is defined by:

(Def. 15) $1 \leqslant \mathrm{n_{WN}}\, g$ and $\mathrm{n_{WN}}\, g + 1 \leqslant \mathrm{len}\, g$ and $g(\mathrm{n_{WN}}\, g) = \text{N-min}\, \widetilde{\mathcal{L}}(g)$.

The functor $\mathrm{n_{EN}}\, g$ yielding a natural number is defined by:

(Def. 16) $1 \leqslant \mathrm{n_{EN}}\, g$ and $\mathrm{n_{EN}}\, g + 1 \leqslant \mathrm{len}\, g$ and $g(\mathrm{n_{EN}}\, g) = \text{N-max}\, \widetilde{\mathcal{L}}(g)$.

Next we state four propositions:

(49) $\mathrm{n_{WN}}\, h \neq \mathrm{n_{WS}}\, h$.

(50) $\mathrm{n_{SW}}\, h \neq \mathrm{n_{SE}}\, h$.

(51) $\mathrm{n_{EN}}\, h \neq \mathrm{n_{ES}}\, h$.

(52) $\mathrm{n_{NW}}\, h \neq \mathrm{n_{NE}}\, h$.

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