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Introduction to the Homotopy Theory

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Summary. The paper introduces some preliminary notions concerning the homotopy theory according to [15]: paths and arcwise connected to topological spaces. The basic operations on paths (addition and reversing) are defined. In the last section the predicate: P, Q *are homotopic* is defined. We also showed some properties of the product of two topological spaces needed to prove reflexivity and symmetry of the above predicate.

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The articles [27], [30], [26], [16], [10], [32], [7], [23], [13], [12], [25], [28], [24], [4], [1], [33], [11], [21], [31], [9], [19], [29], [17], [8], [34], [14], [6], [5], [22], [20], [2], [18], and [3] provide the notation and terminology for this paper.

1. Preliminaries

In this paper T, T_1, T_2, S denote non empty topological spaces.

The scheme *FrCard* deals with a non empty set $\mathcal{A}$, a set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$$\overline{\overline{\{\mathcal{F}(w); w \text{ ranges over elements of } \mathcal{A} : w \in \mathcal{B} \wedge \mathcal{P}[w]\}}} \leqslant \overline{\overline{\mathcal{B}}}$$

for all values of the parameters.

The following proposition is true

(1) Let f be a map from T_1 into S and g be a map from T_2 into S. Suppose that

(i) T_1 is a subspace of T,

(ii) T_2 is a subspace of T,

(iii) $\Omega_{(T_1)} \cup \Omega_{(T_2)} = \Omega_T$,

(iv) T_1 is compact,

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(v) T_2 is compact,
(vi) T is a T_2 space,
(vii) f is continuous,
(viii) g is continuous, and
(ix) for every set p such that $p \in \Omega_{(T_1)} \cap \Omega_{(T_2)}$ holds $f(p) = g(p)$.

Then there exists a map h from T into S such that $h = f{+\cdot}g$ and h is continuous.

Let S, T be non empty topological spaces. One can verify that there exists a map from S into T which is continuous.

One can prove the following proposition

(2) For all non empty topological spaces S, T holds every continuous mapping from S into T is a continuous map from S into T.

Let T be a non empty topological structure. Note that id_T is open and continuous.

Let T be a non empty topological structure. Observe that there exists a map from T into T which is continuous and one-to-one.

We now state the proposition

(3) Let S, T be non empty topological spaces and f be a map from S into T. If f is a homeomorphism, then f^{-1} is open.

2. Paths and arcwise connected spaces

Let T be a topological structure and let a, b be points of T. Let us assume that there exists a map f from $\mathbb{I}$ into T such that f is continuous and $f(0) = a$ and $f(1) = b$. A map from $\mathbb{I}$ into T is said to be a path from a to b if:

(Def. 1) It is continuous and $\mathrm{it}(0) = a$ and $\mathrm{it}(1) = b$.

Next we state the proposition

(4) Let T be a non empty topological space and a be a point of T. Then there exists a map f from $\mathbb{I}$ into T such that f is continuous and $f(0) = a$ and $f(1) = a$.

Let T be a non empty topological space and let a be a point of T. Note that there exists a path from a to a which is continuous.

Let T be a topological structure. We say that T is arcwise connected if and only if:

(Def. 2) For all points a, b of T there exists a map f from $\mathbb{I}$ into T such that f is continuous and $f(0) = a$ and $f(1) = b$.

Let us observe that there exists a topological space which is arcwise connected and non empty.

Let T be an arcwise connected topological structure and let a, b be points of T. Let us note that the path from a to b can be characterized by the following (equivalent) condition:

(Def. 3) It is continuous and it$(0) = a$ and it$(1) = b$.

Let T be an arcwise connected topological structure and let a, b be points of T. Note that every path from a to b is continuous.

Next we state the proposition

(5) For every non empty topological space G_1 such that G_1 is arcwise connected holds G_1 is connected.

Let us mention that every non empty topological space which is arcwise connected is also connected.

3. Basic operations on paths

Let T be a non empty topological space, let a, b, c be points of T, let P be a path from a to b, and let Q be a path from b to c. Let us assume that there exist maps f, g from $\mathbb{I}$ into T such that f is continuous and $f(0) = a$ and $f(1) = b$ and g is continuous and $g(0) = b$ and $g(1) = c$. The functor $P + Q$ yielding a path from a to c is defined by the condition (Def. 4).

(Def. 4) Let t be a point of $\mathbb{I}$ and t' be a real number such that $t = t'$. Then
  - (i) if $0 \leqslant t'$ and $t' \leqslant \frac{1}{2}$, then $(P + Q)(t) = P(2 \cdot t')$, and
  - (ii) if $\frac{1}{2} \leqslant t'$ and $t' \leqslant 1$, then $(P + Q)(t) = Q(2 \cdot t' - 1)$.

Let T be a non empty topological space and let a be a point of T. Note that there exists a path from a to a which is constant.

One can prove the following two propositions:

(6) Let T be a non empty topological space, a be a point of T, and P be a constant path from a to a. Then $P = \mathbb{I} \longmapsto a$.

(7) Let T be a non empty topological space, a be a point of T, and P be a constant path from a to a. Then $P + P = P$.

Let T be a non empty topological space, let a be a point of T, and let P be a constant path from a to a. Observe that $P + P$ is constant.

Let T be a non empty topological space, let a, b be points of T, and let P be a path from a to b. Let us assume that there exists a map f from $\mathbb{I}$ into T such that f is continuous and $f(0) = a$ and $f(1) = b$. The functor $-P$ yields a path from b to a and is defined as follows:

(Def. 5) For every point t of $\mathbb{I}$ and for every real number t' such that $t = t'$ holds $(-P)(t) = P(1 - t')$.

The following proposition is true

(8) Let T be a non empty topological space, a be a point of T, and P be a constant path from a to a. Then $-P = P$.

Let T be a non empty topological space, let a be a point of T, and let P be a constant path from a to a. One can verify that $-P$ is constant.

4. The product of two topological spaces

One can prove the following proposition

(9) Let X, Y be non empty topological spaces, A be a family of subsets of Y, and f be a map from X into Y. Then $f^{-1}(\bigcup A) = \bigcup(f^{-1}(A))$.

Let S_1, S_2, T_1, T_2 be non empty topological spaces, let f be a map from S_1 into S_2, and let g be a map from T_1 into T_2. Then $[\!:f, g:\!]$ is a map from $[\!:S_1, T_1:\!]$ into $[\!:S_2, T_2:\!]$.

Next we state three propositions:

(10) Let S_1, S_2, T_1, T_2 be non empty topological spaces, f be a continuous map from S_1 into T_1, g be a continuous map from S_2 into T_2, and P_1, P_2 be subsets of the carrier of $[\!:T_1, T_2:\!]$. If $P_2 \in \mathrm{BaseAppr}(P_1)$, then $[\!:f, g:\!]^{-1}(P_2)$ is open.

(11) Let S_1, S_2, T_1, T_2 be non empty topological spaces, f be a continuous map from S_1 into T_1, g be a continuous map from S_2 into T_2, and P_2 be a subset of the carrier of $[\!:T_1, T_2:\!]$. If P_2 is open, then $[\!:f, g:\!]^{-1}(P_2)$ is open.

(12) Let S_1, S_2, T_1, T_2 be non empty topological spaces, f be a continuous map from S_1 into T_1, and g be a continuous map from S_2 into T_2. Then $[\!:f, g:\!]$ is continuous.

Let us note that every topological structure which is empty is also T_0.

Let T_1, T_2 be discernible non empty topological spaces. One can check that $[\!:T_1, T_2:\!]$ is discernible.

We now state two propositions:

(13) For all T_0-spaces T_1, T_2 holds $[\!:T_1, T_2:\!]$ is a T_0-space.

(14) Let T_1, T_2 be non empty topological spaces. Suppose T_1 is a T_1 space and T_2 is a T_1 space. Then $[\!:T_1, T_2:\!]$ is a T_1 space.

Let T_1, T_2 be a T_1 space non empty topological spaces. Observe that $[\!:T_1, T_2:\!]$ is a T_1 space.

Let T_1, T_2 be T_2 non empty topological spaces. Observe that $[\!:T_1, T_2:\!]$ is T_2.

Let us note that $\mathbb{I}$ is compact and T_2.

Let us mention that $\mathcal{E}^2_{\mathrm{T}}$ is T_2.

Let T be a non empty arcwise connected topological space, let a, b be points of T, and let P, Q be paths from a to b. We say that P, Q are homotopic if and only if the condition (Def. 6) is satisfied.

(Def. 6) There exists a map f from $[:\mathbb{I}, \mathbb{I}:]$ into T such that
 (i) f is continuous, and
 (ii) for every point s of $\mathbb{I}$ holds $f(s, 0) = P(s)$ and $f(s, 1) = Q(s)$ and for every point t of $\mathbb{I}$ holds $f(0, t) = a$ and $f(1, t) = b$.

Let us notice that the predicate P, Q are homotopic is reflexive and symmetric.

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Some Properties of Real Maps

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Summary. The main goal of the paper is to show logical equivalence of the two definitions of the *open subset*: one from [2] and the other from [23]. This has been used to show that the other two definitions are equivalent: the continuity of the map as in [20] and in [22]. We used this to show that continuous and one-to-one maps are monotone (see theorems 16 and 17 for details).

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The terminology and notation used here are introduced in the following articles: [26], [13], [27], [28], [4], [5], [24], [22], [17], [18], [10], [3], [23], [6], [25], [29], [16], [14], [19], [11], [20], [8], [7], [9], [15], [21], [2], [1], and [12].

1. Preliminaries

One can prove the following four propositions:

(1) For all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ and for every subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that P is an arc from p to q holds P is compact.

(2) For every real number r holds $0 \leqslant r$ and $r \leqslant 1$ iff $r \in$ the carrier of $\mathbb{I}$.

(3) For all points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ and for all real numbers r_1, r_2 such that $(1 - r_1) \cdot p_1 + r_1 \cdot p_2 = (1 - r_2) \cdot p_1 + r_2 \cdot p_2$ holds $r_1 = r_2$ or $p_1 = p_2$.

(4) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_1 \neq p_2$. Then there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction \mathcal{L}(p_1, p_2)$ such that for every real number x such that $x \in [0, 1]$ holds $f(x) = (1 - x) \cdot p_1 + x \cdot p_2$ and f is a homeomorphism and $f(0) = p_1$ and $f(1) = p_2$.

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One can verify that $\mathcal{E}^2_{\mathrm{T}}$ is arcwise connected.

One can check that there exists a subspace of $\mathcal{E}^2_{\mathrm{T}}$ which is compact and non empty.

The following proposition is true

(5) Let a, b be points of $\mathcal{E}^2_{\mathrm{T}}$, f be a path from a to b, P be a non empty compact subspace of $\mathcal{E}^2_{\mathrm{T}}$, and g be a map from $\mathbb{I}$ into P. If f is one-to-one and $g = f$ and $\Omega_P = \operatorname{rng} f$, then g is a homeomorphism.

2. Equivalence of analytical and topological definitions of continuity

We now state a number of propositions:

(6) Let X be a subset of $\mathbb{R}$. Then $X \in$ the open set family of the metric space of real numbers if and only if X is open.

(7) Let f be a map from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$, x be a point of $\mathbb{R}^{\mathbf{1}}$, g be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and x_1 be a real number. If f is continuous at x and $f = g$ and $x = x_1$, then g is continuous in x_1.

(8) Let f be a continuous map from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$ and g be a partial function from $\mathbb{R}$ to $\mathbb{R}$. If $f = g$, then g is continuous on $\mathbb{R}$.

(9) Let f be a continuous one-to-one map from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$. Then

(i) for all points x, y of $\mathbb{I}$ and for all real numbers p, q, f_1, f_2 such that $x = p$ and $y = q$ and $p < q$ and $f_1 = f(x)$ and $f_2 = f(y)$ holds $f_1 < f_2$, or

(ii) for all points x, y of $\mathbb{I}$ and for all real numbers p, q, f_1, f_2 such that $x = p$ and $y = q$ and $p < q$ and $f_1 = f(x)$ and $f_2 = f(y)$ holds $f_1 > f_2$.

(10) Let r, g_1, a, b be real numbers and x be an element of the carrier of $[a,\, b]_{\mathrm{M}}$. If $a \leqslant b$ and $x = r$ and $g_1 > 0$ and $]r - g_1, r + g_1[\; \subseteq [a, b]$, then $]r - g_1, r + g_1[\; = \mathrm{Ball}(x, g_1)$.

(11) Let a, b be real numbers and X be a subset of $\mathbb{R}$. Suppose $a < b$ and $a \notin X$ and $b \notin X$. If $X \in$ the open set family of $[a,\, b]_{\mathrm{M}}$, then X is open.

(12) For every open subset X of $\mathbb{R}$ and for all real numbers a, b such that $X \subseteq [a, b]$ holds $a \notin X$ and $b \notin X$.

(13) Let a, b be real numbers, X be a subset of $\mathbb{R}$, and V be a subset of the carrier of $[a,\, b]_{\mathrm{M}}$. Suppose $a \leqslant b$ and $V = X$. If X is open, then $V \in$ the open set family of $[a,\, b]_{\mathrm{M}}$.

(14) Let a, b, c, d, x_1 be real numbers, f be a map from $[a,\, b]_{\mathrm{T}}$ into $[c,\, d]_{\mathrm{T}}$, x be a point of $[a,\, b]_{\mathrm{T}}$, and g be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a < b$ and $c < d$ and f is continuous at x and $f(a) = c$ and $f(b) = d$ and f is one-to-one and $f = g$ and $x = x_1$. Then $g \restriction [a, b]$ is continuous in x_1.

(15) Let a, b, c, d be real numbers, f be a map from $[a, b]_{\rm T}$ into $[c, d]_{\rm T}$, and g be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is continuous and one-to-one and $a < b$ and $c < d$ and $f = g$ and $f(a) = c$ and $f(b) = d$. Then g is continuous on $[a, b]$.

3. On the monotonicity of continuous maps

One can prove the following propositions:

(16) Let a, b, c, d be real numbers and f be a map from $[a, b]_{\rm T}$ into $[c, d]_{\rm T}$. Suppose $a < b$ and $c < d$ and f is continuous and one-to-one and $f(a) = c$ and $f(b) = d$. Let x, y be points of $[a, b]_{\rm T}$ and p, q, f_1, f_2 be real numbers. If $x = p$ and $y = q$ and $p < q$ and $f_1 = f(x)$ and $f_2 = f(y)$, then $f_1 < f_2$.

(17) Let f be a continuous one-to-one map from $\mathbb{I}$ into $\mathbb{I}$. Suppose $f(0) = 0$ and $f(1) = 1$. Let x, y be points of $\mathbb{I}$ and p, q, f_1, f_2 be real numbers. If $x = p$ and $y = q$ and $p < q$ and $f_1 = f(x)$ and $f_2 = f(y)$, then $f_1 < f_2$.

(18) Let a, b, c, d be real numbers, f be a map from $[a, b]_{\rm T}$ into $[c, d]_{\rm T}$, P be a non empty subset of $[a, b]_{\rm T}$, and P_1, Q_1 be subsets of $\mathbb{R}^{\mathbf{1}}$. Suppose $a < b$ and $c < d$ and $P_1 = P$ and f is continuous and one-to-one and P_1 is compact and $f(a) = c$ and $f(b) = d$ and $f^\circ P = Q_1$. Then $f(\inf(\Omega_{(P_1)})) = \inf(\Omega_{(Q_1)})$.

(19) Let a, b, c, d be real numbers, f be a map from $[a, b]_{\rm T}$ into $[c, d]_{\rm T}$, P, Q be non empty subsets of $[a, b]_{\rm T}$, and P_1, Q_1 be subsets of $\mathbb{R}^{\mathbf{1}}$. Suppose $a < b$ and $c < d$ and $P_1 = P$ and $Q_1 = Q$ and f is continuous and one-to-one and P_1 is compact and $f(a) = c$ and $f(b) = d$ and $f^\circ P = Q$. Then $f(\sup(\Omega_{(P_1)})) = \sup(\Omega_{(Q_1)})$.

(20) For all real numbers a, b such that $a \leqslant b$ holds $\inf[a, b] = a$ and $\sup[a, b] = b$.

(21) Let a, b, c, d, e, f, g, h be real numbers and F be a map from $[a, b]_{\rm T}$ into $[c, d]_{\rm T}$. Suppose $a < b$ and $c < d$ and $e < f$ and $a \leqslant e$ and $f \leqslant b$ and F is a homeomorphism and $F(a) = c$ and $F(b) = d$ and $g = F(e)$ and $h = F(f)$. Then $F^\circ[e, f] = [g, h]$.

(22) Let P, Q be subsets of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose P meets Q and $P \cap Q$ is closed and P is an arc from p_1 to p_2. Then there exists a point E_1 of $\mathcal{E}^2_{\rm T}$ such that

(i) $E_1 \in P \cap Q$, and

(ii) there exists a map g from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T})\restriction P$ and there exists a real number s_2 such that g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_2) = E_1$ and $0 \leqslant s_2$ and $s_2 \leqslant 1$ and for every real number t such that $0 \leqslant t$ and $t < s_2$ holds $g(t) \notin Q$.

(23) Let P, Q be subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P meets Q and $P \cap Q$ is closed and P is an arc from p_1 to p_2. Then there exists a point E_1 of $\mathcal{E}^2_{\mathrm{T}}$ such that

(i) $E_1 \in P \cap Q$, and

(ii) there exists a map g from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction P$ and there exists a real number s_2 such that g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_2) = E_1$ and $0 \leqslant s_2$ and $s_2 \leqslant 1$ and for every real number t such that $1 \geqslant t$ and $t > s_2$ holds $g(t) \notin Q$.

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The Ordering of Points on a Curve. Part I

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Summary. Some auxiliary theorems needed to formalize the proof of the Jordan Curve Theorem according to [25] are proved.

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The articles [26], [29], [13], [1], [22], [24], [31], [2], [4], [5], [11], [28], [20], [12], [16], [23], [9], [8], [27], [10], [30], [15], [17], [18], [14], [19], [21], [6], [7], and [3] provide the terminology and notation for this paper.

1. Preliminaries

The following propositions are true:

(1) For every natural number i_1 such that $1 \leqslant i_1$ holds $i_1 -' 1 < i_1$.

(2) For all natural numbers i, k such that $i + 1 \leqslant k$ holds $1 \leqslant k -' i$.

(3) For all natural numbers i, k such that $1 \leqslant i$ and $1 \leqslant k$ holds $k -' i + 1 \leqslant k$.

(4) For every real number r such that $r \in$ the carrier of $\mathbb{I}$ holds $1 - r \in$ the carrier of $\mathbb{I}$.

(5) For all points p, q, p_1 of $\mathcal{E}^2_{\rm T}$ such that $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $p_1 \in \mathcal{L}(p, q)$ holds if $(p_1)_{\mathbf{2}} = p_{\mathbf{2}}$, then $(p_1)_{\mathbf{1}} = p_{\mathbf{1}}$.

(6) For all points p, q, p_1 of $\mathcal{E}^2_{\rm T}$ such that $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_1 \in \mathcal{L}(p, q)$ holds if $(p_1)_{\mathbf{1}} = p_{\mathbf{1}}$, then $(p_1)_{\mathbf{2}} = p_{\mathbf{2}}$.

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(7) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, F be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction P$, and i be a natural number. Suppose $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and f is a special sequence and $P = \widetilde{\mathcal{L}}(f)$ and F is a homeomorphism and $F(0) = \pi_1 f$ and $F(1) = \pi_{\operatorname{len} f} f$. Then there exist real numbers p_1, p_2 such that $p_1 < p_2$ and $0 \leqslant p_1$ and $p_1 \leqslant 1$ and $0 \leqslant p_2$ and $p_2 \leqslant 1$ and $\mathcal{L}(f, i) = F^\circ[p_1, p_2]$ and $F(p_1) = \pi_i f$ and $F(p_2) = \pi_{i+1} f$.

(8) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, Q, R be non empty subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, F be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction Q$, i be a natural number, and P be a non empty subset of $\mathbb{I}$. Suppose that

(i) f is a special sequence,

(ii) F is a homeomorphism,

(iii) $F(0) = \pi_1 f$,

(iv) $F(1) = \pi_{\operatorname{len} f} f$,

(v) $1 \leqslant i$,

(vi) $i + 1 \leqslant \operatorname{len} f$,

(vii) $F^\circ P = \mathcal{L}(f, i)$,

(viii) $Q = \widetilde{\mathcal{L}}(f)$, and

(ix) $R = \mathcal{L}(f, i)$.

Then there exists a map G from $\mathbb{I}\restriction P$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction R$ such that $G = F\restriction P$ and G is a homeomorphism.

2. Some properties of real intervals

One can prove the following propositions:

(9) For all points p_1, p_2, p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_1 \neq p_2$ and $p \in \mathcal{L}(p_1, p_2)$ holds $\mathrm{LE}(p, p, p_1, p_2)$.

(10) For all points p, p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_1 \neq p_2$ and $p \in \mathcal{L}(p_1, p_2)$ holds $\mathrm{LE}(p_1, p, p_1, p_2)$.

(11) For all points p, p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathcal{L}(p_1, p_2)$ and $p_1 \neq p_2$ holds $\mathrm{LE}(p, p_2, p_1, p_2)$.

(12) For all points p_1, p_2, q_1, q_2, q_3 of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_1 \neq p_2$ and $\mathrm{LE}(q_1, q_2, p_1, p_2)$ and $\mathrm{LE}(q_2, q_3, p_1, p_2)$ holds $\mathrm{LE}(q_1, q_3, p_1, p_2)$.

(13) For all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \neq q$ holds $\mathcal{L}(p, q) = \{p_1; p_1$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $\mathrm{LE}(p, p_1, p, q) \ \wedge\ \mathrm{LE}(p_1, q, p, q)\}$.

(14) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2, then P is an arc from p_2 to p_1.

(15) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, P be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, and i be a natural number. Suppose f is a special sequence and

$1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and $P = \mathcal{L}(f, i)$. Then P is an arc from $\pi_i f$ to $\pi_{i+1} f$.

3. Cutting off sequences

One can prove the following propositions:

(16) Let g_1 be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and i be a natural number. Suppose $1 \leqslant i$ and $i \leqslant \operatorname{len} g_1$ and g_1 is a special sequence. If $\pi_1 g_1 \in \widetilde{\mathcal{L}}(\operatorname{mid}(g_1, i, \operatorname{len} g_1))$, then $i = 1$.

(17) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If f is a special sequence and $p = f(\operatorname{len} f)$, then $\downharpoonright p, f = \langle p, p \rangle$.

(18) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and k be a natural number. If $1 \leqslant k$ and $k \leqslant \operatorname{len} f$, then $\operatorname{mid}(f, k, k) = \langle \pi_k f \rangle$.

(19) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If f is a special sequence and $p = f(1)$, then $\downharpoonleft f, p = \langle p \rangle$.

(20) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$, then $\widetilde{\mathcal{L}}(\downharpoonleft f, p) \subseteq \widetilde{\mathcal{L}}(f)$.

(21) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\operatorname{len} f)$ and f is a special sequence, then $\operatorname{Index}(p, \downharpoonright p, f) = 1$.

(22) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence, then $p \in \widetilde{\mathcal{L}}(\downharpoonright p, f)$.

(23) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence and $p \neq f(1)$, then $p \in \widetilde{\mathcal{L}}(\downharpoonleft f, p)$.

(24) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\operatorname{len} f)$ and f is a special sequence, then $\downharpoonright \downharpoonleft p, f, p = \langle p \rangle$.

(25) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p = f(\operatorname{len} f)$ and f is a special sequence, then $p \in \widetilde{\mathcal{L}}(\downharpoonright q, f)$.

(26) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence, then $p \in \widetilde{\mathcal{L}}(\downharpoonright q, f)$ or $q \in \widetilde{\mathcal{L}}(\downharpoonright p, f)$.

(27) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\operatorname{len} f)$ or $q \neq f(\operatorname{len} f)$ and f is a special sequence. Then $\widetilde{\mathcal{L}}(\downharpoonright \downharpoonleft p, f, q) \subseteq \widetilde{\mathcal{L}}(f)$.

(28) Let f be a non constant standard special circular sequence and i, j be natural numbers. Suppose $1 \leqslant i$ and $j \leqslant \operatorname{len}$ the Go-board of f and $i < j$. Then $\mathcal{L}((\text{the Go-board of } f)_{1, \text{width the Go-board of } f}, (\text{the Go-board of } f)_{i, \text{width the Go-board of } f}) \cap \mathcal{L}((\text{the Go-board of } f)_{j, \text{width the Go-board of } f}, (\text{the Go-board of } f)_{\text{len the Go-board of } f,\ \text{width the Go-board of } f}) = \emptyset$.

(29) Let f be a non constant standard special circular sequence and i, j be natural numbers. Suppose $1 \leqslant i$ and $j \leqslant$ width the Go-board of f and $i < j$. Then $\mathcal{L}((\text{the Go-board of } f)_{\text{len the Go-board of } f,\ 1}, (\text{the Go-board of } f)_{\text{len the Go-board of } f,\ i}) \cap \mathcal{L}((\text{the Go-board of } f)_{\text{len the Go-board of } f,\ j}, (\text{the Go-board of } f)_{\text{len the Go-board of } f,\ \text{width the Go-board of } f}) = \emptyset$.

(30) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If f is a special sequence, then $\downharpoonleft \pi_1 f, f = f$.

(31) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If f is a special sequence, then $\downharpoonright f, \pi_{\text{len } f} f = f$.

(32) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence and $p \neq f(\text{len } f)$, then $p \in \mathcal{L}(\pi_{\text{Index}(p,f)} f, \pi_{\text{Index}(p,f)+1} f)$.

(33) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, p be a point of $\mathcal{E}^2_{\rm T}$, and i be a natural number. If f is a special sequence, then if $\pi_1 f \in \mathcal{L}(f, i)$, then $i = 1$.

(34) Let f be a non constant standard special circular sequence, j be a natural number, and P be a subset of the carrier of $\mathcal{E}^2_{\rm T}$. Suppose $1 \leqslant j$ and $j \leqslant$ width the Go-board of f and $P = \mathcal{L}((\text{the Go-board of } f)_{1,j}, (\text{the Go-board of } f)_{\text{len the Go-board of } f,\ j})$. Then P is a special polygonal arc joining $(\text{the Go-board of } f)_{1,j}$ and $(\text{the Go-board of } f)_{\text{len the Go-board of } f,\ j}$.

(35) Let f be a non constant standard special circular sequence, j be a natural number, and P be a subset of the carrier of $\mathcal{E}^2_{\rm T}$. Suppose $1 \leqslant j$ and $j \leqslant$ len the Go-board of f and $P = \mathcal{L}((\text{the Go-board of } f)_{j,1}, (\text{the Go-board of } f)_{j,\text{width the Go-board of } f})$. Then P is a special polygonal arc joining $(\text{the Go-board of } f)_{j,1}$ and $(\text{the Go-board of } f)_{j,\text{width the Go-board of } f}$.

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The Ordering of Points on a Curve. Part II

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Summary. The proof of the Jordan Curve Theorem according to [14] is continued. The notions of the first and last point of a oriented arc are introduced as well as ordering of points on a curve in $\mathcal{E}^2_T$.

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The papers [15], [18], [10], [1], [13], [20], [2], [3], [4], [8], [17], [11], [9], [12], [6], [5], [16], [7], and [19] provide the terminology and notation for this paper.

1. First and last point of a curve

One can prove the following proposition

(1) Let P, Q be subsets of the carrier of $\mathcal{E}^2_{\rm T}$, p_1, p_2, q_1 be points of $\mathcal{E}^2_{\rm T}$, f be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T})\restriction P$, and s_1 be a real number. Suppose that

(i) P is an arc from p_1 to p_2,
(ii) $q_1 \in P$,
(iii) $q_1 \in Q$,
(iv) $f(s_1) = q_1$,
(v) f is a homeomorphism,
(vi) $f(0) = p_1$,
(vii) $f(1) = p_2$,
(viii) $0 \leqslant s_1$,
(ix) $s_1 \leqslant 1$, and

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(x) for every real number t such that $0 \leqslant t$ and $t < s_1$ holds $f(t) \notin Q$.

Let g be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction P$ and s_2 be a real number. Suppose g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_2) = q_1$ and $0 \leqslant s_2$ and $s_2 \leqslant 1$. Let t be a real number. If $0 \leqslant t$ and $t < s_2$, then $g(t) \notin Q$.

Let P, Q be subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Let us assume that P meets Q and $P \cap Q$ is closed and P is an arc from p_1 to p_2. The functor $\mathrm{FPoint}(P, p_1, p_2, Q)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by the conditions (Def. 1).

(Def. 1)(i) $\mathrm{FPoint}(P, p_1, p_2, Q) \in P \cap Q$, and

(ii) for every map g from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction P$ and for every real number s_2 such that g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_2) = \mathrm{FPoint}(P, p_1, p_2, Q)$ and $0 \leqslant s_2$ and $s_2 \leqslant 1$ and for every real number t such that $0 \leqslant t$ and $t < s_2$ holds $g(t) \notin Q$.

One can prove the following three propositions:

(2) Let P, Q be subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p, p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in P$ and P is an arc from p_1 to p_2 and $Q = \{p\}$, then $\mathrm{FPoint}(P, p_1, p_2, Q) = p$.

(3) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, Q be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 \in Q$ and $P \cap Q$ is closed and P is an arc from p_1 to p_2, then $\mathrm{FPoint}(P, p_1, p_2, Q) = p_1$.

(4) Let P, Q be subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, p_1, p_2, q_1 be points of $\mathcal{E}^2_{\mathrm{T}}$, f be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction P$, and s_1 be a real number. Suppose that

(i) P is an arc from p_1 to p_2,

(ii) $q_1 \in P$,

(iii) $q_1 \in Q$,

(iv) $f(s_1) = q_1$,

(v) f is a homeomorphism,

(vi) $f(0) = p_1$,

(vii) $f(1) = p_2$,

(viii) $0 \leqslant s_1$,

(ix) $s_1 \leqslant 1$, and

(x) for every real number t such that $1 \geqslant t$ and $t > s_1$ holds $f(t) \notin Q$.

Let g be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction P$ and s_2 be a real number. Suppose g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_2) = q_1$ and $0 \leqslant s_2$ and $s_2 \leqslant 1$. Let t be a real number. If $1 \geqslant t$ and $t > s_2$, then $g(t) \notin Q$.

Let P, Q be subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Let us assume that P meets Q and $P \cap Q$ is closed and P is an arc from p_1 to p_2. The functor $\mathrm{LPoint}(P, p_1, p_2, Q)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by the conditions (Def. 2).

(Def. 2)(i) $\mathrm{LPoint}(P, p_1, p_2, Q) \in P \cap Q$, and

(ii) for every map g from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction P$ and for every real number s_2 such that g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_2) =$

LPoint(P, p_1, p_2, Q) and $0 \leqslant s_2$ and $s_2 \leqslant 1$ and for every real number t such that $1 \geqslant t$ and $t > s_2$ holds $g(t) \notin Q$.

One can prove the following propositions:

(5) Let P, Q be subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p, p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in P$ and P is an arc from p_1 to p_2 and $Q = \{p\}$, then LPoint$(P, p_1, p_2, Q) = p$.

(6) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, Q be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_2 \in Q$ and $P \cap Q$ is closed and P is an arc from p_1 to p_2, then LPoint$(P, p_1, p_2, Q) = p_2$.

(7) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, Q be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P \subseteq Q$ and P is closed and an arc from p_1 to p_2. Then FPoint$(P, p_1, p_2, Q) = p_1$ and LPoint$(P, p_1, p_2, Q) = p_2$.

2. The ordering of points on a curve

Let P be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and let p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. We say that LE q_1, q_2, P, p_1, p_2 if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) $q_1 \in P$,

(ii) $q_2 \in P$, and

(iii) for every map g from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction P$ and for all real numbers s_1, s_2 such that g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_1) = q_1$ and $0 \leqslant s_1$ and $s_1 \leqslant 1$ and $g(s_2) = q_2$ and $0 \leqslant s_2$ and $s_2 \leqslant 1$ holds $s_1 \leqslant s_2$.

The following propositions are true:

(8) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$, g be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction P$, and s_1, s_2 be real numbers. Suppose that

(i) P is an arc from p_1 to p_2,

(ii) g is a homeomorphism,

(iii) $g(0) = p_1$,

(iv) $g(1) = p_2$,

(v) $g(s_1) = q_1$,

(vi) $0 \leqslant s_1$,

(vii) $s_1 \leqslant 1$,

(viii) $g(s_2) = q_2$,

(ix) $0 \leqslant s_2$,

(x) $s_2 \leqslant 1$, and

(xi) $s_1 \leqslant s_2$.

Then LE q_1, q_2, P, p_1, p_2.

(9) Let P be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and $q_1 \in P$, then LE q_1, q_1, P, p_1, p_2.

(10) Let P be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2 and $q_1 \in P$. Then LE p_1, q_1, P, p_1, p_2 and LE q_1, p_2, P, p_1, p_2.

(11) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2, then LE p_1, p_2, P, p_1, p_2.

(12) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2 and LE q_1, q_2, P, p_1, p_2 and LE q_2, q_1, P, p_1, p_2. Then $q_1 = q_2$.

(13) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2, q_3 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2 and LE q_1, q_2, P, p_1, p_2 and LE q_2, q_3, P, p_1, p_2. Then LE q_1, q_3, P, p_1, p_2.

(14) Let P be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2 and $q_1 \in P$ and $q_2 \in P$ and $q_1 \neq q_2$. Then LE q_1, q_2, P, p_1, p_2 and not LE q_2, q_1, P, p_1, p_2 or LE q_2, q_1, P, p_1, p_2 and not LE q_1, q_2, P, p_1, p_2.

3. Some properties of the ordering of points on a curve

We now state a number of propositions:

(15) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, Q be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $\widetilde{\mathcal{L}}(f) \cap Q$ is closed and $q \in \widetilde{\mathcal{L}}(f)$ and $q \in Q$. Then LE $\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), \pi_1 f, \pi_{\mathrm{len}\, f} f, Q)$, q, $\widetilde{\mathcal{L}}(f)$, $\pi_1 f$, $\pi_{\mathrm{len}\, f} f$.

(16) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, Q be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $\widetilde{\mathcal{L}}(f) \cap Q$ is closed and $q \in \widetilde{\mathcal{L}}(f)$ and $q \in Q$. Then LE q, $\mathrm{LPoint}(\widetilde{\mathcal{L}}(f), \pi_1 f, \pi_{\mathrm{len}\, f} f, Q)$, $\widetilde{\mathcal{L}}(f)$, $\pi_1 f$, $\pi_{\mathrm{len}\, f} f$.

(17) For all points q_1, q_2, p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_1 \neq p_2$ holds if LE q_1, q_2, $\mathcal{L}(p_1, p_2)$, p_1, p_2, then $\mathrm{LE}(q_1, q_2, p_1, p_2)$.

(18) Let P, Q be subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2 and $P \cap Q \neq \emptyset$ and $P \cap Q$ is closed. Then $\mathrm{FPoint}(P, p_1, p_2, Q) = \mathrm{LPoint}(P, p_2, p_1, Q)$ and $\mathrm{LPoint}(P, p_1, p_2, Q) = \mathrm{FPoint}(P, p_2, p_1, Q)$.

(19) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, Q be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, and i be a natural number. Suppose $\widetilde{\mathcal{L}}(f)$ meets Q and Q is closed and f is a special sequence and $1 \leqslant i$

and $i+1 \leqslant \operatorname{len} f$ and $\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), \pi_1 f, \pi_{\operatorname{len} f} f, Q) \in \mathcal{L}(f,i)$. Then $\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), \pi_1 f, \pi_{\operatorname{len} f} f, Q) = \mathrm{FPoint}(\mathcal{L}(f,i), \pi_i f, \pi_{i+1} f, Q)$.

(20) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, Q be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, and i be a natural number. Suppose $\widetilde{\mathcal{L}}(f)$ meets Q and Q is closed and f is a special sequence and $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and $\mathrm{LPoint}(\widetilde{\mathcal{L}}(f), \pi_1 f, \pi_{\operatorname{len} f} f, Q) \in \mathcal{L}(f,i)$. Then $\mathrm{LPoint}(\widetilde{\mathcal{L}}(f), \pi_1 f, \pi_{\operatorname{len} f} f, Q) = \mathrm{LPoint}(\mathcal{L}(f,i), \pi_i f, \pi_{i+1} f, Q)$.

(21) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and i be a natural number. Suppose $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and f is a special sequence and $\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), \pi_1 f, \pi_{\operatorname{len} f} f, \mathcal{L}(f,i)) \in \mathcal{L}(f,i)$. Then $\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), \pi_1 f, \pi_{\operatorname{len} f} f, \mathcal{L}(f,i)) = \pi_i f$.

(22) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and i be a natural number. Suppose $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and f is a special sequence and $\mathrm{LPoint}(\widetilde{\mathcal{L}}(f), \pi_1 f, \pi_{\operatorname{len} f} f, \mathcal{L}(f,i)) \in \mathcal{L}(f,i)$. Then $\mathrm{LPoint}(\widetilde{\mathcal{L}}(f), \pi_1 f, \pi_{\operatorname{len} f} f, \mathcal{L}(f,i)) = \pi_{i+1} f$.

(23) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and i be a natural number. Suppose f is a special sequence and $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$. Then LE $\pi_i f$, $\pi_{i+1} f$, $\widetilde{\mathcal{L}}(f)$, $\pi_1 f$, $\pi_{\operatorname{len} f} f$.

(24) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and i, k be natural numbers. Suppose f is a special sequence and $1 \leqslant i$ and $i+k+1 \leqslant \operatorname{len} f$. Then LE $\pi_i f$, $\pi_{i+k} f$, $\widetilde{\mathcal{L}}(f)$, $\pi_1 f$, $\pi_{\operatorname{len} f} f$.

(25) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, q be a point of $\mathcal{E}^2_{\mathrm{T}}$, and i be a natural number. Suppose f is a special sequence and $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and $q \in \mathcal{L}(f,i)$. Then LE $\pi_i f$, q, $\widetilde{\mathcal{L}}(f)$, $\pi_1 f$, $\pi_{\operatorname{len} f} f$.

(26) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, q be a point of $\mathcal{E}^2_{\mathrm{T}}$, and i be a natural number. Suppose f is a special sequence and $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and $q \in \mathcal{L}(f,i)$. Then LE q, $\pi_{i+1} f$, $\widetilde{\mathcal{L}}(f)$, $\pi_1 f$, $\pi_{\operatorname{len} f} f$.

(27) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, Q be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, q be a point of $\mathcal{E}^2_{\mathrm{T}}$, and i, j be natural numbers. Suppose that

(i) $\widetilde{\mathcal{L}}(f)$ meets Q,

(ii) f is a special sequence,

(iii) Q is closed,

(iv) $\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), \pi_1 f, \pi_{\operatorname{len} f} f, Q) \in \mathcal{L}(f,i)$,

(v) $1 \leqslant i$,

(vi) $i+1 \leqslant \operatorname{len} f$,

(vii) $q \in \mathcal{L}(f,j)$,

(viii) $1 \leqslant j$,

(ix) $j+1 \leqslant \operatorname{len} f$,

(x) $q \in Q$, and

(xi) $\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), \pi_1 f, \pi_{\operatorname{len} f} f, Q) \neq q$.

Then $i \leqslant j$ and if $i = j$, then $\mathrm{LE}(\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), \pi_1 f, \pi_{\mathrm{len}\, f} f, Q), q, \pi_i f, \pi_{i+1} f)$.

(28) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, Q be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, q be a point of $\mathcal{E}^2_{\mathrm{T}}$, and i, j be natural numbers. Suppose that

(i) $\widetilde{\mathcal{L}}(f)$ meets Q,

(ii) f is a special sequence,

(iii) Q is closed,

(iv) $\mathrm{LPoint}(\widetilde{\mathcal{L}}(f), \pi_1 f, \pi_{\mathrm{len}\, f} f, Q) \in \mathcal{L}(f, i)$,

(v) $1 \leqslant i$,

(vi) $i + 1 \leqslant \mathrm{len}\, f$,

(vii) $q \in \mathcal{L}(f, j)$,

(viii) $1 \leqslant j$,

(ix) $j + 1 \leqslant \mathrm{len}\, f$,

(x) $q \in Q$, and

(xi) $\mathrm{LPoint}(\widetilde{\mathcal{L}}(f), \pi_1 f, \pi_{\mathrm{len}\, f} f, Q) \neq q$.

Then $i \geqslant j$ and if $i = j$, then $\mathrm{LE}(q, \mathrm{LPoint}(\widetilde{\mathcal{L}}(f), \pi_1 f, \pi_{\mathrm{len}\, f} f, Q), \pi_i f, \pi_{i+1} f)$.

(29) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$, and i be a natural number. Suppose $q_1 \in \mathcal{L}(f, i)$ and $q_2 \in \mathcal{L}(f, i)$ and f is a special sequence and $1 \leqslant i$ and $i + 1 \leqslant \mathrm{len}\, f$. If LE q_1, q_2, $\widetilde{\mathcal{L}}(f)$, $\pi_1 f$, $\pi_{\mathrm{len}\, f} f$, then LE q_1, q_2, $\mathcal{L}(f, i)$, $\pi_i f$, $\pi_{i+1} f$.

(30) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $q_1 \in \widetilde{\mathcal{L}}(f)$ and $q_2 \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence and $q_1 \neq q_2$. Then LE q_1, q_2, $\widetilde{\mathcal{L}}(f)$, $\pi_1 f$, $\pi_{\mathrm{len}\, f} f$ if and only if for all natural numbers i, j such that $q_1 \in \mathcal{L}(f, i)$ and $q_2 \in \mathcal{L}(f, j)$ and $1 \leqslant i$ and $i + 1 \leqslant \mathrm{len}\, f$ and $1 \leqslant j$ and $j+1 \leqslant \mathrm{len}\, f$ holds $i \leqslant j$ and if $i = j$, then $\mathrm{LE}(q_1, q_2, \pi_i f, \pi_{i+1} f)$.

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On the Categories Without Uniqueness of cod and dom . Some Properties of the Morphisms and the Functors

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The notation and terminology used here are introduced in the following papers: [9], [4], [10], [16], [2], [3], [1], [7], [8], [11], [15], [5], [12], [13], [6], and [14].

1. Preliminaries

In this paper C denotes a category and o_1, o_2, o_3 denote objects of C.

Let C be a non empty category structure with units and let o be an object of C. Observe that $\langle o, o\rangle$ is non empty.

The following propositions are true:

(1) Let v be a morphism from o_1 to o_2, u be a morphism from o_1 to o_3, and f be a morphism from o_2 to o_3. If $u = f \cdot v$ and $f^{-1} \cdot f = \mathrm{id}_{(o_2)}$ and $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_3\rangle \neq \emptyset$ and $\langle o_3, o_2\rangle \neq \emptyset$, then $v = f^{-1} \cdot u$.

(2) Let v be a morphism from o_2 to o_3, u be a morphism from o_1 to o_3, and f be a morphism from o_1 to o_2. If $u = v \cdot f$ and $f \cdot f^{-1} = \mathrm{id}_{(o_2)}$ and $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_1\rangle \neq \emptyset$ and $\langle o_2, o_3\rangle \neq \emptyset$, then $v = u \cdot f^{-1}$.

(3) For every morphism m from o_1 to o_2 such that $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_1\rangle \neq \emptyset$ and m is iso holds m^{-1} is iso.

(4) For every non empty category structure C with units and for every object o of C holds id_o is epi and mono.

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Let C be a non empty category structure with units and let o be an object of C. One can verify that id_o is epi mono retraction and coretraction.

Let C be a category and let o be an object of C. Note that id_o is iso.

We now state two propositions:

(5) Let f be a morphism from o_1 to o_2 and g, h be morphisms from o_2 to o_1. If $h \cdot f = \mathrm{id}_{(o_1)}$ and $f \cdot g = \mathrm{id}_{(o_2)}$ and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$, then $g = h$.

(6) Suppose that for all objects o_1, o_2 of C holds every morphism from o_1 to o_2 is coretraction. Let a, b be objects of C and g be a morphism from a to b. If $\langle a, b \rangle \neq \emptyset$ and $\langle b, a \rangle \neq \emptyset$, then g is iso.

2. Some properties of the initial and terminal objects

The following propositions are true:

(7) For all morphisms m, m' from o_1 to o_2 such that m is zero and m' is zero and there exists an object of C which is zero holds $m = m'$.

(8) Let C be a non empty category structure, O, A be objects of C, and M be a morphism from O to A. If O is terminal, then M is mono.

(9) Let C be a non empty category structure, O, A be objects of C, and M be a morphism from A to O. If O is initial, then M is epi.

(10) If o_2 is terminal and o_1, o_2 are iso , then o_1 is terminal.

(11) If o_1 is initial and o_1, o_2 are iso , then o_2 is initial.

(12) If o_1 is initial and o_2 is terminal and $\langle o_2, o_1 \rangle \neq \emptyset$, then o_2 is initial and o_1 is terminal.

3. The properties of the functors

One can prove the following propositions:

(13) Let A, B be transitive non empty category structures with units, F be a contravariant functor from A to B, and a be an object of A. Then $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$.

(14) Let C_1, C_2 be non empty category structures and F be a precontravariant functor structure from C_1 to C_2. Then F is full if and only if for all objects o_1, o_2 of C_1 holds $\text{Morph-Map}_F(o_2, o_1)$ is onto.

(15) Let C_1, C_2 be non empty category structures and F be a precontravariant functor structure from C_1 to C_2. Then F is faithful if and only if for all objects o_1, o_2 of C_1 holds $\text{Morph-Map}_F(o_2, o_1)$ is one-to-one.

(16) Let C_1, C_2 be non empty category structures, F be a precovariant functor structure from C_1 to C_2, o_1, o_2 be objects of C_1, and F_1 be a morphism from $F(o_1)$ to $F(o_2)$. Suppose $\langle o_1, o_2\rangle \neq \emptyset$ and F is full and feasible. Then there exists a morphism m from o_1 to o_2 such that $F_1 = F(m)$.

(17) Let C_1, C_2 be non empty category structures, F be a precontravariant functor structure from C_1 to C_2, o_1, o_2 be objects of C_1, and F_1 be a morphism from $F(o_2)$ to $F(o_1)$. Suppose $\langle o_1, o_2\rangle \neq \emptyset$ and F is full and feasible. Then there exists a morphism m from o_1 to o_2 such that $F_1 = F(m)$.

(18) Let A, B be transitive non empty category structures with units, F be a covariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. If $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_1\rangle \neq \emptyset$ and a is retraction, then $F(a)$ is retraction.

(19) Let A, B be transitive non empty category structures with units, F be a covariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. If $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_1\rangle \neq \emptyset$ and a is coretraction, then $F(a)$ is coretraction.

(20) Let A, B be categories, F be a covariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. If $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_1\rangle \neq \emptyset$ and a is iso, then $F(a)$ is iso.

(21) Let A, B be categories, F be a covariant functor from A to B, and o_1, o_2 be objects of A. If o_1, o_2 are iso , then $F(o_1)$, $F(o_2)$ are iso .

(22) Let A, B be transitive non empty category structures with units, F be a contravariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. If $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_1\rangle \neq \emptyset$ and a is retraction, then $F(a)$ is coretraction.

(23) Let A, B be transitive non empty category structures with units, F be a contravariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. If $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_1\rangle \neq \emptyset$ and a is coretraction, then $F(a)$ is retraction.

(24) Let A, B be categories, F be a contravariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. If $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_1\rangle \neq \emptyset$ and a is iso, then $F(a)$ is iso.

(25) Let A, B be categories, F be a contravariant functor from A to B, and o_1, o_2 be objects of A. If o_1, o_2 are iso , then $F(o_2)$, $F(o_1)$ are iso .

(26) Let A, B be transitive non empty category structures with units, F be a covariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. Suppose F is full and faithful and $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_1\rangle \neq \emptyset$ and $F(a)$ is retraction. Then a is retraction.

(27) Let A, B be transitive non empty category structures with units, F

be a covariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. Suppose F is full and faithful and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and $F(a)$ is coretraction. Then a is coretraction.

(28) Let A, B be categories, F be a covariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. Suppose F is full and faithful and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and $F(a)$ is iso. Then a is iso.

(29) Let A, B be categories, F be a covariant functor from A to B, and o_1, o_2 be objects of A. Suppose F is full and faithful and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and $F(o_1)$, $F(o_2)$ are iso . Then o_1, o_2 are iso .

(30) Let A, B be transitive non empty category structures with units, F be a contravariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. Suppose F is full and faithful and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and $F(a)$ is retraction. Then a is coretraction.

(31) Let A, B be transitive non empty category structures with units, F be a contravariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. Suppose F is full and faithful and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and $F(a)$ is coretraction. Then a is retraction.

(32) Let A, B be categories, F be a contravariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. Suppose F is full and faithful and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and $F(a)$ is iso. Then a is iso.

(33) Let A, B be categories, F be a contravariant functor from A to B, and o_1, o_2 be objects of A. Suppose F is full and faithful and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and $F(o_2)$, $F(o_1)$ are iso . Then o_1, o_2 are iso .

4. The subcategories of the morphisms

We now state two propositions:

(34) Let C be a category structure and D be a substructure of C. Suppose the carrier of C = the carrier of D and the arrows of C = the arrows of D. Then D is full.

(35) Let C be a non empty category structure with units and D be a substructure of C. Suppose the carrier of C = the carrier of D and the arrows of C = the arrows of D. Then D is full and id-inheriting.

Let C be a category. Observe that there exists a subcategory of C which is full, non empty, and strict.

Next we state several propositions:

(36) For every non empty subcategory B of C holds every non empty subcategory of B is a non empty subcategory of C.

(37) Let C be a non empty transitive category structure, D be a non empty transitive substructure of C, o_1, o_2 be objects of C, p_1, p_2 be objects of D, m be a morphism from o_1 to o_2, and n be a morphism from p_1 to p_2 such that $p_1 = o_1$ and $p_2 = o_2$ and $m = n$ and $\langle p_1, p_2 \rangle \neq \emptyset$. Then

(i) if m is mono, then n is mono, and

(ii) if m is epi, then n is epi.

(38) Let D be a non empty subcategory of C, o_1, o_2 be objects of C, p_1, p_2 be objects of D, m be a morphism from o_1 to o_2, m_1 be a morphism from o_2 to o_1, n be a morphism from p_1 to p_2, and n_1 be a morphism from p_2 to p_1 such that $p_1 = o_1$ and $p_2 = o_2$ and $m = n$ and $m_1 = n_1$ and $\langle p_1, p_2 \rangle \neq \emptyset$ and $\langle p_2, p_1 \rangle \neq \emptyset$. Then

(i) m is left inverse of m_1 iff n is left inverse of n_1, and

(ii) m is right inverse of m_1 iff n is right inverse of n_1.

(39) Let D be a full non empty subcategory of C, o_1, o_2 be objects of C, p_1, p_2 be objects of D, m be a morphism from o_1 to o_2, and n be a morphism from p_1 to p_2 such that $p_1 = o_1$ and $p_2 = o_2$ and $m = n$ and $\langle p_1, p_2 \rangle \neq \emptyset$ and $\langle p_2, p_1 \rangle \neq \emptyset$. Then

(i) if m is retraction, then n is retraction,

(ii) if m is coretraction, then n is coretraction, and

(iii) if m is iso, then n is iso.

(40) Let D be a non empty subcategory of C, o_1, o_2 be objects of C, p_1, p_2 be objects of D, m be a morphism from o_1 to o_2, and n be a morphism from p_1 to p_2 such that $p_1 = o_1$ and $p_2 = o_2$ and $m = n$ and $\langle p_1, p_2 \rangle \neq \emptyset$ and $\langle p_2, p_1 \rangle \neq \emptyset$. Then

(i) if n is retraction, then m is retraction,

(ii) if n is coretraction, then m is coretraction, and

(iii) if n is iso, then m is iso.

Let C be a category. The functor $\mathrm{AllMono}\, C$ yields a strict non empty transitive substructure of C and is defined by the conditions (Def. 1).

(Def. 1)(i) The carrier of $\mathrm{AllMono}\, C =$ the carrier of C,

(ii) the arrows of $\mathrm{AllMono}\, C \mathrel{\dot{\subseteq}}$ the arrows of C, and

(iii) for all objects o_1, o_2 of C and for every morphism m from o_1 to o_2 holds $m \in$ (the arrows of $\mathrm{AllMono}\, C$)(o_1, o_2) iff $\langle o_1, o_2 \rangle \neq \emptyset$ and m is mono.

Let C be a category. Note that $\mathrm{AllMono}\, C$ is id-inheriting.

Let C be a category. The functor $\mathrm{AllEpi}\, C$ yields a strict non empty transitive substructure of C and is defined by the conditions (Def. 2).

(Def. 2)(i) The carrier of $\mathrm{AllEpi}\, C =$ the carrier of C,

(ii) the arrows of $\mathrm{AllEpi}\, C \mathrel{\dot{\subseteq}}$ the arrows of C, and

(iii) for all objects o_1, o_2 of C and for every morphism m from o_1 to o_2 holds $m \in$ (the arrows of $\mathrm{AllEpi}\, C$)(o_1, o_2) iff $\langle o_1, o_2 \rangle \neq \emptyset$ and m is epi.

Let C be a category. Observe that AllEpi C is id-inheriting.

Let C be a category. The functor AllRetr C yielding a strict non empty transitive substructure of C is defined by the conditions (Def. 3).

(Def. 3)(i) The carrier of AllRetr C = the carrier of C,

(ii) the arrows of AllRetr $C \mathrel{\dot{\subseteq}}$ the arrows of C, and

(iii) for all objects o_1, o_2 of C and for every morphism m from o_1 to o_2 holds $m \in$ (the arrows of AllRetr C)(o_1, o_2) iff $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and m is retraction.

Let C be a category. One can check that AllRetr C is id-inheriting.

Let C be a category. The functor AllCoretr C yielding a strict non empty transitive substructure of C is defined by the conditions (Def. 4).

(Def. 4)(i) The carrier of AllCoretr C = the carrier of C,

(ii) the arrows of AllCoretr $C \mathrel{\dot{\subseteq}}$ the arrows of C, and

(iii) for all objects o_1, o_2 of C and for every morphism m from o_1 to o_2 holds $m \in$ (the arrows of AllCoretr C)(o_1, o_2) iff $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and m is coretraction.

Let C be a category. One can verify that AllCoretr C is id-inheriting.

Let C be a category. The functor AllIso C yields a strict non empty transitive substructure of C and is defined by the conditions (Def. 5).

(Def. 5)(i) The carrier of AllIso C = the carrier of C,

(ii) the arrows of AllIso $C \mathrel{\dot{\subseteq}}$ the arrows of C, and

(iii) for all objects o_1, o_2 of C and for every morphism m from o_1 to o_2 holds $m \in$ (the arrows of AllIso C)(o_1, o_2) iff $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and m is iso.

Let C be a category. Note that AllIso C is id-inheriting.

Next we state a number of propositions:

(41) AllIso C is a non empty subcategory of AllRetr C.

(42) AllIso C is a non empty subcategory of AllCoretr C.

(43) AllCoretr C is a non empty subcategory of AllMono C.

(44) AllRetr C is a non empty subcategory of AllEpi C.

(45) If for all objects o_1, o_2 of C holds every morphism from o_1 to o_2 is mono, then the category structure of C = AllMono C.

(46) If for all objects o_1, o_2 of C holds every morphism from o_1 to o_2 is epi, then the category structure of C = AllEpi C.

(47) Suppose that for all objects o_1, o_2 of C and for every morphism m from o_1 to o_2 holds m is retraction and $\langle o_2, o_1 \rangle \neq \emptyset$. Then the category structure of C = AllRetr C.

(48) Suppose that for all objects o_1, o_2 of C and for every morphism m from o_1 to o_2 holds m is coretraction and $\langle o_2, o_1 \rangle \neq \emptyset$. Then the category structure of C = AllCoretr C.

(49) Suppose that for all objects o_1, o_2 of C and for every morphism m from o_1 to o_2 holds m is iso and $\langle o_2, o_1 \rangle \neq \emptyset$. Then the category structure of $C = \mathrm{AllIso}\, C$.

(50) For all objects o_1, o_2 of $\mathrm{AllMono}\, C$ and for every morphism m from o_1 to o_2 such that $\langle o_1, o_2 \rangle \neq \emptyset$ holds m is mono.

(51) For all objects o_1, o_2 of $\mathrm{AllEpi}\, C$ and for every morphism m from o_1 to o_2 such that $\langle o_1, o_2 \rangle \neq \emptyset$ holds m is epi.

(52) For all objects o_1, o_2 of $\mathrm{AllIso}\, C$ and for every morphism m from o_1 to o_2 such that $\langle o_1, o_2 \rangle \neq \emptyset$ holds m is iso and $m^{-1} \in \langle o_2, o_1 \rangle$.

(53) $\mathrm{AllMono}\,\mathrm{AllMono}\, C = \mathrm{AllMono}\, C$.

(54) $\mathrm{AllEpi}\,\mathrm{AllEpi}\, C = \mathrm{AllEpi}\, C$.

(55) $\mathrm{AllIso}\,\mathrm{AllIso}\, C = \mathrm{AllIso}\, C$.

(56) $\mathrm{AllIso}\,\mathrm{AllMono}\, C = \mathrm{AllIso}\, C$.

(57) $\mathrm{AllIso}\,\mathrm{AllEpi}\, C = \mathrm{AllIso}\, C$.

(58) $\mathrm{AllIso}\,\mathrm{AllRetr}\, C = \mathrm{AllIso}\, C$.

(59) $\mathrm{AllIso}\,\mathrm{AllCoretr}\, C = \mathrm{AllIso}\, C$.

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The loop and Times Macroinstruction for $\mathbf{SCM}_{\mathrm{FSA}}$

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Summary. We implement two macroinstructions loop and Times which iterate macroinstructions of $\mathbf{SCM}_{\mathrm{FSA}}$. In a loop macroinstruction it jumps to the head when the original macroinstruction stops, in a Times macroinstruction it behaves as if the original macroinstrucion repeats n times.

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The articles [22], [29], [16], [8], [12], [30], [13], [14], [11], [7], [9], [28], [15], [17], [23], [20], [21], [27], [24], [25], [1], [10], [19], [26], [5], [6], [4], [2], [3], and [18] provide the terminology and notation for this paper.

1. Preliminaries

Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$ and let P be an initial finite partial state of $\mathbf{SCM}_{\mathrm{FSA}}$. We say that P is pseudo-closed on s if and only if the condition (Def. 1) is satisfied.

(Def. 1) There exists a natural number k such that $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot(P+\cdot\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))(k)} = \mathrm{insloc}(\mathrm{card}\,\mathrm{ProgramPart}(P))$ and for every natural number n such that $n < k$ holds $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot(P+\cdot\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))(n)} \in \mathrm{dom}\,P$.

Let P be an initial finite partial state of $\mathbf{SCM}_{\mathrm{FSA}}$. We say that P is pseudo-paraclosed if and only if:

(Def. 2) For every state s of $\mathbf{SCM}_{\mathrm{FSA}}$ holds P is pseudo-closed on s.

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Let us note that there exists a macro instruction which is pseudo-paraclosed.

Let s be a state of $\mathbf{SCM}_{\rm FSA}$ and let P be an initial finite partial state of $\mathbf{SCM}_{\rm FSA}$. Let us assume that P is pseudo-closed on s.

The functor $\mathrm{pseudo} - \mathrm{LifeSpan}(s, P)$ yielding a natural number is defined as follows:

(Def. 3) $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot(P+\cdot\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))(\mathrm{pseudo}-\mathrm{LifeSpan}(s,P))} = \mathrm{insloc}(\mathrm{card}\,\mathrm{ProgramPart}(P))$ and for every natural number n such that $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot(P+\cdot\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))(n)} \notin \mathrm{dom}\,P$ holds $\mathrm{pseudo} - \mathrm{LifeSpan}(s, P) \leqslant n$.

We now state a number of propositions:

(1) Let s be a state of $\mathbf{SCM}_{\rm FSA}$ and P be an initial finite partial state of $\mathbf{SCM}_{\rm FSA}$. Suppose P is pseudo-closed on s. Let n be a natural number. If $n < \mathrm{pseudo} - \mathrm{LifeSpan}(s, P)$, then $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot(P+\cdot\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))(n)} \in \mathrm{dom}\,P$ and $\mathrm{CurInstr}((\mathrm{Computation}(s+\cdot(P+\cdot\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))(n)) \neq \mathbf{halt}_{\mathbf{SCM}_{\rm FSA}}$.

(2) Let s be a state of $\mathbf{SCM}_{\rm FSA}$ and P be an initial finite partial state of $\mathbf{SCM}_{\rm FSA}$. Suppose P is pseudo-closed on s. Let k be a natural number. Suppose that for every natural number n such that $n \leqslant k$ holds $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot(P+\cdot\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))(n)} \in \mathrm{dom}\,P$. Then $k < \mathrm{pseudo} - \mathrm{LifeSpan}(s, P)$.

(3) Let s be a state of $\mathbf{SCM}_{\rm FSA}$ and I, J be macro instructions. Suppose I is pseudo-closed on s. Let k be a natural number. Suppose $k \leqslant \mathrm{pseudo} - \mathrm{LifeSpan}(s, I)$. Then $(\mathrm{Computation}(s+\cdot(I+\cdot\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))(k)$ and $(\mathrm{Computation}(s+\cdot((I;J)+\cdot\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))(k)$ are equal outside the instruction locations of $\mathbf{SCM}_{\rm FSA}$.

(4) Let s be a state of $\mathbf{SCM}_{\rm FSA}$ and I be a macro instruction. If I is closed on s and halting on s, then $\mathrm{Directed}(I)$ is pseudo-closed on s.

(5) Let s be a state of $\mathbf{SCM}_{\rm FSA}$ and I be a macro instruction. If I is closed on s and halting on s, then $\mathrm{pseudo} - \mathrm{LifeSpan}(s, \mathrm{Directed}(I)) = \mathrm{LifeSpan}(s+\cdot(I+\cdot\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))) + 1$.

(6) For every function f and for every set x such that $x \in \mathrm{dom}\,f$ holds $f+\cdot(x \dot{\longmapsto} f(x)) = f$.

(7) For every instruction-location l of $\mathbf{SCM}_{\rm FSA}$ holds $l + 0 = l$.

(8) For every instruction i of $\mathbf{SCM}_{\rm FSA}$ holds $\mathrm{IncAddr}(i, 0) = i$.

(9) For every programmed finite partial state P of $\mathbf{SCM}_{\rm FSA}$ holds $\mathrm{ProgramPart}(\mathrm{Relocated}(P, 0)) = P$.

(10) For all finite partial states P, Q of $\mathbf{SCM}_{\rm FSA}$ such that $P \subseteq Q$ holds $\mathrm{ProgramPart}(P) \subseteq \mathrm{ProgramPart}(Q)$.

(11) For all programmed finite partial states P, Q of $\mathbf{SCM}_{\rm FSA}$ and for every natural number k such that $P \subseteq Q$ holds $\mathrm{Shift}(P, k) \subseteq \mathrm{Shift}(Q, k)$.

(12) For all finite partial states P, Q of $\mathbf{SCM}_{\mathrm{FSA}}$ and for every natural number k such that $P \subseteq Q$ holds $\mathrm{ProgramPart}(\mathrm{Relocated}(P, k)) \subseteq \mathrm{ProgramPart}(\mathrm{Relocated}(Q, k))$.

(13) Let I, J be macro instructions and k be a natural number. Suppose $\mathrm{card}\, I \leqslant k$ and $k < \mathrm{card}\, I + \mathrm{card}\, J$. Let i be an instruction of $\mathbf{SCM}_{\mathrm{FSA}}$. If $i = J(\mathrm{insloc}(k -' \mathrm{card}\, I))$, then $(I;J)(\mathrm{insloc}(k)) = \mathrm{IncAddr}(i, \mathrm{card}\, I)$.

(14) For every state s of $\mathbf{SCM}_{\mathrm{FSA}}$ such that $s(\mathrm{intloc}(0)) = 1$ and $\mathbf{IC}_s = \mathrm{insloc}(0)$ holds $\mathrm{Initialize}(s) = s$.

(15) For every state s of $\mathbf{SCM}_{\mathrm{FSA}}$ holds $\mathrm{Initialize}(\mathrm{Initialize}(s)) = \mathrm{Initialize}(s)$.

(16) For every state s of $\mathbf{SCM}_{\mathrm{FSA}}$ and for every macro instruction I holds $s{+\cdot}(\mathrm{Initialized}(I){+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0))) =$
$\mathrm{Initialize}(s){+\cdot}(I{+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))$.

(17) For every state s of $\mathbf{SCM}_{\mathrm{FSA}}$ and for every macro instruction I holds $\mathrm{IExec}(I, s) = \mathrm{IExec}(I, \mathrm{Initialize}(s))$.

(18) For every state s of $\mathbf{SCM}_{\mathrm{FSA}}$ and for every macro instruction I such that $s(\mathrm{intloc}(0)) = 1$ holds $s{+\cdot}(I{+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0))) = s{+\cdot}\,\mathrm{Initialized}(I)$.

(19) For every macro instruction I holds $I{+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)) \subseteq \mathrm{Initialized}(I)$.

(20) For every instruction-location l of $\mathbf{SCM}_{\mathrm{FSA}}$ and for every macro instruction I holds $l \in \mathrm{dom}\, I$ iff $l \in \mathrm{dom}\,\mathrm{Initialized}(I)$.

(21) For every state s of $\mathbf{SCM}_{\mathrm{FSA}}$ and for every macro instruction I holds $\mathrm{Initialized}(I)$ is closed on s iff I is closed on $\mathrm{Initialize}(s)$.

(22) For every state s of $\mathbf{SCM}_{\mathrm{FSA}}$ and for every macro instruction I holds $\mathrm{Initialized}(I)$ is halting on s iff I is halting on $\mathrm{Initialize}(s)$.

(23) For every macro instruction I such that for every state s of $\mathbf{SCM}_{\mathrm{FSA}}$ holds I is halting on $\mathrm{Initialize}(s)$ holds $\mathrm{Initialized}(I)$ is halting.

(24) For every macro instruction I such that for every state s of $\mathbf{SCM}_{\mathrm{FSA}}$ holds $\mathrm{Initialized}(I)$ is halting on s holds $\mathrm{Initialized}(I)$ is halting.

(25) For every macro instruction I holds $\mathrm{ProgramPart}(\mathrm{Initialized}(I)) = I$.

(26) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I be a macro instruction, l be an instruction-location of $\mathbf{SCM}_{\mathrm{FSA}}$, and x be a set. If $x \in \mathrm{dom}\, I$, then $I(x) = (s{+\cdot}(I{+\cdot}\,\mathrm{Start\text{-}At}(l)))(x)$.

(27) For every state s of $\mathbf{SCM}_{\mathrm{FSA}}$ such that $s(\mathrm{intloc}(0)) = 1$
holds $\mathrm{Initialize}(s){\restriction}(\mathrm{Int\text{-}Locations} \cup \mathrm{FinSeq\text{-}Locations}) =$
$s{\restriction}(\mathrm{Int\text{-}Locations} \cup \mathrm{FinSeq\text{-}Locations})$.

(28) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I be a macro instruction, a be an integer location, and l be an instruction-location of $\mathbf{SCM}_{\mathrm{FSA}}$. Then $(s{+\cdot}(I{+\cdot}\,\mathrm{Start\text{-}At}(l)))(a) = s(a)$.

(29) For every programmed finite partial state I of $\mathbf{SCM}_{\mathrm{FSA}}$ and for every instruction-location l of $\mathbf{SCM}_{\mathrm{FSA}}$ holds $\mathbf{IC}_{\mathbf{SCM}_{\mathrm{FSA}}} \in \mathrm{dom}(I +\cdot \text{Start-At}(l))$.

(30) For every programmed finite partial state I of $\mathbf{SCM}_{\mathrm{FSA}}$ and for every instruction-location l of $\mathbf{SCM}_{\mathrm{FSA}}$ holds $(I +\cdot \text{Start-At}(l))(\mathbf{IC}_{\mathbf{SCM}_{\mathrm{FSA}}}) = l$.

(31) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, P be a finite partial state of $\mathbf{SCM}_{\mathrm{FSA}}$, and l be an instruction-location of $\mathbf{SCM}_{\mathrm{FSA}}$. Then $\mathbf{IC}_{s +\cdot (P +\cdot \text{Start-At}(l))} = l$.

(32) For every state s of $\mathbf{SCM}_{\mathrm{FSA}}$ and for every instruction i of $\mathbf{SCM}_{\mathrm{FSA}}$ such that $\mathrm{InsCode}(i) \in \{0, 6, 7, 8\}$ holds $\mathrm{Exec}(i, s) \restriction (\text{Int-Locations} \cup \text{FinSeq-Locations}) = s \restriction (\text{Int-Locations} \cup \text{FinSeq-Locations})$.

(33) Let s_1, s_2 be states of $\mathbf{SCM}_{\mathrm{FSA}}$. Suppose that

(i) $s_1(\mathrm{intloc}(0)) = s_2(\mathrm{intloc}(0))$,

(ii) for every read-write integer location a holds $s_1(a) = s_2(a)$, and

(iii) for every finite sequence location f holds $s_1(f) = s_2(f)$.

Then $s_1 \restriction (\text{Int-Locations} \cup \text{FinSeq-Locations}) = s_2 \restriction (\text{Int-Locations} \cup \text{FinSeq-Locations})$.

(34) For every state s of $\mathbf{SCM}_{\mathrm{FSA}}$ and for every programmed finite partial state P of $\mathbf{SCM}_{\mathrm{FSA}}$ holds $(s +\cdot P) \restriction (\text{Int-Locations} \cup \text{FinSeq-Locations}) = s \restriction (\text{Int-Locations} \cup \text{FinSeq-Locations})$.

(35) For all states s, s_3 of $\mathbf{SCM}_{\mathrm{FSA}}$ holds $(s +\cdot s_3 \restriction \text{the instruction locations of } \mathbf{SCM}_{\mathrm{FSA}}) \restriction (\text{Int-Locations} \cup \text{FinSeq-Locations}) = s \restriction (\text{Int-Locations} \cup \text{FinSeq-Locations})$.

(36) For every state s of $\mathbf{SCM}_{\mathrm{FSA}}$ holds $\mathrm{Initialize}(s) \restriction \text{the instruction locations of } \mathbf{SCM}_{\mathrm{FSA}} = s \restriction \text{the instruction locations of } \mathbf{SCM}_{\mathrm{FSA}}$.

(37) Let s, s_3 be states of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a macro instruction. Then $(s_3 +\cdot s \restriction \text{the instruction locations of } \mathbf{SCM}_{\mathrm{FSA}}) \restriction (\text{Int-Locations} \cup \text{FinSeq-Locations}) = s_3 \restriction (\text{Int-Locations} \cup \text{FinSeq-Locations})$.

(38) For every state s of $\mathbf{SCM}_{\mathrm{FSA}}$ holds $\mathrm{IExec}(\mathrm{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}, s) = \mathrm{Initialize}(s) +\cdot \text{Start-At}(\mathrm{insloc}(0))$.

(39) For every state s of $\mathbf{SCM}_{\mathrm{FSA}}$ and for every macro instruction I such that I is closed on s holds $\mathrm{insloc}(0) \in \mathrm{dom}\, I$.

(40) For every state s of $\mathbf{SCM}_{\mathrm{FSA}}$ and for every paraclosed macro instruction I holds $\mathrm{insloc}(0) \in \mathrm{dom}\, I$.

(41) For every instruction i of $\mathbf{SCM}_{\mathrm{FSA}}$ holds $\mathrm{rng}\, \mathrm{Macro}(i) = \{i, \mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}}\}$.

(42) Let s_1, s_2 be states of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a macro instruction. Suppose I is closed on s_1 and $I +\cdot \text{Start-At}(\mathrm{insloc}(0)) \subseteq s_1$. Let n be a natural number. Suppose $\mathrm{ProgramPart}(\mathrm{Relocated}(I, n)) \subseteq s_2$ and $\mathbf{IC}_{(s_2)} = \mathrm{insloc}(n)$ and $s_1 \restriction (\text{Int-Locations} \cup \text{FinSeq-Locations}) = s_2 \restriction (\text{Int-Locations} \cup \text{FinSeq-Locations})$. Let i be a natural number. Then $\mathbf{IC}_{(\mathrm{Computation}(s_1))(i)} + n = \mathbf{IC}_{(\mathrm{Computation}(s_2))(i)}$ and

IncAddr(CurInstr((Computation(s_1))(i)), n) =
CurInstr((Computation(s_2))(i)) and
(Computation(s_1))(i)↾(Int-Locations ∪ FinSeq-Locations) =
(Computation(s_2))(i)↾(Int-Locations ∪ FinSeq-Locations).

(43) Let s_1, s_2 be states of $\mathbf{SCM}_{\rm FSA}$ and I be a macro instruction. Suppose I is closed on s_1 and I+· Start-At(insloc(0)) $\subseteq$ s_1 and I+· Start-At(insloc(0)) $\subseteq s_2$ and s_1↾(Int-Locations ∪ FinSeq-Locations) = s_2↾(Int-Locations ∪ FinSeq-Locations). Let i be a natural number. Then $\mathbf{IC}_{(\mathrm{Computation}(s_1))(i)} = \mathbf{IC}_{(\mathrm{Computation}(s_2))(i)}$ and
CurInstr((Computation(s_1))(i)) = CurInstr((Computation(s_2))(i)) and
(Computation(s_1))(i)↾(Int-Locations ∪ FinSeq-Locations) =
(Computation(s_2))(i)↾(Int-Locations ∪ FinSeq-Locations).

(44) Let s_1, s_2 be states of $\mathbf{SCM}_{\rm FSA}$ and I be a macro instruction. Suppose I is closed on s_1 and halting on s_1 and I+· Start-At(insloc(0)) $\subseteq s_1$ and I+· Start-At(insloc(0)) $\subseteq s_2$ and s_1↾(Int-Locations ∪ FinSeq-Locations) = s_2↾(Int-Locations ∪ FinSeq-Locations). Then LifeSpan(s_1) = LifeSpan(s_2).

(45) Let s_1, s_2 be states of $\mathbf{SCM}_{\rm FSA}$ and I be a macro instruction. Suppose that

(i) s_1(intloc(0)) = 1,

(ii) I is closed on s_1 and halting on s_1,

(iii) for every read-write integer location a holds $s_1(a) = s_2(a)$, and

(iv) for every finite sequence location f holds $s_1(f) = s_2(f)$.

Then IExec(I, s_1)↾(Int-Locations ∪ FinSeq-Locations) =
IExec(I, s_2)↾(Int-Locations ∪ FinSeq-Locations).

(46) Let s_1, s_2 be states of $\mathbf{SCM}_{\rm FSA}$ and I be a macro instruction. Suppose s_1(intloc(0)) = 1 and I is closed on s_1 and halting on s_1 and s_1↾(Int-Locations ∪ FinSeq-Locations) = s_2↾(Int-Locations ∪ FinSeq-Locations).
Then IExec(I, s_1)↾(Int-Locations ∪ FinSeq-Locations) =
IExec(I, s_2)↾(Int-Locations ∪ FinSeq-Locations).

Let I be a macro instruction. Observe that Initialized(I) is initial.

One can prove the following propositions:

(47) Let s be a state of $\mathbf{SCM}_{\rm FSA}$ and I be a macro instruction. Then Initialized(I) is pseudo-closed on s if and only if I is pseudo-closed on Initialize(s).

(48) For every state s of $\mathbf{SCM}_{\rm FSA}$ and for every macro instruction I such that I is pseudo-closed on Initialize(s) holds
pseudo − LifeSpan(s, Initialized(I)) = pseudo − LifeSpan(Initialize(s), I).

(49) For every state s of $\mathbf{SCM}_{\rm FSA}$ and for every macro instruction I such that Initialized(I) is pseudo-closed on s holds
pseudo − LifeSpan(s, Initialized(I)) = pseudo − LifeSpan(Initialize(s), I).

(50) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be an initial finite partial state of $\mathbf{SCM}_{\mathrm{FSA}}$. Suppose I is pseudo-closed on s. Then I is pseudo-closed on $s{+}\cdot(I{+}\cdot\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))$ and $\mathrm{pseudo}-\mathrm{LifeSpan}(s,I) = \mathrm{pseudo}-\mathrm{LifeSpan}(s{+}\cdot(I{+}\cdot\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0))),I)$.

(51) Let s_1, s_2 be states of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a macro instruction. Suppose $I{+}\cdot\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)) \subseteq s_1$ and I is pseudo-closed on s_1. Let n be a natural number. Suppose $\mathrm{ProgramPart}(\mathrm{Relocated}(I,n)) \subseteq s_2$ and $\mathbf{IC}_{(s_2)} = \mathrm{insloc}(n)$ and $s_1\restriction(\mathrm{Int\text{-}Locations}\cup\mathrm{FinSeq\text{-}Locations}) = s_2\restriction(\mathrm{Int\text{-}Locations}\cup\mathrm{FinSeq\text{-}Locations})$. Then

(i) for every natural number i such that $i < \mathrm{pseudo}-\mathrm{LifeSpan}(s_1,I)$ holds $\mathrm{IncAddr}(\mathrm{CurInstr}((\mathrm{Computation}(s_1))(i)),n) =$ $\mathrm{CurInstr}((\mathrm{Computation}(s_2))(i))$, and

(ii) for every natural number i such that $i \leqslant \mathrm{pseudo}-\mathrm{LifeSpan}(s_1,I)$ holds $\mathbf{IC}_{(\mathrm{Computation}(s_1))(i)} + n = \mathbf{IC}_{(\mathrm{Computation}(s_2))(i)}$ and $(\mathrm{Computation}(s_1))(i)\restriction(\mathrm{Int\text{-}Locations}\cup\mathrm{FinSeq\text{-}Locations}) =$ $(\mathrm{Computation}(s_2))(i)\restriction(\mathrm{Int\text{-}Locations}\cup\mathrm{FinSeq\text{-}Locations})$.

(52) Let s_1, s_2 be states of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a macro instruction. Suppose $s_1\restriction(\mathrm{Int\text{-}Locations}\cup\mathrm{FinSeq\text{-}Locations}) =$ $s_2\restriction(\mathrm{Int\text{-}Locations}\cup\mathrm{FinSeq\text{-}Locations})$. If I is pseudo-closed on s_1, then I is pseudo-closed on s_2.

(53) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a macro instruction. Suppose $s(\mathrm{intloc}(0)) = 1$. Then I is pseudo-closed on s if and only if I is pseudo-closed on $\mathrm{Initialize}(s)$.

(54) Let a be an integer location and I, J be macro instructions. Then $\mathrm{insloc}(0) \in \mathrm{dom}\, if = 0(a,I,J)$ and $\mathrm{insloc}(1) \in \mathrm{dom}\, if = 0(a,I,J)$ and $\mathrm{insloc}(0) \in \mathrm{dom}\, if > 0(a,I,J)$ and $\mathrm{insloc}(1) \in \mathrm{dom}\, if > 0(a,I,J)$.

(55) Let a be an integer location and I, J be macro instructions. Then $(if = 0(a,I,J))(\mathrm{insloc}(0)) = \mathbf{if}\ a = 0\ \mathbf{goto}\ \mathrm{insloc}(\mathrm{card}\, J + 3)$ and $(if = 0(a,I,J))(\mathrm{insloc}(1)) = \mathrm{goto}\ \mathrm{insloc}(2)$ and $(if > 0(a,I,J))(\mathrm{insloc}(0)) = \mathbf{if}\ a > 0\ \mathbf{goto}\ \mathrm{insloc}(\mathrm{card}\, J + 3)$ and $(if > 0(a,I,J))(\mathrm{insloc}(1)) = \mathrm{goto}\ \mathrm{insloc}(2)$.

(56) Let a be an integer location, I, J be macro instructions, and n be a natural number. If $n < \mathrm{card}\, I + \mathrm{card}\, J + 3$, then $\mathrm{insloc}(n) \in \mathrm{dom}\, if = 0(a,I,J)$ and $(if = 0(a,I,J))(\mathrm{insloc}(n)) \neq \mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}}$.

(57) Let a be an integer location, I, J be macro instructions, and n be a natural number. If $n < \mathrm{card}\, I + \mathrm{card}\, J + 3$, then $\mathrm{insloc}(n) \in \mathrm{dom}\, if > 0(a,I,J)$ and $(if > 0(a,I,J))(\mathrm{insloc}(n)) \neq \mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}}$.

(58) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a macro instruction. Suppose $\mathrm{Directed}(I)$ is pseudo-closed on s. Then

(i) $I;\mathrm{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}$ is closed on s,

(ii) $I;\mathrm{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}$ is halting on s,

(iii) $\mathrm{LifeSpan}(s{+}\cdot((I;\mathrm{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}){+}\cdot\,\text{Start-At}(\mathrm{insloc}(0)))) =$ $\mathrm{pseudo} - \mathrm{LifeSpan}(s, \mathrm{Directed}(I))$,

(iv) for every natural number n such that $n < \mathrm{pseudo} - \mathrm{LifeSpan}(s, \mathrm{Directed}(I))$ holds $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot(I+\cdot\,\text{Start-At}(\mathrm{insloc}(0)))))(n)} =$ $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot((I;\mathrm{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}})+\cdot\,\text{Start-At}(\mathrm{insloc}(0)))))(n)}$, and

(v) for every natural number n such that $n \leqslant \mathrm{pseudo} - \mathrm{LifeSpan}(s, \mathrm{Directed}(I))$ holds $(\mathrm{Computation}(s{+}\cdot(I{+}\cdot\,\text{Start-At}(\mathrm{insloc}(0)))))(n){\restriction}D =$ $(\mathrm{Computation}(s{+}\cdot((I;\mathrm{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}){+}\cdot\,\text{Start-At}(\mathrm{insloc}(0)))))(n){\restriction}D$.

(59) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a macro instruction. If $\mathrm{Directed}(I)$ is pseudo-closed on s, then $\mathrm{Result}(s{+}\cdot((I;\mathrm{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}){+}\cdot\,\text{Start-At}(\mathrm{insloc}(0)))){\restriction}D =$ $(\mathrm{Computation}(s{+}\cdot(I{+}\cdot\,\text{Start-At}(\mathrm{insloc}(0)))))$ $(\mathrm{pseudo} - \mathrm{LifeSpan}(s, \mathrm{Directed}(I))){\restriction}D$.

(60) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a macro instruction. If $s(\mathrm{intloc}(0)) = 1$ and $\mathrm{Directed}(I)$ is pseudo-closed on s, then $\mathrm{IExec}(I;\mathrm{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}, s){\restriction}D = (\mathrm{Computation}(s{+}\cdot(I{+}\cdot\,\text{Start-At}(\mathrm{insloc}(0)))))$ $(\mathrm{pseudo} - \mathrm{LifeSpan}(s, \mathrm{Directed}(I))){\restriction}D$.

(61) For all macro instructions I, J and for every integer location a holds $(\mathit{if} = 0(a, I, J))(\mathrm{insloc}(\mathrm{card}\, I + \mathrm{card}\, J + 3)) = \mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}}$.

(62) For all macro instructions I, J and for every integer location a holds $(\mathit{if} > 0(a, I, J))(\mathrm{insloc}(\mathrm{card}\, I + \mathrm{card}\, J + 3)) = \mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}}$.

(63) For all macro instructions I, J and for every integer location a holds $(\mathit{if} = 0(a, I, J))(\mathrm{insloc}(\mathrm{card}\, J + 2)) = \mathrm{goto}\ \mathrm{insloc}(\mathrm{card}\, I + \mathrm{card}\, J + 3)$.

(64) For all macro instructions I, J and for every integer location a holds $(\mathit{if} > 0(a, I, J))(\mathrm{insloc}(\mathrm{card}\, J + 2)) = \mathrm{goto}\ \mathrm{insloc}(\mathrm{card}\, I + \mathrm{card}\, J + 3)$.

(65) For every macro instruction J and for every integer location a holds $(\mathit{if} = 0(a, \mathrm{Goto}(\mathrm{insloc}(2)), J))(\mathrm{insloc}(\mathrm{card}\, J + 3)) = \mathrm{goto}\ \mathrm{insloc}(\mathrm{card}\, J + 5)$.

(66) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I, J be macro instructions, and a be a read-write integer location. Suppose $s(a) = 0$ and $\mathrm{Directed}(I)$ is pseudo-closed on s. Then $\mathit{if} = 0(a, I, J)$ is halting on s and $\mathit{if} = 0(a, I, J)$ is closed on s and $\mathrm{LifeSpan}(s{+}\cdot(\mathit{if} = 0(a, I, J){+}\cdot\,\text{Start-At}(\mathrm{insloc}(0)))) =$ $\mathrm{LifeSpan}(s{+}\cdot((I;\mathrm{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}){+}\cdot\,\text{Start-At}(\mathrm{insloc}(0)))) + 1$.

(67) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I, J be macro instructions, and a be a read-write integer location. Suppose $s(\mathrm{intloc}(0)) = 1$ and $s(a) = 0$ and $\mathrm{Directed}(I)$ is pseudo-closed on s. Then $\mathrm{IExec}(\mathit{if} = 0(a, I, J), s){\restriction}(\text{Int-Locations} \cup \text{FinSeq-Locations}) = \mathrm{IExec}(I;\mathrm{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}, s)$ ${\restriction}(\text{Int-Locations} \cup \text{FinSeq-Locations})$.

(68) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I, J be macro instructions, and a be a read-write integer location. Suppose $s(a) > 0$ and Directed(I) is pseudo-closed on s. Then $if > 0(a, I, J)$ is halting on s and $if > 0(a, I, J)$ is closed on s and LifeSpan$(s{+}\cdot(if > 0(a, I, J){+}\cdot \text{Start-At}(\text{insloc}(0)))) =$ LifeSpan$(s{+}\cdot((I;\text{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}){+}\cdot \text{Start-At}(\text{insloc}(0)))) + 1$.

(69) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I, J be macro instructions, and a be a read-write integer location. Suppose $s(\text{intloc}(0)) = 1$ and $s(a) > 0$ and Directed(I) is pseudo-closed on s. Then IExec$(if > 0(a, I, J), s){\restriction}(\text{Int-Locations} \cup \text{FinSeq-Locations}) = \text{IExec}(I;\text{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}, s)$ ${\restriction}(\text{Int-Locations} \cup \text{FinSeq-Locations})$.

(70) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I, J be macro instructions, and a be a read-write integer location. Suppose $s(a) \neq 0$ and Directed(J) is pseudo-closed on s. Then $if = 0(a, I, J)$ is halting on s and $if = 0(a, I, J)$ is closed on s and LifeSpan$(s{+}\cdot(if = 0(a, I, J){+}\cdot \text{Start-At}(\text{insloc}(0)))) =$ LifeSpan$(s{+}\cdot((J;\text{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}){+}\cdot \text{Start-At}(\text{insloc}(0)))) + 3$.

(71) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I, J be macro instructions, and a be a read-write integer location. Suppose $s(\text{intloc}(0)) = 1$ and $s(a) \neq 0$ and Directed(J) is pseudo-closed on s. Then IExec$(if = 0(a, I, J), s){\restriction}(\text{Int-Locations} \cup \text{FinSeq-Locations}) = \text{IExec}(J;\text{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}, s)$ ${\restriction}(\text{Int-Locations} \cup \text{FinSeq-Locations})$.

(72) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I, J be macro instructions, and a be a read-write integer location. Suppose $s(a) \leqslant 0$ and Directed(J) is pseudo-closed on s. Then $if > 0(a, I, J)$ is halting on s and $if > 0(a, I, J)$ is closed on s and LifeSpan$(s{+}\cdot(if > 0(a, I, J){+}\cdot \text{Start-At}(\text{insloc}(0)))) =$ LifeSpan$(s{+}\cdot((J;\text{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}){+}\cdot \text{Start-At}(\text{insloc}(0)))) + 3$.

(73) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I, J be macro instructions, and a be a read-write integer location. Suppose $s(\text{intloc}(0)) = 1$ and $s(a) \leqslant 0$ and Directed(J) is pseudo-closed on s. Then IExec$(if > 0(a, I, J), s){\restriction}(\text{Int-Locations} \cup \text{FinSeq-Locations}) = \text{IExec}(J;\text{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}, s)$ ${\restriction}(\text{Int-Locations} \cup \text{FinSeq-Locations})$.

(74) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I, J be macro instructions, and a be a read-write integer location. Suppose Directed(I) is pseudo-closed on s and Directed(J) is pseudo-closed on s. Then $if = 0(a, I, J)$ is closed on s and $if = 0(a, I, J)$ is halting on s.

(75) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I, J be macro instructions, and a be a read-write integer location. Suppose Directed(I) is pseudo-closed on s and Directed(J) is pseudo-closed on s. Then $if > 0(a, I, J)$ is closed on s and $if > 0(a, I, J)$ is halting on s.

(76) Let I be a macro instruction and a be an integer location. If I does not destroy a, then Directed(I) does not destroy a.

(77) Let i be an instruction of $\mathbf{SCM}_{\rm FSA}$ and a be an integer location. If i does not destroy a, then Macro(i) does not destroy a.

(78) For every integer location a holds $\mathbf{halt}_{\mathbf{SCM}_{\rm FSA}}$ does not refer a.

(79) For all integer locations a, b, c such that $a \neq b$ holds AddTo(c, b) does not refer a.

(80) Let i be an instruction of $\mathbf{SCM}_{\rm FSA}$ and a be an integer location. If i does not refer a, then Macro(i) does not refer a.

(81) Let I, J be macro instructions and a be an integer location. Suppose I does not destroy a and J does not destroy a. Then I;J does not destroy a.

(82) Let J be a macro instruction, i be an instruction of $\mathbf{SCM}_{\rm FSA}$, and a be an integer location. Suppose i does not destroy a and J does not destroy a. Then i;J does not destroy a.

(83) Let I be a macro instruction, j be an instruction of $\mathbf{SCM}_{\rm FSA}$, and a be an integer location. Suppose I does not destroy a and j does not destroy a. Then I;j does not destroy a.

(84) Let i, j be instructions of $\mathbf{SCM}_{\rm FSA}$ and a be an integer location. Suppose i does not destroy a and j does not destroy a. Then i;j does not destroy a.

(85) For every integer location a holds $\mathrm{Stop}_{\mathrm{SCM}_{\rm FSA}}$ does not destroy a.

(86) For every integer location a and for every instruction-location l of $\mathbf{SCM}_{\rm FSA}$ holds Goto(l) does not destroy a.

(87) Let s be a state of $\mathbf{SCM}_{\rm FSA}$ and I be a macro instruction. Suppose I is halting on Initialize(s). Then

 (i) for every read-write integer location a holds $(\mathrm{IExec}(I, s))(a) =$ (Computation(Initialize(s)+$\cdot$(I+$\cdot$ Start-At(insloc(0))))) (LifeSpan(Initialize(s)+$\cdot$(I+$\cdot$ Start-At(insloc(0))))))(a), and

 (ii) for every finite sequence location f holds $(\mathrm{IExec}(I, s))(f) =$ (Computation(Initialize(s)+$\cdot$(I+$\cdot$ Start-At(insloc(0))))) (LifeSpan(Initialize(s)+$\cdot$(I+$\cdot$ Start-At(insloc(0))))))(f).

(88) Let s be a state of $\mathbf{SCM}_{\rm FSA}$, I be a parahalting macro instruction, and a be a read-write integer location. Then $(\mathrm{IExec}(I, s))(a) =$ (Computation(Initialize(s)+$\cdot$(I+$\cdot$ Start-At(insloc(0))))) (LifeSpan(Initialize(s)+$\cdot$(I+$\cdot$ Start-At(insloc(0))))))(a).

(89) Let s be a state of $\mathbf{SCM}_{\rm FSA}$, I be a macro instruction, a be an integer location, and k be a natural number. Suppose I is closed on Initialize(s) and halting on Initialize(s) and I does not destroy a. Then $(\mathrm{IExec}(I, s))(a) =$ (Computation(Initialize(s)+$\cdot$(I+$\cdot$ Start-At(insloc(0)))))(k)(a).

(90) Let s be a state of $\mathbf{SCM}_{\rm FSA}$, I be a parahalting macro instruction, a be an integer location, and k be a natural number. If I does not destroy a,

then $(\mathrm{IExec}(I,s))(a) =$
$(\mathrm{Computation}(\mathrm{Initialize}(s) +\cdot (I +\cdot \text{Start-At}(\mathrm{insloc}(0)))))(k)(a)$.

(91) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I be a parahalting macro instruction, and a be an integer location. If I does not destroy a, then $(\mathrm{IExec}(I,s))(a) = (\mathrm{Initialize}(s))(a)$.

(92) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a keeping 0 macro instruction. Suppose I is halting on $\mathrm{Initialize}(s)$. Then $(\mathrm{IExec}(I,s))(\mathrm{intloc}(0)) = 1$ and for every natural number k holds $(\mathrm{Computation}(\mathrm{Initialize}(s) +\cdot (I +\cdot \text{Start-At}(\mathrm{insloc}(0)))))(k)(\mathrm{intloc}(0)) = 1$.

(93) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I be a macro instruction, and a be an integer location. Suppose I does not destroy a. Let k be a natural number. If $\mathbf{IC}_{(\mathrm{Computation}(s +\cdot (I +\cdot \text{Start-At}(\mathrm{insloc}(0)))))(k)} \in \mathrm{dom}\, I$, then $(\mathrm{Computation}(s +\cdot (I +\cdot \text{Start-At}(\mathrm{insloc}(0)))))(k+1)(a) = (\mathrm{Computation}(s +\cdot (I +\cdot \text{Start-At}(\mathrm{insloc}(0)))))(k)(a)$.

(94) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I be a macro instruction, and a be an integer location. Suppose I does not destroy a. Let m be a natural number. Suppose that for every natural number n such that $n < m$ holds $\mathbf{IC}_{(\mathrm{Computation}(s +\cdot (I +\cdot \text{Start-At}(\mathrm{insloc}(0)))))(n)} \in \mathrm{dom}\, I$. Let n be a natural number. If $n \leqslant m$, then
$(\mathrm{Computation}(s +\cdot (I +\cdot \text{Start-At}(\mathrm{insloc}(0)))))(n)(a) = s(a)$.

(95) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I be a good macro instruction, and m be a natural number. Suppose that for every natural number n such that $n < m$ holds $\mathbf{IC}_{(\mathrm{Computation}(s +\cdot (I +\cdot \text{Start-At}(\mathrm{insloc}(0)))))(n)} \in \mathrm{dom}\, I$. Let n be a natural number. If $n \leqslant m$, then $(\mathrm{Computation}(s +\cdot (I +\cdot \text{Start-At}(\mathrm{insloc}(0)))))(n)(\mathrm{intloc}(0)) = s(\mathrm{intloc}(0))$.

(96) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a good macro instruction. Suppose I is halting on $\mathrm{Initialize}(s)$ and closed on $\mathrm{Initialize}(s)$. Then $(\mathrm{IExec}(I,s))(\mathrm{intloc}(0)) = 1$ and for every natural number k holds $(\mathrm{Computation}(\mathrm{Initialize}(s) +\cdot (I +\cdot \text{Start-At}(\mathrm{insloc}(0)))))(k)(\mathrm{intloc}(0)) = 1$.

(97) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a good macro instruction. Suppose I is closed on s. Let k be a natural number. Then $(\mathrm{Computation}(s +\cdot (I +\cdot \text{Start-At}(\mathrm{insloc}(0)))))(k)(\mathrm{intloc}(0)) = s(\mathrm{intloc}(0))$.

(98) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I be a keeping 0 parahalting macro instruction, and a be a read-write integer location. Suppose I does not destroy a. Then $(\mathrm{Computation}(\mathrm{Initialize}(s) +\cdot ((I;\mathrm{SubFrom}(a,\mathrm{intloc}(0))) +\cdot \text{Start-At}(\mathrm{insloc}(0)))))(\mathrm{LifeSpan}(\mathrm{Initialize}(s) +\cdot ((I;\mathrm{SubFrom}(a,\mathrm{intloc}(0))) +\cdot \text{Start-At}(\mathrm{insloc}(0)))))(a) = s(a) - 1$.

(99) For every instruction i of $\mathbf{SCM}_{\mathrm{FSA}}$ such that i does not destroy $\mathrm{intloc}(0)$ holds $\mathrm{Macro}(i)$ is good.

(100) Let s_1, s_2 be states of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a macro instruction. Suppose I is closed on s_1 and halting on s_1 and $s_1 \restriction D = s_2 \restriction D$. Let k be a natural number. Then

(i) (Computation(s_1+·(I+· Start-At(insloc(0)))))(k) and (Computation(s_2+·(I+· Start-At(insloc(0)))))(k) are equal outside the instruction locations of $\mathbf{SCM}_{\mathrm{FSA}}$, and

(ii) CurInstr((Computation(s_1+·(I+· Start-At(insloc(0)))))(k)) = CurInstr((Computation(s_2+·(I+· Start-At(insloc(0)))))(k)).

(101) Let s_1, s_2 be states of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a macro instruction. Suppose I is closed on s_1 and halting on s_1 and $s_1 \restriction D = s_2 \restriction D$. Then LifeSpan($s_1$+·($I$+· Start-At(insloc(0)))) = LifeSpan(s_2+·(I+· Start-At(insloc(0)))) and Result(s_1+·(I+· Start-At(insloc(0)))) and Result(s_2+·(I+· Start-At (insloc(0)))) are equal outside the instruction locations of $\mathbf{SCM}_{\mathrm{FSA}}$.

(102) Let N be a non empty set with non empty elements, S be a steady-programmed von Neumann definite AMI over N, and s be a state of S. Suppose s is halting. Then there exists a natural number k such that s halts at $\mathbf{IC}_{(\mathrm{Computation}(s))(k)}$.

(103) Let s_1, s_2 be states of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a macro instruction. Suppose that

(i) I is closed on s_1 and halting on s_1,

(ii) I+· Start-At(insloc(0)) $\subseteq s_1$,

(iii) I+· Start-At(insloc(0)) $\subseteq s_2$, and

(iv) there exists a natural number k such that (Computation(s_1))(k) and s_2 are equal outside the instruction locations of $\mathbf{SCM}_{\mathrm{FSA}}$.

Then Result(s_1) and Result(s_2) are equal outside the instruction locations of $\mathbf{SCM}_{\mathrm{FSA}}$.

2. The loop Macroinstruction

Let I be a macro instruction and let k be a natural number. One can verify that IncAddr(I, k) is initial and programmed.

Let I be a macro instruction. The functor loop I yields a halt-free macro instruction and is defined by:

(Def. 4) $\mathrm{loop}\, I = (\mathrm{id}_{\text{the instructions of } \mathbf{SCM}_{\mathrm{FSA}}} +\cdot (\mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}} \mapsto \mathrm{goto\ insloc}(0))) \cdot I$.

Next we state two propositions:

(104) For every macro instruction I holds loop I = Directed(I, insloc(0)).

(105) Let I be a macro instruction and a be an integer location. If I does not destroy a, then loop I does not destroy a.

Let I be a good macro instruction. One can verify that loop I is good.

The following propositions are true:

(106) For every macro instruction I holds $\mathrm{dom}\,\mathrm{loop}\,I = \mathrm{dom}\,I$.

(107) For every macro instruction I holds $\mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}} \notin \mathrm{rng}\,\mathrm{loop}\,I$.

(108) For every macro instruction I and for every set x such that $x \in \mathrm{dom}\,I$ holds if $I(x) \neq \mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}}$, then $(\mathrm{loop}\,I)(x) = I(x)$.

(109) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a macro instruction. Suppose I is closed on s and halting on s. Let m be a natural number. Suppose $m \leqslant \mathrm{LifeSpan}(s{+\cdot}(I{+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0))))$. Then $(\mathrm{Computation}(s{+\cdot}(I{+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))(m)$ and $(\mathrm{Computation}(s{+\cdot}(\mathrm{loop}\,I{+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))(m)$ are equal outside the instruction locations of $\mathbf{SCM}_{\mathrm{FSA}}$.

(110) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a macro instruction. Suppose I is closed on s and halting on s. Let m be a natural number. If $m < \mathrm{LifeSpan}(s{+\cdot}(I{+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0))))$, then $\mathrm{CurInstr}((\mathrm{Computation}(s{+\cdot}(I{+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))(m)) = \mathrm{CurInstr}((\mathrm{Computation}(s{+\cdot}(\mathrm{loop}\,I{+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))(m))$.

(111) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a macro instruction. Suppose I is closed on s and halting on s. Let m be a natural number. If $m \leqslant \mathrm{LifeSpan}(s{+\cdot}(I{+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0))))$, then $\mathrm{CurInstr}((\mathrm{Computation}(s{+\cdot}(\mathrm{loop}\,I{+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))(m)) \neq \mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}}$.

(112) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a macro instruction. If I is closed on s and halting on s, then $\mathrm{CurInstr}((\mathrm{Computation}(s{+\cdot}(\mathrm{loop}\,I{+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))(\mathrm{LifeSpan}(s{+\cdot}(I{+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))) = \mathrm{goto}\ \mathrm{insloc}(0)$.

(113) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a paraclosed macro instruction. Suppose $I{+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)) \subseteq s$ and s is halting. Let m be a natural number. Suppose $m \leqslant \mathrm{LifeSpan}(s)$. Then $(\mathrm{Computation}(s))(m)$ and $(\mathrm{Computation}(s{+\cdot}\,\mathrm{loop}\,I))(m)$ are equal outside the instruction locations of $\mathbf{SCM}_{\mathrm{FSA}}$.

(114) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$ and I be a parahalting macro instruction. Suppose $\mathrm{Initialized}(I) \subseteq s$. Let k be a natural number. If $k \leqslant \mathrm{LifeSpan}(s)$, then $\mathrm{CurInstr}((\mathrm{Computation}(s{+\cdot}\,\mathrm{loop}\,I))(k)) \neq \mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}}$.

3. The Times Macroinstruction

Let a be an integer location and let I be a macro instruction. The functor $\mathrm{Times}(a, I)$ yields a macro instruction and is defined by:

(Def. 5) $\mathrm{Times}(a, I) = if > 0(a, \mathrm{loop}\, if = 0(a, \mathrm{Goto}(\mathrm{insloc}(2)), I;\mathrm{SubFrom}(a, \mathrm{intloc}(0))), \mathrm{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}})$.

The following propositions are true:

(115) For every good macro instruction I and for every read-write integer location a holds $if = 0(a, \mathrm{Goto}(\mathrm{insloc}(2)), I;\mathrm{SubFrom}(a, \mathrm{intloc}(0)))$ is good.

(116) For all macro instructions I, J and for every integer location a holds $(if = 0(a, \mathrm{Goto}(\mathrm{insloc}(2)), I;\mathrm{SubFrom}(a, \mathrm{intloc}(0))))$ $(\mathrm{insloc}(\mathrm{card}(I;\mathrm{SubFrom}(a, \mathrm{intloc}(0))) + 3)) = \mathrm{goto}$ $\mathrm{insloc}(\mathrm{card}(I;\mathrm{SubFrom}(a, \mathrm{intloc}(0))) + 5)$.

(117) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I be a good parahalting macro instruction, and a be a read-write integer location. Suppose I does not destroy a and $s(\mathrm{intloc}(0)) = 1$ and $s(a) > 0$. Then $\mathrm{loop}\, if = 0(a, \mathrm{Goto}(\mathrm{insloc}(2)), I;\mathrm{SubFrom}(a, \mathrm{intloc}(0)))$ is pseudo-closed on s.

(118) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I be a good parahalting macro instruction, and a be a read-write integer location. Suppose I does not destroy a and $s(a) > 0$. Then $\mathrm{Initialized}(\mathrm{loop}\, if = 0(a, \mathrm{Goto}(\mathrm{insloc}(2)), I;\mathrm{SubFrom}(a, \mathrm{intloc}(0))))$ is pseudo-closed on s.

(119) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I be a good parahalting macro instruction, and a be a read-write integer location. Suppose I does not destroy a and $s(\mathrm{intloc}(0)) = 1$. Then $\mathrm{Times}(a, I)$ is closed on s and $\mathrm{Times}(a, I)$ is halting on s.

(120) Let I be a good parahalting macro instruction and a be a read-write integer location. If I does not destroy a, then $\mathrm{Initialized}(\mathrm{Times}(a, I))$ is halting.

(121) Let I, J be macro instructions and a, c be integer locations. Suppose I does not destroy c and J does not destroy c. Then $if = 0(a, I, J)$ does not destroy c and $if > 0(a, I, J)$ does not destroy c.

(122) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I be a good parahalting macro instruction, and a be a read-write integer location. Suppose I does not destroy a and $s(\mathrm{intloc}(0)) = 1$ and $s(a) > 0$. Then there exists a state s_2 of $\mathbf{SCM}_{\mathrm{FSA}}$ and there exists a natural number k such that

(i) $s_2 = s{+\cdot}(\mathrm{loop}\, if = 0(a, \mathrm{Goto}(\mathrm{insloc}(2)), I;\mathrm{SubFrom}(a, \mathrm{intloc}(0))) {+\cdot}\, \mathrm{Start\text{-}At}(\mathrm{insloc}(0)))$,

(ii) $k = \mathrm{LifeSpan}(s{+\cdot}(if = 0(a, \mathrm{Goto}(\mathrm{insloc}(2)), I;\mathrm{SubFrom}(a, \mathrm{intloc}(0))) {+\cdot}\, \mathrm{Start\text{-}At}(\mathrm{insloc}(0)))) + 1$,

(iii) $(\mathrm{Computation}(s_2))(k)(a) = s(a) - 1$,

(iv) $(\mathrm{Computation}(s_2))(k)(\mathrm{intloc}(0)) = 1$,

(v) for every read-write integer location b such that $b \neq a$ holds $(\mathrm{Computation}(s_2))(k)(b) = (\mathrm{IExec}(I, s))(b)$,

(vi) for every finite sequence location f holds $(\mathrm{Computation}(s_2))(k)(f) = (\mathrm{IExec}(I,s))(f)$,

(vii) $\mathbf{IC}_{(\mathrm{Computation}(s_2))(k)} = \mathrm{insloc}(0)$, and

(viii) for every natural number n such that $n \leqslant k$ holds $\mathbf{IC}_{(\mathrm{Computation}(s_2))(n)} \in \mathrm{dom\,loop}\, \mathit{if} = 0(a, \mathrm{Goto}(\mathrm{insloc}(2)), I;\mathrm{SubFrom}(a, \mathrm{intloc}(0)))$.

(123) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I be a good parahalting macro instruction, and a be a read-write integer location. If $s(\mathrm{intloc}(0)) = 1$ and $s(a) \leqslant 0$, then $\mathrm{IExec}(\mathrm{Times}(a,I),s) \restriction (\mathrm{Int\text{-}Locations} \cup \mathrm{FinSeq\text{-}Locations}) = s \restriction (\mathrm{Int\text{-}Locations} \cup \mathrm{FinSeq\text{-}Locations})$.

(124) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I be a good parahalting macro instruction, and a be a read-write integer location. Suppose I does not destroy a and $s(a) > 0$. Then $(\mathrm{IExec}(I;\mathrm{SubFrom}(a, \mathrm{intloc}(0)), s))(a) = s(a) - 1$ and $\mathrm{IExec}(\mathrm{Times}(a,I),s) \restriction (\mathrm{Int\text{-}Locations} \cup \mathrm{FinSeq\text{-}Locations}) = \mathrm{IExec}(\mathrm{Times}(a,I), \mathrm{IExec}(I;\mathrm{SubFrom}(a,\mathrm{intloc}(0)),s)) \restriction (\mathrm{Int\text{-}Locations} \cup \mathrm{FinSeq\text{-}Locations})$.

4. An example

One can prove the following proposition

(125) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$ and a, b, c be read-write integer locations. If $a \neq b$ and $a \neq c$ and $b \neq c$ and $s(a) \geqslant 0$, then $(\mathrm{IExec}(\mathrm{Times}(a, \mathrm{Macro}(\mathrm{AddTo}(b,c))), s))(b) = s(b) + s(c) \cdot s(a)$.

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Algebraic and Arithmetic Lattices. Part II[1]

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Summary. The article is a translation of [13, pp. 89–92]

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The articles [21], [22], [1], [8], [9], [12], [20], [19], [18], [3], [11], [17], [2], [4], [14], [24], [6], [5], [10], [7], [23], [15], and [16] provide the notation and terminology for this paper.

1. Preliminaries

The following propositions are true:

(1) Let R be a relational structure and S be a full relational substructure of R. Then every full relational substructure of S is a full relational substructure of R.

(2) Let X, Y, Z be non empty 1-sorted structures, f be a map from X into Y, and g be a map from Y into Z. If f is onto and g is onto, then $g \cdot f$ is onto.

(3) For every non empty 1-sorted structure X and for every subset Y of the carrier of X holds $(\mathrm{id}_X)^\circ Y = Y$.

(4) For every set X and for every element a of $2^X_\subseteq$ holds $\uparrow a = \{Y; Y$ ranges over subsets of X: $a \subseteq Y\}$.

(5) Let L be an upper-bounded non empty antisymmetric relational structure and a be an element of L. If $\top_L \leqslant a$, then $a = \top_L$.

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(6) Let S, T be non empty posets, g be a map from S into T, and d be a map from T into S. If g is onto and $\langle g, d\rangle$ is Galois, then T and $\mathrm{Im}\, d$ are isomorphic.

(7) Let L_1, L_2, L_3 be non empty posets, g_1 be a map from L_1 into L_2, g_2 be a map from L_2 into L_3, d_1 be a map from L_2 into L_1, and d_2 be a map from L_3 into L_2. If $\langle g_1, d_1\rangle$ is Galois and $\langle g_2, d_2\rangle$ is Galois, then $\langle g_2 \cdot g_1, d_1 \cdot d_2\rangle$ is Galois.

(8) Let L_1, L_2 be non empty posets, f be a map from L_1 into L_2, and f_1 be a map from L_2 into L_1. Suppose $f_1 = (f$ **qua** function$)^{-1}$ and f is isomorphic. Then $\langle f, f_1\rangle$ is Galois and $\langle f_1, f\rangle$ is Galois.

(9) For every set X holds $2^X_\subseteq$ is arithmetic.

Next we state four propositions:

(10) Let L_1, L_2 be up-complete non empty posets and f be a map from L_1 into L_2. If f is isomorphic, then for every element x of L_1 holds $f^\circ{\downarrow}x = {\downarrow}f(x)$.

(11) For all non empty posets L_1, L_2 such that L_1 and L_2 are isomorphic and L_1 is continuous holds L_2 is continuous.

(12) Let L_1, L_2 be lattices. Suppose L_1 and L_2 are isomorphic and L_1 is lower-bounded and arithmetic. Then L_2 is arithmetic.

(13) Let L_1, L_2, L_3 be non empty posets, f be a map from L_1 into L_2, and g be a map from L_2 into L_3. Suppose f is directed-sups-preserving and g is directed-sups-preserving. Then $g \cdot f$ is directed-sups-preserving.

2. Maps Preserving Sup's and Inf's

One can prove the following propositions:

(14) Let L_1, L_2 be non empty relational structures, f be a map from L_1 into L_2, and X be a subset of $\mathrm{Im}\, f$. Then $(f_\circ)^\circ X = X$.

(15) Let X be a set and c be a map from $2^X_\subseteq$ into $2^X_\subseteq$. Suppose c is idempotent and directed-sups-preserving. Then $c_\circ$ is directed-sups-preserving.

(16) Let L be a continuous complete lattice and p be a kernel map from L into L. If p is directed-sups-preserving, then $\mathrm{Im}\, p$ is a continuous lattice.

(17) Let L be a continuous complete lattice and p be a projection map from L into L. If p is directed-sups-preserving, then $\mathrm{Im}\, p$ is a continuous lattice.

(18) Let L be a lower-bounded lattice. Then L is continuous if and only if there exists an arithmetic lower-bounded lattice A such that there exists a map from A into L which is onto, infs-preserving, and directed-sups-preserving.

(19) Let L be a lower-bounded lattice. Then L is continuous if and only if there exists an algebraic lower-bounded lattice A such that there exists

a map from A into L which is onto, infs-preserving, and directed-sups-preserving.

(20) Let L be a lower-bounded lattice. Then L is continuous if and only if there exists a set X and there exists a projection map p from $2^X_\subseteq$ into $2^X_\subseteq$ such that p is directed-sups-preserving and L and $\mathrm{Im}\, p$ are isomorphic.

3. Atoms Elements

Next we state two propositions:

(21) For every non empty relational structure L and for every element x of L holds $x \in \mathrm{PRIME}(L^{\mathrm{op}})$ iff x is co-prime.

(22) Let L be a sup-semilattice and a be an element of L. Then a is co-prime if and only if for all elements x, y of L such that $a \leqslant x \sqcup y$ holds $a \leqslant x$ or $a \leqslant y$.

Let L be a non empty relational structure and let a be an element of L. We say that a is an atom if and only if:

(Def. 1) $\bot_L < a$ and for every element b of L such that $\bot_L < b$ and $b \leqslant a$ holds $b = a$.

Let L be a non empty relational structure. The functor $\mathrm{ATOM}(L)$ yielding a subset of L is defined by:

(Def. 2) For every element x of L holds $x \in \mathrm{ATOM}(L)$ iff x is atom.

The following proposition is true

(23) For every Boolean lattice L and for every element a of L holds a is atom iff a is co-prime and $a \neq \bot_L$.

Let L be a Boolean lattice. Observe that every element of L which is atom is also co-prime.

Next we state several propositions:

(24) For every Boolean lattice L holds $\mathrm{ATOM}(L) = \mathrm{PRIME}(L^{\mathrm{op}}) \setminus \{\bot_L\}$.

(25) For every Boolean lattice L and for all elements x, a of L such that a is atom holds $a \leqslant x$ iff $a \not\leqslant \neg x$.

(26) Let L be a complete Boolean lattice, X be a subset of L, and x be an element of L. Then $x \sqcap \sup X = \bigsqcup_L \{x \sqcap y; y$ ranges over elements of L: $y \in X\}$.

(27) Let L be a lower-bounded antisymmetric non empty relational structure with g.l.b.'s and x, y be elements of L. If x is atom and y is atom and $x \neq y$, then $x \sqcap y = \bot_L$.

(28) Let L be a complete Boolean lattice, x be an element of L, and A be a subset of L. If $A \subseteq \mathrm{ATOM}(L)$, then $x \in A$ iff x is atom and $x \leqslant \sup A$.

(29) Let L be a complete Boolean lattice and X, Y be subsets of L. If $X \subseteq \mathrm{ATOM}(L)$ and $Y \subseteq \mathrm{ATOM}(L)$, then $X \subseteq Y$ iff $\sup X \leqslant \sup Y$.

4. More on the Boolean Lattice

One can prove the following propositions:

(30) For every Boolean lattice L holds L is arithmetic iff there exists a set X such that L and $2^X_\subseteq$ are isomorphic.

(31) For every Boolean lattice L holds L is arithmetic iff L is algebraic.

(32) For every Boolean lattice L holds L is arithmetic iff L is continuous.

(33) For every Boolean lattice L holds L is arithmetic iff L is continuous and L^{op} is continuous.

(34) For every Boolean lattice L holds L is arithmetic iff L is completely-distributive.

(35) Let L be a Boolean lattice. Then L is arithmetic if and only if the following conditions are satisfied:

(i) L is complete, and

(ii) for every element x of L there exists a subset X of L such that $X \subseteq \mathrm{ATOM}(L)$ and $x = \sup X$.

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Projections in n-Dimensional Euclidean Space to Each Coordinates

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Summary. In the n-dimensional Euclidean space $\mathcal{E}^n_{\rm T}$, a projection operator to each coordinate is defined. It is proven that such an operator is linear. Moreover, it is continuous as a mapping from $\mathcal{E}^n_{\rm T}$ to R^1, the carrier of which is a set of all reals. If n is 1, the projection becomes a homeomorphism, which means that $\mathcal{E}^1_{\rm T}$ is homeomorphic to R^1.

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The notation and terminology used in this paper are introduced in the following articles: [30], [35], [34], [20], [1], [37], [33], [27], [12], [29], [11], [26], [23], [36], [2], [8], [9], [5], [32], [3], [18], [17], [25], [15], [10], [14], [31], [16], [19], [22], [7], [24], [13], [21], [4], [6], and [28].

1. Projections

For simplicity, we use the following convention: a, b, s, s_1, r, r_1, r_2 denote real numbers, n, i denote natural numbers, X denotes a non empty topological space, p, p_1, p_2, q denote points of $\mathcal{E}^n_{\rm T}$, P denotes a subset of the carrier of $\mathcal{E}^n_{\rm T}$, and f denotes a map from $\mathcal{E}^n_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$.

Let n, i be natural numbers and let p be an element of the carrier of $\mathcal{E}^n_{\rm T}$. The functor $\mathrm{Proj}(p,i)$ yielding a real number is defined as follows:

(Def. 1) For every finite sequence g of elements of $\mathbb{R}$ such that $g = p$ holds $\mathrm{Proj}(p,i) = \pi_i g$.

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The following propositions are true:

(1) For every i there exists a map f from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ such that for every element p of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ holds $f(p) = \mathrm{Proj}(p, i)$.

(2) For every i such that $i \in \mathrm{Seg}\, n$ holds $\langle \underbrace{0, \ldots, 0}_{n} \rangle(i) = 0$.

(3) For every i such that $i \in \mathrm{Seg}\, n$ holds $\mathrm{Proj}(0_{\mathcal{E}^n_{\mathrm{T}}}, i) = 0$.

(4) For all r, p, i such that $i \in \mathrm{Seg}\, n$ holds $\mathrm{Proj}(r \cdot p, i) = r \cdot \mathrm{Proj}(p, i)$.

(5) For all p, i such that $i \in \mathrm{Seg}\, n$ holds $\mathrm{Proj}(-p, i) = -\mathrm{Proj}(p, i)$.

(6) For all p_1, p_2, i such that $i \in \mathrm{Seg}\, n$ holds $\mathrm{Proj}(p_1 + p_2, i) = \mathrm{Proj}(p_1, i) + \mathrm{Proj}(p_2, i)$.

(7) For all p_1, p_2, i such that $i \in \mathrm{Seg}\, n$ holds $\mathrm{Proj}(p_1 - p_2, i) = \mathrm{Proj}(p_1, i) - \mathrm{Proj}(p_2, i)$.

(8) $\mathrm{len} \langle \underbrace{0, \ldots, 0}_{n} \rangle = n$.

(9) For every i such that $i \leqslant n$ holds $\langle \underbrace{0, \ldots, 0}_{n} \rangle \restriction i = \langle \underbrace{0, \ldots, 0}_{i} \rangle$.

(10) For every i holds $\langle \underbrace{0, \ldots, 0}_{n} \rangle_{\lfloor i} = \langle \underbrace{0, \ldots, 0}_{n -' i} \rangle$.

(11) For every i holds $\sum \langle \underbrace{0, \ldots, 0}_{i} \rangle = 0$.

(12) For every finite sequence w and for all r, i holds $\mathrm{len}(w +\cdot (i, r)) = \mathrm{len}\, w$.

(13) For every finite sequence w of elements of $\mathbb{R}$ and for all r, i such that $i \in \mathrm{Seg}\, \mathrm{len}\, w$ holds $w +\cdot (i, r) = (w \restriction i -' 1) \frown \langle r \rangle \frown (w_{\lfloor i})$.

(14) For all i, r such that $i \in \mathrm{Seg}\, n$ holds $\sum(\langle \underbrace{0, \ldots, 0}_{n} \rangle +\cdot (i, r)) = r$.

(15) For every element q of $\mathcal{R}^n$ and for all p, i such that $i \in \mathrm{Seg}\, n$ and $q = p$ holds $\mathrm{Proj}(p, i) \leqslant |q|$ and $(\mathrm{Proj}(p, i))^{\mathbf{2}} \leqslant |q|^{\mathbf{2}}$.

2. Continuity of Projections

Next we state several propositions:

(16) For all s_1, P, i such that $P = \{p : s_1 > \mathrm{Proj}(p, i)\}$ and $i \in \mathrm{Seg}\, n$ holds P is open.

(17) For all s_1, P, i such that $P = \{p : s_1 < \mathrm{Proj}(p, i)\}$ and $i \in \mathrm{Seg}\, n$ holds P is open.

(18) Let P be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, a, b be real numbers, and given i. Suppose $P = \{p; p$ ranges over elements of the carrier of $\mathcal{E}^n_{\mathrm{T}}$: $a < \mathrm{Proj}(p, i) \ \wedge\ \mathrm{Proj}(p, i) < b\}$ and $i \in \mathrm{Seg}\, n$. Then P is open.

(19) Let a, b be real numbers, f be a map from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$, and given i. Suppose that for every element p of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ holds $f(p) = \mathrm{Proj}(p, i)$. Then $f^{-1}(\{s : a < s \ \wedge \ s < b\}) = \{p; p$ ranges over elements of the carrier of $\mathcal{E}^n_{\mathrm{T}}$: $a < \mathrm{Proj}(p, i) \ \wedge \ \mathrm{Proj}(p, i) < b\}$.

(20) Let M be a metric space and f be a map from X into M_{top}. Suppose that for every real number r and for every element u of the carrier of M and for every subset P of the carrier of M_{top} such that $r > 0$ and $P = \mathrm{Ball}(u, r)$ holds $f^{-1}(P)$ is open. Then f is continuous.

(21) Let u be a point of the metric space of real numbers and r, u_1 be real numbers. If $u_1 = u$ and $r > 0$, then $\mathrm{Ball}(u, r) = \{s : u_1 - r < s \ \wedge \ s < u_1 + r\}$.

(22) Let f be a map from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ and given i. Suppose $i \in \mathrm{Seg}\, n$ and for every element p of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ holds $f(p) = \mathrm{Proj}(p, i)$. Then f is continuous.

3. 1-dimensional and 2-dimensional Cases

The following three propositions are true:

(23) For every s holds $|\langle s \rangle| = \langle |s| \rangle$.

(24) For every element p of the carrier of $\mathcal{E}^1_{\mathrm{T}}$ there exists r such that $p = \langle r \rangle$.

(25) For every element w of the carrier of $\mathcal{E}^1$ there exists r such that $w = \langle r \rangle$.

Let us consider r. The functor $|[r]|$ yields a point of $\mathcal{E}^1_{\mathrm{T}}$ and is defined by:

(Def. 2) $|[r]| = \langle r \rangle$.

The following propositions are true:

(26) For all r, s holds $s \cdot |[r]| = |[s \cdot r]|$.

(27) For all r_1, r_2 holds $|[r_1 + r_2]| = |[r_1]| + |[r_2]|$.

(28) $|[0]| = 0_{\mathcal{E}^1_{\mathrm{T}}}$.

(29) For all r_1, r_2 such that $|[r_1]| = |[r_2]|$ holds $r_1 = r_2$.

(30) For every subset P of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for every real number b such that $P = \{s : s < b\}$ holds P is open.

(31) For every subset P of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $P = \{s : a < s\}$ holds P is open.

(32) For every subset P of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $P = \{s : a < s \ \wedge \ s < b\}$ holds P is open.

(33) For every point u of $\mathcal{E}^1$ and for all real numbers r, u_1 such that $\langle u_1 \rangle = u$ and $r > 0$ holds $\mathrm{Ball}(u, r) = \{\langle s \rangle : u_1 - r < s \ \wedge \ s < u_1 + r\}$.

(34) Let f be a map from $\mathcal{E}^1_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that for every element p of the carrier of $\mathcal{E}^1_{\mathrm{T}}$ holds $f(p) = \mathrm{Proj}(p, 1)$. Then f is a homeomorphism.

(35) For every element p of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{Proj}(p,1) = p_{\mathbf{1}}$ and $\mathrm{Proj}(p,2) = p_{\mathbf{2}}$.

(36) For every element p of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{Proj}(p,1) = (\mathrm{proj1})(p)$ and $\mathrm{Proj}(p,2) = (\mathrm{proj2})(p)$.

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Intermediate Value Theorem and Thickness of Simple Closed Curves

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Summary. Various types of the intermediate value theorem ([25]) are proved. For their special cases, the Bolzano theorem is also proved. Using such a theorem, it is shown that if a curve is a simple closed curve, then it is not horizontally degenerated, neither is it vertically degenerated.

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The articles [29], [33], [28], [16], [1], [27], [34], [6], [7], [4], [8], [32], [22], [35], [11], [10], [24], [2], [5], [31], [17], [3], [12], [13], [14], [15], [18], [19], [21], [26], [23], [30], [9], and [20] provide the notation and terminology for this paper.

1. Intermediate Value Theorems and Bolzano Theorem

For simplicity, we adopt the following convention: a, b, c, d, r_1, r_2, r_3, r, r_4, s_1, s_2 are real numbers, p, q are points of $\mathcal{E}^2_{\mathrm{T}}$, P is a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, and X, Y, Z are non empty topological spaces.

Next we state a number of propositions:

(1) For all a, b, c holds $c \in [a, b]$ iff $a \leqslant c$ and $c \leqslant b$.

(2) Let f be a continuous mapping from X into Y and g be a continuous mapping from Y into Z. Then $g \cdot f$ is a continuous mapping from X into Z.

(3) Let A, B be subsets of the carrier of X. If A is open and B is open and $A \cap B = \emptyset_X$, then A and B are separated.

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(4) Let A, B_1, B_2 be subsets of the carrier of X. Suppose B_1 is open and B_2 is open and $B_1 \cap A \neq \emptyset$ and $B_2 \cap A \neq \emptyset$ and $A \subseteq B_1 \cup B_2$ and $B_1 \cap B_2 = \emptyset$. Then A is not connected.

(5) Let f be a continuous mapping from X into Y and A be a subset of the carrier of X. If A is connected and $A \neq \emptyset$, then $f^\circ A$ is connected.

(6) For all r_1, r_2 such that $r_1 \leqslant r_2$ holds $\Omega_{[(r_1),\, r_2]_{\mathrm{T}}}$ is connected.

(7) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for every a such that $A = \{r : a < r\}$ holds A is open.

(8) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for every a such that $A = \{r : a > r\}$ holds A is open.

(9) Let A be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$ and given a. Suppose $a \notin A$ and there exist b, d such that $b \in A$ and $d \in A$ and $b < a$ and $a < d$. Then A is not connected.

(10) Let X be a non empty topological space, x_1, x_2 be points of X, a, b, d be real numbers, and f be a continuous mapping from X into $\mathbb{R}^{\mathbf{1}}$. Suppose X is connected and $f(x_1) = a$ and $f(x_2) = b$ and $a \leqslant d$ and $d \leqslant b$. Then there exists a point x_3 of X such that $f(x_3) = d$.

(11) Let X be a non empty topological space, x_1, x_2 be points of X, B be a subset of the carrier of X, a, b, d be real numbers, and f be a continuous mapping from X into $\mathbb{R}^{\mathbf{1}}$. Suppose B is connected and $f(x_1) = a$ and $f(x_2) = b$ and $a \leqslant d$ and $d \leqslant b$ and $x_1 \in B$ and $x_2 \in B$. Then there exists a point x_3 of X such that $x_3 \in B$ and $f(x_3) = d$.

(12) Let given r_1, r_2, a, b. Suppose $r_1 < r_2$. Let f be a continuous mapping from $[(r_1),\, r_2]_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ and given d. Suppose $f(r_1) = a$ and $f(r_2) = b$ and $a < d$ and $d < b$. Then there exists r_3 such that $f(r_3) = d$ and $r_1 < r_3$ and $r_3 < r_2$.

(13) Let given r_1, r_2, a, b. Suppose $r_1 < r_2$. Let f be a continuous mapping from $[(r_1),\, r_2]_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ and given d. Suppose $f(r_1) = a$ and $f(r_2) = b$ and $a > d$ and $d > b$. Then there exists r_3 such that $f(r_3) = d$ and $r_1 < r_3$ and $r_3 < r_2$.

(14) Let r_1, r_2 be real numbers, g be a continuous mapping from $[(r_1),\, r_2]_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$, and given s_1, s_2. Suppose $r_1 < r_2$ and $s_1 \cdot s_2 < 0$ and $s_1 = g(r_1)$ and $s_2 = g(r_2)$. Then there exists r_4 such that $g(r_4) = 0$ and $r_1 < r_4$ and $r_4 < r_2$.

(15) Let g be a map from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ and given s_1, s_2. Suppose g is continuous and $g(0) \neq g(1)$ and $s_1 = g(0)$ and $s_2 = g(1)$. Then there exists r_4 such that $0 < r_4$ and $r_4 < 1$ and $g(r_4) = \frac{s_1 + s_2}{2}$.

2. Simple Closed Curves Are Not Flat

Next we state a number of propositions:

(16) For every map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ such that $f = \mathrm{proj1}$ holds f is continuous.

(17) For every map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ such that $f = \mathrm{proj2}$ holds f is continuous.

(18) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction P$. Suppose f is continuous. Then there exists a map g from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ such that g is continuous and for all r, q such that $r \in$ the carrier of $\mathbb{I}$ and $q = f(r)$ holds $q_{\mathbf{1}} = g(r)$.

(19) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction P$. Suppose f is continuous. Then there exists a map g from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ such that g is continuous and for all r, q such that $r \in$ the carrier of $\mathbb{I}$ and $q = f(r)$ holds $q_{\mathbf{2}} = g(r)$.

(20) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is simple closed curve. Then it is not true that there exists r such that for every p such that $p \in P$ holds $p_{\mathbf{2}} = r$.

(21) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is simple closed curve. Then it is not true that there exists r such that for every p such that $p \in P$ holds $p_{\mathbf{1}} = r$.

(22) For every compact non empty subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that C is a simple closed curve holds $\text{N-bound}\,C > \text{S-bound}\,C$.

(23) For every compact non empty subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that C is a simple closed curve holds $\text{E-bound}\,C > \text{W-bound}\,C$.

(24) For every compact non empty subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that P is a simple closed curve holds $\text{S-min}\,P \neq \text{N-max}\,P$.

(25) For every compact non empty subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that P is a simple closed curve holds $\text{W-min}\,P \neq \text{E-max}\,P$.

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The Jónson's Theorem

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The papers [30], [16], [34], [36], [35], [13], [14], [6], [33], [29], [21], [26], [2], [18], [23], [3], [4], [1], [31], [28], [22], [15], [19], [24], [27], [32], [25], [20], [10], [12], [5], [17], [37], [7], [11], [8], [9], and [38] provide the notation and terminology for this paper.

1. Preliminaries

The scheme *RecChoice* deals with a set $\mathcal{A}$ and a ternary predicate $\mathcal{P}$, and states that:

There exists a function f such that $\mathrm{dom}\, f = \mathbb{N}$ and $f(0) = \mathcal{A}$ and for every element n of $\mathbb{N}$ holds $\mathcal{P}[n, f(n), f(n+1)]$

provided the following condition is satisfied:

- For every natural number n and for every set x there exists a set y such that $\mathcal{P}[n, x, y]$.

One can prove the following propositions:

(1) For every function f and for every function yielding function F such that $f = \bigcup \mathrm{rng}\, F$ holds $\mathrm{dom}\, f = \bigcup \mathrm{rng}(\mathrm{dom}_\kappa\, F(\kappa))$.

(2) For all non empty sets A, B holds $[\!:\bigcup A, \bigcup B\,:\!] = \bigcup\{[\!:a, b:\!]; a$ ranges over elements of A, b ranges over elements of B: $a \in A \wedge b \in B\}$.

(3) For every non empty set A such that A is $\subseteq$-linear holds $[\!:\bigcup A, \bigcup A\,:\!] = \bigcup\{[\!:a, a:\!]; a$ ranges over elements of A: $a \in A\}$.

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2. An equivalence lattice of a set

In the sequel X is a non empty set.

Let A be a non empty set. The functor $\mathrm{EqRelPoset}(A)$ yielding a poset is defined as follows:

(Def. 1) $\mathrm{EqRelPoset}(A) = \mathrm{Poset}(\mathrm{EqRelLatt}(A))$.

Let A be a non empty set. One can check that $\mathrm{EqRelPoset}(A)$ is non empty and has g.l.b.'s and l.u.b.'s.

One can prove the following propositions:

(4) Let A be a non empty set and x be a set. Then $x \in$ the carrier of $\mathrm{EqRelPoset}(A)$ if and only if x is an equivalence relation of A.

(5) For every non empty set A and for all elements x, y of the carrier of $\mathrm{EqRelLatt}(A)$ holds $x \sqsubseteq y$ iff $x \subseteq y$.

(6) For every non empty set A and for all elements a, b of $\mathrm{EqRelPoset}(A)$ holds $a \leqslant b$ iff $a \subseteq b$.

(7) For every lattice L and for all elements a, b of $\mathrm{Poset}(L)$ holds $a \sqcap b = {}^{\cdot}a \sqcap {}^{\cdot}b$.

(8) For every non empty set A and for all elements a, b of $\mathrm{EqRelPoset}(A)$ holds $a \sqcap b = a \cap b$.

(9) For every lattice L and for all elements a, b of $\mathrm{Poset}(L)$ holds $a \sqcup b = {}^{\cdot}a \sqcup {}^{\cdot}b$.

(10) Let A be a non empty set, a, b be elements of $\mathrm{EqRelPoset}(A)$, and E_1, E_2 be equivalence relations of A. If $a = E_1$ and $b = E_2$, then $a \sqcup b = E_1 \sqcup E_2$.

(11) Let L be a lattice, X be a set, and b be an element of L. Then $b \leqslant X$ if and only if $b \leqslant X \cap$ the carrier of L.

Let L be a non empty relational structure. Let us observe that L is complete if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let X be a subset of L. Then there exists an element a of L such that $a \leqslant X$ and for every element b of L such that $b \leqslant X$ holds $b \leqslant a$.

Let A be a non empty set. Note that $\mathrm{EqRelPoset}(A)$ is complete.

3. A type of a sublattice of equivalence lattice of a set

Let L_1, L_2 be lattices. One can check that there exists a map from L_1 into L_2 which is meet-preserving and join-preserving.

Let L_1, L_2 be lattices. A homomorphism from L_1 to L_2 is a meet-preserving join-preserving map from L_1 into L_2.

Let L be a lattice. One can check that there exists a relational substructure of L which is meet-inheriting, join-inheriting, and strict.

Let L_1, L_2 be lattices and let f be a homomorphism from L_1 to L_2. Then $\operatorname{Im} f$ is a strict full sublattice of L_2.

We follow the rules: e, e_1, e_2 denote equivalence relations of X and x, y denote sets.

Let us consider X, let f be a non empty finite sequence of elements of X, let us consider x, y, and let R be a binary relation. We say that x and y are joint by f and R if and only if:

(Def. 3) $f(1) = x$ and $f(\operatorname{len} f) = y$ and for every natural number i such that $1 \leqslant i$ and $i < \operatorname{len} f$ holds $\langle f(i),\, f(i+1)\rangle \in R$.

One can prove the following propositions:

(12) Let x be a set, o be a natural number, R be a binary relation, and f be a non empty finite sequence of elements of X. If R is reflexive in X and $f = o \longmapsto x$, then x and x are joint by f and R.

(13) Let x, y, z be sets, R be a binary relation, and f, g be non empty finite sequences of elements of X. Suppose R is reflexive in X and x and y are joint by f and R and y and z are joint by g and R. Then there exists a non empty finite sequence h of elements of X such that $h = f \frown g$ and x and z are joint by h and R.

(14) Let x, y be sets, R be a binary relation, and n, m be natural numbers. Suppose that

(i) $n \leqslant m$,

(ii) R is reflexive in X, and

(iii) there exists a non empty finite sequence f of elements of X such that $\operatorname{len} f = n$ and x and y are joint by f and R.

Then there exists a non empty finite sequence h of elements of X such that $\operatorname{len} h = m$ and x and y are joint by h and R.

Let us consider X and let Y be a sublattice of $\operatorname{EqRelPoset}(X)$. Let us assume that there exists e such that $e \in$ the carrier of Y $e \neq \mathrm{id}_X$. And let us assume that there exists a natural number o such that for all e_1, e_2, x, y such that $e_1 \in$ the carrier of Y and $e_2 \in$ the carrier of Y and $\langle x, y\rangle \in e_1 \sqcup e_2$ there exists a non empty finite sequence F of elements of X such that $\operatorname{len} F = o$ and x and y are joint by F and $e_1 \cup e_2$. The type of Y is a natural number and is defined by the conditions (Def. 4).

(Def. 4)(i) For all e_1, e_2, x, y such that $e_1 \in$ the carrier of Y and $e_2 \in$ the carrier of Y and $\langle x, y\rangle \in e_1 \sqcup e_2$ there exists a non empty finite sequence F of elements of X such that $\operatorname{len} F =$ (the type of Y) $+ 2$ and x and y are joint by F and $e_1 \cup e_2$, and

(ii) there exist e_1, e_2, x, y such that $e_1 \in$ the carrier of Y and $e_2 \in$ the carrier of Y and $\langle x, y\rangle \in e_1 \sqcup e_2$ and it is not true that there exists a non empty finite sequence F of elements of X such that $\operatorname{len} F =$ (the type of Y) $+ 1$ and x and y are joint by F and $e_1 \cup e_2$.

One can prove the following proposition

(15) Let Y be a sublattice of EqRelPoset(X) and n be a natural number. Suppose that

(i) there exists e such that $e \in$ the carrier of Y and $e \neq \mathrm{id}_X$, and

(ii) for all e_1, e_2, x, y such that $e_1 \in$ the carrier of Y and $e_2 \in$ the carrier of Y and $\langle x, y \rangle \in e_1 \sqcup e_2$ there exists a non empty finite sequence F of elements of X such that $\mathrm{len}\, F = n + 2$ and x and y are joint by F and $e_1 \cup e_2$.

Then the type of $Y \leqslant n$.

4. A meet-representation of a lattice

In the sequel A is a non empty set and L is a lower-bounded lattice.

Let us consider A, L.

(Def. 5) A function from $[\!: A, A :\!]$ into the carrier of L is said to be a bifunction from A into L.

Let us consider A, L, let f be a bifunction from A into L, and let x, y be elements of A. Then $f(x, y)$ is an element of L.

Let us consider A, L and let f be a bifunction from A into L. We say that f is symmetric if and only if:

(Def. 6) For all elements x, y of A holds $f(x, y) = f(y, x)$.

We say that f is zeroed if and only if:

(Def. 7) For every element x of A holds $f(x, x) = \bot_L$.

We say that f satisfies triangle inequality if and only if:

(Def. 8) For all elements x, y, z of A holds $f(x, y) \sqcup f(y, z) \geqslant f(x, z)$.

Let us consider A, L. Observe that there exists a bifunction from A into L which is symmetric and zeroed and satisfies triangle inequality.

Let us consider A, L. A distance function of A, L is a symmetric zeroed bifunction from A into L satisfying triangle inequality.

Let us consider A, L and let d be a distance function of A, L. The functor $\alpha(d)$ yielding a map from L into EqRelPoset(A) is defined by the condition (Def. 9).

(Def. 9) Let e be an element of L. Then there exists an equivalence relation E of A such that $E = (\alpha(d))(e)$ and for all elements x, y of A holds $\langle x, y \rangle \in E$ iff $d(x, y) \leqslant e$.

The following two propositions are true:

(16) For every distance function d of A, L holds $\alpha(d)$ is meet-preserving.

(17) For every distance function d of A, L such that d is onto holds $\alpha(d)$ is one-to-one.

5. Jónson's theorem

Let A be a set. The functor A^* is defined as follows:

(Def. 10) $A^* = A \cup \{\{A\}, \{\{A\}\}, \{\{\{A\}\}\}\}$.

Let A be a set. One can verify that A^* is non empty.

Let us consider A, L, let d be a bifunction from A into L, and let q be an element of $[\![A, A,$ the carrier of $L,$ the carrier of $L]\!]$. The functor d_q^* yields a bifunction from A^* into L and is defined by the conditions (Def. 11).

(Def. 11)(i) For all elements u, v of A holds $d_q^*(u, v) = d(u, v)$,

(ii) $d_q^*(\{A\}, \{A\}) = \bot_L$,

(iii) $d_q^*(\{\{A\}\}, \{\{A\}\}) = \bot_L$,

(iv) $d_q^*(\{\{\{A\}\}\}, \{\{\{A\}\}\}) = \bot_L$,

(v) $d_q^*(\{\{A\}\}, \{\{\{A\}\}\}) = q_{\mathbf{3}}$,

(vi) $d_q^*(\{\{\{A\}\}\}, \{\{A\}\}) = q_{\mathbf{3}}$,

(vii) $d_q^*(\{A\}, \{\{A\}\}) = q_{\mathbf{4}}$,

(viii) $d_q^*(\{\{A\}\}, \{A\}) = q_{\mathbf{4}}$,

(ix) $d_q^*(\{A\}, \{\{\{A\}\}\}) = q_{\mathbf{3}} \sqcup q_{\mathbf{4}}$,

(x) $d_q^*(\{\{\{A\}\}\}, \{A\}) = q_{\mathbf{3}} \sqcup q_{\mathbf{4}}$, and

(xi) for every element u of A holds $d_q^*(u, \{A\}) = d(u, q_{\mathbf{1}}) \sqcup q_{\mathbf{3}}$ and $d_q^*(\{A\}, u) = d(u, q_{\mathbf{1}}) \sqcup q_{\mathbf{3}}$ and $d_q^*(u, \{\{A\}\}) = d(u, q_{\mathbf{1}}) \sqcup q_{\mathbf{3}} \sqcup q_{\mathbf{4}}$ and $d_q^*(\{\{A\}\}, u) = d(u, q_{\mathbf{1}}) \sqcup q_{\mathbf{3}} \sqcup q_{\mathbf{4}}$ and $d_q^*(u, \{\{\{A\}\}\}) = d(u, q_{\mathbf{2}}) \sqcup q_{\mathbf{4}}$ and $d_q^*(\{\{\{A\}\}\}, u) = d(u, q_{\mathbf{2}}) \sqcup q_{\mathbf{4}}$.

Next we state several propositions:

(18) Let d be a bifunction from A into L. Suppose d is zeroed. Let q be an element of $[\![A, A,$ the carrier of $L,$ the carrier of $L]\!]$. Then d_q^* is zeroed.

(19) Let d be a bifunction from A into L. Suppose d is symmetric. Let q be an element of $[\![A, A,$ the carrier of $L,$ the carrier of $L]\!]$. Then d_q^* is symmetric.

(20) Let d be a bifunction from A into L. Suppose d is symmetric and satisfies triangle inequality. Let q be an element of $[\![A, A,$ the carrier of $L,$ the carrier of $L]\!]$. If $d(q_{\mathbf{1}}, q_{\mathbf{2}}) \leqslant q_{\mathbf{3}} \sqcup q_{\mathbf{4}}$, then d_q^* satisfies triangle inequality.

(21) For every set A holds $A \subseteq A^*$.

(22) Let d be a bifunction from A into L and q be an element of $[\![A, A,$ the carrier of $L,$ the carrier of $L]\!]$. Then $d \subseteq d_q^*$.

Let us consider A, L and let d be a bifunction from A into L. The functor $\mathrm{DistEsti}(d)$ yields a cardinal number and is defined as follows:

(Def. 12) $\mathrm{DistEsti}(d) \approx \{\langle x, y, a, b\rangle; x$ ranges over elements of A, y ranges over elements of A, a ranges over elements of L, b ranges over elements of L: $d(x, y) \leqslant a \sqcup b\}$.

We now state the proposition

(23) For every distance function d of A, L holds $\mathrm{DistEsti}(d) \neq \emptyset$.

In the sequel T denotes a transfinite sequence and O, O_1, O_2 denote ordinal numbers.

Let us consider A and let us consider O. The functor $\mathrm{ConsecutiveSet}(A, O)$ is defined by the condition (Def. 13).

(Def. 13) There exists a transfinite sequence L_0 such that

(i) $\mathrm{ConsecutiveSet}(A, O) = \mathrm{last}\, L_0$,

(ii) $\mathrm{dom}\, L_0 = \mathrm{succ}\, O$,

(iii) $L_0(\emptyset) = A$,

(iv) for every ordinal number C and for every set z such that $\mathrm{succ}\, C \in \mathrm{succ}\, O$ and $z = L_0(C)$ holds $L_0(\mathrm{succ}\, C) = z^*$, and

(v) for every ordinal number C and for every transfinite sequence L_1 such that $C \in \mathrm{succ}\, O$ and $C \neq \emptyset$ and C is a limit ordinal number and $L_1 = L_0 \restriction C$ holds $L_0(C) = \bigcup \mathrm{rng}\, L_1$.

We now state three propositions:

(24) $\mathrm{ConsecutiveSet}(A, \emptyset) = A$.

(25) $\mathrm{ConsecutiveSet}(A, \mathrm{succ}\, O) = (\mathrm{ConsecutiveSet}(A, O))^*$.

(26) Suppose $O \neq \emptyset$ and O is a limit ordinal number and $\mathrm{dom}\, T = O$ and for every ordinal number O_1 such that $O_1 \in O$ holds $T(O_1) = \mathrm{ConsecutiveSet}(A, O_1)$. Then $\mathrm{ConsecutiveSet}(A, O) = \bigcup \mathrm{rng}\, T$.

Let us consider A and let us consider O. Note that $\mathrm{ConsecutiveSet}(A, O)$ is non empty.

One can prove the following proposition

(27) $A \subseteq \mathrm{ConsecutiveSet}(A, O)$.

Let us consider A, L and let d be a bifunction from A into L. A transfinite sequence of elements of $[\!:A,\ A,\ \text{the carrier of } L,\ \text{the carrier of } L:\!]$ is said to be a sequence of quadruples of d if it satisfies the conditions (Def. 14).

(Def. 14)(i) dom it is a cardinal number,

(ii) it is one-to-one, and

(iii) $\mathrm{rng}\, \mathrm{it} = \{\langle x, y, a, b\rangle; x$ ranges over elements of A, y ranges over elements of A, a ranges over elements of L, b ranges over elements of L: $d(x, y) \leqslant a \sqcup b\}$.

Let us consider A, L, let d be a bifunction from A into L, let q be a sequence of quadruples of d, and let us consider O. Let us assume that $O \in \mathrm{dom}\, q$. The functor $\mathrm{Quadr}(q, O)$ yielding an element of $[\!:\mathrm{ConsecutiveSet}(A, O),\ \mathrm{ConsecutiveSet}(A, O),\ \text{the carrier of } L,\ \text{the carrier of } L:\!]$ is defined as follows:

(Def. 15) $\mathrm{Quadr}(q, O) = q(O)$.

One can prove the following proposition

(28) Let d be a bifunction from A into L and q be a sequence of quadruples of d. Then $O \in \mathrm{DistEsti}(d)$ if and only if $O \in \mathrm{dom}\, q$.

Let us consider A, L and let z be a set. Let us assume that z is a bifunction from A into L. The functor $\mathrm{BiFun}(z, A, L)$ yields a bifunction from A into L and is defined as follows:

(Def. 16) $\mathrm{BiFun}(z, A, L) = z$.

Let us consider A, L, let d be a bifunction from A into L, let q be a sequence of quadruples of d, and let us consider O. The functor $\mathrm{ConsecutiveDelta}(q, O)$ is defined by the condition (Def. 17).

(Def. 17) There exists a transfinite sequence L_0 such that

(i) $\mathrm{ConsecutiveDelta}(q, O) = \mathrm{last}\, L_0$,

(ii) $\mathrm{dom}\, L_0 = \mathrm{succ}\, O$,

(iii) $L_0(\emptyset) = d$,

(iv) for every ordinal number C and for every set z such that $\mathrm{succ}\, C \in \mathrm{succ}\, O$ and $z = L_0(C)$ holds $L_0(\mathrm{succ}\, C) = (\mathrm{BiFun}(z, \mathrm{ConsecutiveSet}(A, C), L))^*_{\mathrm{Quadr}(q,C)}$, and

(v) for every ordinal number C and for every transfinite sequence L_1 such that $C \in \mathrm{succ}\, O$ and $C \neq \emptyset$ and C is a limit ordinal number and $L_1 = L_0 \restriction C$ holds $L_0(C) = \bigcup \mathrm{rng}\, L_1$.

Next we state four propositions:

(29) For every bifunction d from A into L and for every sequence q of quadruples of d holds $\mathrm{ConsecutiveDelta}(q, \emptyset) = d$.

(30) For every bifunction d from A into L and for every sequence q of quadruples of d holds $\mathrm{ConsecutiveDelta}(q, \mathrm{succ}\, O) = (\mathrm{BiFun}(\mathrm{ConsecutiveDelta}(q, O), \mathrm{ConsecutiveSet}(A, O), L))^*_{\mathrm{Quadr}(q,O)}$.

(31) Let d be a bifunction from A into L and q be a sequence of quadruples of d. Suppose $O \neq \emptyset$ and O is a limit ordinal number and $\mathrm{dom}\, T = O$ and for every ordinal number O_1 such that $O_1 \in O$ holds $T(O_1) = \mathrm{ConsecutiveDelta}(q, O_1)$. Then $\mathrm{ConsecutiveDelta}(q, O) = \bigcup \mathrm{rng}\, T$.

(32) If $O_1 \subseteq O_2$, then $\mathrm{ConsecutiveSet}(A, O_1) \subseteq \mathrm{ConsecutiveSet}(A, O_2)$.

Let O be a non empty ordinal number. Note that every element of O is ordinal-like.

Next we state the proposition

(33) Let d be a bifunction from A into L and q be a sequence of quadruples of d. Then $\mathrm{ConsecutiveDelta}(q, O)$ is a bifunction from $\mathrm{ConsecutiveSet}(A, O)$ into L.

Let us consider A, L, let d be a bifunction from A into L, let q be a sequence of quadruples of d, and let us consider O. Then $\mathrm{ConsecutiveDelta}(q, O)$ is a bifunction from $\mathrm{ConsecutiveSet}(A, O)$ into L.

Next we state several propositions:

(34) For every bifunction d from A into L and for every sequence q of quadruples of d holds $d \subseteq \mathrm{ConsecutiveDelta}(q, O)$.

(35) For every bifunction d from A into L and for every sequence q of quadruples of d such that $O_1 \subseteq O_2$ holds $\mathrm{ConsecutiveDelta}(q, O_1) \subseteq \mathrm{ConsecutiveDelta}(q, O_2)$.

(36) Let d be a bifunction from A into L. Suppose d is zeroed. Let q be a sequence of quadruples of d. Then $\mathrm{ConsecutiveDelta}(q, O)$ is zeroed.

(37) Let d be a bifunction from A into L. Suppose d is symmetric. Let q be a sequence of quadruples of d. Then $\mathrm{ConsecutiveDelta}(q, O)$ is symmetric.

(38) Let d be a bifunction from A into L. Suppose d is symmetric and satisfies triangle inequality. Let q be a sequence of quadruples of d. If $O \subseteq \mathrm{DistEsti}(d)$, then $\mathrm{ConsecutiveDelta}(q, O)$ satisfies triangle inequality.

(39) Let d be a distance function of A, L and q be a sequence of quadruples of d. If $O \subseteq \mathrm{DistEsti}(d)$, then $\mathrm{ConsecutiveDelta}(q, O)$ is a distance function of $\mathrm{ConsecutiveSet}(A, O)$, L.

Let us consider A, L and let d be a bifunction from A into L. The functor $\mathrm{NextSet}(d)$ is defined as follows:

(Def. 18) $\mathrm{NextSet}(d) = \mathrm{ConsecutiveSet}(A, \mathrm{DistEsti}(d))$.

Let us consider A, L and let d be a bifunction from A into L. One can check that $\mathrm{NextSet}(d)$ is non empty.

Let us consider A, L, let d be a bifunction from A into L, and let q be a sequence of quadruples of d. The functor $\mathrm{NextDelta}(q)$ is defined as follows:

(Def. 19) $\mathrm{NextDelta}(q) = \mathrm{ConsecutiveDelta}(q, \mathrm{DistEsti}(d))$.

Let us consider A, L, let d be a distance function of A, L, and let q be a sequence of quadruples of d. Then $\mathrm{NextDelta}(q)$ is a distance function of $\mathrm{NextSet}(d)$, L.

Let us consider A, L, let d be a distance function of A, L, let A_1 be a non empty set, and let d_1 be a distance function of A_1, L. We say that (A_1, d_1) is extension of (A, d) if and only if:

(Def. 20) There exists a sequence q of quadruples of d such that $A_1 = \mathrm{NextSet}(d)$ and $d_1 = \mathrm{NextDelta}(q)$.

The following proposition is true

(40) Let d be a distance function of A, L, A_1 be a non empty set, and d_1 be a distance function of A_1, L. Suppose (A_1, d_1) is extension of (A, d). Let x, y be elements of A and a, b be elements of L. Suppose $d(x, y) \leqslant a \sqcup b$. Then there exist elements z_1, z_2, z_3 of A_1 such that $d_1(x, z_1) = a$ and $d_1(z_2, z_3) = a$ and $d_1(z_1, z_2) = b$ and $d_1(z_3, y) = b$.

Let us consider A, L and let d be a distance function of A, L. A function is called an extension sequence of (A, d) if it satisfies the conditions (Def. 21).

(Def. 21)(i) $\mathrm{dom}\,\mathrm{it} = \mathbb{N}$,

(ii) $\mathrm{it}(0) = \langle A, d\rangle$, and

(iii) for every natural number n there exists a non empty set A' and there exists a distance function d' of A', L and there exists a non empty set A_1 and there exists a distance function d_1 of A_1, L such that (A_1, d_1) is extension of (A', d') and $\mathrm{it}(n) = \langle A', d'\rangle$ and $\mathrm{it}(n+1) = \langle A_1, d_1\rangle$.

Next we state two propositions:

(41) Let d be a distance function of A, L, S be an extension sequence of (A, d), and k, l be natural numbers. If $k \leqslant l$, then $S(k)_{\mathbf{1}} \subseteq S(l)_{\mathbf{1}}$.

(42) Let d be a distance function of A, L, S be an extension sequence of (A, d), and k, l be natural numbers. If $k \leqslant l$, then $S(k)_{\mathbf{2}} \subseteq S(l)_{\mathbf{2}}$.

Let us consider L. The functor $\delta_0(L)$ yields a distance function of the carrier of L, L and is defined by:

(Def. 22) For all elements x, y of the carrier of L holds if $x \neq y$, then $(\delta_0(L))(x, y) = x \sqcup y$ and if $x = y$, then $(\delta_0(L))(x, y) = \perp_L$.

We now state two propositions:

(43) $\delta_0(L)$ is onto.

(44) There exists a non empty set A and there exists a homomorphism f from L to $\mathrm{EqRelPoset}(A)$ such that f is one-to-one and the type of $\mathrm{Im}\, f \leqslant 3$.

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Lebesgue's Covering Lemma, Uniform Continuity and Segmentation of Arcs

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Summary. For mappings from a metric space to a metric space, a notion of uniform continuity is defined. If we introduce natural topologies to the metric spaces, a uniformly continuous function becomes continuous. On the other hand, if the domain is compact, a continuous function is uniformly continuous. For this proof, Lebesgue's covering lemma is also proved. An arc, which is homeomorphic to [0,1], can be devided into small segments, as small as one wishes.

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The notation and terminology used in this paper have been introduced in the following articles: [35], [41], [40], [34], [28], [23], [1], [43], [38], [27], [39], [31], [11], [33], [10], [30], [26], [42], [2], [7], [8], [4], [19], [20], [18], [29], [15], [9], [14], [36], [17], [21], [16], [6], [22], [13], [24], [3], [5], [32], [12], [25], and [37].

1. Lebesgue's Covering Lemma

We adopt the following rules: s, s_1, s_2, t, r, r_1, r_2 are real numbers, n, m are natural numbers, and X, Y are non empty metric spaces.

The following two propositions are true:

(1) $t - r - (t - s) = -r + s$ and $t - r - (t - s) = s - r$.

(2) For every r such that $r > 0$ there exists a natural number n such that $n > 0$ and $\frac{1}{n} < r$.

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Let X, Y be non empty metric structures and let f be a map from X into Y. We say that f is uniformly continuous if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let given r. Suppose $0 < r$. Then there exists s such that $0 < s$ and for all elements u_1, u_2 of the carrier of X such that $\rho(u_1, u_2) < s$ holds $\rho(f_{u_1}, f_{u_2}) < r$.

Next we state several propositions:

(3) Let X be a non empty topological space, M be a metric space, and f be a map from X into $M_{\rm top}$. Suppose f is continuous. Let r be a real number, u be an element of the carrier of M, and P be a subset of the carrier of $M_{\rm top}$. If $P = \mathrm{Ball}(u, r)$, then $f^{-1}(P)$ is open.

(4) Let N, M be metric spaces and f be a map from $N_{\rm top}$ into $M_{\rm top}$. Suppose that for every real number r and for every element u of the carrier of N and for every element u_1 of the carrier of M such that $r > 0$ and $u_1 = f(u)$ there exists s such that $s > 0$ and for every element w of the carrier of N and for every element w_1 of the carrier of M such that $w_1 = f(w)$ and $\rho(u, w) < s$ holds $\rho(u_1, w_1) < r$. Then f is continuous.

(5) Let N be a metric space, M be a non empty metric space, and f be a map from $N_{\rm top}$ into $M_{\rm top}$. Suppose f is continuous. Let r be a real number, u be an element of the carrier of N, and u_1 be an element of the carrier of M. Suppose $r > 0$ and $u_1 = f(u)$. Then there exists s such that

(i) $s > 0$, and

(ii) for every element w of the carrier of N and for every element w_1 of the carrier of M such that $w_1 = f(w)$ and $\rho(u, w) < s$ holds $\rho(u_1, w_1) < r$.

(6) Let N, M be non empty metric spaces, f be a map from N into M, and g be a map from $N_{\rm top}$ into $M_{\rm top}$. If $f = g$ and f is uniformly continuous, then g is continuous.

(7) Let N be a non empty metric space and G be a family of subsets of $N_{\rm top}$. Suppose G is a cover of $N_{\rm top}$ and open and $N_{\rm top}$ is compact. Then there exists r such that

(i) $r > 0$, and

(ii) for all elements w_1, w_2 of the carrier of N such that $\rho(w_1, w_2) < r$ there exists a subset G_1 of the carrier of $N_{\rm top}$ such that $w_1 \in G_1$ and $w_2 \in G_1$ and $G_1 \in G$.

2. Uniformity of Continuous Functions on Compact Spaces

Next we state three propositions:

(8) Let N, M be non empty metric spaces, f be a map from N into M, and g be a map from N_{top} into M_{top}. Suppose $g = f$ and N_{top} is compact and g is continuous. Then f is uniformly continuous.

(9) Let g be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$ and f be a map from $[0, 1]_{\mathrm{M}}$ into $\mathcal{E}^n$. If g is continuous and $f = g$, then f is uniformly continuous.

(10) Let P be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, Q be a non empty subset of the carrier of $\mathcal{E}^n$, g be a map from $\mathbb{I}$ into $(\mathcal{E}^n_{\mathrm{T}}){\restriction}P$, and f be a map from $[0, 1]_{\mathrm{M}}$ into $\mathcal{E}^n{\restriction}Q$. If $P = Q$ and g is continuous and $f = g$, then f is uniformly continuous.

3. Segmentation of Arcs

We now state four propositions:

(11) For every map g from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$ there exists a map f from $[0, 1]_{\mathrm{M}}$ into $\mathcal{E}^n$ such that $f = g$.

(12) For every r such that $r \geqslant 0$ holds $\lceil r \rceil \geqslant 0$ and $\lfloor r \rfloor \geqslant 0$ and $\lceil r \rceil$ is a natural number and $\lfloor r \rfloor$ is a natural number.

(13) For all r, s holds $|r - s| = |s - r|$.

(14) For all r_1, r_2, s_1, s_2 such that $r_1 \in [s_1, s_2]$ and $r_2 \in [s_1, s_2]$ holds $|r_1 - r_2| \leqslant s_2 - s_1$.

Let I_1 be a finite sequence of elements of $\mathbb{R}$. We say that I_1 is decreasing if and only if:

(Def. 2) For all n, m such that $n \in \operatorname{dom} I_1$ and $m \in \operatorname{dom} I_1$ and $n < m$ holds $I_1(n) > I_1(m)$.

We now state the proposition

(15) Let e be a real number, g be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$, and p_1, p_2 be elements of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $e > 0$ and g is continuous and one-to-one and $g(0) = p_1$ and $g(1) = p_2$. Then there exists a finite sequence h of elements of $\mathbb{R}$ such that

(i) $h(1) = 1$,

(ii) $h(\operatorname{len} h) = 0$,

(iii) $5 \leqslant \operatorname{len} h$,

(iv) $\operatorname{rng} h \subseteq$ the carrier of $\mathbb{I}$,

(v) h is decreasing, and

(vi) for every natural number i and for every subset Q of the carrier of $\mathbb{I}$ and for every subset W of the carrier of $\mathcal{E}^n$ such that $1 \leqslant i$ and $i < \operatorname{len} h$ and $Q = [\pi_{i+1}h, \pi_i h]$ and $W = g^\circ Q$ holds $\emptyset W < e$.

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On the Rectangular Finite Sequences of the Points of the Plane

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Summary. The article deals with a rather technical concept – rectangular sequences of the points of the plane. We mean by that a finite sequence consisting of five elements, that is circular, i.e. the first element and the fifth one of it are equal, and such that the polygon determined by it is a non degenerated rectangle, with sides parallel to axes. The main result is that for the rectangle determined by such a sequence the left and the right component of the complement of it are different and disjoint.

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The terminology and notation used in this paper are introduced in the following papers: [29], [35], [34], [28], [7], [36], [13], [2], [25], [1], [27], [32], [5], [6], [3], [33], [31], [17], [16], [14], [15], [4], [26], [24], [37], [10], [23], [11], [12], [21], [18], [19], [22], [30], [20], [8], and [9].

1. General preliminaries

One can prove the following proposition

(1) For every trivial set A and for every set B such that $B \subseteq A$ holds B is trivial.

One can verify that every function which is non constant is also non trivial.

Let us observe that every function which is trivial is also constant.

One can prove the following proposition

(2) For every function f such that $\operatorname{rng} f$ is trivial holds f is constant.

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Let f be a constant function. One can verify that $\operatorname{rng} f$ is trivial.

Let us observe that there exists a finite sequence which is non empty and constant.

We now state three propositions:

(3) For all finite sequences f, g such that $f \frown g$ is constant holds f is constant and g is constant.

(4) For all sets x, y such that $\langle x, y\rangle$ is constant holds $x = y$.

(5) For all sets x, y, z such that $\langle x, y, z\rangle$ is constant holds $x = y$ and $y = z$ and $z = x$.

2. Preliminaries (general topology)

One can prove the following four propositions:

(6) Let G_1 be a non empty topological space, A be a subset of the carrier of G_1, and B be a non empty subset of the carrier of G_1. If A is a component of B, then $A \neq \emptyset$.

(7) Let G_1 be a non empty topological space, A be a subset of the carrier of G_1, and B be a non empty subset of the carrier of G_1. If A is a component of B, then $A \subseteq B$.

(8) Let T be a non empty topological space, A be a non empty subset of the carrier of T, and B_1, B_2, C be subsets of the carrier of T. Suppose B_1 is a component of A and B_2 is a component of A and C is a component of A and $B_1 \cup B_2 = A$. Then $C = B_1$ or $C = B_2$.

(9) Let T be a non empty topological space, A be a non empty subset of the carrier of T, and B_1, B_2, C_1, C_2 be subsets of the carrier of T. Suppose B_1 is a component of A and B_2 is a component of A and C_1 is a component of A and C_2 is a component of A and $B_1 \cup B_2 = A$ and $C_1 \cup C_2 = A$. Then $\{B_1, B_2\} = \{C_1, C_2\}$.

3. Preliminaries (the topology of the plane)

We follow the rules: C, C_1, C_2 are non empty compact subsets of $\mathcal{E}^2_{\mathrm{T}}$ and p, q are points of $\mathcal{E}^2_{\mathrm{T}}$.

Next we state the proposition

(10) For all points p, q, r of $\mathcal{E}^2_{\mathrm{T}}$ holds $\widetilde{\mathcal{L}}(\langle p, q, r\rangle) = \mathcal{L}(p, q) \cup \mathcal{L}(q, r)$.

Let n be a natural number and let f be a non trivial finite sequence of elements of $\mathcal{E}^n_{\mathrm{T}}$. Observe that $\widetilde{\mathcal{L}}(f)$ is non empty.

Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Note that $\widetilde{\mathcal{L}}(f)$ is compact.

We now state two propositions:

(11) For all subsets A, B of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ such that $A \subseteq B$ and B is horizontal holds A is horizontal.

(12) For all subsets A, B of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ such that $A \subseteq B$ and B is vertical holds A is vertical.

Let us observe that $\square_{\mathcal{E}^2}$ is special polygonal.

One can check that $\square_{\mathcal{E}^2}$ is non horizontal and non vertical.

One can check that there exists a subset of $\mathcal{E}^2_{\mathrm{T}}$ which is non vertical, non horizontal, non empty, and compact.

4. Special points of a compact non empty subset of the plane

The following propositions are true:

(13) N-min $C \in C$ and N-max $C \in C$.

(14) S-min $C \in C$ and S-max $C \in C$.

(15) W-min $C \in C$ and W-max $C \in C$.

(16) E-min $C \in C$ and E-max $C \in C$.

(17) C is vertical iff W-bound $C =$ E-bound C.

(18) C is horizontal iff S-bound $C =$ N-bound C.

(19) For every C such that NW-corner $C =$ NE-corner C holds C is vertical.

(20) For every C such that SW-corner $C =$ SE-corner C holds C is vertical.

(21) For every C such that NW-corner $C =$ SW-corner C holds C is horizontal.

(22) For every C such that NE-corner $C =$ SE-corner C holds C is horizontal.

In the sequel t, r_1, r_2, s_1, s_2 are real numbers.

The following propositions are true:

(23) W-bound $C \leqslant$ E-bound C.

(24) S-bound $C \leqslant$ N-bound C.

(25) $\mathcal{L}(\text{SE-corner}\, C, \text{NE-corner}\, C) = \{p : p_{\mathbf{1}} = \text{E-bound}\, C \wedge p_{\mathbf{2}} \leqslant \text{N-bound}\, C \wedge p_{\mathbf{2}} \geqslant \text{S-bound}\, C\}$.

(26) $\mathcal{L}(\text{SW-corner}\, C, \text{SE-corner}\, C) = \{p : p_{\mathbf{1}} \leqslant \text{E-bound}\, C \wedge p_{\mathbf{1}} \geqslant \text{W-bound}\, C \wedge p_{\mathbf{2}} = \text{S-bound}\, C\}$.

(27) $\mathcal{L}(\text{NW-corner}\, C, \text{NE-corner}\, C) = \{p : p_{\mathbf{1}} \leqslant \text{E-bound}\, C \wedge p_{\mathbf{1}} \geqslant \text{W-bound}\, C \wedge p_{\mathbf{2}} = \text{N-bound}\, C\}$.

(28) $\mathcal{L}(\text{SW-corner}\, C, \text{NW-corner}\, C) = \{p : p_{\mathbf{1}} = \text{W-bound}\, C \wedge p_{\mathbf{2}} \leqslant \text{N-bound}\, C \wedge p_{\mathbf{2}} \geqslant \text{S-bound}\, C\}$.

(29) $\mathcal{L}(\text{SW-corner}\,C, \text{NW-corner}\,C) \cap \mathcal{L}(\text{NW-corner}\,C, \text{NE-corner}\,C) = \{\text{NW-corner}\,C\}$.

(30) $\mathcal{L}(\text{NW-corner}\,C, \text{NE-corner}\,C) \cap \mathcal{L}(\text{NE-corner}\,C, \text{SE-corner}\,C) = \{\text{NE-corner}\,C\}$.

(31) $\mathcal{L}(\text{SE-corner}\,C, \text{NE-corner}\,C) \cap \mathcal{L}(\text{SW-corner}\,C, \text{SE-corner}\,C) = \{\text{SE-corner}\,C\}$.

(32) $\mathcal{L}(\text{NW-corner}\,C, \text{SW-corner}\,C) \cap \mathcal{L}(\text{SW-corner}\,C, \text{SE-corner}\,C) = \{\text{SW-corner}\,C\}$.

5. Subsets of the plane that are neither vertical nor horizontal

In the sequel D is a non vertical non horizontal non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$.

The following propositions are true:

(33) $\text{W-bound}\,D < \text{E-bound}\,D$.

(34) $\text{S-bound}\,D < \text{N-bound}\,D$.

(35) $\mathcal{L}(\text{SW-corner}\,D, \text{NW-corner}\,D) \cap \mathcal{L}(\text{SE-corner}\,D, \text{NE-corner}\,D) = \emptyset$.

(36) $\mathcal{L}(\text{SW-corner}\,D, \text{SE-corner}\,D) \cap \mathcal{L}(\text{NW-corner}\,D, \text{NE-corner}\,D) = \emptyset$.

6. A special sequence related to a compact non empty subset of the plane

Let us consider C. The functor $\text{SpStSeq}\,C$ yielding a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 1) $\text{SpStSeq}\,C = \langle \text{NW-corner}\,C, \text{NE-corner}\,C, \text{SE-corner}\,C \rangle \frown \langle \text{SW-corner}\,C, \text{NW-corner}\,C \rangle$.

The following propositions are true:

(37) $\pi_1\,\text{SpStSeq}\,C = \text{NW-corner}\,C$.

(38) $\pi_2\,\text{SpStSeq}\,C = \text{NE-corner}\,C$.

(39) $\pi_3\,\text{SpStSeq}\,C = \text{SE-corner}\,C$.

(40) $\pi_4\,\text{SpStSeq}\,C = \text{SW-corner}\,C$.

(41) $\pi_5\,\text{SpStSeq}\,C = \text{NW-corner}\,C$.

(42) $\text{len}\,\text{SpStSeq}\,C = 5$.

(43) $\widetilde{\mathcal{L}}(\text{SpStSeq}\,C) = \mathcal{L}(\text{NW-corner}\,C, \text{NE-corner}\,C) \cup \mathcal{L}(\text{NE-corner}\,C, \text{SE-corner}\,C) \cup (\mathcal{L}(\text{SE-corner}\,C, \text{SW-corner}\,C) \cup \mathcal{L}(\text{SW-corner}\,C, \text{NW-corner}\,C))$.

Let D be a non vertical non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. Note that SpStSeq D is non constant.

Let D be a non horizontal non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. Note that SpStSeq D is non constant.

Let us consider D. One can check that SpStSeq D is special unfolded circular s.c.c. and standard.

Next we state four propositions:

(44) $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\,D) = [.\,\text{W-bound}\,D, \text{E-bound}\,D, \text{S-bound}\,D, \text{N-bound}\,D.]$.

(45) Let T be a non empty topological space, X be a non empty subset of T, and f be a real map of T. Then $\mathrm{rng}(f \restriction X) = f^\circ X$.

(46) Let T be a non empty topological space, X be a non empty compact subset of T, and f be a continuous real map of T. Then $f^\circ X$ is lower bounded.

(47) Let T be a non empty topological space, X be a non empty compact subset of T, and f be a continuous real map of T. Then $f^\circ X$ is upper bounded.

Let us observe that there exists a subset of $\mathbb{R}$ which is non empty, upper bounded, and lower bounded.

We now state a number of propositions:

(48) $\text{W-bound}\,C = \inf((\mathrm{proj1})^\circ C)$.

(49) $\text{S-bound}\,C = \inf((\mathrm{proj2})^\circ C)$.

(50) $\text{N-bound}\,C = \sup((\mathrm{proj2})^\circ C)$.

(51) $\text{E-bound}\,C = \sup((\mathrm{proj1})^\circ C)$.

(52) For all non empty lower bounded subsets A, B of $\mathbb{R}$ holds $\inf(A \cup B) = \min(\inf A, \inf B)$.

(53) For all non empty upper bounded subsets A, B of $\mathbb{R}$ holds $\sup(A \cup B) = \max(\sup A, \sup B)$.

(54) If $C = C_1 \cup C_2$, then $\text{W-bound}\,C = \min(\text{W-bound}\,C_1, \text{W-bound}\,C_2)$.

(55) If $C = C_1 \cup C_2$, then $\text{S-bound}\,C = \min(\text{S-bound}\,C_1, \text{S-bound}\,C_2)$.

(56) If $C = C_1 \cup C_2$, then $\text{N-bound}\,C = \max(\text{N-bound}\,C_1, \text{N-bound}\,C_2)$.

(57) If $C = C_1 \cup C_2$, then $\text{E-bound}\,C = \max(\text{E-bound}\,C_1, \text{E-bound}\,C_2)$.

Let us consider p, q. One can check that $\mathcal{L}(p, q)$ is compact.

One can verify that $\emptyset_{\mathbb{R}}$ is bounded.

Next we state the proposition

(58) $s_1 \in [r_1, r_2]$ iff $r_1 \leqslant s_1$ and $s_1 \leqslant r_2$.

Let us consider r_1, r_2. One can check that $[r_1, r_2]$ is bounded.

Let us observe that every subset of $\mathbb{R}$ which is bounded is also lower bounded and upper bounded and every subset of $\mathbb{R}$ which is lower bounded and upper bounded is also bounded.

The following propositions are true:

(59) If $r_1 \leqslant r_2$, then $t \in [r_1, r_2]$ iff there exists s_1 such that $0 \leqslant s_1$ and $s_1 \leqslant 1$ and $t = s_1 \cdot r_1 + (1 - s_1) \cdot r_2$.

(60) If $p_{\mathbf{1}} \leqslant q_{\mathbf{1}}$, then $(\mathrm{proj1})^\circ \mathcal{L}(p, q) = [p_{\mathbf{1}}, q_{\mathbf{1}}]$.

(61) If $p_{\mathbf{2}} \leqslant q_{\mathbf{2}}$, then $(\mathrm{proj2})^\circ \mathcal{L}(p, q) = [p_{\mathbf{2}}, q_{\mathbf{2}}]$.

(62) If $p_{\mathbf{1}} \leqslant q_{\mathbf{1}}$, then W-bound $\mathcal{L}(p, q) = p_{\mathbf{1}}$.

(63) If $p_{\mathbf{2}} \leqslant q_{\mathbf{2}}$, then S-bound $\mathcal{L}(p, q) = p_{\mathbf{2}}$.

(64) If $p_{\mathbf{2}} \leqslant q_{\mathbf{2}}$, then N-bound $\mathcal{L}(p, q) = q_{\mathbf{2}}$.

(65) If $p_{\mathbf{1}} \leqslant q_{\mathbf{1}}$, then E-bound $\mathcal{L}(p, q) = q_{\mathbf{1}}$.

(66) W-bound $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) =$ W-bound D.

(67) S-bound $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) =$ S-bound D.

(68) N-bound $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) =$ N-bound D.

(69) E-bound $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) =$ E-bound D.

(70) NW-corner $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) =$ NW-corner D.

(71) NE-corner $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) =$ NE-corner D.

(72) SW-corner $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) =$ SW-corner D.

(73) SE-corner $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) =$ SE-corner D.

(74) W-most $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) = \mathcal{L}(\text{SW-corner}\, D, \text{NW-corner}\, D)$.

(75) N-most $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) = \mathcal{L}(\text{NW-corner}\, D, \text{NE-corner}\, D)$.

(76) S-most $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) = \mathcal{L}(\text{SW-corner}\, D, \text{SE-corner}\, D)$.

(77) E-most $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) = \mathcal{L}(\text{SE-corner}\, D, \text{NE-corner}\, D)$.

(78) $(\mathrm{proj2})^\circ \mathcal{L}(\text{SW-corner}\, D, \text{NW-corner}\, D) = [\text{S-bound}\, D, \text{N-bound}\, D]$.

(79) $(\mathrm{proj1})^\circ \mathcal{L}(\text{NW-corner}\, D, \text{NE-corner}\, D) = [\text{W-bound}\, D, \text{E-bound}\, D]$.

(80) $(\mathrm{proj2})^\circ \mathcal{L}(\text{NE-corner}\, D, \text{SE-corner}\, D) = [\text{S-bound}\, D, \text{N-bound}\, D]$.

(81) $(\mathrm{proj1})^\circ \mathcal{L}(\text{SE-corner}\, D, \text{SW-corner}\, D) = [\text{W-bound}\, D, \text{E-bound}\, D]$.

(82) W-min $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) =$ SW-corner D.

(83) W-max $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) =$ NW-corner D.

(84) N-min $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) =$ NW-corner D.

(85) N-max $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) =$ NE-corner D.

(86) E-min $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) =$ SE-corner D.

(87) E-max $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) =$ NE-corner D.

(88) S-min $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) =$ SW-corner D.

(89) S-max $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) =$ SE-corner D.

7. Rectangular finite suequences of the points of the plane

Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. We say that f is rectangular if and only if:

(Def. 2) There exists D such that $f = \mathrm{SpStSeq}\, D$.

Let us consider D. Note that $\mathrm{SpStSeq}\, D$ is rectangular.

Let us mention that there exists a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ which is rectangular.

In the sequel s denotes a rectangular finite sequence of elements of $\mathcal{E}^2_{\rm T}$.

The following proposition is true

(90) $\mathrm{len}\, s = 5$.

Let us note that every finite sequence of elements of $\mathcal{E}^2_{\rm T}$ which is rectangular is also non constant.

One can verify that every non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ which is rectangular is also standard, special, unfolded, circular, and s.c.c..

In the sequel s is a rectangular finite sequence of elements of $\mathcal{E}^2_{\rm T}$.

Next we state four propositions:

(91) $\pi_1 s = \text{N-min}\, \widetilde{\mathcal{L}}(s)$ and $\pi_1 s = \text{W-max}\, \widetilde{\mathcal{L}}(s)$.

(92) $\pi_2 s = \text{N-max}\, \widetilde{\mathcal{L}}(s)$ and $\pi_2 s = \text{E-max}\, \widetilde{\mathcal{L}}(s)$.

(93) $\pi_3 s = \text{S-max}\, \widetilde{\mathcal{L}}(s)$ and $\pi_3 s = \text{E-min}\, \widetilde{\mathcal{L}}(s)$.

(94) $\pi_4 s = \text{S-min}\, \widetilde{\mathcal{L}}(s)$ and $\pi_4 s = \text{W-min}\, \widetilde{\mathcal{L}}(s)$.

8. Jordan property

One can prove the following proposition

(95) If $r_1 < r_2$ and $s_1 < s_2$, then $[.r_1, r_2, s_1, s_2.]$ is Jordan.

Let f be a rectangular finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Observe that $\widetilde{\mathcal{L}}(f)$ is Jordan.

Let S be a subset of the carrier of $\mathcal{E}^2_{\rm T}$. Let us observe that S is Jordan if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) $S^{\rm c} \neq \emptyset$, and

(ii) there exist subsets A_1, A_2 of the carrier of $\mathcal{E}^2_{\rm T}$ such that $S^{\rm c} = A_1 \cup A_2$ and A_1 misses A_2 and $\overline{A_1} \setminus A_1 = \overline{A_2} \setminus A_2$ and A_1 is a component of $S^{\rm c}$ and A_2 is a component of $S^{\rm c}$.

Next we state the proposition

(96) For every rectangular finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ holds $\mathrm{LeftComp}(f)$ misses $\mathrm{RightComp}(f)$.

Let f be a non constant standard special circular sequence. One can verify that $\mathrm{LeftComp}(f)$ is non empty and $\mathrm{RightComp}(f)$ is non empty.

The following proposition is true

(97) For every rectangular finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{LeftComp}(f) \neq \mathrm{RightComp}(f)$.

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On the Order on a Special Polygon

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Summary. The goal of the article is to determine the order of the special points defined in [10] on a special polygon. We restrict ourselves to the clockwise oriented finite sequences (the concept defined in this article) that start in N-min C (C being a compact non empty subset of the plane).

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The papers [28], [33], [27], [7], [15], [29], [34], [1], [5], [6], [3], [32], [8], [30], [16], [17], [2], [25], [4], [19], [18], [26], [11], [12], [13], [14], [21], [20], [22], [9], [24], [23], [10], and [31] provide the terminology and notation for this paper.

1. Preliminaries

One can prove the following propositions:

(1) For all sets A, B, C, p such that $A \cap B \subseteq \{p\}$ and $p \in C$ and C misses B holds $A \cup C$ misses B.

(2) For all sets A, B, C, p such that $A \cap C = \{p\}$ and $p \in B$ and $B \subseteq C$ holds $A \cap B = \{p\}$.

(3) For all sets A, B such that for every set y such that $y \in B$ holds A misses y holds A misses $\bigcup B$.

(4) For all sets A, B such that for all sets x, y such that $x \in A$ and $y \in B$ holds x misses y holds $\bigcup A$ misses $\bigcup B$.

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2. On the finite sequences

We adopt the following convention: i, j, k, m, n denote natural numbers, D denotes a non empty set, and f denotes a finite sequence of elements of D.

The following propositions are true:

(5) For all i, j, k such that $i \leqslant j$ and $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $k \in \operatorname{dom} \operatorname{mid}(f, i, j)$ holds $(k + i) -' 1 \in \operatorname{dom} f$.

(6) For all i, j, k such that $i > j$ and $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $k \in \operatorname{dom} \operatorname{mid}(f, i, j)$ holds $i -' k + 1 \in \operatorname{dom} f$.

(7) For all i, j, k such that $i \leqslant j$ and $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $k \in \operatorname{dom} \operatorname{mid}(f, i, j)$ holds $\pi_k \operatorname{mid}(f, i, j) = \pi_{(k+i)-'1} f$.

(8) For all i, j, k such that $i > j$ and $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $k \in \operatorname{dom} \operatorname{mid}(f, i, j)$ holds $\pi_k \operatorname{mid}(f, i, j) = \pi_{i-'k+1} f$.

(9) If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$, then $\operatorname{len} \operatorname{mid}(f, i, j) \geqslant 1$.

(10) If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $\operatorname{len} \operatorname{mid}(f, i, j) = 1$, then $i = j$.

(11) If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$, then $\operatorname{mid}(f, i, j)$ is non empty.

(12) If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$, then $\pi_1 \operatorname{mid}(f, i, j) = \pi_i f$.

(13) If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$, then $\pi_{\operatorname{len} \operatorname{mid}(f,i,j)} \operatorname{mid}(f, i, j) = \pi_j f$.

3. Compact subsets of the plane

In the sequel X denotes a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$.

One can prove the following four propositions:

(14) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in X$ and $p_{\mathbf{2}} = \text{N-bound}\, X$ holds $p \in \text{N-most}\, X$.

(15) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in X$ and $p_{\mathbf{2}} = \text{S-bound}\, X$ holds $p \in \text{S-most}\, X$.

(16) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in X$ and $p_{\mathbf{1}} = \text{W-bound}\, X$ holds $p \in \text{W-most}\, X$.

(17) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in X$ and $p_{\mathbf{1}} = \text{E-bound}\, X$ holds $p \in \text{E-most}\, X$.

4. Finite sequences on the plane

We now state several propositions:

(18) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ such that $1 \leqslant i$ and $i \leqslant j$ and $j \leqslant \operatorname{len} f$ holds $\widetilde{\mathcal{L}}(\operatorname{mid}(f,i,j)) = \bigcup\{\mathcal{L}(f,k) : i \leqslant k \ \wedge \ k < j\}$.

(19) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ holds $\operatorname{dom} \mathbf{X}\text{-coordinate}(f) = \operatorname{dom} f$.

(20) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ holds $\operatorname{dom} \mathbf{Y}\text{-coordinate}(f) = \operatorname{dom} f$.

(21) For all points a, b, c of $\mathcal{E}^2_{\rm T}$ such that $b \in \mathcal{L}(a,c)$ and $a_{\mathbf{1}} \leqslant b_{\mathbf{1}}$ and $c_{\mathbf{1}} \leqslant b_{\mathbf{1}}$ holds $a = b$ or $b = c$ or $a_{\mathbf{1}} = b_{\mathbf{1}}$ and $c_{\mathbf{1}} = b_{\mathbf{1}}$.

(22) For all points a, b, c of $\mathcal{E}^2_{\rm T}$ such that $b \in \mathcal{L}(a,c)$ and $a_{\mathbf{2}} \leqslant b_{\mathbf{2}}$ and $c_{\mathbf{2}} \leqslant b_{\mathbf{2}}$ holds $a = b$ or $b = c$ or $a_{\mathbf{2}} = b_{\mathbf{2}}$ and $c_{\mathbf{2}} = b_{\mathbf{2}}$.

(23) For all points a, b, c of $\mathcal{E}^2_{\rm T}$ such that $b \in \mathcal{L}(a,c)$ and $a_{\mathbf{1}} \geqslant b_{\mathbf{1}}$ and $c_{\mathbf{1}} \geqslant b_{\mathbf{1}}$ holds $a = b$ or $b = c$ or $a_{\mathbf{1}} = b_{\mathbf{1}}$ and $c_{\mathbf{1}} = b_{\mathbf{1}}$.

(24) For all points a, b, c of $\mathcal{E}^2_{\rm T}$ such that $b \in \mathcal{L}(a,c)$ and $a_{\mathbf{2}} \geqslant b_{\mathbf{2}}$ and $c_{\mathbf{2}} \geqslant b_{\mathbf{2}}$ holds $a = b$ or $b = c$ or $a_{\mathbf{2}} = b_{\mathbf{2}}$ and $c_{\mathbf{2}} = b_{\mathbf{2}}$.

5. The area of a sequence

Let f be a non trivial finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and let g be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. We say that g is in the area of f if and only if:

(Def. 1) For every n such that $n \in \operatorname{dom} g$ holds $\text{W-bound}\,\widetilde{\mathcal{L}}(f) \leqslant (\pi_n g)_{\mathbf{1}}$ and $(\pi_n g)_{\mathbf{1}} \leqslant \text{E-bound}\,\widetilde{\mathcal{L}}(f)$ and $\text{S-bound}\,\widetilde{\mathcal{L}}(f) \leqslant (\pi_n g)_{\mathbf{2}}$ and $(\pi_n g)_{\mathbf{2}} \leqslant \text{N-bound}\,\widetilde{\mathcal{L}}(f)$.

We now state several propositions:

(25) Every non trivial finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ is in the area of f.

(26) Let f be a non trivial finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and g be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose g is in the area of f. Let given i, j. If $i \in \operatorname{dom} g$ and $j \in \operatorname{dom} g$, then $\operatorname{mid}(g,i,j)$ is in the area of f.

(27) Let f be a non trivial finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and given i, j. If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$, then $\operatorname{mid}(f,i,j)$ is in the area of f.

(28) Let f be a non trivial finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and g, h be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose g is in the area of f and h is in the area of f. Then $g \mathbin{^\frown} h$ is in the area of f.

(29) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ holds $\langle \text{NE-corner}\,\widetilde{\mathcal{L}}(f) \rangle$ is in the area of f.

(30) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ holds $\langle \text{NW-corner}\, \widetilde{\mathcal{L}}(f) \rangle$ is in the area of f.

(31) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ holds $\langle \text{SE-corner}\, \widetilde{\mathcal{L}}(f) \rangle$ is in the area of f.

(32) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ holds $\langle \text{SW-corner}\, \widetilde{\mathcal{L}}(f) \rangle$ is in the area of f.

6. Horizontal and vertical connections

Let f be a non trivial finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and let g be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. We say that g is a h.c. for f if and only if:

(Def. 2) g is in the area of f and $(\pi_1 g)_{\mathbf{1}} = \text{W-bound}\, \widetilde{\mathcal{L}}(f)$ and $(\pi_{\text{len}\, g} g)_{\mathbf{1}} = \text{E-bound}\, \widetilde{\mathcal{L}}(f)$.

We say that g is a v.c. for f if and only if:

(Def. 3) g is in the area of f and $(\pi_1 g)_{\mathbf{2}} = \text{S-bound}\, \widetilde{\mathcal{L}}(f)$ and $(\pi_{\text{len}\, g} g)_{\mathbf{2}} = \text{N-bound}\, \widetilde{\mathcal{L}}(f)$.

Next we state the proposition

(33) Let f be a non trivial finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and g, h be S-sequences in $\mathbb{R}^2$. If g is a h.c. for f and h is a v.c. for f, then $\widetilde{\mathcal{L}}(g)$ meets $\widetilde{\mathcal{L}}(h)$.

7. Orientation

Let f be a non trivial finite sequence of elements of $\mathcal{E}^2_{\rm T}$. We say that f is clockwise oriented if and only if:

(Def. 4) $\pi_2 f_{\circlearrowleft}^{\text{N-min}\, \widetilde{\mathcal{L}}(f)} \in \text{N-most}\, \widetilde{\mathcal{L}}(f)$.

The following proposition is true

(34) Let f be a non constant standard special circular sequence. If $\pi_1 f = \text{N-min}\, \widetilde{\mathcal{L}}(f)$, then f is clockwise oriented iff $\pi_2 f \in \text{N-most}\, \widetilde{\mathcal{L}}(f)$.

Let us note that $\square_{\mathcal{E}^2}$ is compact.

We now state several propositions:

(35) $\text{N-bound}\, \square_{\mathcal{E}^2} = 1$.

(36) $\text{W-bound}\, \square_{\mathcal{E}^2} = 0$.

(37) $\text{E-bound}\, \square_{\mathcal{E}^2} = 1$.

(38) $\text{S-bound}\, \square_{\mathcal{E}^2} = 0$.

(39) $\text{N-most}\, \square_{\mathcal{E}^2} = \mathcal{L}([0,1],[1,1])$.

(40) N-min $\square_{\mathcal{E}^2} = [0,1]$.

Let X be a non vertical non horizontal non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. One can verify that SpStSeq X is clockwise oriented.

One can verify that there exists a non constant standard special circular sequence which is clockwise oriented.

One can prove the following propositions:

(41) Let f be a non constant standard special circular sequence and given i, j. Suppose $i > j$ but $1 < j$ and $i \leqslant \operatorname{len} f$ or $1 \leqslant j$ and $i < \operatorname{len} f$. Then $\operatorname{mid}(f, i, j)$ is a S-sequence in $\mathbb{R}^2$.

(42) Let f be a non constant standard special circular sequence and given i, j. Suppose $i < j$ but $1 < i$ and $j \leqslant \operatorname{len} f$ or $1 \leqslant i$ and $j < \operatorname{len} f$. Then $\operatorname{mid}(f, i, j)$ is a S-sequence in $\mathbb{R}^2$.

In the sequel f is a clockwise oriented non constant standard special circular sequence.

One can prove the following propositions:

(43) N-min $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(44) N-max $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(45) S-min $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(46) S-max $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(47) W-min $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(48) W-max $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(49) E-min $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(50) E-max $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(51) If $1 \leqslant i$ and $i \leqslant j$ and $j < m$ and $m \leqslant n$ and $n \leqslant \operatorname{len} f$ and $1 < i$ or $n < \operatorname{len} f$, then $\widetilde{\mathcal{L}}(\operatorname{mid}(f, i, j))$ misses $\widetilde{\mathcal{L}}(\operatorname{mid}(f, m, n))$.

(52) If $1 \leqslant i$ and $i \leqslant j$ and $j < m$ and $m \leqslant n$ and $n \leqslant \operatorname{len} f$ and $1 < i$ or $n < \operatorname{len} f$, then $\widetilde{\mathcal{L}}(\operatorname{mid}(f, i, j))$ misses $\widetilde{\mathcal{L}}(\operatorname{mid}(f, n, m))$.

(53) If $1 \leqslant i$ and $i \leqslant j$ and $j < m$ and $m \leqslant n$ and $n \leqslant \operatorname{len} f$ and $1 < i$ or $n < \operatorname{len} f$, then $\widetilde{\mathcal{L}}(\operatorname{mid}(f, j, i))$ misses $\widetilde{\mathcal{L}}(\operatorname{mid}(f, n, m))$.

(54) If $1 \leqslant i$ and $i \leqslant j$ and $j < m$ and $m \leqslant n$ and $n \leqslant \operatorname{len} f$ and $1 < i$ or $n < \operatorname{len} f$, then $\widetilde{\mathcal{L}}(\operatorname{mid}(f, j, i))$ misses $\widetilde{\mathcal{L}}(\operatorname{mid}(f, m, n))$.

(55) $(\text{N-min}\,\widetilde{\mathcal{L}}(f))_{\mathbf{1}} < (\text{N-max}\,\widetilde{\mathcal{L}}(f))_{\mathbf{1}}$.

(56) N-min $\widetilde{\mathcal{L}}(f) \neq$ N-max $\widetilde{\mathcal{L}}(f)$.

(57) $(\text{E-min}\,\widetilde{\mathcal{L}}(f))_{\mathbf{2}} < (\text{E-max}\,\widetilde{\mathcal{L}}(f))_{\mathbf{2}}$.

(58) E-min $\widetilde{\mathcal{L}}(f) \neq$ E-max $\widetilde{\mathcal{L}}(f)$.

(59) $(\text{S-min}\,\widetilde{\mathcal{L}}(f))_{\mathbf{1}} < (\text{S-max}\,\widetilde{\mathcal{L}}(f))_{\mathbf{1}}$.

(60) S-min $\widetilde{\mathcal{L}}(f) \neq$ S-max $\widetilde{\mathcal{L}}(f)$.

(61) $(\text{W-min}\,\widetilde{\mathcal{L}}(f))_{\mathbf{2}} < (\text{W-max}\,\widetilde{\mathcal{L}}(f))_{\mathbf{2}}$.

(62) W-min $\widetilde{\mathcal{L}}(f) \neq$ W-max $\widetilde{\mathcal{L}}(f)$.

(63) $\mathcal{L}(\text{NW-corner}\,\widetilde{\mathcal{L}}(f), \text{N-min}\,\widetilde{\mathcal{L}}(f))$ misses $\mathcal{L}(\text{N-max}\,\widetilde{\mathcal{L}}(f), \text{NE-corner}\,\widetilde{\mathcal{L}}(f))$.

(64) Let f be a S-sequence in $\mathbb{R}^2$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p \neq \pi_1 f$ but $p_{\mathbf{1}} = (\pi_1 f)_{\mathbf{1}}$ or $p_{\mathbf{2}} = (\pi_1 f)_{\mathbf{2}}$ but $\mathcal{L}(p, \pi_1 f) \cap \widetilde{\mathcal{L}}(f) = \{\pi_1 f\}$. Then $\langle p \rangle \mathbin{^\frown} f$ is a S-sequence in $\mathbb{R}^2$.

(65) Let f be a S-sequence in $\mathbb{R}^2$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p \neq \pi_{\mathrm{len}\,f} f$ but $p_{\mathbf{1}} = (\pi_{\mathrm{len}\,f} f)_{\mathbf{1}}$ or $p_{\mathbf{2}} = (\pi_{\mathrm{len}\,f} f)_{\mathbf{2}}$ but $\mathcal{L}(p, \pi_{\mathrm{len}\,f} f) \cap \widetilde{\mathcal{L}}(f) = \{\pi_{\mathrm{len}\,f} f\}$. Then $f \mathbin{^\frown} \langle p \rangle$ is a S-sequence in $\mathbb{R}^2$.

8. Appending corners

We now state several propositions:

(66) Let given i, j. Suppose $i \in \mathrm{dom}\, f$ and $j \in \mathrm{dom}\, f$ and $\mathrm{mid}(f, i, j)$ is a S-sequence in $\mathbb{R}^2$ and $\pi_j f = \text{N-max}\,\widetilde{\mathcal{L}}(f)$ and $\text{N-max}\,\widetilde{\mathcal{L}}(f) \neq \text{NE-corner}\,\widetilde{\mathcal{L}}(f)$. Then $(\mathrm{mid}(f, i, j)) \mathbin{^\frown} \langle \text{NE-corner}\,\widetilde{\mathcal{L}}(f) \rangle$ is a S-sequence in $\mathbb{R}^2$.

(67) Let given i, j. Suppose $i \in \mathrm{dom}\, f$ and $j \in \mathrm{dom}\, f$ and $\mathrm{mid}(f, i, j)$ is a S-sequence in $\mathbb{R}^2$ and $\pi_j f = \text{E-max}\,\widetilde{\mathcal{L}}(f)$ and $\text{E-max}\,\widetilde{\mathcal{L}}(f) \neq \text{NE-corner}\,\widetilde{\mathcal{L}}(f)$. Then $(\mathrm{mid}(f, i, j)) \mathbin{^\frown} \langle \text{NE-corner}\,\widetilde{\mathcal{L}}(f) \rangle$ is a S-sequence in $\mathbb{R}^2$.

(68) Let given i, j. Suppose $i \in \mathrm{dom}\, f$ and $j \in \mathrm{dom}\, f$ and $\mathrm{mid}(f, i, j)$ is a S-sequence in $\mathbb{R}^2$ and $\pi_j f = \text{S-max}\,\widetilde{\mathcal{L}}(f)$ and $\text{S-max}\,\widetilde{\mathcal{L}}(f) \neq \text{SE-corner}\,\widetilde{\mathcal{L}}(f)$. Then $(\mathrm{mid}(f, i, j)) \mathbin{^\frown} \langle \text{SE-corner}\,\widetilde{\mathcal{L}}(f) \rangle$ is a S-sequence in $\mathbb{R}^2$.

(69) Let given i, j. Suppose $i \in \mathrm{dom}\, f$ and $j \in \mathrm{dom}\, f$ and $\mathrm{mid}(f, i, j)$ is a S-sequence in $\mathbb{R}^2$ and $\pi_j f = \text{E-max}\,\widetilde{\mathcal{L}}(f)$ and $\text{E-max}\,\widetilde{\mathcal{L}}(f) \neq \text{NE-corner}\,\widetilde{\mathcal{L}}(f)$. Then $(\mathrm{mid}(f, i, j)) \mathbin{^\frown} \langle \text{NE-corner}\,\widetilde{\mathcal{L}}(f) \rangle$ is a S-sequence in $\mathbb{R}^2$.

(70) Let given i, j. Suppose $i \in \mathrm{dom}\, f$ and $j \in \mathrm{dom}\, f$ and $\mathrm{mid}(f, i, j)$ is a S-sequence in $\mathbb{R}^2$ and $\pi_i f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$ and $\text{N-min}\,\widetilde{\mathcal{L}}(f) \neq \text{NW-corner}\,\widetilde{\mathcal{L}}(f)$. Then $\langle \text{NW-corner}\,\widetilde{\mathcal{L}}(f) \rangle \mathbin{^\frown} \mathrm{mid}(f, i, j)$ is a S-sequence in $\mathbb{R}^2$.

(71) Let given i, j. Suppose $i \in \mathrm{dom}\, f$ and $j \in \mathrm{dom}\, f$ and $\mathrm{mid}(f, i, j)$ is a S-sequence in $\mathbb{R}^2$ and $\pi_i f = \text{W-min}\,\widetilde{\mathcal{L}}(f)$ and $\text{W-min}\,\widetilde{\mathcal{L}}(f) \neq \text{SW-corner}\,\widetilde{\mathcal{L}}(f)$. Then $\langle \text{SW-corner}\,\widetilde{\mathcal{L}}(f) \rangle \mathbin{^\frown} \mathrm{mid}(f, i, j)$ is a S-sequence in $\mathbb{R}^2$.

Let f be a non constant standard special circular sequence. One can check that $\widetilde{\mathcal{L}}(f)$ is simple closed curve.

9. The order

We now state a number of propositions:

(72) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{N-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < (\text{N-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f$.

(73) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{N-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f > 1$.

(74) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$ and $\text{N-max}\,\widetilde{\mathcal{L}}(f) \neq \text{E-max}\,\widetilde{\mathcal{L}}(f)$, then $(\text{N-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < (\text{E-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f$.

(75) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{E-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < (\text{E-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f$.

(76) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$ and $\text{E-min}\,\widetilde{\mathcal{L}}(f) \neq \text{S-max}\,\widetilde{\mathcal{L}}(f)$, then $(\text{E-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < (\text{S-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f$.

(77) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{S-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < (\text{S-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f$.

(78) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$ and $\text{S-min}\,\widetilde{\mathcal{L}}(f) \neq \text{W-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{S-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < (\text{W-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f$.

(79) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$ and $\text{N-min}\,\widetilde{\mathcal{L}}(f) \neq \text{W-max}\,\widetilde{\mathcal{L}}(f)$, then $(\text{W-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < (\text{W-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f$.

(80) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{W-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < \operatorname{len} f$.

(81) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{W-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < \operatorname{len} f$.

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The Euler's Function

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Summary. This article is concerned with the Euler's function [10] that plays an important role in cryptograms. In the first section, we present some selected theorems on integers. Next, we define the Euler's function. Finally, three theorems relating to the Euler's function are proved. The third theorem concerns two relatively prime integers which make up the Euler's function parameter. In the public key cryptography these two integer values are used as public and secret keys.

MML Identifier: EULER_1.

The notation and terminology used here are introduced in the following papers: [12], [6], [1], [13], [9], [2], [3], [7], [8], [14], [11], [15], [4], and [5].

1. Preliminary

We follow the rules: a, b, c, k, l, m, n are natural numbers and i, j, x, y are integers.

The following propositions are true:

(1) $k \in n$ iff $k < n$.

(2) n and n are relative prime iff $n = 1$.

(3) If $k \neq 0$ and $k < n$ and n is prime, then k and n are relative prime.

(4) n is prime and $k \in \{k_1; k_1$ ranges over natural numbers: n and k_1 are relative prime $\wedge\ k_1 \geqslant 1\ \wedge\ k_1 \leqslant n\}$ if and only if n is prime and $k \in n$ and $k \notin \{0\}$.

(5) For every finite set A and for every set x such that $x \in A$ holds $\overline{\overline{A \setminus \{x\}}} = \overline{\overline{A}} - \overline{\overline{\{x\}}}$.

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(6) If $\gcd(a, b) = 1$, then for every c holds $\gcd(a \cdot c, b \cdot c) = c$.

(7) If $a \neq 0$ and $b \neq 0$ and $c \neq 0$ and $\gcd(a \cdot c, b \cdot c) = c$, then a and b are relative prime.

(8) If $\gcd(a, b) = 1$, then $\gcd(a + b, b) = 1$.

(9) For every c holds $\gcd(a + b \cdot c, b) = \gcd(a, b)$.

(10) Suppose m and n are relative prime. Then there exists k such that
  (i) there exist integers i_0, j_0 such that $k = i_0 \cdot m + j_0 \cdot n$ and $k > 0$, and
  (ii) for every l such that there exist integers i, j such that $l = i \cdot m + j \cdot n$ and $l > 0$ holds $k \leqslant l$.

(11) If m and n are relative prime, then for every k there exist i, j such that $i \cdot m + j \cdot n = k$.

(12) For all non empty finite sets A, B such that there exists a function from A into B which is one-to-one and onto holds $\overline{\overline{A}} = \overline{\overline{B}}$.

(13) For all integers i, k, n such that $n \neq 0$ holds $(i + k \cdot n) \bmod n = i \bmod n$.

(14) If $a \neq 0$ and $b \neq 0$ and $c \neq 0$ and $c \mid a \cdot b$ and a and c are relative prime, then $c \mid b$.

(15) Suppose $a \neq 0$ and $b \neq 0$ and $c \neq 0$ and a and c are relative prime and b and c are relative prime. Then $a \cdot b$ and c are relative prime.

(16) If $x \neq 0$ and $y \neq 0$ and $i > 0$, then $i \cdot x \operatorname{gcd} i \cdot y = i \cdot (x \operatorname{gcd} y)$.

(17) For every x such that $a \neq 0$ and $b \neq 0$ holds $a + x \cdot b \operatorname{gcd} b = a \operatorname{gcd} b$.

2. Definition of Euler's Function

Let n be a natural number. The functor $\mathrm{Euler}\, n$ yields a natural number and is defined as follows:

(Def. 1) $\mathrm{Euler}\, n = \overline{\overline{\{k; k \text{ ranges over natural numbers: } n \text{ and } k \text{ are relative prime} \wedge k \geqslant 1 \wedge k \leqslant n\}}}$.

We now state several propositions:

(18) $\mathrm{Euler}\, 1 = 1$.

(19) $\mathrm{Euler}\, 2 = 1$.

(20) If $n > 1$, then $\mathrm{Euler}\, n \leqslant n - 1$.

(21) If n is prime, then $\mathrm{Euler}\, n = n - 1$.

(22) If $m > 1$ and $n > 1$ and m and n are relative prime, then $\mathrm{Euler}\, m \cdot n = \mathrm{Euler}\, m \cdot \mathrm{Euler}\, n$.

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While Macro Instructions of $\mathbf{SCM}_{\mathrm{FSA}}$

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Summary. The article defines *while macro instructions* based on $\mathbf{SCM}_{\mathrm{FSA}}$. Some theorems about the generalized halting problems of *while macro instructions* are proved.

MML Identifier: `SCMFSA_9`.

The notation and terminology used in this paper are introduced in the following papers: [24], [32], [19], [8], [13], [33], [15], [16], [17], [12], [34], [7], [10], [14], [31], [18], [9], [20], [21], [25], [11], [23], [30], [29], [26], [27], [1], [28], [22], [5], [6], [4], [2], and [3].

The following propositions are true:

(1) For every macro instruction I and for every integer location a holds $\operatorname{card} \mathit{if} = 0(a, I; \mathrm{Goto}(\mathrm{insloc}(0)), \mathrm{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}) = \operatorname{card} I + 6$.

(2) For every macro instruction I and for every integer location a holds $\operatorname{card} \mathit{if} > 0(a, I; \mathrm{Goto}(\mathrm{insloc}(0)), \mathrm{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}) = \operatorname{card} I + 6$.

Let a be an integer location and let I be a macro instruction. The functor $\mathit{while} = 0(a, I)$ yielding a macro instruction is defined as follows:

(Def. 1) $\mathit{while} = 0(a, I) = \mathit{if} = 0(a, I; \mathrm{Goto}(\mathrm{insloc}(0)), \mathrm{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}) {+\cdot} (\mathrm{insloc}(\operatorname{card} I + 4) \mapsto \mathrm{goto}\ \mathrm{insloc}(0))$.

The functor $\mathit{while} > 0(a, I)$ yielding a macro instruction is defined by:

(Def. 2) $\mathit{while} > 0(a, I) = \mathit{if} > 0(a, I; \mathrm{Goto}(\mathrm{insloc}(0)), \mathrm{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}) {+\cdot} (\mathrm{insloc}(\operatorname{card} I + 4) \mapsto \mathrm{goto}\ \mathrm{insloc}(0))$.

The following proposition is true

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(3) For every macro instruction I and for every integer location a holds $\mathrm{card}\, if = 0(a, \mathrm{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}, if > 0(a, \mathrm{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}, I; \mathrm{Goto}(\mathrm{insloc}(0)))) = \mathrm{card}\, I + 11$.

Let a be an integer location and let I be a macro instruction. The functor $while < 0(a, I)$ yields a macro instruction and is defined as follows:

(Def. 3) $while < 0(a, I) = if = 0(a, \mathrm{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}, if > 0(a, \mathrm{Stop}_{\mathrm{SCM}_{\mathrm{FSA}}}, I; \mathrm{Goto}$ $(\mathrm{insloc}(0))))+\cdot(\mathrm{insloc}(\mathrm{card}\, I + 4) \mapsto \mathrm{goto}\ \mathrm{insloc}(0))$.

Next we state a number of propositions:

(4) For every macro instruction I and for every integer location a holds $\mathrm{card}\, while = 0(a, I) = \mathrm{card}\, I + 6$.

(5) For every macro instruction I and for every integer location a holds $\mathrm{card}\, while > 0(a, I) = \mathrm{card}\, I + 6$.

(6) For every macro instruction I and for every integer location a holds $\mathrm{card}\, while < 0(a, I) = \mathrm{card}\, I + 11$.

(7) For every integer location a and for every instruction-location l of $\mathbf{SCM}_{\mathrm{FSA}}$ holds $\mathbf{if}\ a = 0\ \mathbf{goto}\ l \neq \mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}}$.

(8) For every integer location a and for every instruction-location l of $\mathbf{SCM}_{\mathrm{FSA}}$ holds $\mathbf{if}\ a > 0\ \mathbf{goto}\ l \neq \mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}}$.

(9) For every instruction-location l of $\mathbf{SCM}_{\mathrm{FSA}}$ holds $\mathrm{goto}\ l \neq \mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}}$.

(10) Let a be an integer location and I be a macro instruction. Then $\mathrm{insloc}(0) \in \mathrm{dom}\, while = 0(a, I)$ and $\mathrm{insloc}(1) \in \mathrm{dom}\, while = 0(a, I)$ and $\mathrm{insloc}(0) \in \mathrm{dom}\, while > 0(a, I)$ and $\mathrm{insloc}(1) \in \mathrm{dom}\, while > 0(a, I)$.

(11) Let a be an integer location and I be a macro instruction. Then $(while = 0(a, I))(\mathrm{insloc}(0)) = \mathbf{if}\ a = 0\ \mathbf{goto}\ \mathrm{insloc}(4)$ and $(while = 0(a, I))(\mathrm{insloc}(1)) = \mathrm{goto}\ \mathrm{insloc}(2)$ and $(while > 0(a, I))(\mathrm{insloc}(0)) = \mathbf{if}\ a > 0\ \mathbf{goto}\ \mathrm{insloc}(4)$ and $(while > 0(a, I))(\mathrm{insloc}(1)) = \mathrm{goto}\ \mathrm{insloc}(2)$.

(12) Let a be an integer location, I be a macro instruction, and k be a natural number. If $k < 6$, then $\mathrm{insloc}(k) \in \mathrm{dom}\, while = 0(a, I)$.

(13) Let a be an integer location, I be a macro instruction, and k be a natural number. If $k < 6$, then $\mathrm{insloc}(\mathrm{card}\, I + k) \in \mathrm{dom}\, while = 0(a, I)$.

(14) For every integer location a and for every macro instruction I holds $(while = 0(a, I))(\mathrm{insloc}(\mathrm{card}\, I + 5)) = \mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}}$.

(15) For every integer location a and for every macro instruction I holds $(while = 0(a, I))(\mathrm{insloc}(3)) = \mathrm{goto}\ \mathrm{insloc}(\mathrm{card}\, I + 5)$.

(16) For every integer location a and for every macro instruction I holds $(while = 0(a, I))(\mathrm{insloc}(2)) = \mathrm{goto}\ \mathrm{insloc}(3)$.

(17) Let a be an integer location, I be a macro instruction, and k be a natural number. If $k < \mathrm{card}\, I + 6$, then $\mathrm{insloc}(k) \in \mathrm{dom}\, while = 0(a, I)$.

(18) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I be a macro instruction, and a be a read-write integer location. If $s(a) \neq 0$, then $while = 0(a, I)$ is halting on s and $while = 0(a, I)$ is closed on s.

(19) Let a be an integer location, I be a macro instruction, s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, and k be a natural number. Suppose that

(i) I is closed on s and halting on s,

(ii) $k < \mathrm{LifeSpan}(s{+}\cdot(I{+}\cdot \text{Start-At}(\mathrm{insloc}(0))))$,

(iii) $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot(while=0(a,I)+\cdot\,\text{Start-At}(\mathrm{insloc}(0)))))(1+k)} = \mathbf{IC}_{(\mathrm{Computation}(s+\cdot(I+\cdot\,\text{Start-At}(\mathrm{insloc}(0)))))(k)} + 4$, and

(iv) $(\mathrm{Computation}(s{+}\cdot(while = 0(a, I){+}\cdot \text{Start-At}(\mathrm{insloc}(0)))))(1 + k) \restriction(\text{Int-Locations} \cup \text{FinSeq-Locations}) = (\mathrm{Computation}(s{+}\cdot(I{+}\cdot \text{Start-At}(\mathrm{insloc}(0)))))(k)\restriction(\text{Int-Locations} \cup \text{FinSeq-Locations})$.

Then $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot(while=0(a,I)+\cdot\,\text{Start-At}(\mathrm{insloc}(0)))))(1+k+1)} = \mathbf{IC}_{(\mathrm{Computation}(s+\cdot(I+\cdot\,\text{Start-At}(\mathrm{insloc}(0)))))(k+1)} + 4$ and $(\mathrm{Computation}(s{+}\cdot(while = 0(a, I){+}\cdot \text{Start-At}(\mathrm{insloc}(0)))))(1 + k + 1)\restriction(\text{Int-Locations} \cup \text{FinSeq-Locations}) = (\mathrm{Computation}(s{+}\cdot(I{+}\cdot \text{Start-At}(\mathrm{insloc}(0)))))(k + 1)\restriction(\text{Int-Locations} \cup \text{FinSeq-Locations})$.

(20) Let a be an integer location, I be a macro instruction, and s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$. Suppose I is closed on s and halting on s and $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot(while=0(a,I)+\cdot\,\text{Start-At}(\mathrm{insloc}(0)))))(1+\mathrm{LifeSpan}(s+\cdot(I+\cdot\,\text{Start-At}(\mathrm{insloc}(0)))))} = \mathbf{IC}_{(\mathrm{Computation}(s+\cdot(I+\cdot\,\text{Start-At}(\mathrm{insloc}(0)))))(\mathrm{LifeSpan}(s+\cdot(I+\cdot\,\text{Start-At}(\mathrm{insloc}(0)))))} + 4$. Then $\mathrm{CurInstr}((\mathrm{Computation}(s{+}\cdot(while = 0(a, I){+}\cdot \text{Start-At}(\mathrm{insloc}(0)))))(1 + \mathrm{LifeSpan}(s{+}\cdot(I{+}\cdot \text{Start-At}(\mathrm{insloc}(0)))))) = \mathbf{goto}\ \mathrm{insloc}(\mathrm{card}\, I + 4)$.

(21) For every integer location a and for every macro instruction I holds $(while = 0(a, I))(\mathrm{insloc}(\mathrm{card}\, I + 4)) = \mathbf{goto}\ \mathrm{insloc}(0)$.

(22) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I be a macro instruction, and a be a read-write integer location. Suppose I is closed on s and halting on s and $s(a) = 0$. Then $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot(while=0(a,I)+\cdot\,\text{Start-At}(\mathrm{insloc}(0)))))(\mathrm{LifeSpan}(s+\cdot(I+\cdot\,\text{Start-At}(\mathrm{insloc}(0))))+3)} = \mathrm{insloc}(0)$ and for every natural number k such that $k \leqslant \mathrm{LifeSpan}(s{+}\cdot(I{+}\cdot \text{Start-At}(\mathrm{insloc}(0)))) + 3$ holds $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot(while=0(a,I)+\cdot\,\text{Start-At}(\mathrm{insloc}(0)))))(k)} \in \mathrm{dom}\, while = 0(a, I)$.

In the sequel s denotes a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I denotes a macro instruction, and a denotes a read-write integer location.

Let us consider s, I, a. The functor $StepWhile = 0(a, I, s)$ yields a function from $\mathbb{N}$ into $\prod$(the object kind of $\mathbf{SCM}_{\mathrm{FSA}}$) and is defined by the conditions (Def. 4).

(Def. 4)(i) $(StepWhile = 0(a, I, s))(0) = s$, and

(ii) for every natural number i and for every element x of $\prod$(the object kind of $\mathbf{SCM}_{\mathrm{FSA}}$) such that $x = (StepWhile = 0(a, I, s))(i)$

holds $(StepWhile = 0(a, I, s))(i + 1) = (\mathrm{Computation}(x{+\cdot}(while = 0(a, I){+\cdot}s_0)))(\mathrm{LifeSpan}(x{+\cdot}(I{+\cdot}s_0)) + 3)$.

In the sequel k, n are natural numbers.

We now state three propositions:

(23) $(StepWhile = 0(a, I, s))(0) = s$.

(24) $(StepWhile = 0(a, I, s))(k + 1) = (\mathrm{Computation}((StepWhile = 0(a, I, s))(k){+\cdot}(while = 0(a, I){+\cdot}s_0)))(\mathrm{LifeSpan}((StepWhile = 0(a, I, s))(k){+\cdot}(I{+\cdot}s_0)) + 3)$.

(25) $(StepWhile = 0(a, I, s))(k + 1) = (StepWhile = 0(a, I, (StepWhile = 0(a, I, s))(k)))(1)$.

The scheme *MinIndex* deals with a unary functor $\mathcal{F}$ yielding a natural number and a natural number $\mathcal{A}$, and states that:

There exists k such that $\mathcal{F}(k) = 0$ and for every n such that $\mathcal{F}(n) = 0$ holds $k \leqslant n$

provided the parameters meet the following conditions:

- $\mathcal{F}(0) = \mathcal{A}$, and
- For every k holds $\mathcal{F}(k + 1) < \mathcal{F}(k)$ or $\mathcal{F}(k) = 0$.

We now state a number of propositions:

(26) For all functions f, g holds $f{+\cdot}g{+\cdot}g = f{+\cdot}g$.

(27) For all functions f, g, h and for every set D such that $(f{+\cdot}g){\restriction}D = h{\restriction}D$ holds $(h{+\cdot}g){\restriction}D = (f{+\cdot}g){\restriction}D$.

(28) For all functions f, g, h and for every set D such that $f{\restriction}D = h{\restriction}D$ holds $(h{+\cdot}g){\restriction}D = (f{+\cdot}g){\restriction}D$.

(29) For all states s_1, s_2 of $\mathbf{SCM}_{\mathrm{FSA}}$ such that $\mathbf{IC}_{(s_1)} = \mathbf{IC}_{(s_2)}$ and $s_1{\restriction}(\mathrm{Int\text{-}Locations} \cup \mathrm{FinSeq\text{-}Locations}) = s_2{\restriction}(\mathrm{Int\text{-}Locations} \cup \mathrm{FinSeq\text{-}Locations})$ and $s_1{\restriction}I_1 = s_2{\restriction}I_1$ holds $s_1 = s_2$.

(30) Let I be a macro instruction, a be a read-write integer location, and s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$. Then $(StepWhile = 0(a, I, s))(0 + 1) = (\mathrm{Computation}(s{+\cdot}(while = 0(a, I){+\cdot}s_0)))(\mathrm{LifeSpan}(s{+\cdot}(I{+\cdot}s_0)) + 3)$.

(31) Let I be a macro instruction, a be a read-write integer location, s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, and k, n be natural numbers. Suppose $\mathbf{IC}_{(StepWhile=0(a,I,s))(k)} = \mathrm{insloc}(0)$ and $(StepWhile = 0(a, I, s))(k) = (\mathrm{Computation}(s{+\cdot}(while = 0(a, I){+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))(n)$. Then $(StepWhile = 0(a, I, s))(k) = (StepWhile = 0(a, I, s))(k){+\cdot}(while = 0(a, I){+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))$ and $(StepWhile = 0(a, I, s))(k + 1) = (\mathrm{Computation}(s{+\cdot}(while = 0(a, I){+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))(n + (\mathrm{LifeSpan}((StepWhile = 0(a, I, s))(k){+\cdot}(I{+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))) + 3))$.

(32) Let I be a macro instruction, a be a read-write integer location, and s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$. Suppose that

(i) for every natural number k holds I is closed on $(\mathit{StepWhile}=0(a,I,s))(k)$ and halting on $(\mathit{StepWhile}=0(a,I,s))(k)$, and

(ii) there exists a function f from $\prod$(the object kind of $\mathbf{SCM}_{\mathrm{FSA}}$) into $\mathbb{N}$ such that for every natural number k holds $f((\mathit{StepWhile}=0(a,I,s))(k+1)) < f((\mathit{StepWhile}=0(a,I,s))(k))$ or $f((\mathit{StepWhile}=0(a,I,s))(k)) = 0$ but $f((\mathit{StepWhile}=0(a,I,s))(k)) = 0$ iff $(\mathit{StepWhile}=0(a,I,s))(k)(a) \neq 0$.

Then $\mathit{while}=0(a,I)$ is halting on s and $\mathit{while}=0(a,I)$ is closed on s.

(33) Let I be a parahalting macro instruction, a be a read-write integer location, and s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$. Given a function f from $\prod$(the object kind of $\mathbf{SCM}_{\mathrm{FSA}}$) into $\mathbb{N}$ such that let k be a natural number. Then $f((\mathit{StepWhile}=0(a,I,s))(k+1)) < f((\mathit{StepWhile}=0(a,I,s))(k))$ or $f((\mathit{StepWhile}=0(a,I,s))(k)) = 0$ but $f((\mathit{StepWhile}=0(a,I,s))(k)) = 0$ iff $(\mathit{StepWhile}=0(a,I,s))(k)(a) \neq 0$. Then $\mathit{while}=0(a,I)$ is halting on s and $\mathit{while}=0(a,I)$ is closed on s.

(34) Let I be a parahalting macro instruction and a be a read-write integer location. Given a function f from $\prod$(the object kind of $\mathbf{SCM}_{\mathrm{FSA}}$) into $\mathbb{N}$ such that let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$. Then $f((\mathit{StepWhile}=0(a,I,s))(1)) < f(s)$ or $f(s)=0$ but $f(s)=0$ iff $s(a)\neq 0$. Then $\mathit{while}=0(a,I)$ is parahalting.

(35) For all instructions-locations l_1, l_2 of $\mathbf{SCM}_{\mathrm{FSA}}$ and for every integer location a holds $l_1 \dot{\longmapsto} \mathrm{goto}\ l_2$ does not destroy a.

(36) For every instruction i of $\mathbf{SCM}_{\mathrm{FSA}}$ such that i does not destroy $\mathrm{intloc}(0)$ holds $\mathrm{Macro}(i)$ is good.

Let I, J be good macro instructions and let a be an integer location. Note that $\mathit{if}=0(a,I,J)$ is good.

Let I be a good macro instruction and let a be an integer location. One can verify that $\mathit{while}=0(a,I)$ is good.

We now state a number of propositions:

(37) Let a be an integer location, I be a macro instruction, and k be a natural number. If $k<6$, then $\mathrm{insloc}(k)\in \mathrm{dom}\,\mathit{while}>0(a,I)$.

(38) Let a be an integer location, I be a macro instruction, and k be a natural number. If $k<6$, then $\mathrm{insloc}(\mathrm{card}\,I+k)\in \mathrm{dom}\,\mathit{while}>0(a,I)$.

(39) For every integer location a and for every macro instruction I holds $(\mathit{while}>0(a,I))(\mathrm{insloc}(\mathrm{card}\,I+5)) = \mathbf{halt}_{\mathbf{SCM}_{\mathrm{FSA}}}$.

(40) For every integer location a and for every macro instruction I holds $(\mathit{while}>0(a,I))(\mathrm{insloc}(3)) = \mathrm{goto}\ \mathrm{insloc}(\mathrm{card}\,I+5)$.

(41) For every integer location a and for every macro instruction I holds $(\mathit{while}>0(a,I))(\mathrm{insloc}(2)) = \mathrm{goto}\ \mathrm{insloc}(3)$.

(42) Let a be an integer location, I be a macro instruction, and k be a natural

number. If $k < \operatorname{card} I + 6$, then $\operatorname{insloc}(k) \in \operatorname{dom} while > 0(a, I)$.

(43) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I be a macro instruction, and a be a read-write integer location. If $s(a) \leqslant 0$, then $while > 0(a, I)$ is halting on s and $while > 0(a, I)$ is closed on s.

(44) Let a be an integer location, I be a macro instruction, s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, and k be a natural number. Suppose that

(i) I is closed on s and halting on s,

(ii) $k < \mathrm{LifeSpan}(s{+\cdot}(I{+\cdot}\,\text{Start-At}(\operatorname{insloc}(0))))$,

(iii) $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot(while>0(a,I)+\cdot\,\text{Start-At}(\operatorname{insloc}(0)))))(1+k)} = \mathbf{IC}_{(\mathrm{Computation}(s+\cdot(I+\cdot\,\text{Start-At}(\operatorname{insloc}(0)))))(k)} + 4$, and

(iv) $(\mathrm{Computation}(s{+\cdot}(while > 0(a, I){+\cdot}\,\text{Start-At}(\operatorname{insloc}(0)))))(1+k){\restriction}D = (\mathrm{Computation}(s{+\cdot}(I{+\cdot}\,\text{Start-At}(\operatorname{insloc}(0)))))(k){\restriction}D$.

Then $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot(while>0(a,I)+\cdot\,\text{Start-At}(\operatorname{insloc}(0)))))(1+k+1)} = \mathbf{IC}_{(\mathrm{Computation}(s+\cdot(I+\cdot\,\text{Start-At}(\operatorname{insloc}(0)))))(k+1)} + 4$ and $(\mathrm{Computation}(s{+\cdot}(while > 0(a, I){+\cdot}\,\text{Start-At}(\operatorname{insloc}(0)))))(1 + k + 1){\restriction}D = (\mathrm{Computation}(s{+\cdot}(I{+\cdot}\,\text{Start-At}(\operatorname{insloc}(0)))))(k + 1){\restriction}D$.

(45) Let a be an integer location, I be a macro instruction, and s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$. Suppose I is closed on s and halting on s and $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot(while>0(a,I)+\cdot\,\text{Start-At}(\operatorname{insloc}(0)))))(1+\mathrm{LifeSpan}(s+\cdot(I+\cdot\,\text{Start-At}(\operatorname{insloc}(0)))))} = \mathbf{IC}_{(\mathrm{Computation}(s+\cdot(I+\cdot\,\text{Start-At}(\operatorname{insloc}(0)))))(\mathrm{LifeSpan}(s+\cdot(I+\cdot\,\text{Start-At}(\operatorname{insloc}(0)))))} + 4$. Then $\mathrm{CurInstr}((\mathrm{Computation}(s{+\cdot}(while > 0(a, I){+\cdot}\,\text{Start-At}(\operatorname{insloc}(0)))))(1 + \mathrm{LifeSpan}(s{+\cdot}(I{+\cdot}\,\text{Start-At}(\operatorname{insloc}(0)))))) = \text{goto } \operatorname{insloc}(\operatorname{card} I + 4)$.

(46) For every integer location a and for every macro instruction I holds $(while > 0(a, I))(\operatorname{insloc}(\operatorname{card} I + 4)) = \text{goto } \operatorname{insloc}(0)$.

(47) Let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I be a macro instruction, and a be a read-write integer location. Suppose I is closed on s and halting on s and $s(a) > 0$.
Then $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot(while>0(a,I)+\cdot\,\text{Start-At}(\operatorname{insloc}(0)))))(\mathrm{LifeSpan}(s+\cdot(I+\cdot\,\text{Start-At}(\operatorname{insloc}(0))))+3)} = \operatorname{insloc}(0)$ and for every natural number k such that $k \leqslant \mathrm{LifeSpan}(s{+\cdot}(I{+\cdot}\,\text{Start-At}(\operatorname{insloc}(0)))) + 3$ holds $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot(while>0(a,I)+\cdot\,\text{Start-At}(\operatorname{insloc}(0)))))(k)} \in \operatorname{dom} while > 0(a, I)$.

In the sequel s denotes a state of $\mathbf{SCM}_{\mathrm{FSA}}$, I denotes a macro instruction, and a denotes a read-write integer location.

Let us consider s, I, a. The functor $StepWhile > 0(a, I, s)$ yielding a function from $\mathbb{N}$ into $\prod$(the object kind of $\mathbf{SCM}_{\mathrm{FSA}}$) is defined by the conditions (Def. 5).

(Def. 5)(i) $(StepWhile > 0(a, I, s))(0) = s$, and

(ii) for every natural number i and for every element x of $\prod$(the object kind of $\mathbf{SCM}_{\mathrm{FSA}}$) such that $x = (StepWhile > 0(a, I, s))(i)$ holds $(StepWhile > 0(a, I, s))(i + 1) = (\mathrm{Computation}(x{+\cdot}(while > 0(a, I){+\cdot}s_0)))(\mathrm{LifeSpan}(x{+\cdot}(I{+\cdot}s_0)) + 3)$.

One can prove the following propositions:

(48) $(\mathit{StepWhile}>0(a,I,s))(0)=s$.

(49) $(\mathit{StepWhile}>0(a,I,s))(k+1)=(\mathrm{Computation}((\mathit{StepWhile}>0(a,I,s))(k)+\cdot(\mathit{while}>0(a,I)+\cdot s_0)))(\mathrm{LifeSpan}((\mathit{StepWhile}>0(a,I,s))(k)+\cdot(I+\cdot s_0))+3)$.

(50) $(\mathit{StepWhile}>0(a,I,s))(k+1)=(\mathit{StepWhile}>0(a,I,(\mathit{StepWhile}>0(a,I,s))(k)))(1)$.

(51) Let I be a macro instruction, a be a read-write integer location, and s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$. Then $(\mathit{StepWhile}>0(a,I,s))(0+1)=(\mathrm{Computation}(s+\cdot(\mathit{while}>0(a,I)+\cdot s_0)))(\mathrm{LifeSpan}(s+\cdot(I+\cdot s_0))+3)$.

(52) Let I be a macro instruction, a be a read-write integer location, s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$, and k, n be natural numbers. Suppose $\mathbf{IC}_{(\mathit{StepWhile}>0(a,I,s))(k)}=\mathrm{insloc}(0)$ and $(\mathit{StepWhile}>0(a,I,s))(k)=(\mathrm{Computation}(s+\cdot(\mathit{while}>0(a,I)+\cdot\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))(n)$. Then $(\mathit{StepWhile}>0(a,I,s))(k)=(\mathit{StepWhile}>0(a,I,s))(k)+\cdot(\mathit{while}>0(a,I)+\cdot\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))$ and $(\mathit{StepWhile}>0(a,I,s))(k+1)=(\mathrm{Computation}(s+\cdot(\mathit{while}>0(a,I)+\cdot\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0)))))(n+(\mathrm{LifeSpan}((\mathit{StepWhile}>0(a,I,s))(k)+\cdot(I+\cdot\,\mathrm{Start\text{-}At}(\mathrm{insloc}(0))))+3))$.

(53) Let I be a macro instruction, a be a read-write integer location, and s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$. Suppose that

(i) for every natural number k holds I is closed on $(\mathit{StepWhile}>0(a,I,s))(k)$ and halting on $(\mathit{StepWhile}>0(a,I,s))(k)$, and

(ii) there exists a function f from $\prod$(the object kind of $\mathbf{SCM}_{\mathrm{FSA}}$) into $\mathbb{N}$ such that for every natural number k holds $f((\mathit{StepWhile}>0(a,I,s))(k+1))<f((\mathit{StepWhile}>0(a,I,s))(k))$ or $f((\mathit{StepWhile}>0(a,I,s))(k))=0$ but $f((\mathit{StepWhile}>0(a,I,s))(k))=0$ iff $(\mathit{StepWhile}>0(a,I,s))(k)(a)\leqslant 0$.

Then $\mathit{while}>0(a,I)$ is halting on s and $\mathit{while}>0(a,I)$ is closed on s.

(54) Let I be a parahalting macro instruction, a be a read-write integer location, and s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$. Given a function f from $\prod$(the object kind of $\mathbf{SCM}_{\mathrm{FSA}}$) into $\mathbb{N}$ such that let k be a natural number. Then $f((\mathit{StepWhile}>0(a,I,s))(k+1))<f((\mathit{StepWhile}>0(a,I,s))(k))$ or $f((\mathit{StepWhile}>0(a,I,s))(k))=0$ but $f((\mathit{StepWhile}>0(a,I,s))(k))=0$ iff $(\mathit{StepWhile}>0(a,I,s))(k)(a)\leqslant 0$. Then $\mathit{while}>0(a,I)$ is halting on s and $\mathit{while}>0(a,I)$ is closed on s.

(55) Let I be a parahalting macro instruction and a be a read-write integer location. Given a function f from $\prod$(the object kind of $\mathbf{SCM}_{\mathrm{FSA}}$) into $\mathbb{N}$ such that let s be a state of $\mathbf{SCM}_{\mathrm{FSA}}$. Then $f((\mathit{StepWhile}>0(a,I,s))(1))<f(s)$ or $f(s)=0$ but $f(s)=0$ iff $s(a)\leqslant 0$. Then $\mathit{while}>0(a,I)$ is parahalting.

Let I, J be good macro instructions and let a be an integer location. One can verify that $if > 0(a, I, J)$ is good.

Let I be a good macro instruction and let a be an integer location. One can verify that $while > 0(a, I)$ is good.

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A Decomposition of a Simple Closed Curves and the Order of Their Points

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Summary. The goal of the article is to introduce an order on a simple closed curve. To do this, we fix two points on the curve and devide it into two arcs. We prove that such a decomposition is unique. Other auxiliary theorems about arcs are proven for preparation of the proof of the above.

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The papers [41], [46], [45], [40], [26], [1], [49], [44], [37], [12], [39], [10], [36], [32], [48], [2], [7], [8], [4], [20], [21], [34], [33], [29], [11], [43], [28], [19], [35], [16], [9], [15], [42], [18], [22], [17], [6], [23], [27], [3], [31], [5], [38], [13], [25], [47], [14], [30], and [24] provide the notation and terminology for this paper.

1. Middle Points of Arcs

For simplicity, we use the following convention: a, b, c, s, r are real numbers, n is a natural number, p, q are points of $\mathcal{E}^2_{\rm T}$, and P is a subset of the carrier of $\mathcal{E}^2_{\rm T}$.

The following propositions are true:

(1) If $a = \frac{a+b}{2}$, then $a = b$.

(2) If $r \leqslant s$, then $r \leqslant \frac{r+s}{2}$ and $\frac{r+s}{2} \leqslant s$.

(3) Let T_1 be a non empty topological space, P be a subset of the carrier of T_1, A be a subset of the carrier of $T_1 \restriction P$, and B be a subset of the carrier of T_1. If B is closed and $A = B \cap P$, then A is closed.

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(4) Let T_1, T_2 be non empty topological spaces, P be a non empty subset of the carrier of T_2, and f be a map from T_1 into $T_2 \restriction P$. Then

(i) f is a map from T_1 into T_2, and

(ii) for every map f_2 from T_1 into T_2 such that $f_2 = f$ and f is continuous holds f_2 is continuous.

(5) Let r be a real number and P be a subset of the carrier of $\mathcal{E}^2_{\rm T}$. If $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p_{\mathbf{1}} \geqslant r\}$, then P is closed.

(6) Let r be a real number and P be a subset of the carrier of $\mathcal{E}^2_{\rm T}$. If $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p_{\mathbf{1}} \leqslant r\}$, then P is closed.

(7) Let r be a real number and P be a subset of the carrier of $\mathcal{E}^2_{\rm T}$. If $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p_{\mathbf{1}} = r\}$, then P is closed.

(8) Let r be a real number and P be a subset of the carrier of $\mathcal{E}^2_{\rm T}$. If $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p_{\mathbf{2}} \geqslant r\}$, then P is closed.

(9) Let r be a real number and P be a subset of the carrier of $\mathcal{E}^2_{\rm T}$. If $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p_{\mathbf{2}} \leqslant r\}$, then P is closed.

(10) Let r be a real number and P be a subset of the carrier of $\mathcal{E}^2_{\rm T}$. If $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p_{\mathbf{2}} = r\}$, then P is closed.

(11) Let P be a non empty subset of the carrier of $\mathcal{E}^n_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. If P is an arc from p_1 to p_2, then P is connected.

(12) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2, then P is closed.

(13) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2. Then there exists a point q of $\mathcal{E}^2_{\rm T}$ such that $q \in P$ and $q_{\mathbf{1}} = \frac{(p_1)_{\mathbf{1}}+(p_2)_{\mathbf{1}}}{2}$.

(14) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$, Q be a subset of the carrier of $\mathcal{E}^2_{\rm T}$, and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and $Q = \{q : q_{\mathbf{1}} = \frac{(p_1)_{\mathbf{1}}+(p_2)_{\mathbf{1}}}{2}\}$. Then P meets Q and $P \cap Q$ is closed.

(15) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$, Q be a subset of the carrier of $\mathcal{E}^2_{\rm T}$, and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and $Q = \{q : q_{\mathbf{2}} = \frac{(p_1)_{\mathbf{2}}+(p_2)_{\mathbf{2}}}{2}\}$. Then P meets Q and $P \cap Q$ is closed.

Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Let us assume that P is an arc from p_1 to p_2. The functor xMiddle(P, p_1, p_2) yields a point of $\mathcal{E}^2_{\rm T}$ and is defined as follows:

(Def. 1) For every subset Q of the carrier of $\mathcal{E}^2_{\rm T}$ such that $Q = \{q : q_{\mathbf{1}} = \frac{(p_1)_{\mathbf{1}}+(p_2)_{\mathbf{1}}}{2}\}$ holds xMiddle$(P, p_1, p_2) =$ FPoint(P, p_1, p_2, Q).

Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Let us assume that P is an arc from p_1 to p_2. The functor yMiddle(P, p_1, p_2) yields a point of $\mathcal{E}^2_{\rm T}$ and is defined by:

(Def. 2) For every subset Q of the carrier of $\mathcal{E}^2_{\rm T}$ such that $Q = \{q : q_{\mathbf{2}} = \frac{(p_1)_{\mathbf{2}}+(p_2)_{\mathbf{2}}}{2}\}$ holds $\mathrm{yMiddle}(P, p_1, p_2) = \mathrm{FPoint}(P, p_1, p_2, Q)$.

One can prove the following propositions:

(16) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2, then $\mathrm{xMiddle}(P, p_1, p_2) \in P$ and $\mathrm{yMiddle}(P, p_1, p_2) \in P$.

(17) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2, then $p_1 = \mathrm{xMiddle}(P, p_1, p_2)$ iff $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$.

(18) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2, then $p_1 = \mathrm{yMiddle}(P, p_1, p_2)$ iff $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$.

2. Segments of Arcs

The following proposition is true

(19) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2 and LE q_1, q_2, P, p_1, p_2, then LE q_2, q_1, P, p_2, p_1.

Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and let p_1, p_2, q_1 be points of $\mathcal{E}^2_{\rm T}$. The functor $\mathrm{LSegment}(P, p_1, p_2, q_1)$ yields a subset of the carrier of $\mathcal{E}^2_{\rm T}$ and is defined by:

(Def. 3) $\mathrm{LSegment}(P, p_1, p_2, q_1) = \{q : \mathrm{LE}\ q,\ q_1,\ P,\ p_1,\ p_2\}$.

Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and let p_1, p_2, q_1 be points of $\mathcal{E}^2_{\rm T}$. The functor $\mathrm{RSegment}(P, p_1, p_2, q_1)$ yielding a subset of the carrier of $\mathcal{E}^2_{\rm T}$ is defined as follows:

(Def. 4) $\mathrm{RSegment}(P, p_1, p_2, q_1) = \{q : \mathrm{LE}\ q_1,\ q,\ P,\ p_1,\ p_2\}$.

Next we state several propositions:

(20) For every non empty subset P of the carrier of $\mathcal{E}^2_{\rm T}$ and for all points p_1, p_2, q_1 of $\mathcal{E}^2_{\rm T}$ holds $\mathrm{LSegment}(P, p_1, p_2, q_1) \subseteq P$.

(21) For every non empty subset P of the carrier of $\mathcal{E}^2_{\rm T}$ and for all points p_1, p_2, q_1 of $\mathcal{E}^2_{\rm T}$ holds $\mathrm{RSegment}(P, p_1, p_2, q_1) \subseteq P$.

(22) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2, then $\mathrm{LSegment}(P, p_1, p_2, p_1) = \{p_1\}$.

(23) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2 and $q \in P$, then LE q, p_2, P, p_1, p_2.

(24) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2 and $q \in P$, then LE p_1, q, P, p_1, p_2.

(25) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2, then $\mathrm{LSegment}(P, p_1, p_2, p_2) = P$.

(26) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2, then $\mathrm{RSegment}(P, p_1, p_2, p_2) = \{p_2\}$.

(27) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2, then $\mathrm{RSegment}(P, p_1, p_2, p_1) = P$.

(28) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and $q_1 \in P$, then $\mathrm{RSegment}(P, p_1, p_2, q_1) = \mathrm{LSegment}(P, p_2, p_1, q_1)$.

Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and let p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. The functor $\mathrm{Segment}(P, p_1, p_2, q_1, q_2)$ yielding a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 5) $\mathrm{Segment}(P, p_1, p_2, q_1, q_2) = \mathrm{RSegment}(P, p_1, p_2, q_1) \cap \mathrm{LSegment}(P, p_1, p_2, q_2)$.

Next we state four propositions:

(29) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Then $\mathrm{Segment}(P, p_1, p_2, q_1, q_2) = \{q : \mathrm{LE}\ q_1, q, P, p_1, p_2 \wedge \mathrm{LE}\ q, q_2, P, p_1, p_2\}$.

(30) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2. Then LE q_1, q_2, P, p_1, p_2 if and only if LE q_2, q_1, P, p_2, p_1.

(31) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and $q \in P$, then $\mathrm{LSegment}(P, p_1, p_2, q) = \mathrm{RSegment}(P, p_2, p_1, q)$.

(32) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and $q_1 \in P$ and $q_2 \in P$, then $\mathrm{Segment}(P, p_1, p_2, q_1, q_2) = \mathrm{Segment}(P, p_2, p_1, q_2, q_1)$.

3. Decomposition of a Simple Closed Curve Into Two Arcs

Let s be a real number. The functor $\mathrm{VerticalLine}\, s$ yields a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 6) $\mathrm{VerticalLine}\, s = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{1}} = s\}$.

The functor $\mathrm{HorizontalLine}\, s$ yielding a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 7) $\mathrm{HorizontalLine}\, s = \{p : p_{\mathbf{2}} = s\}$.

Next we state several propositions:

(33) For every real number r holds $\mathrm{VerticalLine}\, r$ is closed and $\mathrm{HorizontalLine}\, r$ is closed.

(34) For every real number r and for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ VerticalLine r holds $p_{\mathbf{1}} = r$.

(35) For every real number r and for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ HorizontalLine r holds $p_{\mathbf{2}} = r$.

(36) For every compact non empty subset P of $\mathcal{E}^2_{\rm T}$ holds W-min $P \in P$ and W-max $P \in P$.

(37) For every compact non empty subset P of $\mathcal{E}^2_{\rm T}$ holds N-min $P \in P$ and N-max $P \in P$.

(38) For every compact non empty subset P of $\mathcal{E}^2_{\rm T}$ holds E-min $P \in P$ and E-max $P \in P$.

(39) For every compact non empty subset P of $\mathcal{E}^2_{\rm T}$ holds S-min $P \in P$ and S-max $P \in P$.

(40) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose P is a simple closed curve. Then there exist non empty subsets P_1, P_2 of the carrier of $\mathcal{E}^2_{\rm T}$ such that

(i) P_1 is an arc from W-min P to E-max P,

(ii) P_2 is an arc from E-max P to W-min P,

(iii) $P_1 \cap P_2 = \{\text{W-min}\,P, \text{E-max}\,P\}$,

(iv) $P_1 \cup P_2 = P$, and

(v) $(\text{FPoint}(P_1, \text{W-min}\,P, \text{E-max}\,P, \text{VerticalLine}\,\frac{\text{W-bound}\,P+\text{E-bound}\,P}{2}))_{\mathbf{2}} > (\text{LPoint}(P_2, \text{E-max}\,P, \text{W-min}\,P, \text{VerticalLine}\,\frac{\text{W-bound}\,P+\text{E-bound}\,P}{2}))_{\mathbf{2}}$.

4. Uniqueness of Decomposition of a Simple Closed Curve

One can prove the following propositions:

(41) For every subset P of the carrier of $\mathbb{I}$ such that $P =$ (the carrier of $\mathbb{I}) \setminus \{0, 1\}$ holds P is open.

(42) For all subsets B_1, B_2 of $\mathbb{R}$ such that B_2 is lower bounded and $B_1 \subseteq B_2$ holds B_1 is lower bounded.

(43) For all subsets B_1, B_2 of $\mathbb{R}$ such that B_2 is upper bounded and $B_1 \subseteq B_2$ holds B_1 is upper bounded.

(44) For all r, s holds $]r, s[\cap \{r, s\} = \emptyset$.

(45) For all a, b, c holds $c \in]a, b[$ iff $a < c$ and $c < b$.

(46) For every subset P of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all r, s such that $P =]r, s[$ holds P is open.

(47) Let S be a non empty topological space, P_1, P_2 be subsets of the carrier of S, and P_1' be a subset of the carrier of $S \restriction P_2$. If $P_1 = P_1'$ and $P_1 \neq \emptyset$ and $P_1 \subseteq P_2$, then $S \restriction P_1 = S \restriction P_2 \restriction P_1'$.

(48) For every subset P_7 of the carrier of $\mathbb{I}$ such that $P_7 =$ (the carrier of $\mathbb{I}$) $\setminus \{0, 1\}$ holds $P_7 \neq \emptyset$ and P_7 is connected.

(49) Let P be a non empty subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. If P is an arc from p_1 to p_2, then $p_1 \neq p_2$.

(50) Let P be a non empty subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, Q be a subset of the carrier of $(\mathcal{E}^n_{\mathrm{T}}) \restriction P$, and p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and $Q = P \setminus \{p_1, p_2\}$, then Q is open.

(51) For all points p, q of $\mathcal{E}^n_{\mathrm{T}}$ and for every non empty subset P of $\mathcal{E}^n_{\mathrm{T}}$ such that P is an arc from p to q holds P is compact.

(52) Let P be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, P_1, P_2 be non empty subsets of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, Q be a subset of the carrier of $(\mathcal{E}^n_{\mathrm{T}}) \restriction P$, and p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $p_1 \in P$ and $p_2 \in P$ and P_1 is an arc from p_1 to p_2 and P_2 is an arc from p_1 to p_2 and $P_1 \cup P_2 = P$ and $P_1 \cap P_2 = \{p_1, p_2\}$ and $Q = P_1 \setminus \{p_1, p_2\}$. Then Q is open.

(53) Let P be a non empty subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, Q be a subset of the carrier of $(\mathcal{E}^n_{\mathrm{T}}) \restriction P$, and p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and $Q = P \setminus \{p_1, p_2\}$, then Q is connected.

(54) Let G_1 be a non empty topological space, P_1, P be non empty subsets of the carrier of G_1, Q' be a subset of the carrier of $G_1 \restriction P_1$, and Q be a non empty subset of the carrier of $G_1 \restriction P$. If $P_1 \subseteq P$ and $Q = Q'$ and Q' is connected, then Q is connected.

(55) Let P be a non empty subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2. Then there exists a point p_3 of $\mathcal{E}^n_{\mathrm{T}}$ such that $p_3 \in P$ and $p_3 \neq p_1$ and $p_3 \neq p_2$.

(56) Let P be a non empty subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. If P is an arc from p_1 to p_2, then $P \setminus \{p_1, p_2\} \neq \emptyset$.

(57) Let P_1 be a non empty subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, P be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, Q be a subset of the carrier of $(\mathcal{E}^n_{\mathrm{T}}) \restriction P$, and p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. If P_1 is an arc from p_1 to p_2 and $P_1 \subseteq P$ and $Q = P_1 \setminus \{p_1, p_2\}$, then Q is connected.

(58) Let T, S, V be non empty topological spaces, P_1 be a non empty subset of the carrier of S, P_2 be a subset of the carrier of S, f be a map from T into $S \restriction P_1$, and g be a map from $S \restriction P_2$ into V. Suppose $P_1 \subseteq P_2$ and f is continuous and g is continuous. Then there exists a map h from T into V such that $h = g \cdot f$ and h is continuous.

(59) Let P_1, P_2 be non empty subsets of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. If P_1 is an arc from p_1 to p_2 and P_2 is an arc from p_1 to p_2 and $P_1 \subseteq P_2$, then $P_1 = P_2$.

(60) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, Q be a subset of the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction P$, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is a simple closed

curve and $p_1 \in P$ and $p_2 \in P$ and $p_1 \neq p_2$ and $Q = P \setminus \{p_1, p_2\}$. Then Q is not connected.

(61) Let P, P_1, P_2, P_1', P_2' be non empty subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) P is a simple closed curve,
(ii) P_1 is an arc from p_1 to p_2,
(iii) P_2 is an arc from p_1 to p_2,
(iv) $P_1 \cup P_2 = P$,
(v) $P_1 \cap P_2 = \{p_1, p_2\}$,
(vi) P_1' is an arc from p_1 to p_2,
(vii) P_2' is an arc from p_1 to p_2,
(viii) $P_1' \cup P_2' = P$, and
(ix) $P_1' \cap P_2' = \{p_1, p_2\}$.

Then $P_1 = P_1'$ and $P_2 = P_2'$ or $P_1 = P_2'$ and $P_2 = P_1'$.

5. Lower Arcs and Upper Arcs

One can prove the following propositions:

(62) Let P_1 be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P_1 is an arc from p_1 to p_2, then P_1 is closed.

(63) Let G_1, G_2 be non empty topological spaces, P be a non empty subset of the carrier of G_2, f be a map from G_1 into $G_2 \restriction P$, and f_1 be a map from G_1 into G_2. If $f = f_1$ and f is continuous, then f_1 is continuous.

(64) Let P_1 be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $(p_1)_{\mathbf{1}} \leqslant (p_2)_{\mathbf{1}}$ and P_1 is an arc from p_1 to p_2. Then $P_1 \cap \mathrm{VerticalLine}\,\frac{(p_1)_{\mathbf{1}}+(p_2)_{\mathbf{1}}}{2} \neq \emptyset$ and $P_1 \cap \mathrm{VerticalLine}\,\frac{(p_1)_{\mathbf{1}}+(p_2)_{\mathbf{1}}}{2}$ is closed.

Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Let us assume that P is a simple closed curve. The functor $\mathrm{UpperArc}\,P$ yields a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by the conditions (Def. 8).

(Def. 8)(i) $\mathrm{UpperArc}\,P$ is an arc from $\text{W-min}\,P$ to $\text{E-max}\,P$, and

(ii) there exists a non empty subset P_2 of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ such that P_2 is an arc from $\text{E-max}\,P$ to $\text{W-min}\,P$ and $\mathrm{UpperArc}\,P \cap P_2 = \{\text{W-min}\,P, \text{E-max}\,P\}$ and $\mathrm{UpperArc}\,P \cup P_2 = P$ and $(\mathrm{FPoint}(\mathrm{UpperArc}\,P, \text{W-min}\,P, \text{E-max}\,P,$ $\mathrm{VerticalLine}\,\frac{\text{W-bound}\,P+\text{E-bound}\,P}{2}))_{\mathbf{2}} >$ $(\mathrm{LPoint}(P_2, \text{E-max}\,P, \text{W-min}\,P,$ $\mathrm{VerticalLine}\,\frac{\text{W-bound}\,P+\text{E-bound}\,P}{2}))_{\mathbf{2}}$.

Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Let us assume that P is a simple closed curve. The functor $\mathrm{LowerArc}\,P$ yielding a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 9) LowerArc P is an arc from E-max P to W-min P and UpperArc $P \cap$ LowerArc $P = \{$W-min P, E-max $P\}$ and UpperArc $P \cup$ LowerArc $P = P$ and (FPoint(UpperArc P, W-min P, E-max P, VerticalLine $\frac{\text{W-bound } P + \text{E-bound } P}{2}$))$_{\mathbf{2}}$ > (LPoint(LowerArc P, E-max P, W-min P, VerticalLine $\frac{\text{W-bound } P + \text{E-bound } P}{2}$))$_{\mathbf{2}}$.

The following propositions are true:

(65) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose P is a simple closed curve. Then

(i) UpperArc P is an arc from W-min P to E-max P,
(ii) UpperArc P is an arc from E-max P to W-min P,
(iii) LowerArc P is an arc from E-max P to W-min P,
(iv) LowerArc P is an arc from W-min P to E-max P,
(v) UpperArc $P \cap$ LowerArc $P = \{$W-min P, E-max $P\}$,
(vi) UpperArc $P \cup$ LowerArc $P = P$, and
(vii) (FPoint(UpperArc P, W-min P, E-max P, VerticalLine $\frac{\text{W-bound } P + \text{E-bound } P}{2}$))$_{\mathbf{2}}$ > (LPoint(LowerArc P, E-max P, W-min P, VerticalLine $\frac{\text{W-bound } P + \text{E-bound } P}{2}$))$_{\mathbf{2}}$.

(66) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. If P is a simple closed curve, then LowerArc $P = (P \setminus$ UpperArc $P) \cup \{$W-min P, E-max $P\}$ and UpperArc $P = (P \setminus$ LowerArc $P) \cup \{$W-min P, E-max $P\}$.

(67) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and P_1 be a subset of the carrier of $(\mathcal{E}^2_{\rm T})\restriction P$. If P is a simple closed curve and UpperArc $P \cap P_1 = \{$W-min P, E-max $P\}$ and UpperArc $P \cup P_1 = P$, then $P_1 =$ LowerArc P.

(68) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and P_1 be a subset of the carrier of $(\mathcal{E}^2_{\rm T})\restriction P$. If P is a simple closed curve and $P_1 \cap$ LowerArc $P = \{$W-min P, E-max $P\}$ and $P_1 \cup$ LowerArc $P = P$, then $P_1 =$ UpperArc P.

6. An Order of Points in a Simple Closed Curve

One can prove the following propositions:

(69) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2 and LE q, p_1, P, p_1, p_2, then $q = p_1$.

(70) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2 and LE p_2, q, P, p_1, p_2, then $q = p_2$.

Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and let q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. The predicate LE(q_1, q_2, P) is defined by the conditions (Def. 10).

(Def. 10)(i) $q_1 \in$ UpperArc P and $q_2 \in$ LowerArc P and $q_2 \neq$ W-min P, or

(ii) $q_1 \in$ UpperArc P and $q_2 \in$ UpperArc P and LE q_1, q_2, UpperArc P, W-min P, E-max P, or

(iii) $q_1 \in \mathrm{LowerArc}\,P$ and $q_2 \in \mathrm{LowerArc}\,P$ and $q_2 \neq \text{W-min}\,P$ and LE q_1, q_2, $\mathrm{LowerArc}\,P$, $\text{E-max}\,P$, $\text{W-min}\,P$.

Next we state three propositions:

(71) Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. If P is a simple closed curve and $q \in P$, then $\mathrm{LE}(q, q, P)$.

(72) Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is a simple closed curve and $\mathrm{LE}(q_1, q_2, P)$ and $\mathrm{LE}(q_2, q_1, P)$, then $q_1 = q_2$.

(73) Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and q_1, q_2, q_3 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is a simple closed curve and $\mathrm{LE}(q_1, q_2, P)$ and $\mathrm{LE}(q_2, q_3, P)$, then $\mathrm{LE}(q_1, q_3, P)$.

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The Chinese Remainder Theorem

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Summary. The article is a translation of the first chapters of a book *Wstęp do teorii liczb* (Eng. *Introduction to Number Theory*) by W. Sierpiński, WSiP, Biblioteczka Matematyczna, Warszawa, 1987. The first few pages of this book have already been formalized in MML. We prove the Chinese Remainder Theorem and Thue's Theorem as well as several useful number theory propositions.

MML Identifier: `WSIERP_1`.

The terminology and notation used in this paper are introduced in the following articles: [20], [16], [9], [14], [18], [1], [10], [13], [12], [15], [11], [17], [21], [6], [7], [2], [5], [3], [8], [4], and [19].

For simplicity, we follow the rules: x, y, z, w denote real numbers, a, b, c, d, e, f, g denote natural numbers, k, l, m, n, m_1, n_1 denote integers, and q denotes a rational number.

The following propositions are true:

(1) If $y \neq 0$, then $(\frac{x}{y})^a = \frac{x^a}{y^a}$.

(2) $x^2 = x \cdot x$ and $(-x)^2 = x^2$.

(3) $(-x)^{2 \cdot a} = x^{2 \cdot a}$ and $(-x)^{2 \cdot a+1} = -x^{2 \cdot a+1}$.

(4) If $x \neq 0$, then $x^a_{\mathbb{Z}} = x^a$.

(5) If $x \geqslant 0$ and $y \geqslant 0$ and $d > 0$ and $x^d = y^d$, then $x = y$.

(6) $x > \max(y, z)$ iff $x > y$ and $x > z$.

(7) If $x \leqslant 0$ and $y \geqslant z$, then $y - x \geqslant z$ and $y \geqslant z + x$.

(8) If $x \leqslant 0$ and $y > z$ or $x < 0$ and $y \geqslant z$, then $y > z + x$ and $y - x > z$.

Let us consider a, b. Then $\gcd(a, b)$ is a natural number. Let us observe that the functor $\gcd(a, b)$ is commutative.

Let us consider m, n. Then $m \gcd n$ is an integer. Let us observe that the functor $m \gcd n$ is commutative.

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Let us consider k, a. Then k^a is an integer.

Let us consider a, b. Then a^b is a natural number.

We now state a number of propositions:

(9) If $k \mid m$ and $k \mid n$, then $k \mid m + n$.

(10) If $k \mid m$ and $k \mid n$, then $k \mid m \cdot m_1 + n \cdot n_1$.

(11) If $m \gcd n = 1$ and $k \gcd n = 1$, then $m \cdot k \gcd n = 1$.

(12) If $\gcd(a, b) = 1$ and $\gcd(c, b) = 1$, then $\gcd(a \cdot c, b) = 1$.

(13) $0 \gcd m = |m|$ and $1 \gcd m = 1$.

(14) 1 and k are relative prime.

(15) If k and l are relative prime, then k^a and l are relative prime.

(16) If k and l are relative prime, then k^a and l^b are relative prime.

(17) If $k \gcd l = 1$, then $k \gcd l^b = 1$ and $k^a \gcd l^b = 1$.

(18) $|m| \mid k$ iff $m \mid k$.

(19) If $a \mid b$, then $a^c \mid b^c$.

(20) If $a \mid 1$, then $a = 1$.

(21) If $d \mid a$ and $\gcd(a, b) = 1$, then $\gcd(d, b) = 1$.

(22) If $k \neq 0$, then $k \mid l$ iff $\frac{l}{k}$ is an integer.

(23) If $a \leqslant b - c$, then $a \leqslant b$ and $c \leqslant b$.

In the sequel f_1, f_2, f_3 are finite sequences.

Next we state two propositions:

(24) If $a \in \operatorname{Seg} \operatorname{len} f_2$, then $a \in \operatorname{Seg} \operatorname{len}(f_2 \frown f_3)$.

(25) If $a \in \operatorname{Seg} \operatorname{len} f_3$, then $\operatorname{len} f_2 + a \in \operatorname{Seg} \operatorname{len}(f_2 \frown f_3)$.

Let f_4 be a finite sequence of elements of $\mathbb{R}$ and let us consider a. Then $f_4(a)$ is a real number.

Let f_5 be a finite sequence of elements of $\mathbb{Z}$ and let us consider a. Then $f_5(a)$ is an integer.

Let f_6 be a finite sequence of elements of $\mathbb{N}$ and let us consider a. Then $f_6(a)$ is a natural number.

Let D be a non empty set and let D_1 be a non empty subset of D. We see that the finite sequence of elements of D_1 is a finite sequence of elements of D.

Let D be a non empty set, let D_1 be a non empty subset of D, and let f_7, f_8 be finite sequences of elements of D_1. Then $f_7 \frown f_8$ is a finite sequence of elements of D_1.

Let D be a non empty set and let D_1 be a non empty subset of D. Then $\varepsilon_{(D_1)}$ is an empty finite sequence of elements of D_1.

$\mathbb{Z}$ is a non empty subset of $\mathbb{R}$.

For simplicity, we adopt the following convention: D, D_1 are non empty sets, v_1, v_2, v_3 are sets, f_6 is a finite sequence of elements of $\mathbb{N}$, f_5, f_9 are finite sequences of elements of $\mathbb{Z}$, and f_4 is a finite sequence of elements of $\mathbb{R}$.

Let us consider f_5. Then $\sum f_5$ is an integer. Then $\prod f_5$ is an integer.

Let us consider f_6. Then $\sum f_6$ is a natural number. Then $\prod f_6$ is a natural number.

Let us consider a, f_1. The functor $f_1{\sim}a$ yielding a finite sequence is defined as follows:

(Def. 1)(i) $f_1{\sim}a = f_1$ if $a \notin \operatorname{dom} f_1$,

(ii) $\operatorname{len}(f_1{\sim}a) + 1 = \operatorname{len} f_1$ and for every b holds if $b < a$, then $(f_1{\sim}a)(b) = f_1(b)$ and if $b \geqslant a$, then $(f_1{\sim}a)(b) = f_1(b+1)$, otherwise.

Let us consider D, let us consider a, and let f_1 be a finite sequence of elements of D. Then $f_1{\sim}a$ is a finite sequence of elements of D.

Let us consider D, let D_1 be a non empty subset of D, let us consider a, and let f_1 be a finite sequence of elements of D_1. Then $f_1{\sim}a$ is a finite sequence of elements of D_1.

One can prove the following propositions:

(26) $\langle v_1\rangle{\sim}1 = \varepsilon$ and $\langle v_1, v_2\rangle{\sim}1 = \langle v_2\rangle$ and $\langle v_1, v_2\rangle{\sim}2 = \langle v_1\rangle$ and $\langle v_1, v_2, v_3\rangle{\sim}1 = \langle v_2, v_3\rangle$ and $\langle v_1, v_2, v_3\rangle{\sim}2 = \langle v_1, v_3\rangle$ and $\langle v_1, v_2, v_3\rangle{\sim}3 = \langle v_1, v_2\rangle$.

(27) If $1 \leqslant a$ and $a \leqslant \operatorname{len} f_4$, then $\sum(f_4{\sim}a) + f_4(a) = \sum f_4$.

(28) If $a \in \operatorname{Seg} \operatorname{len} f_6$ and $f_6(a) \neq 0$, then $\frac{\prod f_6}{f_6(a)}$ is a natural number.

(29) $\operatorname{num} q$ and $\operatorname{den} q$ are relative prime.

(30) If $q \neq 0$ and $q = \frac{k}{a}$ and $a \neq 0$ and k and a are relative prime, then $k = \operatorname{num} q$ and $a = \operatorname{den} q$.

(31) If there exists q such that $a = q^b$, then there exists k such that $a = k^b$.

(32) If there exists q such that $a = q^d$, then there exists b such that $a = b^d$.

(33) If $e > 0$ and $a^e \mid b^e$, then $a \mid b$.

(34) There exist m, n such that $\gcd(a, b) = a \cdot m + b \cdot n$.

(35) There exist m_1, n_1 such that $m \gcd n = m \cdot m_1 + n \cdot n_1$.

(36) If $m \mid n \cdot k$ and $m \gcd n = 1$, then $m \mid k$.

(37) If $\gcd(a, b) = 1$ and $a \mid b \cdot c$, then $a \mid c$.

(38) If $a \neq 0$ and $b \neq 0$, then there exist c, d such that $\gcd(a, b) = a \cdot c - b \cdot d$.

(39) If $f > 0$ and $g > 0$ and $\gcd(f, g) = 1$ and $a^f = b^g$, then there exists e such that $a = e^g$ and $b = e^f$.

In the sequel x, y, z, t denote integers.

Next we state several propositions:

(40) There exist x, y such that $m \cdot x + n \cdot y = k$ iff $m \gcd n \mid k$.

(41) Suppose $m \neq 0$ and $n \neq 0$ and $m \cdot m_1 + n \cdot n_1 = k$. Let given x, y. If $m \cdot x + n \cdot y = k$, then there exists t such that $x = m_1 + t \cdot \frac{n}{m \gcd n}$ and $y = n_1 - t \cdot \frac{m}{m \gcd n}$.

(42) If $\gcd(a, b) = 1$ and $a \cdot b = c^d$, then there exist e, f such that $a = e^d$ and $b = f^d$.

(43) For every d such that for every a such that $a \in \operatorname{Seg} \operatorname{len} f_6$ holds $\gcd(f_6(a), d) = 1$ holds $\gcd(\prod f_6, d) = 1$.

(44) Suppose $\operatorname{len} f_6 \geqslant 2$ and for all b, c such that $b \in \operatorname{Seg} \operatorname{len} f_6$ and $c \in \operatorname{Seg} \operatorname{len} f_6$ and $b \neq c$ holds $\gcd(f_6(b), f_6(c)) = 1$. Let given f_5. Suppose $\operatorname{len} f_5 = \operatorname{len} f_6$. Then there exists f_9 such that $\operatorname{len} f_9 = \operatorname{len} f_6$ and for every b such that $b \in \operatorname{Seg} \operatorname{len} f_6$ holds $f_6(b) \cdot f_9(b) + f_5(b) = f_6(1) \cdot f_9(1) + f_5(1)$.

(45) If $x < y$ and $z \geqslant w$ or $x \leqslant y$ and $z > w$ or $x < y$ and $z > w$, then $x - z < y - w$.

(46) If $a \neq 0$ and $a \gcd k = 1$, then there exist b, e such that $0 \neq b$ and $0 \neq e$ and $b \leqslant \sqrt{a}$ and $e \leqslant \sqrt{a}$ and $a \mid k \cdot b + e$ or $a \mid k \cdot b - e$.

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