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Ideals

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Summary. The dual concept to filters (see [2,3]) i.e. ideals of a lattice is introduced.

MML Identifier: `FILTER_2`.

The articles [12], [14], [13], [4], [15], [6], [10], [9], [7], [5], [16], [8], [2], [11], [3], and [1] provide the notation and terminology for this paper.

1. Some Properties of the Restriction of Binary Operations

In this paper D is a non empty set.

We now state several propositions:

(1) Let D be a non empty set, and let S be a non empty subset of D, and let f be a binary operation on D, and let g be a binary operation on S. Suppose $g = f \restriction [\!:S, S:\!]$. Then

(i) if f is commutative, then g is commutative,

(ii) if f is idempotent, then g is idempotent, and

(iii) if f is associative, then g is associative.

(2) Let D be a non empty set, and let S be a non empty subset of D, and let f be a binary operation on D, and let g be a binary operation on S, and let d be an element of D, and let d' be an element of S. Suppose $g = f \restriction [\!:S, S:\!]$ and $d' = d$. Then

(i) if d is a left unity w.r.t. f, then d' is a left unity w.r.t. g,

(ii) if d is a right unity w.r.t. f, then d' is a right unity w.r.t. g, and

(iii) if d is a unity w.r.t. f, then d' is a unity w.r.t. g.

(3) Let D be a non empty set, and let S be a non empty subset of D, and let f_1, f_2 be binary operations on D, and let g_1, g_2 be binary operations on S. Suppose $g_1 = f_1 \restriction [\!:S, S:\!]$ and $g_2 = f_2 \restriction [\!:S, S:\!]$. Then

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(i) if f_1 is left distributive w.r.t. f_2, then g_1 is left distributive w.r.t. g_2, and

(ii) if f_1 is right distributive w.r.t. f_2, then g_1 is right distributive w.r.t. g_2.

(4) Let D be a non empty set, and let S be a non empty subset of D, and let f_1, f_2 be binary operations on D, and let g_1, g_2 be binary operations on S. Suppose $g_1 = f_1 \restriction [: S, S :]$ and $g_2 = f_2 \restriction [: S, S :]$. If f_1 is distributive w.r.t. f_2, then g_1 is distributive w.r.t. g_2.

(5) Let D be a non empty set, and let S be a non empty subset of D, and let f_1, f_2 be binary operations on D, and let g_1, g_2 be binary operations on S. If $g_1 = f_1 \restriction [: S, S :]$ and $g_2 = f_2 \restriction [: S, S :]$, then if f_1 absorbs f_2, then g_1 absorbs g_2.

2. Closed Subsets of a Lattice

Let D be a non empty set and let X_1, X_2 be subsets of D. Let us observe that $X_1 = X_2$ if and only if:

(Def.1) For every element x of D holds $x \in X_1$ iff $x \in X_2$.

For simplicity we follow the rules: L will denote a lattice, p, q, r will denote elements of the carrier of L, p', q' will denote elements of the carrier of L°, and x will be arbitrary.

Next we state several propositions:

(6) Let L_1, L_2 be lattice structures. Suppose the lattice structure of L_1 = the lattice structure of L_2. Then ${L_1}^\circ = {L_2}^\circ$.

(7) $(L^\circ)^\circ$ = the lattice structure of L.

(8) Let L_1, L_2 be non empty lattice structures. Suppose the lattice structure of L_1 = the lattice structure of L_2. Let a_1, b_1 be elements of the carrier of L_1 and let a_2, b_2 be elements of the carrier of L_2. Suppose $a_1 = a_2$ and $b_1 = b_2$. Then $a_1 \sqcup b_1 = a_2 \sqcup b_2$ and $a_1 \sqcap b_1 = a_2 \sqcap b_2$ and $a_1 \sqsubseteq b_1$ iff $a_2 \sqsubseteq b_2$.

(9) Let L_1, L_2 be lower bound lattices. Suppose the lattice structure of L_1 = the lattice structure of L_2. Then $\bot_{(L_1)} = \bot_{(L_2)}$.

(10) Let L_1, L_2 be upper bound lattices. Suppose the lattice structure of L_1 = the lattice structure of L_2. Then $\top_{(L_1)} = \top_{(L_2)}$.

(11) Let L_1, L_2 be complemented lattices. Suppose the lattice structure of L_1 = the lattice structure of L_2. Let a_1, b_1 be elements of the carrier of L_1 and let a_2, b_2 be elements of the carrier of L_2. If $a_1 = a_2$ and $b_1 = b_2$ and a_1 is a complement of b_1, then a_2 is a complement of b_2.

(12) Let L_1, L_2 be Boolean lattices. Suppose the lattice structure of L_1 = the lattice structure of L_2. Let a be an element of the carrier of L_1 and let b be an element of the carrier of L_2. If $a = b$, then $a^{\mathrm{c}} = b^{\mathrm{c}}$.

Let us consider L. A subset of the carrier of L is said to be a closed subset of L if:

(Def.2) For all p, q such that $p \in$ it and $q \in$ it holds $p \sqcap q \in$ it and $p \sqcup q \in$ it.

Let us consider L. Observe that there exists a closed subset of L which is non empty.

The following two propositions are true:

(13) Let X be a subset of the carrier of L. Suppose that for all p, q holds $p \in X$ and $q \in X$ iff $p \sqcap q \in X$. Then X is a closed subset of L.

(14) Let X be a subset of the carrier of L. Suppose that for all p, q holds $p \in X$ and $q \in X$ iff $p \sqcup q \in X$. Then X is a closed subset of L.

Let us consider L. Then $[L)$ is a filter of L. Let p be an element of the carrier of L. Then $[p)$ is a filter of L.

Let us consider L and let D be a non empty subset of the carrier of L. Then $[D)$ is a filter of L.

Let L be a distributive lattice and let F_1, F_2 be filters of L. Then $F_1 \sqcap F_2$ is a filter of L.

Let us consider L. A non empty closed subset of L is called an ideal of L if:

(Def.3) $p \in$ it and $q \in$ it iff $p \sqcup q \in$ it.

Next we state three propositions:

(15) Let X be a non empty subset of the carrier of L. Suppose that for all p, q holds $p \in X$ and $q \in X$ iff $p \sqcup q \in X$. Then X is an ideal of L.

(16) Let L_1, L_2 be lattices. Suppose the lattice structure of $L_1 =$ the lattice structure of L_2. Given x. If x is a filter of L_1, then x is a filter of L_2.

(17) Let L_1, L_2 be lattices. Suppose the lattice structure of $L_1 =$ the lattice structure of L_2. Given x. If x is an ideal of L_1, then x is an ideal of L_2.

Let us consider L, p. The functor p° yielding an element of the carrier of L° is defined by:

(Def.4) $p^\circ = p$.

Let us consider L and let p be an element of the carrier of L°. The functor ${}^\circ p$ yields an element of the carrier of L and is defined as follows:

(Def.5) ${}^\circ p = p$.

Next we state four propositions:

(18) ${}^\circ p^\circ = p$ and $({}^\circ p')^\circ = p'$.

(19) $p \sqcap q = p^\circ \sqcup q^\circ$ and $p \sqcup q = p^\circ \sqcap q^\circ$ and $p' \sqcap q' = {}^\circ p' \sqcup {}^\circ q'$ and $p' \sqcup q' = {}^\circ p' \sqcap {}^\circ q'$.

(20) $p \sqsubseteq q$ iff $q^\circ \sqsubseteq p^\circ$ and $p' \sqsubseteq q'$ iff ${}^\circ q' \sqsubseteq {}^\circ p'$.

(21) x is an ideal of L iff x is a filter of L°.

Let us consider L and let X be a subset of the carrier of L. The functor X° yielding a subset of the carrier of L° is defined as follows:

(Def.6) $X^\circ = X$.

Let us consider L and let X be a subset of the carrier of L°. The functor ${}^\circ X$ yielding a subset of the carrier of L is defined by:

(Def.7) $^\circ X = X$.

Let us consider L and let D be a non empty subset of the carrier of L. Observe that D° is non empty.

Let us consider L and let D be a non empty subset of the carrier of L°. Observe that $^\circ D$ is non empty.

Let us consider L and let S be a closed subset of L. Then S° is a closed subset of L°.

Let us consider L and let S be a non empty closed subset of L. Then S° is a non empty closed subset of L°.

Let us consider L and let S be a closed subset of L°. Then $^\circ S$ is a closed subset of L.

Let us consider L and let S be a non empty closed subset of L°. Then $^\circ S$ is a non empty closed subset of L.

Let us consider L and let F be a filter of L. Then F° is an ideal of L°.

Let us consider L and let F be a filter of L°. Then $^\circ F$ is an ideal of L.

Let us consider L and let I be an ideal of L. Then I° is a filter of L°.

Let us consider L and let I be an ideal of L°. Then $^\circ I$ is a filter of L.

We now state the proposition

(22) Let D be a non empty subset of the carrier of L. Then D is an ideal of L if and only if the following conditions are satisfied:

(i) for all p, q such that $p \in D$ and $q \in D$ holds $p \sqcup q \in D$, and

(ii) for all p, q such that $p \in D$ and $q \sqsubseteq p$ holds $q \in D$.

In the sequel I, J will be ideals of L and F will be a filter of L.

One can prove the following propositions:

(23) If $p \in I$, then $p \sqcap q \in I$ and $q \sqcap p \in I$.

(24) There exists p such that $p \in I$.

(25) If L is lower-bounded, then $\perp_L \in I$.

(26) If L is lower-bounded, then $\{\perp_L\}$ is an ideal of L.

(27) If $\{p\}$ is an ideal of L, then L is lower-bounded.

3. Ideals Generated by Subsets of a Lattice

Next we state the proposition

(28) The carrier of L is an ideal of L.

Let us consider L. The functor $(L]$ yielding an ideal of L is defined as follows:

(Def.8) $(L] =$ the carrier of L.

Let us consider L, p. The functor $(p]$ yields an ideal of L and is defined as follows:

(Def.9) $(p] = \{q : q \sqsubseteq p\}$.

We now state four propositions:

(29) $q \in (p]$ iff $q \sqsubseteq p$.

(30) $(p] = [p^\circ)$ and $(p^\circ] = [p)$.

(31) $p \in (p]$ and $p \sqcap q \in (p]$ and $q \sqcap p \in (p]$.

(32) If L is upper-bounded, then $(L] = (\top_L]$.

Let us consider L, I. We say that I is maximal if and only if:

(Def.10) $I \neq$ the carrier of L and for every J such that $I \subseteq J$ and $J \neq$ the carrier of L holds $I = J$.

One can prove the following four propositions:

(33) I is maximal iff I° is an ultrafilter.

(34) If L is upper-bounded, then for every I such that $I \neq$ the carrier of L there exists J such that $I \subseteq J$ and J is maximal.

(35) If there exists r such that $p \sqcup r \neq p$, then $(p] \neq$ the carrier of L.

(36) If L is upper-bounded and $p \neq \top_L$, then there exists I such that $p \in I$ and I is maximal.

In the sequel D denotes a non empty subset of the carrier of L and D' denotes a non empty subset of the carrier of L°.

Let us consider L, D. The functor $(D]$ yields an ideal of L and is defined as follows:

(Def.11) $D \subseteq (D]$ and for every I such that $D \subseteq I$ holds $(D] \subseteq I$.

We now state two propositions:

(37) $[D^\circ) = (D]$ and $[D) = (D^\circ]$ and $[{}^\circ D') = (D']$ and $[D') = ({}^\circ D']$.

(38) $(I] = I$.

In the sequel D_1, D_2 are non empty subsets of the carrier of L and D_1', D_2' are non empty subsets of the carrier of L°.

The following propositions are true:

(39) If $D_1 \subseteq D_2$, then $(D_1] \subseteq (D_2]$ and $((D]] \subseteq (D]$.

(40) If $p \in D$, then $(p] \subseteq (D]$.

(41) If $D = \{p\}$, then $(D] = (p]$.

(42) If L is upper-bounded and $\top_L \in D$, then $(D] = (L]$ and $(D] =$ the carrier of L.

(43) If L is upper-bounded and $\top_L \in I$, then $I = (L]$ and $I =$ the carrier of L.

Let us consider L, I. We say that I is prime if and only if:

(Def.12) $p \sqcap q \in I$ iff $p \in I$ or $q \in I$.

The following proposition is true

(44) I is prime iff I° is prime.

Let us consider L, D_1, D_2. The functor $D_1 \sqcup D_2$ yielding a non empty subset of the carrier of L is defined by:

(Def.13) $D_1 \sqcup D_2 = \{p \sqcup q : p \in D_1 \ \wedge \ q \in D_2\}$.

We now state four propositions:

(45) $D_1 \sqcup D_2 = {D_1}^\circ \sqcap {D_2}^\circ$ and ${D_1}^\circ \sqcup {D_2}^\circ = D_1 \sqcap D_2$ and $D_1' \sqcup D_2' = {}^\circ D_1' \sqcap {}^\circ D_2'$ and ${}^\circ D_1' \sqcup {}^\circ D_2' = D_1' \sqcap D_2'$.

(46) If $p \in D_1$ and $q \in D_2$, then $p \sqcup q \in D_1 \sqcup D_2$ and $q \sqcup p \in D_1 \sqcup D_2$.

(47) If $x \in D_1 \sqcup D_2$, then there exist p, q such that $x = p \sqcup q$ and $p \in D_1$ and $q \in D_2$.

(48) $D_1 \sqcup D_2 = D_2 \sqcup D_1$.

Let L be a distributive lattice and let I_1, I_2 be ideals of L. Then $I_1 \sqcup I_2$ is an ideal of L.

The following four propositions are true:

(49) $(D_1 \cup D_2] = ((D_1] \cup D_2]$ and $(D_1 \cup D_2] = (D_1 \cup (D_2]]$.

(50) $(I \cup J] = \{r : \bigvee_{p,q}\ r \sqsubseteq p \sqcup q\ \wedge\ p \in I\ \wedge\ q \in J\}$.

(51) $I \subseteq I \sqcup J$ and $J \subseteq I \sqcup J$.

(52) $(I \cup J] = (I \sqcup J]$.

We follow the rules: B denotes a Boolean lattice, I_3, J_1 denote ideals of B, and a, b denote elements of the carrier of B.

The following propositions are true:

(53) L is a complemented lattice iff L° is a complemented lattice.

(54) L is a Boolean lattice iff L° is a Boolean lattice.

Let B be a Boolean lattice. One can verify that B° is Boolean and lattice-like.

In the sequel a' will denote an element of the carrier of (B **qua** lattice)$^\circ$.

The following propositions are true:

(55) $(a^\circ)^{\mathrm{c}} = a^{\mathrm{c}}$ and $({}^\circ a')^{\mathrm{c}} = a'^{\mathrm{c}}$.

(56) $(I_3 \cup J_1] = I_3 \sqcup J_1$.

(57) I_3 is maximal iff $I_3 \neq$ the carrier of B and for every a holds $a \in I_3$ or $a^{\mathrm{c}} \in I_3$.

(58) $I_3 \neq (B]$ and I_3 is prime iff I_3 is maximal.

(59) If I_3 is maximal, then for every a holds $a \in I_3$ iff $a^{\mathrm{c}} \notin I_3$.

(60) If $a \neq b$, then there exists I_3 such that I_3 is maximal but $a \in I_3$ and $b \notin I_3$ or $a \notin I_3$ and $b \in I_3$.

In the sequel P denotes a non empty closed subset of L and o_1, o_2 denote binary operations on P.

One can prove the following two propositions:

(61) (i) (The join operation of L) $\restriction [\![P, P]\!]$ is a binary operation on P, and
(ii) (the meet operation of L) $\restriction [\![P, P]\!]$ is a binary operation on P.

(62) Suppose $o_1 =$ (the join operation of L) $\restriction [\![P, P]\!]$ and $o_2 =$ (the meet operation of L) $\restriction [\![P, P]\!]$. Then o_1 is commutative and associative and o_2 is commutative and associative and o_1 absorbs o_2 and o_2 absorbs o_1.

Let us consider L, p, q. Let us assume that $p \sqsubseteq q$. The functor $[p, q]$ yielding a non empty closed subset of L is defined by:

(Def.14) $[p, q] = \{r : p \sqsubseteq r\ \wedge\ r \sqsubseteq q\}$.

We now state several propositions:

(63) If $p \sqsubseteq q$, then $r \in [p, q]$ iff $p \sqsubseteq r$ and $r \sqsubseteq q$.

(64) If $p \sqsubseteq q$, then $p \in [p, q]$ and $q \in [p, q]$.

(65) $[p, p] = \{p\}$.

(66) If L is upper-bounded, then $[p) = [p, \top_L]$.

(67) If L is lower-bounded, then $(p] = [\bot_L, p]$.

(68) Let L_1, L_2 be lattices, and let F_1 be a filter of L_1, and let F_2 be a filter of L_2. Suppose the lattice structure of L_1 = the lattice structure of L_2 and $F_1 = F_2$. Then $\mathbb{L}_{(F_1)} = \mathbb{L}_{(F_2)}$.

4. Sublattices

Let us consider L. Let us note that the sublattice of L can be characterized by the following (equivalent) condition:

(Def.15) There exist P, o_1, o_2 such that
(i) o_1 = (the join operation of L) $\restriction [\![P, P]\!]$,
(ii) o_2 = (the meet operation of L) $\restriction [\![P, P]\!]$, and
(iii) the lattice structure of it $= \langle P, o_1, o_2 \rangle$.

The following proposition is true

(69) For every sublattice K of L holds every element of the carrier of K is an element of the carrier of L.

Let us consider L, P. The functor $\mathbb{L}^L_P$ yields a strict sublattice of L and is defined as follows:

(Def.16) There exist o_1, o_2 such that o_1 = (the join operation of L) $\restriction [\![P, P]\!]$ and o_2 = (the meet operation of L) $\restriction [\![P, P]\!]$ and $\mathbb{L}^L_P = \langle P, o_1, o_2 \rangle$.

Let us consider L and let l be a sublattice of L. Then l° is a strict sublattice of L°.

Next we state a number of propositions:

(70) $\mathbb{L}_F = \mathbb{L}^L_F$.

(71) $\mathbb{L}^L_P = (\mathbb{L}^{L^\circ}_{P^\circ})^\circ$.

(72) $\mathbb{L}^L_{(L]}$ = the lattice structure of L and $\mathbb{L}^L_{[L)}$ = the lattice structure of L.

(73) (i) The carrier of $\mathbb{L}^L_P = P$,
(ii) the join operation of $\mathbb{L}^L_P$ = (the join operation of L) $\restriction [\![P, P]\!]$, and
(iii) the meet operation of $\mathbb{L}^L_P$ = (the meet operation of L) $\restriction [\![P, P]\!]$.

(74) For all p, q and for all elements p', q' of the carrier of $\mathbb{L}^L_P$ such that $p = p'$ and $q = q'$ holds $p \sqcup q = p' \sqcup q'$ and $p \sqcap q = p' \sqcap q'$.

(75) For all p, q and for all elements p', q' of the carrier of $\mathbb{L}^L_P$ such that $p = p'$ and $q = q'$ holds $p \sqsubseteq q$ iff $p' \sqsubseteq q'$.

(76) If L is lower-bounded, then $\mathbb{L}^L_I$ is lower-bounded.

(77) If L is modular, then $\mathbb{L}^L_P$ is modular.

(78) If L is distributive, then $\mathbb{L}^L_P$ is distributive.

(79) If L is implicative and $p \sqsubseteq q$, then $\mathbb{L}^{L}_{[p,q]}$ is implicative.

(80) $\mathbb{L}^{L}_{(p]}$ is upper-bounded.

(81) $\top_{\mathbb{L}^{L}_{(p]}} = p$.

(82) If L is lower-bounded, then $\mathbb{L}^{L}_{(p]}$ is lower-bounded and $\perp_{\mathbb{L}^{L}_{(p]}} = \perp_{L}$.

(83) If L is lower-bounded, then $\mathbb{L}^{L}_{(p]}$ is bounded.

(84) If $p \sqsubseteq q$, then $\mathbb{L}^{L}_{[p,q]}$ is bounded and $\top_{\mathbb{L}^{L}_{[p,q]}} = q$ and $\perp_{\mathbb{L}^{L}_{[p,q]}} = p$.

(85) If L is a complemented lattice and modular, then $\mathbb{L}^{L}_{(p]}$ is a complemented lattice.

(86) If L is a complemented lattice and modular and $p \sqsubseteq q$, then $\mathbb{L}^{L}_{[p,q]}$ is a complemented lattice.

(87) If L is a Boolean lattice and $p \sqsubseteq q$, then $\mathbb{L}^{L}_{[p,q]}$ is a Boolean lattice.

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Categorial Categories and Slice Categories

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Summary. By categorial categories we mean categories with categories as objects and morphisms of the form (C_1, C_2, F), where C_1 and C_2 are categories and F is a functor from C_1 into C_2.

MML Identifier: `CAT_5`.

The terminology and notation used here are introduced in the following articles: [14], [16], [9], [15], [11], [17], [2], [3], [5], [12], [10], [7], [6], [4], [8], [1], and [13].

1. Categories with Triple-like Morphisms

Let D_1, D_2, D be non empty sets and let x be an element of $[\![[\![D_1, D_2]\!], D]\!]$. Then $x_{\mathbf{1},\mathbf{1}}$ is an element of D_1. Then $x_{\mathbf{1},\mathbf{2}}$ is an element of D_2.

Let D_1, D_2 be non empty sets and let x be an element of $[\![D_1, D_2]\!]$. Then $x_{\mathbf{2}}$ is an element of D_2.

Next we state the proposition

(1) Let C, D be category structures. Suppose the category structure of $C =$ the category structure of D. If C is category-like, then D is category-like.

A category structure has triple-like morphisms if:

(Def.1) For every morphism f of it there exists a set x such that $f = \langle\langle \operatorname{dom} f, \operatorname{cod} f\rangle, x\rangle$.

One can verify that there exists a strict category has triple-like morphisms.

Next we state the proposition

(2) Let C be a category structure with triple-like morphisms and let f be a morphism of C. Then $\operatorname{dom} f = f_{\mathbf{1},\mathbf{1}}$ and $\operatorname{cod} f = f_{\mathbf{1},\mathbf{2}}$ and $f = \langle\langle \operatorname{dom} f, \operatorname{cod} f\rangle, f_{\mathbf{2}}\rangle$.

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Let C be a category structure with triple-like morphisms and let f be a morphism of C. Then $f_{\mathbf{1},\mathbf{1}}$ is an object of C. Then $f_{\mathbf{1},\mathbf{2}}$ is an object of C.

In this article we present several logical schemes. The scheme *CatEx* concerns non empty sets $\mathcal{A}$, $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding arbitrary, and a ternary predicate $\mathcal{P}$, and states that:

There exists a strict category C with triple-like morphisms such that

(i) the objects of $C = \mathcal{A}$,

(ii) for all elements a, b of $\mathcal{A}$ and for every element f of $\mathcal{B}$ such that $\mathcal{P}[a, b, f]$ holds $\langle\langle a, b\rangle, f\rangle$ is a morphism of C,

(iii) for every morphism m of C there exist elements a, b of $\mathcal{A}$ and there exists an element f of $\mathcal{B}$ such that $m = \langle\langle a, b\rangle, f\rangle$ and $\mathcal{P}[a, b, f]$, and

(iv) for all morphisms m_1, m_2 of C and for all elements a_1, a_2, a_3 of $\mathcal{A}$ and for all elements f_1, f_2 of $\mathcal{B}$ such that $m_1 = \langle\langle a_1, a_2\rangle, f_1\rangle$ and $m_2 = \langle\langle a_2, a_3\rangle, f_2\rangle$ holds $m_2 \cdot m_1 = \langle\langle a_1, a_3\rangle, \mathcal{F}(f_2, f_1)\rangle$

provided the parameters meet the following requirements:

- For all elements a, b, c of $\mathcal{A}$ and for all elements f, g of $\mathcal{B}$ such that $\mathcal{P}[a, b, f]$ and $\mathcal{P}[b, c, g]$ holds $\mathcal{F}(g, f) \in \mathcal{B}$ and $\mathcal{P}[a, c, \mathcal{F}(g, f)]$,
- Let a be an element of $\mathcal{A}$. Then there exists an element f of $\mathcal{B}$ such that
  (i) $\mathcal{P}[a, a, f]$, and
  (ii) for every element b of $\mathcal{A}$ and for every element g of $\mathcal{B}$ holds if $\mathcal{P}[a, b, g]$, then $\mathcal{F}(g, f) = g$ and if $\mathcal{P}[b, a, g]$, then $\mathcal{F}(f, g) = g$,
- Let a, b, c, d be elements of $\mathcal{A}$ and let f, g, h be elements of $\mathcal{B}$. If $\mathcal{P}[a, b, f]$ and $\mathcal{P}[b, c, g]$ and $\mathcal{P}[c, d, h]$, then $\mathcal{F}(h, \mathcal{F}(g, f)) = \mathcal{F}(\mathcal{F}(h, g), f)$.

The scheme *CatUniq* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding arbitrary, and a ternary predicate $\mathcal{P}$, and states that:

Let C_1, C_2 be strict categories with triple-like morphisms. Suppose that

(i) the objects of $C_1 = \mathcal{A}$,

(ii) for all elements a, b of $\mathcal{A}$ and for every element f of $\mathcal{B}$ such that $\mathcal{P}[a, b, f]$ holds $\langle\langle a, b\rangle, f\rangle$ is a morphism of C_1,

(iii) for every morphism m of C_1 there exist elements a, b of $\mathcal{A}$ and there exists an element f of $\mathcal{B}$ such that $m = \langle\langle a, b\rangle, f\rangle$ and $\mathcal{P}[a, b, f]$,

(iv) for all morphisms m_1, m_2 of C_1 and for all elements a_1, a_2, a_3 of $\mathcal{A}$ and for all elements f_1, f_2 of $\mathcal{B}$ such that $m_1 = \langle\langle a_1, a_2\rangle, f_1\rangle$ and $m_2 = \langle\langle a_2, a_3\rangle, f_2\rangle$ holds $m_2 \cdot m_1 = \langle\langle a_1, a_3\rangle, \mathcal{F}(f_2, f_1)\rangle$,

(v) the objects of $C_2 = \mathcal{A}$,

(vi) for all elements a, b of $\mathcal{A}$ and for every element f of $\mathcal{B}$ such that $\mathcal{P}[a, b, f]$ holds $\langle\langle a, b\rangle, f\rangle$ is a morphism of C_2,

(vii) for every morphism m of C_2 there exist elements a, b of $\mathcal{A}$ and there exists an element f of $\mathcal{B}$ such that $m = \langle\langle a, b\rangle, f\rangle$ and

$\mathcal{P}[a, b, f]$, and

(viii) for all morphisms m_1, m_2 of C_2 and for all elements a_1, a_2, a_3 of $\mathcal{A}$ and for all elements f_1, f_2 of $\mathcal{B}$ such that $m_1 = \langle\langle a_1, a_2\rangle, f_1\rangle$ and $m_2 = \langle\langle a_2, a_3\rangle, f_2\rangle$ holds $m_2 \cdot m_1 = \langle\langle a_1, a_3\rangle, \mathcal{F}(f_2, f_1)\rangle$.

Then $C_1 = C_2$

provided the parameters meet the following requirement:

- Let a be an element of $\mathcal{A}$. Then there exists an element f of $\mathcal{B}$ such that
  (i) $\mathcal{P}[a, a, f]$, and
  (ii) for every element b of $\mathcal{A}$ and for every element g of $\mathcal{B}$ holds if $\mathcal{P}[a, b, g]$, then $\mathcal{F}(g, f) = g$ and if $\mathcal{P}[b, a, g]$, then $\mathcal{F}(f, g) = g$.

The scheme *FunctorEx* concerns categories $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an object of $\mathcal{B}$, and a unary functor $\mathcal{G}$ yielding a set, and states that:

There exists a functor F from $\mathcal{A}$ to $\mathcal{B}$ such that for every morphism f of $\mathcal{A}$ holds $F(f) = \mathcal{G}(f)$

provided the following conditions are met:

- Let f be a morphism of $\mathcal{A}$. Then $\mathcal{G}(f)$ is a morphism of $\mathcal{B}$ and for every morphism g of $\mathcal{B}$ such that $g = \mathcal{G}(f)$ holds $\operatorname{dom} g = \mathcal{F}(\operatorname{dom} f)$ and $\operatorname{cod} g = \mathcal{F}(\operatorname{cod} f)$,
- For every object a of $\mathcal{A}$ holds $\mathcal{G}(\mathrm{id}_a) = \mathrm{id}_{\mathcal{F}(a)}$,
- For all morphisms f_1, f_2 of $\mathcal{A}$ and for all morphisms g_1, g_2 of $\mathcal{B}$ such that $g_1 = \mathcal{G}(f_1)$ and $g_2 = \mathcal{G}(f_2)$ and $\operatorname{dom} f_2 = \operatorname{cod} f_1$ holds $\mathcal{G}(f_2 \cdot f_1) = g_2 \cdot g_1$.

We now state two propositions:

(3) Let C_1 be a category and let C_2 be a subcategory of C_1. Suppose C_1 is a subcategory of C_2. Then the category structure of C_1 = the category structure of C_2.

(4) For every category C and for every subcategory D of C holds every subcategory of D is a subcategory of C.

Let C_1, C_2 be categories. Let us assume that there exists a category C such that C_1 is a subcategory of C and C_2 is a subcategory of C. And let us assume that there exists an object o_1 of C_1 such that o_1 is an object of C_2. The functor $C_1 \cap C_2$ yields a strict category and is defined by the conditions (Def.2).

(Def.2) (i) The objects of $C_1 \cap C_2$ = (the objects of C_1) $\cap$ (the objects of C_2),
(ii) the morphisms of $C_1 \cap C_2$ = (the morphisms of C_1) $\cap$ (the morphisms of C_2),
(iii) the dom-map of $C_1 \cap C_2$ = (the dom-map of C_1) $\restriction$ (the morphisms of C_2),
(iv) the cod-map of $C_1 \cap C_2$ = (the cod-map of C_1) $\restriction$ (the morphisms of C_2),
(v) the composition of $C_1 \cap C_2$ = (the composition of C_1) $\restriction$ ($[\!:$ the morphisms of C_2, the morphisms of $C_2 :\!]$ **qua** set), and
(vi) the id-map of $C_1 \cap C_2$ = (the id-map of C_1) $\restriction$ (the objects of C_2).

In the sequel C is a category and C_1, C_2 are subcategories of C.

The following propositions are true:

(5) If (the objects of C_1) $\cap$ (the objects of C_2) $\neq \emptyset$, then $C_1 \cap C_2 = C_2 \cap C_1$.

(6) If (the objects of C_1) $\cap$ (the objects of C_2) $\neq \emptyset$, then $C_1 \cap C_2$ is a subcategory of C_1 and $C_1 \cap C_2$ is a subcategory of C_2.

Let C, D be categories and let F be a functor from C to D. The functor $\operatorname{Im} F$ yields a strict subcategory of D and is defined by the conditions (Def.3).

(Def.3) (i) The objects of $\operatorname{Im} F = \operatorname{rng} \operatorname{Obj} F$,

(ii) $\operatorname{rng} F \subseteq$ the morphisms of $\operatorname{Im} F$, and

(iii) for every subcategory E of D such that the objects of $E = \operatorname{rng} \operatorname{Obj} F$ and $\operatorname{rng} F \subseteq$ the morphisms of E holds $\operatorname{Im} F$ is a subcategory of E.

Next we state three propositions:

(7) Let C, D be categories, and let E be a subcategory of D, and let F be a functor from C to D. If $\operatorname{rng} F \subseteq$ the morphisms of E, then F is a functor from C to E.

(8) For all categories C, D holds every functor from C to D is a functor from C to $\operatorname{Im} F$.

(9) Let C, D be categories, and let E be a subcategory of D, and let F be a functor from C to E, and let G be a functor from C to D. If $F = G$, then $\operatorname{Im} F = \operatorname{Im} G$.

2. Categorial Categories

A set is categorial if:

(Def.4) For every set x such that $x \in$ it holds x is a category.

One can check that there exists a non empty set which is categorial. Let us observe that a non empty set is categorial if:

(Def.5) Every element of it is a category.

A category is categorial if it satisfies the conditions (Def.6).

(Def.6) (i) The objects of it is categorial,

(ii) for every object a of it and for every category A such that $a = A$ holds $\mathrm{id}_a = \langle\langle A, A\rangle, \mathrm{id}_A\rangle$,

(iii) for every morphism m of it and for all categories A, B such that $A = \operatorname{dom} m$ and $B = \operatorname{cod} m$ there exists a functor F from A to B such that $m = \langle\langle A, B\rangle, F\rangle$, and

(iv) for all morphisms m_1, m_2 of it and for all categories A, B, C and for every functor F from A to B and for every functor G from B to C such that $m_1 = \langle\langle A, B\rangle, F\rangle$ and $m_2 = \langle\langle B, C\rangle, G\rangle$ holds $m_2 \cdot m_1 = \langle\langle A, C\rangle, G \cdot F\rangle$.

Let us mention that every category which is categorial has triple-like morphisms.

One can prove the following two propositions:

(10) Let C, D be categories. Suppose the category structure of C = the category structure of D. If C is categorial, then D is categorial.

(11) For every category C holds $\dot{\circlearrowleft}(C, \langle\langle C, C\rangle, \mathrm{id}_C\rangle)$ is categorial.

Let us note that there exists a strict category which is categorial.

We now state two propositions:

(12) For every categorial category C holds every object of C is a category.

(13) For every categorial category C and for every morphism f of C holds $\operatorname{dom} f = f_{\mathbf{1},\mathbf{1}}$ and $\operatorname{cod} f = f_{\mathbf{1},\mathbf{2}}$.

Let C be a categorial category and let m be a morphism of C. Then $m_{\mathbf{1},\mathbf{1}}$ is a category. Then $m_{\mathbf{1},\mathbf{2}}$ is a category.

We now state the proposition

(14) Let C_1, C_2 be categorial categories. Suppose the objects of C_1 = the objects of C_2 and the morphisms of C_1 = the morphisms of C_2. Then the category structure of C_1 = the category structure of C_2.

Let C be a categorial category. One can check that every subcategory of C is categorial.

We now state the proposition

(15) Let C, D be categorial categories. Suppose the morphisms of $C \subseteq$ the morphisms of D. Then C is a subcategory of D.

Let a be a set. Let us assume that a is a category. The functor $\operatorname{cat} a$ yields a category and is defined by:

(Def.7) $\operatorname{cat} a = a$.

One can prove the following proposition

(16) For every categorial category C and for every object c of C holds $\operatorname{cat} c = c$.

Let C be a categorial category and let m be a morphism of C. Then $m_{\mathbf{2}}$ is a functor from $\operatorname{cat}\operatorname{dom} m$ to $\operatorname{cat}\operatorname{cod} m$.

Next we state two propositions:

(17) Let X be a categorial non empty set and let Y be a non empty set. Suppose that

(i) for all elements A, B, C of X and for every functor F from A to B and for every functor G from B to C such that $F \in Y$ and $G \in Y$ holds $G \cdot F \in Y$, and

(ii) for every element A of X holds $\mathrm{id}_A \in Y$.

Then there exists a strict categorial category C such that

(iii) the objects of $C = X$, and

(iv) for all elements A, B of X and for every functor F from A to B holds $\langle\langle A, B\rangle, F\rangle$ is a morphism of C iff $F \in Y$.

(18) Let X be a categorial non empty set, and let Y be a non empty set, and let C_1, C_2 be strict categorial categories. Suppose that

(i) the objects of $C_1 = X$,

(ii) for all elements A, B of X and for every functor F from A to B holds $\langle\langle A, B\rangle, F\rangle$ is a morphism of C_1 iff $F \in Y$,

(iii) the objects of $C_2 = X$, and

(iv) for all elements A, B of X and for every functor F from A to B holds $\langle\langle A, B\rangle, F\rangle$ is a morphism of C_2 iff $F \in Y$.

Then $C_1 = C_2$.

A categorial category is full if it satisfies the condition (Def.8).

(Def.8) Let a, b be categories. Suppose a is an object of it and b is an object of it. Let F be a functor from a to b. Then $\langle\langle a, b\rangle, F\rangle$ is a morphism of it.

Let us note that there exists a categorial strict category which is full.

The following propositions are true:

(19) Let C_1, C_2 be full categorial categories. Suppose the objects of C_1 = the objects of C_2. Then the category structure of C_1 = the category structure of C_2.

(20) For every categorial non empty set A there exists a full categorial strict category C such that the objects of $C = A$.

(21) Let C be a categorial category and let D be a full categorial category. Suppose the objects of $C \subseteq$ the objects of D. Then C is a subcategory of D.

(22) Let C be a category, and let D_1, D_2 be categorial categories, and let F_1 be a functor from C to D_1, and let F_2 be a functor from C to D_2. If $F_1 = F_2$, then $\operatorname{Im} F_1 = \operatorname{Im} F_2$.

3. Slice Categories

Let C be a category and let o be an object of C. The functor $\operatorname{Hom}(o)$ yielding a non empty subset of the morphisms of C is defined by:

(Def.9) $\operatorname{Hom}(o) = (\text{the cod-map of } C)^{-1}\{o\}$.

The functor $\operatorname{hom}(o, \square)$ yields a non empty subset of the morphisms of C and is defined by:

(Def.10) $\operatorname{hom}(o, \square) = (\text{the dom-map of } C)^{-1}\{o\}$.

We now state several propositions:

(23) For every category C and for every object a of C and for every morphism f of C holds $f \in \operatorname{Hom}(a)$ iff $\operatorname{cod} f = a$.

(24) For every category C and for every object a of C and for every morphism f of C holds $f \in \operatorname{hom}(a, \square)$ iff $\operatorname{dom} f = a$.

(25) For every category C and for all objects a, b of C holds $\operatorname{hom}(a, b) = \operatorname{hom}(a, \square) \cap \operatorname{Hom}(b)$.

(26) For every category C and for every morphism f of C holds $f \in \operatorname{hom}(\operatorname{dom} f, \square)$ and $f \in \operatorname{Hom}(\operatorname{cod} f)$.

(27) For every category C and for every morphism f of C and for every element g of $\mathrm{Hom}(\mathrm{dom}\, f)$ holds $f \cdot g \in \mathrm{Hom}(\mathrm{cod}\, f)$.

(28) For every category C and for every morphism f of C and for every element g of $\mathrm{hom}(\mathrm{cod}\, f, \square)$ holds $g \cdot f \in \mathrm{hom}(\mathrm{dom}\, f, \square)$.

Let C be a category and let o be an object of C. The functor $\mathrm{SliceCat}(C, o)$ yields a strict category with triple-like morphisms and is defined by the conditions (Def.11).

(Def.11) (i) The objects of $\mathrm{SliceCat}(C, o) = \mathrm{Hom}(o)$,

(ii) for all elements a, b of $\mathrm{Hom}(o)$ and for every morphism f of C such that $\mathrm{dom}\, b = \mathrm{cod}\, f$ and $a = b \cdot f$ holds $\langle\langle a, b\rangle, f\rangle$ is a morphism of $\mathrm{SliceCat}(C, o)$,

(iii) for every morphism m of $\mathrm{SliceCat}(C, o)$ there exist elements a, b of $\mathrm{Hom}(o)$ and there exists a morphism f of C such that $m = \langle\langle a, b\rangle, f\rangle$ and $\mathrm{dom}\, b = \mathrm{cod}\, f$ and $a = b \cdot f$, and

(iv) for all morphisms m_1, m_2 of $\mathrm{SliceCat}(C, o)$ and for all elements a_1, a_2, a_3 of $\mathrm{Hom}(o)$ and for all morphisms f_1, f_2 of C such that $m_1 = \langle\langle a_1, a_2\rangle, f_1\rangle$ and $m_2 = \langle\langle a_2, a_3\rangle, f_2\rangle$ holds $m_2 \cdot m_1 = \langle\langle a_1, a_3\rangle, f_2 \cdot f_1\rangle$.

The functor $\mathrm{SliceCat}(o, C)$ yielding a strict category with triple-like morphisms is defined by the conditions (Def.12).

(Def.12) (i) The objects of $\mathrm{SliceCat}(o, C) = \mathrm{hom}(o, \square)$,

(ii) for all elements a, b of $\mathrm{hom}(o, \square)$ and for every morphism f of C such that $\mathrm{dom}\, f = \mathrm{cod}\, a$ and $f \cdot a = b$ holds $\langle\langle a, b\rangle, f\rangle$ is a morphism of $\mathrm{SliceCat}(o, C)$,

(iii) for every morphism m of $\mathrm{SliceCat}(o, C)$ there exist elements a, b of $\mathrm{hom}(o, \square)$ and there exists a morphism f of C such that $m = \langle\langle a, b\rangle, f\rangle$ and $\mathrm{dom}\, f = \mathrm{cod}\, a$ and $f \cdot a = b$, and

(iv) for all morphisms m_1, m_2 of $\mathrm{SliceCat}(o, C)$ and for all elements a_1, a_2, a_3 of $\mathrm{hom}(o, \square)$ and for all morphisms f_1, f_2 of C such that $m_1 = \langle\langle a_1, a_2\rangle, f_1\rangle$ and $m_2 = \langle\langle a_2, a_3\rangle, f_2\rangle$ holds $m_2 \cdot m_1 = \langle\langle a_1, a_3\rangle, f_2 \cdot f_1\rangle$.

Let C be a category, let o be an object of C, and let m be a morphism of $\mathrm{SliceCat}(C, o)$. Then $m_{\mathbf{2}}$ is a morphism of C. Then $m_{\mathbf{1},1}$ is an element of $\mathrm{Hom}(o)$. Then $m_{\mathbf{1},\mathbf{2}}$ is an element of $\mathrm{Hom}(o)$.

We now state two propositions:

(29) Let C be a category, and let a be an object of C, and let m be a morphism of $\mathrm{SliceCat}(C, a)$. Then $m = \langle\langle m_{\mathbf{1},\mathbf{1}}, m_{\mathbf{1},\mathbf{2}}\rangle, m_{\mathbf{2}}\rangle$ and $\mathrm{dom}(m_{\mathbf{1},\mathbf{2}}) = \mathrm{cod}(m_{\mathbf{2}})$ and $m_{\mathbf{1},\mathbf{1}} = m_{\mathbf{1},\mathbf{2}} \cdot m_{\mathbf{2}}$ and $\mathrm{dom}\, m = m_{\mathbf{1},\mathbf{1}}$ and $\mathrm{cod}\, m = m_{\mathbf{1},\mathbf{2}}$.

(30) Let C be a category, and let o be an object of C, and let f be an element of $\mathrm{Hom}(o)$, and let a be an object of $\mathrm{SliceCat}(C, o)$. If $a = f$, then $\mathrm{id}_a = \langle\langle a, a\rangle, \mathrm{id}_{\mathrm{dom}\, f}\rangle$.

Let C be a category, let o be an object of C, and let m be a morphism of $\mathrm{SliceCat}(o, C)$. Then $m_{\mathbf{2}}$ is a morphism of C. Then $m_{\mathbf{1},1}$ is an element of $\mathrm{hom}(o, \square)$. Then $m_{\mathbf{1},\mathbf{2}}$ is an element of $\mathrm{hom}(o, \square)$.

We now state two propositions:

(31) Let C be a category, and let a be an object of C, and let m be a morphism of $\mathrm{SliceCat}(a, C)$. Then $m = \langle\langle m_{\mathbf{1},\mathbf{1}}, m_{\mathbf{1},\mathbf{2}}\rangle, m_{\mathbf{2}}\rangle$ and $\mathrm{dom}(m_{\mathbf{2}}) = \mathrm{cod}(m_{\mathbf{1},\mathbf{1}})$ and $m_{\mathbf{2}} \cdot m_{\mathbf{1},\mathbf{1}} = m_{\mathbf{1},\mathbf{2}}$ and $\mathrm{dom}\, m = m_{\mathbf{1},\mathbf{1}}$ and $\mathrm{cod}\, m = m_{\mathbf{1},\mathbf{2}}$.

(32) Let C be a category, and let o be an object of C, and let f be an element of $\mathrm{hom}(o, \square)$, and let a be an object of $\mathrm{SliceCat}(o, C)$. If $a = f$, then $\mathrm{id}_a = \langle\langle a, a\rangle, \mathrm{id}_{\mathrm{cod}\, f}\rangle$.

4. Functors Between Slice Categories

Let C be a category and let f be a morphism of C. The functor $\mathrm{SliceFunctor}(f)$ yielding a functor from $\mathrm{SliceCat}(C, \mathrm{dom}\, f)$ to $\mathrm{SliceCat}(C, \mathrm{cod}\, f)$ is defined by:

(Def.13) For every morphism m of $\mathrm{SliceCat}(C, \mathrm{dom}\, f)$ holds $(\mathrm{SliceFunctor}(f))(m) = \langle\langle f \cdot m_{\mathbf{1},\mathbf{1}}, f \cdot m_{\mathbf{1},\mathbf{2}}\rangle, m_{\mathbf{2}}\rangle$.

The functor $\mathrm{SliceContraFunctor}(f)$ yields a functor from $\mathrm{SliceCat}(\mathrm{cod}\, f, C)$ to $\mathrm{SliceCat}(\mathrm{dom}\, f, C)$ and is defined as follows:

(Def.14) For every morphism m of $\mathrm{SliceCat}(\mathrm{cod}\, f, C)$ holds $(\mathrm{SliceContraFunctor}(f))(m) = \langle\langle m_{\mathbf{1},\mathbf{1}} \cdot f, m_{\mathbf{1},\mathbf{2}} \cdot f\rangle, m_{\mathbf{2}}\rangle$.

We now state two propositions:

(33) For every category C and for all morphisms f, g of C such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $\mathrm{SliceFunctor}(g \cdot f) = \mathrm{SliceFunctor}(g) \cdot \mathrm{SliceFunctor}(f)$.

(34) For every category C and for all morphisms f, g of C such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $\mathrm{SliceContraFunctor}(g \cdot f) = \mathrm{SliceContraFunctor}(f) \cdot \mathrm{SliceContraFunctor}(g)$.

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Preliminaries to Circuits, I [1]

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Summary. This article is the first in a series of four articles (continued in [24,23,22]) about modelling circuits by many-sorted algebras.

Here, we introduce some auxiliary notations and prove auxiliary facts about many sorted sets, many sorted functions and trees.

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The articles [29], [33], [18], [4], [30], [1], [34], [13], [17], [31], [28], [14], [25], [16], [15], [8], [5], [7], [9], [6], [3], [2], [27], [19], [20], [26], [21], [11], [10], [12], and [32] provide the terminology and notation for this paper.

1. Varia

One can prove the following proposition

(1) For all sets X, Y holds $X \setminus Y$ misses Y.

In this article we present several logical schemes. The scheme *Fraenkel Subset* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(x) : x$ ranges over elements of $\mathcal{A}$, $\mathcal{P}[x]\}$ is a subset of $\mathcal{B}$

for all values of the parameters.

The scheme *FraenkelFinIm* concerns a finite non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding arbitrary, and a unary predicate $\mathcal{P}$, and states that:

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$\{\mathcal{F}(x) : x$ ranges over elements of $\mathcal{A}$, $\mathcal{P}[x]\}$ is finite

for all values of the parameters.

The following three propositions are true:

(2) For every function f and for arbitrary x, y such that $\operatorname{dom} f = \{x\}$ and $\operatorname{rng} f = \{y\}$ holds $f = \{\langle x, y \rangle\}$.

(3) For all functions f, g, h such that $f \subseteq g$ holds $f +\cdot h \subseteq g +\cdot h$.

(4) For all functions f, g, h such that $f \subseteq g$ and $\operatorname{dom} f$ misses $\operatorname{dom} h$ holds $f \subseteq g +\cdot h$.

Let X be a finite non empty subset of $\mathbb{R}$. The functor $\max X$ yields a real number and is defined as follows:

(Def.1) $\max X \in X$ and for every real number k such that $k \in X$ holds $k \leq \max X$.

Let X be a finite non empty subset of $\mathbb{N}$. The functor $\max X$ yielding a natural number is defined by:

(Def.2) There exists a finite non empty subset Y of $\mathbb{R}$ such that $Y = X$ and $\max X = \max Y$.

2. Many Sorted Sets and Functions

One can prove the following proposition

(5) For every set I and for every many sorted set M_1 indexed by I holds ${M_1}^{\#}(\varepsilon_I) = \{\varepsilon\}$.

The scheme *MSSLambda2Part* deals with a set $\mathcal{A}$, two unary functors $\mathcal{F}$ and $\mathcal{G}$ yielding arbitrary, and a unary predicate $\mathcal{P}$, and states that:

There exists a many sorted set f indexed by $\mathcal{A}$ such that for every element i of $\mathcal{A}$ holds if $i \in \mathcal{A}$, then if $\mathcal{P}[i]$, then $f(i) = \mathcal{F}(i)$ and if not $\mathcal{P}[i]$, then $f(i) = \mathcal{G}(i)$

for all values of the parameters.

Let I be a set. A many sorted set indexed by I is locally-finite if:

(Def.3) For arbitrary i such that $i \in I$ holds $\mathrm{it}(i)$ is finite.

Let I be a set. Observe that there exists a many sorted set indexed by I which is non-empty and locally-finite.

Let I, A be sets. Then $I \longmapsto A$ is a many sorted set indexed by I.

Let I be a set, let M be a many sorted set indexed by I, and let A be a subset of I. Then $M \restriction A$ is a many sorted set indexed by A.

Let M be a non-empty function and let A be a set. One can check that $M \restriction A$ is non-empty.

One can prove the following three propositions:

(6) For every non empty set I and for every non-empty many sorted set B indexed by I holds $\bigcup \operatorname{rng} B$ is non empty.

(7) For every set I holds $\operatorname{uncurry}(I \longmapsto \emptyset) = \emptyset$.

(8) Let I be a non empty set, and let A be a set, and let B be a non-empty many sorted set indexed by I, and let F be a many sorted function from $I \longmapsto A$ into B. Then $\operatorname{dom} \operatorname{commute}(F) = A$.

Now we present two schemes. The scheme *LambdaRecCorrD* concerns a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

(i) There exists a function f from $\mathbb{N}$ into $\mathcal{A}$ such that $f(0) = \mathcal{B}$ and for every natural number i and for every element x of $\mathcal{A}$ such that $x = f(i)$ holds $f(i+1) = \mathcal{F}(i, x)$, and

(ii) for all functions f_1, f_2 from $\mathbb{N}$ into $\mathcal{A}$ such that $f_1(0) = \mathcal{B}$ and for every natural number i and for every element x of $\mathcal{A}$ such that $x = f_1(i)$ holds $f_1(i+1) = \mathcal{F}(i, x)$ and $f_2(0) = \mathcal{B}$ and for every natural number i and for every element x of $\mathcal{A}$ such that $x = f_2(i)$ holds $f_2(i+1) = \mathcal{F}(i, x)$ holds $f_1 = f_2$

for all values of the parameters.

The scheme *LambdaMSFD* concerns a non empty set $\mathcal{A}$, a subset $\mathcal{B}$ of $\mathcal{A}$, many sorted sets $\mathcal{C}$, $\mathcal{D}$ indexed by $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding arbitrary, and states that:

There exists a many sorted function f from $\mathcal{C}$ into $\mathcal{D}$ such that for every element i of $\mathcal{A}$ such that $i \in \mathcal{B}$ holds $f(i) = \mathcal{F}(i)$

provided the following requirement is met:

- For every element i of $\mathcal{A}$ such that $i \in \mathcal{B}$ holds $\mathcal{F}(i)$ is a function from $\mathcal{C}(i)$ into $\mathcal{D}(i)$.

Let F be a non-empty function and let f be a function. Observe that $F \cdot f$ is non-empty.

Let I be a set and let M_1 be a non-empty many sorted set indexed by I. Note that every element of $\prod M_1$ is function-like and relation-like.

One can prove the following propositions:

(9) Let I be a set, and let f be a non-empty many sorted set indexed by I, and let g be a function, and let s be an element of $\prod f$. Suppose $\operatorname{dom} g \subseteq \operatorname{dom} f$ and for arbitrary x such that $x \in \operatorname{dom} g$ holds $g(x) \in f(x)$. Then $s +\cdot g$ is an element of $\prod f$.

(10) Let A, B be non empty sets, and let C be a non-empty many sorted set indexed by A, and let I_1 be a many sorted function from $A \longmapsto B$ into C, and let b be an element of B. Then there exists a many sorted set c indexed by A such that $c = (\operatorname{commute}(I_1))(b)$ and $c \in C$.

(11) Let I be a set, and let M be a many sorted set indexed by I, and let x, g be functions. If $x \in \prod M$, then $x \cdot g \in \prod(M \cdot g)$.

(12) For every natural number n and for arbitrary a holds $\prod(n \mapsto \{a\}) = \{n \mapsto a\}$.

3. Trees

We follow the rules: T, T_1 will denote finite trees, t, p will denote elements of T, and t_1 will denote an element of T_1.

Let D be a non empty set. Note that every element of $\mathrm{FinTrees}(D)$ is finite.

Let T be a finite decorated tree and let t be an element of $\mathrm{dom}\, T$. Observe that $T \restriction t$ is finite.

We now state the proposition

(13) $T \restriction p \approx \{t : p \preceq t\}$.

Let T be a finite decorated tree, let t be an element of $\mathrm{dom}\, T$, and let T_1 be a finite decorated tree. Note that $T(t/T_1)$ is finite.

Next we state a number of propositions:

(14) $T(p/T_1) = \{t : p \not\preceq t\} \cup \{p \mathbin{^\frown} t_1\}$.

(15) For every finite sequence f of elements of $\mathbb{N}$ such that $f \in T(p/T_1)$ and $p \preceq f$ there exists t_1 such that $f = p \mathbin{^\frown} t_1$.

(16) For every tree yielding finite sequence p and for every natural number k such that $k+1 \in \mathrm{dom}\, p$ holds $\overbrace{p} \restriction \langle k \rangle = p(k+1)$.

(17) Let q be a decorated tree yielding finite sequence and let k be a natural number. If $k+1 \in \mathrm{dom}\, q$, then $\langle k \rangle \in \overbrace{\mathop{\mathrm{dom}}_{\kappa} q(\kappa)}$.

(18) Let p, q be tree yielding finite sequences and let k be a natural number. Suppose $\mathrm{len}\, p = \mathrm{len}\, q$ and $k+1 \in \mathrm{dom}\, p$ and for every natural number i such that $i \in \mathrm{dom}\, p$ and $i \neq k+1$ holds $p(i) = q(i)$. Let t be a tree. If $q(k+1) = t$, then $\overbrace{q} = \overbrace{p}(\langle k \rangle / t)$.

(19) Let e_1, e_2 be finite decorated trees, and let x be arbitrary, and let k be a natural number, and let p be a decorated tree yielding finite sequence. Suppose $\langle k \rangle \in \mathrm{dom}\, e_1$ and $e_1 = x\text{-tree}(p)$. Then there exists a decorated tree yielding finite sequence q such that $e_1(\langle k \rangle / e_2) = x\text{-tree}(q)$ and $\mathrm{len}\, q = \mathrm{len}\, p$ and $q(k+1) = e_2$ and for every natural number i such that $i \in \mathrm{dom}\, p$ and $i \neq k+1$ holds $q(i) = p(i)$.

(20) For every finite tree T and for every element p of T such that $p \neq \varepsilon$ holds $\mathrm{card}(T \restriction p) < \mathrm{card}\, T$.

(21) For every finite function f holds $\mathrm{card}\, f = \mathrm{card}\, \mathrm{dom}\, f$.

(22) For all finite trees T, T_1 and for every element p of T holds $\mathrm{card}(T(p/T_1)) + \mathrm{card}(T \restriction p) = \mathrm{card}\, T + \mathrm{card}\, T_1$.

(23) For all finite decorated trees T, T_1 and for every element p of $\mathrm{dom}\, T$ holds $\mathrm{card}(T(p/T_1)) + \mathrm{card}(T \restriction p) = \mathrm{card}\, T + \mathrm{card}\, T_1$.

Let x be arbitrary. One can check that the root tree of x is finite.

We now state the proposition

(24) For arbitrary x holds card (the root tree of x) $= 1$.

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Minimization of Finite State Machines [1]

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Summary. We have formalized deterministic finite state machines closely following the textbook [9], pp. 88–119 up to the minimization theorem. In places, we have changed the approach presented in the book as it turned out to be too specific and inconvenient. Our work also revealed several minor mistakes in the book. After defining a structure for an outputless finite state machine, we have derived the structures for the transition assigned output machine (Mealy) and state assigned output machine (Moore). The machines are then proved similar, in the sense that for any Mealy (Moore) machine there exists a Moore (Mealy) machine producing essentially the same response for the same input. The rest of work is then done for Mealy machines. Next, we define equivalence of machines, equivalence and k-equivalence of states, and characterize a process of constructing for a given Mealy machine, the machine equivalent to it in which no two states are equivalent. The final, minimization theorem states:

> **Theorem 4.5:** Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be reduced, connected finite-state machines. Then the state graphs of $\mathbf{M}_1$ and $\mathbf{M}_2$ are isomorphic if and only if $\mathbf{M}_1$ and $\mathbf{M}_2$ are equivalent.

and it is the last theorem in this article.

MML Identifier: `FSM_1`.

The papers [19], [23], [10], [2], [21], [13], [16], [8], [20], [18], [24], [5], [6], [7], [22], [3], [4], [1], [14], [17], [12], [11], and [15] provide the terminology and notation for this paper.

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1. Preliminaries

For simplicity we adopt the following convention: m, n, i, k will denote natural numbers, D will denote a non empty set, d will denote an element of D, and d_1, d_2 will denote finite sequences of elements of D.

Next we state several propositions:

(1) If $m < n$, then there exists a natural number p such that $n = m + p$ and $1 \leq p$.

(2) If $i \in \mathrm{Seg}\, n$, then $i + m \in \mathrm{Seg}(n + m)$.

(3) If $i > 0$ and $i + m \in \mathrm{Seg}(n + m)$, then $i \in \mathrm{Seg}\, n$ and $i \in \mathrm{Seg}(n + m)$.

(4) If $k < i$, then there exists a natural number j such that $j = i - k$ and $1 \leq j$.

(5) If $1 \leq \mathrm{len}\, d_1$, then there exist d, d_2 such that $d = d_1(1)$ and $d_1 = \langle d \rangle ^\frown d_2$.

(6) If $i \in \mathrm{dom}\, d_1$, then $(\langle d \rangle ^\frown d_1)(i + 1) = d_1(i)$.

For simplicity we adopt the following rules: S is a set, D_1, D_2 are non empty sets, f_1 is a function from S into D_1, and f_2 is a function from D_1 into D_2.

One can prove the following propositions:

(7) If f_1 is bijective and f_2 is bijective, then $f_2 \cdot f_1$ is bijective.

(8) For every set Y and for all equivalence relations E_1, E_2 of Y such that $\mathrm{Classes}\, E_1 = \mathrm{Classes}\, E_2$ holds $E_1 = E_2$.

(9) For every non empty set W holds every partition of W is non empty.

(10) For every finite set Z holds every partition of Z is finite.

Let W be a non empty set. Note that every partition of W is non empty.

Let Z be a finite set. Note that every partition of Z is finite.

Let X be a non empty finite set. Observe that there exists a partition of X which is non empty and finite.

We adopt the following rules: X, A will be non empty finite sets, P_1 will be a partition of X, and P_2, P_3 will be partitions of A.

We now state several propositions:

(11) For every set P_4 such that $P_4 \in P_1$ there exists an element x of X such that $x \in P_4$.

(12) $\mathrm{card}\, P_1 \leq \mathrm{card}\, X$.

(13) If P_2 is finer than P_3, then $\mathrm{card}\, P_3 \leq \mathrm{card}\, P_2$.

(14) If P_2 is finer than P_3, then for every element p_2 of P_3 there exists an element p_1 of P_2 such that $p_1 \subseteq p_2$.

(15) If P_2 is finer than P_3 and $\mathrm{card}\, P_2 = \mathrm{card}\, P_3$, then $P_2 = P_3$.

2. Definitions and Terminology

Let I_1 be a non empty set. We consider FSM over I_1 as systems
$\langle$ states, a Tran, a InitS $\rangle$,
where the states constitute a finite non empty set, the Tran is a function from $[\!:$ the states, $I_1 :\!]$ into the states, and the InitS is an element of the states.

Let I_1 be a non empty set and let f_3 be a FSM over I_1. A state of f_3 is an element of the states of f_3.

For simplicity we follow a convention: I_1, O_1 are non empty sets, f_3 is a FSM over I_1, s is an element of I_1, w, w_1, w_2 are finite sequences of elements of I_1, q, q', q_1, q_2 are states of f_3, and q_3 is a finite sequence of elements of the states of f_3.

Let us consider I_1, f_3, s, q. The functor s-succ(q) yielding a state of f_3 is defined by:

(Def.1) $s\text{-succ}(q) = (\text{the Tran of } f_3)(\langle q, s\rangle)$.

Let us consider I_1, f_3, q, w. The functor (q, w)-admissible yields a finite sequence of elements of the states of f_3 and is defined by the conditions (Def.2).

(Def.2) (i) $(q, w)\text{-admissible}(1) = q$,

(ii) $\text{len}((q, w)\text{-admissible}) = \text{len}\, w + 1$, and

(iii) for every i such that $1 \leq i$ and $i \leq \text{len}\, w$ there exists an element w_3 of I_1 and there exist states q_4, q_5 of f_3 such that $w_3 = w(i)$ and $q_4 = (q, w)\text{-admissible}(i)$ and $q_5 = (q, w)\text{-admissible}(i + 1)$ and $w_3\text{-succ}(q_4) = q_5$.

The following proposition is true

(16) $(q, \varepsilon_{(I_1)})\text{-admissible} = \langle q\rangle$.

Let us consider I_1, f_3, w, q_1, q_2. The predicate $q_1 \xrightarrow{w} q_2$ is defined as follows:

(Def.3) $(q_1, w)\text{-admissible}(\text{len}\, w + 1) = q_2$.

We now state the proposition

(17) $q \xrightarrow{\varepsilon_{(I_1)}} q$.

Let us consider I_1, f_3, w, q_3. We say that q_3 is admissible for w if and only if:

(Def.4) There exists q_1 such that $q_1 = q_3(1)$ and $(q_1, w)\text{-admissible} = q_3$.

We now state the proposition

(18) $\langle q\rangle$ is admissible for $\varepsilon_{(I_1)}$.

Let us consider I_1, f_3, q, w. The functor w-succ(q) yields a state of f_3 and is defined by:

(Def.5) $q \xrightarrow{w} w\text{-succ}(q)$.

One can prove the following propositions:

(19) $(q, w)\text{-admissible}(\text{len}((q, w)\text{-admissible})) = q'$ iff $q \xrightarrow{w} q'$.

(20) For every k such that $1 \leq k$ and $k \leq \text{len}\, w_1$ holds $(q_1, w_1 \frown w_2)\text{-admissible}(k) = (q_1, w_1)\text{-admissible}(k)$.

(21) If $q_1 \xrightarrow{w_1} q_2$, then $(q_1, w_1 \frown w_2)$-admissible$(\mathrm{len}\, w_1 + 1) = q_2$.

(22) If $q_1 \xrightarrow{w_1} q_2$, then for every k such that $1 \leq k$ and $k \leq \mathrm{len}\, w_2 + 1$ holds $(q_1, w_1 \frown w_2)$-admissible$(\mathrm{len}\, w_1 + k) = (q_2, w_2)$-admissible$(k)$.

(23) If $q_1 \xrightarrow{w_1} q_2$, then $(q_1, w_1 {\frown} w_2)$-admissible $= ((q_1, w_1)$-admissible$_{\restriction \mathrm{len}\, w_1 + 1})^{\frown}$ (q_2, w_2)-admissible.

3. Mealy and Moore Machines

Let I_1, O_1 be non empty sets. We consider Mealy-FSM over I_1, O_1 as extensions of FSM over I_1 as systems

⟨ states, a Tran, a OFun, a InitS ⟩,

where the states constitute a finite non empty set, the Tran is a function from $[\!:$ the states, $I_1 :\!]$ into the states, the OFun is a function from $[\!:$ the states, $I_1 :\!]$ into O_1, and the InitS is an element of the states. We introduce Moore-FSM over I_1, O_1 which are extensions of FSM over I_1 and are systems

⟨ states, a Tran, a OFun, a InitS ⟩,

where the states constitute a finite non empty set, the Tran is a function from $[\!:$ the states, $I_1 :\!]$ into the states, the OFun is a function from the states into O_1, and the InitS is an element of the states.

For simplicity we adopt the following convention: t_1, t_2, t_3, t_4 will denote Mealy-FSM over I_1, O_1, s_1 will denote a Moore-FSM over I_1, O_1, q_6 will denote a state of s_1, q, q_1, q_2, q_7, q_8, q_9, q_{10}, q_1', q_{11}, q_{12}, q_{13} will denote states of t_1, q_{14}, q_{15} will denote states of t_2, and q_{21}, q_{22} will denote states of t_3.

Let us consider I_1, O_1, t_1, q_{11}, w. The functor (q_{11}, w)-response yields a finite sequence of elements of O_1 and is defined as follows:

(Def.6) $\mathrm{len}((q_{11}, w)$-response$) = \mathrm{len}\, w$ and for every i such that $i \in \mathrm{dom}\, w$ holds (q_{11}, w)-response$(i) =$ (the OFun of t_1)$(\langle\!\langle (q_{11}, w)$-admissible$(i)$, $w(i)\rangle\!\rangle)$.

The following proposition is true

(24) $(q_{11}, \varepsilon_{(I_1)})$-response $= \varepsilon_{(O_1)}$.

Let us consider I_1, O_1, s_1, q_6, w. The functor (q_6, w)-response yields a finite sequence of elements of O_1 and is defined by:

(Def.7) $\mathrm{len}((q_6, w)$-response$) = \mathrm{len}\, w + 1$ and for every i such that $i \in \mathrm{Seg}(\mathrm{len}\, w + 1)$ holds (q_6, w)-response$(i) =$ (the OFun of s_1)$((q_6, w)$-admissible$(i))$.

One can prove the following propositions:

(25) (q_6, w)-response$(1) =$ (the OFun of s_1)(q_6).

(26) If $q_{12} \xrightarrow{w_1} q_{13}$, then $(q_{12}, w_1 \frown w_2)$-response $= (q_{12}, w_1)$-response $\frown$ (q_{13}, w_2)-response.

(27) If $q_{14} \xrightarrow{w_1} q_{15}$ and $q_{21} \xrightarrow{w_1} q_{22}$ and (q_{15}, w_2)-response $\neq$ (q_{22}, w_2)-response, then $(q_{14}, w_1 \frown w_2)$-response $\neq (q_{21}, w_1 \frown w_2)$-response.

In the sequel O_2 is a finite non empty set, t_5 is a Mealy-FSM over I_1, O_2, and s_2 is a Moore-FSM over I_1, O_2.

Let us consider I_1, O_1, t_1, s_1. We say that t_1 is similar to s_1 if and only if the condition (Def.8) is satisfied.

(Def.8) Let t_6 be a finite sequence of elements of I_1. Then $\langle$(the OFun of s_1)(the InitS of s_1)$\rangle$ $^\frown$ (the InitS of t_1, t_6)-response = (the InitS of s_1, t_6)-response.

The following propositions are true:

(28) There exists t_1 which is similar to s_1.

(29) There exists s_2 such that t_5 is similar to s_2.

4. Equivalence of States and Machines

Let us consider I_1, O_1, t_2, t_3. We say that t_2 and t_3 are equivalent if and only if:

(Def.9) For every w holds (the InitS of t_2, w)-response = (the InitS of t_3, w)-response.

Let us observe that the predicate introduced above is reflexive and symmetric.

We now state the proposition

(30) If t_2 and t_3 are equivalent and t_3 and t_4 are equivalent, then t_2 and t_4 are equivalent.

Let us consider I_1, O_1, t_1, q_8, q_9. We say that q_8 and q_9 are equivalent if and only if:

(Def.10) For every w holds (q_8, w)-response = (q_9, w)-response.

We now state several propositions:

(31) q and q are equivalent.

(32) If q_1 and q_2 are equivalent, then q_2 and q_1 are equivalent.

(33) If q_1 and q_2 are equivalent and q_2 and q_7 are equivalent, then q_1 and q_7 are equivalent.

(34) If q'_1 = (the Tran of t_1)($\langle q_8, s\rangle$), then for every i such that $i \in \mathrm{Seg}(\mathrm{len}\, w+1)$ holds $(q_8, \langle s\rangle ^\frown w)$-admissible$(i+1) = (q'_1, w)$-admissible$(i)$.

(35) If q'_1 = (the Tran of t_1)($\langle q_8, s\rangle$), then $(q_8, \langle s\rangle ^\frown w)$-response = $\langle$(the OFun of t_1)($\langle q_8, s\rangle$)$\rangle$ $^\frown$ (q'_1, w)-response.

(36) q_8 and q_9 are equivalent if and only if for every s holds (the OFun of t_1)($\langle q_8, s\rangle$) = (the OFun of t_1)($\langle q_9, s\rangle$) and (the Tran of t_1)($\langle q_8, s\rangle$) and (the Tran of t_1)($\langle q_9, s\rangle$) are equivalent.

(37) Suppose q_8 and q_9 are equivalent. Given w, i. Suppose $i \in \mathrm{dom}\, w$. Then there exist states q_{16}, q_{17} of t_1 such that $q_{16} = (q_8, w)$-admissible(i) and $q_{17} = (q_9, w)$-admissible(i) and q_{16} and q_{17} are equivalent.

Let us consider I_1, O_1, t_1, q_8, q_9, k. We say that q_8 and q_9 are k-equivalent if and only if:

(Def.11) For every w such that $\operatorname{len} w \leq k$ holds (q_8, w)-response $=$ (q_9, w)-response.

One can prove the following propositions:

(38) q_8 and q_8 are k-equivalent.

(39) If q_8 and q_9 are k-equivalent, then q_9 and q_8 are k-equivalent.

(40) If q_8 and q_9 are k-equivalent and q_9 and q_{10} are k-equivalent, then q_8 and q_{10} are k-equivalent.

(41) If q_8 and q_9 are equivalent, then q_8 and q_9 are k-equivalent.

(42) q_8 and q_9 are 0-equivalent.

(43) If q_8 and q_9 are $k+m$-equivalent, then q_8 and q_9 are k-equivalent.

(44) Suppose $1 \leq k$. Then q_8 and q_9 are k-equivalent if and only if the following conditions are satisfied:

(i) q_8 and q_9 are 1-equivalent, and

(ii) for every element s of I_1 and for every natural number k_1 such that $k_1 = k-1$ holds (the Tran of t_1)($\langle q_8, s\rangle$) and (the Tran of t_1)($\langle q_9, s\rangle$) are k_1-equivalent.

Let us consider I_1, O_1, t_1, i. The functor i-EqS-Rel(t_1) yielding an equivalence relation of the states of t_1 is defined as follows:

(Def.12) For all q_8, q_9 holds $\langle q_8, q_9\rangle \in i$-EqS-Rel($t_1$) iff q_8 and q_9 are i-equivalent.

Let us consider I_1, O_1, t_1, i. The functor i-EqS-Part(t_1) yields a non empty finite partition of the states of t_1 and is defined by:

(Def.13) i-EqS-Part(t_1) = Classes(i-EqS-Rel(t_1)).

One can prove the following propositions:

(45) $(k+1)$-EqS-Part(t_1) is finer than k-EqS-Part(t_1).

(46) If Classes(k-EqS-Rel(t_1)) = Classes($(k+1)$-EqS-Rel(t_1)), then for every m holds Classes($(k+m)$-EqS-Rel(t_1)) = Classes(k-EqS-Rel(t_1)).

(47) If k-EqS-Part(t_1) = $(k+1)$-EqS-Part(t_1), then for every m holds $(k+m)$-EqS-Part(t_1) = k-EqS-Part(t_1).

(48) If $(k+1)$-EqS-Part(t_1) $\neq$ k-EqS-Part(t_1), then for every i such that $i \leq k$ holds $(i+1)$-EqS-Part(t_1) $\neq$ i-EqS-Part(t_1).

(49) k-EqS-Part(t_1) = $(k+1)$-EqS-Part(t_1) or card(k-EqS-Part(t_1)) $<$ card($(k+1)$-EqS-Part(t_1)).

(50) $[q]_{0\text{-EqS-Rel}(t_1)}$ = the states of t_1.

(51) 0-EqS-Part(t_1) = {the states of t_1}.

(52) If $n+1$ = card (the states of t_1), then $(n+1)$-EqS-Part(t_1) = n-EqS-Part(t_1).

Let us consider I_1, O_1, t_1. A partition of the states of t_1 is final if:

(Def.14) For all q_8, q_9 holds q_8 and q_9 are equivalent iff there exists an element X of it such that $q_8 \in X$ and $q_9 \in X$.

Next we state three propositions:

(53) If k-EqS-Part(t_1) is final, then $(k+1)$-EqS-Rel(t_1) = k-EqS-Rel(t_1).

(54) k-EqS-Part$(t_1) = (k+1)$-EqS-Part(t_1) iff k-EqS-Part(t_1) is final.

(55) If $n + 1 = \mathrm{card}$ (the states of t_1), then there exists a natural number k such that $k \leq n$ and k-EqS-Part(t_1) is final.

Let us consider I_1, O_1, t_1. The functor final-Partition(t_1) yields a partition of the states of t_1 and is defined by:

(Def.15) final-Partition(t_1) is final.

We now state the proposition

(56) If $n + 1 = \mathrm{card}$ (the states of t_1), then final-Partition$(t_1) =$ n-EqS-Part(t_1).

5. The Reduction of a Mealy Machine

In the sequel r_1 will be a Mealy-FSM over I_1, O_1, q_{18} will be a state of r_1, and q_{19} will be an element of final-Partition(t_1).

Let us consider I_1, O_1, t_1, q_{19}, s. The functor (s, q_{19})-C-succ yields an element of final-Partition(t_1) and is defined by:

(Def.16) There exist q, n such that $q \in q_{19}$ and $n + 1 = \mathrm{card}$ (the states of t_1) and (s, q_{19})-C-succ $= [(\text{the Tran of } t_1)(\langle q, s\rangle)]_{n\text{-EqS-Rel}(t_1)}$.

Let us consider I_1, O_1, t_1, q_{19}, s. The functor (q_{19}, s)-C-response yielding an element of O_1 is defined by:

(Def.17) There exists q such that $q \in q_{19}$ and (q_{19}, s)-C-response = (the OFun of t_1)$(\langle q, s\rangle)$.

Let us consider I_1, O_1, t_1. The reduction of t_1 yielding a strict Mealy-FSM over I_1, O_1 is defined by the conditions (Def.18).

(Def.18) (i) The states of the reduction of t_1 = final-Partition(t_1),

(ii) for every state Q of the reduction of t_1 and for all s, q such that $q \in Q$ holds (the Tran of t_1)$(\langle q, s\rangle) \in$ (the Tran of the reduction of t_1)$(\langle Q, s\rangle)$ and (the OFun of t_1)$(\langle q, s\rangle) =$ (the OFun of the reduction of t_1)$(\langle Q, s\rangle)$, and

(iii) the InitS of $t_1 \in$ the InitS of the reduction of t_1.

The following two propositions are true:

(57) If $r_1 =$ the reduction of t_1 and $q \in q_{18}$, then for every k such that $k \in \mathrm{Seg}(\mathrm{len}\, w + 1)$ holds (q, w)-admissible$(k) \in (q_{18}, w)$-admissible(k).

(58) t_1 and the reduction of t_1 are equivalent.

6. Machine Isomorphism

In the sequel q_{20}, q_{23} will denote states of r_1 and T_1 will denote a function from the states of t_2 into the states of t_3.

Let us consider I_1, O_1, t_2, t_3. We say that t_2 and t_3 are isomorphic if and only if the condition (Def.19) is satisfied.

(Def.19) There exists T_1 such that
(i) T_1 is bijective,
(ii) T_1(the InitS of t_2) = the InitS of t_3, and
(iii) for all q_{14}, s holds T_1((the Tran of t_2)($\langle q_{14}, s\rangle$)) = (the Tran of t_3)($\langle T_1(q_{14}), s\rangle$) and (the OFun of t_2)($\langle q_{14}, s\rangle$) = (the OFun of t_3)($\langle T_1(q_{14}), s\rangle$).

Let us observe that the predicate introduced above is reflexive and symmetric.

We now state four propositions:

(59) If t_2 and t_3 are isomorphic and t_3 and t_4 are isomorphic, then t_2 and t_4 are isomorphic.

(60) Suppose that for every state q of t_2 and for every s holds T_1((the Tran of t_2)($\langle q, s\rangle$)) = (the Tran of t_3)($\langle T_1(q), s\rangle$). Given k. If $1 \leq k$ and $k \leq \operatorname{len} w + 1$, then $T_1((q_{14}, w)\text{-admissible}(k)) = (T_1(q_{14}), w)\text{-admissible}(k)$.

(61) Suppose that
(i) T_1(the InitS of t_2) = the InitS of t_3, and
(ii) for every state q of t_2 and for every s holds T_1((the Tran of t_2)($\langle q, s\rangle$)) = (the Tran of t_3)($\langle T_1(q), s\rangle$) and (the OFun of t_2)($\langle q, s\rangle$) = (the OFun of t_3)($\langle T_1(q), s\rangle$).

Then q_{14} and q_{15} are equivalent if and only if $T_1(q_{14})$ and $T_1(q_{15})$ are equivalent.

(62) If r_1 = the reduction of t_1 and $q_{20} \neq q_{23}$, then q_{20} and q_{23} are not equivalent.

7. Reduced and Connected Machines

Let I_1, O_1 be non empty sets. A Mealy-FSM over I_1, O_1 is reduced if:

(Def.20) For all states q_8, q_9 of it such that $q_8 \neq q_9$ holds q_8 and q_9 are not equivalent.

One can prove the following proposition

(63) The reduction of t_1 is reduced.

Let us consider I_1, O_1. Note that there exists a Mealy-FSM over I_1, O_1 which is reduced.

In the sequel R_1 will denote a reduced Mealy-FSM over I_1, O_1.

Next we state two propositions:

(64) R_1 and the reduction of R_1 are isomorphic.

(65) t_1 is reduced iff there exists a Mealy-FSM M over I_1, O_1 such that t_1 and the reduction of M are isomorphic.

Let us consider I_1, O_1, t_1. A state of t_1 is accessible if:

(Def.21) There exists w such that the InitS of $t_1 \xrightarrow{w}$ it.

Let us consider I_1, O_1. A Mealy-FSM over I_1, O_1 is connected if:

(Def.22) Every state of it is accessible.

Let us consider I_1, O_1. One can check that there exists a Mealy-FSM over I_1, O_1 which is connected.

In the sequel C_1, C_2, C_3 will be connected Mealy-FSM over I_1, O_1.

We now state the proposition

(66) The reduction of C_1 is connected.

Let us consider I_1, O_1. Note that there exists a Mealy-FSM over I_1, O_1 which is connected and reduced.

Let us consider I_1, O_1, t_1. The functor accessible-States(t_1) yields a finite non empty set and is defined as follows:

(Def.23) accessible-States(t_1) $= \{q : q$ ranges over states of t_1, q is accessible$\}$.

The following propositions are true:

(67) accessible-States(t_1) $\subseteq$ the states of t_1 and for every q holds $q \in$ accessible-States(t_1) iff q is accessible.

(68) (The Tran of t_1) $\restriction$ $[\!:$ accessible-States(t_1), $I_1 :\!]$ is a function from $[\!:$ accessible-States(t_1), $I_1 :\!]$ into accessible-States(t_1).

(69) Let c_1 be a function from $[\!:$ accessible-States(t_1), $I_1 :\!]$ into accessible-States(t_1), and let c_2 be a function from $[\!:$ accessible-States(t_1), $I_1 :\!]$ into O_1, and let c_3 be an element of accessible-States(t_1). Suppose $c_1 =$ (the Tran of t_1) $\restriction$ $[\!:$ accessible-States(t_1), $I_1 :\!]$ and $c_2 =$ (the OFun of t_1) $\restriction$ $[\!:$ accessible-States(t_1), $I_1 :\!]$ and $c_3 =$ the InitS of t_1. Then t_1 and Mealy-FSM$\langle$accessible-States$(t_1), c_1, c_2, c_3\rangle$ are equivalent.

(70) There exists C_1 such that
  (i) the Tran of $C_1 =$ (the Tran of t_1) $\restriction$ $[\!:$ accessible-States(t_1), $I_1 :\!]$,
  (ii) the OFun of $C_1 =$ (the OFun of t_1) $\restriction$ $[\!:$ accessible-States(t_1), $I_1 :\!]$,
  (iii) the InitS of $C_1 =$ the InitS of t_1, and
  (iv) t_1 and C_1 are equivalent.

8. Machine Union

Let us consider I_1, O_1, t_2, t_3. The functor Mealy-U(t_2, t_3) yields a strict Mealy-FSM over I_1, O_1 and is defined by the conditions (Def.24).

(Def.24) (i) The states of Mealy-U$(t_2, t_3) =$ (the states of t_2) $\cup$ (the states of t_3),
  (ii) the Tran of Mealy-U$(t_2, t_3) =$ (the Tran of t_2) $+\cdot$ (the Tran of t_3),
  (iii) the OFun of Mealy-U$(t_2, t_3) =$ (the OFun of t_2) $+\cdot$ (the OFun of t_3), and
  (iv) the InitS of Mealy-U$(t_2, t_3) =$ the InitS of t_2.

One can prove the following propositions:

(71) If $t_1 =$ Mealy-U(t_2, t_3) and (the states of t_2) $\cap$ (the states of t_3) $= \emptyset$ and $q_{14} = q$, then (q_{14}, w)-admissible $= (q, w)$-admissible.

(72) If $t_1 = \text{Mealy-U}(t_2, t_3)$ and (the states of t_2) $\cap$ (the states of t_3) $= \emptyset$ and $q_{14} = q$, then (q_{14}, w)-response $= (q, w)$-response.

(73) If $t_1 = \text{Mealy-U}(t_2, t_3)$ and (the states of t_2) $\cap$ (the states of t_3) $= \emptyset$ and $q_{21} = q$, then (q_{21}, w)-admissible $= (q, w)$-admissible.

(74) If $t_1 = \text{Mealy-U}(t_2, t_3)$ and (the states of t_2) $\cap$ (the states of t_3) $= \emptyset$ and $q_{21} = q$, then (q_{21}, w)-response $= (q, w)$-response.

In the sequel R_2, R_3 will be reduced Mealy-FSM over I_1, O_1.

The following proposition is true

(75) Suppose $t_1 = \text{Mealy-U}(R_2, R_3)$ and (the states of R_2) $\cap$ (the states of R_3) $= \emptyset$ and R_2 and R_3 are equivalent. Then there exists a state Q of the reduction of t_1 such that the InitS of $R_2 \in Q$ and the InitS of $R_3 \in Q$ and $Q =$ the InitS of the reduction of t_1.

For simplicity we follow a convention: C_4, C_5 will denote connected reduced Mealy-FSM over I_1, O_1, c_{11}, c_{12} will denote states of C_4, c_{21}, c_{22} will denote states of C_5, and q_{24}, q_{25} will denote states of t_1.

The following propositions are true:

(76) Suppose that
- (i) $c_{11} = q_{24}$,
- (ii) $c_{12} = q_{25}$,
- (iii) (the states of C_4) $\cap$ (the states of C_5) $= \emptyset$,
- (iv) C_4 and C_5 are equivalent,
- (v) $t_1 = \text{Mealy-U}(C_4, C_5)$, and
- (vi) c_{11} and c_{12} are not equivalent.

Then q_{24} and q_{25} are not equivalent.

(77) Suppose that
- (i) $c_{21} = q_{24}$,
- (ii) $c_{22} = q_{25}$,
- (iii) (the states of C_4) $\cap$ (the states of C_5) $= \emptyset$,
- (iv) C_4 and C_5 are equivalent,
- (v) $t_1 = \text{Mealy-U}(C_4, C_5)$, and
- (vi) c_{21} and c_{22} are not equivalent.

Then q_{24} and q_{25} are not equivalent.

(78) Suppose (the states of C_4) $\cap$ (the states of C_5) $= \emptyset$ and C_4 and C_5 are equivalent and $t_1 = \text{Mealy-U}(C_4, C_5)$. Let Q be a state of the reduction of t_1. Then there do not exist elements q_1, q_2 of Q such that $q_1 \in$ the states of C_4 and $q_2 \in$ the states of C_4 and $q_1 \neq q_2$.

(79) Suppose (the states of C_4) $\cap$ (the states of C_5) $= \emptyset$ and C_4 and C_5 are equivalent and $t_1 = \text{Mealy-U}(C_4, C_5)$. Let Q be a state of the reduction of t_1. Then there do not exist elements q_1, q_2 of Q such that $q_1 \in$ the states of C_5 and $q_2 \in$ the states of C_5 and $q_1 \neq q_2$.

(80) Suppose (the states of C_4) $\cap$ (the states of C_5) $= \emptyset$ and C_4 and C_5 are equivalent and $t_1 = \text{Mealy-U}(C_4, C_5)$. Let Q be a state of the reduction of t_1. Then there exist elements q_1, q_2 of Q such that $q_1 \in$ the states of

C_4 and $q_2 \in$ the states of C_5 and for every element q of Q holds $q = q_1$ or $q = q_2$.

9. The Minimization Theorem

We now state several propositions:

(81) There exist Mealy-FSM f_4, f_5 over I_1, O_1 such that (the states of f_4) $\cap$ (the states of f_5) $= \emptyset$ and f_4 and t_2 are isomorphic and f_5 and t_3 are isomorphic.

(82) If t_2 and t_3 are isomorphic, then t_2 and t_3 are equivalent.

(83) If (the states of C_4)$\cap$(the states of C_5) $= \emptyset$ and C_4 and C_5 are equivalent, then C_4 and C_5 are isomorphic.

(84) If C_2 and C_3 are equivalent, then the reduction of C_2 and the reduction of C_3 are isomorphic.

(85) Let M_1, M_2 be connected reduced Mealy-FSM over I_1, O_1. Then M_1 and M_2 are isomorphic if and only if M_1 and M_2 are equivalent.

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Subtrees [1]

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Summary. The concepts of root tree, the set of successors of a node in decorated tree and sets of subtrees are introduced.

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The notation and terminology used here are introduced in the following papers: [16], [17], [15], [3], [18], [12], [13], [9], [14], [11], [7], [2], [1], [4], [6], [8], [5], and [10].

1. Root Tree and Successors of Node in Decorated Tree

One can check that every tree which is finite is also finite-order.

The following propositions are true:

(1) For every decorated tree t holds $t \restriction \varepsilon_{\mathbb{N}} = t$.

(2) For every tree t and for all finite sequences p, q of elements of $\mathbb{N}$ such that $p \mathbin{^\frown} q \in t$ holds $t \restriction (p \mathbin{^\frown} q) = t \restriction p \restriction q$.

(3) Let t be a decorated tree and let p, q be finite sequences of elements of $\mathbb{N}$. If $p \mathbin{^\frown} q \in \operatorname{dom} t$, then $t \restriction (p \mathbin{^\frown} q) = t \restriction p \restriction q$.

A decorated tree is root if:

(Def.1) $\operatorname{dom} \mathrm{it} =$ the elementary tree of 0.

Let us note that every decorated tree which is root is also finite.

The following three propositions are true:

(4) For every decorated tree t holds t is root iff $\varepsilon \in \operatorname{Leaves}(\operatorname{dom} t)$.

(5) For every tree t and for every element p of t holds $t \restriction p =$ the elementary tree of 0 iff $p \in \operatorname{Leaves}(t)$.

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(6) For every decorated tree t and for every node p of t holds $t \upharpoonright p$ is root iff $p \in \mathrm{Leaves}(\mathrm{dom}\, t)$.

Let us mention that there exists a decorated tree which is root and there exists a decorated tree which is finite and non root.

Let x be a set. Note that the root tree of x is finite and root.

A tree is finite-branching if:

(Def.2) For every element x of it holds $\mathrm{succ}\, x$ is finite.

Let us mention that every tree which is finite-order is also finite-branching.

Let us note that there exists a tree which is finite.

A decorated tree is finite-order if:

(Def.3) dom it is finite-order.

A decorated tree is finite-branching if:

(Def.4) dom it is finite-branching.

One can check that every decorated tree which is finite is also finite-order and every decorated tree which is finite-order is also finite-branching.

Let us observe that there exists a decorated tree which is finite.

Let t be a finite-order decorated tree. One can verify that $\mathrm{dom}\, t$ is finite-order.

Let t be a finite-branching decorated tree. Note that $\mathrm{dom}\, t$ is finite-branching.

Let t be a finite-branching tree and let p be an element of t. Note that $\mathrm{succ}\, p$ is finite.

The scheme *FinOrdSet* concerns a unary functor $\mathcal{F}$ yielding a set and a finite set $\mathcal{A}$, and states that:

For every natural number n holds $\mathcal{F}(n) \in \mathcal{A}$ iff $n < \mathrm{card}\, \mathcal{A}$

provided the parameters have the following properties:

- For every set x such that $x \in \mathcal{A}$ there exists a natural number n such that $x = \mathcal{F}(n)$,
- For all natural numbers i, j such that $i < j$ and $\mathcal{F}(j) \in \mathcal{A}$ holds $\mathcal{F}(i) \in \mathcal{A}$,
- For all natural numbers i, j such that $\mathcal{F}(i) = \mathcal{F}(j)$ holds $i = j$.

Let X be a set. One can verify that there exists a finite sequence of elements of X which is one-to-one and empty.

The following proposition is true

(7) Let t be a finite-branching tree, and let p be an element of t, and let n be a natural number. Then $p \mathbin{\frown} \langle n \rangle \in \mathrm{succ}\, p$ if and only if $n < \mathrm{card}\, \mathrm{succ}\, p$.

Let t be a finite-branching tree and let p be an element of t. The functor $\mathrm{Succ}\, p$ yielding an one-to-one finite sequence of elements of t is defined by:

(Def.5) $\mathrm{len}\, \mathrm{Succ}\, p = \mathrm{card}\, \mathrm{succ}\, p$ and $\mathrm{rng}\, \mathrm{Succ}\, p = \mathrm{succ}\, p$ and for every natural number i such that $i < \mathrm{len}\, \mathrm{Succ}\, p$ holds $(\mathrm{Succ}\, p)(i+1) = p \mathbin{\frown} \langle i \rangle$.

Let t be a finite-branching decorated tree and let p be a finite sequence. Let us assume that $p \in \mathrm{dom}\, t$. The functor $\mathrm{succ}(t, p)$ yielding a finite sequence is defined by:

(Def.6) There exists an element q of $\operatorname{dom} t$ such that $q = p$ and $\operatorname{succ}(t, p) = t \cdot \operatorname{Succ} q$.

One can prove the following two propositions:

(8) Let t be a finite-branching decorated tree. Then there exists a set x and there exists a decorated tree yielding finite sequence p such that $t = x\text{-tree}(p)$.

(9) For every finite decorated tree t and for every node p of t holds $t \restriction p$ is finite.

Let t be a finite decorated tree and let p be a node of t. Observe that $t \restriction p$ is finite.

The following proposition is true

(10) For every finite tree t and for every element p of t such that $t = t \restriction p$ holds $p = \varepsilon$.

Let D be a non empty set and let S be a non empty subset of $\operatorname{FinTrees}(D)$. One can verify that every element of S is finite.

2. Set of Subtrees of Decorated Tree

Let t be a decorated tree. The functor $\operatorname{Subtrees}(t)$ yielding a constituted of decorated trees non empty set is defined by:

(Def.7) $\operatorname{Subtrees}(t) = \{t \restriction p : p \text{ ranges over nodes of } t\}$.

Let D be a non empty set and let t be a tree decorated with elements of D. Then $\operatorname{Subtrees}(t)$ is a non empty subset of $\operatorname{Trees}(D)$.

Let D be a non empty set and let t be a finite tree decorated with elements of D. Then $\operatorname{Subtrees}(t)$ is a non empty subset of $\operatorname{FinTrees}(D)$.

Let t be a finite decorated tree. One can verify that every element of $\operatorname{Subtrees}(t)$ is finite.

In the sequel x denotes a set and t, t_1, t_2 denote decorated trees.

One can prove the following propositions:

(11) $x \in \operatorname{Subtrees}(t)$ iff there exists a node n of t such that $x = t \restriction n$.

(12) $t \in \operatorname{Subtrees}(t)$.

(13) If t_1 is finite and $\operatorname{Subtrees}(t_1) = \operatorname{Subtrees}(t_2)$, then $t_1 = t_2$.

(14) For every node n of t holds $\operatorname{Subtrees}(t \restriction n) \subseteq \operatorname{Subtrees}(t)$.

Let t be a decorated tree. The functor $\operatorname{FixedSubtrees}(t)$ yields a non empty subset of $[\!:\operatorname{dom} t, \operatorname{Subtrees}(t):\!]$ and is defined as follows:

(Def.8) $\operatorname{FixedSubtrees}(t) = \{\langle p, t \restriction p \rangle : p \text{ ranges over nodes of } t\}$.

Next we state three propositions:

(15) $x \in \operatorname{FixedSubtrees}(t)$ iff there exists a node n of t such that $x = \langle n, t \restriction n \rangle$.

(16) $\langle \varepsilon, t \rangle \in \operatorname{FixedSubtrees}(t)$.

(17) If $\operatorname{FixedSubtrees}(t_1) = \operatorname{FixedSubtrees}(t_2)$, then $t_1 = t_2$.

Let t be a decorated tree and let C be a set. The functor C-Subtrees(t) yielding a subset of Subtrees(t) is defined as follows:

(Def.9) C-Subtrees$(t) = \{t \upharpoonright p : p$ ranges over nodes of t, $p \notin \mathrm{Leaves}(\mathrm{dom}\, t) \vee t(p) \in C\}$.

In the sequel C denotes a set.

We now state two propositions:

(18) $x \in C$-Subtrees(t) iff there exists a node n of t such that $x = t \upharpoonright n$ but $n \notin \mathrm{Leaves}(\mathrm{dom}\, t)$ or $t(n) \in C$.

(19) C-Subtrees(t) is empty iff t is root and $t(\varepsilon) \notin C$.

Let t be a finite decorated tree and let C be a set. The functor C-ImmediateSubtrees(t) yields a function from C-Subtrees(t) into $(\mathrm{Subtrees}(t))^*$ and is defined by the condition (Def.10).

(Def.10) Let d be a decorated tree. Suppose $d \in C$-Subtrees(t). Let p be a finite sequence of elements of Subtrees(t). If $p = (C\text{-ImmediateSubtrees}(t))(d)$, then $d = d(\varepsilon)$-tree(p).

3. Set of Subtrees of Set of Decorated Tree

Let X be a constituted of decorated trees non empty set. The functor Subtrees(X) yielding a constituted of decorated trees non empty set is defined by:

(Def.11) Subtrees$(X) = \{t \upharpoonright p : t$ ranges over elements of X, p ranges over nodes of $t\}$.

Let D be a non empty set and let X be a non empty subset of Trees(D). Then Subtrees(X) is a non empty subset of Trees(D).

Let D be a non empty set and let X be a non empty subset of FinTrees(D). Then Subtrees(X) is a non empty subset of FinTrees(D).

In the sequel X, Y will be non empty constituted of decorated trees sets.

We now state three propositions:

(20) $x \in \mathrm{Subtrees}(X)$ iff there exists an element t of X and there exists a node n of t such that $x = t \upharpoonright n$.

(21) If $t \in X$, then $t \in \mathrm{Subtrees}(X)$.

(22) If $X \subseteq Y$, then $\mathrm{Subtrees}(X) \subseteq \mathrm{Subtrees}(Y)$.

Let t be a decorated tree. Observe that $\{t\}$ is non empty and constituted of decorated trees.

Next we state two propositions:

(23) $\mathrm{Subtrees}(\{t\}) = \mathrm{Subtrees}(t)$.

(24) $\mathrm{Subtrees}(X) = \bigcup\{\mathrm{Subtrees}(t) : t$ ranges over elements of $X\}$.

Let X be a constituted of decorated trees non empty set and let C be a set. The functor C-Subtrees(X) yields a subset of Subtrees(X) and is defined as follows:

(Def.12) C-Subtrees$(X) = \{t \restriction p : t$ ranges over elements of X, p ranges over nodes of t, $p \notin \mathrm{Leaves}(\mathrm{dom}\, t) \ \vee \ t(p) \in C\}$.

We now state four propositions:

(25) $x \in C$-Subtrees(X) iff there exists an element t of X and there exists a node n of t such that $x = t \restriction n$ but $n \notin \mathrm{Leaves}(\mathrm{dom}\, t)$ or $t(n) \in C$.

(26) C-Subtrees(X) is empty iff for every element t of X holds t is root and $t(\varepsilon) \notin C$.

(27) C-Subtrees$(\{t\}) = C$-Subtrees(t).

(28) C-Subtrees$(X) = \bigcup\{C$-Subtrees$(t) : t$ ranges over elements of $X\}$.

Let X be a non empty constituted of decorated trees set. Let us assume that every element of X is finite. Let C be a set. The functor C-ImmediateSubtrees(X) yields a function from C-Subtrees(X) into $(\mathrm{Subtrees}(X))^*$ and is defined by the condition (Def.13).

(Def.13) Let d be a decorated tree. Suppose $d \in C$-Subtrees(X). Let p be a finite sequence of elements of Subtrees(X). If $p = (C$-ImmediateSubtrees$(X))(d)$, then $d = d(\varepsilon)$-tree(p).

Let t be a tree. Observe that there exists an element of t which is empty.

We now state four propositions:

(29) For every finite decorated tree t and for every element p of $\mathrm{dom}\, t$ holds $\mathrm{len}\,\mathrm{succ}(t, p) = \mathrm{len}\,\mathrm{Succ}\, p$ and $\mathrm{dom}\,\mathrm{succ}(t, p) = \mathrm{dom}\,\mathrm{Succ}\, p$.

(30) For every finite tree yielding finite sequence p and for every empty element n of $\widehat{p}$ holds $\mathrm{card}\,\mathrm{succ}\, n = \mathrm{len}\, p$.

(31) Let t be a finite decorated tree, and let x be a set, and let p be a decorated tree yielding finite sequence. Suppose $t = x$-tree(p). Let n be an empty element of $\mathrm{dom}\, t$. Then $\mathrm{succ}(t, n) =$ the roots of p.

(32) For every finite decorated tree t and for every node p of t and for every node q of $t \restriction p$ holds $\mathrm{succ}(t, p \frown q) = \mathrm{succ}(t \restriction p, q)$.

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Terms Over Many Sorted Universal Algebra [1]

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Summary. Pure terms (without constants) over a signature of many sorted universal algebra and terms with constants from algebra are introduced. Facts on evaluation of a term in some valuation are proved.

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The articles [19], [22], [2], [20], [23], [11], [9], [12], [14], [3], [5], [6], [21], [1], [13], [7], [4], [8], [18], [17], [10], [15], and [16] provide the terminology and notation for this paper.

1. Terms over a Signature and over an Algebra

Let I be a non empty set, let X be a non-empty many sorted set indexed by I, and let i be an element of I. Note that $X(i)$ is non empty.

In the sequel S will be a non void non empty many sorted signature and V will be a non-empty many sorted set indexed by the carrier of S.

Let us consider S, V. The functor S-Terms(V) yielding a non empty subset of FinTrees(the carrier of DTConMSA(V)) is defined as follows:

(Def.1) S-Terms$(V) = \mathrm{TS}(\mathrm{DTConMSA}(V))$.

Let us consider S, V. A term of S over V is an element of S-Terms(V).

In the sequel A denotes an algebra over S and t denotes a term of S over V.

Let us consider S, V and let o be an operation symbol of S. Then $\mathrm{Sym}(o, V)$ is a nonterminal of DTConMSA(V).

Let us consider S, V and let s_1 be a nonterminal of DTConMSA(V). A finite sequence of elements of S-Terms(V) is called an argument sequence of s_1 if:

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(Def.2) It is a subtree sequence joinable by s_1.

We now state the proposition

(1) Let o be an operation symbol of S and let a be a finite sequence. Then $\langle o$, the carrier of $S\rangle$-tree$(a) \in S$-Terms(V) and a is decorated tree yielding if and only if a is an argument sequence of $\mathrm{Sym}(o, V)$.

The scheme *TermInd* concerns a non void non empty many sorted signature $\mathcal{A}$, a non-empty many sorted set $\mathcal{B}$ indexed by the carrier of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

For every term t of $\mathcal{A}$ over $\mathcal{B}$ holds $\mathcal{P}[t]$

provided the parameters satisfy the following conditions:

- For every sort symbol s of $\mathcal{A}$ and for every element v of $\mathcal{B}(s)$ holds $\mathcal{P}$[the root tree of $\langle v, s\rangle$],
- Let o be an operation symbol of $\mathcal{A}$ and let p be an argument sequence of $\mathrm{Sym}(o, \mathcal{B})$. Suppose that for every term t of $\mathcal{A}$ over $\mathcal{B}$ such that $t \in \mathrm{rng}\, p$ holds $\mathcal{P}[t]$. Then $\mathcal{P}[\langle o$, the carrier of $\mathcal{A}\rangle$-tree$(p)]$.

Let us consider S, A, V. A term of A over V is a term of S over (the sorts of $A) \cup (V)$.

Let us consider S, A, V and let o be an operation symbol of S. An argument sequence of o, A, and V is an argument sequence of $\mathrm{Sym}(o,$ (the sorts of $A) \cup (V))$.

The scheme *CTermInd* concerns a non void non empty many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, a non-empty many sorted set $\mathcal{C}$ indexed by the carrier of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

For every term t of $\mathcal{B}$ over $\mathcal{C}$ holds $\mathcal{P}[t]$

provided the following requirements are met:

- For every sort symbol s of $\mathcal{A}$ and for every element x of (the sorts of $\mathcal{B})(s)$ holds $\mathcal{P}$[the root tree of $\langle x, s\rangle$],
- For every sort symbol s of $\mathcal{A}$ and for every element v of $\mathcal{C}(s)$ holds $\mathcal{P}$[the root tree of $\langle v, s\rangle$],
- Let o be an operation symbol of $\mathcal{A}$ and let p be an argument sequence of o, $\mathcal{B}$, and $\mathcal{C}$. Suppose that for every term t of $\mathcal{B}$ over $\mathcal{C}$ such that $t \in \mathrm{rng}\, p$ holds $\mathcal{P}[t]$. Then $\mathcal{P}[\mathrm{Sym}(o,$ (the sorts of $\mathcal{B}) \cup \mathcal{C})$-tree$(p)]$.

Let us consider S, V, t and let p be a node of t. Then $t(p)$ is a symbol of DTConMSA(V).

Let us consider S, V. Observe that every term of S over V is finite.

Next we state several propositions:

(2) (i) There exists a sort symbol s of S and there exists an element v of $V(s)$ such that $t(\varepsilon) = \langle v, s\rangle$, or

(ii) $t(\varepsilon) \in [\!:$ the operation symbols of S, {the carrier of S} $:\!]$.

(3) Let t be a term of A over V. Then

(i) there exists a sort symbol s of S and there exists a set x such that $x \in$ (the sorts of $A)(s)$ and $t(\varepsilon) = \langle x, s\rangle$, or

(ii) there exists a sort symbol s of S and there exists an element v of $V(s)$ such that $t(\varepsilon) = \langle v, s\rangle$, or

(iii) $t(\varepsilon) \in [\!:$ the operation symbols of S, {the carrier of S} $:\!]$.

(4) For every sort symbol s of S and for every element v of $V(s)$ holds the root tree of $\langle v, s\rangle$ is a term of S over V.

(5) For every sort symbol s of S and for every element v of $V(s)$ such that $t(\varepsilon) = \langle v, s\rangle$ holds $t =$ the root tree of $\langle v, s\rangle$.

(6) Let s be a sort symbol of S and let x be a set. Suppose $x \in$ (the sorts of A)(s). Then the root tree of $\langle x, s\rangle$ is a term of A over V.

(7) Let t be a term of A over V, and let s be a sort symbol of S, and let x be a set. If $x \in$ (the sorts of A)(s) and $t(\varepsilon) = \langle x, s\rangle$, then $t =$ the root tree of $\langle x, s\rangle$.

(8) For every sort symbol s of S and for every element v of $V(s)$ holds the root tree of $\langle v, s\rangle$ is a term of A over V.

(9) Let t be a term of A over V, and let s be a sort symbol of S, and let v be an element of $V(s)$. If $t(\varepsilon) = \langle v, s\rangle$, then $t =$ the root tree of $\langle v, s\rangle$.

(10) Let o be an operation symbol of S. Suppose $t(\varepsilon) = \langle o$, the carrier of $S\rangle$. Then there exists an argument sequence a of $\mathrm{Sym}(o, V)$ such that $t = \langle o$, the carrier of $S\rangle$-tree(a).

Let us consider S, let A be a non-empty algebra over S, let us consider V, let s be a sort symbol of S, and let x be an element of (the sorts of A)(s). The functor $x_{A,V}$ yielding a term of A over V is defined as follows:

(Def.3) $x_{A,V} =$ the root tree of $\langle x, s\rangle$.

Let us consider S, A, V, let s be a sort symbol of S, and let v be an element of $V(s)$. The functor v_A yields a term of A over V and is defined as follows:

(Def.4) $v_A =$ the root tree of $\langle v, s\rangle$.

Let us consider S, V, let s_1 be a nonterminal of DTConMSA(V), and let p be an argument sequence of s_1. Then s_1-tree(p) is a term of S over V.

The scheme *TermInd2* concerns a non void non empty many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, a non-empty many sorted set $\mathcal{C}$ indexed by the carrier of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

For every term t of $\mathcal{B}$ over $\mathcal{C}$ holds $\mathcal{P}[t]$

provided the following conditions are satisfied:

- For every sort symbol s of $\mathcal{A}$ and for every element x of (the sorts of $\mathcal{B}$)(s) holds $\mathcal{P}[x_{\mathcal{B},\mathcal{C}}]$,
- For every sort symbol s of $\mathcal{A}$ and for every element v of $\mathcal{C}(s)$ holds $\mathcal{P}[v_{\mathcal{B}}]$,
- Let o be an operation symbol of $\mathcal{A}$ and let p be an argument sequence of $\mathrm{Sym}(o,$ (the sorts of $\mathcal{B}) \cup \mathcal{C})$. Suppose that for every term t of $\mathcal{B}$ over $\mathcal{C}$ such that $t \in \operatorname{rng} p$ holds $\mathcal{P}[t]$. Then $\mathcal{P}[\mathrm{Sym}(o,$ (the sorts of $\mathcal{B}) \cup \mathcal{C})$-tree$(p)]$.

2. Sort of a Term

One can prove the following three propositions:

(11) For every term t of S over V there exists a sort symbol s of S such that $t \in \mathrm{FreeSort}(V, s)$.

(12) For every sort symbol s of S holds $\mathrm{FreeSort}(V, s) \subseteq S\text{-Terms}(V)$.

(13) $S\text{-Terms}(V) = \bigcup \mathrm{FreeSorts}(V)$.

Let us consider S, V, t. The sort of t yields a sort symbol of S and is defined by:

(Def.5) $t \in \mathrm{FreeSort}(V, \text{the sort of } t)$.

One can prove the following propositions:

(14) Let s be a sort symbol of S and let v be an element of $V(s)$. If $t =$ the root tree of $\langle v, s\rangle$, then the sort of $t = s$.

(15) Let t be a term of A over V, and let s be a sort symbol of S, and let x be a set. Suppose $x \in$ (the sorts of A)(s) and $t =$ the root tree of $\langle x, s\rangle$. Then the sort of $t = s$.

(16) Let t be a term of A over V, and let s be a sort symbol of S, and let v be an element of $V(s)$. If $t =$ the root tree of $\langle v, s\rangle$, then the sort of $t = s$.

(17) Let o be an operation symbol of S. Suppose $t(\varepsilon) = \langle o$, the carrier of $S\rangle$. Then the sort of $t =$ the result sort of o.

(18) Let A be a non-empty algebra over S, and let s be a sort symbol of S, and let x be an element of (the sorts of A)(s). Then the sort of $x_{A,V} = s$.

(19) For every sort symbol s of S and for every element v of $V(s)$ holds the sort of $v_A = s$.

(20) Let o be an operation symbol of S and let p be an argument sequence of $\mathrm{Sym}(o, V)$. Then the sort of ($\mathrm{Sym}(o, V)$-tree(p) **qua** term of S over V) = the result sort of o.

3. Argument Sequence

We now state several propositions:

(21) Let o be an operation symbol of S and let a be a finite sequence of elements of $S\text{-Terms}(V)$. Then a is an argument sequence of $\mathrm{Sym}(o, V)$ if and only if $\mathrm{Sym}(o, V) \Rightarrow$ the roots of a.

(22) Let o be an operation symbol of S and let a be an argument sequence of $\mathrm{Sym}(o, V)$. Then $\mathrm{len}\, a = \mathrm{len}\,\mathrm{Arity}(o)$ and $\mathrm{dom}\, a = \mathrm{dom}\,\mathrm{Arity}(o)$ and for every natural number i such that $i \in \mathrm{dom}\, a$ holds $a(i)$ is a term of S over V.

(23) Let o be an operation symbol of S, and let a be an argument sequence of $\mathrm{Sym}(o, V)$, and let i be a natural number. Suppose $i \in \mathrm{dom}\, a$. Let t be a term of S over V. Suppose $t = a(i)$. Then

(i) $t = \pi_i(a$ **qua** finite sequence of elements of S-$\mathrm{Terms}(V)$ **qua** non empty set),

(ii) the sort of $t = \mathrm{Arity}(o)(i)$, and

(iii) the sort of $t = \pi_i\, \mathrm{Arity}(o)$.

(24) Let o be an operation symbol of S and let a be a finite sequence. Suppose that

(i) $\mathrm{len}\, a = \mathrm{len}\, \mathrm{Arity}(o)$ or $\mathrm{dom}\, a = \mathrm{dom}\, \mathrm{Arity}(o)$, and

(ii) for every natural number i such that $i \in \mathrm{dom}\, a$ there exists a term t of S over V such that $t = a(i)$ and the sort of $t = \mathrm{Arity}(o)(i)$ or for every natural number i such that $i \in \mathrm{dom}\, a$ there exists a term t of S over V such that $t = a(i)$ and the sort of $t = \pi_i\, \mathrm{Arity}(o)$.

Then a is an argument sequence of $\mathrm{Sym}(o, V)$.

(25) Let o be an operation symbol of S and let a be a finite sequence of elements of S-$\mathrm{Terms}(V)$. Suppose that

(i) $\mathrm{len}\, a = \mathrm{len}\, \mathrm{Arity}(o)$ or $\mathrm{dom}\, a = \mathrm{dom}\, \mathrm{Arity}(o)$, and

(ii) for every natural number i such that $i \in \mathrm{dom}\, a$ and for every term t of S over V such that $t = a(i)$ holds the sort of $t = \mathrm{Arity}(o)(i)$ or for every natural number i such that $i \in \mathrm{dom}\, a$ and for every term t of S over V such that $t = a(i)$ holds the sort of $t = \pi_i\, \mathrm{Arity}(o)$.

Then a is an argument sequence of $\mathrm{Sym}(o, V)$.

(26) Let S be a non void non empty many sorted signature and let V_1, V_2 be non-empty many sorted sets indexed by the carrier of S. If $V_1 \subseteq V_2$, then every term of S over V_1 is a term of S over V_2.

(27) Let S be a non void non empty many sorted signature, and let A be an algebra over S, and let V be a non-empty many sorted set indexed by the carrier of S. Then every term of S over V is a term of A over V.

4. Compound Terms

Let S be a non void non empty many sorted signature and let V be a non-empty many sorted set indexed by the carrier of S. A term of S over V is said to be a compound term of S over V if:

(Def.6) $\mathrm{It}(\varepsilon) \in [\!:$ the operation symbols of S, {the carrier of S} $:\!]$.

Let S be a non void non empty many sorted signature and let V be a non-empty many sorted set indexed by the carrier of S. A non empty subset of S-$\mathrm{Terms}(V)$ is called a set with a compound term of S over V if:

(Def.7) There exists a compound term t of S over V such that $t \in \mathrm{it}$.

Next we state two propositions:

(28) If t is not root, then t is a compound term of S over V.

(29) For every node p of t holds $t \upharpoonright p$ is a term of S over V.

Let S be a non void non empty many sorted signature, let V be a non-empty many sorted set indexed by the carrier of S, let t be a term of S over V, and let p be a node of t. Then $t \upharpoonright p$ is a term of S over V.

5. Evaluation of Terms

Let S be a non void non empty many sorted signature and let A be an algebra over S. A non-empty many sorted set indexed by the carrier of S is said to be a variables family of A if:

(Def.8) It misses the sorts of A.

We now state the proposition

(30) Let V be a variables family of A, and let s be a sort symbol of S, and let x be a set. If $x \in$ (the sorts of A)(s), then for every element v of $V(s)$ holds $x \neq v$.

Let S be a non void non empty many sorted signature, let A be a non-empty algebra over S, let V be a non-empty many sorted set indexed by the carrier of S, let t be a term of A over V, let f be a many sorted function from V into the sorts of A, and let v_1 be a finite decorated tree. We say that v_1 is an evaluation of t w.r.t. f if and only if the conditions (Def.9) are satisfied.

(Def.9) (i) $\operatorname{dom} v_1 = \operatorname{dom} t$, and

(ii) for every node p of v_1 holds for every sort symbol s of S and for every element v of $V(s)$ such that $t(p) = \langle v, s \rangle$ holds $v_1(p) = f(s)(v)$ and for every sort symbol s of S and for every element x of (the sorts of A)(s) such that $t(p) = \langle x, s \rangle$ holds $v_1(p) = x$ and for every operation symbol o of S such that $t(p) = \langle o, \text{the carrier of } S \rangle$ holds $v_1(p) = (\mathrm{Den}(o, A))(\mathrm{succ}(v_1, p))$.

For simplicity we follow the rules: S will be a non void non empty many sorted signature, A will be a non-empty algebra over S, V will be a variables family of A, t will be a term of A over V, and f will be a many sorted function from V into the sorts of A.

We now state several propositions:

(31) Let s be a sort symbol of S and let x be an element of (the sorts of A)(s). Suppose $t =$ the root tree of $\langle x, s \rangle$. Then the root tree of x is an evaluation of t w.r.t. f.

(32) Let s be a sort symbol of S and let v be an element of $V(s)$. Suppose $t =$ the root tree of $\langle v, s \rangle$. Then the root tree of $f(s)(v)$ is an evaluation of t w.r.t. f.

(33) Let o be an operation symbol of S, and let p be an argument sequence of o, A, and V, and let q be a decorated tree yielding finite sequence. Suppose that

(i) $\operatorname{len} q = \operatorname{len} p$, and

(ii) for every natural number i and for every term t of A over V such that $i \in \operatorname{dom} p$ and $t = p(i)$ there exists a finite decorated tree v_1 such that $v_1 = q(i)$ and v_1 is an evaluation of t w.r.t. f.

Then there exists a finite decorated tree v_1 such that $v_1 = (\mathrm{Den}(o, A))$(the roots of q)-tree(q) and v_1 is an evaluation of Sym(o, (the sorts of A) $\cup$ (V))-tree(p) **qua** term of A over V w.r.t. f.

(34) Let t be a term of A over V and let e be a finite decorated tree. Suppose e is an evaluation of t w.r.t. f. Let p be a node of t and let n be a node of e. If $n = p$, then $e \restriction n$ is an evaluation of $t \restriction p$ w.r.t. f.

(35) Let o be an operation symbol of S, and let p be an argument sequence of o, A, and V, and let v_1 be a finite decorated tree. Suppose v_1 is an evaluation of Sym(o, (the sorts of A) $\cup$ (V))-tree(p) **qua** term of A over V w.r.t. f. Then there exists a decorated tree yielding finite sequence q such that

(i) $\operatorname{len} q = \operatorname{len} p$,

(ii) $v_1 = (\mathrm{Den}(o, A))$(the roots of q)-tree(q), and

(iii) for every natural number i and for every term t of A over V such that $i \in \operatorname{dom} p$ and $t = p(i)$ there exists a finite decorated tree v_1 such that $v_1 = q(i)$ and v_1 is an evaluation of t w.r.t. f.

(36) There exists finite decorated tree which is an evaluation of t w.r.t. f.

(37) Let e_1, e_2 be finite decorated trees. Suppose e_1 is an evaluation of t w.r.t. f and e_2 is an evaluation of t w.r.t. f. Then $e_1 = e_2$.

(38) Let v_1 be a finite decorated tree. Suppose v_1 is an evaluation of t w.r.t. f. Then $v_1(\varepsilon) \in$ (the sorts of A)(the sort of t).

Let S be a non void non empty many sorted signature, let A be a non-empty algebra over S, let V be a variables family of A, let t be a term of A over V, and let f be a many sorted function from V into the sorts of A. The functor $t\,^{@}\,f$ yields an element of (the sorts of A)(the sort of t) and is defined as follows:

(Def.10) There exists a finite decorated tree v_1 such that v_1 is an evaluation of t w.r.t. f and $t\,^{@}\,f = v_1(\varepsilon)$.

In the sequel t denotes a term of A over V.

We now state several propositions:

(39) For every finite decorated tree v_1 such that v_1 is an evaluation of t w.r.t. f holds $t\,^{@}\,f = v_1(\varepsilon)$.

(40) Let v_1 be a finite decorated tree. Suppose v_1 is an evaluation of t w.r.t. f. Let p be a node of t. Then $v_1(p) = t \restriction p\,^{@}\,f$.

(41) For every sort symbol s of S and for every element x of (the sorts of A)(s) holds $x_{A,V}\,^{@}\,f = x$.

(42) For every sort symbol s of S and for every element v of $V(s)$ holds $v_A\,^{@}\,f = f(s)(v)$.

(43) Let o be an operation symbol of S, and let p be an argument sequence of o, A, and V, and let q be a finite sequence. Suppose that

(i) $\operatorname{len} q = \operatorname{len} p$, and

(ii) for every natural number i such that $i \in \operatorname{dom} p$ and for every term t of A over V such that $t = p(i)$ holds $q(i) = t\,^{@}\,f$.
Then $(\mathrm{Sym}(o, (\text{the sorts of } A) \cup (V))\text{-tree}(p)$ **qua** term of A over $V)^{@}(f) = (\mathrm{Den}(o, A))(q)$.

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On the Decomposition of the Continuity

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Summary. This article is devoted to functions of general topological spaces. A function from X to Y is A-continuous if the counterimage of every open set V of Y belongs to A, where A is a collection of subsets of X. We give the following characteristics of the continuity, called decomposition of continuity: A function f is continuous if and only if it is both A-continuous and B-continuous.

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The articles [14], [12], [2], [1], [3], [10], [6], [8], [11], [5], [13], [9], [15], [7], and [4] provide the notation and terminology for this paper.

Let T be a topological space. A subset of the carrier of T is called an α-set of T if:

(Def.1) It $\subseteq \operatorname{Int} \overline{\operatorname{Int} \mathrm{it}}$.

A subset of the carrier of T is semi-open if:

(Def.2) It $\subseteq \overline{\operatorname{Int} \mathrm{it}}$.

A subset of the carrier of T is pre-open if:

(Def.3) It $\subseteq \operatorname{Int} \overline{\mathrm{it}}$.

A subset of the carrier of T is pre-semi-open if:

(Def.4) It $\subseteq \overline{\operatorname{Int} \overline{\mathrm{it}}}$.

A subset of the carrier of T is semi-pre-open if:

(Def.5) It $\subseteq \overline{\operatorname{Int} \mathrm{it}} \cup \operatorname{Int} \overline{\mathrm{it}}$.

Let T be a topological space and let B be a subset of the carrier of T. The functor $\mathrm{sInt}(B)$ yielding a subset of the carrier of T is defined as follows:

(Def.6) $\mathrm{sInt}(B) = B \cap \overline{\operatorname{Int} B}$.

The functor $\mathrm{pInt}(B)$ yielding a subset of the carrier of T is defined as follows:

(Def.7) $\mathrm{pInt}(B) = B \cap \operatorname{Int} \overline{B}$.

The functor $\alpha\mathrm{Int}(B)$ yielding a subset of the carrier of T is defined as follows:

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(Def.8) $\alpha\mathrm{Int}(B) = B \cap \mathrm{Int}\,\overline{\mathrm{Int}\,B}$.

The functor psInt(B) yields a subset of the carrier of T and is defined as follows:

(Def.9) $\mathrm{psInt}(B) = B \cap \overline{\mathrm{Int}\,\overline{B}}$.

Let T be a topological space and let B be a subset of the carrier of T. The functor spInt(B) yields a subset of the carrier of T and is defined by:

(Def.10) $\mathrm{spInt}(B) = \mathrm{sInt}(B) \cup \mathrm{pInt}(B)$.

Let T be a topological space. The functor T^α yields a family of subsets of the carrier of T and is defined as follows:

(Def.11) $T^\alpha = \{B : B$ ranges over subsets of the carrier of T, B is anα-set of $T\}$.

The functor SO(T) yielding a family of subsets of the carrier of T is defined by:

(Def.12) $\mathrm{SO}(T) = \{B : B$ ranges over subsets of the carrier of T, B is semi-open$\}$.

The functor PO(T) yielding a family of subsets of the carrier of T is defined as follows:

(Def.13) $\mathrm{PO}(T) = \{B : B$ ranges over subsets of the carrier of T, B is pre-open$\}$.

The functor SPO(T) yielding a family of subsets of the carrier of T is defined as follows:

(Def.14) $\mathrm{SPO}(T) = \{B : B$ ranges over subsets of the carrier of T, B is semi-pre-open$\}$.

The functor PSO(T) yields a family of subsets of the carrier of T and is defined by:

(Def.15) $\mathrm{PSO}(T) = \{B : B$ ranges over subsets of the carrier of T, B is pre-semi-open$\}$.

The functor $D(c,\alpha)(T)$ yielding a family of subsets of the carrier of T is defined as follows:

(Def.16) $D(c,\alpha)(T) = \{B : B$ ranges over subsets of the carrier of T, $\mathrm{Int}\,B = \alpha\mathrm{Int}(B)\}$.

The functor $D(c,p)(T)$ yielding a family of subsets of the carrier of T is defined by:

(Def.17) $D(c,p)(T) = \{B : B$ ranges over subsets of the carrier of T, $\mathrm{Int}\,B = \mathrm{pInt}(B)\}$.

The functor $D(c,s)(T)$ yielding a family of subsets of the carrier of T is defined by:

(Def.18) $D(c,s)(T) = \{B : B$ ranges over subsets of the carrier of T, $\mathrm{Int}\,B = \mathrm{sInt}(B)\}$.

The functor $D(c,ps)(T)$ yielding a family of subsets of the carrier of T is defined as follows:

(Def.19) $D(c,ps)(T) = \{B : B$ ranges over subsets of the carrier of T, $\mathrm{Int}\,B = \mathrm{psInt}(B)\}$.

The functor $D(\alpha,p)(T)$ yields a family of subsets of the carrier of T and is defined as follows:

(Def.20) $D(\alpha,p)(T) = \{B : B$ ranges over subsets of the carrier of T, $\alpha\mathrm{Int}(B) = \mathrm{pInt}(B)\}$.

The functor $D(\alpha,s)(T)$ yielding a family of subsets of the carrier of T is defined as follows:

(Def.21) $D(\alpha,s)(T) = \{B : B$ ranges over subsets of the carrier of T, $\alpha\mathrm{Int}(B) = \mathrm{sInt}(B)\}$.

The functor $D(\alpha,ps)(T)$ yields a family of subsets of the carrier of T and is defined as follows:

(Def.22) $D(\alpha,ps)(T) = \{B : B$ ranges over subsets of the carrier of T, $\alpha\mathrm{Int}(B) = \mathrm{psInt}(B)\}$.

The functor $D(p,sp)(T)$ yielding a family of subsets of the carrier of T is defined by:

(Def.23) $D(p,sp)(T) = \{B : B$ ranges over subsets of the carrier of T, $\mathrm{pInt}(B) = \mathrm{spInt}(B)\}$.

The functor $D(p,ps)(T)$ yielding a family of subsets of the carrier of T is defined by:

(Def.24) $D(p,ps)(T) = \{B : B$ ranges over subsets of the carrier of T, $\mathrm{pInt}(B) = \mathrm{psInt}(B)\}$.

The functor $D(sp,ps)(T)$ yields a family of subsets of the carrier of T and is defined as follows:

(Def.25) $D(sp,ps)(T) = \{B : B$ ranges over subsets of the carrier of T, $\mathrm{spInt}(B) = \mathrm{psInt}(B)\}$.

In the sequel T will be a topological space and B will be a subset of the carrier of T.

One can prove the following propositions:

(1) $\alpha\mathrm{Int}(B) = \mathrm{pInt}(B)$ iff $\mathrm{sInt}(B) = \mathrm{psInt}(B)$.

(2) B is an α-set of T iff $B = \alpha\mathrm{Int}(B)$.

(3) B is semi-open iff $B = \mathrm{sInt}(B)$.

(4) B is pre-open iff $B = \mathrm{pInt}(B)$.

(5) B is pre-semi-open iff $B = \mathrm{psInt}(B)$.

(6) B is semi-pre-open iff $B = \mathrm{spInt}(B)$.

(7) $T^\alpha \cap D(c,\alpha)(T) =$ the topology of T.

(8) $\mathrm{SO}(T) \cap D(c,s)(T) =$ the topology of T.

(9) $\mathrm{PO}(T) \cap D(c,p)(T) =$ the topology of T.

(10) $\mathrm{PSO}(T) \cap D(c,ps)(T) =$ the topology of T.

(11) $\mathrm{PO}(T) \cap D(\alpha,p)(T) = T^\alpha$.

(12) $\mathrm{SO}(T) \cap D(\alpha,s)(T) = T^\alpha$.

(13) $\mathrm{PSO}(T) \cap D(\alpha,ps)(T) = T^\alpha$.

(14) $\mathrm{SPO}(T) \cap D(p,sp)(T) = \mathrm{PO}(T)$.

(15) $\mathrm{PSO}(T) \cap D(p,ps)(T) = \mathrm{PO}(T)$.

(16) $\mathrm{PSO}(T) \cap D(\alpha,p)(T) = \mathrm{SO}(T)$.

(17) $\mathrm{PSO}(T) \cap D(sp, ps)(T) = \mathrm{SPO}(T)$.

Let X, Y be topological spaces and let f be a mapping from X into Y. We say that f is s -continuous if and only if:

(Def.26) For every subset G of the carrier of Y such that G is open holds $f^{-1}G \in \mathrm{SO}(X)$.

We say that f is p -continuous if and only if:

(Def.27) For every subset G of the carrier of Y such that G is open holds $f^{-1}G \in \mathrm{PO}(X)$.

We say that f is α -continuous if and only if:

(Def.28) For every subset G of the carrier of Y such that G is open holds $f^{-1}G \in X^{\alpha}$.

We say that f is ps -continuous if and only if:

(Def.29) For every subset G of the carrier of Y such that G is open holds $f^{-1}G \in \mathrm{PSO}(X)$.

We say that f is sp -continuous if and only if:

(Def.30) For every subset G of the carrier of Y such that G is open holds $f^{-1}G \in \mathrm{SPO}(X)$.

We say that f is (c, α) -continuous if and only if:

(Def.31) For every subset G of the carrier of Y such that G is open holds $f^{-1}G \in D(c, \alpha)(X)$.

We say that f is (c, s) -continuous if and only if:

(Def.32) For every subset G of the carrier of Y such that G is open holds $f^{-1}G \in D(c, s)(X)$.

We say that f is (c, p) -continuous if and only if:

(Def.33) For every subset G of the carrier of Y such that G is open holds $f^{-1}G \in D(c, p)(X)$.

We say that f is (c, ps) -continuous if and only if:

(Def.34) For every subset G of the carrier of Y such that G is open holds $f^{-1}G \in D(c, ps)(X)$.

We say that f is (α, p) -continuous if and only if:

(Def.35) For every subset G of the carrier of Y such that G is open holds $f^{-1}G \in D(\alpha, p)(X)$.

We say that f is (α, s) -continuous if and only if:

(Def.36) For every subset G of the carrier of Y such that G is open holds $f^{-1}G \in D(\alpha, s)(X)$.

We say that f is (α, ps) -continuous if and only if:

(Def.37) For every subset G of the carrier of Y such that G is open holds $f^{-1}G \in D(\alpha, ps)(X)$.

We say that f is (p, ps) -continuous if and only if:

(Def.38) For every subset G of the carrier of Y such that G is open holds $f^{-1}G \in D(p, ps)(X)$.

We say that f is (p, sp) -continuous if and only if:

(Def.39) For every subset G of the carrier of Y such that G is open holds $f^{-1}G \in D(p, sp)(X)$.

We say that f is (sp, ps) -continuous if and only if:

(Def.40) For every subset G of the carrier of Y such that G is open holds $f^{-1}G \in D(sp, ps)(X)$.

In the sequel X, Y will denote topological spaces and f will denote a mapping from X into Y.

The following propositions are true:

(18) f is α -continuous iff f is p -continuous and (α, p) -continuous.

(19) f is α -continuous iff f is s -continuous and (α, s) -continuous.

(20) f is α -continuous iff f is ps -continuous and (α, ps) -continuous.

(21) f is p -continuous iff f is sp -continuous and (p, sp) -continuous.

(22) f is p -continuous iff f is ps -continuous and (p, ps) -continuous.

(23) f is s -continuous iff f is ps -continuous and (α, p) -continuous.

(24) f is sp -continuous iff f is ps -continuous and (sp, ps) -continuous.

(25) f is continuous iff f is α -continuous and (c, α) -continuous.

(26) f is continuous iff f is s -continuous and (c, s) -continuous.

(27) f is continuous iff f is p -continuous and (c, p) -continuous.

(28) f is continuous iff f is ps -continuous and (c, ps) -continuous.

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A Scheme for Extensions of Homomorphisms of Many Sorted Algebras

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Summary. The aim of this workis to provide a bridge between the theory of context-free grammars developed in [11], [6] and universally free manysorted algebras([17]. The third scheme proved in the article allows to prove that two homomorphisms equal on the set of free generators are equal. The first scheme is a slight modification of the scheme in [6] and the second is rather technical, but since it was useful for me, perhaps it might be useful for somebody else. The concept of flattening of a many sorted function F between two manysorted sets A and B (with common set of indices I) is introduced for A with mutually disjoint components (pairwise disjoint function – the concept introduced in [16]). This is a function on the union of A, that is equal to F on every component of A. A trivial many sorted algebra over a signature S is defined with sorts being singletons of corresponding sort symbols. It has mutually disjoint sorts.

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The notation and terminology used in this paper are introduced in the following articles: [20], [23], [24], [8], [9], [21], [5], [7], [14], [16], [3], [22], [2], [4], [1], [15], [11], [6], [10], [19], [13], [18], [17], and [12].

One can prove the following proposition

(1) For all functions f, g such that $g \in \prod f$ holds $\operatorname{rng} g \subseteq \bigcup f$.

The scheme *DTConstrUniq* concerns a non empty tree construction structure $\mathcal{A}$, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and functions $\mathcal{C}$, $\mathcal{D}$ from $\mathrm{TS}(\mathcal{A})$ into $\mathcal{B}$, and states that:

$\mathcal{C} = \mathcal{D}$

provided the parameters meet the following conditions:

- For every symbol t of $\mathcal{A}$ such that $t \in$ the terminals of $\mathcal{A}$ holds $\mathcal{C}(\text{the root tree of } t) = \mathcal{F}(t)$,

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- Let n_1 be a symbol of $\mathcal{A}$ and let t_1 be a finite sequence of elements of TS($\mathcal{A}$). Suppose $n_1 \Rightarrow$ the roots of t_1. Let x be a finite sequence of elements of $\mathcal{B}$. If $x = \mathcal{C} \cdot t_1$, then $\mathcal{C}(n_1\text{-tree}(t_1)) = \mathcal{G}(n_1, t_1, x)$,
- For every symbol t of $\mathcal{A}$ such that $t \in$ the terminals of $\mathcal{A}$ holds $\mathcal{D}$(the root tree of t) $= \mathcal{F}(t)$,
- Let n_1 be a symbol of $\mathcal{A}$ and let t_1 be a finite sequence of elements of TS($\mathcal{A}$). Suppose $n_1 \Rightarrow$ the roots of t_1. Let x be a finite sequence of elements of $\mathcal{B}$. If $x = \mathcal{D} \cdot t_1$, then $\mathcal{D}(n_1\text{-tree}(t_1)) = \mathcal{G}(n_1, t_1, x)$.

The following two propositions are true:

(2) Let S be a non void non empty many sorted signature, and let X be a many sorted set indexed by the carrier of S, and let o, b be arbitrary. Suppose $\langle o, b\rangle \in$ REL(X). Then
 (i) $o \in [\!:$ the operation symbols of S, {the carrier of S} $:\!]$, and
 (ii) $b \in ([\!:$ the operation symbols of S, {the carrier of S} $:\!] \cup \bigcup \mathrm{coprod}(X))^*$.

(3) Let S be a non void non empty many sorted signature, and let X be a many sorted set indexed by the carrier of S, and let o be an operation symbol of S, and let b be a finite sequence. Suppose $\langle\langle o$, the carrier of $S\rangle$, $b\rangle \in$ REL(X). Then
 (i) $\mathrm{len}\, b = \mathrm{len}\, \mathrm{Arity}(o)$, and
 (ii) for arbitrary x such that $x \in \mathrm{dom}\, b$ holds if $b(x) \in [\!:$ the operation symbols of S, {the carrier of S} $:\!]$, then for every operation symbol o_1 of S such that $\langle o_1$, the carrier of $S\rangle = b(x)$ holds the result sort of $o_1 = \mathrm{Arity}(o)(x)$ and if $b(x) \in \bigcup \mathrm{coprod}(X)$, then $b(x) \in \mathrm{coprod}(\mathrm{Arity}(o)(x), X)$.

Let I be a non empty set and let M be a non-empty many sorted set indexed by I. Observe that $\mathrm{rng}\, M$ is non empty and has non empty elements.

Let D be a non empty set with non empty elements. Note that $\bigcup D$ is non empty.

Let I be a set. One can check that every many sorted set indexed by I which is empty is also pairwise disjoint.

Let I be a set. Observe that there exists a many sorted set indexed by I which is pairwise disjoint.

Let I be a non empty set, let X be a pairwise disjoint many sorted set indexed by I, let D be a non-empty many sorted set indexed by I, and let F be a many sorted function from X into D. The functor Flatten(F) yields a function from $\bigcup X$ into $\bigcup D$ and is defined by:

(Def.1) For every element i of I and for arbitrary x such that $x \in X(i)$ holds $(\mathrm{Flatten}(F))(x) = F(i)(x)$.

The following proposition is true

(4) Let I be a non empty set, and let X be a pairwise disjoint many sorted set indexed by I, and let D be a non-empty many sorted set indexed by I, and let F_1, F_2 be many sorted functions from X into D. If $\mathrm{Flatten}(F_1) = \mathrm{Flatten}(F_2)$, then $F_1 = F_2$.

Let S be a non empty many sorted signature and let A be an algebra over S. We say that A is pairwise disjoint if and only if:

(Def.2) The sorts of A is pairwise disjoint.

Let S be a non empty many sorted signature. The functor SingleAlg(S) yields a strict algebra over S and is defined by:

(Def.3) For arbitrary i such that $i \in$ the carrier of S holds (the sorts of SingleAlg(S))(i) = $\{i\}$.

Let S be a non empty many sorted signature. Note that there exists an algebra over S which is non-empty and pairwise disjoint.

Let S be a non empty many sorted signature. Observe that SingleAlg(S) is non-empty and pairwise disjoint.

Let S be a non empty many sorted signature and let A be a pairwise disjoint algebra over S. Observe that the sorts of A is pairwise disjoint.

The following proposition is true

(5) Let S be a non void non empty many sorted signature, and let o be an operation symbol of S, and let A_1 be a non-empty pairwise disjoint algebra over S, and let A_2 be a non-empty algebra over S, and let f be a many sorted function from A_1 into A_2, and let a be an element of $\mathrm{Args}(o, A_1)$. Then $\mathrm{Flatten}(f) \cdot a = f\#a$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. Observe that FreeSorts(X) is pairwise disjoint.

The scheme *FreeSortUniq* deals with a non void non empty many sorted signature $\mathcal{A}$, non-empty many sorted sets $\mathcal{B}$, $\mathcal{C}$ indexed by the carrier of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\bigcup\mathcal{C}$, a ternary functor $\mathcal{G}$ yielding an element of $\bigcup\mathcal{C}$, and many sorted functions $\mathcal{D}$, $\mathcal{E}$ from FreeSorts($\mathcal{B}$) into $\mathcal{C}$, and states that:

$$\mathcal{D} = \mathcal{E}$$

provided the following conditions are satisfied:

- Let o be an operation symbol of $\mathcal{A}$, and let t_1 be an element of $\mathrm{Args}(o, \mathrm{Free}(\mathcal{B}))$, and let x be a finite sequence of elements of $\bigcup\mathcal{C}$. If $x = \mathrm{Flatten}(\mathcal{D}) \cdot t_1$, then $\mathcal{D}$(the result sort of o)$((\mathrm{Den}(o, \mathrm{Free}(\mathcal{B})))(t_1)) = \mathcal{G}(o, t_1, x)$,
- For every sort symbol s of $\mathcal{A}$ and for arbitrary y such that $y \in \mathrm{FreeGenerator}(s, \mathcal{B})$ holds $\mathcal{D}(s)(y) = \mathcal{F}(y)$,
- Let o be an operation symbol of $\mathcal{A}$, and let t_1 be an element of $\mathrm{Args}(o, \mathrm{Free}(\mathcal{B}))$, and let x be a finite sequence of elements of $\bigcup\mathcal{C}$. If $x = \mathrm{Flatten}(\mathcal{E}) \cdot t_1$, then $\mathcal{E}$(the result sort of o)$((\mathrm{Den}(o, \mathrm{Free}(\mathcal{B})))(t_1)) = \mathcal{G}(o, t_1, x)$,
- For every sort symbol s of $\mathcal{A}$ and for arbitrary y such that $y \in \mathrm{FreeGenerator}(s, \mathcal{B})$ holds $\mathcal{E}(s)(y) = \mathcal{F}(y)$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. Note that Free(X) is non-empty.

Let S be a non void non empty many sorted signature, let o be an operation symbol of S, and let A be a non-empty algebra over S. Note that $\mathrm{Args}(o, A)$ is non empty and Result(o, A) is non empty.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. Note that the sorts of Free(X) is pairwise disjoint.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. One can verify that Free(X) is pairwise disjoint.

The scheme *ExtFreeGen* deals with a non void non empty many sorted signature $\mathcal{A}$, a non-empty many sorted set $\mathcal{B}$ indexed by the carrier of $\mathcal{A}$, a non-empty algebra $\mathcal{C}$ over $\mathcal{A}$, many sorted functions $\mathcal{D}$, $\mathcal{E}$ from Free$(\mathcal{B})$ into $\mathcal{C}$, and a ternary predicate $\mathcal{P}$, and states that:

$\mathcal{D} = \mathcal{E}$

provided the following conditions are satisfied:

- $\mathcal{D}$ is a homomorphism of Free$(\mathcal{B})$ into $\mathcal{C}$,
- For every sort symbol s of $\mathcal{A}$ and for arbitrary x, y such that $y \in$ FreeGenerator$(s, \mathcal{B})$ holds $\mathcal{D}(s)(y) = x$ iff $\mathcal{P}[s, x, y]$,
- $\mathcal{E}$ is a homomorphism of Free$(\mathcal{B})$ into $\mathcal{C}$,
- For every sort symbol s of $\mathcal{A}$ and for arbitrary x, y such that $y \in$ FreeGenerator$(s, \mathcal{B})$ holds $\mathcal{E}(s)(y) = x$ iff $\mathcal{P}[s, x, y]$.

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The Correspondence Between Homomorphisms of Universal Algebra & Many Sorted Algebra

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Summary. The aim of the article is to check the compatibility of the homomorphism of universal algebras introduced in [13] and the corresponding concept for many sorted algebras introduced in [14].

MML Identifier: `MSUHOM_1`.

The articles [22], [25], [26], [28], [8], [9], [11], [21], [23], [3], [12], [10], [1], [19], [6], [27], [18], [15], [2], [5], [4], [16], [7], [24], [13], [14], [17], and [20] provide the notation and terminology for this paper.

For simplicity we follow the rules: U_1, U_2, U_3 denote universal algebras, n denotes a natural number, A denotes a non empty set, and h denotes a function from U_1 into U_2.

The following propositions are true:

(1) For all functions f, g and for every set C such that $\operatorname{rng} f \subseteq C$ holds $(g \restriction C) \cdot f = g \cdot f$.

(2) For every set I and for every subset C of I holds $C^* \subseteq I^*$.

(3) For every function f and for every set C such that f is function yielding holds $f \restriction C$ is function yielding.

(4) For every set I and for every subset C of I and for every many sorted set M indexed by I holds $(M \restriction C)^{\#} = M^{\#} \restriction C^*$.

Let us consider A, n and let a be an element of A. Then $n \mapsto a$ is a finite sequence of elements of A.

Let S, S' be non empty many sorted signatures. The predicate $S \leq S'$ is defined by the conditions (Def.1).

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(Def.1) (i) The carrier of $S \subseteq$ the carrier of S',

(ii) the operation symbols of $S \subseteq$ the operation symbols of S',

(iii) (the arity of S') $\restriction$ (the operation symbols of S) = the arity of S, and

(iv) (the result sort of S') $\restriction$ (the operation symbols of S) = the result sort of S.

Let us note that this predicate is reflexive.

Next we state four propositions:

(5) For all non empty many sorted signatures S, S', S'' such that $S \leq S'$ and $S' \leq S''$ holds $S \leq S''$.

(6) For all strict non empty many sorted signatures S, S' such that $S \leq S'$ and $S' \leq S$ holds $S = S'$.

(7) Let g be a function, and let a be an element of A, and let k be a natural number. If $1 \leq k$ and $k \leq n$, then $(a \dot{\longmapsto} g)(\pi_k(n \mapsto a)) = g$.

(8) Let I be a set, and let I_0 be a subset of I, and let A, B be many sorted sets indexed by I, and let F be a many sorted function from A into B, and let A_0, B_0 be many sorted sets indexed by I_0. Suppose $A_0 = A \restriction I_0$ and $B_0 = B \restriction I_0$. Then $F \restriction I_0$ is a many sorted function from A_0 into B_0.

Let S, S' be strict non void non empty many sorted signatures and let A be a non-empty strict algebra over S'. Let us assume that $S \leq S'$. The functor $(A \operatorname{over} S)$ yielding a non-empty strict algebra over S is defined by the conditions (Def.2).

(Def.2) (i) The sorts of $(A \operatorname{over} S)$ = (the sorts of A) $\restriction$ (the carrier of S), and

(ii) the characteristics of $(A \operatorname{over} S)$ = (the characteristics of A) $\restriction$ (the operation symbols of S).

We now state two propositions:

(9) For every strict non void non empty many sorted signature S and for every non-empty strict algebra A over S holds $A = (A \operatorname{over} S)$.

(10) For all U_1, U_2 such that U_1 and U_2 are similar holds $\mathrm{MSSign}(U_1) = \mathrm{MSSign}(U_2)$.

Let U_1, U_2 be universal algebras and let h be a function from U_1 into U_2. Let us assume that $\mathrm{MSSign}(U_1) = \mathrm{MSSign}(U_2)$. The functor $\mathrm{MSAlg}(h)$ yielding a many sorted function from $\mathrm{MSAlg}(U_1)$ into $(\mathrm{MSAlg}(U_2) \operatorname{over} \mathrm{MSSign}(U_1))$ is defined by:

(Def.3) $\mathrm{MSAlg}(h) = \{0\} \longmapsto h$.

The following propositions are true:

(11) Given U_1, U_2, h. Suppose U_1 and U_2 are similar. Let o be an operation symbol of $\mathrm{MSSign}(U_1)$. Then $(\mathrm{MSAlg}(h))$(the result sort of o) $= h$.

(12) For every operation symbol o of $\mathrm{MSSign}(U_1)$ holds $\mathrm{Den}(o, \mathrm{MSAlg}(U_1)) =$ (the characteristic of U_1)(o).

(13) For every operation symbol o of $\mathrm{MSSign}(U_1)$ holds $\mathrm{Den}(o, \mathrm{MSAlg}(U_1))$ is an operation of U_1.

(14) For every operation symbol o of $\mathrm{MSSign}(U_1)$ holds every element of $\mathrm{Args}(o, \mathrm{MSAlg}(U_1))$ is a finite sequence of elements of the carrier of U_1.

(15) Given U_1, U_2, h. Suppose U_1 and U_2 are similar. Let o be an operation symbol of $\mathrm{MSSign}(U_1)$ and let y be an element of $\mathrm{Args}(o, \mathrm{MSAlg}(U_1))$. Then $\mathrm{MSAlg}(h)\#y = h \cdot y$.

(16) If h is a homomorphism of U_1 into U_2, then $\mathrm{MSAlg}(h)$ is a homomorphism of $\mathrm{MSAlg}(U_1)$ into $(\mathrm{MSAlg}(U_2)\,\mathrm{over}\,\mathrm{MSSign}(U_1))$.

(17) If U_1 and U_2 are similar, then $\mathrm{MSAlg}(h)$ is a many sorted set indexed by $\{0\}$.

(18) If h is an epimorphism of U_1 onto U_2, then $\mathrm{MSAlg}(h)$ is an epimorphism of $\mathrm{MSAlg}(U_1)$ onto $(\mathrm{MSAlg}(U_2)\,\mathrm{over}\,\mathrm{MSSign}(U_1))$.

(19) If h is a monomorphism of U_1 into U_2, then $\mathrm{MSAlg}(h)$ is a monomorphism of $\mathrm{MSAlg}(U_1)$ into $(\mathrm{MSAlg}(U_2)\,\mathrm{over}\,\mathrm{MSSign}(U_1))$.

(20) If h is an isomorphism of U_1 and U_2, then $\mathrm{MSAlg}(h)$ is an isomorphism of $\mathrm{MSAlg}(U_1)$ and $(\mathrm{MSAlg}(U_2)\,\mathrm{over}\,\mathrm{MSSign}(U_1))$.

(21) Given U_1, U_2, h. Suppose U_1 and U_2 are similar. Suppose $\mathrm{MSAlg}(h)$ is a homomorphism of $\mathrm{MSAlg}(U_1)$ into $(\mathrm{MSAlg}(U_2)\,\mathrm{over}\,\mathrm{MSSign}(U_1))$. Then h is a homomorphism of U_1 into U_2.

(22) Given U_1, U_2, h. Suppose U_1 and U_2 are similar. Suppose $\mathrm{MSAlg}(h)$ is an epimorphism of $\mathrm{MSAlg}(U_1)$ onto $(\mathrm{MSAlg}(U_2)\,\mathrm{over}\,\mathrm{MSSign}(U_1))$. Then h is an epimorphism of U_1 onto U_2.

(23) Given U_1, U_2, h. Suppose U_1 and U_2 are similar. Suppose $\mathrm{MSAlg}(h)$ is a monomorphism of $\mathrm{MSAlg}(U_1)$ into $(\mathrm{MSAlg}(U_2)\,\mathrm{over}\,\mathrm{MSSign}(U_1))$. Then h is a monomorphism of U_1 into U_2.

(24) Given U_1, U_2, h. Suppose U_1 and U_2 are similar. Suppose $\mathrm{MSAlg}(h)$ is an isomorphism of $\mathrm{MSAlg}(U_1)$ and $(\mathrm{MSAlg}(U_2)\,\mathrm{over}\,\mathrm{MSSign}(U_1))$. Then h is an isomorphism of U_1 and U_2.

(25) $\mathrm{MSAlg}(\mathrm{id}_{(\text{the carrier of } U_1)}) = \mathrm{id}_{(\text{the sorts of } \mathrm{MSAlg}(U_1))}$.

(26) Given U_1, U_2, U_3. Suppose U_1 and U_2 are similar and U_2 and U_3 are similar. Let h_1 be a function from U_1 into U_2 and let h_2 be a function from U_2 into U_3. Then $\mathrm{MSAlg}(h_2) \circ \mathrm{MSAlg}(h_1) = \mathrm{MSAlg}(h_2 \cdot h_1)$.

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Preliminaries to Circuits, II [1]

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Summary. This article is the second in a series of four articles (started with [20] and continued in [19,18]) about modelling circuits by many sorted algebras.

First, we introduce some additional terminology for many sorted signatures. The vertices of such signatures are divided into input vertices and inner vertices. A many sorted signature is called *circuit like* if each sort is a result sort of at most one operation. Next, we introduce some notions for many sorted algebras and many sorted free algebras. Free envelope of an algebra is a free algebra generated by the sorts of the algebra. Evaluation of an algebra is defined as a homomorphism from the free envelope of the algebra into the algebra. We define depth of elements of free many sorted algebras.

A many sorted signature is said to be monotonic if every finitely generated algebra over it is locally finite (finite in each sort). Monotonic signatures are used (see [19,18]) in modelling backbones of circuits without directed cycles.

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The papers [24], [28], [25], [1], [29], [12], [15], [7], [13], [5], [2], [4], [6], [3], [23], [17], [22], [11], [21], [9], [10], [8], [14], [26], [30], [16], [27], and [20] provide the notation and terminology for this paper.

1. Many Sorted Signatures

Let S be a many sorted signature. A vertex of S is an element of the carrier of S.

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Let S be a non empty many sorted signature.

The functor SortsWithConstants(S) yielding a subset of the carrier of S is defined as follows:

(Def.1) (i) SortsWithConstants(S) $= \{v : v$ ranges over sort symbols of S, v has constants$\}$ if S is non void,

(ii) SortsWithConstants(S) $= \emptyset$, otherwise.

Let G be a non empty many sorted signature. The functor InputVertices(G) yields a subset of the carrier of G and is defined by:

(Def.2) InputVertices(G) = (the carrier of G) $\setminus$ rng (the result sort of G).

The functor InnerVertices(G) yielding a subset of the carrier of G is defined by:

(Def.3) InnerVertices(G) = rng (the result sort of G).

Next we state several propositions:

(1) For every void non empty many sorted signature G holds InputVertices(G) = the carrier of G.

(2) Let G be a non void non empty many sorted signature and let v be a vertex of G. Suppose $v \in$ InputVertices(G). Then it is not true that there exists an operation symbol o of G such that the result sort of $o = v$.

(3) For every non empty many sorted signature G holds InputVertices(G) $\cup$ InnerVertices(G) = the carrier of G.

(4) For every non empty many sorted signature G holds InputVertices(G) misses InnerVertices(G).

(5) For every non empty many sorted signature G holds SortsWithConstants(G) $\subseteq$ InnerVertices(G).

(6) For every non empty many sorted signature G holds InputVertices(G) misses SortsWithConstants(G).

A non empty many sorted signature has input vertices if:

(Def.4) InputVertices(it) $\neq \emptyset$.

Let us note that there exists a non empty many sorted signature which is non void and has input vertices.

Let G be a non empty many sorted signature with input vertices. Note that InputVertices(G) is non empty.

Let G be a non void non empty many sorted signature. Then InnerVertices(G) is a non empty subset of the carrier of G.

Let S be a non empty many sorted signature and let M_1 be a non-empty algebra over S. A many sorted set indexed by InputVertices(S) is said to be an input assignment of M_1 if:

(Def.5) For every vertex v of S such that $v \in$ InputVertices(S) holds it(v) $\in$ (the sorts of M_1)(v).

Let S be a non empty many sorted signature. We say that S is circuit-like if and only if the condition (Def.6) is satisfied.

(Def.6) Let S' be a non void non empty many sorted signature. Suppose $S' = S$. Let o_1, o_2 be operation symbols of S'. If the result sort of $o_1 =$ the result sort of o_2, then $o_1 = o_2$.

Let us observe that every non empty many sorted signature which is void is also circuit-like.

Let us note that there exists a non empty many sorted signature which is non void circuit-like and strict.

Let I_1 be a circuit-like non void non empty many sorted signature and let v be a vertex of I_1. Let us assume that $v \in \mathrm{InnerVertices}(I_1)$. The action at v yielding an operation symbol of I_1 is defined as follows:

(Def.7) The result sort of the action at $v = v$.

2. Free Many Sorted Algebras

Next we state the proposition

(7) Let S be a non void non empty many sorted signature, and let A be an algebra over S, and let o be an operation symbol of S, and let p be a finite sequence. Suppose $\mathrm{len}\, p = \mathrm{len}\, \mathrm{Arity}(o)$ and for every natural number k such that $k \in \mathrm{dom}\, p$ holds $p(k) \in$ (the sorts of A)($\pi_k \mathrm{Arity}(o)$). Then $p \in \mathrm{Args}(o, A)$.

Let S be a non void non empty many sorted signature and let M_1 be a non-empty algebra over S. The functor $\mathrm{FreeEnvelope}(M_1)$ yielding a free strict non-empty algebra over S is defined as follows:

(Def.8) $\mathrm{FreeEnvelope}(M_1) = \mathrm{Free}(\text{the sorts of } M_1)$.

One can prove the following proposition

(8) Let S be a non void non empty many sorted signature and let M_1 be a non-empty algebra over S. Then $\mathrm{FreeGenerator}(\text{the sorts of } M_1)$ is a free generator set of $\mathrm{FreeEnvelope}(M_1)$.

Let S be a non void non empty many sorted signature and let M_1 be a non-empty algebra over S. The functor $\mathrm{Eval}(M_1)$ yielding a many sorted function from $\mathrm{FreeEnvelope}(M_1)$ into M_1 is defined by the conditions (Def.9).

(Def.9) (i) $\mathrm{Eval}(M_1)$ is a homomorphism of $\mathrm{FreeEnvelope}(M_1)$ into M_1, and

(ii) for every sort symbol s of S and for arbitrary x, y such that $y \in \mathrm{FreeSort}(\text{the sorts of } M_1, s)$ and $y =$ the root tree of $\langle x, s\rangle$ and $x \in$ (the sorts of M_1)(s) holds $(\mathrm{Eval}(M_1))(s)(y) = x$.

One can prove the following proposition

(9) Let S be a non void non empty many sorted signature and let A be a non-empty algebra over S. Then the sorts of A is a generator set of A.

Let S be a non empty many sorted signature. An algebra over S is finitely-generated if:

(Def.10) (i) For every non void non empty many sorted signature S' such that $S' = S$ and for every algebra A over S' such that $A =$ it holds there exists generator set of A which is locally-finite if S is not void,

(ii) the sorts of it is locally-finite, otherwise.

Let S be a non empty many sorted signature. An algebra over S is locally-finite if:

(Def.11) The sorts of it is locally-finite.

Let S be a non empty many sorted signature. Observe that every non-empty algebra over S which is locally-finite is also finitely-generated.

Let S be a non empty many sorted signature. The trivial algebra of S yields a strict algebra over S and is defined by:

(Def.12) The sorts of the trivial algebra of $S =$ (the carrier of S) $\longmapsto \{0\}$.

Let S be a non empty many sorted signature. Observe that there exists an algebra over S which is locally-finite non-empty and strict.

A non empty many sorted signature is monotonic if:

(Def.13) Every finitely-generated non-empty algebra over it is locally-finite.

One can verify that there exists a non empty many sorted signature which is non void finite monotonic and circuit-like.

The following propositions are true:

(10) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set indexed by the carrier of S, and let v be a sort symbol of S. Then every element of the sorts of $\mathrm{Free}(X)(v)$ is a finite decorated tree.

(11) Let S be a non void non empty many sorted signature and let X be a non-empty locally-finite many sorted set indexed by the carrier of S. Then $\mathrm{Free}(X)$ is finitely-generated.

(12) Let S be a non void non empty many sorted signature, and let A be a non-empty algebra over S, and let v be a vertex of S, and let e be an element of (the sorts of $\mathrm{FreeEnvelope}(A)$)(v). Suppose $v \in \mathrm{InputVertices}(S)$. Then there exists an element x of (the sorts of A)(v) such that $e =$ the root tree of $\langle x, v\rangle$.

(13) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set indexed by the carrier of S, and let o be an operation symbol of S, and let p be a decorated tree yielding finite sequence. Suppose $\langle o,$ the carrier of $S\rangle$-tree$(p) \in$ (the sorts of $\mathrm{Free}(X)$)(the result sort of o). Then $\operatorname{len} p = \operatorname{len} \mathrm{Arity}(o)$.

(14) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set indexed by the carrier of S, and let o be an operation symbol of S, and let p be a decorated tree yielding finite sequence. Suppose $\langle o,$ the carrier of $S\rangle$-tree$(p) \in$ (the sorts of $\mathrm{Free}(X)$)(the result sort of o). Let i be a natural number. If $i \in \operatorname{dom} \mathrm{Arity}(o)$, then $p(i) \in$ (the sorts of $\mathrm{Free}(X)$)$(\mathrm{Arity}(o)(i))$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, and let v be a vertex of S. One can check that every element of the sorts of $\mathrm{Free}(X)(v)$ is finite non empty function-like and relation-like.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, and let v be a vertex of S. Note that there exists an element of the sorts of $\mathrm{Free}(X)(v)$ which is function-like and relation-like.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, and let v be a vertex of S. Observe that every function-like relation-like element of the sorts of $\mathrm{Free}(X)(v)$ is decorated tree-like.

Let I_1 be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of I_1, and let v be a vertex of I_1. Observe that there exists an element of the sorts of $\mathrm{Free}(X)(v)$ which is finite.

We now state the proposition

(15) Let S be a non void non empty many sorted signature, and let X be a non-empty many sorted set indexed by the carrier of S, and let v be a vertex of S, and let o be an operation symbol of S, and let e be an element of (the sorts of $\mathrm{Free}(X)$)(v). Suppose $v \in \mathrm{InnerVertices}(S)$ and $e(\varepsilon) = \langle o, \text{the carrier of } S\rangle$. Then there exists a decorated tree yielding finite sequence p such that $\mathrm{len}\, p = \mathrm{len}\, \mathrm{Arity}(o)$ and for every natural number i such that $i \in \mathrm{dom}\, p$ holds $p(i) \in$ (the sorts of $\mathrm{Free}(X)$)$(\mathrm{Arity}(o)(i))$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, let v be a sort symbol of S, and let e be an element of (the sorts of $\mathrm{Free}(X)$)(v). The functor $\mathrm{depth}(e)$ yielding a natural number is defined by:

(Def.14) There exists a finite decorated tree d_1 and there exists a finite tree t such that $d_1 = e$ and $t = \mathrm{dom}\, d_1$ and $\mathrm{depth}(e) = \mathrm{height}\, t$.

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On the Group of Automorphisms of Universal Algebra & Many Sorted Algebra

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Summary. The aim of the article is to check the compatibility of the automorphisms of universal algebras introduced in [8] and the corresponding concept for many sorted algebras introduced in [9].

MML Identifier: `AUTALG_1`.

The notation and terminology used in this paper have been introduced in the following articles: [2], [17], [20], [21], [5], [6], [4], [14], [16], [11], [13], [18], [19], [1], [10], [3], [8], [12], [15], [9], and [7].

1. On the Group of Automorphisms of Universal Algebra

In this paper U_1 denotes a universal algebra and f, g denote functions from U_1 into U_1.

One can prove the following proposition

(1) $\mathrm{id}_{(\text{the carrier of } U_1)}$ is an isomorphism of U_1 and U_1.

Let us consider U_1. The functor $\mathrm{UAAut}(U_1)$ yields a non empty set of functions from the carrier of U_1 to the carrier of U_1 and is defined by the conditions (Def.1).

(Def.1) (i) Every element of $\mathrm{UAAut}(U_1)$ is a function from U_1 into U_1, and

(ii) for every function h from U_1 into U_1 holds $h \in \mathrm{UAAut}(U_1)$ iff h is an isomorphism of U_1 and U_1.

Next we state several propositions:

(2) $\mathrm{UAAut}(U_1) \subseteq (\text{the carrier of } U_1)^{\text{the carrier of } U_1}$.

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(3) For every f holds $f \in \mathrm{UAAut}(U_1)$ iff f is an isomorphism of U_1 and U_1.

(4) $\mathrm{id}_{(\text{the carrier of } U_1)} \in \mathrm{UAAut}(U_1)$.

(5) For all f, g such that f is an element of $\mathrm{UAAut}(U_1)$ and $g = f^{-1}$ holds g is an isomorphism of U_1 and U_1.

(6) For every element f of $\mathrm{UAAut}(U_1)$ holds $f^{-1} \in \mathrm{UAAut}(U_1)$.

(7) For all elements f_1, f_2 of $\mathrm{UAAut}(U_1)$ holds $f_1 \cdot f_2 \in \mathrm{UAAut}(U_1)$.

Let us consider U_1. The functor $\mathrm{UAAutComp}(U_1)$ yields a binary operation on $\mathrm{UAAut}(U_1)$ and is defined as follows:

(Def.2) For all elements x, y of $\mathrm{UAAut}(U_1)$ holds $(\mathrm{UAAutComp}(U_1))(x, y) = y \cdot x$.

Let us consider U_1. The functor $\mathrm{UAAutGroup}(U_1)$ yielding a group is defined by:

(Def.3) $\mathrm{UAAutGroup}(U_1) = \langle \mathrm{UAAut}(U_1), \mathrm{UAAutComp}(U_1) \rangle$.

Let us consider U_1. Note that $\mathrm{UAAutGroup}(U_1)$ is strict.

The following propositions are true:

(8) Let x, y be elements of the carrier of $\mathrm{UAAutGroup}(U_1)$ and let f, g be elements of $\mathrm{UAAut}(U_1)$. If $x = f$ and $y = g$, then $x \cdot y = g \cdot f$.

(9) $\mathrm{id}_{(\text{the carrier of } U_1)} = 1_{\mathrm{UAAutGroup}(U_1)}$.

(10) For every element f of $\mathrm{UAAut}(U_1)$ and for every element g of the carrier of $\mathrm{UAAutGroup}(U_1)$ such that $f = g$ holds $f^{-1} = g^{-1}$.

2. Some Properties of Many Sorted Functions

In the sequel I is a set and A, B, C are many sorted sets indexed by I.

Let us consider I, A, B. We say that A is transformable to B if and only if:

(Def.4) For arbitrary i such that $i \in I$ holds if $B(i) = \emptyset$, then $A(i) = \emptyset$.

Let us observe that the predicate introduced above is reflexive.

Next we state several propositions:

(11) If A is transformable to B and B is transformable to C, then A is transformable to C.

(12) For arbitrary x and for every many sorted set A indexed by $\{x\}$ holds $A = \{x\} \longmapsto A(x)$.

(13) For all function yielding functions F, G, H holds $(H \circ G) \circ F = H \circ (G \circ F)$.

(14) Let A, B be non-empty many sorted sets indexed by I and let F be a many sorted function from A into B. If F is "1-1" and "onto", then F^{-1} is "1-1" and "onto".

(15) Let A, B be non-empty many sorted sets indexed by I and let F be a many sorted function from A into B. If F is "1-1" and "onto", then $(F^{-1})^{-1} = F$.

(16) For all function yielding functions F, G such that F is "1-1" and G is "1-1" holds $G \circ F$ is "1-1".

(17) Let B, C be non-empty many sorted sets indexed by I, and let F be a many sorted function from A into B, and let G be a many sorted function from B into C. If F is "onto" and G is "onto", then $G \circ F$ is "onto".

(18) Let A, B, C be non-empty many sorted sets indexed by I, and let F be a many sorted function from A into B, and let G be a many sorted function from B into C. Suppose F is "1-1" and "onto" and G is "1-1" and "onto". Then $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.

(19) Let A, B be non-empty many sorted sets indexed by I, and let F be a many sorted function from A into B, and let G be a many sorted function from B into A. If F is "1-1" and "onto" and $G \circ F = \mathrm{id}_A$, then $G = F^{-1}$.

3. On the Group of Automorphisms of Many Sorted Algebra

In the sequel S will be a non void non empty many sorted signature and U_2, U_3 will be non-empty algebras over S.

Let us consider I, A, B. The functor $\mathrm{MSFuncs}(A, B)$ yields a many sorted set indexed by I and is defined as follows:

(Def.5) For arbitrary i such that $i \in I$ holds $(\mathrm{MSFuncs}(A, B))(i) = B(i)^{A(i)}$.

One can prove the following propositions:

(20) Let h be a many sorted set indexed by I. If $h = \mathrm{MSFuncs}(A, B)$, then for arbitrary i such that $i \in I$ holds $h(i) = B(i)^{A(i)}$.

(21) Let A, B be many sorted sets indexed by I. Suppose A is transformable to B. Let x be arbitrary. If $x \in \prod \mathrm{MSFuncs}(A, B)$, then x is a many sorted function from A into B.

(22) Let A, B be many sorted sets indexed by I. Suppose A is transformable to B. Let g be a many sorted function from A into B. Then $g \in \prod \mathrm{MSFuncs}(A, B)$.

(23) For all many sorted sets A, B indexed by I such that A is transformable to B holds $\mathrm{MSFuncs}(A, B)$ is non-empty.

Let us consider I, A, B. Let us assume that A is transformable to B. A non empty set is said to be a set of manysorted functions from A into B if:

(Def.6) For arbitrary x such that $x \in$ it holds x is a many sorted function from A into B.

Let us consider I, A. Note that $\mathrm{MSFuncs}(A, A)$ is non-empty.

Let us consider S, U_2, U_3. A set of manysorted functions from U_2 into U_3 is a set of manysorted functions from the sorts of U_2 into the sorts of U_3.

Let I be a set and let D be a many sorted set indexed by I. Note that there exists a set of manysorted functions from D into D which is non empty.

We now state four propositions:

(24) id_A is "onto".

(25) id_A is "1-1".

(26) $\mathrm{id}_{(\text{the sorts of } U_2)}$ is an isomorphism of U_2 and U_2.

(27) $\mathrm{id}_{(\text{the sorts of } U_2)} \in \prod \mathrm{MSFuncs}(\text{the sorts of } U_2, \text{the sorts of } U_2)$.

Let us consider S, U_2. The functor $\mathrm{MSAAut}(U_2)$ yielding a set of manysorted functions from the sorts of U_2 into the sorts of U_2 is defined by the conditions (Def.7).

(Def.7) (i) Every element of $\mathrm{MSAAut}(U_2)$ is a many sorted function from U_2 into U_2, and

(ii) for every many sorted function h from U_2 into U_2 holds $h \in \mathrm{MSAAut}(U_2)$ iff h is an isomorphism of U_2 and U_2.

One can prove the following propositions:

(28) For every many sorted function F from U_2 into U_2 holds $F \in \mathrm{MSAAut}(U_2)$ iff F is an isomorphism of U_2 and U_2.

(29) For every element f of $\mathrm{MSAAut}(U_2)$ holds $f \in \prod \mathrm{MSFuncs}(\text{the sorts of } U_2, \text{the sorts of } U_2)$.

(30) $\mathrm{MSAAut}(U_2) \subseteq \prod \mathrm{MSFuncs}(\text{the sorts of } U_2, \text{the sorts of } U_2)$.

(31) $\mathrm{id}_{(\text{the sorts of } U_2)} \in \mathrm{MSAAut}(U_2)$.

(32) For every element f of $\mathrm{MSAAut}(U_2)$ holds $f^{-1} \in \mathrm{MSAAut}(U_2)$.

(33) For all elements f_1, f_2 of $\mathrm{MSAAut}(U_2)$ holds $f_1 \circ f_2 \in \mathrm{MSAAut}(U_2)$.

(34) For every many sorted function F from $\mathrm{MSAlg}(U_1)$ into $\mathrm{MSAlg}(U_1)$ and for every element f of $\mathrm{UAAut}(U_1)$ such that $F = \{0\} \longmapsto f$ holds $F \in \mathrm{MSAAut}(\mathrm{MSAlg}(U_1))$.

Let us consider S, U_2. The functor $\mathrm{MSAAutComp}(U_2)$ yields a binary operation on $\mathrm{MSAAut}(U_2)$ and is defined as follows:

(Def.8) For all elements x, y of $\mathrm{MSAAut}(U_2)$ holds $(\mathrm{MSAAutComp}(U_2))(x, y) = y \circ x$.

Let us consider S, U_2. The functor $\mathrm{MSAAutGroup}(U_2)$ yields a group and is defined by:

(Def.9) $\mathrm{MSAAutGroup}(U_2) = \langle \mathrm{MSAAut}(U_2), \mathrm{MSAAutComp}(U_2) \rangle$.

Let us consider S, U_2. Observe that $\mathrm{MSAAutGroup}(U_2)$ is strict.

The following three propositions are true:

(35) Let x, y be elements of the carrier of $\mathrm{MSAAutGroup}(U_2)$ and let f, g be elements of $\mathrm{MSAAut}(U_2)$. If $x = f$ and $y = g$, then $x \cdot y = g \circ f$.

(36) $\mathrm{id}_{(\text{the sorts of } U_2)} = 1_{\mathrm{MSAAutGroup}(U_2)}$.

(37) For every element f of $\mathrm{MSAAut}(U_2)$ and for every element g of $\mathrm{MSAAutGroup}(U_2)$ such that $f = g$ holds $f^{-1} = g^{-1}$.

4. On the Relationship of Automorphisms of 1-sorted and Many Sorted Algebras

Next we state several propositions:

(38) Let U_4, U_5 be universal algebras. Suppose U_4 and U_5 are similar. Let F be a many sorted function from $\mathrm{MSAlg}(U_4)$ into $(\mathrm{MSAlg}(U_5)\,\mathrm{over}\,\mathrm{MSSign}(U_4))$. Then $F(0)$ is a function from U_4 into U_5.

(39) For every element f of $\mathrm{UAAut}(U_1)$ holds $\{0\} \longmapsto f$ is a many sorted function from $\mathrm{MSAlg}(U_1)$ into $\mathrm{MSAlg}(U_1)$.

(40) Let h be a function. Suppose $\mathrm{dom}\, h = \mathrm{UAAut}(U_1)$ and for arbitrary x such that $x \in \mathrm{UAAut}(U_1)$ holds $h(x) = \{0\} \longmapsto x$. Then h is a homomorphism from $\mathrm{UAAutGroup}(U_1)$ to $\mathrm{MSAAutGroup}(\mathrm{MSAlg}(U_1))$.

(41) Let h be a homomorphism from $\mathrm{UAAutGroup}(U_1)$ to $\mathrm{MSAAutGroup}(\mathrm{MSAlg}(U_1))$. Suppose that for arbitrary x such that $x \in \mathrm{UAAut}(U_1)$ holds $h(x) = \{0\} \longmapsto x$. Then h is an isomorphism.

(42) $\mathrm{UAAutGroup}(U_1)$ and $\mathrm{MSAAutGroup}(\mathrm{MSAlg}(U_1))$ are isomorphic.

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Introduction to Circuits, I [1]

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Summary. This article is the third in a series of four articles (preceded by [19,20] and continued in [18]) about modelling circuits by many sorted algebras.

A circuit is defined as a locally-finite algebra over a circuit-like many sorted signature. For circuits we define notions of input function and of circuit state which are later used (see [18]) to define circuit computations. For circuits over monotonic signatures we introduce notions of vertex size and vertex depth that characterize certain graph properties of circuit's signature in terms of elements of its free envelope algebra. The depth of a finite circuit is defined as the maximal depth over its vertices.

MML Identifier: `CIRCUIT1`.

The terminology and notation used in this paper are introduced in the following papers: [24], [27], [3], [16], [28], [12], [9], [29], [15], [25], [1], [7], [26], [13], [2], [4], [6], [8], [5], [14], [10], [23], [22], [11], [17], [21], [19], and [20].

1. Circuit State

Let S be a non void circuit-like non empty many sorted signature. A circuit of S is a locally-finite algebra over S.

In the sequel I_1 will denote a circuit-like non void non empty many sorted signature.

Let us consider I_1 and let S_1 be a non-empty circuit of I_1. The functor Set-Constants(S_1) yielding a many sorted set indexed by SortsWithConstants(I_1) is defined as follows:

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(Def.1) For every vertex x of I_1 such that $x \in \operatorname{dom} \text{Set-Constants}(S_1)$ holds $(\text{Set-Constants}(S_1))(x) \in \text{Constants}(S_1, x)$.

The following proposition is true

(1) Given I_1, and let S_1 be a non-empty circuit of I_1, and let v be a vertex of I_1, and let e be an element of (the sorts of S_1)(v). If $v \in \text{SortsWithConstants}(I_1)$ and $e \in \text{Constants}(S_1, v)$, then $(\text{Set-Constants}(S_1))(v) = e$.

Let us consider I_1 and let C_1 be a circuit of I_1. An input function of C_1 is a many sorted function from $\text{InputVertices}(I_1) \longmapsto \mathbb{N}$ into (the sorts of C_1) $\restriction$ InputVertices(I_1).

The following proposition is true

(2) Given I_1, and let S_1 be a non-empty circuit of I_1, and let I_2 be an input function of S_1, and let n be a natural number. If I_1 has input vertices, then $(\text{commute}(I_2))(n)$ is an input assignment of S_1.

Let us consider I_1. Let us assume that I_1 has input vertices. Let S_1 be a non-empty circuit of I_1, let I_2 be an input function of S_1, and let n be a natural number. The functor n-*th*-input(I_2) yields an input assignment of S_1 and is defined by:

(Def.2) $n\text{-}th\text{-input}(I_2) = (\text{commute}(I_2))(n)$.

The following proposition is true

(3) Given I_1, and let S_1 be a non-empty circuit of I_1, and let I_2 be an input function of S_1, and let n be a natural number. If I_1 has input vertices, then $n\text{-}th\text{-input}(I_2) = (\text{commute}(I_2))(n)$.

Let us consider I_1 and let S_1 be a circuit of I_1. A state of S_1 is an element of $\prod$(the sorts of S_1).

The following propositions are true:

(4) For every I_1 and for every non-empty circuit S_1 of I_1 and for every state s of S_1 holds $\operatorname{dom} s =$ the carrier of I_1.

(5) Given I_1, and let S_1 be a non-empty circuit of I_1, and let s be a state of S_1, and let v be a vertex of I_1. Then $s(v) \in$ (the sorts of S_1)(v).

Let us consider I_1, let S_1 be a non-empty circuit of I_1, let s be a state of S_1, and let o be an operation symbol of I_1. The functor o depends-on-in s yields an element of $\text{Args}(o, S_1)$ and is defined as follows:

(Def.3) $o \,\text{depends-on-in}\, s = s \cdot \text{Arity}(o)$.

In the sequel I_1 will be a monotonic circuit-like non void non empty many sorted signature.

The following proposition is true

(6) Given I_1, and let S_1 be a locally-finite non-empty algebra over I_1, and let v, w be vertices of I_1, and let e_1 be an element of (the sorts of FreeEnvelope(S_1))(v), and let q_1 be a decorated tree yielding finite sequence. Suppose $v \in \text{InnerVertices}(I_1)$ and $e_1 = \langle$the action at v, the carrier of $I_1\rangle$-tree(q_1). Let k be a natural number. If $k \in \operatorname{dom} q_1$ and

$q_1(k) \in$ (the sorts of FreeEnvelope(S_1))(w), then $w = \pi_k$ Arity(the action at v).

Let us consider I_1, let S_1 be a locally-finite non-empty algebra over I_1, and let v be a vertex of I_1. Note that every element of the sorts of FreeEnvelope$(S_1)(v)$ is finite non empty function-like and relation-like.

Let us consider I_1, let S_1 be a locally-finite non-empty algebra over I_1, and let v be a vertex of I_1. Observe that every element of the sorts of FreeEnvelope$(S_1)(v)$ is decorated tree-like.

Next we state four propositions:

(7) Given I_1, and let S_1 be a locally-finite non-empty algebra over I_1, and let v, w be vertices of I_1, and let e_1 be an element of (the sorts of FreeEnvelope(S_1))(v), and let e_2 be an element of (the sorts of FreeEnvelope(S_1))(w), and let q_1 be a decorated tree yielding finite sequence, and let k_1 be a natural number. Suppose $v \in$ InnerVertices$(I_1) \setminus$ SortsWithConstants(I_1) and $e_1 = \langle$the action at v, the carrier of $I_1\rangle$-tree(q_1) and $k_1 + 1 \in \operatorname{dom} q_1$ and $q_1(k_1 + 1) \in$ (the sorts of FreeEnvelope(S_1))(w). Then $e_1(\langle k_1\rangle / e_2) \in$ (the sorts of FreeEnvelope(S_1))(v).

(8) Given I_1, and let A be a locally-finite non-empty algebra over I_1, and let v be an element of the carrier of I_1, and let e be an element of (the sorts of FreeEnvelope(A))(v). Suppose $1 < \operatorname{card} e$. Then there exists an operation symbol o of I_1 such that $e(\varepsilon) = \langle o$, the carrier of $I_1\rangle$.

(9) Let I_1 be a non void circuit-like non empty many sorted signature, and let S_1 be a non-empty circuit of I_1, and let s be a state of S_1, and let o be an operation symbol of I_1. Then $(\mathrm{Den}(o, S_1))(o\,\text{depends-on-in}\,s) \in$ (the sorts of S_1)(the result sort of o).

(10) Given I_1, and let A be a non-empty circuit of I_1, and let v be a vertex of I_1, and let e be an element of (the sorts of FreeEnvelope(A))(v). Suppose $e(\varepsilon) = \langle$the action at v, the carrier of $I_1\rangle$. Then there exists a decorated tree yielding finite sequence p such that $e = \langle$the action at v, the carrier of $I_1\rangle$-tree(p).

2. Vertex Size

Let I_1 be a monotonic non void non empty many sorted signature, let A be a locally-finite non-empty algebra over I_1, and let v be a sort symbol of I_1. One can verify that (the sorts of FreeEnvelope(A))(v) is finite.

Let us consider I_1, let A be a locally-finite non-empty algebra over I_1, and let v be a sort symbol of I_1. The functor size(v, A) yielding a natural number is defined as follows:

(Def.4) There exists a finite non empty subset s of $\mathbb{N}$ such that $s = \{\operatorname{card} t : t$ ranges over elements of (the sorts of FreeEnvelope(A))$(v)\}$ and size$(v, A) = \max s$.

Next we state four propositions:

(11) Given I_1, and let A be a locally-finite non-empty algebra over I_1, and let v be an element of the carrier of I_1. Then $\mathrm{size}(v, A) = 1$ if and only if $v \in \mathrm{InputVertices}(I_1) \cup \mathrm{SortsWithConstants}(I_1)$.

(12) Given I_1, and let S_1 be a locally-finite non-empty algebra over I_1, and let v, w be vertices of I_1, and let e_1 be an element of (the sorts of $\mathrm{FreeEnvelope}(S_1))(v)$, and let e_2 be an element of (the sorts of $\mathrm{FreeEnvelope}(S_1))(w)$, and let q_1 be a decorated tree yielding finite sequence. Suppose $v \in \mathrm{InnerVertices}(I_1) \setminus \mathrm{SortsWithConstants}(I_1)$ and $\mathrm{card}\, e_1 = \mathrm{size}(v, S_1)$ and $e_1 = \langle$the action at $v,$ the carrier of $I_1\rangle$-tree(q_1) and $e_2 \in \mathrm{rng}\, q_1$. Then $\mathrm{card}\, e_2 = \mathrm{size}(w, S_1)$.

(13) Given I_1, and let A be a locally-finite non-empty algebra over I_1, and let v be a vertex of I_1, and let e be an element of (the sorts of $\mathrm{FreeEnvelope}(A))(v)$. Suppose $v \in \mathrm{InnerVertices}(I_1) \setminus \mathrm{SortsWithConstants}(I_1)$ and $\mathrm{card}\, e = \mathrm{size}(v, A)$. Then there exists a decorated tree yielding finite sequence q such that $e = \langle$the action at $v,$ the carrier of $I_1\rangle$-tree(q).

(14) Given I_1, and let A be a locally-finite non-empty algebra over I_1, and let v be a vertex of I_1, and let e be an element of (the sorts of $\mathrm{FreeEnvelope}(A))(v)$. Suppose $v \in \mathrm{InnerVertices}(I_1) \setminus \mathrm{SortsWithConstants}(I_1)$ and $\mathrm{card}\, e = \mathrm{size}(v, A)$. Then there exists an operation symbol o of I_1 such that $e(\varepsilon) = \langle o,$ the carrier of $I_1\rangle$.

Let S be a non void non empty many sorted signature, let A be a locally-finite non-empty algebra over S, let v be a sort symbol of S, and let e be an element of (the sorts of $\mathrm{FreeEnvelope}(A))(v)$. The functor $\mathrm{depth}(e)$ yielding a natural number is defined as follows:

(Def.5) There exists an element e' of (the sorts of Free(the sorts of A))(v) such that $e = e'$ and $\mathrm{depth}(e) = \mathrm{depth}(e')$.

The following propositions are true:

(15) Given I_1, and let A be a locally-finite non-empty algebra over I_1, and let v, w be elements of the carrier of I_1. If $v \in \mathrm{InnerVertices}(I_1)$ and $w \in \mathrm{rng}\, \mathrm{Arity}$(the action at v), then $\mathrm{size}(w, A) < \mathrm{size}(v, A)$.

(16) For every I_1 and for every locally-finite non-empty algebra A over I_1 and for every sort symbol v of I_1 holds $\mathrm{size}(v, A) > 0$.

(17) Given I_1, and let A be a non-empty circuit of I_1, and let v be a vertex of I_1, and let e be an element of (the sorts of $\mathrm{FreeEnvelope}(A))(v)$, and let p be a decorated tree yielding finite sequence. Suppose that

(i) $v \in \mathrm{InnerVertices}(I_1)$,

(ii) $e = \langle$the action at $v,$ the carrier of $I_1\rangle$-tree(p), and

(iii) for every natural number k such that $k \in \mathrm{dom}\, p$ there exists an element e_3 of (the sorts of $\mathrm{FreeEnvelope}(A))(\pi_k\, \mathrm{Arity}$(the action at v)) such that $e_3 = p(k)$ and $\mathrm{card}\, e_3 = \mathrm{size}(\pi_k\, \mathrm{Arity}(\text{the action at } v), A)$.

Then $\mathrm{card}\, e = \mathrm{size}(v, A)$.

3. Vertex and Circuit Depth

Let S be a monotonic non void non empty many sorted signature, let A be a locally-finite non-empty algebra over S, and let v be a sort symbol of S. The functor $\mathrm{depth}(v, A)$ yields a natural number and is defined by:

(Def.6) There exists a finite non empty subset s of $\mathbb{N}$ such that $s = \{\mathrm{depth}(t) : t$ ranges over elements of (the sorts of FreeEnvelope$(A))(v)\}$ and $\mathrm{depth}(v, A) = \max s$.

Let I_1 be a finite monotonic circuit-like non void non empty many sorted signature and let A be a non-empty circuit of I_1. The functor $\mathrm{depth}(A)$ yielding a natural number is defined by the condition (Def.7).

(Def.7) There exists a finite non empty subset D_1 of $\mathbb{N}$ such that $D_1 = \{\mathrm{depth}(v, A) : v$ ranges over elements of the carrier of I_1, $v \in$ the carrier of $I_1\}$ and $\mathrm{depth}(A) = \max D_1$.

The following three propositions are true:

(18) Let I_1 be a finite monotonic circuit-like non void non empty many sorted signature, and let A be a non-empty circuit of I_1, and let v be a vertex of I_1. Then $\mathrm{depth}(v, A) \leq \mathrm{depth}(A)$.

(19) Given I_1, and let A be a non-empty circuit of I_1, and let v be a vertex of I_1. Then $\mathrm{depth}(v, A) = 0$ if and only if $v \in \mathrm{InputVertices}(I_1)$ or $v \in \mathrm{SortsWithConstants}(I_1)$.

(20) Given I_1, and let A be a locally-finite non-empty algebra over I_1, and let v, v_1 be sort symbols of I_1. If $v \in \mathrm{InnerVertices}(I_1)$ and $v_1 \in \mathrm{rng}\,\mathrm{Arity}($the action at $v)$, then $\mathrm{depth}(v_1, A) < \mathrm{depth}(v, A)$.

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The Cantor Set [1]

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Summary. The aim of the paper is to define some basic notions of the theory of topological spaces like basis and prebasis, and to prove their simple properties. The definition of the Cantor set is given in terms of countable product of $\{0,1\}$ and a collection of its subsets to serve as a prebasis.

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The papers [13], [16], [15], [9], [17], [2], [3], [6], [14], [12], [10], [5], [4], [1], [7], [11], and [8] provide the terminology and notation for this paper.

Let Y be a set and let x be a non empty set. Observe that $Y \longmapsto x$ is non-empty.

Let X be arbitrary and let A be a family of subsets of X. The functor UniCl(A) yields a family of subsets of X and is defined by:

(Def.1) For every subset x of X holds $x \in \mathrm{UniCl}(A)$ iff there exists a family Y of subsets of X such that $Y \subseteq A$ and $x = \bigcup Y$.

Let X be a topological structure. A family of subsets of the carrier of X is called a basis of X if:

(Def.2) It $\subseteq$ the topology of X and the topology of $X \subseteq \mathrm{UniCl}(\mathrm{it})$.

We now state three propositions:

(1) For arbitrary X and for every family A of subsets of X holds $A \subseteq \mathrm{UniCl}(A)$.

(2) For every topological structure S holds the topology of S is a basis of S.

(3) For every topological structure S holds the topology of S is open.

Let M be arbitrary and let B be a family of subsets of M. The functor Intersect(B) yielding a subset of M is defined by:

[1] The present work had been completed while the first author's visit to Białystok in winter 1994-95.

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(Def.3) (i) Intersect$(B) = \bigcap B$ if $B \neq \emptyset$,

(ii) Intersect$(B) = M$, otherwise.

Let X be arbitrary and let A be a family of subsets of X. The functor FinMeetCl(A) yielding a family of subsets of X is defined by the condition (Def.4).

(Def.4) Let x be a subset of X. Then $x \in$ FinMeetCl(A) if and only if there exists a family Y of subsets of X such that $Y \subseteq A$ and Y is finite and $x =$ Intersect(Y).

One can prove the following proposition

(4) For arbitrary X and for every family A of subsets of X holds $A \subseteq$ FinMeetCl(A).

Let T be a topological space. Note that the topology of T is non empty.

The following propositions are true:

(5) For every topological space T holds the topology of $T =$ FinMeetCl(the topology of T).

(6) For every topological space T holds the topology of $T =$ UniCl(the topology of T).

(7) For every topological space T holds the topology of $T =$ UniCl(FinMeetCl(the topology of T)).

(8) For arbitrary X and for every family A of subsets of X holds $X \in$ FinMeetCl(A).

(9) For arbitrary X and for all families A, B of subsets of X such that $A \subseteq B$ holds UniCl$(A) \subseteq$ UniCl(B).

(10) Let X be arbitrary, and let R be a family of subsets of X, and let x be arbitrary. Suppose $x \in X$. Then $x \in$ Intersect(R) if and only if for arbitrary Y such that $Y \in R$ holds $x \in Y$.

(11) For arbitrary X and for all families H, J of subsets of X such that $H \subseteq J$ holds Intersect$(J) \subseteq$ Intersect(H).

(12) Let X be arbitrary, and let R be a non empty family of subsets of 2^X, and let F be a family of subsets of X. If $F = \{$Intersect$(x) : x$ ranges over elements of $R\}$, then Intersect$(F) =$ Intersect$(\bigcup R)$.

Let X, Y be arbitrary, let A be a family of subsets of X, let F be a function from Y into 2^A, and let x be arbitrary. Then $F(x)$ is a family of subsets of X.

We now state four propositions:

(13) For arbitrary X and for every family A of subsets of X holds FinMeetCl$(A) =$ FinMeetCl(FinMeetCl(A)).

(14) Let X be arbitrary, and let A be a family of subsets of X, and let a, b be arbitrary. If $a \in$ FinMeetCl(A) and $b \in$ FinMeetCl(A), then $a \cap b \in$ FinMeetCl(A).

(15) Let X be arbitrary, and let A be a family of subsets of X, and let a, b be arbitrary. If $a \subseteq$ FinMeetCl(A) and $b \subseteq$ FinMeetCl(A), then $a \Cap b \subseteq$ FinMeetCl(A).

(16) For arbitrary X and for all families A, B of subsets of X such that $A \subseteq B$ holds $\mathrm{FinMeetCl}(A) \subseteq \mathrm{FinMeetCl}(B)$.

Let X be arbitrary and let A be a family of subsets of X. Observe that $\mathrm{FinMeetCl}(A)$ is non empty.

One can prove the following proposition

(17) For every non empty set X and for every family A of subsets of X holds $\langle X, \mathrm{UniCl}(\mathrm{FinMeetCl}(A))\rangle$ is topological space-like.

Let X be a topological structure. A family of subsets of the carrier of X is said to be a prebasis of X if:

(Def.5) It $\subseteq$ the topology of X and there exists a basis F of X such that $F \subseteq \mathrm{FinMeetCl}(\mathrm{it})$.

We now state three propositions:

(18) For every non empty set X holds every family of subsets of X is a basis of $\langle X, \mathrm{UniCl}(Y)\rangle$.

(19) Let T_1, T_2 be strict topological spaces and let P be a prebasis of T_1. Suppose the carrier of T_1 = the carrier of T_2 and P is a prebasis of T_2. Then $T_1 = T_2$.

(20) For every non empty set X holds every family of subsets of X is a prebasis of $\langle X, \mathrm{UniCl}(\mathrm{FinMeetCl}(Y))\rangle$.

The strict topological space the Cantor set is defined by the conditions (Def.6).

(Def.6) (i) The carrier of the Cantor set $= \prod(\mathbb{N} \longmapsto \{0,1\})$, and

(ii) there exists a prebasis P of the Cantor set such that for every subset X of $\prod(\mathbb{N} \longmapsto \{0,1\})$ holds $X \in P$ iff there exist natural numbers N, n such that for every element F of $\prod(\mathbb{N} \longmapsto \{0,1\})$ holds $F \in X$ iff $F(N) = n$.

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Logical Equivalence of Formulae [1]

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The notation and terminology used here are introduced in the following papers: [11], [9], [10], [8], [1], [12], [4], [2], [7], [5], [3], and [6].

For simplicity we adopt the following rules: p, q, r, s, p_1, q_1 are elements of CQC-WFF, X, Y, Z, X_1, X_2 are subsets of CQC-WFF, h is a formula, and x, y are bound variables.

One can prove the following four propositions:

(1) If $p \in X$, then $X \vdash p$.

(2) If $X \subseteq \operatorname{Cn} Y$, then $\operatorname{Cn} X \subseteq \operatorname{Cn} Y$.

(3) If $X \vdash p$ and $\{p\} \vdash q$, then $X \vdash q$.

(4) If $X \vdash p$ and $X \subseteq Y$, then $Y \vdash p$.

Let p, q be elements of CQC-WFF. The predicate $p \vdash q$ is defined by:

(Def.1) $\{p\} \vdash q$.

We now state two propositions:

(5) $p \vdash p$.

(6) If $p \vdash q$ and $q \vdash r$, then $p \vdash r$.

Let X, Y be subsets of CQC-WFF. The predicate $X \vdash Y$ is defined as follows:

(Def.2) For every element p of CQC-WFF such that $p \in Y$ holds $X \vdash p$.

We now state several propositions:

(7) $X \vdash Y$ iff $Y \subseteq \operatorname{Cn} X$.

(8) $X \vdash X$.

(9) If $X \vdash Y$ and $Y \vdash Z$, then $X \vdash Z$.

(10) $X \vdash \{p\}$ iff $X \vdash p$.

[1]This work has been done while the author visited Warsaw University in Białystok, in winter 1994–1995.

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(11) $\{p\} \vdash \{q\}$ iff $p \vdash q$.

(12) If $X \subseteq Y$, then $Y \vdash X$.

(13) $X \vdash \mathrm{Taut}$.

(14) $\emptyset_{\mathrm{CQC}} \vdash \mathrm{Taut}$.

Let X be a subset of CQC-WFF. The predicate $\vdash X$ is defined by:

(Def.3) For every element p of CQC-WFF such that $p \in X$ holds $\vdash p$.

We now state three propositions:

(15) $\vdash X$ iff $\emptyset_{\mathrm{CQC}} \vdash X$.

(16) $\vdash \mathrm{Taut}$.

(17) $\vdash X$ iff $X \subseteq \mathrm{Taut}$.

Let us consider X, Y. The predicate $X \mathrel{\vdash\!\dashv} Y$ is defined by:

(Def.4) For every p holds $X \vdash p$ iff $Y \vdash p$.

Let us observe that this predicate is reflexive and symmetric.

The following propositions are true:

(18) $X \mathrel{\vdash\!\dashv} Y$ iff $X \vdash Y$ and $Y \vdash X$.

(19) If $X \mathrel{\vdash\!\dashv} Y$ and $Y \mathrel{\vdash\!\dashv} Z$, then $X \mathrel{\vdash\!\dashv} Z$.

(20) $X \mathrel{\vdash\!\dashv} Y$ iff $\mathrm{Cn}\, X = \mathrm{Cn}\, Y$.

(21) $\mathrm{Cn}\, X \cup \mathrm{Cn}\, Y \subseteq \mathrm{Cn}(X \cup Y)$.

(22) $\mathrm{Cn}(X \cup Y) = \mathrm{Cn}(\mathrm{Cn}\, X \cup \mathrm{Cn}\, Y)$.

(23) $X \mathrel{\vdash\!\dashv} \mathrm{Cn}\, X$.

(24) $X \cup Y \mathrel{\vdash\!\dashv} \mathrm{Cn}\, X \cup \mathrm{Cn}\, Y$.

(25) If $X_1 \mathrel{\vdash\!\dashv} X_2$, then $X_1 \cup Y \mathrel{\vdash\!\dashv} X_2 \cup Y$.

(26) If $X_1 \mathrel{\vdash\!\dashv} X_2$ and $X_1 \cup Y \vdash Z$, then $X_2 \cup Y \vdash Z$.

(27) If $X_1 \mathrel{\vdash\!\dashv} X_2$ and $Y \vdash X_1$, then $Y \vdash X_2$.

Let p, q be elements of CQC-WFF. The predicate $p \mathrel{\vdash\!\dashv} q$ is defined by:

(Def.5) $p \vdash q$ and $q \vdash p$.

Let us observe that the predicate defined above is reflexive and symmetric.

We now state a number of propositions:

(28) If $p \mathrel{\vdash\!\dashv} q$ and $q \mathrel{\vdash\!\dashv} r$, then $p \mathrel{\vdash\!\dashv} r$.

(29) $p \mathrel{\vdash\!\dashv} q$ iff $\{p\} \mathrel{\vdash\!\dashv} \{q\}$.

(30) If $p \mathrel{\vdash\!\dashv} q$ and $X \vdash p$, then $X \vdash q$.

(31) $\{p, q\} \mathrel{\vdash\!\dashv} \{p \wedge q\}$.

(32) $p \wedge q \mathrel{\vdash\!\dashv} q \wedge p$.

(33) $X \vdash p \wedge q$ iff $X \vdash p$ and $X \vdash q$.

(34) If $p \mathrel{\vdash\!\dashv} q$ and $r \mathrel{\vdash\!\dashv} s$, then $p \wedge r \mathrel{\vdash\!\dashv} q \wedge s$.

(35) $X \vdash \forall_x p$ iff $X \vdash p$.

(36) $\forall_x p \mathrel{\vdash\!\dashv} p$.

(37) If $p \mathrel{\vdash\!\dashv} q$, then $\forall_x p \mathrel{\vdash\!\dashv} \forall_y q$.

Let p, q be elements of CQC-WFF. We say that p is an universal closure of q if and only if the conditions (Def.6) are satisfied.

(Def.6) (i) p is closed, and

(ii) there exists a natural number n such that $1 \leq n$ and there exists a finite sequence L such that $\operatorname{len} L = n$ and $L(1) = q$ and $L(n) = p$ and for every natural number k such that $1 \leq k$ and $k < n$ there exists a bound variable x and there exists an element r of CQC-WFF such that $r = L(k)$ and $L(k+1) = \forall_x r$.

One can prove the following propositions:

(38) If p is an universal closure of q, then $p \vdash\dashv q$.

(39) If $\vdash p \Rightarrow q$, then $p \vdash q$.

(40) If $X \vdash p \Rightarrow q$, then $X \cup \{p\} \vdash q$.

(41) If p is closed and $p \vdash q$, then $\vdash p \Rightarrow q$.

(42) If p_1 is an universal closure of p, then $X \cup \{p\} \vdash q$ iff $X \vdash p_1 \Rightarrow q$.

(43) If p is closed and $p \vdash q$, then $\neg q \vdash \neg p$.

(44) If p is closed and $X \cup \{p\} \vdash q$, then $X \cup \{\neg q\} \vdash \neg p$.

(45) If p is closed and $\neg p \vdash \neg q$, then $q \vdash p$.

(46) If p is closed and $X \cup \{\neg p\} \vdash \neg q$, then $X \cup \{q\} \vdash p$.

(47) If p is closed and q is closed, then $p \vdash q$ iff $\neg q \vdash \neg p$.

(48) If p_1 is an universal closure of p and q_1 is an universal closure of q, then $p \vdash q$ iff $\neg q_1 \vdash \neg p_1$.

(49) If p_1 is an universal closure of p and q_1 is an universal closure of q, then $p \vdash\dashv q$ iff $\neg p_1 \vdash\dashv \neg q_1$.

Let p, q be elements of CQC-WFF. The predicate $p \equiv q$ is defined by:

(Def.7) $\vdash p \Leftrightarrow q$.

Let us observe that this predicate is reflexive and symmetric.

One can prove the following propositions:

(50) $p \equiv q$ iff $\vdash p \Rightarrow q$ and $\vdash q \Rightarrow p$.

(51) If $p \equiv q$ and $q \equiv r$, then $p \equiv r$.

(52) If $p \equiv q$, then $p \vdash\dashv q$.

(53) $p \equiv q$ iff $\neg p \equiv \neg q$.

(54) If $p \equiv q$ and $r \equiv s$, then $p \wedge r \equiv q \wedge s$.

(55) If $p \equiv q$ and $r \equiv s$, then $p \Rightarrow r \equiv q \Rightarrow s$.

(56) If $p \equiv q$ and $r \equiv s$, then $p \vee r \equiv q \vee s$.

(57) If $p \equiv q$ and $r \equiv s$, then $p \Leftrightarrow r \equiv q \Leftrightarrow s$.

(58) If $p \equiv q$, then $\forall_x p \equiv \forall_x q$.

(59) If $p \equiv q$, then $\exists_x p \equiv \exists_x q$.

(60) For all sets X, Y, Z such that $Y \cap Z = \emptyset$ holds $(X \setminus Y) \cup Z = (X \cup Z) \setminus Y$.

(61) Let k be a natural number, and let l be a list of variables of the length k, and let a be a free variable, and let x be a bound variable. Then $\mathrm{snb}(l) \subseteq \mathrm{snb}(l[a \dot{\longmapsto} x])$.

(62) Let k be a natural number, and let l be a list of variables of the length k, and let a be a free variable, and let x be a bound variable. Then $\mathrm{snb}(l[a \dot{\longmapsto} x]) \subseteq \mathrm{snb}(l) \cup \{x\}$.

(63) For every h holds $\mathrm{snb}(h) \subseteq \mathrm{snb}(h(x))$.

(64) For every h holds $\mathrm{snb}(h(x)) \subseteq \mathrm{snb}(h) \cup \{x\}$.

(65) If $p = h(x)$ and $x \neq y$ and $y \notin \mathrm{snb}(h)$, then $y \notin \mathrm{snb}(p)$.

(66) If $p = h(x)$ and $q = h(y)$ and $x \notin \mathrm{snb}(h)$ and $y \notin \mathrm{snb}(h)$, then $\forall_x p \equiv \forall_y q$.

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Some Properties of Restrictions of Finite Sequences

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Summary. The aim of the paper is to define some basic notions of restrictions of finite sequences.

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The notation and terminology used in this paper are introduced in the following papers: [12], [15], [11], [14], [9], [2], [16], [5], [6], [3], [13], [1], [4], [7], [10], and [8].

In this paper i, j, k, k_1, k_2, n are natural numbers.

The following propositions are true:

(1) If $i \leq n$, then $(n-i)+1$ is a natural number.

(2) If $i \in \mathrm{Seg}\, n$, then $(n-i)+1 \in \mathrm{Seg}\, n$.

(3) For every function f and for arbitrary x, y such that $f^{-1}\{y\} = \{x\}$ holds $x \in \mathrm{dom}\, f$ and $y \in \mathrm{rng}\, f$ and $f(x) = y$.

(4) For every function f holds f is one-to-one iff for arbitrary x such that $x \in \mathrm{dom}\, f$ holds $f^{-1}\{f(x)\} = \{x\}$.

(5) For every function f and for arbitrary y_1, y_2 such that f is one-to-one and $y_1 \in \mathrm{rng}\, f$ and $y_2 \in \mathrm{rng}\, f$ and $f^{-1}\{y_1\} = f^{-1}\{y_2\}$ holds $y_1 = y_2$.

Let x be arbitrary. Note that $\langle x \rangle$ is non empty.

Let us note that every set which is empty is also trivial.

Let x be arbitrary. Note that $\langle x \rangle$ is trivial. Let y be arbitrary. Observe that $\langle x, y \rangle$ is non trivial.

One can verify that there exists a finite sequence which is one-to-one and non empty.

Next we state three propositions:

(6) For every non empty finite sequence f holds $1 \in \mathrm{dom}\, f$ and $\mathrm{len}\, f \in \mathrm{dom}\, f$.

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(7) For every non empty finite sequence f there exists i such that $i+1 = \operatorname{len} f$.

(8) For arbitrary x and for every finite sequence f holds $\operatorname{len}(\langle x \rangle \frown f) = 1 + \operatorname{len} f$.

The scheme *domSeqLambda* concerns a natural number $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding arbitrary, and states that:

There exists a finite sequence p such that $\operatorname{len} p = \mathcal{A}$ and for every k such that $k \in \operatorname{dom} p$ holds $p(k) = \mathcal{F}(k)$

for all values of the parameters.

We now state four propositions:

(9) For every set X such that $X \subseteq \operatorname{Seg} n$ and $1 \leq i$ and $i \leq j$ and $j \leq \operatorname{len} \operatorname{Sgm} X$ and $k_1 = (\operatorname{Sgm} X)(i)$ and $k_2 = (\operatorname{Sgm} X)(j)$ holds $k_1 \leq k_2$.

(10) For every finite sequence f and for arbitrary p, q such that $p \in \operatorname{rng} f$ and $q \in \operatorname{rng} f$ and $p \leftarrow\!\!\shortmid f = q \leftarrow\!\!\shortmid f$ holds $p = q$.

(11) For all finite sequences f, g such that $n+1 \in \operatorname{dom} f$ and $g = f \restriction \operatorname{Seg} n$ holds $f \restriction \operatorname{Seg}(n+1) = g \frown \langle f(n+1) \rangle$.

(12) For every one-to-one finite sequence f such that $i \in \operatorname{dom} f$ holds $f(i) \leftarrow\!\!\shortmid f = i$.

We adopt the following rules: D is a non empty set, p, q are elements of D, and f, g are finite sequences of elements of D.

Let us consider D. One can verify that there exists a finite sequence of elements of D which is one-to-one and non empty.

One can prove the following propositions:

(13) If $\operatorname{dom} f = \operatorname{dom} g$ and for every i such that $i \in \operatorname{dom} f$ holds $\pi_i f = \pi_i g$, then $f = g$.

(14) If $\operatorname{len} f = \operatorname{len} g$ and for every k such that $1 \leq k$ and $k \leq \operatorname{len} f$ holds $\pi_k f = \pi_k g$, then $f = g$.

(15) If $\operatorname{len} f = 1$, then $f = \langle \pi_1 f \rangle$.

(16) $\pi_1(\langle p \rangle \frown f) = p$.

(18)[1] $\operatorname{len}(f \restriction i) \leq \operatorname{len} f$.

(19) $\operatorname{len}(f \restriction i) \leq i$.

(20) $\operatorname{dom}(f \restriction i) \subseteq \operatorname{dom} f$.

(21) $\operatorname{rng}(f \restriction i) \subseteq \operatorname{rng} f$.

Let us consider D, f. Observe that $f \restriction 0$ is empty.

Next we state three propositions:

(22) If $\operatorname{len} f \leq i$, then $f \restriction i = f$.

(23) If f is non empty, then $f \restriction 1 = \langle \pi_1 f \rangle$.

(24) If $i+1 = \operatorname{len} f$, then $f = (f \restriction i) \frown \langle \pi_{\operatorname{len} f} f \rangle$.

Let us consider i, D and let f be an one-to-one finite sequence of elements of D. One can verify that $f \restriction i$ is one-to-one.

[1]The proposition (17) has been removed.

The following propositions are true:

(25) If $i \leq \operatorname{len} f$, then $(f \frown g) \restriction i = f \restriction i$.

(26) $(f \frown g) \restriction \operatorname{len} f = f$.

(27) If $p \in \operatorname{rng} f$, then $(f \leftarrow p) \frown \langle p \rangle = f \restriction p \leftrightarrow f$.

(28) $\operatorname{len}(f_{\lfloor i}) \leq \operatorname{len} f$.

(29) If $i \in \operatorname{dom}(f_{\lfloor n})$, then $n + i \in \operatorname{dom} f$.

(30) If $i \in \operatorname{dom}(f_{\lfloor n})$, then $\pi_i f_{\lfloor n} = \pi_{n+i} f$.

(31) $f_{\lfloor 0} = f$.

(32) If f is non empty, then $f = \langle \pi_1 f \rangle \frown (f_{\lfloor 1})$.

(33) If $i + 1 = \operatorname{len} f$, then $f_{\lfloor i} = \langle \pi_{\operatorname{len} f} f \rangle$.

(34) If $j + 1 = i$ and $i \in \operatorname{dom} f$, then $\langle \pi_i f \rangle \frown (f_{\lfloor i}) = f_{\lfloor j}$.

(35) If $\operatorname{len} f \leq i$, then $f_{\lfloor i}$ is empty.

(36) $\operatorname{rng}(f_{\lfloor n}) \subseteq \operatorname{rng} f$.

Let us consider i, D and let f be an one-to-one finite sequence of elements of D. Note that $f_{\lfloor i}$ is one-to-one.

The following propositions are true:

(37) If f is one-to-one, then $\operatorname{rng}(f \restriction n)$ misses $\operatorname{rng}(f_{\lfloor n})$.

(38) If $p \in \operatorname{rng} f$, then $f \rightarrow p = f_{\lfloor p \leftrightarrow f}$.

(39) $(f \frown g)_{\lfloor \operatorname{len} f + i} = g_{\lfloor i}$.

(40) $(f \frown g)_{\lfloor \operatorname{len} f} = g$.

(41) If $p \in \operatorname{rng} f$, then $\pi_{p \leftrightarrow f} f = p$.

(42) If $i \in \operatorname{dom} f$, then $(\pi_i f) \leftrightarrow f \leq i$.

(43) If $p \in \operatorname{rng}(f \restriction i)$, then $p \leftrightarrow (f \restriction i) = p \leftrightarrow f$.

(44) If $i \in \operatorname{dom} f$ and f is one-to-one, then $(\pi_i f) \leftrightarrow f = i$.

Let us consider D, f and let p be arbitrary. The functor $f -: p$ yielding a finite sequence of elements of D is defined as follows:

(Def.1) $f -: p = f \restriction p \leftrightarrow f$.

One can prove the following propositions:

(45) If $p \in \operatorname{rng} f$, then $\operatorname{len}(f -: p) = p \leftrightarrow f$.

(46) If $p \in \operatorname{rng} f$ and $i \in \operatorname{Seg}(p \leftrightarrow f)$, then $\pi_i(f -: p) = \pi_i f$.

(47) If $p \in \operatorname{rng} f$, then $\pi_1(f -: p) = \pi_1 f$.

(48) If $p \in \operatorname{rng} f$, then $\pi_{p \leftrightarrow f}(f -: p) = p$.

(49) If $q \in \operatorname{rng} f$ and $p \in \operatorname{rng} f$ and $q \leftrightarrow f \leq p \leftrightarrow f$, then $q \in \operatorname{rng}(f -: p)$.

(50) If $p \in \operatorname{rng} f$, then $f -: p$ is non empty.

(51) $\operatorname{rng}(f -: p) \subseteq \operatorname{rng} f$.

Let us consider D, p and let f be an one-to-one finite sequence of elements of D. Observe that $f -: p$ is one-to-one.

Let us consider D, f, p. The functor $f :- p$ yielding a finite sequence of elements of D is defined by:

(Def.2) $f :- p = \langle p \rangle \frown (f_{\lfloor p \leftrightarrow f})$.

We now state three propositions:

(52) If $p \in \operatorname{rng} f$, then there exists i such that $i+1 = p \leftrightarrow f$ and $f :- p = f_{\restriction i}$.

(53) If $p \in \operatorname{rng} f$, then $\operatorname{len}(f :- p) = (\operatorname{len} f - p \leftrightarrow f) + 1$.

(54) If $p \in \operatorname{rng} f$ and $j + 1 \in \operatorname{dom}(f :- p)$, then $j + p \leftrightarrow f \in \operatorname{dom} f$.

Let us consider D, p, f. One can check that $f :- p$ is non empty.

Next we state several propositions:

(55) If $p \in \operatorname{rng} f$ and $j + 1 \in \operatorname{dom}(f :- p)$, then $\pi_{j+1}(f :- p) = \pi_{j+p \leftrightarrow f} f$.

(56) $\pi_1(f :- p) = p$.

(57) If $p \in \operatorname{rng} f$, then $\pi_{\operatorname{len}(f:-p)}(f :- p) = \pi_{\operatorname{len} f} f$.

(58) If $p \in \operatorname{rng} f$, then $\operatorname{rng}(f :- p) \subseteq \operatorname{rng} f$.

(59) If $p \in \operatorname{rng} f$ and f is one-to-one, then $f :- p$ is one-to-one.

Let f be a finite sequence. The functor $\operatorname{Rev}(f)$ yielding a finite sequence is defined by:

(Def.3) $\operatorname{len} \operatorname{Rev}(f) = \operatorname{len} f$ and for every i such that $i \in \operatorname{dom} \operatorname{Rev}(f)$ holds $(\operatorname{Rev}(f))(i) = f((\operatorname{len} f - i) + 1)$.

One can prove the following propositions:

(60) For every finite sequence f holds $\operatorname{dom} f = \operatorname{dom} \operatorname{Rev}(f)$ and $\operatorname{rng} f = \operatorname{rng} \operatorname{Rev}(f)$.

(61) For every finite sequence f such that $i \in \operatorname{dom} f$ holds $(\operatorname{Rev}(f))(i) = f((\operatorname{len} f - i) + 1)$.

(62) For every finite sequence f and for all natural numbers i, j such that $i \in \operatorname{dom} f$ and $i + j = \operatorname{len} f + 1$ holds $j \in \operatorname{dom} \operatorname{Rev}(f)$.

Let f be an empty finite sequence. Observe that $\operatorname{Rev}(f)$ is empty.

Next we state three propositions:

(63) For arbitrary x holds $\operatorname{Rev}(\langle x \rangle) = \langle x \rangle$.

(64) For arbitrary x_1, x_2 holds $\operatorname{Rev}(\langle x_1, x_2 \rangle) = \langle x_2, x_1 \rangle$.

(65) For every non empty finite sequence f holds $f(1) = (\operatorname{Rev}(f))(\operatorname{len} f)$ and $f(\operatorname{len} f) = (\operatorname{Rev}(f))(1)$.

Let f be an one-to-one finite sequence. Note that $\operatorname{Rev}(f)$ is one-to-one.

The following two propositions are true:

(66) For every finite sequence f and for arbitrary x holds $\operatorname{Rev}(f \frown \langle x \rangle) = \langle x \rangle \frown \operatorname{Rev}(f)$.

(67) For all finite sequences f, g holds $\operatorname{Rev}(f \frown g) = (\operatorname{Rev}(g)) \frown \operatorname{Rev}(f)$.

Let us consider D, f. Then $\operatorname{Rev}(f)$ is a finite sequence of elements of D.

We now state two propositions:

(68) If f is non empty, then $\pi_1 f = \pi_{\operatorname{len} f} \operatorname{Rev}(f)$ and $\pi_{\operatorname{len} f} f = \pi_1 \operatorname{Rev}(f)$.

(69) If $i \in \operatorname{dom} f$ and $i + j = \operatorname{len} f + 1$, then $\pi_i f = \pi_j \operatorname{Rev}(f)$.

Let us consider D, f, p, n. The functor $\operatorname{Ins}(f, n, p)$ yielding a finite sequence of elements of D is defined as follows:

(Def.4) $\operatorname{Ins}(f, n, p) = (f \upharpoonright n) \frown \langle p \rangle \frown (f_{\restriction n})$.

One can prove the following propositions:

(70) $\mathrm{Ins}(f, 0, p) = \langle p \rangle \frown f$.

(71) If $\mathrm{len}\, f \leq n$, then $\mathrm{Ins}(f, n, p) = f \frown \langle p \rangle$.

(72) $\mathrm{len}\,\mathrm{Ins}(f, n, p) = \mathrm{len}\, f + 1$.

(73) $\mathrm{rng}\,\mathrm{Ins}(f, n, p) = \{p\} \cup \mathrm{rng}\, f$.

Let us consider D, f, n, p. Observe that $\mathrm{Ins}(f, n, p)$ is non empty.

The following propositions are true:

(74) $p \in \mathrm{rng}\,\mathrm{Ins}(f, n, p)$.

(75) If $i \in \mathrm{dom}(f \restriction n)$, then $\pi_i\,\mathrm{Ins}(f, n, p) = \pi_i f$.

(76) If $n \leq \mathrm{len}\, f$, then $\pi_{n+1}\,\mathrm{Ins}(f, n, p) = p$.

(77) If $n + 1 \leq i$ and $i \leq \mathrm{len}\, f$, then $\pi_{i+1}\,\mathrm{Ins}(f, n, p) = \pi_i f$.

(78) If $1 \leq n$ and f is non empty, then $\pi_1\,\mathrm{Ins}(f, n, p) = \pi_1 f$.

(79) If f is one-to-one and $p \notin \mathrm{rng}\, f$, then $\mathrm{Ins}(f, n, p)$ is one-to-one.

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Special Polygons

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The papers [22], [26], [21], [25], [13], [1], [14], [27], [4], [5], [2], [23], [3], [10], [24], [19], [15], [18], [7], [9], [8], [20], [11], [12], [17], [16], and [6] provide the notation and terminology for this paper.

1. Segments in $\mathcal{E}^2_{\rm T}$

For simplicity we adopt the following convention: P, P_1, P_2 will be subsets of the carrier of $\mathcal{E}^2_{\rm T}$, f, f_1, f_2, g will be finite sequences of elements of $\mathcal{E}^2_{\rm T}$, p, p_1, p_2, q, q_1, q_2 will be points of $\mathcal{E}^2_{\rm T}$, r_1, r_2, r'_1, r'_2 will be real numbers, and i, j, k, n will be natural numbers.

Next we state a number of propositions:

(1) If $[r_1, r_2] = [r'_1, r'_2]$, then $r_1 = r'_1$ and $r_2 = r'_2$.

(2) If $i + j = \operatorname{len} f$, then $\mathcal{L}(f, i) = \mathcal{L}(\operatorname{Rev}(f), j)$.

(3) If $i + 1 \leq \operatorname{len}(f \restriction n)$, then $\mathcal{L}(f \restriction n, i) = \mathcal{L}(f, i)$.

(4) If $n \leq \operatorname{len} f$ and $1 \leq i$, then $\mathcal{L}(f_{\downharpoonright n}, i) = \mathcal{L}(f, n + i)$.

(5) If $1 \leq i$ and $i + 1 \leq \operatorname{len} f - n$, then $\mathcal{L}(f_{\downharpoonright n}, i) = \mathcal{L}(f, n + i)$.

(6) If $i + 1 \leq \operatorname{len} f$, then $\mathcal{L}(f \frown g, i) = \mathcal{L}(f, i)$.

(7) If $1 \leq i$, then $\mathcal{L}(f \frown g, \operatorname{len} f + i) = \mathcal{L}(g, i)$.

(8) If f is non empty and g is non empty, then $\mathcal{L}(f \frown g, \operatorname{len} f) = \mathcal{L}(\pi_{\operatorname{len} f} f, \pi_1 g)$.

(9) If $i + 1 \leq \operatorname{len}(f -: p)$, then $\mathcal{L}(f -: p, i) = \mathcal{L}(f, i)$.

(10) If $p \in \operatorname{rng} f$ and $1 \leq i + 1$, then $\mathcal{L}(f :- p, i + 1) = \mathcal{L}(f, i + p \looparrowleft f)$.

(11) $\widetilde{\mathcal{L}}(\varepsilon_{(\text{the carrier of } \mathcal{E}^2_{\rm T})}) = \emptyset$.

(12) $\widetilde{\mathcal{L}}(\langle p \rangle) = \emptyset$.

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(13) If $p \in \widetilde{\mathcal{L}}(f)$, then there exists i such that $1 \leq i$ and $i+1 \leq \operatorname{len} f$ and $p \in \mathcal{L}(f,i)$.

(14) If $p \in \widetilde{\mathcal{L}}(f)$, then there exists i such that $1 \leq i$ and $i+1 \leq \operatorname{len} f$ and $p \in \mathcal{L}(\pi_i f, \pi_{i+1} f)$.

(15) If $1 \leq i$ and $i+1 \leq \operatorname{len} f$ and $p \in \mathcal{L}(\pi_i f, \pi_{i+1} f)$, then $p \in \widetilde{\mathcal{L}}(f)$.

(16) If $1 \leq i$ and $i+1 \leq \operatorname{len} f$, then $\mathcal{L}(\pi_i f, \pi_{i+1} f) \subseteq \widetilde{\mathcal{L}}(f)$.

(17) If $p \in \mathcal{L}(f,i)$, then $p \in \widetilde{\mathcal{L}}(f)$.

(18) If $\operatorname{len} f \geq 2$, then $\operatorname{rng} f \subseteq \widetilde{\mathcal{L}}(f)$.

(19) If f is non empty, then $\widetilde{\mathcal{L}}(f \frown \langle p \rangle) = \widetilde{\mathcal{L}}(f) \cup \mathcal{L}(\pi_{\operatorname{len} f} f, p)$.

(20) If f is non empty, then $\widetilde{\mathcal{L}}(\langle p \rangle \frown f) = \mathcal{L}(p, \pi_1 f) \cup \widetilde{\mathcal{L}}(f)$.

(21) $\widetilde{\mathcal{L}}(\langle p, q \rangle) = \mathcal{L}(p,q)$.

(22) $\widetilde{\mathcal{L}}(f) = \widetilde{\mathcal{L}}(\operatorname{Rev}(f))$.

(23) If f_1 is non empty and f_2 is non empty, then $\widetilde{\mathcal{L}}(f_1 \frown f_2) = \widetilde{\mathcal{L}}(f_1) \cup \mathcal{L}(\pi_{\operatorname{len} f_1} f_1, \pi_1 f_2) \cup \widetilde{\mathcal{L}}(f_2)$.

(25)[1] If $q \in \operatorname{rng} f$, then $\widetilde{\mathcal{L}}(f) = \widetilde{\mathcal{L}}(f -\!\!: q) \cup \widetilde{\mathcal{L}}(f :\!\!- q)$.

(26) If $p \in \mathcal{L}(f,n)$, then $\widetilde{\mathcal{L}}(f) = \widetilde{\mathcal{L}}(\operatorname{Ins}(f,n,p))$.

2. Special Sequences in $\mathcal{E}^2_{\mathrm{T}}$

One can verify the following observations:

* there exists a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$
* every finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ is one-to-one unfolded s.n.c. special and non trivial,
* every finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ which is one-to-one unfolded s.n.c. special and non trivial has and
* every finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ is non empty.

Let us note that there exists a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ which is one-to-one unfolded s.n.c. special and non trivial.

We now state the proposition

(27) If $\operatorname{len} f \leq 2$, then f is unfolded.

Let f be an unfolded finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let us consider n. Note that $f \restriction n$ is unfolded and $f_{\restriction n}$ is unfolded.

One can prove the following proposition

(28) If $p \in \operatorname{rng} f$ and f is unfolded, then $f :\!\!- p$ is unfolded.

Let f be an unfolded finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let us consider p. Observe that $f -\!\!: p$ is unfolded.

Next we state several propositions:

[1] The proposition (24) has been removed.

(29) If f is unfolded, then $\mathrm{Rev}(f)$ is unfolded.

(30) If g is unfolded and $\mathcal{L}(p, \pi_1 g) \cap \mathcal{L}(g, 1) = \{\pi_1 g\}$, then $\langle p \rangle \frown g$ is unfolded.

(31) If f is unfolded and $k+1 = \mathrm{len}\, f$ and $\mathcal{L}(f, k) \cap \mathcal{L}(\pi_{\mathrm{len}\, f} f, p) = \{\pi_{\mathrm{len}\, f} f\}$, then $f \frown \langle p \rangle$ is unfolded.

(32) Suppose f is unfolded and g is unfolded and $k+1 = \mathrm{len}\, f$ and $\mathcal{L}(f, k) \cap \mathcal{L}(\pi_{\mathrm{len}\, f} f, \pi_1 g) = \{\pi_{\mathrm{len}\, f} f\}$ and $\mathcal{L}(\pi_{\mathrm{len}\, f} f, \pi_1 g) \cap \mathcal{L}(g, 1) = \{\pi_1 g\}$. Then $f \frown g$ is unfolded.

(33) If f is unfolded and $p \in \mathcal{L}(f, n)$, then $\mathrm{Ins}(f, n, p)$ is unfolded.

(34) If $\mathrm{len}\, f \leq 2$, then f is s.n.c..

Let f be a s.n.c. finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let us consider n. Observe that $f \restriction n$ is s.n.c. and $f_{\lfloor n}$ is s.n.c..

Let f be a s.n.c. finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let us consider p. Note that $f -: p$ is s.n.c..

We now state four propositions:

(35) If $p \in \mathrm{rng}\, f$ and f is s.n.c., then $f :- p$ is s.n.c..

(36) If f is s.n.c., then $\mathrm{Rev}(f)$ is s.n.c..

(37) Suppose that

(i) f is s.n.c.,

(ii) g is s.n.c.,

(iii) $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \emptyset$,

(iv) for every i such that $1 \leq i$ and $i+2 \leq \mathrm{len}\, f$ holds $\mathcal{L}(f, i) \cap \mathcal{L}(\pi_{\mathrm{len}\, f} f, \pi_1 g) = \emptyset$, and

(v) for every i such that $2 \leq i$ and $i+1 \leq \mathrm{len}\, g$ holds $\mathcal{L}(g, i) \cap \mathcal{L}(\pi_{\mathrm{len}\, f} f, \pi_1 g) = \emptyset$.

Then $f \frown g$ is s.n.c..

(38) If f is unfolded and s.n.c. and $p \in \mathcal{L}(f, n)$ and $p \notin \mathrm{rng}\, f$, then $\mathrm{Ins}(f, n, p)$ is s.n.c..

Let us observe that $\varepsilon_{(\text{the carrier of } \mathcal{E}^2_{\mathrm{T}})}$ is special.

Next we state two propositions:

(39) $\langle p \rangle$ is special.

(40) If $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$, then $\langle p, q \rangle$ is special.

Let f be a special finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let us consider n. Note that $f \restriction n$ is special and $f_{\lfloor n}$ is special.

We now state the proposition

(41) If $p \in \mathrm{rng}\, f$ and f is special, then $f :- p$ is special.

Let f be a special finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let us consider p. Observe that $f -: p$ is special.

The following four propositions are true:

(42) If f is special, then $\mathrm{Rev}(f)$ is special.

(44)[2] If f is special and $p \in \mathcal{L}(f, n)$, then $\mathrm{Ins}(f, n, p)$ is special.

[2]The proposition (43) has been removed.

(45) If $q \in \operatorname{rng} f$ and $1 \neq q \looparrowleft f$ and $q \looparrowleft f \neq \operatorname{len} f$ and f is unfolded and s.n.c., then $\widetilde{\mathcal{L}}(f -: q) \cap \widetilde{\mathcal{L}}(f :- q) = \{q\}$.

(46) If $p \neq q$ and if $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$, then $\langle p, q \rangle$

a S-sequence in $\mathbb{R}^2$ is a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$.

The following propositions are true:

(47) For every S-sequence f in $\mathbb{R}^2$ holds $\operatorname{Rev}(f)$

(48) For every S-sequence f in $\mathbb{R}^2$ such that $i \in \operatorname{dom} f$ holds $\pi_i f \in \widetilde{\mathcal{L}}(f)$.

(49) If $p \neq q$ and if $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$, then $\mathcal{L}(p, q)$

(50) For every S-sequence f in $\mathbb{R}^2$ such that $p \in \operatorname{rng} f$ and $p \looparrowleft f \neq 1$ holds $f -: p$

(51) For every S-sequence f in $\mathbb{R}^2$ such that $p \in \operatorname{rng} f$ and $p \looparrowleft f \neq \operatorname{len} f$ holds $f :- p$

(52) For every S-sequence f in $\mathbb{R}^2$ such that $p \in \mathcal{L}(f, i)$ and $p \notin \operatorname{rng} f$ holds $\operatorname{Ins}(f, i, p)$

3. Special Polygons in $\mathcal{E}^2_{\mathrm{T}}$

Let us mention that there exists a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and every subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ is non empty.

The following proposition is true

(53) If P is a special polygonal arc joining p_1 and p_2, then P is a special polygonal arc joining p_2 and p_1.

Let us consider p_1, p_2, P. We say that p_1 and p_2 split P if and only if the conditions (Def.1) are satisfied.

(Def.1) (i) $p_1 \neq p_2$, and

(ii) there exist S-sequences f_1, f_2 in $\mathbb{R}^2$ such that $p_1 = \pi_1 f_1$ and $p_1 = \pi_1 f_2$ and $p_2 = \pi_{\operatorname{len} f_1} f_1$ and $p_2 = \pi_{\operatorname{len} f_2} f_2$ and $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) = \{p_1, p_2\}$ and $P = \widetilde{\mathcal{L}}(f_1) \cup \widetilde{\mathcal{L}}(f_2)$.

We now state four propositions:

(54) If p_1 and p_2 split P, then p_2 and p_1 split P.

(55) If p_1 and p_2 split P and $q \in P$ and $q \neq p_1$, then p_1 and q split P.

(56) If p_1 and p_2 split P and $q \in P$ and $q \neq p_2$, then q and p_2 split P.

(57) If p_1 and p_2 split P and $q_1 \in P$ and $q_2 \in P$ and $q_1 \neq q_2$, then q_1 and q_2 split P.

Let us observe that a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ is special polygon if:

(Def.2) There exist p_1, p_2 such that p_1 and p_2 split it.

We introduce special polygonal as a synonym of special polygon.

Let us consider r_1, r_2, r'_1, r'_2. The functor $[.r_1, r_2, r'_1, r'_2.]$ yields a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by the condition (Def.3).

(Def.3) $[.r_1, r_2, r'_1, r'_2.] = \{p : p_{\mathbf{1}} = r_1 \wedge p_{\mathbf{2}} \leq r'_2 \wedge p_{\mathbf{2}} \geq r'_1 \vee p_{\mathbf{1}} \leq r_2 \wedge p_{\mathbf{1}} \geq r_1 \wedge p_{\mathbf{2}} = r'_2 \vee p_{\mathbf{1}} \leq r_2 \wedge p_{\mathbf{1}} \geq r_1 \wedge p_{\mathbf{2}} = r'_1 \vee p_{\mathbf{1}} = r_2 \wedge p_{\mathbf{2}} \leq r'_2 \wedge p_{\mathbf{2}} \geq r'_1\}$.

One can prove the following propositions:

(58) If $r_1 < r_2$ and $r'_1 < r'_2$, then $[.r_1, r_2, r'_1, r'_2.] = \mathcal{L}([r_1, r'_1], [r_1, r'_2]) \cup \mathcal{L}([r_1, r'_2], [r_2, r'_2]) \cup (\mathcal{L}([r_2, r'_2], [r_2, r'_1]) \cup \mathcal{L}([r_2, r'_1], [r_1, r'_1]))$.

(59) If $r_1 < r_2$ and $r'_1 < r'_2$, then $[.r_1, r_2, r'_1, r'_2.]$ is special polygonal.

(60) $\square_{\mathcal{E}^2} = [.0, 1, 0, 1.]$.

(61) $\square_{\mathcal{E}^2}$ is special polygonal.

One can verify the following observations:

* there exists a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ which is special polygonal,
* every subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ which is special polygonal is also non empty, and
* every subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ which is special polygonal is also non trivial.

A special polygon in $\mathbb{R}^2$ is a special polygonal subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$.

We now state four propositions:

(62) If P is then P is compact.

(63) Every special polygon in $\mathbb{R}^2$ is compact.

(64) If P is special polygonal, then for all p_1, p_2 such that $p_1 \neq p_2$ and $p_1 \in P$ and $p_2 \in P$ holds p_1 and p_2 split P.

(65) Suppose P is special polygonal. Given p_1, p_2. Suppose $p_1 \neq p_2$ and $p_1 \in P$ and $p_2 \in P$. Then there exist P_1, P_2 such that

(i) P_1 is a special polygonal arc joining p_1 and p_2,
(ii) P_2 is a special polygonal arc joining p_1 and p_2,
(iii) $P_1 \cap P_2 = \{p_1, p_2\}$, and
(iv) $P = P_1 \cup P_2$.

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The One-Dimensional Lebesgue Measure

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Summary. The paper is the crowning of a series of articles written in the Mizar language, being a formalization of notions needed for the description of the one-dimensional Lebesgue measure. The formalization of the notion as classical as the Lebesgue measure determines the powers of the PC Mizar system as a tool for the strict, precise notation and verification of the correctness of deductive theories. Following the successive articles [6], [8], [10], [11] constructed so that the final one should include the definition and the basic properties of the Lebesgue measure, we observe one of the paths relatively simple in the sense of the definition, enabling us the formal introduction of this notion. This way, although toilsome, since such is the nature of formal theories, is greatly instructive. It brings home the proper succession of the introduction of the definitions of intermediate notions and points out to those elements of the theory which determine the essence of the complexity of the notion being introduced.

The paper includes the definition of the σ-field of Lebesgue measurable sets, the definition of the Lebesgue measure and the basic set of the theorems describing its properties.

MML Identifier: `MEASURE7`.

The terminology and notation used in this paper are introduced in the following articles: [21], [24], [20], [25], [14], [12], [13], [2], [19], [3], [17], [6], [8], [10], [9], [5], [7], [18], [11], [23], [1], [4], [16], [22], and [15].

The following propositions are true:

(1) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that for every natural number n holds $F(n) = 0_{\overline{\mathbb{R}}}$ holds $\sum F = 0_{\overline{\mathbb{R}}}$.

(2) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is non-negative and for every natural number n holds $F(n) \leq (\operatorname{Ser} F)(n)$.

(3) Let F, G, H be functions from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose G is non-negative and H is non-negative. Suppose that for every natural number n holds $F(n) = G(n) + H(n)$. Let n be a natural number. Then $(\operatorname{Ser} F)(n) = (\operatorname{Ser} G)(n) + (\operatorname{Ser} H)(n)$.

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(4) Let F, G, H be functions from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose that for every natural number n holds $F(n) = G(n) + H(n)$. If G is non-negative and H is non-negative, then $\sum F \leq \sum G + \sum H$.

(5) Let F, G be functions from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose F is non-negative and for every natural number n holds $F(n) = G(n)$. Let n be a natural number. Then $(\operatorname{Ser} F)(n) = (\operatorname{Ser} G)(n)$.

(6) Let F, G be functions from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose F is non-negative and for every natural number n holds $F(n) \leq G(n)$. Let n be a natural number. Then $(\operatorname{Ser} F)(n) \leq \sum G$.

(7) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is non-negative and for every natural number n holds $(\operatorname{Ser} F)(n) \leq \sum F$.

Let S be a non empty subset of $\mathbb{N}$, let H be a function from S into $\mathbb{N}$, and let n be an element of S. Then $H(n)$ is a natural number.

Let G be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$, let S be a non empty subset of $\mathbb{N}$, and let H be a function from S into $\mathbb{N}$. The functor $\operatorname{On}(G, H)$ yields a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and is defined as follows:

(Def.1) For every element n of $\mathbb{N}$ holds if $n \in S$, then $(\operatorname{On}(G, H))(n) = G(H(n))$ and if $n \notin S$, then $(\operatorname{On}(G, H))(n) = 0_{\overline{\mathbb{R}}}$.

Next we state several propositions:

(8) Let G be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose G is non-negative. Let S be a non empty subset of $\mathbb{N}$ and let H be a function from S into $\mathbb{N}$. Then $\operatorname{On}(G, H)$ is non-negative.

(9) Let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose F is non-negative. Let n, k be natural numbers. If $n \leq k$, then $(\operatorname{Ser} F)(n) \leq (\operatorname{Ser} F)(k)$.

(10) Let k be a natural number and let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose F is non-negative. Suppose that for every natural number n such that $n \neq k$ holds $F(n) = 0_{\overline{\mathbb{R}}}$. Then
  (i) for every natural number n such that $n < k$ holds $(\operatorname{Ser} F)(n) = 0_{\overline{\mathbb{R}}}$, and
  (ii) for every natural number n such that $k \leq n$ holds $(\operatorname{Ser} F)(n) = F(k)$.

(11) Let G be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose G is non-negative. Let S be a non empty subset of $\mathbb{N}$ and let H be a function from S into $\mathbb{N}$. If H is one-to-one and $\operatorname{rng} H = \mathbb{N}$, then $\sum \operatorname{On}(G, H) \leq \sum G$.

(12) Let F, G be functions from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose F is non-negative and G is non-negative. Let S be a non empty subset of $\mathbb{N}$ and let H be a function from S into $\mathbb{N}$. Suppose H is one-to-one and $\operatorname{rng} H = \mathbb{N}$. Suppose that for every natural number k holds if $k \in S$, then $F(k) = G(H(k))$ and if $k \notin S$, then $F(k) = 0_{\overline{\mathbb{R}}}$. Then $\sum F \leq \sum G$.

Let A be a subset of $\mathbb{R}$. A function from $\mathbb{N}$ into $2^{\mathbb{R}}$ is said to be an interval covering of A if:

(Def.2) $A \subseteq \bigcup \operatorname{rng} \mathrm{it}$ and for every natural number n holds $\mathrm{it}(n)$ is an interval.

Let A be a subset of $\mathbb{R}$, let F be an interval covering of A, and let n be a natural number. Then $F(n)$ is an interval.

Let F be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$. A function from $\mathbb{N}$ into $(2^{\mathbb{R}})^{\mathbb{N}}$ is said to be an interval covering of F if:

(Def.3) For every natural number n holds it(n) is an interval covering of $F(n)$.

Let A be a subset of $\mathbb{R}$ and let F be an interval covering of A. The functor $(F)\,\mathrm{vol}$ yields a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and is defined by:

(Def.4) For every natural number n holds $(F)\,\mathrm{vol}(n) = \mathrm{vol}(F(n))$.

The following proposition is true

(13) For every subset A of $\mathbb{R}$ and for every interval covering F of A holds $(F)\,\mathrm{vol}$ is non-negative.

Let F be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$, let H be an interval covering of F, and let n be a natural number. Then $H(n)$ is an interval covering of $F(n)$.

Let F be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$ and let G be an interval covering of F. The functor $(G)\,\mathrm{vol}$ yields a function from $\mathbb{N}$ into $\overline{\mathbb{R}}^{\mathbb{N}}$ and is defined by:

(Def.5) For every natural number n holds $(G)\,\mathrm{vol}(n) = (G(n))\,\mathrm{vol}$.

Let A be a subset of $\mathbb{R}$ and let F be an interval covering of A. The functor $\mathrm{vol}(F)$ yields a *Real number* and is defined as follows:

(Def.6) $\mathrm{vol}(F) = \sum((F)\,\mathrm{vol})$.

Let F be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$ and let G be an interval covering of F. The functor $\mathrm{vol}(G)$ yielding a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ is defined by:

(Def.7) For every natural number n holds $(\mathrm{vol}(G))(n) = \mathrm{vol}(G(n))$.

One can prove the following proposition

(14) Let F be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$, and let G be an interval covering of F, and let n be a natural number. Then $0_{\overline{\mathbb{R}}} \leq (\mathrm{vol}(G))(n)$.

Let A be a subset of $\mathbb{R}$. The functor $\mathrm{Svc}(A)$ yielding a non empty subset of $\overline{\mathbb{R}}$ is defined by:

(Def.8) For every *Real number* x holds $x \in \mathrm{Svc}(A)$ iff there exists an interval covering F of A such that $x = \mathrm{vol}(F)$.

Let A be an element of $2^{\mathbb{R}}$. The functor $\mathbb{C}^{A}$ yields an element of $\overline{\mathbb{R}}$ and is defined as follows:

(Def.9) $\mathbb{C}^{A} = \inf \mathrm{Svc}(A)$.

The function OSMeas from $2^{\mathbb{R}}$ into $\overline{\mathbb{R}}$ is defined by:

(Def.10) For every subset A of $\mathbb{R}$ holds $(\mathrm{OSMeas})(A) = \inf \mathrm{Svc}(A)$.

Let F be a function from $\mathbb{N}$ into $\mathbb{N}$ and let n be a natural number. Then $F(n)$ is a natural number.

Let x, y be *Real numbers*. Then $\{x, y\}$ is a subset of $\overline{\mathbb{R}}$.

Let H be a function from $\mathbb{N}$ into $[\!: \mathbb{N}, \mathbb{N} :\!]$. The functor $\mathrm{pr1}(H)$ yielding a function from $\mathbb{N}$ into $\mathbb{N}$ is defined by:

(Def.11) For every element n of $\mathbb{N}$ there exists an element s of $\mathbb{N}$ such that $H(n) = \langle \mathrm{pr1}(H)(n), s \rangle$.

Let H be a function from $\mathbb{N}$ into $[\!: \mathbb{N}, \mathbb{N} :\!]$. The functor $\mathrm{pr2}(H)$ yielding a function from $\mathbb{N}$ into $\mathbb{N}$ is defined by:

(Def.12) For every element n of $\mathbb{N}$ holds $H(n) = \langle \mathrm{pr1}(H)(n), \mathrm{pr2}(H)(n) \rangle$.

Let F be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$, let G be an interval covering of F, and let H be a function from $\mathbb{N}$ into $[:\mathbb{N}, \mathbb{N}:]$. Let us assume that H is one-to-one and $\mathrm{rng}\, H = [:\mathbb{N}, \mathbb{N}:]$. The functor $\mathrm{On}(G, H)$ yields an interval covering of $\bigcup \mathrm{rng}\, F$ and is defined by:

(Def.13) For every element n of $\mathbb{N}$ holds $(\mathrm{On}(G, H))(n) = G(\mathrm{pr1}(H)(n))(\mathrm{pr2}(H)(n))$.

Next we state three propositions:

(15) Let H be a function from $\mathbb{N}$ into $[:\mathbb{N}, \mathbb{N}:]$. Suppose H is one-to-one and $\mathrm{rng}\, H = [:\mathbb{N}, \mathbb{N}:]$. Let k be a natural number. Then there exists a natural number m such that for every function F from $\mathbb{N}$ into $2^{\mathbb{R}}$ and for every interval covering G of F holds $(\mathrm{Ser}((\mathrm{On}(G, H))\, \mathrm{vol}))(k) \leq (\mathrm{Ser}\, \mathrm{vol}(G))(m)$.

(16) For every function F from $\mathbb{N}$ into $2^{\mathbb{R}}$ and for every interval covering G of F holds $\inf \mathrm{Svc}(\bigcup \mathrm{rng}\, F) \leq \sum \mathrm{vol}(G)$.

(17)[1] OSMeas is a Caratheodor's measure on $\mathbb{R}$.

OSMeas is a Caratheodor's measure on $\mathbb{R}$.

The functor L_μ-σFIELD is a σ-field of subsets of $\mathbb{R}$ and is defined by:

(Def.14) L_μ-$\sigma\mathrm{FIELD} = \sigma$-Field(OSMeas).

The σ-measure L_μ on L_μ-σFIELD is defined by:

(Def.15) $L_\mu = \sigma$-Meas(OSMeas).

The following propositions are true:

(18) L_μ is complete on L_μ-σFIELD.

(19) L_μ is a measure on L_μ-σFIELD.

(20) $\emptyset \in L_\mu$-σFIELD and $\mathbb{R} \in L_\mu$-σFIELD.

(21) For every set A such that $A \in L_\mu$-σFIELD holds $\mathbb{R} \setminus A \in L_\mu$-$\sigma$FIELD.

(22) For all sets A, B such that $A \in L_\mu$-σFIELD and $B \in L_\mu$-σFIELD holds $A \cup B \in L_\mu$-σFIELD.

(23) For all sets A, B such that $A \in L_\mu$-σFIELD and $B \in L_\mu$-σFIELD holds $A \cap B \in L_\mu$-σFIELD.

(24) For all sets A, B such that $A \in L_\mu$-σFIELD and $B \in L_\mu$-σFIELD holds $A \setminus B \in L_\mu$-σFIELD.

(25) For every family T of measurable sets of L_μ-σFIELD holds $\bigcap T \in L_\mu$-σFIELD and $\bigcup T \in L_\mu$-σFIELD.

(27)[2] For every denumerable family M of subsets of $\mathbb{R}$ such that $M \subseteq L_\mu$-σFIELD holds $\bigcap M \in L_\mu$-σFIELD.

(28) For all elements A, B of L_μ-σFIELD such that $A \cap B = \emptyset$ holds $L_\mu(A \cup B) = L_\mu(A) + L_\mu(B)$.

(29) For all elements A, B of L_μ-σFIELD such that $A \subseteq B$ holds $L_\mu(A) \leq L_\mu(B)$.

[1]**Editiorial footnote**: The repetition below is caused by the fact that the first sentence is the translation of a Mizar theorem, and the second one – of a Mizar redefinition.

[2]The proposition (26) has been removed.

(30) For all elements A, B of L_μ-σFIELD such that $A \subseteq B$ and $L_\mu(A) < +\infty$ holds $L_\mu(B \setminus A) = L_\mu(B) - L_\mu(A)$.

(31) For all elements A, B of L_μ-σFIELD holds $L_\mu(A \cup B) \leq L_\mu(A) + L_\mu(B)$.

(32) L_μ is non-negative and $L_\mu(\emptyset) = 0_{\overline{\mathbb{R}}}$ and for every sequence F of separated subsets of L_μ-σFIELD holds $\sum(L_\mu \cdot F) = L_\mu(\bigcup \operatorname{rng} F)$.

(33) For every function F from $\mathbb{N}$ into L_μ-σFIELD such that for every element n of $\mathbb{N}$ holds $F(n) \subseteq F(n+1)$ holds $L_\mu(\bigcup \operatorname{rng} F) = \sup \operatorname{rng}(L_\mu \cdot F)$.

(34) Let F be a function from $\mathbb{N}$ into L_μ-σFIELD. Suppose for every element n of $\mathbb{N}$ holds $F(n+1) \subseteq F(n)$ and $L_\mu(F(0)) < +\infty$. Then $L_\mu(\bigcap \operatorname{rng} F) = \inf \operatorname{rng}(L_\mu \cdot F)$.

(35) Let T be a family of measurable sets of L_μ-σFIELD. Suppose that for every set A such that $A \in T$ holds A is a set of measure zero w.r.t. L_μ. Then $\bigcup T$ is a set of measure zero w.r.t. L_μ.

(36) Let T be a family of measurable sets of L_μ-σFIELD. Given a set A such that $A \in T$ and A is a set of measure zero w.r.t. L_μ. Then $\bigcap T$ is a set of measure zero w.r.t. L_μ.

(37) Let T be a family of measurable sets of L_μ-σFIELD. Suppose that for every set A such that $A \in T$ holds A is a set of measure zero w.r.t. L_μ. Then $\bigcap T$ is a set of measure zero w.r.t. L_μ.

(38) Let A be an element of L_μ-σFIELD and let B be a set of measure zero w.r.t. L_μ. If $A \subseteq B$, then A is a set of measure zero w.r.t. L_μ.

(39) Let A, B be sets of measure zero w.r.t. L_μ. Then
  (i) $A \cup B$ is a set of measure zero w.r.t. L_μ,
  (ii) $A \cap B$ is a set of measure zero w.r.t. L_μ, and
  (iii) $A \setminus B$ is a set of measure zero w.r.t. L_μ.

(40) Let A be an element of L_μ-σFIELD and let B be a set of measure zero w.r.t. L_μ. Then $L_\mu(A \cup B) = L_\mu(A)$ and $L_\mu(A \cap B) = 0_{\overline{\mathbb{R}}}$ and $L_\mu(A \setminus B) = L_\mu(A)$.

(41) (i) $\emptyset$ is measurable w.r.t. L_μ,
  (ii) $\mathbb{R}$ is measurable w.r.t. L_μ, and
  (iii) for all sets A, B such that A is measurable w.r.t. L_μ and B is measurable w.r.t. L_μ holds $\mathbb{R} \setminus A$ is measurable w.r.t. L_μ and $A \cup B$ is measurable w.r.t. L_μ and $A \cap B$ is measurable w.r.t. L_μ.

(42) Let T be a denumerable family of subsets of $\mathbb{R}$. Suppose that for every set A such that $A \in T$ holds A is measurable w.r.t. L_μ. Then $\bigcup T$ is measurable w.r.t. L_μ and $\bigcap T$ is measurable w.r.t. L_μ.

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Categories without Uniqueness of cod and dom

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Summary. Category theory had been formalized in Mizar quite early [8]. This had been done closely to the handbook of S. McLane [11]. In this paper we use a different approach. Category is a triple

$$\langle O, \{\langle o_1, o_2 \rangle\}_{o_1, o_2 \in O}, \{\circ_{o_1, o_2, o_3}\}_{o_1, o_2, o_3 \in O} \rangle$$

where $\circ_{o_1,o_2,o_3} : \langle o_2, o_3\rangle \times \langle o_1, o_2 \rangle \to \langle o_1, o_3 \rangle$ that satisfies usual conditions (associativity and the existence of the identities). This approach is closer to the way in which categories are presented in homological algebra (e.g. [1], pp.58-59). We do not assume that $\langle o_1, o_2\rangle$'s are mutually disjoint. If f is simultaneously a morphism from o_1 to o_2 and o_1' to o_2 ($o_1 \neq o_1'$) than different compositions are used ($\circ_{o_1,o_2,o_3}$ or $\circ_{o_1',o_2,o_3}$) to compose it with a morphism g from o_2 to o_3. The operation $g \cdot f$ has actually six arguments (two visible and four hidden: three objects and the category).

We introduce some simple properties of categories. Perhaps more than necessary. It is partially caused by the formalization. The functional categories are characterized by the following properties:

- quasi-functional that means that morphisms are functions (rather meaningless, if it stands alone)
- semi-functional that means that the composition of morphism is the composition of functions, provided they are functions.
- pseudo-functional that means that the composition of morphisms is the composition of functions.

For categories pseudo-functional is just quasi-functional and semi-functional, but we work in a bit more general setting. Similarly the concept of a discrete category is split into two:

- quasi-discrete that means that $\langle o_1, o_2\rangle$ is empty for $o_1 \neq o_2$ and
- pseudo-discrete that means that $\langle o, o\rangle$ is trivial, i.e. consists of the identity only, in a category.

We plan to follow Semadeni-Wiweger book [14], in the development the category theory in Mizar. However, the beginning is not very close to [14], because of the approach adopted and because we work in Tarski-Grothendieck set theory.

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The terminology and notation used in this paper have been introduced in the following articles: [19], [21], [20], [15], [22], [2], [6], [7], [3], [13], [5], [10], [4], [16], [9], [18], [12], and [17].

1. Preliminaries

One can prove the following proposition

(1) For every non empty set A and for all sets B, C, D such that $[: A, B :] \subseteq [: C, D :]$ or $[: B, A :] \subseteq [: D, C :]$ holds $B \subseteq D$.

In the sequel i, j, k, x are arbitrary.

Let A be a functional set. Observe that every subset of A is functional.

Let f be a function yielding function and let C be a set. Observe that $f \restriction C$ is function yielding.

Let f be a function. One can verify that $\{f\}$ is functional.

Next we state four propositions:

(2) For every set A holds $\mathrm{id}_A \in A^A$.

(3) $\emptyset^\emptyset = \{\mathrm{id}_\emptyset\}$.

(4) For all sets A, B, C and for all functions f, g such that $f \in B^A$ and $g \in C^B$ holds $g \cdot f \in C^A$.

(5) For all sets A, B, C such that $B^A \neq \emptyset$ and $C^B \neq \emptyset$ holds $C^A \neq \emptyset$.

Let A, B be sets. One can check that B^A is functional.

We now state two propositions:

(6) For all sets A, B and for every function f such that $f \in B^A$ holds $\operatorname{dom} f = A$ and $\operatorname{rng} f \subseteq B$.

(7) Let A, B be sets, and let F be a many sorted set indexed by $[: B, A :]$, and let C be a subset of A, and let D be a subset of B, and let x, y be arbitrary. If $x \in C$ and $y \in D$, then $F(y, x) = (F \restriction [: D, C :])(y, x)$.

In this article we present several logical schemes. The scheme *MSSLambdaD* deals with a non empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding arbitrary, and states that:

> There exists a many sorted set M indexed by $\mathcal{A}$ such that for every element i of $\mathcal{A}$ holds $M(i) = \mathcal{F}(i)$

for all values of the parameters.

The scheme *MSSLambda2* deals with sets $\mathcal{A}$, $\mathcal{B}$ and a binary functor $\mathcal{F}$ yielding arbitrary, and states that:

> There exists a many sorted set M indexed by $[: \mathcal{A}, \mathcal{B} :]$ such that for all i, j such that $i \in \mathcal{A}$ and $j \in \mathcal{B}$ holds $M(i, j) = \mathcal{F}(i, j)$

for all values of the parameters.

The scheme *MSSLambda2D* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$ and a binary functor $\mathcal{F}$ yielding arbitrary, and states that:

> There exists a many sorted set M indexed by $[\!:\mathcal{A}, \mathcal{B}:\!]$ such that for every element i of $\mathcal{A}$ and for every element j of $\mathcal{B}$ holds $M(i, j) = \mathcal{F}(i,j)$

for all values of the parameters.

The scheme *MSSLambda3* concerns sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a ternary functor $\mathcal{F}$ yielding arbitrary, and states that:

> There exists a many sorted set M indexed by $[\!:\mathcal{A}, \mathcal{B}, \mathcal{C}:\!]$ such that for all i, j, k such that $i \in \mathcal{A}$ and $j \in \mathcal{B}$ and $k \in \mathcal{C}$ holds $M(i, j, k) = \mathcal{F}(i,j,k)$

for all values of the parameters.

The scheme *MSSLambda3D* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a ternary functor $\mathcal{F}$ yielding arbitrary, and states that:

> There exists a many sorted set M indexed by $[\!:\mathcal{A}, \mathcal{B}, \mathcal{C}:\!]$ such that for every element i of $\mathcal{A}$ and for every element j of $\mathcal{B}$ and for every element k of $\mathcal{C}$ holds $M(i, j, k) = \mathcal{F}(i,j,k)$

for all values of the parameters.

One can prove the following propositions:

(8) Let A, B be sets and let N, M be many sorted sets indexed by $[\!:A, B:\!]$. If for all i, j such that $i \in A$ and $j \in B$ holds $N(i, j) = M(i, j)$, then $M = N$.

(9) Let A, B be non empty sets and let N, M be many sorted sets indexed by $[\!:A, B:\!]$. Suppose that for every element i of A and for every element j of B holds $N(i, j) = M(i, j)$. Then $M = N$.

(10) Let A be a set and let N, M be many sorted sets indexed by $[\!:A, A, A:\!]$. Suppose that for all i, j, k such that $i \in A$ and $j \in A$ and $k \in A$ holds $N(i, j, k) = M(i, j, k)$. Then $M = N$.

(11) $[\langle i,j\rangle \mapsto k] = \langle i, j\rangle \dot{\longmapsto} k$.

(12) $[\langle i,j\rangle \mapsto k](i, j) = k$.

2. Graphs

We consider graphs as extensions of 1-sorted structure as systems

⟨ a carrier, arrows ⟩,

where the carrier is a set and the arrows constitute a many sorted set indexed by $[\!:$the carrier, the carrier$:\!]$.

Let G be a graph.

(Def.1) An element of the carrier of G is called an object of G.

Let G be a graph and let o_1, o_2 be objects of G. The functor $\langle o_1, o_2\rangle$ is defined as follows:

(Def.2) $\langle o_1, o_2\rangle =$ (the arrows of G)(o_1, o_2).

Let G be a graph and let o_1, o_2 be objects of G.

(Def.3) An element of $\langle o_1, o_2 \rangle$ is said to be a morphism from o_1 to o_2.

Let G be a graph. We say that G is transitive if and only if:

(Def.4) For all objects o_1, o_2, o_3 of G such that $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$ holds $\langle o_1, o_3 \rangle \neq \emptyset$.

3. Many Sorted Binary Compositions

Let I be a set and let G be a many sorted set indexed by $[\!: I, I :\!]$. The functor $\{\!|G|\!\}$ yields a many sorted set indexed by $[\!: I, I, I :\!]$ and is defined as follows:

(Def.5) For all i, j, k such that $i \in I$ and $j \in I$ and $k \in I$ holds $(\{\!|G|\!\})(i, j, k) = G(i, k)$.

Let H be a many sorted set indexed by $[\!: I, I :\!]$. The functor $\{\!|G, H|\!\}$ yielding a many sorted set indexed by $[\!: I, I, I :\!]$ is defined by:

(Def.6) For all i, j, k such that $i \in I$ and $j \in I$ and $k \in I$ holds $(\{\!|G, H|\!\})(i, j, k) = [\!: H(j, k), G(i, j) :\!]$.

Let I be a set and let G be a many sorted set indexed by $[\!: I, I :\!]$. A binary composition of G is a many sorted function from $\{\!|G, G|\!\}$ into $\{\!|G|\!\}$.

Let I be a non empty set, let G be a many sorted set indexed by $[\!: I, I :\!]$, let o be a binary composition of G, and let i, j, k be elements of I. Then $o(i, j, k)$ is a function from $[\!: G(j, k), G(i, j) :\!]$ into $G(i, k)$.

Let I be a non empty set and let G be a many sorted set indexed by $[\!: I, I :\!]$. A binary composition of G is associative if it satisfies the condition (Def.7).

(Def.7) Let i, j, k, l be elements of I and let f, g, h be arbitrary. Suppose $f \in G(i, j)$ and $g \in G(j, k)$ and $h \in G(k, l)$. Then $\mathrm{it}(i, k, l)(h, \mathrm{it}(i, j, k)(g, f)) = \mathrm{it}(i, j, l)(\mathrm{it}(j, k, l)(h, g), f)$.

A binary composition of G has right units if it satisfies the condition (Def.8).

(Def.8) Let i be an element of I. Then there exists arbitrary e such that $e \in G(i, i)$ and for every element j of I and for arbitrary f such that $f \in G(i, j)$ holds $\mathrm{it}(i, i, j)(f, e) = f$.

A binary composition of G has left units if it satisfies the condition (Def.9).

(Def.9) Let j be an element of I. Then there exists arbitrary e such that $e \in G(j, j)$ and for every element i of I and for arbitrary f such that $f \in G(i, j)$ holds $\mathrm{it}(i, j, j)(e, f) = f$.

4. Categories

We introduce category structures which are extensions of graph and are systems

$\langle$ a carrier, arrows, a composition $\rangle$, where the carrier is a set, the arrows constitute a many sorted set indexed by $[\!:$ the carrier, the carrier $:\!]$, and the composition is a binary composition of the arrows.

Let us observe that there exists a category structure which is strict and non empty.

Let C be a non empty category structure and let o_1, o_2, o_3 be objects of C. Let us assume that $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$ and $\langle o_1, o_3 \rangle \neq \emptyset$. Let f be a morphism from o_1 to o_2 and let g be a morphism from o_2 to o_3. The functor $g \cdot f$ yields a morphism from o_1 to o_3 and is defined by:

(Def.10) $g \cdot f = (\text{the composition of } C)(o_1,\, o_2,\, o_3)(g,\, f)$.

A function is compositional if:

(Def.11) If $x \in \operatorname{dom} \mathrm{it}$, then there exist functions f, g such that $x = \langle g,\, f \rangle$ and $\mathrm{it}(x) = g \cdot f$.

Let A, B be functional sets. Observe that there exists a many sorted function of $[\!:A,\, B:\!]$ which is compositional.

Next we state the proposition

(13) Let A, B be functional sets, and let F be a compositional many sorted set indexed by $[\!:A,\, B:\!]$, and let g, f be functions. If $g \in A$ and $f \in B$, then $F(g,\, f) = g \cdot f$.

Let A, B be functional sets.

(Def.12) $\mathrm{FuncComp}(A, B)$ is a compositional many sorted function of $[\!:B,\, A:\!]$.

The following propositions are true:

(14) For all sets A, B, C holds $\operatorname{rng} \mathrm{FuncComp}(B^A, C^B) \subseteq C^A$.

(15) For every set o holds $\mathrm{FuncComp}(\{\mathrm{id}_o\}, \{\mathrm{id}_o\}) = [\langle \mathrm{id}_o, \mathrm{id}_o \rangle \mapsto \mathrm{id}_o]$.

(16) For all functional sets A, B and for every subset A_1 of A and for every subset B_1 of B holds $\mathrm{FuncComp}(A_1, B_1) = \mathrm{FuncComp}(A, B) \restriction [\!:B_1,\, A_1:\!]$.

Let C be a non empty category structure. We say that C is quasi-functional if and only if:

(Def.13) For all objects a_1, a_2 of C holds $\langle a_1, a_2 \rangle \subseteq {a_2}^{a_1}$.

We say that C is semi-functional if and only if the condition (Def.14) is satisfied.

(Def.14) Let a_1, a_2, a_3 be objects of C. Suppose $\langle a_1, a_2 \rangle \neq \emptyset$ and $\langle a_2, a_3 \rangle \neq \emptyset$ and $\langle a_1, a_3 \rangle \neq \emptyset$. Let f be a morphism from a_1 to a_2, and let g be a morphism from a_2 to a_3, and let f', g' be functions. If $f = f'$ and $g = g'$, then $g \cdot f = g' \cdot f'$.

We say that C is pseudo-functional if and only if:

(Def.15) For all objects o_1, o_2, o_3 of C holds $(\text{the composition of } C)(o_1,\, o_2,\, o_3) = \mathrm{FuncComp}({o_2}^{o_1}, {o_3}^{o_2}) \restriction [\!:\langle o_2, o_3 \rangle,\, \langle o_1, o_2 \rangle:\!]$.

Let X be a non empty set, let A be a many sorted set indexed by $[\!:X,\, X:\!]$, and let C be a binary composition of A. Note that $\langle X, A, C \rangle$ is non empty.

Let us observe that there exists a non empty category structure which is strict and pseudo-functional.

One can prove the following propositions:

(17) Let C be a non empty category structure and let a_1, a_2, a_3 be objects of C. Suppose if $\langle a_1, a_3\rangle = \emptyset$, then $\langle a_1, a_2\rangle = \emptyset$ or $\langle a_2, a_3\rangle = \emptyset$. Then (the composition of C)(a_1, a_2, a_3) is a function from $[\!: \langle a_2, a_3\rangle, \langle a_1, a_2\rangle :\!]$ into $\langle a_1, a_3\rangle$.

(18) Let C be a pseudo-functional non empty category structure and let a_1, a_2, a_3 be objects of C. Suppose $\langle a_1, a_2\rangle \neq \emptyset$ and $\langle a_2, a_3\rangle \neq \emptyset$ and $\langle a_1, a_3\rangle \neq \emptyset$. Let f be a morphism from a_1 to a_2, and let g be a morphism from a_2 to a_3, and let f', g' be functions. If $f = f'$ and $g = g'$, then $g \cdot f = g' \cdot f'$.

Let A be a non empty set. The functor Ens_A yielding a strict pseudo-functional non empty category structure is defined as follows:

(Def.16) The carrier of $\mathrm{Ens}_A = A$ and for all objects a_1, a_2 of Ens_A holds $\langle a_1, a_2\rangle = {a_2}^{a_1}$.

Let C be a non empty category structure. We say that C is associative if and only if:

(Def.17) The composition of C is associative.

We say that C has units if and only if:

(Def.18) The composition of C has left units and right units.

Let us mention that there exists a non empty category structure which is transitive associative and strict and has units.

The following propositions are true:

(19) Let C be a transitive non empty category structure and let a_1, a_2, a_3 be objects of C. Then (the composition of C)(a_1, a_2, a_3) is a function from $[\!: \langle a_2, a_3\rangle, \langle a_1, a_2\rangle :\!]$ into $\langle a_1, a_3\rangle$.

(20) Let C be a transitive non empty category structure and let a_1, a_2, a_3 be objects of C. Then dom (the composition of C)(a_1, a_2, a_3) $= [\!: \langle a_2, a_3\rangle, \langle a_1, a_2\rangle :\!]$ and rng (the composition of C)(a_1, a_2, a_3) $\subseteq \langle a_1, a_3\rangle$.

(21) For every non empty category structure C with units and for every object o of C holds $\langle o, o\rangle \neq \emptyset$.

Let A be a non empty set. Observe that Ens_A is transitive and associative and has units.

Let us mention that every non empty category structure which is quasi-functional semi-functional and transitive is also pseudo-functional and every non empty category structure which is pseudo-functional and transitive and has units is also quasi-functional and semi-functional.

A category is a transitive associative non empty category structure with units.

5. Identities

One can prove the following proposition

(22) Let C be a transitive non empty category structure and let o_1, o_2, o_3 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$. Let f be a morphism from o_1 to o_2 and let g be a morphism from o_2 to o_3. Then $g \cdot f =$ (the composition of C)$(o_1, o_2, o_3)(g, f)$.

Let C be a non empty category structure with units and let o be an object of C. The functor id_o yielding a morphism from o to o is defined by:

(Def.19) For every object o' of C such that $\langle o, o' \rangle \neq \emptyset$ and for every morphism a from o to o' holds $a \cdot \mathrm{id}_o = a$.

One can prove the following three propositions:

(23) For every non empty category structure C with units and for every object o of C holds $\mathrm{id}_o \in \langle o, o \rangle$.

(24) Let C be a non empty category structure with units and let o_1, o_2 be objects of C. If $\langle o_1, o_2 \rangle \neq \emptyset$, then for every morphism a from o_1 to o_2 holds $\mathrm{id}_{(o_2)} \cdot a = a$.

(25) Let C be an associative transitive non empty category structure and let o_1, o_2, o_3, o_4 be objects of C. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$ and $\langle o_3, o_4 \rangle \neq \emptyset$. Let a be a morphism from o_1 to o_2, and let b be a morphism from o_2 to o_3, and let c be a morphism from o_3 to o_4. Then $c \cdot (b \cdot a) = (c \cdot b) \cdot a$.

6. Discrete categories

Let C be a category structure. We say that C is quasi-discrete if and only if:

(Def.20) For all objects i, j of C such that $\langle i, j \rangle \neq \emptyset$ holds $i = j$.

We say that C is pseudo-discrete if and only if:

(Def.21) For every object i of C holds $\langle i, i \rangle$ is trivial.

One can prove the following proposition

(26) Let C be a non empty category structure with units. Then C is pseudo-discrete if and only if for every object o of C holds $\langle o, o \rangle = \{\mathrm{id}_o\}$.

Let us observe that every category structure which is trivial is also quasi-discrete.

One can prove the following proposition

(27) Ens_1 is pseudo-discrete and trivial.

Let us note that there exists a category which is pseudo-discrete trivial and strict.

Let us observe that there exists a category which is quasi-discrete pseudo-discrete trivial and strict.

A discrete category is a quasi-discrete pseudo-discrete category.

Let A be a non empty set. The functor $\mathrm{DiscrCat}(A)$ yields a quasi-discrete strict non empty category structure and is defined by:

(Def.22) The carrier of DiscrCat(A) $= A$ and for every object i of DiscrCat(A) holds $\langle i, i\rangle = \{\mathrm{id}_i\}$.

One can verify that every category structure which is quasi-discrete is also transitive.

One can prove the following propositions:

(28) Let A be a non empty set and let o_1, o_2, o_3 be objects of DiscrCat(A). If $o_1 \neq o_2$ or $o_2 \neq o_3$, then (the composition of DiscrCat(A))(o_1, o_2, o_3) $= \emptyset$.

(29) For every non empty set A and for every object o of DiscrCat(A) holds (the composition of DiscrCat(A))(o, o, o) $= [\langle \mathrm{id}_o, \mathrm{id}_o\rangle \mapsto \mathrm{id}_o]$.

Let A be a non empty set. Note that DiscrCat(A) is pseudo-functional pseudo-discrete and associative and has units.

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Extensions of Mappings on Generator Set

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Summary. The aim of the article is to prove the fact that if extensions of mappings on generator set are equal then these mappings are equal. The article contains the properties of epimorphisms & monomorphisms between Many Sorted Algebras.

MML Identifier: `EXTENS_1`.

The articles [15], [17], [18], [6], [16], [8], [7], [1], [2], [3], [14], [5], [11], [13], [4], [10], [9], and [12] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity we adopt the following convention: S will be a non void non empty many sorted signature, U_1, U_2, U_3 will be non-empty algebras over S, I will be a set, A will be a many sorted set indexed by I, and B, C will be non-empty many sorted sets indexed by I.

We now state four propositions:

(1) For every binary relation R and for all sets X, Y such that $X \subseteq Y$ holds $(R \restriction Y)^\circ X = R^\circ X$.

(2) Let A be a set, and let B, C be non empty sets, and let f be a function from A into B, and let g be a function from B into C, and let X be a subset of A. Then $(g \cdot f) \restriction X = g \cdot (f \restriction X)$.

(3) For every function yielding function f holds $\mathrm{dom}(\mathrm{dom}_\kappa f(\kappa)) = \mathrm{dom} f$.

(4) For every function yielding function f holds $\mathrm{dom}(\mathrm{rng}_\kappa f(\kappa)) = \mathrm{dom} f$.

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2. Facts about Many Sorted Functions

Next we state several propositions:

(5) Let F be a many sorted function from A into B and let X be a many sorted subset of A. If $A \subseteq X$, then $F \restriction X = F$.

(6) Let A, B be many sorted sets indexed by I, and let M be a many sorted subset of A, and let F be a many sorted function from A into B. Then $F^\circ M \subseteq F^\circ A$.

(7) Let F be a many sorted function from A into B and let M_1, M_2 be many sorted subsets of A. If $M_1 \subseteq M_2$, then $(F \restriction M_2)^\circ M_1 = F^\circ M_1$.

(8) Let F be a many sorted function from A into B, and let G be a many sorted function from B into C, and let X be a many sorted subset of A. Then $(G \circ F) \restriction X = G \circ (F \restriction X)$.

(9) Let A, B be many sorted sets indexed by I. Suppose A is transformable to B. Let F be a many sorted function from A into B and let C be a many sorted set indexed by I. Suppose B is a many sorted subset of C. Then F is a many sorted function from A into C.

(10) Let F be a many sorted function from A into B and let X be a many sorted subset of A. If F is "1-1", then $F \restriction X$ is "1-1".

3. dom's & rng's of Many Sorted Functions

Let us consider I and let F be a many sorted function of I. Then $\mathrm{dom}_\kappa F(\kappa)$ is a many sorted set indexed by I.

Let us consider I and let F be a many sorted function of I. Then $\mathrm{rng}_\kappa F(\kappa)$ is a many sorted set indexed by I.

We now state several propositions:

(11) For every many sorted function F from A into B and for every many sorted subset X of A holds $\mathrm{dom}_\kappa F \restriction X(\kappa) \subseteq \mathrm{dom}_\kappa F(\kappa)$.

(12) For every many sorted function F from A into B and for every many sorted subset X of A holds $\mathrm{rng}_\kappa F \restriction X(\kappa) \subseteq \mathrm{rng}_\kappa F(\kappa)$.

(13) Let A, B be many sorted sets indexed by I and let F be a many sorted function from A into B. Then F is "onto" if and only if $\mathrm{rng}_\kappa F(\kappa) = B$.

(14) For every non-empty many sorted set X indexed by the carrier of S holds $\mathrm{rng}_\kappa \mathrm{Reverse}(X)(\kappa) = X$.

(15) Let F be a many sorted function from A into B, and let G be a many sorted function from B into C, and let X be a non-empty many sorted subset of B. If $\mathrm{rng}_\kappa F(\kappa) \subseteq X$, then $(G \restriction X) \circ F = G \circ F$.

4. Other properties of "onto" & "1-1"

Next we state two propositions:

(16) Let F be a many sorted function from A into B. Then F is "onto" if and only if for every C and for all many sorted functions G, H from B into C such that $G \circ F = H \circ F$ holds $G = H$.

(17) Let F be a many sorted function from A into B. Suppose A is non-empty and B is non-empty. Then F is "1-1" if and only if for every many sorted set C indexed by I and for all many sorted functions G, H from C into A such that $F \circ G = F \circ H$ holds $G = H$.

5. Extensions of Mappings on Generator Set

We now state three propositions:

(18) Let X be a non-empty many sorted set indexed by the carrier of S and let h_1, h_2 be many sorted functions from $\mathrm{Free}(X)$ into U_1. Suppose h_1 is a homomorphism of $\mathrm{Free}(X)$ into U_1 and h_2 is a homomorphism of $\mathrm{Free}(X)$ into U_1 and $h_1 \restriction \mathrm{FreeGenerator}(X) = h_2 \restriction \mathrm{FreeGenerator}(X)$. Then $h_1 = h_2$.

(19) Let F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2. Suppose F is an epimorphism of U_1 onto U_2. Let U_3 be a non-empty algebra over S and let h_1, h_2 be many sorted functions from U_2 into U_3. Suppose h_1 is a homomorphism of U_2 into U_3 and h_2 is a homomorphism of U_2 into U_3. If $h_1 \circ F = h_2 \circ F$, then $h_1 = h_2$.

(20) Let F be a many sorted function from U_2 into U_3. Suppose F is a homomorphism of U_2 into U_3. Then F is a monomorphism of U_2 into U_3 if and only if for every non-empty algebra U_1 over S and for all many sorted functions h_1, h_2 from U_1 into U_2 such that h_1 is a homomorphism of U_1 into U_2 and h_2 is a homomorphism of U_1 into U_2 holds if $F \circ h_1 = F \circ h_2$, then $h_1 = h_2$.

Let us consider S, U_1. Note that there exists a generator set of U_1 which is non-empty.

We now state three propositions:

(21) For all non-empty subsets A, B of U_1 such that A is a many sorted subset of B holds $\mathrm{Gen}(A)$ is a subalgebra of $\mathrm{Gen}(B)$.

(22) Let U_2 be a non-empty subalgebra of U_1, and let B_1 be a non-empty subset of U_1, and let B_2 be a subset of U_2. If $B_1 = B_2$, then $\mathrm{Gen}(B_1) = \mathrm{Gen}(B_2)$.

(23) Let U_1 be a strict non-empty algebra over S, and let U_2 be a non-empty algebra over S, and let G_1 be a non-empty generator set of U_1, and let h_1, h_2 be many sorted functions from U_1 into U_2. Suppose h_1 is

a homomorphism of U_1 into U_2 and h_2 is a homomorphism of U_1 into U_2 and $h_1 \restriction G_1 = h_2 \restriction G_1$. Then $h_1 = h_2$.

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Introduction to Circuits, II [1]

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Summary. This article is the last in a series of four articles (preceded by [23,22,21]) about modelling circuits by many sorted algebras.

The notion of a circuit computation is defined as a sequence of circuit states. For a state of a circuit the next state is given by executing operations at circuit vertices in the current state, according to denotations of the operations. The values at input vertices at each state of a computation are provided by an external sequence of input values. The process of how input values propagate through a circuit is described in terms of a homomorphism of the free envelope algebra of the circuit into itself. We prove that every computation of a circuit over a finite monotonic signature and with constant input values stabilizes after executing the number of steps equal to the depth of the circuit.

MML Identifier: `CIRCUIT2`.

The articles [27], [30], [31], [12], [13], [18], [14], [3], [9], [16], [5], [7], [4], [28], [1], [6], [29], [2], [15], [10], [26], [19], [25], [11], [20], [17], [24], [23], [22], [21], and [8] provide the terminology and notation for this paper.

1. Circuit Inputs

In this paper I_1 will be a monotonic circuit-like non void non empty many sorted signature.

The following proposition is true

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(1) Let X be a non-empty many sorted set indexed by the carrier of I_1, and let H be a many sorted function from $\mathrm{Free}(X)$ into $\mathrm{Free}(X)$, and let H_1 be a function yielding function, and let v be a sort symbol of I_1, and let p be a decorated tree yielding finite sequence, and let t be an element of (the sorts of $\mathrm{Free}(X)$)(v). Suppose that

(i) $v \in \mathrm{InnerVertices}(I_1)$,

(ii) $t = \langle$the action at v, the carrier of $I_1\rangle$-tree(p),

(iii) H is a homomorphism of $\mathrm{Free}(X)$ into $\mathrm{Free}(X)$, and

(iv) $H_1 = H \cdot \mathrm{Arity}$(the action at v).

Then there exists a decorated tree yielding finite sequence H_2 such that $H_2 = H_1 \looparrowleft p$ and $H(v)(t) = \langle$the action at v, the carrier of $I_1\rangle$-tree(H_2).

Let us consider I_1, let S_1 be a non-empty circuit of I_1, let s be a state of S_1, and let i_1 be an input assignment of S_1. Then $s +\cdot i_1$ is a state of S_1.

Let us consider I_1, let A be a non-empty circuit of I_1, and let i_1 be an input assignment of A. The functor $\mathrm{FixInput}(i_1)$ yields a many sorted function from FreeGenerator(the sorts of A) into the sorts of $\mathrm{FreeEnvelope}(A)$ and is defined by the condition (Def.1).

(Def.1) Let v be a vertex of I_1. Then

(i) if $v \in \mathrm{InputVertices}(I_1)$, then $(\mathrm{FixInput}(i_1))(v) = \mathrm{FreeGenerator}(v,$the sorts of $A) \longmapsto$ the root tree of $\langle i_1(v), v\rangle$,

(ii) if $v \in \mathrm{SortsWithConstants}(I_1)$, then $(\mathrm{FixInput}(i_1))(v) = \mathrm{FreeGenerator}(v,$the sorts of $A) \longmapsto$ the root tree of $\langle$the action at v, the carrier of $I_1\rangle$, and

(iii) if $v \in \mathrm{InnerVertices}(I_1) \setminus \mathrm{SortsWithConstants}(I_1)$, then $(\mathrm{FixInput}(i_1))(v) = \mathrm{id}_{\mathrm{FreeGenerator}(v,\text{the sorts of } A)}$.

Let us consider I_1, let A be a non-empty circuit of I_1, and let i_1 be an input assignment of A. The functor $\mathrm{FixInputExt}(i_1)$ yields a many sorted function from $\mathrm{FreeEnvelope}(A)$ into $\mathrm{FreeEnvelope}(A)$ and is defined by:

(Def.2) $\mathrm{FixInputExt}(i_1)$ is a homomorphism of $\mathrm{FreeEnvelope}(A)$ into $\mathrm{FreeEnvelope}(A)$ and $\mathrm{FixInput}(i_1) \subseteq \mathrm{FixInputExt}(i_1)$.

The following propositions are true:

(2) Let A be a non-empty circuit of I_1, and let i_1 be an input assignment of A, and let v be a vertex of I_1, and let e be an element of (the sorts of $\mathrm{FreeEnvelope}(A)$)(v), and let x be arbitrary. If $v \in \mathrm{InnerVertices}(I_1) \setminus \mathrm{SortsWithConstants}(I_1)$ and $e =$ the root tree of $\langle x, v\rangle$, then $(\mathrm{FixInputExt}(i_1))(v)(e) = e$.

(3) Let A be a non-empty circuit of I_1, and let i_1 be an input assignment of A, and let v be a vertex of I_1, and let x be an element of (the sorts of A)(v). If $v \in \mathrm{InputVertices}(I_1)$, then $(\mathrm{FixInputExt}(i_1))(v)$(the root tree of $\langle x, v\rangle$) = the root tree of $\langle i_1(v), v\rangle$.

(4) Let A be a non-empty circuit of I_1, and let i_1 be an input assignment of A, and let v be a vertex of I_1, and let e be an element of (the sorts

of FreeEnvelope(A))(v), and let p, q be decorated tree yielding finite sequences. Suppose that

(i) $v \in \mathrm{InnerVertices}(I_1)$,

(ii) $e = \langle$the action at v, the carrier of $I_1\rangle$-tree(p),

(iii) $\mathrm{dom}\, p = \mathrm{dom}\, q$, and

(iv) for every natural number k such that $k \in \mathrm{dom}\, p$ holds $q(k) = (\mathrm{FixInputExt}(i_1))(\pi_k \,\mathrm{Arity}(\text{the action at } v))(p(k))$.

Then $(\mathrm{FixInputExt}(i_1))(v)(e) = \langle$the action at v, the carrier of $I_1\rangle$-tree(q).

(5) Let A be a non-empty circuit of I_1, and let i_1 be an input assignment of A, and let v be a vertex of I_1, and let e be an element of (the sorts of FreeEnvelope(A))(v). Suppose $v \in \mathrm{SortsWithConstants}(I_1)$. Then $(\mathrm{FixInputExt}(i_1))(v)(e) =$ the root tree of $\langle$the action at v, the carrier of $I_1\rangle$.

(6) Let A be a non-empty circuit of I_1, and let i_1 be an input assignment of A, and let v be a vertex of I_1, and let e, e_1 be elements of (the sorts of FreeEnvelope(A))(v), and let t, t_1 be decorated trees. If $t = e$ and $t_1 = e_1$ and $e_1 = (\mathrm{FixInputExt}(i_1))(v)(e)$, then $\mathrm{dom}\, t = \mathrm{dom}\, t_1$.

(7) Let A be a non-empty circuit of I_1, and let i_1 be an input assignment of A, and let v be a vertex of I_1, and let e, e_1 be elements of (the sorts of FreeEnvelope(A))(v). If $e_1 = (\mathrm{FixInputExt}(i_1))(v)(e)$, then $\mathrm{card}\, e = \mathrm{card}\, e_1$.

Let us consider I_1, let S_1 be a non-empty circuit of I_1, let v be a vertex of I_1, and let i_1 be an input assignment of S_1. The functor $\mathrm{InputGenTree}(v, i_1)$ yields an element of (the sorts of FreeEnvelope(S_1))(v) and is defined by:

(Def.3) There exists an element e of (the sorts of FreeEnvelope(S_1))(v) such that $\mathrm{card}\, e = \mathrm{size}(v, S_1)$ and $\mathrm{InputGenTree}(v, i_1) = (\mathrm{FixInputExt}(i_1))(v)(e)$.

We now state two propositions:

(8) Let S_1 be a non-empty circuit of I_1, and let v be a vertex of I_1, and let i_1 be an input assignment of S_1. Then $\mathrm{InputGenTree}(v, i_1) = (\mathrm{FixInputExt}(i_1))(v)(\mathrm{InputGenTree}(v, i_1))$.

(9) Let S_1 be a non-empty circuit of I_1, and let v be a vertex of I_1, and let i_1 be an input assignment of S_1, and let p be a decorated tree yielding finite sequence. Suppose that

(i) $v \in \mathrm{InnerVertices}(I_1)$,

(ii) $\mathrm{dom}\, p = \mathrm{dom}\,\mathrm{Arity}(\text{the action at } v)$, and

(iii) for every natural number k such that $k \in \mathrm{dom}\, p$ holds $p(k) = \mathrm{InputGenTree}(\pi_k \,\mathrm{Arity}(\text{the action at } v), i_1)$.

Then $\mathrm{InputGenTree}(v, i_1) = \langle$the action at v, the carrier of $I_1\rangle$-tree(p).

Let us consider I_1, let S_1 be a non-empty circuit of I_1, let v be a vertex of I_1, and let i_1 be an input assignment of S_1. The functor $\mathrm{InputGenValue}(v, i_1)$ yields an element of (the sorts of S_1)(v) and is defined by:

(Def.4) $\mathrm{InputGenValue}(v, i_1) = (\mathrm{Eval}(S_1))(v)(\mathrm{InputGenTree}(v, i_1))$.

The following propositions are true:

(10) Let S_1 be a non-empty circuit of I_1, and let v be a vertex of I_1, and let i_1 be an input assignment of S_1. If $v \in \mathrm{InputVertices}(I_1)$, then $\mathrm{InputGenValue}(v, i_1) = i_1(v)$.

(11) Let S_1 be a non-empty circuit of I_1, and let v be a vertex of I_1, and let i_1 be an input assignment of S_1. If $v \in \mathrm{SortsWithConstants}(I_1)$, then $\mathrm{InputGenValue}(v, i_1) = (\text{Set-Constants}(S_1))(v)$.

2. Circuit Computations

Let I_1 be a circuit-like non void non empty many sorted signature, let S_1 be a non-empty circuit of I_1, and let s be a state of S_1. The functor $\mathrm{Following}(s)$ yielding a state of S_1 is defined by the condition (Def.5).

(Def.5) Let v be a vertex of I_1. Then if $v \in \mathrm{InputVertices}(I_1)$, then $(\mathrm{Following}(s))(v) = s(v)$ and if $v \in \mathrm{InnerVertices}(I_1)$, then $(\mathrm{Following}(s))(v) = (\mathrm{Den}(\text{the action at } v, S_1))((\text{the action at } v) \text{ depends-on-in } s)$.

Next we state the proposition

(12) Let S_1 be a non-empty circuit of I_1, and let s be a state of S_1, and let i_1 be an input assignment of S_1. If $i_1 \subseteq s$, then $i_1 \subseteq \mathrm{Following}(s)$.

Let I_1 be a circuit-like non void non empty many sorted signature and let S_1 be a non-empty circuit of I_1. A state of S_1 is stable if:

(Def.6) $\mathrm{It} = \mathrm{Following}(\mathrm{it})$.

Let us consider I_1, let S_1 be a non-empty circuit of I_1, let s be a state of S_1, and let i_1 be an input assignment of S_1. The functor $\mathrm{Following}(s, i_1)$ yielding a state of S_1 is defined by:

(Def.7) $\mathrm{Following}(s, i_1) = \mathrm{Following}(s +\cdot i_1)$.

Let us consider I_1, let S_1 be a non-empty circuit of I_1, let I_2 be an input function of S_1, and let s be a state of S_1. The functor $\mathrm{InitialComp}(s, I_2)$ yielding a state of S_1 is defined as follows:

(Def.8) $\mathrm{InitialComp}(s, I_2) = s +\cdot (0\text{-}th\text{-input}(I_2)) +\cdot \text{Set-Constants}(S_1)$.

Let us consider I_1, let S_1 be a non-empty circuit of I_1, let I_2 be an input function of S_1, and let s be a state of S_1. The functor $\mathrm{Computation}(s, I_2)$ yielding a function from $\mathbb{N}$ into $\prod(\text{the sorts of } S_1)$ is defined by the conditions (Def.9).

(Def.9) (i) $(\mathrm{Computation}(s, I_2))(0) = \mathrm{InitialComp}(s, I_2)$, and

(ii) for every natural number i and for every state x of S_1 such that $x = (\mathrm{Computation}(s, I_2))(i)$ holds $(\mathrm{Computation}(s, I_2))(i + 1) = \mathrm{Following}(x, (i + 1)\text{-}th\text{-input}(I_2))$.

In the sequel S_1 denotes a non-empty circuit of I_1, s denotes a state of S_1, and i_1 denotes an input assignment of S_1.

Next we state the proposition

(13) Let k be a natural number. Suppose that for every vertex v of I_1 such that $\mathrm{depth}(v, S_1) \leq k$ holds $s(v) = \mathrm{InputGenValue}(v, i_1)$. Let v_1 be a vertex of I_1. If $\mathrm{depth}(v_1, S_1) \leq k + 1$, then $(\mathrm{Following}(s))(v_1) = \mathrm{InputGenValue}(v_1, i_1)$.

For simplicity we adopt the following convention: I_1 is a finite monotonic circuit-like non void non empty many sorted signature, S_1 is a non-empty circuit of I_1, I_2 is an input function of S_1, s is a state of S_1, and i_1 is an input assignment of S_1.

We now state several propositions:

(14) If $\mathrm{commute}(I_2)$ is constant and $\mathrm{InputVertices}(I_1)$ is non empty, then for all s, i_1 such that $i_1 = (\mathrm{commute}(I_2))(0)$ and for every natural number k holds $i_1 \subseteq (\mathrm{Computation}(s, I_2))(k)$.

(15) Let n be a natural number. Suppose $\mathrm{commute}(I_2)$ is constant and $\mathrm{InputVertices}(I_1)$ is non empty and $(\mathrm{Computation}(s, I_2))(n)$ is stable. Let m be a natural number. If $n \leq m$, then $(\mathrm{Computation}(s, I_2))(n) = (\mathrm{Computation}(s, I_2))(m)$.

(16) Suppose $\mathrm{commute}(I_2)$ is constant and $\mathrm{InputVertices}(I_1)$ is non empty. Given s, i_1. Suppose $i_1 = (\mathrm{commute}(I_2))(0)$. Let k be a natural number and let v be a vertex of I_1. If $\mathrm{depth}(v, S_1) \leq k$, then $((\mathrm{Computation}(s, I_2))(k)$ **qua** element of $\prod(\text{the sorts of } S_1))(v) = \mathrm{InputGenValue}(v, i_1)$.

(17) Suppose $\mathrm{commute}(I_2)$ is constant and $\mathrm{InputVertices}(I_1)$ is non empty and $i_1 = (\mathrm{commute}(I_2))(0)$. Let s be a state of S_1 and let v be a vertex of I_1. Then $((\mathrm{Computation}(s, I_2))(\mathrm{depth}(S_1))$ **qua** state of $S_1)(v) = \mathrm{InputGenValue}(v, i_1)$.

(18) If $\mathrm{commute}(I_2)$ is constant and $\mathrm{InputVertices}(I_1)$ is non empty, then for every state s of S_1 holds $(\mathrm{Computation}(s, I_2))(\mathrm{depth}(S_1))$ is stable.

(19) If $\mathrm{commute}(I_2)$ is constant and $\mathrm{InputVertices}(I_1)$ is non empty, then for all states s_1, s_2 of S_1 holds $(\mathrm{Computation}(s_1, I_2))(\mathrm{depth}(S_1)) = (\mathrm{Computation}(s_2, I_2))(\mathrm{depth}(S_1))$.

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Definitions and Basic Properties of Boolean & Union of Many Sorted Sets

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Summary. In the first part of this article I have proved theorems about boolean of many sorted sets which are corresponded to theorems about boolean of sets, whereas the second part of this article contains propositions about union of many sorted sets. Boolean as well as union of many sorted sets are defined as boolean and union on every sorts.

MML Identifier: `MBOOLEAN`.

The terminology and notation used here are introduced in the following articles: [11], [12], [13], [2], [3], [5], [9], [4], [1], [10], [7], [6], and [8].

1. Boolean of Many Sorted Sets

We follow a convention: I will denote a set, A, B, X, Y will denote many sorted sets indexed by I, and x, y will be arbitrary.

Let us consider I, A. The functor 2^A yielding a many sorted set indexed by I is defined as follows:

(Def.1) For arbitrary i such that $i \in I$ holds $2^A(i) = 2^{A(i)}$.

Let us consider I, A. Note that 2^A is non-empty.

One can prove the following propositions:

(1) $X = 2^Y$ iff for every A holds $A \in X$ iff $A \subseteq Y$.

(2) $2^{\emptyset_I} = I \longmapsto \{\emptyset\}$.

(3) $2^{I \longmapsto x} = I \longmapsto 2^x$.

(4) $2^{I \longmapsto \{x\}} = I \longmapsto \{\emptyset, \{x\}\}$.

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(5) $\emptyset_I \in 2^A$.

(6) If $A \subseteq B$, then $2^A \subseteq 2^B$.

(7) $2^A \cup 2^B \subseteq 2^{A \cup B}$.

(8) If $2^A \cup 2^B = 2^{A \cup B}$, then for arbitrary i such that $i \in I$ holds $A(i) \subseteq B(i)$ or $B(i) \subseteq A(i)$.

(9) $2^{A \cap B} = 2^A \cap 2^B$.

(10) $2^{A \setminus B} \subseteq (I \longmapsto \{\emptyset\}) \cup (2^A \setminus 2^B)$.

(11) $X \in 2^{A \setminus B}$ iff $X \subseteq A$ and X misses B.

(12) $2^{A \setminus B} \cup 2^{B \setminus A} \subseteq 2^{A \dot{-} B}$.

(13) $X \in 2^{A \dot{-} B}$ iff $X \subseteq A \cup B$ and X misses $A \cap B$.

(14) If $X \in 2^A$ and $Y \in 2^A$, then $X \cup Y \in 2^A$.

(15) If $X \in 2^A$ or $Y \in 2^A$, then $X \cap Y \in 2^A$.

(16) If $X \in 2^A$, then $X \setminus Y \in 2^A$.

(17) If $X \in 2^A$ and $Y \in 2^A$, then $X \dot{-} Y \in 2^A$.

(18) $[\![X, Y]\!] \subseteq 2^{2^{X \cup Y}}$.

(19) $X \subseteq A$ iff $X \in 2^A$.

(20) $\mathrm{MSFuncs}(A, B) \subseteq 2^{[\![A, B]\!]}$.

2. Union of Many Sorted Sets

Let us consider I, A. The functor $\bigcup A$ yields a many sorted set indexed by I and is defined as follows:

(Def.2) For arbitrary i such that $i \in I$ holds $(\bigcup A)(i) = \bigcup A(i)$.

Let us consider I. Observe that $\bigcup(\emptyset_I)$ is empty yielding.

We now state a number of propositions:

(21) $A \in \bigcup X$ iff there exists Y such that $A \in Y$ and $Y \in X$.

(22) $\bigcup(\emptyset_I) = \emptyset_I$.

(23) $\bigcup(I \longmapsto x) = I \longmapsto \bigcup x$.

(24) $\bigcup(I \longmapsto \{x\}) = I \longmapsto x$.

(25) $\bigcup(I \longmapsto \{\{x\}, \{y\}\}) = I \longmapsto \{x, y\}$.

(26) If $X \in A$, then $X \subseteq \bigcup A$.

(27) If $A \subseteq B$, then $\bigcup A \subseteq \bigcup B$.

(28) $\bigcup(A \cup B) = \bigcup A \cup \bigcup B$.

(29) $\bigcup(A \cap B) \subseteq \bigcup A \cap \bigcup B$.

(30) $\bigcup(2^A) = A$.

(31) $A \subseteq 2^{\bigcup A}$.

(32) If $\bigcup Y \subseteq A$ and $X \in Y$, then $X \subseteq A$.

(33) Let Z be a many sorted set indexed by I and let A be a non-empty many sorted set indexed by I. Suppose that for every many sorted set X indexed by I such that $X \in A$ holds $X \subseteq Z$. Then $\bigcup A \subseteq Z$.

(34) Let B be a many sorted set indexed by I and let A be a non-empty many sorted set indexed by I. Suppose that for every many sorted set X indexed by I such that $X \in A$ holds $X \cap B = \emptyset_I$. Then $\bigcup A \cap B = \emptyset_I$.

(35) Let A, B be many sorted sets indexed by I. Suppose $A \cup B$ is non-empty. Suppose that for all many sorted sets X, Y indexed by I such that $X \neq Y$ and $X \in A \cup B$ and $Y \in A \cup B$ holds $X \cap Y = \emptyset_I$. Then $\bigcup(A \cap B) = \bigcup A \cap \bigcup B$.

(36) Let A, X be many sorted sets indexed by I and let B be a non-empty many sorted set indexed by I. Suppose $X \subseteq \bigcup(A \cup B)$ and for every many sorted set Y indexed by I such that $Y \in B$ holds $Y \cap X = \emptyset_I$. Then $X \subseteq \bigcup A$.

(37) Let A be a locally-finite non-empty many sorted set indexed by I. Suppose that for all many sorted sets X, Y indexed by I such that $X \in A$ and $Y \in A$ holds $X \subseteq Y$ or $Y \subseteq X$. Then $\bigcup A \in A$.

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Combining of Circuits [1]

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Summary. We continue the formalisation of circuits started in [15,14,13,12]. Our goal was to work out the notation of combining circuits which could be employed to prove the properties of real circuits.

MML Identifier: `CIRCCOMB`.

The terminology and notation used in this paper are introduced in the following papers: [20], [23], [21], [25], [5], [3], [4], [9], [6], [16], [8], [7], [17], [22], [1], [2], [24], [10], [19], [11], [18], [15], [14], [13], and [12].

1. Combining of Many Sorted Signatures

Let S be a many sorted signature. A gate of S is an element of the operation symbols of S.

Let A be a set and let X be a set. Then $A \longmapsto X$ is a many sorted set indexed by A.

Let A be a set and let X be a non empty set. One can check that $A \longmapsto X$ is non-empty.

Let A be a set and let f be a function. One can verify that $A \longmapsto f$ is function yielding.

Let f, g be non-empty functions. Note that $f +\cdot g$ is non-empty.

Let A, B be sets, let f be a many sorted set indexed by A, and let g be a many sorted set indexed by B. Then $f +\cdot g$ is a many sorted set indexed by $A \cup B$.

We now state several propositions:

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(1) For all functions f_1, f_2, g_1, g_2 such that $\operatorname{rng} g_1 \subseteq \operatorname{dom} f_1$ and $\operatorname{rng} g_2 \subseteq \operatorname{dom} f_2$ and $f_1 \approx f_2$ holds $(f_1 +\cdot f_2) \cdot (g_1 +\cdot g_2) = f_1 \cdot g_1 +\cdot f_2 \cdot g_2$.

(2) For all functions f_1, f_2, g such that $\operatorname{rng} g \subseteq \operatorname{dom} f_1$ and $\operatorname{rng} g \subseteq \operatorname{dom} f_2$ and $f_1 \approx f_2$ holds $f_1 \cdot g = f_2 \cdot g$.

(3) Let A, B be sets, and let f be a many sorted set indexed by A, and let g be a many sorted set indexed by B. If $f \subseteq g$, then $f^{\#} \subseteq g^{\#}$.

(4) For all sets X, Y, x, y holds $X \longmapsto x \approx Y \longmapsto y$ iff $x = y$ or X misses Y.

(5) For all functions f, g, h such that $f \approx g$ and $g \approx h$ and $h \approx f$ holds $f +\cdot g \approx h$.

(6) For every set X and for every non empty set Y and for every finite sequence p of elements of X holds $(X \longmapsto Y)^{\#}(p) = Y^{\operatorname{len} p}$.

Let A be a set, let f_1, g_1 be non-empty many sorted sets indexed by A, let B be a set, let f_2, g_2 be non-empty many sorted sets indexed by B, let h_1 be a many sorted function from f_1 into g_1, and let h_2 be a many sorted function from f_2 into g_2. Then $h_1 +\cdot h_2$ is a many sorted function from $f_1 +\cdot f_2$ into $g_1 +\cdot g_2$.

Let S_1, S_2 be many sorted signatures. The predicate $S_1 \approx S_2$ is defined by:

(Def.1) The arity of $S_1 \approx$ the arity of S_2 and the result sort of $S_1 \approx$ the result sort of S_2.

Let us notice that this predicate is reflexive and symmetric.

Let S_1, S_2 be non empty many sorted signatures. The functor $S_1 +\cdot S_2$ yielding a strict non empty many sorted signature is defined by the conditions (Def.2).

(Def.2) (i) The carrier of $S_1 +\cdot S_2 =$ (the carrier of S_1) $\cup$ (the carrier of S_2),

(ii) the operation symbols of $S_1 +\cdot S_2 =$ (the operation symbols of S_1) $\cup$ (the operation symbols of S_2),

(iii) the arity of $S_1 +\cdot S_2 =$ (the arity of S_1) $+\cdot$ (the arity of S_2), and

(iv) the result sort of $S_1 +\cdot S_2 =$ (the result sort of S_1) $+\cdot$ (the result sort of S_2).

The following propositions are true:

(7) For all non empty many sorted signatures S_1, S_2, S_3 such that $S_1 \approx S_2$ and $S_2 \approx S_3$ and $S_3 \approx S_1$ holds $S_1 +\cdot S_2 \approx S_3$.

(8) For every non empty many sorted signature S holds $S +\cdot S =$ the many sorted signature of S.

(9) For all non empty many sorted signatures S_1, S_2 such that $S_1 \approx S_2$ holds $S_1 +\cdot S_2 = S_2 +\cdot S_1$.

(10) For all non empty many sorted signatures S_1, S_2, S_3 holds $(S_1 +\cdot S_2) +\cdot S_3 = S_1 +\cdot (S_2 +\cdot S_3)$.

One can verify that there exists a function which is one-to-one.

Next we state four propositions:

(11) Let f be an one-to-one function and let S_1, S_2 be circuit-like non empty many sorted signatures. Suppose the result sort of $S_1 \subseteq f$ and the result

sort of $S_2 \subseteq f$. Then $S_1 +\cdot S_2$ is circuit-like.

(12) For all circuit-like non empty many sorted signatures S_1, S_2 such that $\mathrm{InnerVertices}(S_1)$ misses $\mathrm{InnerVertices}(S_2)$ holds $S_1 +\cdot S_2$ is circuit-like.

(13) For all non empty many sorted signatures S_1, S_2 such that S_1 is not void or S_2 is not void holds $S_1 +\cdot S_2$ is non void.

(14) For all finite non empty many sorted signatures S_1, S_2 holds $S_1 +\cdot S_2$ is finite.

Let S_1 be a non void non empty many sorted signature and let S_2 be a non empty many sorted signature. Observe that $S_1 +\cdot S_2$ is non void and $S_2 +\cdot S_1$ is non void.

We now state several propositions:

(15) For all non empty many sorted signatures S_1, S_2 such that $S_1 \approx S_2$ holds $\mathrm{InnerVertices}(S_1 +\cdot S_2) = \mathrm{InnerVertices}(S_1) \cup \mathrm{InnerVertices}(S_2)$ and $\mathrm{InputVertices}(S_1 +\cdot S_2) \subseteq \mathrm{InputVertices}(S_1) \cup \mathrm{InputVertices}(S_2)$.

(16) For all non empty many sorted signatures S_1, S_2 and for every vertex v_2 of S_2 such that $v_2 \in \mathrm{InputVertices}(S_1 +\cdot S_2)$ holds $v_2 \in \mathrm{InputVertices}(S_2)$.

(17) Let S_1, S_2 be non empty many sorted signatures. If $S_1 \approx S_2$, then for every vertex v_1 of S_1 such that $v_1 \in \mathrm{InputVertices}(S_1 +\cdot S_2)$ holds $v_1 \in \mathrm{InputVertices}(S_1)$.

(18) Let S_1 be a non empty many sorted signature, and let S_2 be a non void non empty many sorted signature, and let o_2 be an operation symbol of S_2, and let o be an operation symbol of $S_1 +\cdot S_2$. Suppose $o_2 = o$. Then $\mathrm{Arity}(o) = \mathrm{Arity}(o_2)$ and the result sort of $o =$ the result sort of o_2.

(19) Let S_1 be a non empty many sorted signature and let S_2, S be circuit-like non void non empty many sorted signatures. Suppose $S = S_1 +\cdot S_2$. Let v_2 be a vertex of S_2. Suppose $v_2 \in \mathrm{InnerVertices}(S_2)$. Let v be a vertex of S. If $v_2 = v$, then $v \in \mathrm{InnerVertices}(S)$ and the action at $v =$ the action at v_2.

(20) Let S_1 be a non void non empty many sorted signature and let S_2 be a non empty many sorted signature. Suppose $S_1 \approx S_2$. Let o_1 be an operation symbol of S_1 and let o be an operation symbol of $S_1 +\cdot S_2$. Suppose $o_1 = o$. Then $\mathrm{Arity}(o) = \mathrm{Arity}(o_1)$ and the result sort of $o =$ the result sort of o_1.

(21) Let S_1, S be circuit-like non void non empty many sorted signatures and let S_2 be a non empty many sorted signature. Suppose $S_1 \approx S_2$ and $S = S_1 +\cdot S_2$. Let v_1 be a vertex of S_1. Suppose $v_1 \in \mathrm{InnerVertices}(S_1)$. Let v be a vertex of S. If $v_1 = v$, then $v \in \mathrm{InnerVertices}(S)$ and the action at $v =$ the action at v_1.

2. Combinig of Circuits

Let S_1, S_2 be non empty many sorted signatures, let A_1 be an algebra over S_1, and let A_2 be an algebra over S_2. The predicate $A_1 \approx A_2$ is defined by:

(Def.3) $S_1 \approx S_2$ and the sorts of $A_1 \approx$ the sorts of A_2 and the characteristics of $A_1 \approx$ the characteristics of A_2.

Let S_1, S_2 be non empty many sorted signatures, let A_1 be a non-empty algebra over S_1, and let A_2 be a non-empty algebra over S_2. Let us assume that the sorts of $A_1 \approx$ the sorts of A_2. The functor $A_1 +\cdot A_2$ yields a strict non-empty algebra over $S_1 +\cdot S_2$ and is defined by the conditions (Def.4).

(Def.4) (i) The sorts of $A_1 +\cdot A_2 =$ (the sorts of A_1) $+\cdot$ (the sorts of A_2), and
(ii) the characteristics of $A_1 +\cdot A_2 =$ (the characteristics of A_1) $+\cdot$ (the characteristics of A_2).

The following propositions are true:

(22) For every non void non empty many sorted signature S and for every algebra A over S holds $A \approx A$.

(23) Let S_1, S_2 be non void non empty many sorted signatures, and let A_1 be an algebra over S_1, and let A_2 be an algebra over S_2. If $A_1 \approx A_2$, then $A_2 \approx A_1$.

(24) Let S_1, S_2, S_3 be non empty many sorted signatures, and let A_1 be a non-empty algebra over S_1, and let A_2 be a non-empty algebra over S_2, and let A_3 be an algebra over S_3. If $A_1 \approx A_2$ and $A_2 \approx A_3$ and $A_3 \approx A_1$, then $A_1 +\cdot A_2 \approx A_3$.

(25) Let S be a strict non empty many sorted signature and let A be a non-empty algebra over S. Then $A +\cdot A =$ the algebra of A.

(26) Let S_1, S_2 be non empty many sorted signatures, and let A_1 be a non-empty algebra over S_1, and let A_2 be a non-empty algebra over S_2. If $A_1 \approx A_2$, then $A_1 +\cdot A_2 = A_2 +\cdot A_1$.

(27) Let S_1, S_2, S_3 be non empty many sorted signatures, and let A_1 be a non-empty algebra over S_1, and let A_2 be a non-empty algebra over S_2, and let A_3 be a non-empty algebra over S_3. Suppose that
(i) the sorts of $A_1 \approx$ the sorts of A_2,
(ii) the sorts of $A_2 \approx$ the sorts of A_3, and
(iii) the sorts of $A_3 \approx$ the sorts of A_1.
Then $(A_1 +\cdot A_2) +\cdot A_3 = A_1 +\cdot (A_2 +\cdot A_3)$.

(28) Let S_1, S_2 be non empty many sorted signatures, and let A_1 be a locally-finite non-empty algebra over S_1, and let A_2 be a locally-finite non-empty algebra over S_2. If the sorts of $A_1 \approx$ the sorts of A_2, then $A_1 +\cdot A_2$ is locally-finite.

(29) For all non-empty functions f, g and for every element x of $\prod f$ and for every element y of $\prod g$ holds $x +\cdot y \in \prod(f +\cdot g)$.

(30) For all non-empty functions f, g and for every element x of $\prod(f +\cdot g)$ holds $x \restriction \operatorname{dom} g \in \prod g$.

(31) For all non-empty functions f, g such that $f \approx g$ and for every element x of $\prod(f +\cdot g)$ holds $x \restriction \operatorname{dom} f \in \prod f$.

(32) Let S_1, S_2 be non empty many sorted signatures, and let A_1 be a non-empty algebra over S_1, and let s_1 be an element of $\prod$ (the sorts of A_1), and let A_2 be a non-empty algebra over S_2, and let s_2 be an element of $\prod$ (the sorts of A_2). If the sorts of $A_1 \approx$ the sorts of A_2, then $s_1 +\cdot s_2 \in \prod$ (the sorts of $A_1 +\cdot A_2$).

(33) Let S_1, S_2 be non empty many sorted signatures, and let A_1 be a non-empty algebra over S_1, and let A_2 be a non-empty algebra over S_2. Suppose the sorts of $A_1 \approx$ the sorts of A_2. Let s be an element of $\prod$ (the sorts of $A_1 +\cdot A_2$). Then $s \restriction$ (the carrier of S_1) $\in \prod$ (the sorts of A_1) and $s \restriction$ (the carrier of S_2) $\in \prod$ (the sorts of A_2).

(34) Let S_1, S_2 be non void non empty many sorted signatures, and let A_1 be a non-empty algebra over S_1, and let A_2 be a non-empty algebra over S_2. Suppose the sorts of $A_1 \approx$ the sorts of A_2. Let o be an operation symbol of $S_1 +\cdot S_2$ and let o_2 be an operation symbol of S_2. If $o = o_2$, then $\mathrm{Den}(o, A_1 +\cdot A_2) = \mathrm{Den}(o_2, A_2)$.

(35) Let S_1, S_2 be non void non empty many sorted signatures, and let A_1 be a non-empty algebra over S_1, and let A_2 be a non-empty algebra over S_2. Suppose the sorts of $A_1 \approx$ the sorts of A_2 and the characteristics of $A_1 \approx$ the characteristics of A_2. Let o be an operation symbol of $S_1 +\cdot S_2$ and let o_1 be an operation symbol of S_1. If $o = o_1$, then $\mathrm{Den}(o, A_1 +\cdot A_2) = \mathrm{Den}(o_1, A_1)$.

(36) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose $S = S_1 +\cdot S_2$. Let A_1 be a non-empty circuit of S_1, and let A_2 be a non-empty circuit of S_2, and let A be a non-empty circuit of S, and let s be a state of A, and let s_2 be a state of A_2. Suppose $s_2 = s \restriction$ (the carrier of S_2). Let g be a gate of S and let g_2 be a gate of S_2. If $g = g_2$, then g depends-on-in $s = g_2$ depends-on-in s_2.

(37) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose $S = S_1 +\cdot S_2$ and $S_1 \approx S_2$. Let A_1 be a non-empty circuit of S_1, and let A_2 be a non-empty circuit of S_2, and let A be a non-empty circuit of S, and let s be a state of A, and let s_1 be a state of A_1. Suppose $s_1 = s \restriction$ (the carrier of S_1). Let g be a gate of S and let g_1 be a gate of S_1. If $g = g_1$, then g depends-on-in $s = g_1$ depends-on-in s_1.

(38) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose $S = S_1 +\cdot S_2$. Let A_1 be a non-empty circuit of S_1, and let A_2 be a non-empty circuit of S_2, and let A be a non-empty circuit of S. Suppose $A_1 \approx A_2$ and $A = A_1 +\cdot A_2$. Let s be a state of A and let v be a vertex of S. Then

(i) for every state s_1 of A_1 such that $s_1 = s \restriction$ (the carrier of S_1) holds if

$v \in \mathrm{InnerVertices}(S_1)$ or $v \in$ the carrier of S_1 and $v \in \mathrm{InputVertices}(S)$, then $(\mathrm{Following}(s))(v) = (\mathrm{Following}(s_1))(v)$, and

(ii) for every state s_2 of A_2 such that $s_2 = s \restriction$ (the carrier of S_2) holds if $v \in \mathrm{InnerVertices}(S_2)$ or $v \in$ the carrier of S_2 and $v \in \mathrm{InputVertices}(S)$, then $(\mathrm{Following}(s))(v) = (\mathrm{Following}(s_2))(v)$.

(39) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose $\mathrm{InnerVertices}(S_1)$ misses $\mathrm{InputVertices}(S_2)$ and $S = S_1 +\cdot S_2$. Let A_1 be a non-empty circuit of S_1, and let A_2 be a non-empty circuit of S_2, and let A be a non-empty circuit of S. Suppose $A_1 \approx A_2$ and $A = A_1 +\cdot A_2$. Let s be a state of A, and let s_1 be a state of A_1, and let s_2 be a state of A_2. Suppose $s_1 = s \restriction$ (the carrier of S_1) and $s_2 = s \restriction$ (the carrier of S_2). Then $\mathrm{Following}(s) = \mathrm{Following}(s_1) +\cdot \mathrm{Following}(s_2)$.

(40) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose $\mathrm{InnerVertices}(S_2)$ misses $\mathrm{InputVertices}(S_1)$ and $S = S_1 +\cdot S_2$. Let A_1 be a non-empty circuit of S_1, and let A_2 be a non-empty circuit of S_2, and let A be a non-empty circuit of S. Suppose $A_1 \approx A_2$ and $A = A_1 +\cdot A_2$. Let s be a state of A, and let s_1 be a state of A_1, and let s_2 be a state of A_2. Suppose $s_1 = s \restriction$ (the carrier of S_1) and $s_2 = s \restriction$ (the carrier of S_2). Then $\mathrm{Following}(s) = \mathrm{Following}(s_2) +\cdot \mathrm{Following}(s_1)$.

(41) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose $\mathrm{InputVertices}(S_1) \subseteq \mathrm{InputVertices}(S_2)$ and $S = S_1 +\cdot S_2$. Let A_1 be a non-empty circuit of S_1, and let A_2 be a non-empty circuit of S_2, and let A be a non-empty circuit of S. Suppose $A_1 \approx A_2$ and $A = A_1 +\cdot A_2$. Let s be a state of A, and let s_1 be a state of A_1, and let s_2 be a state of A_2. Suppose $s_1 = s \restriction$ (the carrier of S_1) and $s_2 = s \restriction$ (the carrier of S_2). Then $\mathrm{Following}(s) = \mathrm{Following}(s_2) +\cdot \mathrm{Following}(s_1)$.

(42) Let S_1, S_2, S be non void circuit-like non empty many sorted signatures. Suppose $\mathrm{InputVertices}(S_2) \subseteq \mathrm{InputVertices}(S_1)$ and $S = S_1 +\cdot S_2$. Let A_1 be a non-empty circuit of S_1, and let A_2 be a non-empty circuit of S_2, and let A be a non-empty circuit of S. Suppose $A_1 \approx A_2$ and $A = A_1 +\cdot A_2$. Let s be a state of A, and let s_1 be a state of A_1, and let s_2 be a state of A_2. Suppose $s_1 = s \restriction$ (the carrier of S_1) and $s_2 = s \restriction$ (the carrier of S_2). Then $\mathrm{Following}(s) = \mathrm{Following}(s_1) +\cdot \mathrm{Following}(s_2)$.

3. Signatures with One Operation

Let A, B be non empty sets and let a be an element of A. Then $B \longmapsto a$ is a function from B into A.

Let f be a set, let p be a finite sequence, and let x be a set. The functor $\mathrm{1GateCircStr}(p, f, x)$ yields a non void strict many sorted signature and is defined by the conditions (Def.5).

(Def.5) (i) The carrier of $\mathrm{1GateCircStr}(p, f, x) = \mathrm{rng}\, p \cup \{x\}$,

(ii) the operation symbols of $\mathrm{1GateCircStr}(p, f, x) = \{\langle p, f\rangle\}$,

(iii) (the arity of 1GateCircStr(p, f, x))($\langle p,\ f\rangle$) $= p$, and

(iv) (the result sort of 1GateCircStr(p, f, x))($\langle p,\ f\rangle$) $= x$.

Let f be a set, let p be a finite sequence, and let x be a set. Note that 1GateCircStr(p, f, x) is non empty.

The following propositions are true:

(43) Let f, x be sets and let p be a finite sequence. Then the arity of 1GateCircStr$(p, f, x) = \{\langle p,\ f\rangle\} \longmapsto p$ and the result sort of 1GateCircStr$(p, f, x) = \{\langle p,\ f\rangle\} \longmapsto x$.

(44) Let f, x be sets, and let p be a finite sequence, and let g be a gate of 1GateCircStr(p, f, x). Then $g = \langle p,\ f\rangle$ and Arity$(g) = p$ and the result sort of $g = x$.

(45) For all sets f, x and for every finite sequence p holds InputVertices (1GateCircStr(p, f, x)) $= \operatorname{rng} p \setminus \{x\}$ and InnerVertices (1GateCircStr(p, f, x)) $= \{x\}$.

Let f be a set and let p be a finite sequence. The functor 1GateCircStr(p, f) yielding a non void strict many sorted signature is defined by the conditions (Def.6).

(Def.6) (i) The carrier of 1GateCircStr$(p, f) = \operatorname{rng} p \cup \{\langle p,\ f\rangle\}$,

(ii) the operation symbols of 1GateCircStr$(p, f) = \{\langle p,\ f\rangle\}$,

(iii) (the arity of 1GateCircStr(p, f))($\langle p,\ f\rangle$) $= p$, and

(iv) (the result sort of 1GateCircStr(p, f))($\langle p,\ f\rangle$) $= \langle p,\ f\rangle$.

Let f be a set and let p be a finite sequence. Note that 1GateCircStr(p, f) is non empty.

One can prove the following propositions:

(46) For every set f and for every finite sequence p holds 1GateCircStr$(p, f) =$ 1GateCircStr$(p, f, \langle p,\ f\rangle)$.

(47) Let f be a set and let p be a finite sequence. Then the arity of 1GateCircStr$(p, f) = \{\langle p,\ f\rangle\} \longmapsto p$ and the result sort of 1GateCircStr$(p, f) = \{\langle p,\ f\rangle\} \longmapsto \langle p,\ f\rangle$.

(48) Let f be a set, and let p be a finite sequence, and let g be a gate of 1GateCircStr(p, f). Then $g = \langle p,\ f\rangle$ and Arity$(g) = p$ and the result sort of $g = g$.

(49) For every set f and for every finite sequence p holds InputVertices (1GateCircStr(p, f)) $= \operatorname{rng} p$ and InnerVertices(1GateCircStr(p, f)) $= \{\langle p,\ f\rangle\}$.

(50) For every set f and for every finite sequence p and for every set x such that $x \in \operatorname{rng} p$ holds $\operatorname{rk}(x) \in \operatorname{rk}(\langle p,\ f\rangle)$.

(51) For every set f and for all finite sequences p, q holds 1GateCircStr$(p, f) \approx$ 1GateCircStr(q, f).

4. Unsplit Condition

A many sorted signature is unsplit if:

(Def.7) The result sort of it $= \mathrm{id}_{(\text{the operation symbols of it})}$.

A many sorted signature has arity held in gates if:

(Def.8) For every set g such that $g \in$ the operation symbols of it holds $g = \langle$(the arity of it)(g), $g_{\mathbf{2}}\rangle$.

A many sorted signature has Boolean denotation held in gates if it satisfies the condition (Def.9).

(Def.9) Let g be a set. Suppose $g \in$ the operation symbols of it. Let p be a finite sequence. Suppose $p =$ (the arity of it)(g). Then there exists a function f from $\mathit{Boolean}^{\operatorname{len} p}$ into $\mathit{Boolean}$ such that $g = \langle g_{\mathbf{1}}, f\rangle$.

Let S be a non empty many sorted signature. An algebra over S has denotation held in gates if:

(Def.10) For every set g such that $g \in$ the operation symbols of S holds $g = \langle g_{\mathbf{1}}$, (the characteristics of it)$(g)\rangle$.

A non empty many sorted signature has denotation held in gates if:

(Def.11) There exists algebra over it which has denotation held in gates.

One can verify that every non empty many sorted signature which has Boolean denotation held in gates has also denotation held in gates.

The following two propositions are true:

(52) Let S be a non empty many sorted signature. Then S is unsplit if and only if for every set o such that $o \in$ the operation symbols of S holds (the result sort of S)$(o) = o$.

(53) Let S be a non empty many sorted signature. Suppose S is unsplit. Then the operation symbols of $S \subseteq$ the carrier of S.

Let us note that every non empty many sorted signature which is unsplit is also circuit-like.

The following proposition is true

(54) For every set f and for every finite sequence p holds 1GateCircStr(p, f) is unsplit and has arity held in gates.

Let f be a set and let p be a finite sequence. Observe that 1GateCircStr(p, f) is unsplit and has arity held in gates.

Let us observe that there exists a many sorted signature which is unsplit non void strict and non empty and has arity held in gates.

One can prove the following propositions:

(55) For all unsplit non empty many sorted signatures S_1, S_2 with arity held in gates holds $S_1 \approx S_2$.

(56) Let S_1, S_2 be non empty many sorted signatures, and let A_1 be an algebra over S_1, and let A_2 be an algebra over S_2. Suppose A_1 has de-

notation held in gates and A_2 has denotation held in gates. Then the characteristics of $A_1 \approx$ the characteristics of A_2.

(57) For all unsplit non empty many sorted signatures S_1, S_2 holds $S_1 +\cdot S_2$ is unsplit.

Let S_1, S_2 be unsplit non empty many sorted signatures. Observe that $S_1 +\cdot S_2$ is unsplit.

We now state the proposition

(58) For all non empty many sorted signatures S_1, S_2 with arity held in gates holds $S_1 +\cdot S_2$ has arity held in gates.

Let S_1, S_2 be non empty many sorted signatures with arity held in gates. Note that $S_1 +\cdot S_2$ has arity held in gates.

The following proposition is true

(59) Let S_1, S_2 be non empty many sorted signatures. Suppose S_1 has Boolean denotation held in gates and S_2 has Boolean denotation held in gates. Then $S_1 +\cdot S_2$ has Boolean denotation held in gates.

5. One Gate Circuits

Let n be a natural number. A finite sequence is said to be a finite sequence with length n if:

(Def.12) $\operatorname{len} \mathrm{it} = n$.

Let n be a natural number, let X, Y be non empty sets, let f be a function from X^n into Y, let p be a finite sequence with length n, and let x be a set. Let us assume that if $x \in \operatorname{rng} p$, then $X = Y$. The functor $\mathrm{1GateCircuit}(p, f, x)$ yielding a strict non-empty algebra over $\mathrm{1GateCircStr}(p, f, x)$ is defined by:

(Def.13) The sorts of $\mathrm{1GateCircuit}(p, f, x) = (\operatorname{rng} p \longmapsto X) +\cdot (\{x\} \longmapsto Y)$ and (the characteristics of $\mathrm{1GateCircuit}(p, f, x))(\langle p,\ f \rangle) = f$.

Let n be a natural number, let X be a non empty set, let f be a function from X^n into X, and let p be a finite sequence with length n. The functor $\mathrm{1GateCircuit}(p, f)$ yielding a strict non-empty algebra over $\mathrm{1GateCircStr}(p, f)$ is defined as follows:

(Def.14) The sorts of $\mathrm{1GateCircuit}(p, f) =$ (the carrier of $\mathrm{1GateCircStr}(p, f)) \longmapsto (X)$ and (the characteristics of $\mathrm{1GateCircuit}(p, f))(\langle p,\ f \rangle) = f$.

Next we state the proposition

(60) Let n be a natural number, and let X be a non empty set, and let f be a function from X^n into X, and let p be a finite sequence with length n. Then $\mathrm{1GateCircuit}(p, f)$ has denotation held in gates and $\mathrm{1GateCircStr}(p, f)$ has denotation held in gates.

Let n be a natural number, let X be a non empty set, let f be a function from X^n into X, and let p be a finite sequence with length n. One can verify that 1GateCircuit(p, f) has denotation held in gates and 1GateCircStr(p, f) has denotation held in gates.

One can prove the following proposition

(61) Let n be a natural number, and let p be a finite sequence with length n, and let f be a function from $Boolean^n$ into $Boolean$. Then 1GateCircStr(p, f) has Boolean denotation held in gates.

Let n be a natural number, let f be a function from $Boolean^n$ into $Boolean$, and let p be a finite sequence with length n. Note that 1GateCircStr(p, f) has Boolean denotation held in gates.

One can check that there exists a many sorted signature which is non empty and has Boolean denotation held in gates.

Let S_1, S_2 be non empty many sorted signatures with Boolean denotation held in gates. Observe that $S_1 +\cdot S_2$ has Boolean denotation held in gates.

One can prove the following proposition

(62) Let n be a natural number, and let X be a non empty set, and let f be a function from X^n into X, and let p be a finite sequence with length n. Then the characteristics of 1GateCircuit$(p, f) = \{\langle p, f\rangle\} \longmapsto f$ and for every vertex v of 1GateCircStr(p, f) holds (the sorts of 1GateCircuit$(p, f))(v) = X$.

Let n be a natural number, let X be a non empty finite set, let f be a function from X^n into X, and let p be a finite sequence with length n. One can check that 1GateCircuit(p, f) is locally-finite.

Next we state two propositions:

(63) Let n be a natural number, and let X be a non empty set, and let f be a function from X^n into X, and let p, q be finite sequences with length n. Then 1GateCircuit$(p, f) \approx$ 1GateCircuit(q, f).

(64) Let n be a natural number, and let X be a finite non empty set, and let f be a function from X^n into X, and let p be a finite sequence with length n, and let s be a state of 1GateCircuit(p, f). Then (Following$(s))(\langle p, f\rangle) = f(s \cdot p)$.

Let X be a non empty set. Observe that there exists a non empty subset of X which is finite.

6. Boolean Circuits

Boolean is a finite non empty subset of $\mathbb{N}$.

Let S be a non empty many sorted signature. An algebra over S is Boolean if:

(Def.15) For every vertex v of S holds (the sorts of it)$(v) = Boolean$.

Next we state the proposition

(65) Let S be a non empty many sorted signature and let A be an algebra over S. Then A is Boolean if and only if the sorts of $A =$ (the carrier of $S) \longmapsto \mathit{Boolean}$.

Let S be a non empty many sorted signature. Note that every algebra over S which is Boolean is also non-empty and locally-finite.

One can prove the following three propositions:

(66) Let S be a non empty many sorted signature and let A be an algebra over S. Then A is Boolean if and only if rng (the sorts of $A) \subseteq \{\mathit{Boolean}\}$.

(67) Let S_1, S_2 be non empty many sorted signatures, and let A_1 be an algebra over S_1, and let A_2 be an algebra over S_2. Suppose A_1 is Boolean and A_2 is Boolean. Then the sorts of $A_1 \approx$ the sorts of A_2.

(68) Let S_1, S_2 be unsplit non empty many sorted signatures with arity held in gates, and let A_1 be an algebra over S_1, and let A_2 be an algebra over S_2. Suppose A_1 is Boolean and has denotation held in gates and A_2 is Boolean and has denotation held in gates. Then $A_1 \approx A_2$.

Let S be a non empty many sorted signature. One can check that there exists a strict algebra over S which is Boolean.

We now state three propositions:

(69) Let n be a natural number, and let f be a function from $\mathit{Boolean}^n$ into $\mathit{Boolean}$, and let p be a finite sequence with length n. Then $\mathrm{1GateCircuit}(p, f)$ is Boolean.

(70) Let S_1, S_2 be non empty many sorted signatures, and let A_1 be a Boolean algebra over S_1, and let A_2 be a Boolean algebra over S_2. Then $A_1 +\cdot A_2$ is Boolean.

(71) Let S_1, S_2 be non empty many sorted signatures, and let A_1 be a non-empty algebra over S_1, and let A_2 be a non-empty algebra over S_2. Suppose A_1 has denotation held in gates and A_2 has denotation held in gates and the sorts of $A_1 \approx$ the sorts of A_2. Then $A_1 +\cdot A_2$ has denotation held in gates.

Let us observe that there exists a non empty many sorted signature which is unsplit non void and strict and has arity held in gates, denotation held in gates, and Boolean denotation held in gates.

Let S be a non empty many sorted signature with Boolean denotation held in gates. Note that there exists a strict algebra over S which is Boolean and has denotation held in gates.

Let S_1, S_2 be unsplit non void non empty many sorted signatures with Boolean denotation held in gates, let A_1 be a Boolean circuit of S_1 with denotation held in gates, and let A_2 be a Boolean circuit of S_2 with denotation held in gates. One can verify that $A_1 +\cdot A_2$ is Boolean and has denotation held in gates.

Let n be a natural number, let X be a finite non empty set, let f be a function from X^n into X, and let p be a finite sequence with length n. Observe that there exists a circuit of $\mathrm{1GateCircStr}(p, f)$ which is strict and non-empty and has denotation held in gates.

Let n be a natural number, let X be a finite non empty set, let f be a function from X^n into X, and let p be a finite sequence with length n. Note that 1GateCircuit(p, f) has denotation held in gates.

One can prove the following proposition

(72) Let S_1, S_2 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates, and let A_1 be a Boolean circuit of S_1 with denotation held in gates, and let A_2 be a Boolean circuit of S_2 with denotation held in gates, and let s be a state of $A_1 +\cdot A_2$, and let v be a vertex of $S_1 +\cdot S_2$. Then

(i) for every state s_1 of A_1 such that $s_1 = s \restriction$ (the carrier of S_1) holds if $v \in$ InnerVertices(S_1) or $v \in$ the carrier of S_1 and $v \in$ InputVertices$(S_1 +\cdot S_2)$, then (Following(s))(v) = (Following(s_1))(v), and

(ii) for every state s_2 of A_2 such that $s_2 = s \restriction$ (the carrier of S_2) holds if $v \in$ InnerVertices(S_2) or $v \in$ the carrier of S_2 and $v \in$ InputVertices$(S_1 +\cdot S_2)$, then (Following(s))(v) = (Following(s_2))(v).

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