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Basic Properties of Objects and Morphisms

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The articles [7], [9], [10], [1], [3], [4], [2], [8], [6], and [5] provide the notation and terminology for this paper.

Let C be a non empty category structure with units, let o_1, o_2 be objects of C, let A be a morphism from o_1 to o_2, and let B be a morphism from o_2 to o_1. We say that A is left inverse of B if and only if:

(Def. 1) $A \cdot B = \mathrm{id}_{(o_2)}$.

We introduce B is right inverse of A as a synonym of A is left inverse of B.

Let C be a non empty category structure with units, let o_1, o_2 be objects of C, and let A be a morphism from o_1 to o_2. We say that A is retraction if and only if:

(Def. 2) There exists a morphism from o_2 to o_1 which is right inverse of A.

Let C be a non empty category structure with units, let o_1, o_2 be objects of C, and let A be a morphism from o_1 to o_2. We say that A is coretraction if and only if:

(Def. 3) There exists a morphism from o_2 to o_1 which is left inverse of A.

Next we state the proposition

(1) Let C be a non empty category structure with units and o be an object of C. Then id_o is retraction and id_o is coretraction.

Let C be a category and let o_1, o_2 be objects of C. Let us assume that $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2. Let us assume that A is retraction and coretraction. The functor A^{-1} yields a morphism from o_2 to o_1 and is defined by:

(Def. 4) A^{-1} is left inverse of A and A^{-1} is right inverse of A.

We now state three propositions:

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(2) Let C be a category and o_1, o_2 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2. If A is retraction and coretraction, then $A^{-1} \cdot A = \mathrm{id}_{(o_1)}$ and $A \cdot A^{-1} = \mathrm{id}_{(o_2)}$.

(3) Let C be a category and o_1, o_2 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2. If A is retraction and coretraction, then $(A^{-1})^{-1} = A$.

(4) For every category C and for every object o of C holds $(\mathrm{id}_o)^{-1} = \mathrm{id}_o$.

Let C be a category, let o_1, o_2 be objects of C, and let A be a morphism from o_1 to o_2. We say that A is iso if and only if:

(Def. 5) $A \cdot A^{-1} = \mathrm{id}_{(o_2)}$ and $A^{-1} \cdot A = \mathrm{id}_{(o_1)}$.

One can prove the following three propositions:

(5) Let C be a category, o_1, o_2 be objects of C, and A be a morphism from o_1 to o_2. If A is iso, then A is retraction and coretraction.

(6) Let C be a category and o_1, o_2 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2. Then A is iso if and only if A is retraction and coretraction.

(7) Let C be a category, o_1, o_2, o_3 be objects of C, A be a morphism from o_1 to o_2, and B be a morphism from o_2 to o_3. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$ and $\langle o_3, o_1 \rangle \neq \emptyset$ and A is iso and B is iso. Then $B \cdot A$ is iso and $(B \cdot A)^{-1} = A^{-1} \cdot B^{-1}$.

Let C be a category and let o_1, o_2 be objects of C. We say that o_1, o_2 are iso if and only if:

(Def. 6) $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and there exists a morphism from o_1 to o_2 which is iso.

Let us note that the predicate o_1, o_2 are iso is reflexive and symmetric.

One can prove the following proposition

(8) Let C be a category and o_1, o_2, o_3 be objects of C. If o_1, o_2 are iso and o_2, o_3 are iso , then o_1, o_3 are iso .

Let C be a non empty category structure, let o_1, o_2 be objects of C, and let A be a morphism from o_1 to o_2. We say that A is mono if and only if:

(Def. 7) For every object o of C such that $\langle o, o_1 \rangle \neq \emptyset$ and for all morphisms B, C from o to o_1 such that $A \cdot B = A \cdot C$ holds $B = C$.

Let C be a non empty category structure, let o_1, o_2 be objects of C, and let A be a morphism from o_1 to o_2. We say that A is epi if and only if:

(Def. 8) For every object o of C such that $\langle o_2, o \rangle \neq \emptyset$ and for all morphisms B, C from o_2 to o such that $B \cdot A = C \cdot A$ holds $B = C$.

We now state a number of propositions:

(9) Let C be an associative transitive non empty category structure and o_1, o_2, o_3 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$. Let A be a

morphism from o_1 to o_2 and B be a morphism from o_2 to o_3. If A is mono and B is mono, then $B \cdot A$ is mono.

(10) Let C be an associative transitive non empty category structure and o_1, o_2, o_3 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2 and B be a morphism from o_2 to o_3. If A is epi and B is epi, then $B \cdot A$ is epi.

(11) Let C be an associative transitive non empty category structure and o_1, o_2, o_3 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2 and B be a morphism from o_2 to o_3. If $B \cdot A$ is mono, then A is mono.

(12) Let C be an associative transitive non empty category structure and o_1, o_2, o_3 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2 and B be a morphism from o_2 to o_3. If $B \cdot A$ is epi, then B is epi.

(13) Let X be a non empty set and o_1, o_2 be objects of Ens_X. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2 and F be a function from o_1 into o_2. If $F = A$, then A is mono iff F is one-to-one.

(14) Let X be a non empty set with non empty elements and o_1, o_2 be objects of Ens_X. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2 and F be a function from o_1 into o_2. If $F = A$, then A is epi iff F is onto.

(15) Let C be a category and o_1, o_2 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2. If A is retraction, then A is epi.

(16) Let C be a category and o_1, o_2 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2. If A is coretraction, then A is mono.

(17) Let C be a category and o_1, o_2 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2. If A is iso, then A is mono and epi.

(18) Let C be a category and o_1, o_2, o_3 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$ and $\langle o_3, o_1 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2 and B be a morphism from o_2 to o_3. If A is retraction and B is retraction, then $B \cdot A$ is retraction.

(19) Let C be a category and o_1, o_2, o_3 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$ and $\langle o_3, o_1 \rangle \neq \emptyset$. Let A be a morphism from o_1 to o_2 and B be a morphism from o_2 to o_3. If A is coretraction and B is coretraction, then $B \cdot A$ is coretraction.

(20) Let C be a category, o_1, o_2 be objects of C, and A be a morphism from o_1 to o_2. If A is retraction and mono and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$, then A is iso.

(21) Let C be a category, o_1, o_2 be objects of C, and A be a morphism from o_1 to o_2. If A is coretraction and epi and $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_1\rangle \neq \emptyset$, then A is iso.

(22) Let C be a category, o_1, o_2, o_3 be objects of C, A be a morphism from o_1 to o_2, and B be a morphism from o_2 to o_3. Suppose $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_3\rangle \neq \emptyset$ and $\langle o_3, o_1\rangle \neq \emptyset$ and $B \cdot A$ is retraction. Then B is retraction.

(23) Let C be a category, o_1, o_2, o_3 be objects of C, A be a morphism from o_1 to o_2, and B be a morphism from o_2 to o_3. Suppose $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_3\rangle \neq \emptyset$ and $\langle o_3, o_1\rangle \neq \emptyset$ and $B \cdot A$ is coretraction. Then A is coretraction.

(24) Let C be a category. Suppose that for all objects o_1, o_2 of C holds every morphism from o_1 to o_2 is retraction. Let a, b be objects of C and A be a morphism from a to b. If $\langle a, b\rangle \neq \emptyset$ and $\langle b, a\rangle \neq \emptyset$, then A is iso.

Let C be a non empty category structure with units and let o be an object of C. Note that there exists a morphism from o to o which is mono, epi, retraction, and coretraction.

Let C be a category and let o be an object of C. Observe that there exists a morphism from o to o which is mono, epi, iso, retraction, and coretraction.

Let C be a category, let o be an object of C, and let A, B be mono morphisms from o to o. Note that $A \cdot B$ is mono.

Let C be a category, let o be an object of C, and let A, B be epi morphisms from o to o. Observe that $A \cdot B$ is epi.

Let C be a category, let o be an object of C, and let A, B be iso morphisms from o to o. One can verify that $A \cdot B$ is iso.

Let C be a category, let o be an object of C, and let A, B be retraction morphisms from o to o. Observe that $A \cdot B$ is retraction.

Let C be a category, let o be an object of C, and let A, B be coretraction morphisms from o to o. One can check that $A \cdot B$ is coretraction.

Let C be a graph and let o be an object of C. We say that o is initial if and only if:

(Def. 9) For every object o_1 of C there exists a morphism M from o to o_1 such that $M \in \langle o, o_1\rangle$ and $\langle o, o_1\rangle$ is trivial.

One can prove the following two propositions:

(25) Let C be a graph and o be an object of C. Then o is initial if and only if for every object o_1 of C there exists a morphism M from o to o_1 such that $M \in \langle o, o_1\rangle$ and for every morphism M_1 from o to o_1 such that $M_1 \in \langle o, o_1\rangle$ holds $M = M_1$.

(26) For every category C and for all objects o_1, o_2 of C such that o_1 is initial and o_2 is initial holds o_1, o_2 are iso .

Let C be a graph and let o be an object of C. We say that o is terminal if and only if:

(Def. 10) For every object o_1 of C there exists a morphism M from o_1 to o such that $M \in \langle o_1, o \rangle$ and $\langle o_1, o \rangle$ is trivial.

Next we state two propositions:

(27) Let C be a graph and o be an object of C. Then o is terminal if and only if for every object o_1 of C there exists a morphism M from o_1 to o such that $M \in \langle o_1, o \rangle$ and for every morphism M_1 from o_1 to o such that $M_1 \in \langle o_1, o \rangle$ holds $M = M_1$.

(28) For every category C and for all objects o_1, o_2 of C such that o_1 is terminal and o_2 is terminal holds o_1, o_2 are iso .

Let C be a graph and let o be an object of C. We say that o is zero if and only if:

(Def. 11) o is initial and terminal.

We now state the proposition

(29) For every category C and for all objects o_1, o_2 of C such that o_1 is zero and o_2 is zero holds o_1, o_2 are iso .

Let C be a non empty category structure, let o_1, o_2 be objects of C, and let M be a morphism from o_1 to o_2. We say that M is zero if and only if the condition (Def. 12) is satisfied.

(Def. 12) Let o be an object of C. Suppose o is zero. Let A be a morphism from o_1 to o and B be a morphism from o to o_2. Then $M = B \cdot A$.

We now state the proposition

(30) Let C be a category, o_1, o_2, o_3 be objects of C, M_1 be a morphism from o_1 to o_2, and M_2 be a morphism from o_2 to o_3. If M_1 is zero and M_2 is zero, then $M_2 \cdot M_1$ is zero.

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Abian's Fixed Point Theorem[1]

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Summary. A. Abian [1] proved the following theorem:

> Let f be a mapping from a finite set D. Then f has a fixed point if and only if D is not a union of three mutually disjoint sets A, B and C such that
>
> $$A \cap f[A] = B \cap f[B] = C \cap f[C] = \emptyset.$$

(The range of f is not necessarily the subset of its domain). The proof of the sufficiency is by induction on the number of elements of D. A.Mąkowski and K.Wiśniewski [12] have shown that the assumption of finiteness is superfluous. They proved their version of the theorem for f being a function from D into D. In the proof, the required partition was constructed and the construction used the axiom of choice. Their main point was to demonstrate that the use of this axiom in the proof is essential. We have proved in Mizar the generalized version of Abian's theorem, i.e. without assuming finiteness of D. We have simplified the proof from [12] which uses well-ordering principle and transfinite ordinals—our proof does not use these notions but otherwise is based on their idea (we employ choice functions).

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The terminology and notation used here are introduced in the following articles: [18], [21], [9], [6], [19], [17], [7], [13], [8], [22], [3], [4], [5], [16], [20], [2], [14], [10], [11], and [15].

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1. Preliminaries

For simplicity, we adopt the following rules: x, y, E, E_1, E_2, E_3 are sets, s_1 is a family of subsets of E, f is a function from E into E, and k, l, n are natural numbers.

Let i be an integer. We say that i is even if and only if:

(Def. 1) There exists an integer j such that $i = 2 \cdot j$.

We introduce i is odd as an antonym of i is even.

Let n be a natural number. Let us observe that n is even if and only if:

(Def. 2) There exists k such that $n = 2 \cdot k$.

We introduce n is odd as an antonym of n is even.

One can check the following observations:

* there exists a natural number which is even,
* there exists a natural number which is odd,
* there exists an integer which is even, and
* there exists an integer which is odd.

One can prove the following proposition

(1) For every integer i holds i is odd iff there exists an integer j such that $i = 2 \cdot j + 1$.

Let i be an integer. Note that $2 \cdot i$ is even.

Let i be an even integer. Note that $i + 1$ is odd.

Let i be an odd integer. Observe that $i + 1$ is even.

Let i be an even integer. One can verify that $i - 1$ is odd.

Let i be an odd integer. Note that $i - 1$ is even.

Let i be an even integer and let j be an integer. One can check that $i \cdot j$ is even and $j \cdot i$ is even.

Let i, j be odd integers. Note that $i \cdot j$ is odd.

Let i, j be even integers. One can check that $i + j$ is even.

Let i be an even integer and let j be an odd integer. Note that $i + j$ is odd and $j + i$ is odd.

Let i, j be odd integers. Observe that $i + j$ is even.

Let i be an even integer and let j be an odd integer. Observe that $i - j$ is odd and $j - i$ is odd.

Let i, j be odd integers. One can verify that $i - j$ is even.

Let us consider E, f, n. Then f^n is a function from E into E.

Let A be a set and let B be a set with a non-empty element. One can verify that there exists a function from A into B which is non-empty.

Let A be a non empty set, let B be a set with a non-empty element, let f be a non-empty function from A into B, and let a be an element of A. One can verify that $f(a)$ is non empty.

Let X be a non empty set. Note that 2^X has a non-empty element.

We now state two propositions:

(2) For every non empty subset S of $\mathbb{N}$ such that $0 \in S$ holds $\min S = 0$.

(3) For every non empty set E and for every function f from E into E and for every element x of E holds $f^0(x) = x$.

Let f be a function. We say that f has a fixpoint if and only if:

(Def. 3) There exists x which is a fixpoint of f.

We introduce f has no fixpoint as an antonym of f has a fixpoint.

Let X be a set and let x be an element of X. We say that x is covering if and only if:

(Def. 4) $\bigcup x = \bigcup\bigcup X$.

One can prove the following proposition

(4) s_1 is covering iff $\bigcup s_1 = E$.

Let us consider E. One can verify that there exists a family of subsets of E which is non empty, finite, and covering.

2. Abian's Theorem

One can prove the following proposition

(5) Let E be a set, f be a function from E into E, and s_1 be a non empty covering family of subsets of E such that for every element X of s_1 holds X misses $f^\circ X$. Then f has no fixpoint.

Let us consider E, f. The functor $f_\equiv$ yielding an equivalence relation of E is defined by:

(Def. 5) For all x, y such that $x \in E$ and $y \in E$ holds $\langle x, y\rangle \in f_\equiv$ iff there exist k, l such that $f^k(x) = f^l(y)$.

One can prove the following three propositions:

(6) Let E be a non empty set, f be a function from E into E, c be an element of $\mathrm{Classes}(f_\equiv)$, and e be an element of c. Then $f(e) \in c$.

(7) Let E be a non empty set, f be a function from E into E, c be an element of $\mathrm{Classes}(f_\equiv)$, e be an element of c, and given n. Then $f^n(e) \in c$.

(8) Let E be a non empty set and f be a function from E into E. Suppose f has no fixpoint. Then there exist E_1, E_2, E_3 such that $E_1 \cup E_2 \cup E_3 = E$ and $f^\circ E_1$ misses E_1 and $f^\circ E_2$ misses E_2 and $f^\circ E_3$ misses E_3.

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On Same Equivalents of Well-foundedness[1]

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Summary. Four statements equivalent to well-foundedness (well-founded induction, existence of recursively defined functions, uniqueness of recursively defined functions, and absence of descending ω-chains) have been proved in Mizar and the proofs were mechanically checked for correctness. It seems not to be widely known that the existence (without the uniqueness assumption) of recursively defined functions implies well-foundedness. In the proof we used regular cardinals, a fairly advanced notion of set theory. This work was inspired by T. Franzen's paper [17]. Franzen's proofs were written by a mathematician having an argument with a computer scientist. We were curious about the effort needed to formalize Franzen's proofs given the state of the Mizar Mathematical Library at that time (July 1996). The formalization went quite smoothly once the mathematics was sorted out.

MML Identifier: `WELLFND1`.

The articles [23], [3], [25], [14], [26], [11], [19], [27], [13], [12], [21], [4], [6], [5], [16], [2], [1], [24], [22], [9], [10], [20], [7], [15], [18], and [8] provide the terminology and notation for this paper.

1. Preliminaries

Let R be a 1-sorted structure, let X be a set, and let p be a partial function from the carrier of R to X. Then $\mathrm{dom}\, p$ is a subset of R.

Next we state two propositions:

(1) For every set X and for all functions f, g such that $f \subseteq g$ and $X \subseteq \mathrm{dom}\, f$ holds $f \restriction X = g \restriction X$.

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(2) Let X be a functional set. Suppose that for all functions f, g such that $f \in X$ and $g \in X$ holds $f \approx g$. Then $\bigcup X$ is a function.

The scheme *PFSeparation* concerns sets $\mathcal{A}$, $\mathcal{B}$ and a unary predicate $\mathcal{P}$, and states that:

> There exists a subset P_1 of $\mathcal{A} \dot{\rightarrow} \mathcal{B}$ such that for every partial function p_1 from $\mathcal{A}$ to $\mathcal{B}$ holds $p_1 \in P_1$ iff $\mathcal{P}[p_1]$

for all values of the parameters.

Let X be a set. Observe that X^+ is non empty.

Let us note that there exists an aleph which is regular.

One can prove the following two propositions:

(3) For every regular aleph M and for every set X such that $X \subseteq M$ and $\overline{\overline{X}} \in M$ holds $\sup X \in M$.

(4) For every relational structure R and for every set x holds (the internal relation of R)-Seg(x) $\subseteq$ the carrier of R.

Let R be a relational structure and let X be a subset of R. Let us observe that X is lower if and only if:

(Def. 1) For all sets x, y such that $x \in X$ and $\langle y, x\rangle \in$ the internal relation of R holds $y \in X$.

Next we state two propositions:

(5) Let R be a relational structure, X be a subset of R, and x be a set. If X is lower and $x \in X$, then (the internal relation of R)-Seg(x) $\subseteq X$.

(6) Let R be a relational structure, X be a lower subset of R, Y be a subset of R, and x be a set. If $Y = X \cup \{x\}$ and (the internal relation of R)-Seg(x) $\subseteq X$, then Y is lower.

2. Well Founded Relational Structures

Let R be a relational structure. We say that R is well founded if and only if:

(Def. 2) The internal relation of R is well founded in the carrier of R.

Let us mention that there exists a relational structure which is non empty and well founded.

Let R be a relational structure and let X be a subset of R. We say that X is well founded if and only if:

(Def. 3) The internal relation of R is well founded in X.

Let R be a relational structure. Note that there exists a subset of R which is well founded.

Let R be a relational structure. The functor WF-Part(R) yielding a subset of R is defined by:

(Def. 4) WF-Part(R) $= \bigcup\{S, S$ ranges over subsets of R: S is well founded and lower$\}$.

Let R be a relational structure. One can verify that WF-Part(R) is lower and well founded.

One can prove the following four propositions:

(7) Let R be a non empty relational structure and x be an element of the carrier of R. Then $\{x\}$ is a well founded subset of R.

(8) Let R be a relational structure and X, Y be well founded subsets of R. If X is lower, then $X \cup Y$ is a well founded subset of R.

(9) For every relational structure R holds R is well founded iff WF-Part(R) = the carrier of R.

(10) Let R be a non empty relational structure and x be an element of the carrier of R. If (the internal relation of R)-Seg(x) $\subseteq$ WF-Part(R), then $x \in$ WF-Part(R).

The scheme *WFMin* deals with a non empty relational structure $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

There exists an element x of $\mathcal{A}$ such that $\mathcal{P}[x]$ and it is not true that there exists an element y of $\mathcal{A}$ such that $x \neq y$ and $\mathcal{P}[y]$ and $\langle y, x\rangle \in$ the internal relation of $\mathcal{A}$

provided the parameters meet the following requirements:

- $\mathcal{P}[\mathcal{B}]$, and
- $\mathcal{A}$ is well founded.

We now state the proposition

(11) Let R be a non empty relational structure. Then R is well founded if and only if for every set S such that for every element x of the carrier of R such that (the internal relation of R)-Seg(x) $\subseteq S$ holds $x \in S$ holds the carrier of $R \subseteq S$.

The scheme *WFInduction* deals with a non empty relational structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every element x of $\mathcal{A}$ holds $\mathcal{P}[x]$

provided the parameters meet the following conditions:

- Let x be an element of $\mathcal{A}$. Suppose that for every element y of $\mathcal{A}$ such that $y \neq x$ and $\langle y, x\rangle \in$ the internal relation of $\mathcal{A}$ holds $\mathcal{P}[y]$. Then $\mathcal{P}[x]$, and
- $\mathcal{A}$ is well founded.

Let R be a non empty relational structure, let V be a non empty set, let H be a function from [: the carrier of R, (the carrier of R)$\dot{\rightarrow}V$:] into V, and let F be a function. We say that F is recursively expressed by H if and only if:

(Def. 5) For every element x of the carrier of R holds $F(x) = H(\langle x, F{\restriction}$(the internal relation of R)-Seg(x)$\rangle)$.

One can prove the following propositions:

(12) Let R be a non empty relational structure. Then R is well founded if and only if for every non empty set V and for every function H from $[\!:$ the carrier of R, (the carrier of $R) \dot{\rightarrow} V :\!]$ into V holds there exists a function from the carrier of R into V which is recursively expressed by H.

(13) Let R be a non empty relational structure and V be a non trivial set. Suppose that for every function H from $[\!:$ the carrier of R, (the carrier of $R) \dot{\rightarrow} V :\!]$ into V and for all functions F_1, F_2 from the carrier of R into V such that F_1 is recursively expressed by H and F_2 is recursively expressed by H holds $F_1 = F_2$. Then R is well founded.

(14) Let R be a non empty well founded relational structure, V be a non empty set, H be a function from $[\!:$ the carrier of R, (the carrier of $R) \dot{\rightarrow} V :\!]$ into V, and F_1, F_2 be functions from the carrier of R into V. Suppose F_1 is recursively expressed by H and F_2 is recursively expressed by H. Then $F_1 = F_2$.

Let S be a set. Let us assume that *contradiction*.[2]

(Def. 6) choose(S) is an element of S.

Let R be a relational structure and let f be a sequence of R. We say that f is descending if and only if:

(Def. 7) For every natural number n holds $f(n+1) \neq f(n)$ and $\langle f(n+1), f(n) \rangle \in$ the internal relation of R.

One can prove the following proposition

(15) For every non empty relational structure R holds R is well founded iff there exists no sequence of R which is descending.

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[2]This definition is absolutely permissive, i.e. we assume a *contradiction*, but we are interested only in the type of the functor 'choose'.

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Algebraic and Arithmetic Lattices. Part I[1]

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Summary. We formalize [10, pp.87–89]

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The papers [17], [21], [20], [16], [14], [9], [22], [19], [6], [7], [15], [18], [1], [2], [11], [24], [4], [8], [5], [23], [12], [3], and [13] provide the terminology and notation for this paper.

1. Preliminaries

The scheme *LambdaCD* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, a unary functor $\mathcal{G}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

> There exists a function f such that $\operatorname{dom} f = \mathcal{A}$ and for every element x of $\mathcal{A}$ holds if $\mathcal{P}[x]$, then $f(x) = \mathcal{F}(x)$ and if not $\mathcal{P}[x]$, then $f(x) = \mathcal{G}(x)$

for all values of the parameters.

The following propositions are true:

(1) Let L be a non empty reflexive transitive relational structure and x, y be elements of L. If $x \leqslant y$, then $\mathrm{compactbelow}(x) \subseteq \mathrm{compactbelow}(y)$.

(2) For every non empty reflexive relational structure L and for every element x of L holds $\mathrm{compactbelow}(x)$ is a subset of $\mathrm{CompactSublatt}(L)$.

(3) For every relational structure L and for every relational substructure S of L holds every subset of S is a subset of L.

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(4) For every non empty reflexive transitive relational structure L with l.u.b.'s holds the carrier of L is an ideal of L.

(5) Let L_1 be a lower-bounded non empty reflexive antisymmetric relational structure and L_2 be a non empty reflexive antisymmetric relational structure. Suppose the relational structure of L_1 = the relational structure of L_2 and L_1 is up-complete. Then the carrier of CompactSublatt(L_1) = the carrier of CompactSublatt(L_2).

2. Algebraic and Arithmetic Lattices

Next we state three propositions:

(6) For every algebraic lower-bounded lattice L holds every continuous subframe of L is algebraic.

(7) Let X, E be sets and L be a continuous subframe of $2^X_\subseteq$. Then $E \in$ the carrier of CompactSublatt(L) if and only if there exists an element F of $2^X_\subseteq$ such that F is finite and $E = \bigcap\{Y, Y$ ranges over elements of L: $F \subseteq Y\}$ and $F \subseteq E$.

(8) For every lower-bounded sup-semilattice L holds $\langle \mathrm{Ids}(L), \subseteq\rangle$ is a continuous subframe of $2_\subseteq^{\text{the carrier of } L}$.

Let L be a non empty reflexive transitive relational structure. Observe that there exists an ideal of L which is principal.

One can prove the following propositions:

(9) For every lower-bounded sup-semilattice L and for every non empty directed subset X of $\langle \mathrm{Ids}(L), \subseteq\rangle$ holds $\sup X = \bigcup X$.

(10) For every lower-bounded sup-semilattice S holds $\langle \mathrm{Ids}(S), \subseteq\rangle$ is algebraic.

(11) Let S be a lower-bounded sup-semilattice and x be an element of $\langle \mathrm{Ids}(S), \subseteq\rangle$. Then x is compact if and only if x is a principal ideal of S.

(12) Let S be a lower-bounded sup-semilattice and x be an element of $\langle \mathrm{Ids}(S), \subseteq\rangle$. Then x is compact if and only if there exists an element a of S such that $x = \downarrow a$.

(13) Let L be a lower-bounded sup-semilattice and f be a map from L into CompactSublatt$(\langle \mathrm{Ids}(L), \subseteq\rangle)$. If for every element x of L holds $f(x) = \downarrow x$, then f is isomorphic.

(14) For every lower-bounded lattice S holds $\langle \mathrm{Ids}(S), \subseteq\rangle$ is arithmetic.

(15) For every lower-bounded sup-semilattice L holds CompactSublatt(L) is a lower-bounded sup-semilattice.

(16) Let L be an algebraic lower-bounded sup-semilattice and f be a map from L into $\langle \mathrm{Ids}(\mathrm{CompactSublatt}(L)), \subseteq \rangle$. If for every element x of L holds $f(x) = \mathrm{compactbelow}(x)$, then f is isomorphic.

(17) Let L be an algebraic lower-bounded sup-semilattice and x be an element of L. Then $\mathrm{compactbelow}(x)$ is a principal ideal of $\mathrm{CompactSublatt}(L)$ if and only if x is compact.

3. Maps

We now state three propositions:

(18) Let L_1, L_2 be non empty relational structures, X be a subset of L_1, x be an element of L_1, and f be a map from L_1 into L_2. If f is isomorphic, then $x \leqslant X$ iff $f(x) \leqslant f^\circ X$.

(19) Let L_1, L_2 be non empty relational structures, X be a subset of L_1, x be an element of L_1, and f be a map from L_1 into L_2. If f is isomorphic, then $x \geqslant X$ iff $f(x) \geqslant f^\circ X$.

(20) Let L_1, L_2 be non empty antisymmetric relational structures and f be a map from L_1 into L_2. If f is isomorphic, then f is infs-preserving and sups-preserving.

Let L_1, L_2 be non empty antisymmetric relational structures. Note that every map from L_1 into L_2 which is isomorphic is also infs-preserving and sups-preserving.

We now state a number of propositions:

(21) Let L_1, L_2, L_3 be non empty transitive antisymmetric relational structures and f be a map from L_1 into L_2. Suppose f is infs-preserving. Suppose L_2 is a full infs-inheriting relational substructure of L_3 and L_3 is complete. Then there exists a map g from L_1 into L_3 such that $f = g$ and g is infs-preserving.

(22) Let L_1, L_2, L_3 be non empty transitive antisymmetric relational structures and f be a map from L_1 into L_2. Suppose f is monotone and directed-sups-preserving. Suppose L_2 is a full directed-sups-inheriting relational substructure of L_3 and L_3 is complete. Then there exists a map g from L_1 into L_3 such that $f = g$ and g is directed-sups-preserving.

(23) For every lower-bounded sup-semilattice L holds $\langle \mathrm{Ids}(\mathrm{CompactSublatt}(L)), \subseteq \rangle$ is a continuous subframe of $2_\subseteq^{\text{the carrier of CompactSublatt}(L)}$.

(24) Let L be an algebraic lower-bounded lattice. Then there exists a map g from L into $2_\subseteq^{\text{the carrier of CompactSublatt}(L)}$ such that
  (i) g is infs-preserving, directed-sups-preserving, and one-to-one, and
  (ii) for every element x of L holds $g(x) = \mathrm{compactbelow}(x)$.

(25) Let I be a non empty set and J be a relational structure yielding non-empty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is an algebraic lower-bounded lattice. Then $\prod J$ is an algebraic lower-bounded lattice.

(26) Let L_1, L_2 be non empty relational structures. Suppose the relational structure of L_1 = the relational structure of L_2. Then L_1 and L_2 are isomorphic.

(27) Let L_1, L_2 be up-complete non empty posets and f be a map from L_1 into L_2. Suppose f is isomorphic. Let x, y be elements of L_1. Then $x \ll y$ if and only if $f(x) \ll f(y)$.

(28) Let L_1, L_2 be up-complete non empty posets and f be a map from L_1 into L_2. Suppose f is isomorphic. Let x be an element of L_1. Then x is compact if and only if $f(x)$ is compact.

(29) Let L_1, L_2 be up-complete non empty posets and f be a map from L_1 into L_2. If f is isomorphic, then for every element x of L_1 holds $f^\circ \operatorname{compactbelow}(x) = \operatorname{compactbelow}(f(x))$.

(30) For all non empty posets L_1, L_2 such that L_1 and L_2 are isomorphic and L_1 is up-complete holds L_2 is up-complete.

(31) For all non empty posets L_1, L_2 such that L_1 and L_2 are isomorphic and L_1 is complete and satisfies axiom K holds L_2 satisfies axiom K.

(32) Let L_1, L_2 be sup-semilattices. Suppose L_1 and L_2 are isomorphic and L_1 is lower-bounded and algebraic. Then L_2 is algebraic.

(33) For every continuous lower-bounded sup-semilattice L holds $\operatorname{SupMap}(L)$ is infs-preserving and sups-preserving.

(34) Let L be a lower-bounded lattice. Then L is algebraic if and only if there exists a set X and there exists a full relational substructure S of $2^X_\subseteq$ such that S is infs-inheriting and directed-sups-inheriting and L and S are isomorphic.

(35) Let L be a lower-bounded lattice. Then L is algebraic if and only if there exists a set X and there exists a closure map c from $2^X_\subseteq$ into $2^X_\subseteq$ such that c is directed-sups-preserving and L and $\operatorname{Im} c$ are isomorphic.

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Subsequences of Standard Special Circular Sequences in $\mathcal{E}^2_{\mathrm{T}}$

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Summary. It is known that a standard special circular sequence in $\mathcal{E}^2_{\mathrm{T}}$ properly defines a special polygon. We are interested in a part of such a sequence. It is shown that if the first point and the last point of the subsequence are different, it becomes a special polygonal sequence. The concept of „a part of" is introduced, and the subsequence having this property can be characterized by using „mid" function. For such subsequences, the concepts of „Upper" and „Lower" parts are introduced.

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The notation and terminology used here are introduced in the following papers: [16], [19], [8], [1], [14], [20], [2], [3], [18], [4], [6], [7], [11], [10], [13], [15], [5], [17], [9], and [12].

1. Preliminaries

We adopt the following convention: $i, i_1, i_2, i_3, j, k, n$ denote natural numbers and r_1, r_2, s, s_1 denote real numbers.

The following propositions are true:

(1) If $n -' i = 0$, then $n \leqslant i$.

(2) If $i \leqslant j$, then $(j + k) -' i = (j + k) - i$.

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(3) If $i \leqslant j$, then $(j+k) -' i = j -' i + k$.

(4) If $i_1 \neq 0$ and $i_2 = i_3 \cdot i_1$, then $i_3 \leqslant i_2$.

(5) If $i_1 < i_2$, then $i_1 \div i_2 = 0$.

(6) If $0 < j$ and $j < i$ and $i < j + j$, then $i \bmod j \neq 0$.

(7) If $0 < j$ and $j \leqslant i$ and $i < j+j$, then $i \bmod j = i - j$ and $i \bmod j = i -' j$.

(8) If $0 < j$, then $(j+j) \bmod j = 0$ and $k \cdot j \bmod j = 0$.

(9) If $0 < k$ and $k \leqslant j$ and $k \bmod j = 0$, then $k = j$.

(10) $(r_1 + s_1 + r_2) - s_1 = r_1 + r_2$ and $(r_1 - s_1) + r_2 + s_1 = r_1 + r_2$ and $(r_1 + s_1) - r_2 - s_1 = r_1 - r_2$ and $(r_1 - s_1 - r_2) + s_1 = r_1 - r_2$.

(11) $r_1 - r_1 - r_2 = -r_2$ and $(-r_1 + r_1) - r_2 = -r_2$ and $r_1 - r_2 - r_1 = -r_2$ and $(-r_1 - r_2) + r_1 = -r_2$.

(12) If $0 < s$ and if $s \cdot r_1 \leqslant s \cdot r_2$ or $r_1 \cdot s \leqslant r_2 \cdot s$, then $r_1 \leqslant r_2$.

(13) If $0 < s$ and if $s \cdot r_1 < s \cdot r_2$ or $r_1 \cdot s < r_2 \cdot s$, then $r_1 < r_2$.

2. Some facts about cutting of finite sequences

In the sequel D denotes a non empty set, f_1 denotes a finite sequence of elements of D, and f denotes a non constant standard special circular sequence.

We now state a number of propositions:

(14) For every f_1 such that f_1 is circular and $1 \leqslant \operatorname{len} f_1$ holds $f_1(1) = f_1(\operatorname{len} f_1)$.

(15) For all f_1, i_1, i_2 such that $i_1 \leqslant i_2$ holds $f_1 \restriction i_1 \restriction i_2 = f_1 \restriction i_1$ and $f_1 \restriction i_2 \restriction i_1 = f_1 \restriction i_1$.

(16) $\varepsilon_D \restriction i = \varepsilon_D$.

(17) $\operatorname{Rev}(\varepsilon_D) = \varepsilon_D$.

(18) For all f_1, k such that $k < \operatorname{len} f_1$ holds $(f_1)_{\lfloor k}(\operatorname{len}((f_1)_{\lfloor k})) = f_1(\operatorname{len} f_1)$ and $\pi_{\operatorname{len}((f_1)_{\lfloor k})}(f_1)_{\lfloor k} = \pi_{\operatorname{len} f_1} f_1$.

(19) Let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and given i. If g is a special sequence and $i + 1 < \operatorname{len} g$, then $g_{\lfloor i}$ is a special sequence.

(20) For all f_1, i_1, i_2 such that $1 \leqslant i_2$ and $i_2 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} f_1$ holds $\operatorname{len} \operatorname{mid}(f_1, i_2, i_1) = i_1 -' i_2 + 1$.

(21) For all f_1, i_1, i_2 such that $1 \leqslant i_2$ and $i_2 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} f_1$ holds $\operatorname{len} \operatorname{mid}(f_1, i_1, i_2) = i_1 -' i_2 + 1$.

(22) For all f_1, i_1, i_2, j such that $1 \leqslant i_1$ and $i_1 \leqslant i_2$ and $i_2 \leqslant \operatorname{len} f_1$ holds $(\operatorname{mid}(f_1, i_1, i_2))(\operatorname{len} \operatorname{mid}(f_1, i_1, i_2)) = f_1(i_2)$.

(23) For all f_1, i_1, i_2, j such that $1 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} f_1$ and $1 \leqslant i_2$ and $i_2 \leqslant \operatorname{len} f_1$ holds $(\operatorname{mid}(f_1, i_1, i_2))(\operatorname{len} \operatorname{mid}(f_1, i_1, i_2)) = f_1(i_2)$.

(24) For all f_1, i_1, i_2, j such that $1 \leqslant i_2$ and $i_2 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} f_1$ and $1 \leqslant j$ and $j \leqslant i_1 -' i_2 + 1$ holds $(\operatorname{mid}(f_1, i_1, i_2))(j) = f_1(i_1 -' j + 1)$.

(25) Let given f_1, i_1, i_2. Suppose $1 \leqslant i_2$ and $i_2 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} f_1$ and $1 \leqslant j$ and $j \leqslant i_1 -' i_2 + 1$. Then $(\operatorname{mid}(f_1, i_1, i_2))(j) = (\operatorname{mid}(f_1, i_2, i_1))((((i_1 - i_2) + 1) - j) + 1)$ and $(((i_1 - i_2) + 1) - j) + 1 = (i_1 -' i_2 + 1) -' j + 1$.

(26) Let given f_1, i_1, i_2. Suppose $1 \leqslant i_1$ and $i_1 \leqslant i_2$ and $i_2 \leqslant \operatorname{len} f_1$ and $1 \leqslant j$ and $j \leqslant i_2 -' i_1 + 1$. Then $(\operatorname{mid}(f_1, i_1, i_2))(j) = (\operatorname{mid}(f_1, i_2, i_1))((((i_2 - i_1) + 1) - j) + 1)$ and $(((i_2 - i_1) + 1) - j) + 1 = (i_2 -' i_1 + 1) -' j + 1$.

(27) For all f_1, k such that $1 \leqslant k$ and $k \leqslant \operatorname{len} f_1$ holds $\operatorname{mid}(f_1, k, k) = \langle \pi_k f_1 \rangle$ and $\operatorname{len} \operatorname{mid}(f_1, k, k) = 1$.

(28) $\operatorname{mid}(f_1, 0, 0) = f_1 \restriction 1$.

(29) For all f_1, k such that $\operatorname{len} f_1 < k$ holds $\operatorname{mid}(f_1, k, k) = \varepsilon_D$.

(30) For all f_1, i_1, i_2 holds $\operatorname{mid}(f_1, i_1, i_2) = \operatorname{Rev}(\operatorname{mid}(f_1, i_2, i_1))$.

(31) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and given i_1, i_2, i. If $1 \leqslant i_1$ and $i_1 < i_2$ and $i_2 \leqslant \operatorname{len} f$ and $1 \leqslant i$ and $i < i_2 -' i_1 + 1$, then $\mathcal{L}(\operatorname{mid}(f, i_1, i_2), i) = \mathcal{L}(f, (i + i_1) -' 1)$.

(32) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and given i_1, i_2, i. If $1 \leqslant i_1$ and $i_1 < i_2$ and $i_2 \leqslant \operatorname{len} f$ and $1 \leqslant i$ and $i < i_2 -' i_1 + 1$, then $\mathcal{L}(\operatorname{mid}(f, i_2, i_1), i) = \mathcal{L}(f, i_2 -' i)$.

3. Dividing of special circular sequences into parts

Let n be a natural number and let f be a finite sequence. The functor $\operatorname{S\_Drop}(n, f)$ yields a natural number and is defined by:

(Def. 1) $$\operatorname{S\_Drop}(n, f) = \begin{cases} n \bmod \operatorname{len} f -' 1, & \text{if } n \bmod \operatorname{len} f -' 1 \neq 0, \\ \operatorname{len} f -' 1, & \text{otherwise.} \end{cases}$$

Next we state three propositions:

(33) For every finite sequence f such that $0 < \operatorname{len} f -' 1$ holds $\operatorname{S\_Drop}(\operatorname{len} f -' 1, f) = \operatorname{len} f -' 1$.

(34) For every natural number n and for every finite sequence f such that $1 \leqslant n$ and $n \leqslant \operatorname{len} f -' 1$ holds $\operatorname{S\_Drop}(n, f) = n$.

(35) Let n be a natural number and f be a finite sequence. If $\operatorname{len} f > 1$ or $\operatorname{len} f -' 1 > 0$, then $\operatorname{S\_Drop}(n, f) = \operatorname{S\_Drop}(n + \operatorname{len} f -' 1, f)$ and $\operatorname{S\_Drop}(n, f) = \operatorname{S\_Drop}(\operatorname{len} f -' 1 + n, f)$.

Let f be a non constant standard special circular sequence, let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and let i_1, i_2 be natural numbers. We say that g is a right part of f from i_1 to i_2 if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) $1 \leqslant i_1$,

(ii) $i_1 + 1 \leqslant \operatorname{len} f$,
(iii) $1 \leqslant i_2$,
(iv) $i_2 + 1 \leqslant \operatorname{len} f$,
(v) $g(\operatorname{len} g) = f(i_2)$,
(vi) $1 \leqslant \operatorname{len} g$,
(vii) $\operatorname{len} g < \operatorname{len} f$, and
(viii) for every natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} g$ holds $g(i) = f(\mathrm{S\_Drop}((i_1 + i) -' 1, f))$.

Let f be a non constant standard special circular sequence, let g be a finite sequence of elements of $\mathcal{E}^2_\mathrm{T}$, and let i_1, i_2 be natural numbers. We say that g is a left part of f from i_1 to i_2 if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) $1 \leqslant i_1$,
(ii) $i_1 + 1 \leqslant \operatorname{len} f$,
(iii) $1 \leqslant i_2$,
(iv) $i_2 + 1 \leqslant \operatorname{len} f$,
(v) $g(\operatorname{len} g) = f(i_2)$,
(vi) $1 \leqslant \operatorname{len} g$,
(vii) $\operatorname{len} g < \operatorname{len} f$, and
(viii) for every natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} g$ holds $g(i) = f(\mathrm{S\_Drop}((\operatorname{len} f + i_1) -' i, f))$.

Let f be a non constant standard special circular sequence, let g be a finite sequence of elements of $\mathcal{E}^2_\mathrm{T}$, and let i_1, i_2 be natural numbers. We say that g is a part of f from i_1 to i_2 if and only if:

(Def. 4) g is a right part of f from i_1 to i_2 or a left part of f from i_1 to i_2.

We now state a number of propositions:

(36) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_\mathrm{T}$, and i_1, i_2 be natural numbers. Suppose g is a part of f from i_1 to i_2. Then
(i) $1 \leqslant i_1$,
(ii) $i_1 + 1 \leqslant \operatorname{len} f$,
(iii) $1 \leqslant i_2$,
(iv) $i_2 + 1 \leqslant \operatorname{len} f$,
(v) $g(\operatorname{len} g) = f(i_2)$,
(vi) $1 \leqslant \operatorname{len} g$,
(vii) $\operatorname{len} g < \operatorname{len} f$, and
(viii) for every natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} g$ holds $g(i) = f(\mathrm{S\_Drop}((i_1 + i) -' 1, f))$ or for every natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} g$ holds $g(i) = f(\mathrm{S\_Drop}((\operatorname{len} f + i_1) -' i, f))$.

(37) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_\mathrm{T}$, and i_1, i_2 be natural numbers. Suppose g is

a right part of f from i_1 to i_2 and $i_1 \leqslant i_2$. Then $\operatorname{len} g = i_2 -' i_1 + 1$ and $g = \operatorname{mid}(f, i_1, i_2)$.

(38) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a right part of f from i_1 to i_2 and $i_1 > i_2$. Then $\operatorname{len} g = (\operatorname{len} f + i_2) -' i_1$ and $g = (\operatorname{mid}(f, i_1, \operatorname{len} f -' 1)) \frown (f{\restriction}i_2)$ and $g = (\operatorname{mid}(f, i_1, \operatorname{len} f -' 1)) \frown \operatorname{mid}(f, 1, i_2)$.

(39) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a left part of f from i_1 to i_2 and $i_1 \geqslant i_2$. Then $\operatorname{len} g = i_1 -' i_2 + 1$ and $g = \operatorname{mid}(f, i_1, i_2)$.

(40) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a left part of f from i_1 to i_2 and $i_1 < i_2$. Then $\operatorname{len} g = (\operatorname{len} f + i_1) -' i_2$ and $g = (\operatorname{mid}(f, i_1, 1)) \frown \operatorname{mid}(f, \operatorname{len} f -' 1, i_2)$.

(41) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a right part of f from i_1 to i_2. Then $\operatorname{Rev}(g)$ is a left part of f from i_2 to i_1.

(42) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a left part of f from i_1 to i_2. Then $\operatorname{Rev}(g)$ is a right part of f from i_2 to i_1.

(43) Let f be a non constant standard special circular sequence and i_1, i_2 be natural numbers. If $1 \leqslant i_1$ and $i_1 \leqslant i_2$ and $i_2 < \operatorname{len} f$, then $\operatorname{mid}(f, i_1, i_2)$ is a right part of f from i_1 to i_2.

(44) Let f be a non constant standard special circular sequence and i_1, i_2 be natural numbers. If $1 \leqslant i_1$ and $i_1 \leqslant i_2$ and $i_2 < \operatorname{len} f$, then $\operatorname{mid}(f, i_2, i_1)$ is a left part of f from i_2 to i_1.

(45) Let f be a non constant standard special circular sequence and i_1, i_2 be natural numbers. Suppose $1 \leqslant i_2$ and $i_1 > i_2$ and $i_1 < \operatorname{len} f$. Then $(\operatorname{mid}(f, i_1, \operatorname{len} f -' 1)) \frown \operatorname{mid}(f, 1, i_2)$ is a right part of f from i_1 to i_2.

(46) Let f be a non constant standard special circular sequence and i_1, i_2 be natural numbers. Suppose $1 \leqslant i_1$ and $i_1 < i_2$ and $i_2 < \operatorname{len} f$. Then $(\operatorname{mid}(f, i_1, 1)) \frown \operatorname{mid}(f, \operatorname{len} f -' 1, i_2)$ is a left part of f from i_1 to i_2.

(47) Let h be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and given i_1, i_2. If $1 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} h$ and $1 \leqslant i_2$ and $i_2 \leqslant \operatorname{len} h$, then $\widetilde{\mathcal{L}}(\operatorname{mid}(h, i_1, i_2)) \subseteq \widetilde{\mathcal{L}}(h)$.

(48) Let g be a finite sequence of elements of D. Then g is one-to-one if and only if for all i_1, i_2 such that $1 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} g$ and $1 \leqslant i_2$ and $i_2 \leqslant \operatorname{len} g$ and $g(i_1) = g(i_2)$ or $\pi_{i_1} g = \pi_{i_2} g$ holds $i_1 = i_2$.

(49) Let f be a non constant standard special circular sequence and given i_2. If $1 < i_2$ and $i_2 + 1 \leqslant \operatorname{len} f$, then $f{\restriction}i_2$ is a special sequence.

(50) Let f be a non constant standard special circular sequence and given i_2. If $1 \leqslant i_2$ and $i_2 + 1 < \operatorname{len} f$, then $f_{\restriction i_2}$ is a special sequence.

(51) Let f be a non constant standard special circular sequence and given i_1, i_2. If $1 \leqslant i_1$ and $i_1 < i_2$ and $i_2 + 1 \leqslant \operatorname{len} f$, then $\operatorname{mid}(f, i_1, i_2)$ is a special sequence.

(52) Let f be a non constant standard special circular sequence and given i_1, i_2. If $1 < i_1$ and $i_1 < i_2$ and $i_2 \leqslant \operatorname{len} f$, then $\operatorname{mid}(f, i_1, i_2)$ is a special sequence.

(53) For all points p_0, p, q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_0 \in \mathcal{L}(p, q_1)$ and $p_0 \in \mathcal{L}(p, q_2)$ and $p \neq p_0$ holds $q_1 \in \mathcal{L}(p, q_2)$ or $q_2 \in \mathcal{L}(p, q_1)$.

(54) For every non constant standard special circular sequence f holds $\mathcal{L}(f, 1) \cap \mathcal{L}(f, \operatorname{len} f -' 1) = \{f(1)\}$.

(55) Let f be a non constant standard special circular sequence, i_1, i_2 be natural numbers, and g_1, g_2 be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $1 \leqslant i_1$ and $i_1 < i_2$ and $i_2 < \operatorname{len} f$ and $g_1 = \operatorname{mid}(f, i_1, i_2)$ and $g_2 = (\operatorname{mid}(f, i_1, 1)) \frown \operatorname{mid}(f, \operatorname{len} f -' 1, i_2)$. Then $\widetilde{\mathcal{L}}(g_1) \cap \widetilde{\mathcal{L}}(g_2) = \{f(i_1), f(i_2)\}$ and $\widetilde{\mathcal{L}}(g_1) \cup \widetilde{\mathcal{L}}(g_2) = \widetilde{\mathcal{L}}(f)$.

(56) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a right part of f from i_1 to i_2 and $i_1 < i_2$. Then $\widetilde{\mathcal{L}}(g)$ is a special polygonal arc joining $\pi_{i_1} f$ and $\pi_{i_2} f$.

(57) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a left part of f from i_1 to i_2 and $i_1 > i_2$. Then $\widetilde{\mathcal{L}}(g)$ is a special polygonal arc joining $\pi_{i_1} f$ and $\pi_{i_2} f$.

(58) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a right part of f from i_1 to i_2 and $i_1 \neq i_2$. Then $\widetilde{\mathcal{L}}(g)$ is a special polygonal arc joining $\pi_{i_1} f$ and $\pi_{i_2} f$.

(59) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a left part of f from i_1 to i_2 and $i_1 \neq i_2$. Then $\widetilde{\mathcal{L}}(g)$ is a special polygonal arc joining $\pi_{i_1} f$ and $\pi_{i_2} f$.

(60) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a part of f from i_1 to i_2 and $i_1 \neq i_2$. Then $\widetilde{\mathcal{L}}(g)$ is a special polygonal arc joining $\pi_{i_1} f$ and $\pi_{i_2} f$.

(61) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a part of f from i_1 to i_2 and $g(1) \neq g(\operatorname{len} g)$. Then $\widetilde{\mathcal{L}}(g)$ is a special polygonal

arc joining $\pi_{i_1} f$ and $\pi_{i_2} f$.

(62) Let f be a non constant standard special circular sequence and i_1, i_2 be natural numbers. Suppose $1 \leqslant i_1$ and $i_1 + 1 \leqslant \operatorname{len} f$ and $1 \leqslant i_2$ and $i_2 + 1 \leqslant \operatorname{len} f$ and $i_1 \neq i_2$. Then there exist finite sequences g_1, g_2 of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that

(i) g_1 is a part of f from i_1 to i_2,

(ii) g_2 is a part of f from i_1 to i_2,

(iii) $\widetilde{\mathcal{L}}(g_1) \cap \widetilde{\mathcal{L}}(g_2) = \{f(i_1), f(i_2)\}$,

(iv) $\widetilde{\mathcal{L}}(g_1) \cup \widetilde{\mathcal{L}}(g_2) = \widetilde{\mathcal{L}}(f)$,

(v) $\widetilde{\mathcal{L}}(g_1)$ is a special polygonal arc joining $\pi_{i_1} f$ and $\pi_{i_2} f$,

(vi) $\widetilde{\mathcal{L}}(g_2)$ is a special polygonal arc joining $\pi_{i_1} f$ and $\pi_{i_2} f$, and

(vii) for every finite sequence g of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that g is a part of f from i_1 to i_2 holds $g = g_1$ or $g = g_2$.

In the sequel g_1, g_2 are finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$.

We now state several propositions:

(63) Let f be a non constant standard special circular sequence and P be a non empty subset of the carrier of $(\mathcal{E}^2_{\mathrm{T}})$.If $P = \widetilde{\mathcal{L}}(f)$, then P is a simple closed curve.

(64) Let f be a non constant standard special circular sequence and given g_1, g_2. Suppose g_1 is a right part of f from i_1 to i_2 and g_2 is a right part of f from i_1 to i_2. Then $g_1 = g_2$.

(65) Let f be a non constant standard special circular sequence and given g_1, g_2. Suppose g_1 is a left part of f from i_1 to i_2 and g_2 is a left part of f from i_1 to i_2. Then $g_1 = g_2$.

(66) Let f be a non constant standard special circular sequence and given g_1, g_2. Suppose $i_1 \neq i_2$ and g_1 is a right part of f from i_1 to i_2 and g_2 is a left part of f from i_1 to i_2. Then $g_1(2) \neq g_2(2)$.

(67) Let f be a non constant standard special circular sequence and given g_1, g_2. Suppose $i_1 \neq i_2$ and g_1 is a part of f from i_1 to i_2 and g_2 is a part of f from i_1 to i_2 and $g_1(2) = g_2(2)$. Then $g_1 = g_2$.

Let f be a non constant standard special circular sequence and let i_1, i_2 be natural numbers. Let us assume that $1 \leqslant i_1$ and $i_1 + 1 \leqslant \operatorname{len} f$ and $1 \leqslant i_2$ and $i_2 + 1 \leqslant \operatorname{len} f$ and $i_1 \neq i_2$. The functor $\mathrm{Lower}(f, i_1, i_2)$ yields a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by the conditions (Def. 5).

(Def. 5)(i) $\mathrm{Lower}(f, i_1, i_2)$ is a part of f from i_1 to i_2,

(ii) if $(\pi_{i_1+1} f)_{\mathbf{1}} < (\pi_{i_1} f)_{\mathbf{1}}$ or $(\pi_{i_1+1} f)_{\mathbf{2}} < (\pi_{i_1} f)_{\mathbf{2}}$, then $(\mathrm{Lower}(f, i_1, i_2))(2) = f(i_1 + 1)$, and

(iii) if $(\pi_{i_1+1} f)_{\mathbf{1}} \geqslant (\pi_{i_1} f)_{\mathbf{1}}$ and $(\pi_{i_1+1} f)_{\mathbf{2}} \geqslant (\pi_{i_1} f)_{\mathbf{2}}$, then $(\mathrm{Lower}(f, i_1, i_2))(2) = f(\mathrm{S\_Drop}(i_1 -' 1, f))$.

The functor $\mathrm{Upper}(f, i_1, i_2)$ yielding a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ is defined by the conditions (Def. 6).

(Def. 6)(i) $\mathrm{Upper}(f, i_1, i_2)$ is a part of f from i_1 to i_2,

(ii) if $(\pi_{i_1+1}f)_{\mathbf{1}} > (\pi_{i_1}f)_{\mathbf{1}}$ or $(\pi_{i_1+1}f)_{\mathbf{2}} > (\pi_{i_1}f)_{\mathbf{2}}$, then $(\mathrm{Upper}(f, i_1, i_2))(2) = f(i_1 + 1)$, and

(iii) if $(\pi_{i_1+1}f)_{\mathbf{1}} \leqslant (\pi_{i_1}f)_{\mathbf{1}}$ and $(\pi_{i_1+1}f)_{\mathbf{2}} \leqslant (\pi_{i_1}f)_{\mathbf{2}}$, then $(\mathrm{Upper}(f, i_1, i_2))(2) = f(\mathrm{S\_Drop}(i_1 -' 1, f))$.

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Lattice of Substitutions

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The articles [8], [6], [5], [7], [1], [9], [2], [4], [11], [3], and [10] provide the terminology and notation for this paper.

1. Preliminaries

In this paper V, C are sets.

Let us consider V, C. The functor SubstitutionSet(V,C) yielding a subset of $\mathrm{Fin}(V \dot{\rightarrow} C)$ is defined as follows:

(Def. 1) $\mathrm{SubstitutionSet}(V,C) = \{A, A$ ranges over elements of $\mathrm{Fin}(V \dot{\rightarrow} C) : \bigwedge_{s,t:\,\mathrm{element\ of}\ V \dot{\rightarrow} C} (s \in A \wedge t \in A \wedge s \subseteq t \Rightarrow s = t)\}$.

Next we state two propositions:

(1) $\emptyset \in \mathrm{SubstitutionSet}(V,C)$.

(2) $\{\emptyset\} \in \mathrm{SubstitutionSet}(V,C)$.

Let us consider V, C. One can check that SubstitutionSet(V,C) is non empty.

Let us consider V, C and let A, B be elements of SubstitutionSet(V,C). Then $A \cup B$ is an element of $\mathrm{Fin}(V \dot{\rightarrow} C)$.

Let us consider V, C. Note that there exists an element of SubstitutionSet(V,C) which is non empty.

Let us consider V, C. Note that every element of SubstitutionSet(V,C) is finite.

Let us consider V, C and let A be an element of $\mathrm{Fin}(V \dot{\rightarrow} C)$. The functor $\Box^{\mathrm{c}}{}_A$ yields an element of SubstitutionSet(V,C) and is defined by:

(Def. 2) $\Box^{\mathrm{c}}{}_A = \{t, t$ ranges over elements of $V \dot{\rightarrow} C : \bigwedge_{s:\,\mathrm{element\ of}\ V \dot{\rightarrow} C} (s \in A \wedge s \subseteq t \Leftrightarrow s = t)\}$.

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Let us consider V, C and let A be a non empty element of SubstitutionSet(V,C). Note that every element of A is function-like and relation-like.

Let us consider V, C. One can verify that every element of $V \dot{\rightarrow} C$ is function-like and relation-like.

Let us consider V, C and let A, B be elements of $\mathrm{Fin}(V \dot{\rightarrow} C)$. The functor $A \frown B$ yields an element of $\mathrm{Fin}(V \dot{\rightarrow} C)$ and is defined as follows:

(Def. 3) $A \frown B = \{s \cup t, s$ ranges over elements of $V \dot{\rightarrow} C, t$ ranges over elements of $V \dot{\rightarrow} C : s \in A \ \wedge \ t \in B \ \wedge \ s \approx t\}$.

In the sequel A, B, D are elements of $\mathrm{Fin}(V \dot{\rightarrow} C)$.

One can prove the following propositions:

(3) $A \frown B = B \frown A$.

(4) If $B = \{\emptyset\}$, then $A \frown B = A$.

(5) For all sets a, b such that $B \in$ SubstitutionSet(V,C) and $a \in B$ and $b \in B$ and $a \subseteq b$ holds $a = b$.

(6) For every set a such that $a \in \Box^{\mathrm{c}}{}_{B}$ holds $a \in B$ and for every set b such that $b \in B$ and $b \subseteq a$ holds $b = a$.

(7) For every set a such that $a \in B$ and for every set b such that $b \in B$ and $b \subseteq a$ holds $b = a$ holds $a \in \Box^{\mathrm{c}}{}_{B}$.

(8) $\Box^{\mathrm{c}}{}_{A} \subseteq A$.

(9) If $A = \emptyset$, then $\Box^{\mathrm{c}}{}_{A} = \emptyset$.

(10) For every set b such that $b \in B$ there exists a set c such that $c \subseteq b$ and $c \in \Box^{\mathrm{c}}{}_{B}$.

(11) For every element K of SubstitutionSet(V,C) holds $\Box^{\mathrm{c}}{}_{K} = K$.

(12) $\Box^{\mathrm{c}}{}_{A \cup B} \subseteq \Box^{\mathrm{c}}{}_{A} \cup B$.

(13) $\Box^{\mathrm{c}}{}_{\Box^{\mathrm{c}}{}_{A} \cup B} = \Box^{\mathrm{c}}{}_{A \cup B}$.

(14) If $A \subseteq B$, then $A \frown D \subseteq B \frown D$.

(15) For every set a such that $a \in A \frown B$ there exist sets b, c such that $b \in A$ and $c \in B$ and $a = b \cup c$.

(16) For all elements b, c of $V \dot{\rightarrow} C$ such that $b \in A$ and $c \in B$ and $b \approx c$ holds $b \cup c \in A \frown B$.

(17) $\Box^{\mathrm{c}}{}_{A \frown B} \subseteq (\Box^{\mathrm{c}}{}_{A}) \frown B$.

(18) If $A \subseteq B$, then $D \frown A \subseteq D \frown B$.

(19) $\Box^{\mathrm{c}}{}_{(\Box^{\mathrm{c}}{}_{A}) \frown B} = \Box^{\mathrm{c}}{}_{A \frown B}$.

(20) $\Box^{\mathrm{c}}{}_{A \frown (\Box^{\mathrm{c}}{}_{B})} = \Box^{\mathrm{c}}{}_{A \frown B}$.

(21) For all elements K, L, M of $\mathrm{Fin}(V \dot{\rightarrow} C)$ holds $K \frown (L \frown M) = (K \frown L) \frown M$.

(22) For all elements K, L, M of $\mathrm{Fin}(V \dot{\rightarrow} C)$ holds $K \frown (L \cup M) = K \frown L \cup K \frown M$.

(23) $B \subseteq B \frown B$.

(24) $\Box^{c}{}_{A \frown A} = \Box^{c}{}_{A}$.

(25) For every element K of SubstitutionSet(V, C) holds $\Box^{c}{}_{K \frown K} = K$.

2. Definition of the lattice

Let us consider V, C. The functor SubstLatt(V, C) yielding a strict lattice structure is defined by the conditions (Def. 4).

(Def. 4)(i) The carrier of SubstLatt(V, C) = SubstitutionSet(V, C), and

(ii) for all elements A, B of SubstitutionSet(V, C) holds (the join operation of SubstLatt(V, C))$(A, B) = \Box^{c}{}_{A \cup B}$ and (the meet operation of SubstLatt(V, C))$(A, B) = \Box^{c}{}_{A \frown B}$.

Let us consider V, C. One can verify that SubstLatt(V, C) is non empty.

Let us consider V, C. Note that SubstLatt(V, C) is lattice-like.

Let us consider V, C. Observe that SubstLatt(V, C) is distributive and bounded.

One can prove the following two propositions:

(26) $\bot_{\mathrm{SubstLatt}(V, C)} = \emptyset$.

(27) $\top_{\mathrm{SubstLatt}(V, C)} = \{\emptyset\}$.

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Equations in Many Sorted Algebras

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Summary. This paper is preparation to prove Birkhoff's Theorem. Some properties of many sorted algebras are proved. The last section of this work shows that every equation valid in a many sorted algebra is also valid in each subalgebra, and each image of it. Moreover for a family of many sorted algebras $(A_i : i \in I)$ if every equation is valid in each A_i, $i \in I$ then is also valid in product $\prod(A_i : i \in I)$.

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The articles [23], [28], [10], [29], [6], [9], [7], [24], [11], [4], [8], [1], [2], [25], [26], [18], [19], [27], [20], [5], [12], [16], [17], [13], [22], [21], [15], [14], and [3] provide the notation and terminology for this paper.

1. On the Functions and Many Sorted Functions

In this paper I is a set.

Next we state several propositions:

(1) Let A be a set, B, C be non empty sets, f be a function from A into B, and g be a function from B into C. If $\mathrm{rng}(g \cdot f) = C$, then $\mathrm{rng}\, g = C$.

(2) Let A be a many sorted set indexed by I, B, C be non-empty many sorted sets indexed by I, f be a many sorted function from A into B, and g be a many sorted function from B into C. If $g \circ f$ is onto, then g is onto.

(3) Let A, B be non empty sets, C, y be sets, and f be a function. If $f \in (C^B)^A$ and $y \in B$, then $\mathrm{dom}(\mathrm{commute}(f))(y) = A$ and $\mathrm{rng}(\mathrm{commute}(f))(y) \subseteq C$.

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(4) For every many sorted set A indexed by I there exists a non-empty many sorted set B indexed by I such that $A \subseteq B$.

(5) Let A, B be many sorted sets indexed by I. Suppose A is transformable to B. Let f be a many sorted function indexed by I. If $\mathrm{dom}_\kappa f(\kappa) = A$ and $\mathrm{rng}_\kappa f(\kappa) \subseteq B$, then f is a many sorted function from A into B.

(6) Let A, B be many sorted sets indexed by I, F be a many sorted function from A into B, C, E be many sorted subsets indexed by A, and D be a many sorted subset indexed by C. If $E = D$, then $F \restriction C \restriction D = F \restriction E$.

(7) Let B be a non-empty many sorted set indexed by I, C be a many sorted set indexed by I, A be a many sorted subset indexed by C, and F be a many sorted function from A into B. Then there exists a many sorted function G from C into B such that $G \restriction A = F$.

Let I be a set, let A be a many sorted set indexed by I, and let F be a many sorted function indexed by I. The functor $F^{-1}(A)$ yielding a many sorted set indexed by I is defined as follows:

(Def. 1) For every set i such that $i \in I$ holds $(F^{-1}(A))(i) = F(i)^{-1}(A(i))$.

We now state a number of propositions:

(8) Let A, B, C be many sorted sets indexed by I and F be a many sorted function from A into B. Then $F^\circ C$ is a many sorted subset indexed by B.

(9) Let A, B, C be many sorted sets indexed by I and F be a many sorted function from A into B. Then $F^{-1}(C)$ is a many sorted subset indexed by A.

(10) Let A, B be many sorted sets indexed by I and F be a many sorted function from A into B. If F is onto, then $F^\circ A = B$.

(11) Let A, B be many sorted sets indexed by I and F be a many sorted function from A into B. If A is transformable to B, then $F^{-1}(B) = A$.

(12) Let A be a many sorted set indexed by I and F be a many sorted function indexed by I. If $A \subseteq \mathrm{rng}_\kappa F(\kappa)$, then $F^\circ F^{-1}(A) = A$.

(13) For every many sorted function f indexed by I and for every many sorted set X indexed by I holds $f^\circ X \subseteq \mathrm{rng}_\kappa f(\kappa)$.

(14) For every many sorted function f indexed by I holds $f^\circ (\mathrm{dom}_\kappa f(\kappa)) = \mathrm{rng}_\kappa f(\kappa)$.

(15) For every many sorted function f indexed by I holds $f^{-1}(\mathrm{rng}_\kappa f(\kappa)) = \mathrm{dom}_\kappa f(\kappa)$.

(16) For every many sorted set A indexed by I holds $(\mathrm{id}_A)^\circ A = A$.

(17) For every many sorted set A indexed by I holds $(\mathrm{id}_A)^{-1}(A) = A$.

2. On the Many Sorted Algebras

In the sequel S denotes a non empty non void many sorted signature and U_0, U_1 denote non-empty algebras over S.

One can prove the following propositions:

(18) For every algebra A over S holds the algebra of A is a subalgebra of A.

(19) Every algebra A over S is a subalgebra of the algebra of A.

(20) Let U_0 be an algebra over S, A be a subalgebra of U_0, o be an operation symbol of S, and x be a set. If $x \in \mathrm{Args}(o, A)$, then $x \in \mathrm{Args}(o, U_0)$.

(21) Let U_0 be an algebra over S, A be a subalgebra of U_0, o be an operation symbol of S, and x be a set. If $x \in \mathrm{Args}(o, A)$, then $(\mathrm{Den}(o, A))(x) = (\mathrm{Den}(o, U_0))(x)$.

(22) Let F be an algebra family of I over S, B be a subalgebra of $\prod F$, o be an operation symbol of S, and x be a set. If $x \in \mathrm{Args}(o, B)$, then $(\mathrm{Den}(o, B))(x)$ is a function and $(\mathrm{Den}(o, \prod F))(x)$ is a function.

Let S be a non void non empty many sorted signature, let A be an algebra over S, and let B be a subalgebra of A. The functor SuperAlgebraSet(B) is defined by the condition (Def. 2).

(Def. 2) Let x be a set. Then $x \in \mathrm{SuperAlgebraSet}(B)$ if and only if there exists a strict subalgebra C of A such that $x = C$ and B is a subalgebra of C.

Let S be a non void non empty many sorted signature, let A be an algebra over S, and let B be a subalgebra of A. Note that SuperAlgebraSet(B) is non empty.

Let S be a non empty non void many sorted signature. One can verify that there exists an algebra over S which is strict, non-empty, and free.

Let S be a non empty non void many sorted signature, let A be a non-empty algebra over S, and let X be a non-empty locally-finite subset of A. One can verify that $\mathrm{Gen}(X)$ is finitely-generated.

Let S be a non empty non void many sorted signature and let A be a non-empty algebra over S. Note that there exists a subalgebra of A which is strict, non-empty, and finitely-generated.

Let S be a non empty non void many sorted signature and let A be a feasible algebra over S. Note that there exists a subalgebra of A which is feasible.

Next we state several propositions:

(23) Let A be an algebra over S, C be a subalgebra of A, and D be a many sorted subset indexed by the sorts of A. Suppose $D =$ the sorts of C. Let h be a many sorted function from A into U_0 and g be a many sorted function from C into U_0. Suppose $g = h \restriction D$. Let o be an operation symbol of S, x be an element of $\mathrm{Args}(o, A)$, and y be an element of $\mathrm{Args}(o, C)$. If $\mathrm{Args}(o, C) \neq \emptyset$ and $x = y$, then $h\#x = g\#y$.

(24) Let A be a feasible algebra over S, C be a feasible subalgebra of A, and D be a many sorted subset indexed by the sorts of A. Suppose $D =$ the sorts of C. Let h be a many sorted function from A into U_0. Suppose h is a homomorphism of A into U_0. Let g be a many sorted function from C into U_0. If $g = h \restriction D$, then g is a homomorphism of C into U_0.

(25) Let B be a strict non-empty algebra over S, G be a generator set of U_0, H be a non-empty generator set of B, and f be a many sorted function from U_0 into B. Suppose $H \subseteq f \circ G$ and f is a homomorphism of U_0 into B. Then f is an epimorphism of U_0 onto B.

(26) Let W be a strict free non-empty algebra over S and F be a many sorted function from U_0 into U_1. Suppose F is an epimorphism of U_0 onto U_1. Let G be a many sorted function from W into U_1. Suppose G is a homomorphism of W into U_1. Then there exists a many sorted function H from W into U_0 such that H is a homomorphism of W into U_0 and $G = F \circ H$.

(27) Let I be a non empty finite set, A be a non-empty algebra over S, and F be an algebra family of I over S. Suppose that for every element i of I there exists a strict non-empty finitely-generated subalgebra C of A such that $C = F(i)$. Then there exists a strict non-empty finitely-generated subalgebra B of A such that for every element i of I holds $F(i)$ is a subalgebra of B.

(28) Let A, B be strict non-empty finitely-generated subalgebras of U_0. Then there exists a strict non-empty finitely-generated subalgebra M of U_0 such that A is a subalgebra of M and B is a subalgebra of M.

(29) Let S_1 be a non empty non void many sorted signature, A_1 be a non-empty algebra over S_1, and C be a set. Suppose $C = \{A, A$ ranges over elements of Subalgebras(A_1): $\bigvee_{R:\ \text{strict non-empty finitely-generated subalgebra of}\ A_1}$ $R = A\}$. Let F be an algebra family of C over S_1. Suppose that for every set c such that $c \in C$ holds $c = F(c)$. Then there exists a strict non-empty subalgebra P_1 of $\prod F$ such that there exists a many sorted function from P_1 into A_1 which is an epimorphism of P_1 onto A_1.

(30) Let U_0 be a feasible free algebra over S, A be a free generator set of U_0, and Z be a subset of U_0. If $Z \subseteq A$ and $\mathrm{Gen}(Z)$ is feasible, then $\mathrm{Gen}(Z)$ is free.

3. Equations in Many Sorted Algebras

Let S be a non empty non void many sorted signature. The functor $\mathrm{T}_S(\mathbb{N})$ yielding an algebra over S is defined by:

(Def. 3) $\mathrm{T}_S(\mathbb{N}) = \mathrm{Free}((\text{the carrier of } S) \longmapsto \mathbb{N})$.

Let S be a non empty non void many sorted signature. Note that $\mathrm{T}_S(\mathbb{N})$ is strict non-empty and free.

Let S be a non empty non void many sorted signature. The equations of S constitute a many sorted set indexed by the carrier of S and is defined by:

(Def. 4) The equations of $S = [\![\text{the sorts of } \mathrm{T}_S(\mathbb{N}), \text{the sorts of } \mathrm{T}_S(\mathbb{N})]\!]$.

Let S be a non empty non void many sorted signature. Observe that the equations of S is non-empty.

Let S be a non empty non void many sorted signature. A set of equations of S is a many sorted subset indexed by the equations of S.

In the sequel s denotes a sort symbol of S, e denotes an element of (the equations of S)(s), and E denotes a set of equations of S.

Let S be a non empty non void many sorted signature, let s be a sort symbol of S, and let x, y be elements of (the sorts of $\mathrm{T}_S(\mathbb{N})$)(s). Then $\langle x, y\rangle$ is an element of (the equations of S)(s). We introduce x=y as a synonym of $\langle x, y\rangle$.

Next we state two propositions:

(31) $e_{\mathbf{1}} \in (\text{the sorts of } \mathrm{T}_S(\mathbb{N}))(s)$.

(32) $e_{\mathbf{2}} \in (\text{the sorts of } \mathrm{T}_S(\mathbb{N}))(s)$.

Let S be a non empty non void many sorted signature, let A be an algebra over S, let s be a sort symbol of S, and let e be an element of (the equations of S)(s). The predicate $A \models e$ is defined by:

(Def. 5) For every many sorted function h from $\mathrm{T}_S(\mathbb{N})$ into A such that h is a homomorphism of $\mathrm{T}_S(\mathbb{N})$ into A holds $h(s)(e_{\mathbf{1}}) = h(s)(e_{\mathbf{2}})$.

Let S be a non empty non void many sorted signature, let A be an algebra over S, and let E be a set of equations of S. The predicate $A \models E$ is defined as follows:

(Def. 6) For every sort symbol s of S and for every element e of (the equations of S)(s) such that $e \in E(s)$ holds $A \models e$.

We now state several propositions:

(33) For every strict non-empty subalgebra U_2 of U_0 such that $U_0 \models e$ holds $U_2 \models e$.

(34) For every strict non-empty subalgebra U_2 of U_0 such that $U_0 \models E$ holds $U_2 \models E$.

(35) If U_0 and U_1 are isomorphic and $U_0 \models e$, then $U_1 \models e$.

(36) If U_0 and U_1 are isomorphic and $U_0 \models E$, then $U_1 \models E$.

(37) For every congruence R of U_0 such that $U_0 \models e$ holds $U_0/R \models e$.

(38) For every congruence R of U_0 such that $U_0 \models E$ holds $U_0/R \models E$.

(39) Let F be an algebra family of I over S. Suppose that for every set i such that $i \in I$ there exists an algebra A over S such that $A = F(i)$ and $A \models e$. Then $\prod F \models e$.

(40) Let F be an algebra family of I over S. Suppose that for every set i such that $i \in I$ there exists an algebra A over S such that $A = F(i)$ and $A \models E$. Then $\prod F \models E$.

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Category of Functors Between Alternative Categories

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The notation and terminology used in this paper are introduced in the following articles: [9], [13], [5], [10], [7], [15], [1], [3], [4], [2], [6], [8], [11], [14], and [12].

1. Preliminaries

Let A be a transitive non empty category structure with units and let B be a non empty category structure with units. Observe that every functor from A to B is feasible and id-preserving.

Let A be a transitive non empty category structure with units and let B be a non empty category structure with units. One can check the following observations:

* every functor from A to B which is covariant is also precovariant and comp-preserving,
* every functor from A to B which is precovariant and comp-preserving is also covariant,
* every functor from A to B which is contravariant is also precontravariant and comp-reversing, and
* every functor from A to B which is precontravariant and comp-reversing is also contravariant.

The following proposition is true

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(2)[1] Let A, B be transitive non empty category structures with units, F be a covariant functor from A to B, and a be an object of A. Then $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$.

2. Transformations

Let A, B be transitive non empty category structures with units and let F_1, F_2 be covariant functors from A to B. We say that F_1 is transformable to F_2 if and only if:

(Def. 1) For every object a of A holds $\langle F_1(a), F_2(a)\rangle \neq \emptyset$.

Let us note that the predicate F_1 is transformable to F_2 is reflexive.

One can prove the following proposition

(4)[2] Let A, B be transitive non empty category structures with units and F, F_1, F_2 be covariant functors from A to B. Suppose F is transformable to F_1 and F_1 is transformable to F_2. Then F is transformable to F_2.

Let A, B be transitive non empty category structures with units and let F_1, F_2 be covariant functors from A to B. Let us assume that F_1 is transformable to F_2. A many sorted set indexed by the carrier of A is said to be a transformation from F_1 to F_2 if:

(Def. 2) For every object a of A holds $\mathrm{it}(a)$ is a morphism from $F_1(a)$ to $F_2(a)$.

Let A, B be transitive non empty category structures with units and let F be a covariant functor from A to B. The functor id_F yielding a transformation from F to F is defined by:

(Def. 3) For every object a of A holds $\mathrm{id}_F(a) = \mathrm{id}_{F(a)}$.

Let A, B be transitive non empty category structures with units and let F_1, F_2 be covariant functors from A to B. Let us assume that F_1 is transformable to F_2. Let t be a transformation from F_1 to F_2 and let a be an object of A. The functor $t[a]$ yielding a morphism from $F_1(a)$ to $F_2(a)$ is defined as follows:

(Def. 4) $t[a] = t(a)$.

Let A, B be transitive non empty category structures with units and let F, F_1, F_2 be covariant functors from A to B. Let us assume that F is transformable to F_1 and F_1 is transformable to F_2. Let t_1 be a transformation from F to F_1 and let t_2 be a transformation from F_1 to F_2. The functor $t_2 \circ t_1$ yielding a transformation from F to F_2 is defined by:

(Def. 5) For every object a of A holds $(t_2 \circ t_1)[a] = t_2[a] \cdot t_1[a]$.

We now state four propositions:

[1]The proposition (1) has been removed.

[2]The proposition (3) has been removed.

(5) Let A, B be transitive non empty category structures with units and F_1, F_2 be covariant functors from A to B. Suppose F_1 is transformable to F_2. Let t_1, t_2 be transformations from F_1 to F_2. If for every object a of A holds $t_1[a] = t_2[a]$, then $t_1 = t_2$.

(6) Let A, B be transitive non empty category structures with units, F be a covariant functor from A to B, and a be an object of A. Then $\mathrm{id}_F[a] = \mathrm{id}_{F(a)}$.

(7) Let A, B be transitive non empty category structures with units and F_1, F_2 be covariant functors from A to B. Suppose F_1 is transformable to F_2. Let t be a transformation from F_1 to F_2. Then $\mathrm{id}_{(F_2)} \circ t = t$ and $t \circ \mathrm{id}_{(F_1)} = t$.

(8) Let A, B be categories and F, F_1, F_2, F_3 be covariant functors from A to B. Suppose F is transformable to F_1 and F_1 is transformable to F_2 and F_2 is transformable to F_3. Let t_1 be a transformation from F to F_1, t_2 be a transformation from F_1 to F_2, and t_3 be a transformation from F_2 to F_3. Then $(t_3 \circ t_2) \circ t_1 = t_3 \circ (t_2 \circ t_1)$.

3. Natural Transformations

Let A, B be transitive non empty category structures with units and let F_1, F_2 be covariant functors from A to B. We say that F_1 is naturally transformable to F_2 if and only if the conditions (Def. 6) are satisfied.

(Def. 6)(i) F_1 is transformable to F_2, and

(ii) there exists a transformation t from F_1 to F_2 such that for all objects a, b of A such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $t[b] \cdot F_1(f) = F_2(f) \cdot t[a]$.

We now state two propositions:

(9) For all transitive non empty category structures A, B with units holds every covariant functor F from A to B is naturally transformable to F.

(10) Let A, B be categories and F, F_1, F_2 be covariant functors from A to B. Suppose F is naturally transformable to F_1 and F_1 is naturally transformable to F_2. Then F is naturally transformable to F_2.

Let A, B be transitive non empty category structures with units and let F_1, F_2 be covariant functors from A to B. Let us assume that F_1 is naturally transformable to F_2. A transformation from F_1 to F_2 is called a natural transformation from F_1 to F_2 if:

(Def. 7) For all objects a, b of A such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds it $[b] \cdot F_1(f) = F_2(f) \cdot \mathrm{it}[a]$.

Let A, B be transitive non empty category structures with units and let F be a covariant functor from A to B. Then id_F is a natural transformation from F to F.

Let A, B be categories and let F, F_1, F_2 be covariant functors from A to B. Let us assume that F is naturally transformable to F_1 and F_1 is naturally transformable to F_2. Let t_1 be a natural transformation from F to F_1 and let t_2 be a natural transformation from F_1 to F_2. The functor $t_2 \circ t_1$ yielding a natural transformation from F to F_2 is defined by:

(Def. 8) $t_2 \circ t_1 = t_2 \circ t_1$.

We now state three propositions:

(11) Let A, B be transitive non empty category structures with units and F_1, F_2 be covariant functors from A to B. Suppose F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2. Then $\mathrm{id}_{(F_2)} \circ t = t$ and $t \circ \mathrm{id}_{(F_1)} = t$.

(12) Let A, B be transitive non empty category structures with units and F, F_1, F_2 be covariant functors from A to B. Suppose F is naturally transformable to F_1 and F_1 is naturally transformable to F_2. Let t_1 be a natural transformation from F to F_1, t_2 be a natural transformation from F_1 to F_2, and a be an object of A. Then $(t_2 \circ t_1)[a] = t_2[a] \cdot t_1[a]$.

(13) Let A, B be categories, F, F_1, F_2, F_3 be covariant functors from A to B, t be a natural transformation from F to F_1, and t_1 be a natural transformation from F_1 to F_2. Suppose F is naturally transformable to F_1 and F_1 is naturally transformable to F_2 and F_2 is naturally transformable to F_3. Let t_3 be a natural transformation from F_2 to F_3. Then $(t_3 \circ t_1) \circ t = t_3 \circ (t_1 \circ t)$.

4. Category of Functors

Let I be a set and let A, B be many sorted sets indexed by I. The functor B^A yields a set and is defined as follows:

(Def. 9)(i) For every set x holds $x \in B^A$ iff x is a many sorted function from A into B if for every set i such that $i \in I$ holds if $B(i) = \emptyset$, then $A(i) = \emptyset$,

(ii) $B^A = \emptyset$, otherwise.

Let A, B be transitive non empty category structures with units. The functor $\mathrm{Funct}(A, B)$ yields a set and is defined as follows:

(Def. 10) For every set x holds $x \in \mathrm{Funct}(A, B)$ iff x is a covariant strict functor from A to B.

Let A, B be categories. The functor B^A yields a strict non empty transitive category structure and is defined by the conditions (Def. 11).

(Def. 11)(i) The carrier of $B^A = \mathrm{Funct}(A, B)$,

(ii) for all strict covariant functors F, G from A to B and for every set x holds $x \in$ (the arrows of B^A)(F, G) iff F is naturally transformable to G and x is a natural transformation from F to G, and

(iii) for all strict covariant functors F, G, H from A to B such that F is naturally transformable to G and G is naturally transformable to H and for every natural transformation t_1 from F to G and for every natural transformation t_2 from G to H there exists a function f such that $f =$ (the composition of B^A)(F, G, H) and $f(t_2, t_1) = t_2 \circ t_1$.

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Yoneda Embedding

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The articles [7], [8], [10], [1], [2], [3], [4], [6], [5], and [9] provide the notation and terminology for this paper.

In this paper A is a category, a is an object of A, and f is a morphism of A.

Let us consider A. The functor $\mathrm{EnsHom}\,A$ yields a category and is defined by:

(Def. 1) $\mathrm{EnsHom}\,A = \mathbf{Ens}_{\mathrm{Hom}(A)}$.

Next we state two propositions:

(1) Let f, g be functions and m_1, m_2 be morphisms of $\mathrm{EnsHom}\,A$. If $\mathrm{cod}\,m_1 = \mathrm{dom}\,m_2$ and $\langle\langle \mathrm{dom}\,m_1, \mathrm{cod}\,m_1\rangle, f\rangle = m_1$ and $\langle\langle \mathrm{dom}\,m_2, \mathrm{cod}\,m_2\rangle, g\rangle = m_2$, then $\langle\langle \mathrm{dom}\,m_1, \mathrm{cod}\,m_2\rangle, g \cdot f\rangle = m_2 \cdot m_1$.

(2) $\mathrm{hom}(a, -)$ is a functor from A to $\mathrm{EnsHom}\,A$.

Let us consider A, a. The functor $\mathrm{hom}^{\mathrm{F}}(a, -)$ yields a functor from A to $\mathrm{EnsHom}\,A$ and is defined by:

(Def. 2) $\mathrm{hom}^{\mathrm{F}}(a, -) = \mathrm{hom}(a, -)$.

One can prove the following proposition

(3) For every morphism f of A holds $\mathrm{hom}^{\mathrm{F}}(\mathrm{cod}\,f, -)$ is naturally transformable to $\mathrm{hom}^{\mathrm{F}}(\mathrm{dom}\,f, -)$.

Let us consider A, f. The functor $\mathrm{hom}^{\mathrm{F}}(f, -)$ yields a natural transformation from $\mathrm{hom}^{\mathrm{F}}(\mathrm{cod}\,f, -)$ to $\mathrm{hom}^{\mathrm{F}}(\mathrm{dom}\,f, -)$ and is defined by:

(Def. 3) For every object o of A holds $(\mathrm{hom}^{\mathrm{F}}(f, -))(o) = \langle\langle \mathrm{hom}(\mathrm{cod}\,f, o), \mathrm{hom}(\mathrm{dom}\,f, o)\rangle, \mathrm{hom}(f, \mathrm{id}_o)\rangle$.

Next we state the proposition

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(4) For every element f of the morphisms of A holds $\langle\langle \hom^{\mathrm{F}}(\operatorname{cod} f, -), \hom^{\mathrm{F}}(\operatorname{dom} f, -)\rangle, \hom^{\mathrm{F}}(f, -)\rangle$ is an element of the morphisms of $(\mathrm{EnsHom}\, A)^{A}$.

Let us consider A. The functor Yoneda A yielding a contravariant functor from A into $(\mathrm{EnsHom}\, A)^{A}$ is defined by:

(Def. 4) For every morphism f of A holds $(\mathrm{Yoneda}\, A)(f) = \langle\langle \hom^{\mathrm{F}}(\operatorname{cod} f, -), \hom^{\mathrm{F}}(\operatorname{dom} f, -)\rangle, \hom^{\mathrm{F}}(f, -)\rangle$.

Let A, B be categories, let F be a contravariant functor from A into B, and let c be an object of A. The functor $F(c)$ yields an object of B and is defined as follows:

(Def. 5) $F(c) = (\mathrm{Obj}\, F)(c)$.

Next we state the proposition

(5) For every functor F from A to $(\mathrm{EnsHom}\, A)^{A}$ such that $\mathrm{Obj}\, F$ is one-to-one and F is faithful holds F is one-to-one.

Let C, D be categories and let T be a contravariant functor from C into D. We say that T is faithful if and only if:

(Def. 6) For all objects c, c' of C such that $\hom(c, c') \neq \emptyset$ and for all morphisms f_1, f_2 from c to c' such that $T(f_1) = T(f_2)$ holds $f_1 = f_2$.

The following three propositions are true:

(6) Let F be a contravariant functor from A into $(\mathrm{EnsHom}\, A)^{A}$. If $\mathrm{Obj}\, F$ is one-to-one and F is faithful, then F is one-to-one.

(7) Yoneda A is faithful.

(8) Yoneda A is one-to-one.

Let C, D be categories and let T be a contravariant functor from C into D. We say that T is full if and only if the condition (Def. 7) is satisfied.

(Def. 7) Let c, c' be objects of C. Suppose $\hom(T(c'), T(c)) \neq \emptyset$. Let g be a morphism from $T(c')$ to $T(c)$. Then $\hom(c, c') \neq \emptyset$ and there exists a morphism f from c to c' such that $g = T(f)$.

The following proposition is true

(9) Yoneda A is full.

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The Correctness of the Generic Algorithms of Brown and Henrici Concerning Addition and Multiplication in Fraction Fields

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Summary. We prove the correctness of the generic algorithms of Brown and Henrici concerning addition and multiplication in fraction fields of gcd-domains. For that we first prove some basic facts about divisibility in integral domains and introduce the concept of amplesets. After that we are able to define gcd-domains and to prove the theorems of Brown and Henrici which are crucial for the correctness of the algorithms. In the last section we define Mizar functions mirroring their input/output behaviour and prove properties of these functions that ensure the correctness of the algorithms.

MML Identifier: GCD_1.

The papers [4], [6], [5], [3], [1], and [2] provide the notation and terminology for this paper.

1. Basics

In this paper R denotes an integral domain and a, b, c denote elements of the carrier of R.

The following proposition is true

(1) For all elements a, b, c of the carrier of R such that $a \neq 0_R$ holds if $a \cdot b = a \cdot c$, then $b = c$ and if $b \cdot a = c \cdot a$, then $b = c$.

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Let R be an integral domain and let x, y be elements of the carrier of R. We say that x divides y if and only if:

(Def. 1) There exists an element z of the carrier of R such that $y = x \cdot z$.

Let us notice that the predicate x divides y is reflexive.

Let R be an integral domain and let x be an element of the carrier of R. We say that x is unital if and only if:

(Def. 2) x divides 1_R.

Let R be an integral domain and let x, y be elements of the carrier of R. We say that x is associated to y if and only if:

(Def. 3) x divides y and y divides x.

Let us observe that the predicate x is associated to y is reflexive and symmetric. We introduce x is not associated to y as an antonym of x is associated to y.

Let R be an integral domain and let x, y be elements of the carrier of R. Let us assume that y divides x. And let us assume that $y \neq 0_R$. The functor $\frac{x}{y}$ yielding an element of the carrier of R is defined as follows:

(Def. 4) $\frac{x}{y} \cdot y = x$.

One can prove the following propositions:

(2) For all elements a, b, c of the carrier of R such that a divides b and b divides c holds a divides c.

(3) Let a, b, c, d be elements of the carrier of R. If b divides a and d divides c, then $b \cdot d$ divides $a \cdot c$.

(4) Let a, b, c be elements of the carrier of R. If a is associated to b and b is associated to c, then a is associated to c.

(5) For all elements a, b, c of the carrier of R such that a divides b holds $c \cdot a$ divides $c \cdot b$.

(6) For all elements a, b of the carrier of R holds a divides $a \cdot b$ and b divides $a \cdot b$.

(7) For all elements a, b, c of the carrier of R such that a divides b holds a divides $b \cdot c$.

(8) Let a, b be elements of the carrier of R. If b divides a and $b \neq 0_R$, then $\frac{a}{b} = 0_R$ iff $a = 0_R$.

(9) For every element a of the carrier of R such that $a \neq 0_R$ holds $\frac{a}{a} = 1_R$.

(10) For every element a of the carrier of R holds $\frac{a}{1_R} = a$.

(11) Let a, b, c be elements of the carrier of R such that $c \neq 0_R$. Then
  (i) if c divides $a \cdot b$ and c divides a, then $\frac{a \cdot b}{c} = \frac{a}{c} \cdot b$, and
  (ii) if c divides $a \cdot b$ and c divides b, then $\frac{a \cdot b}{c} = a \cdot \frac{b}{c}$.

(12) Let a, b, c be elements of the carrier of R. Suppose $c \neq 0_R$ and c divides a and c divides b and c divides $a + b$. Then $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$.

(13) Let a, b, c be elements of the carrier of R. Suppose $c \neq 0_R$ and c divides a and c divides b. Then $\frac{a}{c} = \frac{b}{c}$ if and only if $a = b$.

(14) Let a, b, c, d be elements of the carrier of R. Suppose $b \neq 0_R$ and $d \neq 0_R$ and b divides a and d divides c. Then $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$.

(15) For all elements a, b, c of the carrier of R such that $a \neq 0_R$ and $a \cdot b$ divides $a \cdot c$ holds b divides c.

(16) For every element a of the carrier of R such that a is associated to 0_R holds $a = 0_R$.

(17) For all elements a, b of the carrier of R such that $a \neq 0_R$ and $a \cdot b = a$ holds $b = 1_R$.

(18) Let a, b be elements of the carrier of R. Then a is associated to b if and only if there exists c such that c is unital and $a \cdot c = b$.

(19) For all elements a, b, c of the carrier of R such that $c \neq 0_R$ and $c \cdot a$ is associated to $c \cdot b$ holds a is associated to b.

2. AmpleSets

Let R be an integral domain and let a be an element of the carrier of R. The functor $\mathrm{Classes}\, a$ yields a subset of the carrier of R and is defined as follows:

(Def. 5) For every element b of the carrier of R holds $b \in \mathrm{Classes}\, a$ iff b is associated to a.

Let R be an integral domain and let a be an element of the carrier of R. Note that $\mathrm{Classes}\, a$ is non empty.

We now state the proposition

(20) For all elements a, b of the carrier of R such that $\mathrm{Classes}\, a \cap \mathrm{Classes}\, b \neq \emptyset$ holds $\mathrm{Classes}\, a = \mathrm{Classes}\, b$.

Let R be an integral domain. The functor $\mathrm{Classes}\, R$ yielding a family of subsets of the carrier of R is defined by the condition (Def. 6).

(Def. 6) Let A be a subset of the carrier of R. Then $A \in \mathrm{Classes}\, R$ if and only if there exists an element a of the carrier of R such that $A = \mathrm{Classes}\, a$.

Let R be an integral domain. One can check that $\mathrm{Classes}\, R$ is non empty.

We now state the proposition

(21) For every subset X of the carrier of R such that $X \in \mathrm{Classes}\, R$ holds X is non empty.

Let R be an integral domain. A non empty subset of the carrier of R is said to be an amp set of R if it satisfies the conditions (Def. 7).

(Def. 7)(i) For every element a of the carrier of R holds there exists an element of it which is associated to a, and

(ii) for all elements x, y of it such that $x \neq y$ holds x is not associated to y.

Let R be an integral domain. A non empty subset of the carrier of R is called an AmpleSet of R if:

(Def. 8) It is an amp set of R and $1_R \in$ it.

In the sequel A_1 denotes an AmpleSet of R.

The following propositions are true:

(22) Let A_1 be an AmpleSet of R. Then
(i) $1_R \in A_1$,
(ii) for every element a of the carrier of R holds there exists an element of A_1 which is associated to a, and
(iii) for all elements x, y of A_1 such that $x \neq y$ holds x is not associated to y.

(23) For all elements x, y of A_1 such that x is associated to y holds $x = y$.

(24) For every AmpleSet A_1 of R holds 0_R is an element of A_1.

Let R be an integral domain, let A_1 be an AmpleSet of R, and let x be an element of the carrier of R. The functor $\mathrm{NF}(x, A_1)$ yields an element of the carrier of R and is defined as follows:

(Def. 9) $\mathrm{NF}(x, A_1) \in A_1$ and $\mathrm{NF}(x, A_1)$ is associated to x.

The following propositions are true:

(25) For every AmpleSet A_1 of R holds $\mathrm{NF}(0_R, A_1) = 0_R$ and $\mathrm{NF}(1_R, A_1) = 1_R$.

(26) For every AmpleSet A_1 of R and for every element a of the carrier of R holds $a \in A_1$ iff $a = \mathrm{NF}(a, A_1)$.

Let R be an integral domain and let A_1 be an AmpleSet of R. We say that A_1 is multiplicative if and only if:

(Def. 10) For all elements x, y of A_1 holds $x \cdot y \in A_1$.

The following proposition is true

(27) Let A_1 be an AmpleSet of R. Suppose A_1 is multiplicative. Let x, y be elements of A_1. If y divides x and $y \neq 0_R$, then $\frac{x}{y} \in A_1$.

3. GCD-Domains

Let R be an integral domain. We say that R is gcd-like if and only if the condition (Def. 11) is satisfied.

(Def. 11) Let x, y be elements of the carrier of R. Then there exists an element z of the carrier of R such that
(i) z divides x,

(ii) z divides y, and
(iii) for every element z_1 of the carrier of R such that z_1 divides x and z_1 divides y holds z_1 divides z.

Let us note that there exists an integral domain which is gcd-like.

A gcdDomain is a gcd-like integral domain.

Let R be a gcdDomain, let A_1 be an AmpleSet of R, and let x, y be elements of the carrier of R. The functor $\gcd_{A_1}(x, y)$ yielding an element of the carrier of R is defined by the conditions (Def. 12).

(Def. 12)(i) $\gcd_{A_1}(x, y) \in A_1$,
(ii) $\gcd_{A_1}(x, y)$ divides x,
(iii) $\gcd_{A_1}(x, y)$ divides y, and
(iv) for every element z of the carrier of R such that z divides x and z divides y holds z divides $\gcd_{A_1}(x, y)$.

In the sequel R is a gcdDomain.

The following propositions are true:

(28) Let A_1 be an AmpleSet of R and a, b be elements of the carrier of R. Then $\gcd_{A_1}(a, b)$ divides a and $\gcd_{A_1}(a, b)$ divides b.

(29) Let A_1 be an AmpleSet of R and a, b, c be elements of the carrier of R. If c divides $\gcd_{A_1}(a, b)$, then c divides a and c divides b.

(30) For every AmpleSet A_1 of R and for all elements a, b of the carrier of R holds $\gcd_{A_1}(a, b) = \gcd_{A_1}(b, a)$.

(31) For every AmpleSet A_1 of R and for every element a of the carrier of R holds $\gcd_{A_1}(a, 0_R) = \mathrm{NF}(a, A_1)$ and $\gcd_{A_1}(0_R, a) = \mathrm{NF}(a, A_1)$.

(32) For every AmpleSet A_1 of R holds $\gcd_{A_1}(0_R, 0_R) = 0_R$.

(33) For every AmpleSet A_1 of R and for every element a of the carrier of R holds $\gcd_{A_1}(a, 1_R) = 1_R$ and $\gcd_{A_1}(1_R, a) = 1_R$.

(34) Let A_1 be an AmpleSet of R and a, b be elements of the carrier of R. Then $\gcd_{A_1}(a, b) = 0_R$ if and only if $a = 0_R$ and $b = 0_R$.

(35) Let A_1 be an AmpleSet of R and a, b, c be elements of the carrier of R. Suppose b is associated to c. Then $\gcd_{A_1}(a, b)$ is associated to $\gcd_{A_1}(a, c)$ and $\gcd_{A_1}(b, a)$ is associated to $\gcd_{A_1}(c, a)$.

(36) For every AmpleSet A_1 of R and for all elements a, b, c of the carrier of R holds $\gcd_{A_1}(\gcd_{A_1}(a, b), c) = \gcd_{A_1}(a, \gcd_{A_1}(b, c))$.

(37) For every AmpleSet A_1 of R and for all elements a, b, c of the carrier of R holds $\gcd_{A_1}(a \cdot c, b \cdot c)$ is associated to $c \cdot (\gcd_{A_1}(a, b))$.

(38) For every AmpleSet A_1 of R and for all elements a, b, c of the carrier of R such that $\gcd_{A_1}(a, b) = 1_R$ holds $\gcd_{A_1}(a, b \cdot c) = \gcd_{A_1}(a, c)$.

(39) Let A_1 be an AmpleSet of R and a, b, c be elements of the carrier of R. If $c = \gcd_{A_1}(a, b)$ and $c \neq 0_R$, then $\gcd_{A_1}(\frac{a}{c}, \frac{b}{c}) = 1_R$.

(40) For every AmpleSet A_1 of R and for all elements a, b, c of the carrier of R holds $\gcd_{A_1}(a + b \cdot c, c) = \gcd_{A_1}(a, c)$.

4. The Theorems of Brown and Henrici

The following propositions are true:

(41) Let A_1 be an AmpleSet of R and r_1, r_2, s_1, s_2 be elements of the carrier of R. Suppose $\gcd_{A_1}(r_1, r_2) = 1_R$ and $\gcd_{A_1}(s_1, s_2) = 1_R$ and $r_2 \neq 0_R$ and $s_2 \neq 0_R$. Then $\gcd_{A_1}(r_1 \cdot \frac{s_2}{\gcd_{A_1}(r_2,s_2)} + s_1 \cdot \frac{r_2}{\gcd_{A_1}(r_2,s_2)}, r_2 \cdot \frac{s_2}{\gcd_{A_1}(r_2,s_2)}) = \gcd_{A_1}(r_1 \cdot \frac{s_2}{\gcd_{A_1}(r_2,s_2)} + s_1 \cdot \frac{r_2}{\gcd_{A_1}(r_2,s_2)}, \gcd_{A_1}(r_2, s_2))$.

(42) Let A_1 be an AmpleSet of R and r_1, r_2, s_1, s_2 be elements of the carrier of R. Suppose $\gcd_{A_1}(r_1, r_2) = 1_R$ and $\gcd_{A_1}(s_1, s_2) = 1_R$ and $r_2 \neq 0_R$ and $s_2 \neq 0_R$. Then $\gcd_{A_1}(\frac{r_1}{\gcd_{A_1}(r_1,s_2)} \cdot \frac{s_1}{\gcd_{A_1}(s_1,r_2)}, \frac{r_2}{\gcd_{A_1}(s_1,r_2)} \cdot \frac{s_2}{\gcd_{A_1}(r_1,s_2)}) = 1_R$.

5. Correctness of the Algorithms

Let R be a gcdDomain, let A_1 be an AmpleSet of R, and let x, y be elements of the carrier of R. We say that x, y are canonical wrt A_1 if and only if:

(Def. 13) $\gcd_{A_1}(x, y) = 1_R$.

Next we state the proposition

(43) Let A_1, A_1' be AmpleSet of R and x, y be elements of the carrier of R. Then x, y are canonical wrt A_1 if and only if x, y are canonical wrt A_1'.

Let R be a gcdDomain and let x, y be elements of the carrier of R. We say that x canonical y if and only if:

(Def. 14) There exists an AmpleSet A_1 of R such that $\gcd_{A_1}(x, y) = 1_R$.

Let us observe that the predicate x canonical y is symmetric.

Next we state the proposition

(44) Let A_1 be an AmpleSet of R and x, y be elements of the carrier of R. If x canonical y, then $\gcd_{A_1}(x, y) = 1_R$.

Let R be a gcdDomain, let A_1 be an AmpleSet of R, and let x, y be elements of the carrier of R. We say that x, y are normalized wrt A_1 if and only if:

(Def. 15) $\gcd_{A_1}(x, y) = 1_R$ and $y \in A_1$ and $y \neq 0_R$.

Let R be a gcdDomain, let A_1 be an AmpleSet of R, and let r_1, r_2, s_1, s_2 be elements of the carrier of R. Let us assume that r_1 canonical r_2 and s_1 canonical s_2 and $r_2 = \mathrm{NF}(r_2, A_1)$ and $s_2 = \mathrm{NF}(s_2, A_1)$. The functor $\mathrm{add1}_{A_1}(r_1, r_2, s_1, s_2)$ yielding an element of the carrier of R is defined as follows:

(Def. 16) $$\mathrm{add1}_{A_1}(r_1, r_2, s_1, s_2) = \begin{cases} s_1, \ \text{if } r_1 = 0_R, \\ r_1, \ \text{if } s_1 = 0_R, \\ r_1 \cdot s_2 + r_2 \cdot s_1, \ \text{if } \gcd_{A_1}(r_2, s_2) = 1_R, \\ 0_R, \ \text{if } r_1 \cdot \frac{s_2}{\gcd_{A_1}(r_2,s_2)} + s_1 \cdot \frac{r_2}{\gcd_{A_1}(r_2,s_2)} = 0_R, \\ \frac{r_1 \cdot \frac{s_2}{\gcd_{A_1}(r_2,s_2)} + s_1 \cdot \frac{r_2}{\gcd_{A_1}(r_2,s_2)}}{\gcd_{A_1}(r_1 \cdot \frac{s_2}{\gcd_{A_1}(r_2,s_2)} + s_1 \cdot \frac{r_2}{\gcd_{A_1}(r_2,s_2)}, \gcd_{A_1}(r_2,s_2))}, \\ \text{otherwise.} \end{cases}$$

Let R be a gcdDomain, let A_1 be an AmpleSet of R, and let r_1, r_2, s_1, s_2 be elements of the carrier of R. Let us assume that r_1 canonical r_2 and s_1 canonical s_2 and $r_2 = \mathrm{NF}(r_2, A_1)$ and $s_2 = \mathrm{NF}(s_2, A_1)$. The functor $\mathrm{add2}_{A_1}(r_1, r_2, s_1, s_2)$ yields an element of the carrier of R and is defined by:

(Def. 17) $$\mathrm{add2}_{A_1}(r_1, r_2, s_1, s_2) = \begin{cases} s_2, \ \text{if } r_1 = 0_R, \\ r_2, \ \text{if } s_1 = 0_R, \\ r_2 \cdot s_2, \ \text{if } \gcd_{A_1}(r_2, s_2) = 1_R, \\ 1_R, \ \text{if } r_1 \cdot \frac{s_2}{\gcd_{A_1}(r_2,s_2)} + s_1 \cdot \frac{r_2}{\gcd_{A_1}(r_2,s_2)} = 0_R, \\ \frac{r_2 \cdot \frac{s_2}{\gcd_{A_1}(r_2,s_2)}}{\gcd_{A_1}(r_1 \cdot \frac{s_2}{\gcd_{A_1}(r_2,s_2)} + s_1 \cdot \frac{r_2}{\gcd_{A_1}(r_2,s_2)}, \gcd_{A_1}(r_2,s_2))}, \\ \text{otherwise.} \end{cases}$$

We now state two propositions:

(45) Let A_1 be an AmpleSet of R and r_1, r_2, s_1, s_2 be elements of the carrier of R. Suppose A_1 is multiplicative and r_1, r_2 are normalized wrt A_1 and s_1, s_2 are normalized wrt A_1. Then $\mathrm{add1}_{A_1}(r_1, r_2, s_1, s_2)$, $\mathrm{add2}_{A_1}(r_1, r_2, s_1, s_2)$ are normalized wrt A_1.

(46) Let A_1 be an AmpleSet of R and r_1, r_2, s_1, s_2 be elements of the carrier of R. Suppose A_1 is multiplicative and r_1, r_2 are normalized wrt A_1 and s_1, s_2 are normalized wrt A_1. Then $\mathrm{add1}_{A_1}(r_1, r_2, s_1, s_2) \cdot (r_2 \cdot s_2) = \mathrm{add2}_{A_1}(r_1, r_2, s_1, s_2) \cdot (r_1 \cdot s_2 + s_1 \cdot r_2)$.

Let R be a gcdDomain, let A_1 be an AmpleSet of R, and let r_1, r_2, s_1, s_2 be elements of the carrier of R. The functor $\mathrm{mult1}_{A_1}(r_1, r_2, s_1, s_2)$ yields an element of the carrier of R and is defined as follows:

(Def. 18) $$\mathrm{mult1}_{A_1}(r_1, r_2, s_1, s_2) = \begin{cases} 0_R, \ \text{if } r_1 = 0_R \text{ or } s_1 = 0_R, \\ r_1 \cdot s_1, \ \text{if } r_2 = 1_R \text{ and } s_2 = 1_R, \\ \frac{r_1 \cdot s_1}{\gcd_{A_1}(r_1,s_2)}, \ \text{if } s_2 \neq 0_R \text{ and } r_2 = 1_R, \\ \frac{r_1 \cdot s_1}{\gcd_{A_1}(s_1,r_2)}, \ \text{if } r_2 \neq 0_R \text{ and } s_2 = 1_R, \\ \frac{r_1}{\gcd_{A_1}(r_1,s_2)} \cdot \frac{s_1}{\gcd_{A_1}(s_1,r_2)}, \ \text{otherwise.} \end{cases}$$

Let R be a gcdDomain, let A_1 be an AmpleSet of R, and let r_1, r_2, s_1, s_2 be elements of the carrier of R. Let us assume that r_1 canonical r_2 and s_1 canonical s_2 and $r_2 = \mathrm{NF}(r_2, A_1)$ and $s_2 = \mathrm{NF}(s_2, A_1)$. The functor $\mathrm{mult2}_{A_1}(r_1, r_2, s_1, s_2)$ yields an element of the carrier of R and is defined as follows:

(Def. 19) $$\mathrm{mult2}_{A_1}(r_1, r_2, s_1, s_2) = \begin{cases} 1_R, \text{ if } r_1 = 0_R \text{ or } s_1 = 0_R, \\ 1_R, \text{ if } r_2 = 1_R \text{ and } s_2 = 1_R, \\ \frac{s_2}{\gcd_{A_1}(r_1, s_2)}, \text{ if } s_2 \neq 0_R \text{ and } r_2 = 1_R, \\ \frac{r_2}{\gcd_{A_1}(s_1, r_2)}, \text{ if } r_2 \neq 0_R \text{ and } s_2 = 1_R, \\ \frac{r_2}{\gcd_{A_1}(s_1, r_2)} \cdot \frac{s_2}{\gcd_{A_1}(r_1, s_2)}, \text{ otherwise.} \end{cases}$$

The following two propositions are true:

(47) Let A_1 be an AmpleSet of R and r_1, r_2, s_1, s_2 be elements of the carrier of R. Suppose A_1 is multiplicative and r_1, r_2 are normalized wrt A_1 and s_1, s_2 are normalized wrt A_1. Then $\mathrm{mult1}_{A_1}(r_1, r_2, s_1, s_2)$, $\mathrm{mult2}_{A_1}(r_1, r_2, s_1, s_2)$ are normalized wrt A_1.

(48) Let A_1 be an AmpleSet of R and r_1, r_2, s_1, s_2 be elements of the carrier of R. Suppose A_1 is multiplicative and r_1, r_2 are normalized wrt A_1 and s_1, s_2 are normalized wrt A_1. Then $\mathrm{mult1}_{A_1}(r_1, r_2, s_1, s_2) \cdot (r_2 \cdot s_2) = \mathrm{mult2}_{A_1}(r_1, r_2, s_1, s_2) \cdot (r_1 \cdot s_1)$.

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Birkhoff Theorem for Many Sorted Algebras

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Summary. In this article Birkhoff Variety Theorem for many sorted algebras is proved. A class of algebras is represented by predicate $\mathcal{P}$. Notation $\mathcal{P}[A]$, where A is an algebra, means that A is in class $\mathcal{P}$. All algebras in our class are many sorted over many sorted signature S. The properties of varieties:

- a class $\mathcal{P}$ of algebras is abstract
- a class $\mathcal{P}$ of algebras is closed under subalgebras
- a class $\mathcal{P}$ of algebras is closed under congruences
- a class $\mathcal{P}$ of algebras is closed under products

are published in this paper as:

- for all non-empty algebras A, B over S such that A and B are_isomorphic and $\mathcal{P}[A]$ holds $\mathcal{P}[B]$
- for every non-empty algebra A over S and for strict non-empty subalgebra B of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[B]$
- for every non-empty algebra A over S and for every congruence R of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[A/R]$
- Let I be a set and F be an algebra family of I over $\mathcal{A}$. Suppose that for every set i such that $i \in I$ there exists an algebra A over $\mathcal{A}$ such that $A = F(i)$ and $\mathcal{P}[A]$. Then$\mathcal{P}[\prod F]$.

This paper is formalization of parts of [29].

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The notation and terminology used in this paper have been introduced in the following articles: [24], [28], [20], [5], [30], [25], [3], [4], [22], [31], [1], [23], [26], [15], [27], [2], [6], [13], [10], [21], [18], [16], [19], [14], [11], [8], [7], [9], [17], and [12].

Let S be a non empty non void many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, let A be a non-empty algebra over S, and let F be a many sorted function from X into the sorts of A. The functor $F^{\#}$ yielding a many sorted function from $\mathrm{Free}(X)$ into A is defined by:

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(Def. 1) $F^{\#}$ is a homomorphism of $\mathrm{Free}(X)$ into A and $F^{\#} \restriction \mathrm{FreeGenerator}(X) = F \circ \mathrm{Reverse}(X)$.

We now state the proposition

(1) Let S be a non empty non void many sorted signature, A be a non-empty algebra over S, X be a non-empty many sorted set indexed by the carrier of S, and F be a many sorted function from X into the sorts of A. Then $\mathrm{rng}_\kappa F(\kappa) \subseteq \mathrm{rng}_\kappa F^{\#}(\kappa)$.

In this article we present several logical schemes. The scheme *ExFreeAlg 1* concerns a non empty non void many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a strict non-empty algebra A over $\mathcal{A}$ and there exists a many sorted function F from $\mathcal{B}$ into A such that

(i) $\mathcal{P}[A]$,

(ii) F is an epimorphism of $\mathcal{B}$ onto A, and

(iii) for every non-empty algebra B over $\mathcal{A}$ and for every many sorted function G from $\mathcal{B}$ into B such that G is a homomorphism of $\mathcal{B}$ into B and $\mathcal{P}[B]$ there exists a many sorted function H from A into B such that H is a homomorphism of A into B and $H \circ F = G$ and for every many sorted function K from A into B such that $K \circ F = G$ holds $H = K$

provided the following conditions are met:

- For all non-empty algebras A, B over $\mathcal{A}$ such that A and B are isomorphic and $\mathcal{P}[A]$ holds $\mathcal{P}[B]$,
- For every non-empty algebra A over $\mathcal{A}$ and for every strict non-empty subalgebra B of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[B]$, and
- Let I be a set and F be an algebra family of I over $\mathcal{A}$. Suppose that for every set i such that $i \in I$ there exists an algebra A over $\mathcal{A}$ such that $A = F(i)$ and $\mathcal{P}[A]$. Then $\mathcal{P}[\prod F]$.

The scheme *ExFreeAlg 2* concerns a non empty non void many sorted signature $\mathcal{A}$, a non-empty many sorted set $\mathcal{B}$ indexed by the carrier of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a strict non-empty algebra A over $\mathcal{A}$ and there exists a many sorted function F from $\mathcal{B}$ into the sorts of A such that

(i) $\mathcal{P}[A]$, and

(ii) for every non-empty algebra B over $\mathcal{A}$ and for every many sorted function G from $\mathcal{B}$ into the sorts of B such that $\mathcal{P}[B]$ there exists a many sorted function H from A into B such that H is a homomorphism of A into B and $H \circ F = G$ and for every many sorted function K from A into B such that K is a homomorphism of A into B and $K \circ F = G$ holds $H = K$

provided the following requirements are met:

- For all non-empty algebras A, B over $\mathcal{A}$ such that A and B are isomorphic and $\mathcal{P}[A]$ holds $\mathcal{P}[B]$,
- For every non-empty algebra A over $\mathcal{A}$ and for every strict non-empty subalgebra B of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[B]$, and
- Let I be a set and F be an algebra family of I over $\mathcal{A}$. Suppose that for every set i such that $i \in I$ there exists an algebra A over $\mathcal{A}$ such that $A = F(i)$ and $\mathcal{P}[A]$. Then $\mathcal{P}[\prod F]$.

The scheme *Ex hash* concerns a non empty non void many sorted signature $\mathcal{A}$, non-empty algebras $\mathcal{B}$, $\mathcal{C}$ over $\mathcal{A}$, a many sorted function $\mathcal{D}$ from the carrier of $\mathcal{A} \longmapsto \mathbb{N}$ into the sorts of $\mathcal{B}$, a many sorted function $\mathcal{E}$ from the carrier of $\mathcal{A} \longmapsto \mathbb{N}$ into the sorts of $\mathcal{C}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a many sorted function H from $\mathcal{B}$ into $\mathcal{C}$ such that H is a homomorphism of $\mathcal{B}$ into $\mathcal{C}$ and $\mathcal{E}^{\#} = H \circ \mathcal{D}^{\#}$

provided the parameters have the following properties:

- $\mathcal{P}[\mathcal{C}]$, and
- Let C be a non-empty algebra over $\mathcal{A}$ and G be a many sorted function from (the carrier of $\mathcal{A}$) $\longmapsto \mathbb{N}$ into the sorts of C. Suppose $\mathcal{P}[C]$. Then there exists a many sorted function h from $\mathcal{B}$ into C such that h is a homomorphism of $\mathcal{B}$ into C and $G = h \circ \mathcal{D}$.

The scheme *EqTerms* concerns a non empty non void many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, a many sorted function $\mathcal{C}$ from the carrier of $\mathcal{A} \longmapsto \mathbb{N}$ into the sorts of $\mathcal{B}$, a sort symbol $\mathcal{D}$ of $\mathcal{A}$, elements $\mathcal{E}$, $\mathcal{F}$ of the sorts of $\mathrm{T}_{\mathcal{A}}(\mathbb{N})(\mathcal{D})$, and a unary predicate $\mathcal{P}$, and states that:

For every non-empty algebra B over $\mathcal{A}$ such that $\mathcal{P}[B]$ holds $B \models \langle \mathcal{E}, \mathcal{F} \rangle$

provided the parameters have the following properties:

- $\mathcal{C}^{\#}(\mathcal{D})(\mathcal{E}) = \mathcal{C}^{\#}(\mathcal{D})(\mathcal{F})$, and
- Let C be a non-empty algebra over $\mathcal{A}$ and G be a many sorted function from (the carrier of $\mathcal{A}$) $\longmapsto \mathbb{N}$ into the sorts of C. Suppose $\mathcal{P}[C]$. Then there exists a many sorted function h from $\mathcal{B}$ into C such that h is a homomorphism of $\mathcal{B}$ into C and $G = h \circ \mathcal{C}$.

The scheme *FreeIsGen* deals with a non empty non void many sorted signature $\mathcal{A}$, a non-empty many sorted set $\mathcal{B}$ indexed by the carrier of $\mathcal{A}$, a strict non-empty algebra $\mathcal{C}$ over $\mathcal{A}$, a many sorted function $\mathcal{D}$ from $\mathcal{B}$ into the sorts of $\mathcal{C}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{D} \circ \mathcal{B}$ is a non-empty generator set of $\mathcal{C}$

provided the parameters satisfy the following conditions:

- Let C be a non-empty algebra over $\mathcal{A}$ and G be a many sorted function from $\mathcal{B}$ into the sorts of C. Suppose $\mathcal{P}[C]$. Then there exists a many sorted function H from $\mathcal{C}$ into C such that
  (i) H is a homomorphism of $\mathcal{C}$ into C,
  (ii) $H \circ \mathcal{D} = G$, and

(iii) for every many sorted function K from $\mathcal{C}$ into C such that K is a homomorphism of $\mathcal{C}$ into C and $K \circ \mathcal{D} = G$ holds $H = K$,

- $\mathcal{P}[\mathcal{C}]$, and
- For every non-empty algebra A over $\mathcal{A}$ and for every strict non-empty subalgebra B of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[B]$.

The scheme *Hash is onto* deals with a non empty non void many sorted signature $\mathcal{A}$, a strict non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, a many sorted function $\mathcal{C}$ from the carrier of $\mathcal{A} \longmapsto \mathbb{N}$ into the sorts of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{C}^{\#}$ is an epimorphism of Free((the carrier of $\mathcal{A}$) $\longmapsto \mathbb{N}$) onto $\mathcal{B}$

provided the following conditions are satisfied:

- Let C be a non-empty algebra over $\mathcal{A}$ and G be a many sorted function from (the carrier of $\mathcal{A}$) $\longmapsto \mathbb{N}$ into the sorts of C. Suppose $\mathcal{P}[C]$. Then there exists a many sorted function H from $\mathcal{B}$ into C such that
  (i) H is a homomorphism of $\mathcal{B}$ into C,
  (ii) $H \circ \mathcal{C} = G$, and
  (iii) for every many sorted function K from $\mathcal{B}$ into C such that K is a homomorphism of $\mathcal{B}$ into C and $K \circ \mathcal{C} = G$ holds $H = K$,
- $\mathcal{P}[\mathcal{B}]$, and
- For every non-empty algebra A over $\mathcal{A}$ and for every strict non-empty subalgebra B of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[B]$.

The scheme *FinGenAlgInVar* concerns a non empty non void many sorted signature $\mathcal{A}$, a strict finitely-generated non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, a non-empty algebra $\mathcal{C}$ over $\mathcal{A}$, a many sorted function $\mathcal{D}$ from the carrier of $\mathcal{A} \longmapsto \mathbb{N}$ into the sorts of $\mathcal{C}$, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the parameters satisfy the following conditions:

- $\mathcal{Q}[\mathcal{B}]$,
- $\mathcal{P}[\mathcal{C}]$,
- Let C be a non-empty algebra over $\mathcal{A}$ and G be a many sorted function from (the carrier of $\mathcal{A}$) $\longmapsto \mathbb{N}$ into the sorts of C. Suppose $\mathcal{Q}[C]$. Then there exists a many sorted function h from $\mathcal{C}$ into C such that h is a homomorphism of $\mathcal{C}$ into C and $G = h \circ \mathcal{D}$,
- For all non-empty algebras A, B over $\mathcal{A}$ such that A and B are isomorphic and $\mathcal{P}[A]$ holds $\mathcal{P}[B]$, and
- For every non-empty algebra A over $\mathcal{A}$ and for every congruence R of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[A/R]$.

The scheme *QuotEpi* concerns a non empty non void many sorted signature $\mathcal{A}$, non-empty algebras $\mathcal{B}$, $\mathcal{C}$ over $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{C}]$

provided the following conditions are satisfied:

- There exists a many sorted function from $\mathcal{B}$ into $\mathcal{C}$ which is an epimorphism of $\mathcal{B}$ onto $\mathcal{C}$,
- $\mathcal{P}[\mathcal{B}]$,
- For all non-empty algebras A, B over $\mathcal{A}$ such that A and B are isomorphic and $\mathcal{P}[A]$ holds $\mathcal{P}[B]$, and
- For every non-empty algebra A over $\mathcal{A}$ and for every congruence R of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[A/R]$.

The scheme *AllFinGen* deals with a non empty non void many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the parameters satisfy the following conditions:

- For every strict non-empty finitely-generated subalgebra B of $\mathcal{B}$ holds $\mathcal{P}[B]$,
- For all non-empty algebras A, B over $\mathcal{A}$ such that A and B are isomorphic and $\mathcal{P}[A]$ holds $\mathcal{P}[B]$,
- For every non-empty algebra A over $\mathcal{A}$ and for every strict non-empty subalgebra B of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[B]$,
- For every non-empty algebra A over $\mathcal{A}$ and for every congruence R of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[A/R]$, and
- Let I be a set and F be an algebra family of I over $\mathcal{A}$. Suppose that for every set i such that $i \in I$ there exists an algebra A over $\mathcal{A}$ such that $A = F(i)$ and $\mathcal{P}[A]$. Then $\mathcal{P}[\prod F]$.

The scheme *FreeInModIsInVar 1* deals with a non empty non void many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\mathcal{Q}[\mathcal{B}]$

provided the following requirements are met:

- Let A be a non-empty algebra over $\mathcal{A}$. Then $\mathcal{Q}[A]$ if and only if for every sort symbol s of $\mathcal{A}$ and for every element e of (the equations of $\mathcal{A})(s)$ such that for every non-empty algebra B over $\mathcal{A}$ such that $\mathcal{P}[B]$ holds $B \models e$ holds $A \models e$, and
- $\mathcal{P}[\mathcal{B}]$.

The scheme *FreeInModIsInVar* deals with a non empty non void many sorted signature $\mathcal{A}$, a strict non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, a many sorted function $\mathcal{C}$ from the carrier of $\mathcal{A} \longmapsto \mathbb{N}$ into the sorts of $\mathcal{B}$, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the parameters meet the following conditions:

- Let A be a non-empty algebra over $\mathcal{A}$. Then $\mathcal{Q}[A]$ if and only if for every sort symbol s of $\mathcal{A}$ and for every element e of (the equations of $\mathcal{A})(s)$ such that for every non-empty algebra B over $\mathcal{A}$ such that $\mathcal{P}[B]$ holds $B \models e$ holds $A \models e$,

- Let C be a non-empty algebra over $\mathcal{A}$ and G be a many sorted function from (the carrier of $\mathcal{A}$) $\longmapsto \mathbb{N}$ into the sorts of C. Suppose $\mathcal{Q}[C]$. Then there exists a many sorted function H from $\mathcal{B}$ into C such that
  - (i) H is a homomorphism of $\mathcal{B}$ into C,
  - (ii) $H \circ \mathcal{C} = G$, and
  - (iii) for every many sorted function K from $\mathcal{B}$ into C such that K is a homomorphism of $\mathcal{B}$ into C and $K \circ \mathcal{C} = G$ holds $H = K$,
- $\mathcal{Q}[\mathcal{B}]$,
- For all non-empty algebras A, B over $\mathcal{A}$ such that A and B are isomorphic and $\mathcal{P}[A]$ holds $\mathcal{P}[B]$,
- For every non-empty algebra A over $\mathcal{A}$ and for every strict non-empty subalgebra B of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[B]$, and
- Let I be a set and F be an algebra family of I over $\mathcal{A}$. Suppose that for every set i such that $i \in I$ there exists an algebra A over $\mathcal{A}$ such that $A = F(i)$ and $\mathcal{P}[A]$. Then $\mathcal{P}[\prod F]$.

The scheme *Birkhoff* deals with a non empty non void many sorted signature $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

There exists a set E of equations of $\mathcal{A}$ such that for every non-empty algebra A over $\mathcal{A}$ holds $\mathcal{P}[A]$ iff $A \models E$

provided the parameters meet the following conditions:

- For all non-empty algebras A, B over $\mathcal{A}$ such that A and B are isomorphic and $\mathcal{P}[A]$ holds $\mathcal{P}[B]$,
- For every non-empty algebra A over $\mathcal{A}$ and for every strict non-empty subalgebra B of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[B]$,
- For every non-empty algebra A over $\mathcal{A}$ and for every congruence R of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[A/R]$, and
- Let I be a set and F be an algebra family of I over $\mathcal{A}$. Suppose that for every set i such that $i \in I$ there exists an algebra A over $\mathcal{A}$ such that $A = F(i)$ and $\mathcal{P}[A]$. Then $\mathcal{P}[\prod F]$.

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Algebraic Operation on Subsets of Many Sorted Sets

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The terminology and notation used in this paper are introduced in the following papers: [14], [17], [13], [12], [18], [3], [4], [1], [6], [5], [15], [16], [2], [11], [9], [7], [8], and [10].

1. Preliminaries

Let S be a non empty 1-sorted structure. One can verify that the 1-sorted structure of S is non empty.

We now state three propositions:

(1) For every non empty set I and for all many sorted sets M, N indexed by I holds $M{+}{\cdot}N = N$.

(2) Let I be a set, M, N be many sorted sets indexed by I, and F be a family of many sorted subsets indexed by M. If $N \in F$, then $\bigcap |{:}F{:}| \subseteq N$.

(3) Let S be a non void non empty many sorted signature, M_1 be a strict non-empty algebra over S, and F be a family of many sorted subsets indexed by the sorts of M_1. Suppose $F \subseteq \mathrm{SubSorts}(M_1)$. Let B be a subset of M_1. If $B = \bigcap |{:}F{:}|$, then B is operations closed.

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2. Relationships between Subsets Families

Let I be a set, let M be a many sorted set indexed by I, let B be a family of many sorted subsets indexed by M, and let A be a family of many sorted subsets indexed by M. We say that A is finer than B if and only if:

(Def. 1) For every set a such that $a \in A$ there exists a set b such that $b \in B$ and $a \subseteq b$.

Let us observe that the predicate A is finer than B is reflexive. We say that B is coarser than A if and only if:

(Def. 2) For every set b such that $b \in B$ there exists a set a such that $a \in A$ and $a \subseteq b$.

Let us notice that the predicate B is coarser than A is reflexive.

We now state two propositions:

(4) Let I be a set, M be a many sorted set indexed by I, and A, B, C be families of many sorted subsets indexed by M. If A is finer than B and B is finer than C, then A is finer than C.

(5) Let I be a set, M be a many sorted set indexed by I, and A, B, C be families of many sorted subsets indexed by M. If A is coarser than B and B is coarser than C, then A is coarser than C.

Let I be a non empty set and let M be a many sorted set indexed by I. The functor $\mathrm{supp}(M)$ yielding a set is defined by:

(Def. 3) $\mathrm{supp}(M) = \{x, x$ ranges over elements of I: $M(x) \neq \emptyset\}$.

We now state four propositions:

(6) For every non empty set I and for every non-empty many sorted set M indexed by I holds $M = \emptyset_I {+\cdot} M \restriction \mathrm{supp}(M)$.

(7) Let I be a non empty set and M_2, M_3 be non-empty many sorted sets indexed by I. If $\mathrm{supp}(M_2) = \mathrm{supp}(M_3)$ and $M_2 \restriction \mathrm{supp}(M_2) = M_3 \restriction \mathrm{supp}(M_3)$, then $M_2 = M_3$.

(8) Let I be a non empty set, M be a many sorted set indexed by I, and x be an element of I. If $x \notin \mathrm{supp}(M)$, then $M(x) = \emptyset$.

(9) Let I be a non empty set, M be a many sorted set indexed by I, x be an element of $\mathrm{Bool}(M)$, i be an element of I, and y be a set. Suppose $y \in x(i)$. Then there exists an element a of $\mathrm{Bool}(M)$ such that $y \in a(i)$ and a is locally-finite and $\mathrm{supp}(a)$ is finite and $a \subseteq x$.

Let I be a set, let M be a many sorted set indexed by I, and let A be a family of many sorted subsets indexed by M. The functor $\mathrm{MSUnion}(A)$ yielding a many sorted subset indexed by M is defined by:

(Def. 4) For every set i such that $i \in I$ holds $(\mathrm{MSUnion}(A))(i) = \bigcup\{f(i), f$ ranges over elements of $\mathrm{Bool}(M)$: $f \in A\}$.

Let I be a set, let M be a many sorted set indexed by I, and let B be a non empty family of many sorted subsets indexed by M. We see that the element of B is a many sorted set indexed by I.

Let I be a set, let M be a many sorted set indexed by I, and let A be an empty family of many sorted subsets indexed by M. One can check that MSUnion(A) is empty yielding.

We now state the proposition

(10) Let I be a set, M be a many sorted set indexed by I, and A be a family of many sorted subsets indexed by M. Then $\mathrm{MSUnion}(A) = \bigcup |{:}A{:}|$.

Let I be a set, let M be a many sorted set indexed by I, and let A, B be families of many sorted subsets indexed by M. Then $A \cup B$ is a family of many sorted subsets indexed by M.

The following propositions are true:

(11) Let I be a set, M be a many sorted set indexed by I, and A, B be families of many sorted subsets indexed by M. Then $\mathrm{MSUnion}(A \cup B) = \mathrm{MSUnion}(A) \cup \mathrm{MSUnion}(B)$.

(12) Let I be a set, M be a many sorted set indexed by I, and A, B be families of many sorted subsets indexed by M. If $A \subseteq B$, then $\mathrm{MSUnion}(A) \subseteq \mathrm{MSUnion}(B)$.

Let I be a set, let M be a many sorted set indexed by I, and let A, B be families of many sorted subsets indexed by M. Then $A \cap B$ is a family of many sorted subsets indexed by M.

One can prove the following propositions:

(13) Let I be a set, M be a many sorted set indexed by I, and A, B be families of many sorted subsets indexed by M. Then $\mathrm{MSUnion}(A \cap B) \subseteq \mathrm{MSUnion}(A) \cap \mathrm{MSUnion}(B)$.

(14) Let I be a set, M be a many sorted set indexed by I, and A_1 be a set. Suppose that for every set x such that $x \in A_1$ holds x is a family of many sorted subsets indexed by M. Let A, B be families of many sorted subsets indexed by M. Suppose $B = \{\mathrm{MSUnion}(X), X$ ranges over families of many sorted subsets indexed by M: $X \in A_1\}$ and $A = \bigcup A_1$. Then $\mathrm{MSUnion}(B) = \mathrm{MSUnion}(A)$.

(15) Let I be a non empty set, M, N be many sorted sets indexed by I, and A be a family of many sorted subsets indexed by M. If for every many sorted set x indexed by I holds $x \subseteq N$, then $\mathrm{MSUnion}(A) \subseteq N$.

3. Algebraic Operation on Subsets of Many Sorted Sets

Let I be a non empty set, let M be a many sorted set indexed by I, and let S be a set operation in M. We say that S is algebraic if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let x be an element of $\mathrm{Bool}(M)$. Suppose $x = S(x)$. Then there exists a family A of many sorted subsets indexed by M such that $A = \{S(a), a$ ranges over elements of $\mathrm{Bool}(M)$: a is locally-finite $\wedge$ $\mathrm{supp}(a)$ is finite $\wedge$ $a \subseteq x\}$ and $x = \mathrm{MSUnion}(A)$.

Let I be a non empty set and let M be a many sorted set indexed by I. Note that there exists a set operation in M which is algebraic, reflexive, monotonic, and idempotent.

Let S be a non empty 1-sorted structure and let I_1 be a closure system of S. We say that I_1 is algebraic if and only if:

(Def. 6) $\mathrm{ClOp}(I_1)$ is algebraic.

Let S be a non void non empty many sorted signature and let M_1 be a non-empty algebra over S. The functor $\mathrm{SubAlgCl}(M_1)$ yields a strict closure system structure over S and is defined by:

(Def. 7) The sorts of $\mathrm{SubAlgCl}(M_1)$ = the sorts of M_1 and the family of $\mathrm{SubAlgCl}(M_1) = \mathrm{SubSorts}(M_1)$.

One can prove the following proposition

(16) Let S be a non void non empty many sorted signature and M_1 be a strict non-empty algebra over S. Then $\mathrm{SubSorts}(M_1)$ is an absolutely-multiplicative family of many sorted subsets indexed by the sorts of M_1.

Let S be a non void non empty many sorted signature and let M_1 be a strict non-empty algebra over S. Note that $\mathrm{SubAlgCl}(M_1)$ is absolutely-multiplicative.

Let S be a non void non empty many sorted signature and let M_1 be a strict non-empty algebra over S. Observe that $\mathrm{SubAlgCl}(M_1)$ is algebraic.

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Convergence and the Limit of Complex Sequences. Serieses

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The papers [5], [4], [8], [6], [2], [7], [10], [12], [3], [1], [9], [13], and [11] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we adopt the following convention: r_1, r_2, r_3 are sequences of real numbers, s_1, s_2, s_3 are complex sequences, k, n, m are natural numbers, and p, r are elements of $\mathbb{R}$.

The following propositions are true:

(1) $(n+1)+0i \neq 0_{\mathbb{C}}$ and $0+(n+1)i \neq 0_{\mathbb{C}}$.

(2) If for every n holds $r_1(n) = 0$, then for every m holds $(\sum_{\alpha=0}^{\kappa} |r_1|(\alpha))_{\kappa\in\mathbb{N}}(m) = 0$.

(3) If for every n holds $r_1(n) = 0$, then r_1 is absolutely summable.

Let us note that there exists a sequence of real numbers which is absolutely summable.

One can check that every sequence of real numbers which is summable is also convergent.

One can verify that every sequence of real numbers which is absolutely summable is also summable.

One can check that there exists a sequence of real numbers which is absolutely summable.

Next we state several propositions:

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(4) Suppose r_1 is convergent. Let given p. Suppose $0 < p$. Then there exists n such that for all natural numbers m, l such that $n \leqslant m$ and $n \leqslant l$ holds $|r_1(m) - r_1(l)| < p$.

(5) If for every n holds $r_1(n) \leqslant p$, then for all natural numbers n, l holds $(\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n+l) - (\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n) \leqslant p \cdot l$.

(6) If for every n holds $r_1(n) \leqslant p$, then for every n holds $(\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n) \leqslant p \cdot (n+1)$.

(7) If for every n such that $n \leqslant m$ holds $r_2(n) \leqslant p \cdot r_3(n)$, then $(\sum_{\alpha=0}^{\kappa}(r_2)(\alpha))_{\kappa\in\mathbb{N}}(m) \leqslant p \cdot (\sum_{\alpha=0}^{\kappa}(r_3)(\alpha))_{\kappa\in\mathbb{N}}(m)$.

(8) Suppose that for every n such that $n \leqslant m$ holds $r_2(n) \leqslant p \cdot r_3(n)$. Let given n. Suppose $n \leqslant m$. Let l be a natural number. If $n + l \leqslant m$, then $(\sum_{\alpha=0}^{\kappa}(r_2)(\alpha))_{\kappa\in\mathbb{N}}(n+l) - (\sum_{\alpha=0}^{\kappa}(r_2)(\alpha))_{\kappa\in\mathbb{N}}(n) \leqslant p \cdot ((\sum_{\alpha=0}^{\kappa}(r_3)(\alpha))_{\kappa\in\mathbb{N}}(n+l) - (\sum_{\alpha=0}^{\kappa}(r_3)(\alpha))_{\kappa\in\mathbb{N}}(n))$.

(9) If for every n holds $0 \leqslant r_1(n)$, then for all n, m such that $n \leqslant m$ holds $|(\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n)$ and for every n holds $|(\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(10) If s_2 is convergent and s_3 is convergent and $\lim(s_2 - s_3) = 0_{\mathbb{C}}$, then $\lim s_2 = \lim s_3$.

2. The Operations on Complex Sequences

In the sequel z denotes an element of $\mathbb{C}$ and N_1 denotes an increasing sequence of naturals.

Let z be an element of $\mathbb{C}$. The functor $(z^{\kappa})_{\kappa\in\mathbb{N}}$ yielding a complex sequence is defined as follows:

(Def. 1) $(z^{\kappa})_{\kappa\in\mathbb{N}}(0) = 1_{\mathbb{C}}$ and for every n holds $(z^{\kappa})_{\kappa\in\mathbb{N}}(n+1) = (z^{\kappa})_{\kappa\in\mathbb{N}}(n) \cdot z$.

Let z be an element of $\mathbb{C}$ and let n be a natural number. The functor $z_{\mathbb{N}}^{n}$ yielding an element of $\mathbb{C}$ is defined by:

(Def. 2) $z_{\mathbb{N}}^{n} = (z^{\kappa})_{\kappa\in\mathbb{N}}(n)$.

The following proposition is true

(11) $z_{\mathbb{N}}^{0} = 1_{\mathbb{C}}$.

Let c be a complex sequence. The functor $\Re(c)$ yields a sequence of real numbers and is defined as follows:

(Def. 3) For every n holds $\Re(c)(n) = \Re(c(n))$.

Let c be a complex sequence. The functor $\Im(c)$ yielding a sequence of real numbers is defined as follows:

(Def. 4) For every n holds $\Im(c)(n) = \Im(c(n))$.

We now state a number of propositions:

(12) $|z| \leqslant |\Re(z)| + |\Im(z)|$.

(13) $|\Re(z)| \leqslant |z|$ and $|\Im(z)| \leqslant |z|$.

(14) $\Re(s_2) = \Re(s_3)$ and $\Im(s_2) = \Im(s_3)$ iff $s_2 = s_3$.

(15) $\Re(s_2) + \Re(s_3) = \Re(s_2 + s_3)$ and $\Im(s_2) + \Im(s_3) = \Im(s_2 + s_3)$.

(16) $-\Re(s_1) = \Re(-s_1)$ and $-\Im(s_1) = \Im(-s_1)$.

(17) $r \cdot \Re(z) = \Re((r + 0i) \cdot z)$ and $r \cdot \Im(z) = \Im((r + 0i) \cdot z)$.

(18) $\Re(s_2) - \Re(s_3) = \Re(s_2 - s_3)$ and $\Im(s_2) - \Im(s_3) = \Im(s_2 - s_3)$.

(19) $r\,\Re(s_1) = \Re((r + 0i)\, s_1)$ and $r\,\Im(s_1) = \Im((r + 0i)\, s_1)$.

(20) $\Re(z\, s_1) = \Re(z)\,\Re(s_1) - \Im(z)\,\Im(s_1)$ and $\Im(z\, s_1) = \Re(z)\,\Im(s_1) + \Im(z)\,\Re(s_1)$.

(21) $\Re(s_2\, s_3) = \Re(s_2)\,\Re(s_3) - \Im(s_2)\,\Im(s_3)$ and $\Im(s_2\, s_3) = \Re(s_2)\,\Im(s_3) + \Im(s_2)\,\Re(s_3)$.

Let s_1 be a complex sequence and let N_1 be an increasing sequence of naturals. The functor $s_1\, N_1$ yielding a complex sequence is defined by:

(Def. 5) For every n holds $(s_1\, N_1)(n) = s_1(N_1(n))$.

Next we state the proposition

(22) $\Re(s_1\, N_1) = \Re(s_1) \cdot N_1$ and $\Im(s_1\, N_1) = \Im(s_1) \cdot N_1$.

Let s_1 be a complex sequence and let k be a natural number. The functor $s_1 \uparrow k$ yields a complex sequence and is defined by:

(Def. 6) For every n holds $(s_1 \uparrow k)(n) = s_1(n + k)$.

The following proposition is true

(23) $\Re(s_1) \uparrow k = \Re(s_1 \uparrow k)$ and $\Im(s_1) \uparrow k = \Im(s_1 \uparrow k)$.

Let s_1 be a complex sequence. The functor $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ yields a complex sequence and is defined as follows:

(Def. 7) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(0) = s_1(0)$ and for every n holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) + s_1(n + 1)$.

Let s_1 be a complex sequence. The functor $\sum s_1$ yields an element of $\mathbb{C}$ and is defined as follows:

(Def. 8) $\sum s_1 = \lim((\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})$.

Next we state a number of propositions:

(24) If for every n holds $s_1(n) = 0_{\mathbb{C}}$, then for every m holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m) = 0_{\mathbb{C}}$.

(25) If for every n holds $s_1(n) = 0_{\mathbb{C}}$, then for every m holds $(\sum_{\alpha=0}^{\kappa}|s_1|(\alpha))_{\kappa\in\mathbb{N}}(m) = 0$.

(26) $(\sum_{\alpha=0}^{\kappa}\Re(s_1)(\alpha))_{\kappa\in\mathbb{N}} = \Re((\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})$ and $(\sum_{\alpha=0}^{\kappa}\Im(s_1)(\alpha))_{\kappa\in\mathbb{N}} = \Im((\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})$.

(27) $(\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}} + (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_2 + s_3)(\alpha))_{\kappa\in\mathbb{N}}$.

(28) $(\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_2 - s_3)(\alpha))_{\kappa\in\mathbb{N}}$.

(29) $(\sum_{\alpha=0}^{\kappa}(z\, s_1)(\alpha))_{\kappa\in\mathbb{N}} = z\,(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$.

(30) $|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(k)| \leqslant (\sum_{\alpha=0}^{\kappa}|s_1|(\alpha))_{\kappa\in\mathbb{N}}(k)$.

(31) $|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n)| \leqslant |(\sum_{\alpha=0}^{\kappa}|s_1|(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}|s_1|(\alpha))_{\kappa\in\mathbb{N}}(n)|$.

(32) $(\sum_{\alpha=0}^{\kappa}\Re(s_1)(\alpha))_{\kappa\in\mathbb{N}}\uparrow k = \Re((\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}\uparrow k)$ and $(\sum_{\alpha=0}^{\kappa}\Im(s_1)(\alpha))_{\kappa\in\mathbb{N}}\uparrow k = \Im((\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}\uparrow k)$.

(33) If for every n holds $s_2(n) = s_1(0)$, then $(\sum_{\alpha=0}^{\kappa}(s_1\uparrow 1)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}\uparrow 1 - s_2$.

(34) $(\sum_{\alpha=0}^{\kappa}|s_1|(\alpha))_{\kappa\in\mathbb{N}}$ is non-decreasing.

Let s_1 be a complex sequence. Note that $(\sum_{\alpha=0}^{\kappa}|s_1|(\alpha))_{\kappa\in\mathbb{N}}$ is non-decreasing.

Next we state three propositions:

(35) If for every n such that $n \leqslant m$ holds $s_2(n) = s_3(n)$, then $(\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}}(m) = (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}}(m)$.

(36) If $1_{\mathbb{C}} \neq z$, then for every n holds $(\sum_{\alpha=0}^{\kappa}((z^{\kappa})_{\kappa\in\mathbb{N}})(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{1_{\mathbb{C}} - z_{\mathbb{N}}^{n+1}}{1_{\mathbb{C}} - z}$.

(37) If $z \neq 1_{\mathbb{C}}$ and for every n holds $s_1(n+1) = z \cdot s_1(n)$, then for every n holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) = s_1(0) \cdot \frac{1_{\mathbb{C}} - z_{\mathbb{N}}^{n+1}}{1_{\mathbb{C}} - z}$.

3. Convergence of Complex Sequences

Next we state four propositions:

(38) Let a, b be sequences of real numbers and c be a complex sequence. Suppose that for every n holds $\Re(c(n)) = a(n)$ and $\Im(c(n)) = b(n)$. Then a is convergent and b is convergent if and only if c is convergent.

(39) Let a, b be convergent sequences of real numbers and c be a complex sequence. Suppose that for every n holds $\Re(c(n)) = a(n)$ and $\Im(c(n)) = b(n)$. Then c is convergent and $\lim c = \lim a + \lim b i$.

(40) Let a, b be sequences of real numbers and c be a convergent complex sequence. Suppose that for every n holds $\Re(c(n)) = a(n)$ and $\Im(c(n)) = b(n)$. Then a is convergent and b is convergent and $\lim a = \Re(\lim c)$ and $\lim b = \Im(\lim c)$.

(41) For every convergent complex sequence c holds $\Re(c)$ is convergent and $\Im(c)$ is convergent and $\lim \Re(c) = \Re(\lim c)$ and $\lim \Im(c) = \Im(\lim c)$.

Let c be a convergent complex sequence. Observe that $\Re(c)$ is convergent and $\Im(c)$ is convergent.

The following propositions are true:

(42) Let c be a complex sequence. Suppose $\Re(c)$ is convergent and $\Im(c)$ is convergent. Then c is convergent and $\Re(\lim c) = \lim \Re(c)$ and $\Im(\lim c) = \lim \Im(c)$.

(43) If $0 < |z|$ and $|z| < 1$ and $s_1(0) = z$ and for every n holds $s_1(n+1) = s_1(n) \cdot z$, then s_1 is convergent and $\lim s_1 = 0_{\mathbb{C}}$.

(44) If $|z| < 1$ and for every n holds $s_1(n) = z_{\mathbb{N}}^{n+1}$, then s_1 is convergent and $\lim s_1 = 0_{\mathbb{C}}$.

(45) If $r > 0$ and there exists m such that for every n such that $n \geqslant m$ holds $|s_1(n)| \geqslant r$, then $|s_1|$ is not convergent or $\lim |s_1| \neq 0$.

(46) s_1 is convergent iff for every p such that $0 < p$ there exists n such that for every m such that $n \leqslant m$ holds $|s_1(m) - s_1(n)| < p$.

(47) Suppose s_1 is convergent. Let given p. Suppose $0 < p$. Then there exists n such that for all natural numbers m, l such that $n \leqslant m$ and $n \leqslant l$ holds $|s_1(m) - s_1(l)| < p$.

(48) If for every n holds $|s_1(n)| \leqslant r_1(n)$ and r_1 is convergent and $\lim r_1 = 0$, then s_1 is convergent and $\lim s_1 = 0_{\mathbb{C}}$.

4. Summable and Absolutely Summable Complex Sequences

Let s_1, s_2 be complex sequences. We say that s_1 is a subsequence of s_2 if and only if:

(Def. 9) There exists N_1 such that $s_1 = s_2\, N_1$.

Next we state three propositions:

(49) If s_1 is a subsequence of s_2, then $\Re(s_1)$ is a subsequence of $\Re(s_2)$ and $\Im(s_1)$ is a subsequence of $\Im(s_2)$.

(50) If s_1 is a subsequence of s_2 and s_2 is a subsequence of s_3, then s_1 is a subsequence of s_3.

(51) If s_1 is bounded, then there exists s_2 which is a subsequence of s_1 and convergent.

Let s_1 be a complex sequence. We say that s_1 is summable if and only if:

(Def. 10) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}}$ is convergent.

Let us observe that there exists a complex sequence which is summable.

Let s_1 be a summable complex sequence. Observe that $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}}$ is convergent.

Let us consider s_1. We say that s_1 is absolutely summable if and only if:

(Def. 11) $|s_1|$ is summable.

One can prove the following proposition

(52) If for every n holds $s_1(n) = 0_{\mathbb{C}}$, then s_1 is absolutely summable.

Let us observe that there exists a complex sequence which is absolutely summable.

Let s_1 be an absolutely summable complex sequence. Observe that $|s_1|$ is summable.

The following proposition is true

(53) If s_1 is summable, then s_1 is convergent and $\lim s_1 = 0_{\mathbb{C}}$.

One can verify that every complex sequence which is summable is also convergent.

We now state the proposition

(54) If s_1 is summable, then $\Re(s_1)$ is summable and $\Im(s_1)$ is summable and $\sum s_1 = \sum \Re(s_1) + \sum \Im(s_1)i$.

Let s_1 be a summable complex sequence. One can verify that $\Re(s_1)$ is summable and $\Im(s_1)$ is summable.

We now state two propositions:

(55) If s_2 is summable and s_3 is summable, then $s_2 + s_3$ is summable and $\sum(s_2 + s_3) = \sum s_2 + \sum s_3$.

(56) If s_2 is summable and s_3 is summable, then $s_2 - s_3$ is summable and $\sum(s_2 - s_3) = \sum s_2 - \sum s_3$.

Let s_2, s_3 be summable complex sequences. One can check that $s_2 + s_3$ is summable and $s_2 - s_3$ is summable.

The following proposition is true

(57) If s_1 is summable, then $z\, s_1$ is summable and $\sum(z\, s_1) = z \cdot \sum s_1$.

Let z be an element of $\mathbb{C}$ and let s_1 be a summable complex sequence. One can check that $z\, s_1$ is summable.

The following two propositions are true:

(58) If $\Re(s_1)$ is summable and $\Im(s_1)$ is summable, then s_1 is summable and $\sum s_1 = \sum \Re(s_1) + \sum \Im(s_1)i$.

(59) If s_1 is summable, then for every n holds $s_1 \uparrow n$ is summable.

Let s_1 be a summable complex sequence and let n be a natural number. Note that $s_1 \uparrow n$ is summable.

One can prove the following propositions:

(60) If there exists n such that $s_1 \uparrow n$ is summable, then s_1 is summable.

(61) If s_1 is summable, then for every n holds $\sum s_1 = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) + \sum(s_1 \uparrow (n+1))$.

(62) $(\sum_{\alpha=0}^{\kappa} |s_1|(\alpha))_{\kappa\in\mathbb{N}}$ is upper bounded iff s_1 is absolutely summable.

Let s_1 be an absolutely summable complex sequence. One can check that $(\sum_{\alpha=0}^{\kappa} |s_1|(\alpha))_{\kappa\in\mathbb{N}}$ is upper bounded.

One can prove the following two propositions:

(63) s_1 is summable iff for every p such that $0 < p$ there exists n such that for every m such that $n \leqslant m$ holds $|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n)| < p$.

(64) If s_1 is absolutely summable, then s_1 is summable.

One can check that every complex sequence which is absolutely summable is also summable.

Let us note that there exists a complex sequence which is absolutely summable.

The following propositions are true:

(65) If $|z| < 1$, then $(z^\kappa)_{\kappa\in\mathbb{N}}$ is summable and $\sum((z^\kappa)_{\kappa\in\mathbb{N}}) = \frac{1_\mathbb{C}}{1_\mathbb{C}-z}$.

(66) If $|z| < 1$ and for every n holds $s_1(n+1) = z \cdot s_1(n)$, then s_1 is summable and $\sum s_1 = \frac{s_1(0)}{1_\mathbb{C}-z}$.

(67) If r_2 is summable and there exists m such that for every n such that $m \leqslant n$ holds $|s_3(n)| \leqslant r_2(n)$, then s_3 is absolutely summable.

(68) Suppose for every n holds $0 \leqslant |s_2|(n)$ and $|s_2|(n) \leqslant |s_3|(n)$ and s_3 is absolutely summable. Then s_2 is absolutely summable and $\sum |s_2| \leqslant \sum |s_3|$.

(69) If for every n holds $|s_1|(n) > 0$ and there exists m such that for every n such that $n \geqslant m$ holds $\frac{|s_1|(n+1)}{|s_1|(n)} \geqslant 1$, then s_1 is not absolutely summable.

(70) If for every n holds $r_2(n) = \sqrt[n]{|s_1|(n)}$ and r_2 is convergent and $\lim r_2 < 1$, then s_1 is absolutely summable.

(71) If for every n holds $r_2(n) = \sqrt[n]{|s_1|(n)}$ and there exists m such that for every n such that $m \leqslant n$ holds $r_2(n) \geqslant 1$, then $|s_1|$ is not summable.

(72) If for every n holds $r_2(n) = \sqrt[n]{|s_1|(n)}$ and r_2 is convergent and $\lim r_2 > 1$, then s_1 is not absolutely summable.

(73) Suppose $|s_1|$ is non-increasing and for every n holds $r_2(n) = 2^n \cdot |s_1|$(the n-th power of 2). Then s_1 is absolutely summable if and only if r_2 is summable.

(74) If $p > 1$ and for every n such that $n \geqslant 1$ holds $|s_1|(n) = \frac{1}{n^p}$, then s_1 is absolutely summable.

(75) If $p \leqslant 1$ and for every n such that $n \geqslant 1$ holds $|s_1|(n) = \frac{1}{n^p}$, then s_1 is not absolutely summable.

(76) If for every n holds $s_1(n) \neq 0_\mathbb{C}$ and $r_2(n) = \frac{|s_1|(n+1)}{|s_1|(n)}$ and r_2 is convergent and $\lim r_2 < 1$, then s_1 is absolutely summable.

(77) If for every n holds $s_1(n) \neq 0_\mathbb{C}$ and there exists m such that for every n such that $n \geqslant m$ holds $\frac{|s_1|(n+1)}{|s_1|(n)} \geqslant 1$, then s_1 is not absolutely summable.

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The Steinitz Theorem and the Dimension of a Real Linear Space

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Summary. Finite-dimensional real linear spaces are defined. The dimension of such spaces is the cardinality of a basis. Obviously, each two basis have the same cardinality. We prove the Steinitz theorem and the Exchange Lemma. We also investigate some fundamental facts involving the dimension of real linear spaces.

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The notation and terminology used here are introduced in the following papers: [10], [19], [9], [7], [2], [20], [4], [5], [18], [1], [6], [3], [13], [15], [8], [17], [12], [16], [14], and [11].

1. Prelimiaries

For simplicity, we follow the rules: V denotes a real linear space, W denotes a subspace of V, x denotes a set, n denotes a natural number, v denotes a vector of V, K_1, K_2 denote linear combinations of V, and X denotes a subset of the carrier of V.

We now state a number of propositions:

(1) If X is linearly independent and the support of $K_1 \subseteq X$ and the support of $K_2 \subseteq X$ and $\sum K_1 = \sum K_2$, then $K_1 = K_2$.

(2) Let V be a real linear space and A be a subset of V. If A is linearly independent, then there exists a basis I of V such that $A \subseteq I$.

(3) Let L be a linear combination of V and x be a vector of V. Then $x \in$ the support of L if and only if there exists v such that $x = v$ and $L(v) \neq 0$.

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(4) For every finite set X such that $n \leqslant \overline{\overline{X}}$ there exists a finite subset A of X such that $\overline{\overline{A}} = n$.

(5) Let L be a linear combination of V, F, G be finite sequences of elements of the carrier of V, and P be a permutation of $\operatorname{dom} F$. If $G = F \cdot P$, then $\sum(L\,F) = \sum(L\,G)$.

(6) Let L be a linear combination of V and F be a finite sequence of elements of the carrier of V. If the support of L misses $\operatorname{rng} F$, then $\sum(L\,F) = 0_V$.

(7) Let F be a finite sequence of elements of the carrier of V. Suppose F is one-to-one. Let L be a linear combination of V. If the support of $L \subseteq \operatorname{rng} F$, then $\sum(L\,F) = \sum L$.

(8) Let L be a linear combination of V and F be a finite sequence of elements of the carrier of V. Then there exists a linear combination K of V such that the support of $K = \operatorname{rng} F \cap$ the support of L and $L\,F = K\,F$.

(9) Let L be a linear combination of V, A be a subset of V, and F be a finite sequence of elements of the carrier of V. Suppose $\operatorname{rng} F \subseteq$ the carrier of $\operatorname{Lin}(A)$. Then there exists a linear combination K of A such that $\sum(L\,F) = \sum K$.

(10) Let L be a linear combination of V and A be a subset of V. Suppose the support of $L \subseteq$ the carrier of $\operatorname{Lin}(A)$. Then there exists a linear combination K of A such that $\sum L = \sum K$.

(11) Let L be a linear combination of V. Suppose the support of $L \subseteq$ the carrier of W. Let K be a linear combination of W. Suppose $K = L{\restriction}$the carrier of W. Then the support of $L =$ the support of K and $\sum L = \sum K$.

(12) Let K be a linear combination of W. Then there exists a linear combination L of V such that the support of $K =$ the support of L and $\sum K = \sum L$.

(13) Let L be a linear combination of V. Suppose the support of $L \subseteq$ the carrier of W. Then there exists a linear combination K of W such that the support of $K =$ the support of L and $\sum K = \sum L$.

(14) For every basis I of V and for every vector v of V holds $v \in \operatorname{Lin}(I)$.

(15) Let A be a subset of W. Suppose A is linearly independent. Then there exists a subset B of V such that B is linearly independent and $B = A$.

(16) Let A be a subset of V. Suppose A is linearly independent and $A \subseteq$ the carrier of W. Then there exists a subset B of W such that B is linearly independent and $B = A$.

(17) For every basis A of W there exists a basis B of V such that $A \subseteq B$.

(18) Let A be a subset of V. Suppose A is linearly independent. Let v be a vector of V. If $v \in A$, then for every subset B of V such that $B = A \setminus \{v\}$ holds $v \notin \operatorname{Lin}(B)$.

(19) Let I be a basis of V and A be a non empty subset of V. Suppose A misses I. Let B be a subset of V. If $B = I \cup A$, then B is linearly-dependent.

(20) For every subset A of V such that $A \subseteq$ the carrier of W holds $\mathrm{Lin}(A)$ is a subspace of W.

(21) For every subset A of V and for every subset B of W such that $A = B$ holds $\mathrm{Lin}(A) = \mathrm{Lin}(B)$.

2. The Steinitz Theorem

Next we state two propositions:

(22) Let A, B be finite subsets of V and v be a vector of V. Suppose $v \in \mathrm{Lin}(A \cup B)$ and $v \notin \mathrm{Lin}(B)$. Then there exists a vector w of V such that $w \in A$ and $w \in \mathrm{Lin}(((A \cup B) \setminus \{w\}) \cup \{v\})$.

(23) Let A, B be finite subsets of V. Suppose the RLS structure of $V = \mathrm{Lin}(A)$ and B is linearly independent. Then $\overline{\overline{B}} \leqslant \overline{\overline{A}}$ and there exists a finite subset C of V such that $C \subseteq A$ and $\overline{\overline{C}} = \overline{\overline{A}} - \overline{\overline{B}}$ and the RLS structure of $V = \mathrm{Lin}(B \cup C)$.

3. Finite Dimensional Vector Spaces

Let V be a real linear space. We say that V is finite dimensional if and only if:

(Def. 1) There exists a finite subset of the carrier of V which is a basis of V.

Let us observe that there exists a real linear space which is strict and finite dimensional.

Let V be a real linear space. Let us observe that V is finite dimensional if and only if:

(Def. 2) There exists a finite subset of V which is a basis of V.

We now state several propositions:

(24) If V is finite dimensional, then every basis of V is finite.

(25) If V is finite dimensional, then for every subset A of V such that A is linearly independent holds A is finite.

(26) If V is finite dimensional, then for all bases A, B of V holds $\overline{\overline{A}} = \overline{\overline{B}}$.

(27) $\mathbf{0}_V$ is finite dimensional.

(28) If V is finite dimensional, then W is finite dimensional.

Let V be a real linear space. One can check that there exists a subspace of V which is finite dimensional and strict.

Let V be a finite dimensional real linear space. Observe that every subspace of V is finite dimensional.

Let V be a finite dimensional real linear space. Note that there exists a subspace of V which is strict.

4. The Dimension of a Vector Space

Let V be a real linear space. Let us assume that V is finite dimensional. The functor $\dim(V)$ yields a natural number and is defined as follows:

(Def. 3) For every basis I of V holds $\dim(V) = \overline{\overline{I}}$.

We use the following convention: V is a finite dimensional real linear space, W, W_1, W_2 are subspaces of V, and u, v are vectors of V.

Next we state a number of propositions:

(29) $\dim(W) \leqslant \dim(V)$.

(30) For every subset A of V such that A is linearly independent holds $\overline{\overline{A}} = \dim(\mathrm{Lin}(A))$.

(31) $\dim(V) = \dim(\Omega_V)$.

(32) $\dim(V) = \dim(W)$ iff $\Omega_V = \Omega_W$.

(33) $\dim(V) = 0$ iff $\Omega_V = \mathbf{0}_V$.

(34) $\dim(V) = 1$ iff there exists v such that $v \neq 0_V$ and $\Omega_V = \mathrm{Lin}(\{v\})$.

(35) $\dim(V) = 2$ iff there exist u, v such that $u \neq v$ and $\{u, v\}$ is linearly independent and $\Omega_V = \mathrm{Lin}(\{u, v\})$.

(36) $\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2)$.

(37) $\dim(W_1 \cap W_2) \geqslant (\dim(W_1) + \dim(W_2)) - \dim(V)$.

(38) If V is the direct sum of W_1 and W_2, then $\dim(V) = \dim(W_1) + \dim(W_2)$.

(39) $n \leqslant \dim(V)$ iff there exists a strict subspace W of V such that $\dim(W) = n$.

Let V be a finite dimensional real linear space and let n be a natural number. The functor $\mathrm{Sub}_n(V)$ yields a set and is defined as follows:

(Def. 4) $x \in \mathrm{Sub}_n(V)$ iff there exists a strict subspace W of V such that $W = x$ and $\dim(W) = n$.

The following propositions are true:

(40) If $n \leqslant \dim(V)$, then $\mathrm{Sub}_n(V)$ is non empty.

(41) If $\dim(V) < n$, then $\mathrm{Sub}_n(V) = \emptyset$.

(42) $\mathrm{Sub}_n(W) \subseteq \mathrm{Sub}_n(V)$.

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Euler Circuits and Paths[1]

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Summary. We prove the Euler theorem on existence of Euler circuits and paths in multigraphs.

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The notation and terminology used in this paper are introduced in the following papers: [19], [23], [13], [10], [22], [24], [6], [9], [7], [4], [8], [2], [20], [12], [3], [5], [21], [1], [14], [15], [11], [16], [17], and [18].

1. Preliminaries

Let D be a set, let T be a non empty set of finite sequences of D, and let S be a non empty subset of T. We see that the element of S is a finite sequence of elements of D.

Let i, j be even integers. One can verify that $i - j$ is even.

We now state two propositions:

(1) For all integers i, j holds i is even iff j is even iff $i - j$ is even.

(2) Let p be a finite sequence and m, n, a be natural numbers. Suppose $a \in \mathrm{dom}\langle p(m), \ldots, p(n)\rangle$. Then there exists a natural number k such that $k \in \mathrm{dom}\, p$ and $p(k) = \langle p(m), \ldots, p(n)\rangle(a)$ and $k + 1 = m + a$ and $m \leqslant k$ and $k \leqslant n$.

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Let G be a graph. A vertex of G is an element of the vertices of G.

For simplicity, we follow the rules: G denotes a graph, v, v_1, v_2 denote vertices of G, c, c_1, c_2 denote chains of G, p, p_1, p_2 denote paths of G, v_3, v_4, v_5 denote finite sequences of elements of the vertices of G, e, X denote sets, and n, m denote natural numbers.

One can prove the following propositions:

(3) If v_3 is vertex sequence of c, then v_3 is non empty.

(4) If c is cyclic and v_3 is vertex sequence of c, then $v_3(1) = v_3(\operatorname{len} v_3)$.

(5) If $n \in \operatorname{dom} p$ and $m \in \operatorname{dom} p$ and $n \neq m$, then $p(n) \neq p(m)$.

(6) ε is a path of G.

(7) If $e \in$ the edges of G, then $\langle e \rangle$ is a path of G.

(8) $\langle p(m), \ldots, p(n) \rangle$ is a path of G.

(9) Suppose $\operatorname{rng} p_1$ misses $\operatorname{rng} p_2$ and v_4 is vertex sequence of p_1 and v_5 is vertex sequence of p_2 and $v_4(\operatorname{len} v_4) = v_5(1)$. Then $p_1 \frown p_2$ is a path of G.

(10) p is one-to-one.

(11) If $c_1 \frown c_2$ is a path of G, then $\operatorname{rng} c_1$ misses $\operatorname{rng} c_2$.

(12) If $c = \varepsilon$, then c is cyclic.

Let G be a graph. Observe that there exists a path of G which is cyclic.

Next we state several propositions:

(13) For every cyclic path p of G holds $\langle p(m+1), \ldots, p(\operatorname{len} p) \rangle \frown \langle p(1), \ldots, p(m) \rangle$ is a cyclic path of G.

(14) If $m+1 \in \operatorname{dom} p$, then $\operatorname{len}(\langle p(m+1), \ldots, p(\operatorname{len} p) \rangle \frown \langle p(1), \ldots, p(m) \rangle) = \operatorname{len} p$ and $\operatorname{rng}(\langle p(m+1), \ldots, p(\operatorname{len} p) \rangle \frown \langle p(1), \ldots, p(m) \rangle) = \operatorname{rng} p$ and $(\langle p(m+1), \ldots, p(\operatorname{len} p) \rangle \frown \langle p(1), \ldots, p(m) \rangle)(1) = p(m+1)$.

(15) For every cyclic path p of G such that $n \in \operatorname{dom} p$ there exists a cyclic path p' of G such that $p'(1) = p(n)$ and $\operatorname{len} p' = \operatorname{len} p$ and $\operatorname{rng} p' = \operatorname{rng} p$.

(16) Let s, t be vertices of G. Suppose $s =$ (the source of $G)(e)$ and $t =$ (the target of $G)(e)$. Then $\langle t, s \rangle$ is vertex sequence of $\langle e \rangle$.

(17) Suppose $e \in$ the edges of G and v_3 is vertex sequence of c and $v_3(\operatorname{len} v_3) =$ (the source of $G)(e)$. Then
  (i) $c \frown \langle e \rangle$ is a chain of G, and
  (ii) there exists a finite sequence v_1' of elements of the vertices of G such that $v_1' = v_3 \frown\!\frown \langle$(the source of $G)(e)$, (the target of $G)(e)\rangle$ and v_1' is vertex sequence of $c \frown \langle e \rangle$ and $v_1'(1) = v_3(1)$ and $v_1'(\operatorname{len} v_1') =$ (the target of $G)(e)$.

(18) Suppose $e \in$ the edges of G and v_3 is vertex sequence of c and $v_3(\operatorname{len} v_3) =$ (the target of $G)(e)$. Then
  (i) $c \frown \langle e \rangle$ is a chain of G, and
  (ii) there exists a finite sequence v_1' of elements of the vertices of G such that $v_1' = v_3 \frown\!\frown \langle$(the target of $G)(e)$, (the source of $G)(e)\rangle$ and v_1' is vertex

sequence of $c^\frown\langle e\rangle$ and $v_1'(1) = v_3(1)$ and $v_1'(\mathrm{len}\, v_1') =$ (the source of G)(e).

(19) Suppose v_3 is vertex sequence of c. Let n be a natural number. Suppose $n \in \mathrm{dom}\, c$. Then

(i) $v_3(n) =$ (the target of G)($c(n)$) and $v_3(n+1) =$ (the source of G)($c(n)$), or

(ii) $v_3(n) =$ (the source of G)($c(n)$) and $v_3(n+1) =$ (the target of G)($c(n)$).

(20) If v_3 is vertex sequence of c and $e \in \mathrm{rng}\, c$, then (the target of G)(e) $\in \mathrm{rng}\, v_3$ and (the source of G)(e) $\in \mathrm{rng}\, v_3$.

Let G be a graph and let X be a set. Then G-VSet(X) is a subset of the vertices of G.

One can prove the following propositions:

(21) G-VSet($\emptyset$) $= \emptyset$.

(22) If $e \in$ the edges of G and $e \in X$, then G-VSet(X) is non empty.

(23) G is connected if and only if for all v_1, v_2 such that $v_1 \neq v_2$ there exist c, v_3 such that c is non empty and v_3 is vertex sequence of c and $v_3(1) = v_1$ and $v_3(\mathrm{len}\, v_3) = v_2$.

(24) Let G be a connected graph, X be a set, and v be a vertex of G. Suppose X meets the edges of G and $v \notin G$-VSet(X). Then there exists a vertex v' of G and there exists an element e of the edges of G such that $v' \in G$-VSet(X) but $e \notin X$ but $v' =$ (the target of G)(e) or $v' =$ (the source of G)(e).

2. Degree of a vertex

Let G be a graph, let v be a vertex of G, and let X be a set. The functor EdgesIn(v, X) yields a subset of the edges of G and is defined as follows:

(Def. 1) For every set e holds $e \in$ EdgesIn(v, X) iff $e \in$ the edges of G and $e \in X$ and (the target of G)(e) $= v$.

The functor EdgesOut(v, X) yields a subset of the edges of G and is defined as follows:

(Def. 2) For every set e holds $e \in$ EdgesOut(v, X) iff $e \in$ the edges of G and $e \in X$ and (the source of G)(e) $= v$.

Let G be a graph, let v be a vertex of G, and let X be a set. The functor EdgesAt(v, X) yields a subset of the edges of G and is defined as follows:

(Def. 3) EdgesAt(v, X) $=$ EdgesIn(v, X) $\cup$ EdgesOut(v, X).

Let G be a finite graph, let v be a vertex of G, and let X be a set. One can check the following observations:

* EdgesIn(v, X) is finite,

* EdgesOut(v, X) is finite, and
* EdgesAt(v, X) is finite.

Let G be a graph, let v be a vertex of G, and let X be an empty set. One can verify the following observations:

* EdgesIn(v, X) is empty,
* EdgesOut(v, X) is empty, and
* EdgesAt(v, X) is empty.

Let G be a graph and let v be a vertex of G. The functor EdgesIn v yields a subset of the edges of G and is defined as follows:

(Def. 4) EdgesIn v = EdgesIn(v, the edges of G).

The functor EdgesOut v yields a subset of the edges of G and is defined by:

(Def. 5) EdgesOut v = EdgesOut(v, the edges of G).

One can prove the following propositions:

(25) EdgesIn$(v, X) \subseteq$ EdgesIn v.

(26) EdgesOut$(v, X) \subseteq$ EdgesOut v.

Let G be a finite graph and let v be a vertex of G. Note that EdgesIn v is finite and EdgesOut v is finite.

For simplicity, we follow the rules: G denotes a finite graph, v denotes a vertex of G, c denotes a chain of G, v_3 denotes a finite sequence of elements of the vertices of G, and X_1, X_2 denote sets.

One can prove the following two propositions:

(27) card EdgesIn v = EdgIn(v).

(28) card EdgesOut v = EdgOut(v).

Let G be a finite graph, let v be a vertex of G, and let X be a set. The functor Degree(v, X) yields a natural number and is defined as follows:

(Def. 6) Degree(v, X) = card EdgesIn(v, X) + card EdgesOut(v, X).

The following propositions are true:

(29) The degree of v = Degree(v, the edges of G).

(30) If Degree$(v, X) \neq 0$, then EdgesAt(v, X) is non empty.

(31) Suppose $e \in$ the edges of G but $e \notin X$ but v = (the target of G)(e) or v = (the source of G)(e). Then the degree of $v \neq$ Degree(v, X).

(32) If $X_2 \subseteq X_1$, then card EdgesIn$(v, X_1 \setminus X_2)$ = card EdgesIn(v, X_1) − card EdgesIn(v, X_2).

(33) If $X_2 \subseteq X_1$, then card EdgesOut$(v, X_1 \setminus X_2)$ = card EdgesOut(v, X_1) − card EdgesOut(v, X_2).

(34) If $X_2 \subseteq X_1$, then Degree$(v, X_1 \setminus X_2)$ = Degree(v, X_1) − Degree(v, X_2).

(35) EdgesIn(v, X) = EdgesIn($v, X \cap$ the edges of G) and EdgesOut(v, X) = EdgesOut($v, X \cap$ the edges of G).

(36) $\text{Degree}(v, X) = \text{Degree}(v, X \cap \text{the edges of } G)$.

(37) If c is non empty and v_3 is vertex sequence of c, then $v \in \operatorname{rng} v_3$ iff $\text{Degree}(v, \operatorname{rng} c) \neq 0$.

(38) For every non empty finite connected graph G and for every vertex v of G holds the degree of $v \neq 0$.

3. Adding an edge to a graph

Let G be a graph and let v_1, v_2 be vertices of G. The functor $\text{AddNewEdge}(v_1, v_2)$ yielding a strict graph is defined by the conditions (Def. 7).

(Def. 7)(i) The vertices of $\text{AddNewEdge}(v_1, v_2) =$ the vertices of G,

(ii) the edges of $\text{AddNewEdge}(v_1, v_2) = (\text{the edges of } G) \cup \{\text{the edges of } G\}$,

(iii) the source of $\text{AddNewEdge}(v_1, v_2) = (\text{the source of } G) +\cdot ((\text{the edges of } G) \dot{\longmapsto} (v_1))$, and

(iv) the target of $\text{AddNewEdge}(v_1, v_2) = (\text{the target of } G) +\cdot ((\text{the edges of } G) \dot{\longmapsto} (v_2))$.

Let G be a finite graph and let v_1, v_2 be vertices of G. Observe that $\text{AddNewEdge}(v_1, v_2)$ is finite.

For simplicity, we adopt the following rules: G is a graph, v, v_1, v_2 are vertices of G, c is a chain of G, p is a path of G, v_3 is a finite sequence of elements of the vertices of G, v' is a vertex of $\text{AddNewEdge}(v_1, v_2)$, p' is a path of $\text{AddNewEdge}(v_1, v_2)$, and v_1' is a finite sequence of elements of the vertices of $\text{AddNewEdge}(v_1, v_2)$.

We now state a number of propositions:

(39)(i) The edges of $G \in$ the edges of $\text{AddNewEdge}(v_1, v_2)$,

(ii) the edges of $G = (\text{the edges of } \text{AddNewEdge}(v_1, v_2)) \setminus \{\text{the edges of } G\}$,

(iii) $(\text{the source of } \text{AddNewEdge}(v_1, v_2))(\text{the edges of } G) = v_1$, and

(iv) $(\text{the target of } \text{AddNewEdge}(v_1, v_2))(\text{the edges of } G) = v_2$.

(40) Suppose $e \in$ the edges of G. Then $(\text{the source of } \text{AddNewEdge}(v_1, v_2))(e) = (\text{the source of } G)(e)$ and $(\text{the target of } \text{AddNewEdge}(v_1, v_2))(e) = (\text{the target of } G)(e)$.

(41) If $v_1' = v_3$ and v_3 is vertex sequence of c, then v_1' is vertex sequence of c.

(42) c is a chain of $\text{AddNewEdge}(v_1, v_2)$.

(43) p is a path of $\text{AddNewEdge}(v_1, v_2)$.

(44) If $v' = v_1$ and $v_1 \neq v_2$, then $\text{EdgesIn}(v', X) = \text{EdgesIn}(v_1, X)$.

(45) If $v' = v_2$ and $v_1 \neq v_2$, then $\text{EdgesOut}(v', X) = \text{EdgesOut}(v_2, X)$.

(46) If $v' = v_1$ and $v_1 \neq v_2$ and the edges of $G \in X$, then $\mathrm{EdgesOut}(v', X) = \mathrm{EdgesOut}(v_1, X) \cup \{\text{the edges of } G\}$ and $\mathrm{EdgesOut}(v_1, X) \cap \{\text{the edges of } G\} = \emptyset$.

(47) If $v' = v_2$ and $v_1 \neq v_2$ and the edges of $G \in X$, then $\mathrm{EdgesIn}(v', X) = \mathrm{EdgesIn}(v_2, X) \cup \{\text{the edges of } G\}$ and $\mathrm{EdgesIn}(v_2, X) \cap \{\text{the edges of } G\} = \emptyset$.

(48) If $v' = v$ and $v \neq v_1$ and $v \neq v_2$, then $\mathrm{EdgesIn}(v', X) = \mathrm{EdgesIn}(v, X)$.

(49) If $v' = v$ and $v \neq v_1$ and $v \neq v_2$, then $\mathrm{EdgesOut}(v', X) = \mathrm{EdgesOut}(v, X)$.

(50) If the edges of $G \notin \mathrm{rng}\, p'$, then p' is a path of G.

(51) If the edges of $G \notin \mathrm{rng}\, p'$ and $v_3 = v'_1$ and v'_1 is vertex sequence of p', then v_3 is vertex sequence of p'.

Let G be a connected graph and let v_1, v_2 be vertices of G. One can check that $\mathrm{AddNewEdge}(v_1, v_2)$ is connected.

For simplicity, we adopt the following rules: G is a finite graph, v, v_1, v_2 are vertices of G, v_3 is a finite sequence of elements of the vertices of G, and v' is a vertex of $\mathrm{AddNewEdge}(v_1, v_2)$.

We now state two propositions:

(52) If $v' = v$ and $v_1 \neq v_2$ and $v = v_1$ or $v = v_2$ and the edges of $G \in X$, then $\mathrm{Degree}(v', X) = \mathrm{Degree}(v, X) + 1$.

(53) If $v' = v$ and $v \neq v_1$ and $v \neq v_2$, then $\mathrm{Degree}(v', X) = \mathrm{Degree}(v, X)$.

4. Some properties of and operations on cycles

The following two propositions are true:

(54) For every cyclic path c of G holds $\mathrm{Degree}(v, \mathrm{rng}\, c)$ is even.

(55) Let c be a path of G. Suppose c is non cyclic and v_3 is vertex sequence of c. Then $\mathrm{Degree}(v, \mathrm{rng}\, c)$ is even if and only if $v \neq v_3(1)$ and $v \neq v_3(\mathrm{len}\, v_3)$.

In the sequel G is a graph, v is a vertex of G, and v_3 is a finite sequence of elements of the vertices of G.

Let G be a graph. The functor G-CycleSet yields a non empty set of finite sequences of the edges of G and is defined as follows:

(Def. 8) For every set x holds $x \in G$-CycleSet iff x is a cyclic path of G.

One can prove the following propositions:

(56) ε is an element of G-CycleSet.

(57) Let c be an element of G-CycleSet. Suppose $v \in G\text{-VSet}(\mathrm{rng}\, c)$. Then $\{c', c'$ ranges over elements of G-CycleSet: $\mathrm{rng}\, c' = \mathrm{rng}\, c \ \wedge \ \bigvee_{v_3} (v_3$ is vertex sequence of $c' \ \wedge \ v_3(1) = v)\}$ is a non empty subset of G-CycleSet.

Let us consider G, v and let c be an element of G-CycleSet. Let us assume that $v \in G$-VSet(rng c). The functor $c^{v}_{\circlearrowleft}$ yields an element of G-CycleSet and is defined as follows:

(Def. 9) $c^{v}_{\circlearrowleft} = \mathrm{choose}(\{c', c'$ ranges over elements of G-CycleSet: $\mathrm{rng}\, c' = \mathrm{rng}\, c \ \wedge$ $\bigvee_{v_3}$ (v_3 is vertex sequence of $c' \ \wedge \ v_3(1) = v)\})$.

Let G be a graph and let c_1, c_2 be elements of G-CycleSet. Let us assume that G-VSet(rng c_1) meets G-VSet(rng c_2) and rng c_1 misses rng c_2. The functor CatCycles(c_1, c_2) yields an element of G-CycleSet and is defined as follows:

(Def. 10) There exists a vertex v of G such that $v = \mathrm{choose}((G\text{-VSet}(\mathrm{rng}\, c_1)) \cap (G\text{-VSet}(\mathrm{rng}\, c_2)))$ and $\mathrm{CatCycles}(c_1, c_2) = ({c_1}^{v}_{\circlearrowleft}) \frown {c_2}^{v}_{\circlearrowleft}$.

The following proposition is true

(58) Let G be a graph and c_1, c_2 be elements of G-CycleSet. Suppose G-VSet(rng c_1) meets G-VSet(rng c_2) but rng c_1 misses rng c_2 but $c_1 \neq \varepsilon$ or $c_2 \neq \varepsilon$. Then CatCycles(c_1, c_2) is non empty.

In the sequel G denotes a finite graph, v denotes a vertex of G, and v_3 denotes a finite sequence of elements of the vertices of G.

Let us consider G, v and let X be a set. Let us assume that $\mathrm{Degree}(v, X) \neq 0$. The functor X-PathSet(v) yielding a non empty set of finite sequences of the edges of G is defined as follows:

(Def. 11) X-PathSet$(v) = \{c, c$ ranges over elements of X^*: c is a path of $G \wedge c$ is non empty $\wedge \bigvee_{v_3}$ (v_3 is vertex sequence of $c \ \wedge \ v_3(1) = v)\}$.

One can prove the following proposition

(59) For every element p of X-PathSet(v) and for every finite set Y such that $Y =$ the edges of G and $\mathrm{Degree}(v, X) \neq 0$ holds $\mathrm{len}\, p \leqslant \mathrm{card}\, Y$.

Let us consider G, v and let X be a set. Let us assume that for every vertex v_1 of G holds $\mathrm{Degree}(v_1, X)$ is even and $\mathrm{Degree}(v, X) \neq 0$. The functor X-CycleSet$v$ yielding a non empty subset of G-CycleSet is defined as follows:

(Def. 12) X-CycleSet$v = \{c, c$ ranges over elements of G-CycleSet: $\mathrm{rng}\, c \subseteq X \wedge c$ is non empty $\wedge \bigvee_{v_3}$ (v_3 is vertex sequence of $c \ \wedge \ v_3(1) = v)\}$.

Next we state two propositions:

(60) If $\mathrm{Degree}(v, X) \neq 0$ and for every v holds $\mathrm{Degree}(v, X)$ is even, then for every element c of X-CycleSet$v$ holds c is non empty and $\mathrm{rng}\, c \subseteq X$ and $v \in G$-VSet(rng c).

(61) Let G be a finite connected graph and c be an element of G-CycleSet. Suppose $\mathrm{rng}\, c \neq$ the edges of G and c is non empty. Then $\{v', v'$ ranges over vertices of G: $v' \in G$-VSet(rng c) $\wedge$ the degree of $v' \neq \mathrm{Degree}(v', \mathrm{rng}\, c)\}$ is a non empty subset of the vertices of G.

Let G be a finite connected graph and let c be an element of G-CycleSet. Let us assume that $\mathrm{rng}\, c \neq$ the edges of G and c is non empty. The functor

ExtendCycle c yields an element of G-CycleSet and is defined by the condition (Def. 13).

(Def. 13) There exists an element c' of G-CycleSet and there exists a vertex v of G such that $v = \mathrm{choose}(\{v', v'$ ranges over vertices of G: $v' \in G\text{-VSet}(\mathrm{rng}\, c) \wedge$ the degree of $v' \neq \mathrm{Degree}(v', \mathrm{rng}\, c)\})$ and $c' = \mathrm{choose}(((\text{the edges of } G) \setminus \mathrm{rng}\, c)\text{-CycleSet}v)$ and $\mathrm{ExtendCycle}\, c = \mathrm{CatCycles}(c, c')$.

One can prove the following proposition

(62) Let G be a finite connected graph and c be an element of G-CycleSet. Suppose $\mathrm{rng}\, c \neq$ the edges of G and c is non empty and for every vertex v of G holds the degree of v is even. Then ExtendCycle c is non empty and $\mathrm{card}\,\mathrm{rng}\, c < \mathrm{card}\,\mathrm{rng}\,\mathrm{ExtendCycle}\, c$.

5. Euler circuits and paths

Let G be a graph and let p be a path of G. We say that p is Eulerian if and only if:

(Def. 14) $\mathrm{rng}\, p =$ the edges of G.

We now state three propositions:

(63) Let G be a connected graph, p be a path of G, and v_3 be a finite sequence of elements of the vertices of G. Suppose p is Eulerian and v_3 is vertex sequence of p. Then $\mathrm{rng}\, v_3 =$ the vertices of G.

(64) Let G be a finite connected graph. Then there exists a cyclic path of G which is Eulerian if and only if for every vertex v of G holds the degree of v is even.

(65) Let G be a finite connected graph. Then there exists a path of G which is non cyclic and Eulerian if and only if there exist vertices v_1, v_2 of G such that $v_1 \neq v_2$ and for every vertex v of G holds the degree of v is even iff $v \neq v_1$ and $v \neq v_2$.

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Bounding Boxes for Compact Sets in $\mathcal{E}^2$

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Summary. We define pseudocompact topological spaces and prove that every compact space is pseudocompact. We also solve an exercise from [16] p.225 that the for a topological space X the following are equivalent:

- Every continuous real map from X is bounded (i.e. X is pseudocompact).
- Every continuous real map from X attains minimum.
- Every continuous real map from X attains maximum.

Finally, for a compact set in E^2 we define its bounding rectangle and introduce a collection of notions associated with the box.

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The papers [25], [30], [19], [7], [29], [24], [18], [17], [27], [21], [23], [10], [1], [26], [31], [3], [4], [14], [12], [13], [11], [22], [15], [20], [6], [5], [2], [8], [9], and [28] provide the notation and terminology for this paper.

1. Preliminaries

Let X be a set. Let us observe that X has non empty elements if and only if:

(Def. 1) $0 \notin X$.

We introduce X is without zero as a synonym of X has non empty elements. We introduce X has zero as an antonym of X has non empty elements.

Let us observe that $\mathbb{R}$ has zero and $\mathbb{N}$ has zero.

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Let us observe that there exists a set which is non empty and without zero and there exists a set which is non empty and has zero.

Let us observe that there exists a subset of $\mathbb{R}$ which is non empty and without zero and there exists a subset of $\mathbb{R}$ which is non empty and has zero.

Next we state the proposition

(1) For every set F such that F is non empty and $\subseteq$-linear and has non empty elements holds F is centered.

Let F be a set. Note that every family of subsets of F which is non empty and $\subseteq$-linear and has non empty elements is also centered.

Let A, B be sets and let f be a function from A into B. Then $\operatorname{rng} f$ is a subset of B.

Let X, Y be non empty sets and let f be a function from X into Y. Note that $f^\circ X$ is non empty.

Let X, Y be sets and let f be a function from X into Y. The functor ${}^{-1}f$ yields a function from 2^Y into 2^X and is defined by:

(Def. 2) For every subset y of Y holds $({}^{-1}f)(y) = f^{-1}(y)$.

We now state the proposition

(2) Let X, Y, x be sets, S be a subset of 2^Y, and f be a function from X into Y. If $x \in \bigcap(({}^{-1}f)^\circ S)$, then $f(x) \in \bigcap S$.

We follow the rules: p, q, r, r_1, r_2, s, t are real numbers, s_1 is a sequence of real numbers, and X, Y are subsets of $\mathbb{R}$.

One can prove the following propositions:

(3) If $|r| + |s| = 0$, then $r = 0$ and $s = 0$.

(4) If $r < s$ and $s < t$, then $|s| < |r| + |t|$.

(5) If $-s < r$ and $r < s$, then $|r| < s$.

(6) If s_1 is convergent and non-zero and $\lim s_1 = 0$, then ${s_1}^{-1}$ is non bounded.

(7) $\operatorname{rng} s_1$ is bounded iff s_1 is bounded.

Next we state four propositions:

(8) Let X be a non empty subset of $\mathbb{R}$ and given r. Suppose X is lower bounded. Then $r = \inf X$ if and only if the following conditions are satisfied:

(i) for every p such that $p \in X$ holds $p \geqslant r$, and

(ii) for every q such that for every p such that $p \in X$ holds $p \geqslant q$ holds $r \geqslant q$.

(9) Let X be a non empty subset of $\mathbb{R}$ and given r. Suppose X is upper bounded. Then $r = \sup X$ if and only if the following conditions are satisfied:

(i) for every p such that $p \in X$ holds $p \leqslant r$, and

(ii) for every q such that for every p such that $p \in X$ holds $p \leqslant q$ holds $r \leqslant q$.

(10) For every non empty subset X of $\mathbb{R}$ and for every subset Y of $\mathbb{R}$ such that $X \subseteq Y$ and Y is lower bounded holds $\inf Y \leqslant \inf X$.

(11) For every non empty subset X of $\mathbb{R}$ and for every subset Y of $\mathbb{R}$ such that $X \subseteq Y$ and Y is upper bounded holds $\sup X \leqslant \sup Y$.

Let X be a subset of $\mathbb{R}$. We say that X has maximum if and only if:

(Def. 3) X is upper bounded and $\sup X \in X$.

We say that X has minimum if and only if:

(Def. 4) X is lower bounded and $\inf X \in X$.

One can verify that there exists a subset of $\mathbb{R}$ which is non empty, closed, and bounded.

Let R be a family of subsets of $\mathbb{R}$. We say that R is open if and only if:

(Def. 5) For every subset X of $\mathbb{R}$ such that $X \in R$ holds X is open.

We say that R is closed if and only if:

(Def. 6) For every subset X of $\mathbb{R}$ such that $X \in R$ holds X is closed.

Let X be a subset of $\mathbb{R}$. The functor $-X$ yielding a subset of $\mathbb{R}$ is defined by:

(Def. 7) $-X = \{-r : r \in X\}$.

Next we state the proposition

(12) $r \in X$ iff $-r \in -X$.

Let X be a non empty subset of $\mathbb{R}$. One can check that $-X$ is non empty.

One can prove the following propositions:

(13) $--X = X$.

(14) X is upper bounded iff $-X$ is lower bounded.

(15) X is lower bounded iff $-X$ is upper bounded.

(16) For every non empty subset X of $\mathbb{R}$ such that X is lower bounded holds $\inf X = -\sup(-X)$.

(17) For every non empty subset X of $\mathbb{R}$ such that X is upper bounded holds $\sup X = -\inf(-X)$.

(18) X is closed iff $-X$ is closed.

Let X be a subset of $\mathbb{R}$ and let p be a real number. The functor $p+X$ yields a subset of $\mathbb{R}$ and is defined by:

(Def. 8) $p + X = \{p + r : r \in X\}$.

One can prove the following proposition

(19) $r \in X$ iff $s + r \in s + X$.

Let X be a non empty subset of $\mathbb{R}$ and let s be a real number. Observe that $s + X$ is non empty.

One can prove the following propositions:

(20) $X = 0 + X$.

(21) $s + (t + X) = (s + t) + X$.

(22) X is upper bounded iff $s + X$ is upper bounded.

(23) X is lower bounded iff $s + X$ is lower bounded.

(24) For every non empty subset X of $\mathbb{R}$ such that X is lower bounded holds $\inf(s + X) = s + \inf X$.

(25) For every non empty subset X of $\mathbb{R}$ such that X is upper bounded holds $\sup(s + X) = s + \sup X$.

(26) X is closed iff $s + X$ is closed.

Let X be a subset of $\mathbb{R}$. The functor $\operatorname{Inv} X$ yielding a subset of $\mathbb{R}$ is defined by:

(Def. 9) $\operatorname{Inv} X = \{\frac{1}{r} : r \in X\}$.

The following proposition is true

(27) For every without zero subset X of $\mathbb{R}$ such that $r \neq 0$ holds $r \in X$ iff $\frac{1}{r} \in \operatorname{Inv} X$.

Let X be a non empty without zero subset of $\mathbb{R}$. One can verify that $\operatorname{Inv} X$ is non empty and without zero.

Let X be a without zero subset of $\mathbb{R}$. One can verify that $\operatorname{Inv} X$ is without zero.

The following propositions are true:

(28) For every without zero subset X of $\mathbb{R}$ holds $\operatorname{Inv} \operatorname{Inv} X = X$.

(29) For every without zero subset X of $\mathbb{R}$ such that X is closed and bounded holds $\operatorname{Inv} X$ is closed.

(30) For every family Z of subsets of $\mathbb{R}$ such that Z is closed holds $\bigcap Z$ is closed.

Let X be a subset of $\mathbb{R}$. The functor $\overline{X}$ yielding a subset of $\mathbb{R}$ is defined by:

(Def. 10) $\overline{X} = \bigcap\{A, A$ ranges over elements of $2^{\mathbb{R}}$: $X \subseteq A \ \wedge \ A$ is closed$\}$.

Let X be a subset of $\mathbb{R}$. Observe that $\overline{X}$ is closed.

Next we state several propositions:

(31) For every closed subset Y of $\mathbb{R}$ such that $X \subseteq Y$ holds $\overline{X} \subseteq Y$.

(32) $X \subseteq \overline{X}$.

(33) X is closed iff $X = \overline{X}$.

(34) $\overline{\emptyset_{\mathbb{R}}} = \emptyset$.

(35) $\overline{\Omega_{\mathbb{R}}} = \mathbb{R}$.

(36) $\overline{X} = \overline{\overline{X}}$.

(37) If $X \subseteq Y$, then $\overline{X} \subseteq \overline{Y}$.

(38) $r \in \overline{X}$ iff for every open subset O of $\mathbb{R}$ such that $r \in O$ holds $O \cap X$ is non empty.

(39) If $r \in \overline{X}$, then there exists s_1 such that $\operatorname{rng} s_1 \subseteq X$ and s_1 is convergent and $\lim s_1 = r$.

2. Functions into Reals

Let A be a set, let f be a function from A into $\mathbb{R}$, and let a be a set. Then $f(a)$ is a real number.

Let X be a set and let f be a function from X into $\mathbb{R}$. We say that f is lower bounded if and only if:

(Def. 11) $f^\circ X$ is lower bounded.

We say that f is upper bounded if and only if:

(Def. 12) $f^\circ X$ is upper bounded.

Let X be a set and let f be a function from X into $\mathbb{R}$. We say that f is bounded if and only if:

(Def. 13) f is lower bounded and upper bounded.

We say that f has maximum if and only if:

(Def. 14) $f^\circ X$ has maximum.

We say that f has minimum if and only if:

(Def. 15) $f^\circ X$ has minimum.

Let X be a set. One can check that every function from X into $\mathbb{R}$ which is bounded is also lower bounded and upper bounded and every function from X into $\mathbb{R}$ which is lower bounded and upper bounded is also bounded.

Let X be a set and let f be a function from X into $\mathbb{R}$. The functor $-f$ yields a function from X into $\mathbb{R}$ and is defined as follows:

(Def. 16) For every set p such that $p \in X$ holds $(-f)(p) = -f(p)$.

The following propositions are true:

(40) For all sets X, A and for every function f from X into $\mathbb{R}$ holds $(-f)^\circ A = -f^\circ A$.

(41) For every set X and for every function f from X into $\mathbb{R}$ holds $--f = f$.

(42) For every non empty set X and for every function f from X into $\mathbb{R}$ holds f has minimum iff $-f$ has maximum.

(43) For every non empty set X and for every function f from X into $\mathbb{R}$ holds f has maximum iff $-f$ has minimum.

(44) For every set X and for every subset A of $\mathbb{R}$ and for every function f from X into $\mathbb{R}$ holds $(-f)^{-1}(A) = f^{-1}(-A)$.

Let X be a set, let r be a real number, and let f be a function from X into $\mathbb{R}$. The functor $r + f$ yielding a function from X into $\mathbb{R}$ is defined as follows:

(Def. 17) For every set p such that $p \in X$ holds $(r + f)(p) = r + f(p)$.

One can prove the following two propositions:

(45) For all sets X, A and for every function f from X into $\mathbb{R}$ and for every real number s holds $(s+f)^\circ A = s + f^\circ A$.

(46) For every set X and for every subset A of $\mathbb{R}$ and for every function f from X into $\mathbb{R}$ and for every s holds $(s+f)^{-1}(A) = f^{-1}(-s+A)$.

Let X be a set and let f be a function from X into $\mathbb{R}$. The functor $\operatorname{Inv} f$ yields a function from X into $\mathbb{R}$ and is defined by:

(Def. 18) For every set p such that $p \in X$ holds $(\operatorname{Inv} f)(p) = \frac{1}{f(p)}$.

We now state the proposition

(47) Let X be a set, A be a without zero subset of $\mathbb{R}$, and f be a function from X into $\mathbb{R}$. If $0 \notin \operatorname{rng} f$, then $(\operatorname{Inv} f)^{-1}(A) = f^{-1}(\operatorname{Inv} A)$.

3. Real maps

Let T be a 1-sorted structure.

(Def. 19) A function from the carrier of T into $\mathbb{R}$ is called a real map of T.

Let T be a non empty 1-sorted structure. Note that there exists a real map of T which is bounded.

In this article we present several logical schemes. The scheme *NonUniqExRF* deals with a non empty topological structure $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

> There exists a real map f of $\mathcal{A}$ such that for every element x of the carrier of $\mathcal{A}$ holds $\mathcal{P}[x, f(x)]$

provided the parameters meet the following requirement:

- For every set x such that $x \in$ the carrier of $\mathcal{A}$ there exists r such that $\mathcal{P}[x, r]$.

The scheme *LambdaRF* deals with a non empty topological structure $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a real number, and states that:

> There exists a real map f of $\mathcal{A}$ such that for every element x of the carrier of $\mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

for all values of the parameters.

Let T be a 1-sorted structure, let f be a real map of T, and let P be a set. Then $f^{-1}(P)$ is a subset of T.

Let T be a 1-sorted structure and let f be a real map of T. The functor $\inf f$ yielding a real number is defined by:

(Def. 20) $\inf f = \inf(f^\circ(\text{the carrier of } T))$.

The functor $\sup f$ yields a real number and is defined by:

(Def. 21) $\sup f = \sup(f^\circ(\text{the carrier of } T))$.

Next we state three propositions:

(48) Let T be a non empty topological space and f be a lower bounded real map of T. Then $r = \inf f$ if and only if the following conditions are satisfied:

(i) for every point p of T holds $f(p) \geqslant r$, and

(ii) for every real number q such that for every point p of T holds $f(p) \geqslant q$ holds $r \geqslant q$.

(49) Let T be a non empty topological space and f be an upper bounded real map of T. Then $r = \sup f$ if and only if the following conditions are satisfied:

(i) for every point p of T holds $f(p) \leqslant r$, and

(ii) for every real number q such that for every point p of T holds $f(p) \leqslant q$ holds $r \leqslant q$.

(50) For every non empty 1-sorted structure T and for every bounded real map f of T holds $\inf f \leqslant \sup f$.

Let T be a 1-sorted structure and let f be a real map of T. The functor $-f$ yielding a real map of T is defined by:

(Def. 22) $-f = -f$.

Let T be a 1-sorted structure, let r be a real number, and let f be a real map of T. The functor $r + f$ yields a real map of T and is defined by:

(Def. 23) $r + f = r + f$.

Let T be a 1-sorted structure and let f be a real map of T. The functor $\operatorname{Inv} f$ yields a real map of T and is defined by:

(Def. 24) $\operatorname{Inv} f = \operatorname{Inv} f$.

Let T be a topological structure and let f be a real map of T. We say that f is continuous if and only if:

(Def. 25) For every subset Y of $\mathbb{R}$ such that Y is closed holds $f^{-1}(Y)$ is closed.

Let T be a non empty topological space. Note that there exists a real map of T which is continuous.

Let T be a non empty topological space and let S be a non empty subspace of T. One can check that there exists a real map of S which is continuous.

In the sequel T is a topological space and f is a real map of T.

Next we state several propositions:

(51) f is continuous iff for every subset Y of $\mathbb{R}$ such that Y is open holds $f^{-1}(Y)$ is open.

(52) If f is continuous, then $-f$ is continuous.

(53) If f is continuous, then $r + f$ is continuous.

(54) If f is continuous and $0 \notin \operatorname{rng} f$, then $\operatorname{Inv} f$ is continuous.

(55) For every family R of subsets of $\mathbb{R}$ such that f is continuous and R is open holds $({}^{-1}f)^{\circ}R$ is open.

(56) For every family R of subsets of $\mathbb{R}$ such that f is continuous and R is closed holds $({}^{-1}f)^{\circ}R$ is closed.

Let T be a non empty topological space, let X be a subset of T, and let f be a real map of T. The functor $f \restriction X$ yielding a real map of $T \restriction X$ is defined as follows:

(Def. 26) $f \restriction X = f \restriction X$.

Let T be a non empty topological space. One can check that there exists a subset of T which is compact and non empty.

Let T be a non empty topological space, let f be a continuous real map of T, and let X be a compact non empty subset of T. Note that $f \restriction X$ is continuous.

Let T be a non empty topological space and let P be a compact non empty subset of T. Note that $T \restriction P$ is compact.

4. Pseudocompact spaces

We now state two propositions:

(57) Let T be a non empty topological space. Then for every real map f of T such that f is continuous holds f has maximum if and only if for every real map f of T such that f is continuous holds f has minimum.

(58) Let T be a non empty topological space. Then for every real map f of T such that f is continuous holds f is bounded if and only if for every real map f of T such that f is continuous holds f has maximum.

Let T be a topological space. We say that T is pseudocompact if and only if:

(Def. 27) For every real map f of T such that f is continuous holds f is bounded.

Let us mention that every non empty topological space which is compact is also pseudocompact.

Let us mention that there exists a topological space which is compact and non empty.

Let T be a pseudocompact non empty topological space. One can check that every real map of T which is continuous is also bounded and has maximum and minimum.

We now state two propositions:

(59) Let T be a non empty topological space, X, Y be non empty compact subsets of T, and f be a continuous real map of T. If $X \subseteq Y$, then $\inf(f \restriction Y) \leqslant \inf(f \restriction X)$.

(60) Let T be a non empty topological space, X, Y be non empty compact subsets of T, and f be a continuous real map of T. If $X \subseteq Y$, then $\sup(f \restriction X) \leqslant \sup(f \restriction Y)$.

5. Bounding boxes for compact sets in $\mathcal{E}^2$

Let n be a natural number and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. Note that $\mathcal{L}(p_1, p_2)$ is compact.

One can prove the following proposition

(61) For every natural number n and for all compact subsets X, Y of $\mathcal{E}^n_{\mathrm{T}}$ holds $X \cap Y$ is compact.

In the sequel p is a point of $\mathcal{E}^2_{\mathrm{T}}$, P is a subset of $\mathcal{E}^2_{\mathrm{T}}$, and X is a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$.

The real map proj1 of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 28) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\mathrm{proj1})(p) = p_{\mathbf{1}}$.

The real map proj2 of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 29) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\mathrm{proj2})(p) = p_{\mathbf{2}}$.

One can prove the following propositions:

(62) $(\mathrm{proj1})^{-1}(]r, s[) = \{[r_1, r_2] : r < r_1 \ \wedge \ r_1 < s\}$.

(63) For all r, s such that $P = \{[r_1, r_2] : r < r_1 \ \wedge \ r_1 < s\}$ holds P is open.

(64) $(\mathrm{proj2})^{-1}(]r, s[) = \{[r_1, r_2] : r < r_2 \ \wedge \ r_2 < s\}$.

(65) For all r, s such that $P = \{[r_1, r_2] : r < r_2 \ \wedge \ r_2 < s\}$ holds P is open.

One can verify that proj1 is continuous and proj2 is continuous.

One can prove the following two propositions:

(66) For every non empty subset X of $\mathcal{E}^2_{\mathrm{T}}$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in X$ holds $(\mathrm{proj1} \restriction X)(p) = p_{\mathbf{1}}$.

(67) For every non empty subset X of $\mathcal{E}^2_{\mathrm{T}}$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in X$ holds $(\mathrm{proj2} \restriction X)(p) = p_{\mathbf{2}}$.

Let X be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. The functor W-bound X yielding a real number is defined by:

(Def. 30) $\text{W-bound}\,X = \inf(\mathrm{proj1} \restriction X)$.

The functor N-bound X yielding a real number is defined as follows:

(Def. 31) $\text{N-bound}\,X = \sup(\mathrm{proj2} \restriction X)$.

The functor E-bound X yielding a real number is defined by:

(Def. 32) $\text{E-bound}\,X = \sup(\mathrm{proj1} \restriction X)$.

The functor S-bound X yielding a real number is defined by:

(Def. 33) $\text{S-bound}\,X = \inf(\mathrm{proj2} \restriction X)$.

We now state the proposition

(68) If $p \in X$, then W-bound $X \leqslant p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leqslant$ E-bound X and S-bound $X \leqslant p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leqslant$ N-bound X.

Let X be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. The functor SW-corner X yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 34) SW-corner $X = [$W-bound $X,$ S-bound $X]$.

The functor NW-corner X yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 35) NW-corner $X = [$W-bound $X,$ N-bound $X]$.

The functor NE-corner X yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 36) NE-corner $X = [$E-bound $X,$ N-bound $X]$.

The functor SE-corner X yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 37) SE-corner $X = [$E-bound $X,$ S-bound $X]$.

Let X be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. The functor W-most X yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 38) W-most $X = \mathcal{L}($SW-corner $X,$ NW-corner $X) \cap X$.

The functor N-most X yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 39) N-most $X = \mathcal{L}($NW-corner $X,$ NE-corner $X) \cap X$.

The functor E-most X yields a subset of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 40) E-most $X = \mathcal{L}($SE-corner $X,$ NE-corner $X) \cap X$.

The functor S-most X yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 41) S-most $X = \mathcal{L}($SW-corner $X,$ SE-corner $X) \cap X$.

Let X be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. One can check the following observations:

* W-most X is non empty and compact,
* N-most X is non empty and compact,
* E-most X is non empty and compact, and
* S-most X is non empty and compact.

Let X be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. The functor W-min X yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 42) W-min $X = [$W-bound $X, \inf(\mathrm{proj2} \restriction$ W-most $X)]$.

The functor W-max X yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 43) W-max $X = [$W-bound $X, \sup(\mathrm{proj2} \restriction$ W-most $X)]$.

The functor N-min X yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 44) N-min $X = [\inf(\mathrm{proj1} \restriction$ N-most $X),$ N-bound $X]$.

The functor N-max X yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 45) N-max $X = [\sup(\mathrm{proj1} \restriction$ N-most $X),$ N-bound $X]$.

The functor E-max X yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 46) $\text{E-max}\, X = [\text{E-bound}\, X, \sup(\text{proj2} \restriction \text{E-most}\, X)]$.

The functor $\text{E-min}\, X$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 47) $\text{E-min}\, X = [\text{E-bound}\, X, \inf(\text{proj2} \restriction \text{E-most}\, X)]$.

The functor $\text{S-max}\, X$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 48) $\text{S-max}\, X = [\sup(\text{proj1} \restriction \text{S-most}\, X), \text{S-bound}\, X]$.

The functor $\text{S-min}\, X$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 49) $\text{S-min}\, X = [\inf(\text{proj1} \restriction \text{S-most}\, X), \text{S-bound}\, X]$.

Next we state a number of propositions:

(69) $(\text{SW-corner}\, X)_{\mathbf{1}} = \text{W-bound}\, X$ and $(\text{W-min}\, X)_{\mathbf{1}} = \text{W-bound}\, X$ and $(\text{W-max}\, X)_{\mathbf{1}} = \text{W-bound}\, X$ and $(\text{NW-corner}\, X)_{\mathbf{1}} = \text{W-bound}\, X$.

(70) $(\text{SW-corner}\, X)_{\mathbf{1}} = (\text{NW-corner}\, X)_{\mathbf{1}}$ and $(\text{SW-corner}\, X)_{\mathbf{1}} = (\text{W-min}\, X)_{\mathbf{1}}$ and $(\text{SW-corner}\, X)_{\mathbf{1}} = (\text{W-max}\, X)_{\mathbf{1}}$ and $(\text{W-min}\, X)_{\mathbf{1}} = (\text{W-max}\, X)_{\mathbf{1}}$ and $(\text{W-min}\, X)_{\mathbf{1}} = (\text{NW-corner}\, X)_{\mathbf{1}}$ and $(\text{W-max}\, X)_{\mathbf{1}} = (\text{NW-corner}\, X)_{\mathbf{1}}$.

(71) $(\text{SW-corner}\, X)_{\mathbf{2}} = \text{S-bound}\, X$ and $(\text{W-min}\, X)_{\mathbf{2}} = \inf(\text{proj2} \restriction \text{W-most}\, X)$ and $(\text{W-max}\, X)_{\mathbf{2}} = \sup(\text{proj2} \restriction \text{W-most}\, X)$ and $(\text{NW-corner}\, X)_{\mathbf{2}} = \text{N-bound}\, X$.

(72) $(\text{SW-corner}\, X)_{\mathbf{2}} \leqslant (\text{W-min}\, X)_{\mathbf{2}}$ and $(\text{SW-corner}\, X)_{\mathbf{2}} \leqslant (\text{W-max}\, X)_{\mathbf{2}}$ and $(\text{SW-corner}\, X)_{\mathbf{2}} \leqslant (\text{NW-corner}\, X)_{\mathbf{2}}$ and $(\text{W-min}\, X)_{\mathbf{2}} \leqslant (\text{W-max}\, X)_{\mathbf{2}}$ and $(\text{W-min}\, X)_{\mathbf{2}} \leqslant (\text{NW-corner}\, X)_{\mathbf{2}}$ and $(\text{W-max}\, X)_{\mathbf{2}} \leqslant (\text{NW-corner}\, X)_{\mathbf{2}}$.

(73) If $p \in \text{W-most}\, X$, then $p_{\mathbf{1}} = (\text{W-min}\, X)_{\mathbf{1}}$ and $(\text{W-min}\, X)_{\mathbf{2}} \leqslant p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leqslant (\text{W-max}\, X)_{\mathbf{2}}$.

(74) $\text{W-most}\, X \subseteq \mathcal{L}(\text{W-min}\, X, \text{W-max}\, X)$.

(75) $\mathcal{L}(\text{W-min}\, X, \text{W-max}\, X) \subseteq \mathcal{L}(\text{SW-corner}\, X, \text{NW-corner}\, X)$.

(76) $\text{W-min}\, X \in \text{W-most}\, X$ and $\text{W-max}\, X \in \text{W-most}\, X$.

(77) $\mathcal{L}(\text{SW-corner}\, X, \text{W-min}\, X) \cap X = \{\text{W-min}\, X\}$ and $\mathcal{L}(\text{W-max}\, X, \text{NW-corner}\, X) \cap X = \{\text{W-max}\, X\}$.

(78) If $\text{W-min}\, X = \text{W-max}\, X$, then $\text{W-most}\, X = \{\text{W-min}\, X\}$.

(79) $(\text{NW-corner}\, X)_{\mathbf{2}} = \text{N-bound}\, X$ and $(\text{N-min}\, X)_{\mathbf{2}} = \text{N-bound}\, X$ and $(\text{N-max}\, X)_{\mathbf{2}} = \text{N-bound}\, X$ and $(\text{NE-corner}\, X)_{\mathbf{2}} = \text{N-bound}\, X$.

(80) $(\text{NW-corner}\, X)_{\mathbf{2}} = (\text{NE-corner}\, X)_{\mathbf{2}}$ and $(\text{NW-corner}\, X)_{\mathbf{2}} = (\text{N-min}\, X)_{\mathbf{2}}$ and $(\text{NW-corner}\, X)_{\mathbf{2}} = (\text{N-max}\, X)_{\mathbf{2}}$ and $(\text{N-min}\, X)_{\mathbf{2}} = (\text{N-max}\, X)_{\mathbf{2}}$ and $(\text{N-min}\, X)_{\mathbf{2}} = (\text{NE-corner}\, X)_{\mathbf{2}}$ and $(\text{N-max}\, X)_{\mathbf{2}} = (\text{NE-corner}\, X)_{\mathbf{2}}$.

(81) $(\text{NW-corner}\, X)_{\mathbf{1}} = \text{W-bound}\, X$ and $(\text{N-min}\, X)_{\mathbf{1}} = \inf(\text{proj1} \restriction \text{N-most}\, X)$ and $(\text{N-max}\, X)_{\mathbf{1}} = \sup(\text{proj1} \restriction \text{N-most}\, X)$ and $(\text{NE-corner}\, X)_{\mathbf{1}} = \text{E-bound}\, X$.

(82) $(\text{NW-corner}\, X)_{\mathbf{1}} \leqslant (\text{N-min}\, X)_{\mathbf{1}}$ and $(\text{NW-corner}\, X)_{\mathbf{1}} \leqslant (\text{N-max}\, X)_{\mathbf{1}}$ and $(\text{NW-corner}\, X)_{\mathbf{1}} \leqslant (\text{NE-corner}\, X)_{\mathbf{1}}$ and $(\text{N-min}\, X)_{\mathbf{1}} \leqslant (\text{N-max}\, X)_{\mathbf{1}}$ and $(\text{N-min}\, X)_{\mathbf{1}} \leqslant (\text{NE-corner}\, X)_{\mathbf{1}}$ and $(\text{N-max}\, X)_{\mathbf{1}} \leqslant (\text{NE-corner}\, X)_{\mathbf{1}}$.

(83) If $p \in$ N-most X, then $p_{\mathbf{2}} = (\text{N-min}\,X)_{\mathbf{2}}$ and $(\text{N-min}\,X)_{\mathbf{1}} \leqslant p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leqslant (\text{N-max}\,X)_{\mathbf{1}}$.

(84) N-most $X \subseteq \mathcal{L}(\text{N-min}\,X, \text{N-max}\,X)$.

(85) $\mathcal{L}(\text{N-min}\,X, \text{N-max}\,X) \subseteq \mathcal{L}(\text{NW-corner}\,X, \text{NE-corner}\,X)$.

(86) N-min $X \in$ N-most X and N-max $X \in$ N-most X.

(87) $\mathcal{L}(\text{NW-corner}\,X, \text{N-min}\,X) \cap X = \{\text{N-min}\,X\}$ and $\mathcal{L}(\text{N-max}\,X, \text{NE-corner}\,X) \cap X = \{\text{N-max}\,X\}$.

(88) If N-min $X =$ N-max X, then N-most $X = \{\text{N-min}\,X\}$.

(89) $(\text{SE-corner}\,X)_{\mathbf{1}} = \text{E-bound}\,X$ and $(\text{E-min}\,X)_{\mathbf{1}} = \text{E-bound}\,X$ and $(\text{E-max}\,X)_{\mathbf{1}} = \text{E-bound}\,X$ and $(\text{NE-corner}\,X)_{\mathbf{1}} = \text{E-bound}\,X$.

(90) $(\text{SE-corner}\,X)_{\mathbf{1}} = (\text{NE-corner}\,X)_{\mathbf{1}}$ and $(\text{SE-corner}\,X)_{\mathbf{1}} = (\text{E-min}\,X)_{\mathbf{1}}$ and $(\text{SE-corner}\,X)_{\mathbf{1}} = (\text{E-max}\,X)_{\mathbf{1}}$ and $(\text{E-min}\,X)_{\mathbf{1}} = (\text{E-max}\,X)_{\mathbf{1}}$ and $(\text{E-min}\,X)_{\mathbf{1}} = (\text{NE-corner}\,X)_{\mathbf{1}}$ and $(\text{E-max}\,X)_{\mathbf{1}} = (\text{NE-corner}\,X)_{\mathbf{1}}$.

(91) $(\text{SE-corner}\,X)_{\mathbf{2}} = \text{S-bound}\,X$ and $(\text{E-min}\,X)_{\mathbf{2}} = \inf(\text{proj2} \restriction \text{E-most}\,X)$ and $(\text{E-max}\,X)_{\mathbf{2}} = \sup(\text{proj2} \restriction \text{E-most}\,X)$ and $(\text{NE-corner}\,X)_{\mathbf{2}} = \text{N-bound}\,X$.

(92) $(\text{SE-corner}\,X)_{\mathbf{2}} \leqslant (\text{E-min}\,X)_{\mathbf{2}}$ and $(\text{SE-corner}\,X)_{\mathbf{2}} \leqslant (\text{E-max}\,X)_{\mathbf{2}}$ and $(\text{SE-corner}\,X)_{\mathbf{2}} \leqslant (\text{NE-corner}\,X)_{\mathbf{2}}$ and $(\text{E-min}\,X)_{\mathbf{2}} \leqslant (\text{E-max}\,X)_{\mathbf{2}}$ and $(\text{E-min}\,X)_{\mathbf{2}} \leqslant (\text{NE-corner}\,X)_{\mathbf{2}}$ and $(\text{E-max}\,X)_{\mathbf{2}} \leqslant (\text{NE-corner}\,X)_{\mathbf{2}}$.

(93) If $p \in$ E-most X, then $p_{\mathbf{1}} = (\text{E-min}\,X)_{\mathbf{1}}$ and $(\text{E-min}\,X)_{\mathbf{2}} \leqslant p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leqslant (\text{E-max}\,X)_{\mathbf{2}}$.

(94) E-most $X \subseteq \mathcal{L}(\text{E-min}\,X, \text{E-max}\,X)$.

(95) $\mathcal{L}(\text{E-min}\,X, \text{E-max}\,X) \subseteq \mathcal{L}(\text{SE-corner}\,X, \text{NE-corner}\,X)$.

(96) E-min $X \in$ E-most X and E-max $X \in$ E-most X.

(97) $\mathcal{L}(\text{SE-corner}\,X, \text{E-min}\,X) \cap X = \{\text{E-min}\,X\}$ and $\mathcal{L}(\text{E-max}\,X, \text{NE-corner}\,X) \cap X = \{\text{E-max}\,X\}$.

(98) If E-min $X =$ E-max X, then E-most $X = \{\text{E-min}\,X\}$.

(99) $(\text{SW-corner}\,X)_{\mathbf{2}} = \text{S-bound}\,X$ and $(\text{S-min}\,X)_{\mathbf{2}} = \text{S-bound}\,X$ and $(\text{S-max}\,X)_{\mathbf{2}} = \text{S-bound}\,X$ and $(\text{SE-corner}\,X)_{\mathbf{2}} = \text{S-bound}\,X$.

(100) $(\text{SW-corner}\,X)_{\mathbf{2}} = (\text{SE-corner}\,X)_{\mathbf{2}}$ and $(\text{SW-corner}\,X)_{\mathbf{2}} = (\text{S-min}\,X)_{\mathbf{2}}$ and $(\text{SW-corner}\,X)_{\mathbf{2}} = (\text{S-max}\,X)_{\mathbf{2}}$ and $(\text{S-min}\,X)_{\mathbf{2}} = (\text{S-max}\,X)_{\mathbf{2}}$ and $(\text{S-min}\,X)_{\mathbf{2}} = (\text{SE-corner}\,X)_{\mathbf{2}}$ and $(\text{S-max}\,X)_{\mathbf{2}} = (\text{SE-corner}\,X)_{\mathbf{2}}$.

(101) $(\text{SW-corner}\,X)_{\mathbf{1}} = \text{W-bound}\,X$ and $(\text{S-min}\,X)_{\mathbf{1}} = \inf(\text{proj1} \restriction \text{S-most}\,X)$ and $(\text{S-max}\,X)_{\mathbf{1}} = \sup(\text{proj1} \restriction \text{S-most}\,X)$ and $(\text{SE-corner}\,X)_{\mathbf{1}} = \text{E-bound}\,X$.

(102) $(\text{SW-corner}\,X)_{\mathbf{1}} \leqslant (\text{S-min}\,X)_{\mathbf{1}}$ and $(\text{SW-corner}\,X)_{\mathbf{1}} \leqslant (\text{S-max}\,X)_{\mathbf{1}}$ and $(\text{SW-corner}\,X)_{\mathbf{1}} \leqslant (\text{SE-corner}\,X)_{\mathbf{1}}$ and $(\text{S-min}\,X)_{\mathbf{1}} \leqslant (\text{S-max}\,X)_{\mathbf{1}}$ and $(\text{S-min}\,X)_{\mathbf{1}} \leqslant (\text{SE-corner}\,X)_{\mathbf{1}}$ and $(\text{S-max}\,X)_{\mathbf{1}} \leqslant (\text{SE-corner}\,X)_{\mathbf{1}}$.

(103) If $p \in \text{S-most}\,X$, then $p_{\mathbf{2}} = (\text{S-min}\,X)_{\mathbf{2}}$ and $(\text{S-min}\,X)_{\mathbf{1}} \leqslant p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leqslant (\text{S-max}\,X)_{\mathbf{1}}$.

(104) $\text{S-most}\,X \subseteq \mathcal{L}(\text{S-min}\,X, \text{S-max}\,X)$.

(105) $\mathcal{L}(\text{S-min}\,X, \text{S-max}\,X) \subseteq \mathcal{L}(\text{SW-corner}\,X, \text{SE-corner}\,X)$.

(106) $\text{S-min}\,X \in \text{S-most}\,X$ and $\text{S-max}\,X \in \text{S-most}\,X$.

(107) $\mathcal{L}(\text{SW-corner}\,X, \text{S-min}\,X) \cap X = \{\text{S-min}\,X\}$ and $\mathcal{L}(\text{S-max}\,X, \text{SE-corner}\,X) \cap X = \{\text{S-max}\,X\}$.

(108) If $\text{S-min}\,X = \text{S-max}\,X$, then $\text{S-most}\,X = \{\text{S-min}\,X\}$.

(109) If $\text{W-max}\,X = \text{N-min}\,X$, then $\text{W-max}\,X = \text{NW-corner}\,X$.

(110) If $\text{N-max}\,X = \text{E-max}\,X$, then $\text{N-max}\,X = \text{NE-corner}\,X$.

(111) If $\text{E-min}\,X = \text{S-max}\,X$, then $\text{E-min}\,X = \text{SE-corner}\,X$.

(112) If $\text{S-min}\,X = \text{W-min}\,X$, then $\text{S-min}\,X = \text{SW-corner}\,X$.

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The Scott Topology. Part II[1]

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Summary. Mizar formalization of pp. 105–108 of [15] which continues [34]. We found a simplification for the proof of Corollary 1.15, in the last case, see the proof in the Mizar article for details.

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The terminology and notation used in this paper are introduced in the following articles: [30], [37], [10], [2], [25], [14], [29], [38], [8], [9], [35], [3], [1], [36], [27], [39], [13], [26], [31], [17], [28], [18], [12], [4], [16], [41], [19], [20], [33], [6], [32], [5], [11], [21], [7], [40], [23], [24], [22], and [34].

1. Preliminaries

The following propositions are true:

(1) Let X be a set and F be a finite family of subsets of X. Then there exists a finite family G of subsets of X such that $G \subseteq F$ and $\bigcup G = \bigcup F$ and for every subset g of X such that $g \in G$ holds $g \not\subseteq \bigcup(G \setminus \{g\})$.

(2) Let S be a 1-sorted structure and X be a subset of the carrier of S. Then $-X =$ the carrier of S if and only if X is empty.

(3) Let R be an antisymmetric transitive non empty relational structure with g.l.b.'s and x, y be elements of R. Then $\downarrow(x \sqcap y) = \downarrow x \cap \downarrow y$.

(4) Let R be an antisymmetric transitive non empty relational structure with l.u.b.'s and x, y be elements of R. Then $\uparrow(x \sqcup y) = \uparrow x \cap \uparrow y$.

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(5) Let L be a complete antisymmetric non empty relational structure and X be a lower subset of L. If $\sup X \in X$, then $X = {\downarrow}\sup X$.

(6) Let L be a complete antisymmetric non empty relational structure and X be an upper subset of L. If $\inf X \in X$, then $X = {\uparrow}\inf X$.

(7) Let R be a non empty reflexive transitive relational structure and x, y be elements of R. Then $x \ll y$ if and only if ${\uparrow}y \subseteq$ ↟x.

(8) Let R be a non empty reflexive transitive relational structure and x, y be elements of R. Then $x \ll y$ if and only if ${\downarrow}x \subseteq$ ↡y.

(9) Let R be a complete reflexive antisymmetric non empty relational structure and x be an element of R. Then $\sup$ ↡$x \leqslant x$ and $x \leqslant \inf$ ↟x.

(10) For every lower-bounded antisymmetric non empty relational structure L holds ${\uparrow}(\bot_L) =$ the carrier of L.

(11) For every upper-bounded antisymmetric non empty relational structure L holds ${\downarrow}(\top_L) =$ the carrier of L.

(12) For every poset P with l.u.b.'s and for all elements x, y of P holds ↟$x \sqcup$ ↟$y \subseteq {\uparrow}(x \sqcup y)$.

(13) For every poset P with g.l.b.'s and for all elements x, y of P holds ↡$x \sqcap$ ↡$y \subseteq {\downarrow}(x \sqcap y)$.

(14) Let R be a non empty poset with l.u.b.'s and l be an element of R. Then l is co-prime if and only if for all elements x, y of R such that $l \leqslant x \sqcup y$ holds $l \leqslant x$ or $l \leqslant y$.

(15) For every complete non empty poset P and for every non empty subset V of P holds ${\downarrow}\inf V = \bigcap\{{\downarrow}u, u$ ranges over elements of P: $u \in V\}$.

(16) For every complete non empty poset P and for every non empty subset V of P holds ${\uparrow}\sup V = \bigcap\{{\uparrow}u, u$ ranges over elements of P: $u \in V\}$.

Let L be a sup-semilattice and let x be an element of L.

Note that compactbelow(x) is directed.

We now state four propositions:

(17) Let T be a non empty topological space, S be an irreducible subset of T, and V be an element of $\langle$the topology of T, $\subseteq\rangle$. If $V = -S$, then V is prime.

(18) Let T be a non empty topological space and x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. Then $x \sqcup y = x \cup y$ and $x \sqcap y = x \cap y$.

(19) Let T be a non empty topological space and V be an element of $\langle$the topology of T, $\subseteq\rangle$. Then V is prime if and only if for all elements X, Y of $\langle$the topology of T, $\subseteq\rangle$ such that $X \cap Y \subseteq V$ holds $X \subseteq V$ or $Y \subseteq V$.

(20) Let T be a non empty topological space and V be an element of $\langle$the topology of T, $\subseteq\rangle$. Then V is co-prime if and only if for all elements X, Y of $\langle$the topology of T, $\subseteq\rangle$ such that $V \subseteq X \cup Y$ holds $V \subseteq X$ or $V \subseteq Y$.

Let T be a non empty topological space. One can check that $\langle$the topology of T, $\subseteq\rangle$ is distributive.

The following propositions are true:

(21) Let T be a non empty topological space, L be a TopLattice, t be a point of T, l be a point of L, and X be a family of subsets of the carrier of L. Suppose the topological structure of T = the topological structure of L and $t = l$ and X is a basis of l. Then X is a basis of t.

(22) Let L be a TopLattice and x be an element of L. Suppose that for every subset X of L such that X is open holds X is upper. Then $\uparrow x$ is compact.

2. The Scott topology[2]

For simplicity, we use the following convention: L is a complete Scott TopLattice, x is an element of L, X, Y are subsets of L, V, W are elements of $\langle \sigma(L), \subseteq \rangle$, and V_1 is a subset of $\langle \sigma(L), \subseteq \rangle$.

Let L be a complete lattice. One can check that $\sigma(L)$ is non empty.

The following four propositions are true:

(23) $\sigma(L) =$ the topology of L.

(24) $X \in \sigma(L)$ iff X is open.

(25) For every filtered subset X of L such that $V_1 = \{-\downarrow x : x \in X\}$ holds V_1 is directed.

(26) If X is open and $x \in X$, then $\inf X \ll x$.

Let R be a non empty reflexive relational structure and let f be a map from $[\!: R, R :\!]$ into R. We say that f is jointly Scott-continuous if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let T be a non empty topological space. Suppose the topological structure of T = ConvergenceSpace(the Scott convergence of R). Then there exists a map f_1 from $[\!: T, T :\!]$ into T such that $f_1 = f$ and f_1 is continuous.

One can prove the following propositions:

(27) If $V = X$, then V is co-prime iff X is filtered and upper.

(28) If $V = X$ and there exists x such that $X = -\downarrow x$, then V is prime and $V \neq$ the carrier of L.

(29) If $V = X$ and $\sqcup_L$ is jointly Scott-continuous and V is prime and $V \neq$ the carrier of L, then there exists x such that $X = -\downarrow x$.

(30) If L is continuous, then $\sqcup_L$ is jointly Scott-continuous.

(31) If $\sqcup_L$ is jointly Scott-continuous, then L is sober.

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[2] $\sigma(L) =$ sigma L, as defined in [34, p. 316, Def. 12] and $\sqcup_L =$ sup_op(L), as defined in [21, p. 163, Def. 5].

(32) If L is continuous, then L is compact, locally-compact, sober, and Baire.

(33) If L is continuous and $X \in \sigma(L)$, then $X = \bigcup\{\uparrow x : x \in X\}$.

(34) If for every X such that $X \in \sigma(L)$ holds $X = \bigcup\{\uparrow x : x \in X\}$, then L is continuous.

(35) If L is continuous, then there exists a basis B of x such that for every X such that $X \in B$ holds X is open and filtered.

(36) If L is continuous, then $\langle \sigma(L), \subseteq \rangle$ is continuous.

(37) Suppose for every x there exists a basis B of x such that for every Y such that $Y \in B$ holds Y is open and filtered and $\langle \sigma(L), \subseteq \rangle$ is continuous. Then $x = \bigsqcup_L \{\inf X : x \in X \ \wedge \ X \in \sigma(L)\}$.

(38) If for every x holds $x = \bigsqcup_L \{\inf X : x \in X \ \wedge \ X \in \sigma(L)\}$, then L is continuous.

(39) The following statements are equivalent

(i) for every x there exists a basis B of x such that for every Y such that $Y \in B$ holds Y is open and filtered,

(ii) for every V there exists V_1 such that $V = \sup V_1$ and for every W such that $W \in V_1$ holds W is co-prime.

(40) For every V there exists V_1 such that $V = \sup V_1$ and for every W such that $W \in V_1$ holds W is co-prime and $\langle \sigma(L), \subseteq \rangle$ is continuous if and only if $\langle \sigma(L), \subseteq \rangle$ is completely-distributive.

(41) $\langle \sigma(L), \subseteq \rangle$ is completely-distributive iff $\langle \sigma(L), \subseteq \rangle$ is continuous and $(\langle \sigma(L), \subseteq \rangle)^{\rm op}$ is continuous.

(42) If L is algebraic, then there exists a basis B of L such that $B = \{\uparrow x : x \in$ the carrier of CompactSublatt$(L)\}$.

(43) Given a basis B of L such that $B = \{\uparrow x : x \in$ the carrier of CompactSublatt$(L)\}$. Then $\langle \sigma(L), \subseteq \rangle$ is algebraic and for every V there exists V_1 such that $V = \sup V_1$ and for every W such that $W \in V_1$ holds W is co-prime.

(44) Suppose $\langle \sigma(L), \subseteq \rangle$ is algebraic and for every V there exists V_1 such that $V = \sup V_1$ and for every W such that $W \in V_1$ holds W is co-prime. Then there exists a basis B of L such that $B = \{\uparrow x : x \in$ the carrier of CompactSublatt$(L)\}$.

(45) If there exists a basis B of L such that $B = \{\uparrow x : x \in$ the carrier of CompactSublatt$(L)\}$, then L is algebraic.

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