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On the Trivial Many Sorted Algebras and Many Sorted Congruences

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Summary. This paper contains properties of many sorted functions between two many sorted sets. Other theorems describe trivial many sorted algebras. In the last section there are theorems about many sorted congruences, which are defined on many sorted algebras. I have also proved facts about natural epimorphism.

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The articles [35], [38], [10], [39], [41], [27], [40], [7], [29], [8], [9], [3], [11], [32], [6], [36], [12], [31], [2], [37], [30], [1], [4], [34], [33], [5], [13], [20], [28], [22], [23], [25], [26], [21], [17], [15], [19], [14], [18], [16], and [24] provide the terminology and notation for this paper.

1. Preliminaries

In this paper a, I will be sets and S will be a non empty non void many sorted signature.

The scheme *MSSExD* deals with a non empty set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

> There exists a many sorted set f indexed by $\mathcal{A}$ such that for every element i of $\mathcal{A}$ holds $\mathcal{P}[i, f(i)]$

provided the parameters meet the following condition:

- For every element i of $\mathcal{A}$ there exists a set j such that $\mathcal{P}[i, j]$.

Let I be a set and let M be a many sorted set indexed by I. Note that there exists an element of $\mathrm{Bool}(M)$ which is locally-finite.

Let I be a set and let M be a non-empty many sorted set indexed by I. Note that there exists a many sorted subset of M which is non-empty and locally-finite.

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Let S be a non empty non void many sorted signature, let A be a non-empty algebra over S, and let o be an operation symbol of S. One can verify that every element of $\mathrm{Args}(o, A)$ is finite sequence-like.

Let S be a non void non empty many sorted signature, let I be a set, let s be a sort symbol of S, and let F be an algebra family of I over S. Note that every element of $\mathrm{SORTS}(F)(s)$ is function-like and relation-like.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. Note that $\mathrm{FreeGenerator}(X)$ is free and non-empty.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. One can verify that $\mathrm{Free}(X)$ is free.

Let S be a non empty non void many sorted signature and let A, B be non-empty algebras over S. One can check that $[\!:A, B:\!]$ is non-empty.

The following propositions are true:

(1) For all sets X, Y and for every function f such that $a \in \mathrm{dom}\, f$ and $f(a) \in [\!:X, Y:\!]$ holds $f(a) = \langle \mathrm{pr1}(f)(a), \mathrm{pr2}(f)(a) \rangle$.

(2) For every non empty set X and for every set Y and for every function f from X into $\{Y\}$ holds $\mathrm{rng}\, f = \{Y\}$.

(3) For every non empty finite set A there exists a function f from $\mathbb{N}$ into A such that $\mathrm{rng}\, f = A$.

(4) $\mathrm{Classes}(\nabla_I) \subseteq \{I\}$.

(5) For every non empty set I holds $\mathrm{Classes}(\nabla_I) = \{I\}$.

(6) There exists a many sorted set A indexed by I such that $\{A\} = I \longmapsto \{a\}$.

(7) For every many sorted set A indexed by I there exists a non-empty many sorted set B indexed by I such that $A \subseteq B$.

(8) Let M be a non-empty many sorted set indexed by I and let B be a locally-finite many sorted subset of M. Then there exists a non-empty locally-finite many sorted subset C of M such that $B \subseteq C$.

(9) For all many sorted sets A, B indexed by I and for all many sorted functions F, G from A into $\{B\}$ holds $F = G$.

(10) For every non-empty many sorted set A indexed by I and for every many sorted set B indexed by I holds every many sorted function from A into $\{B\}$ is onto.

(11) Let A be a many sorted set indexed by I and let B be a non-empty many sorted set indexed by I. Then every many sorted function from $\{A\}$ into B is "1-1".

(12) For every non-empty many sorted set X indexed by the carrier of S holds $\mathrm{Reverse}(X)$ is "1-1".

(13) For every non-empty locally-finite many sorted set A indexed by I holds there exists many sorted function from $I \longmapsto \mathbb{N}$ into A which is onto.

(14) Let S be a non empty many sorted signature, and let A be a non-empty algebra over S, and let f, g be elements of $\prod$(the sorts of A). Suppose that for every set i holds (proj(the sorts of A, i))(f) = (proj(the sorts of A, i))(g). Then $f = g$.

(15) Let I be a non empty set, and let s be an element of the carrier of S, and let A be an algebra family of I over S, and let f, g be elements of $\prod \mathrm{Carrier}(A, s)$. If for every element a of I holds $(\mathrm{proj}(\mathrm{Carrier}(A, s), a))(f) = (\mathrm{proj}(\mathrm{Carrier}(A, s), a))(g)$, then $f = g$.

(16) Let A, B be non-empty algebras over S, and let C be a strict non-empty subalgebra of A, and let h_1 be a many sorted function from B into C. Suppose h_1 is a homomorphism of B into C. Let h_2 be a many sorted function from B into A. If $h_1 = h_2$, then h_2 is a homomorphism of B into A.

(17) Let A, B be non-empty algebras over S and let F be a many sorted function from A into B. If F is a monomorphism of A into B, then A and $\mathrm{Im}\, F$ are isomorphic.

(18) Let A, B be non-empty algebras over S and let F be a many sorted function from A into B. Suppose F is onto. Let o be an operation symbol of S and let x be an element of $\mathrm{Args}(o, B)$. Then there exists an element y of $\mathrm{Args}(o, A)$ such that $F\#y = x$.

(19) Let A be a non-empty algebra over S, and let o be an operation symbol of S, and let x be an element of $\mathrm{Args}(o, A)$. Then $(\mathrm{Den}(o, A))(x) \in$ (the sorts of A)(the result sort of o).

(20) Let A, B, C be non-empty algebras over S, and let F_1 be a many sorted function from A into B, and let F_2 be a many sorted function from A into C. Suppose F_1 is an epimorphism of A onto B and F_2 is a homomorphism of A into C. Let G be a many sorted function from B into C. If $G \circ F_1 = F_2$, then G is a homomorphism of B into C.

In the sequel A, M will be many sorted sets indexed by I and B, C will be non-empty many sorted sets indexed by I.

Let I be a set, let A be a many sorted set indexed by I, let B, C be non-empty many sorted sets indexed by I, and let F be a many sorted function from A into $[\![B, C]\!]$. The functor $\mathrm{Mpr1}(F)$ yields a many sorted function from A into B and is defined as follows:

(Def. 1) For every set i such that $i \in I$ holds $(\mathrm{Mpr1}(F))(i) = \mathrm{pr1}(F(i))$.

The functor $\mathrm{Mpr2}(F)$ yielding a many sorted function from A into C is defined by:

(Def. 2) For every set i such that $i \in I$ holds $(\mathrm{Mpr2}(F))(i) = \mathrm{pr2}(F(i))$.

One can prove the following four propositions:

(21) For every many sorted function F from A into $[\![I \longmapsto \{a\}, I \longmapsto \{a\}]\!]$ holds $\mathrm{Mpr1}(F) = \mathrm{Mpr2}(F)$.

(22) For every many sorted function F from A into $[\![B, C]\!]$ such that F is onto holds $\mathrm{Mpr1}(F)$ is onto.

(23) For every many sorted function F from A into $[\![B, C]\!]$ such that F is onto holds $\mathrm{Mpr2}(F)$ is onto.

(24) Let F be a many sorted function from A into $[\![B, C]\!]$. If $M \in \mathrm{dom}_\kappa F(\kappa)$, then for every set i such that $i \in I$ holds $(F \leftarrow\!\!\!\!\!\!\!\curvearrowright M)(i) = \langle\!\langle ((\mathrm{Mpr1}(F)) \leftarrow\!\!\!\!\!\!\!\curvearrowright M)(i), ((\mathrm{Mpr2}(F)) \leftarrow\!\!\!\!\!\!\!\curvearrowright M)(i) \rangle\!\rangle$.

2. On the Trivial Many Sorted Algebras

Let S be a non empty many sorted signature. Note that the sorts of the trivial algebra of S is locally-finite and non-empty.

Let S be a non empty many sorted signature. Note that the trivial algebra of S is locally-finite and non-empty.

We now state three propositions:

(25) Let A be a non-empty algebra over S, and let F be a many sorted function from A into the trivial algebra of S, and let o be an operation symbol of S, and let x be an element of $\mathrm{Args}(o, A)$. Then F(the result sort of o)$((\mathrm{Den}(o, A))(x)) = 0$ and $(\mathrm{Den}(o,$ the trivial algebra of $S))(F\#x) = 0$.

(26) For every non-empty algebra A over S holds every many sorted function from A into the trivial algebra of S is an epimorphism of A onto the trivial algebra of S.

(27) Let A be an algebra over S. Given a many sorted set X indexed by the carrier of S such that the sorts of $A = \{X\}$. Then A and the trivial algebra of S are isomorphic.

3. On the Many Sorted Congruences

One can prove the following propositions:

(28) For every non-empty algebra A over S holds every congruence of A is a many sorted subset of $[\![$the sorts of A, the sorts of A $]\!]$.

(29) Let A be a non-empty algebra over S, and let R be a subset of the carrier of $\mathrm{CongrLatt}(A)$, and let F be a family of many sorted subsets of $[\![$the sorts of A, the sorts of A $]\!]$. If $R = F$, then $\bigcap |{:}F{:}|$ is a congruence of A.

(30) Let A be a non-empty algebra over S and let C be a congruence of A. Suppose $C = [\![$the sorts of A, the sorts of A $]\!]$. Then the sorts of $\mathrm{QuotMSAlg}(C) = \{$the sorts of A $\}$.

(31) Let A, B be non-empty algebras over S and let F be a many sorted function from A into B. If F is a homomorphism of A into B, then $\mathrm{MSHomQuot}(F) \circ \mathrm{MSNatHom}(A, \mathrm{Congruence}(F)) = F$.

(32) Let A be a non-empty algebra over S, and let C be a congruence of A, and let s be a sort symbol of S, and let a be an element of (the sorts of QuotMSAlg(C))(s). Then there exists an element x of (the sorts of A)(s) such that $a = [x]_C$.

(33) Let A be an algebra over S and let C_1, C_2 be equivalence many sorted relations of A. Suppose $C_1 \subseteq C_2$. Let i be an element of the carrier of S and let x, y be elements of (the sorts of A)(i). If $\langle x, y \rangle \in C_1(i)$, then $[x]_{(C_1)} \subseteq [y]_{(C_2)}$ and if A is non-empty, then $[y]_{(C_1)} \subseteq [x]_{(C_2)}$.

(34) Let A be a non-empty algebra over S, and let C be a congruence of A, and let s be a sort symbol of S, and let x, y be elements of (the sorts of A)(s). Then $(\mathrm{MSNatHom}(A, C))(s)(x) = (\mathrm{MSNatHom}(A, C))(s)(y)$ if and only if $\langle x, y \rangle \in C(s)$.

(35) Let A be a non-empty algebra over S, and let C_1, C_2 be congruences of A, and let G be a many sorted function from QuotMSAlg(C_1) into QuotMSAlg(C_2). Suppose that for every element i of the carrier of S and for every element x of (the sorts of QuotMSAlg(C_1))(i) and for every element x_1 of (the sorts of A)(i) such that $x = [x_1]_{(C_1)}$ holds $G(i)(x) = [x_1]_{(C_2)}$. Then $G \circ \mathrm{MSNatHom}(A, C_1) = \mathrm{MSNatHom}(A, C_2)$.

(36) Let A be a non-empty algebra over S, and let C_1, C_2 be congruences of A, and let G be a many sorted function from QuotMSAlg(C_1) into QuotMSAlg(C_2). Suppose that for every element i of the carrier of S and for every element x of (the sorts of QuotMSAlg(C_1))(i) and for every element x_1 of (the sorts of A)(i) such that $x = [x_1]_{(C_1)}$ holds $G(i)(x) = [x_1]_{(C_2)}$. Then G is an epimorphism of QuotMSAlg(C_1) onto QuotMSAlg(C_2).

(37) Let A, B be non-empty algebras over S and let F be a many sorted function from A into B. Suppose F is a homomorphism of A into B. Let C_1 be a congruence of A. Suppose $C_1 \subseteq \mathrm{Congruence}(F)$. Then there exists a many sorted function H from QuotMSAlg(C_1) into B such that H is a homomorphism of QuotMSAlg(C_1) into B and $F = H \circ \mathrm{MSNatHom}(A, C_1)$.

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