Many-sorted Sets

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Summary. The article deals with parameterized families of sets. When treated in a similar way as sets (due to systematic overloading notation used for sets) they are called many sorted sets. For instance, if \( x \) and \( X \) are two many-sorted sets (with the same set of indices \( I \)) then relation \( x \in X \) is defined as \( \forall i \in I x_i \in X_i \).

I was prompted by a remark in a paper by Tarlecki and Wirsing: "Throughout the paper we deal with many-sorted sets, functions, relations etc. ... We feel free to use any standard set-theoretic notation without explicit use of indices" [3, p.97]. The aim of this work was to check the feasibility of such approach in Mizar. It works.

Let us observe some peculiarities:
- empty set (i.e. the many sorted set with empty set of indices) belongs to itself (theorem 133),
- we get two different inclusions \( X \subseteq Y \) iff \( \forall i \in I X_i \subseteq Y_i \) and \( X \subseteq Y \) iff \( \forall x \in X \Rightarrow x \in Y \) equivalent only for sets that yield non empty values.

Therefore the care is advised.

MML Identifier: PBOOLE.

The articles [5], [1], [4], and [2] provide the terminology and notation for this paper.

1. Preliminaries

In the sequel \( i, e \) will be arbitrary.

A function is empty yielding if:

(Def.1) For every \( i \) such that \( i \in \text{dom} \) it holds \( \text{it}(i) \) is empty.

A function is non empty set yielding if:

(Def.2) For every \( i \) such that \( i \in \text{dom} \) it holds \( \text{it}(i) \) is non empty.
Next we state two propositions:

(1) For every function $f$ such that $f$ is non empty yielding holds $\text{rng } f$ has non empty elements.

(2) For every function $f$ holds $f$ is empty yielding iff $f = \emptyset$ or $\text{rng } f = \{\emptyset\}$.

In the sequel $I$ denotes a set.
Let us consider $I$. A function is said to be a many sorted set of $I$ if:

(Def.3) \[ \text{dom } it = I. \]

In the sequel $x, y, z, X, Y, Z, V$ are many sorted sets of $I$.

The scheme Kuratowski Function deals with a set $A$ and a unary functor $F$ yielding arbitrary, and states that:

There exists a many sorted set $f$ of $A$ such that for every $e$ such that $e \in A$ holds $f(e) \in F(e)$

provided the following requirement is met:

- For every $e$ such that $e \in A$ holds $F(e) \neq \emptyset$.

Let us consider $I, X, Y$. The predicate $X \in Y$ is defined by:

(Def.4) For every $i$ such that $i \in I$ holds $X(i) \in Y(i)$.

The predicate $X \subseteq Y$ is defined by:

(Def.5) For every $i$ such that $i \in I$ holds $X(i) \subseteq Y(i)$.

The scheme $P$ Separation deals with a set $A$, a many sorted set $B$ of $A$, and a binary predicate $P$, and states that:

There exists a many sorted set $X$ of $A$ such that for every set $i$ holds if $i \in A$, then for every $e$ holds $e \in X(i)$ iff $e \in B(i)$ and $P[i, e]$ for all values of the parameters.

One can prove the following proposition

(3) If for every $i$ such that $i \in I$ holds $X(i) = Y(i)$, then $X = Y$.

Let us consider $I$. The functor $\emptyset_I$ yields a many sorted set of $I$ and is defined by:

(Def.6) $\emptyset_I = I \longmapsto \emptyset$.

Let us consider $X, Y$. The functor $X \cup Y$ yielding a many sorted set of $I$ is defined by:

(Def.7) For every $i$ such that $i \in I$ holds $(X \cup Y)(i) = X(i) \cup Y(i)$.

The functor $X \cap Y$ yielding a many sorted set of $I$ is defined by:

(Def.8) For every $i$ such that $i \in I$ holds $(X \cap Y)(i) = X(i) \cap Y(i)$.

The functor $X \setminus Y$ yields a many sorted set of $I$ and is defined as follows:

(Def.9) For every $i$ such that $i \in I$ holds $(X \setminus Y)(i) = X(i) \setminus Y(i)$.

We say that $X$ overlaps $Y$ if and only if:

(Def.10) For every $i$ such that $i \in I$ holds $X(i)$ meets $Y(i)$.

We say that $X$ misses $Y$ if and only if:

(Def.11) For every $i$ such that $i \in I$ holds $X(i)$ misses $Y(i)$. 
Let us consider $I$, $X$, $Y$. The functor $X \rhd Y$ yielding a many sorted set of $I$ is defined as follows:

(Def.12) \( X \rhd Y = (X \setminus Y) \cup (Y \setminus X) \).

Next we state several propositions:

(4) For every $i$ such that $i \in I$ holds $(X \rhd Y)(i) = X(i) \rhd Y(i)$.

(5) For every $i$ such that $i \in I$ holds $\emptyset_I(i) = \emptyset$.

(6) If for every $i$ such that $i \in I$ holds $X(i) = \emptyset$, then $X = \emptyset_I$.

(7) If $x \in X$ or $x \in Y$, then $x \in X \cup Y$.

(8) $x \in X \cap Y$ iff $x \in X$ and $x \in Y$.

(9) If $x \in X$ and $X \subseteq Y$, then $x \in Y$.

(10) If $x \in X$ and $x \in Y$, then $X$ overlaps $Y$.

(11) If $X$ overlaps $Y$, then there exists $x$ such that $x \in X$ and $x \in Y$.

(12) If $x \in X \setminus Y$, then $x \in X$.

2. LATTICE PROPERTIES OF MANY SORTED SETS

One can prove the following proposition

(13) $X \subseteq X$.

Let us consider $I$, $X$, $Y$. Let us observe that $X = Y$ if and only if:

(Def.13) $X \subseteq Y$ and $Y \subseteq X$.

Next we state a number of propositions:

(14) If $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$.

(15) If $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$.

(16) $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$.

(17) $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$.

(18) If $X \subseteq Z$ and $Y \subseteq Z$, then $X \cup Y \subseteq Z$.

(19) If $Z \subseteq X$ and $Z \subseteq Y$, then $Z \subseteq X \cap Y$.

(20) If $X \subseteq Y$, then $X \cup Z \subseteq Y \cup Z$ and $Z \cup X \subseteq Z \cup Y$.

(21) If $X \subseteq Y$, then $X \cap Z \subseteq Y \cap Z$ and $Z \cap X \subseteq Z \cap Y$.

(22) If $X \subseteq Y$ and $Z \subseteq V$, then $X \cup Z \subseteq Y \cup V$.

(23) If $X \subseteq Y$ and $Z \subseteq V$, then $X \cap Z \subseteq Y \cap V$.

(24) If $X \subseteq Y$, then $X \cup Y = Y$ and $Y \cup X = Y$.

(25) If $X \subseteq Y$, then $X \cap Y = X$ and $Y \cap X = X$.

(26) $X \cap Y \subseteq X \cup Z$.

(27) If $X \subseteq Z$, then $X \cup Y \cap Z = (X \cup Y) \cap Z$.

(28) $X = Y \cup Z$ iff $Y \subseteq X$ and $Z \subseteq X$ and for every $V$ such that $Y \subseteq V$ and $Z \subseteq V$ holds $X \subseteq V$.

(29) $X = Y \cap Z$ iff $X \subseteq Y$ and $X \subseteq Z$ and for every $V$ such that $V \subseteq Y$ and $V \subseteq Z$ holds $V \subseteq X$. 
(30) \( X \cup X = X \).
(31) \( X \cap X = X \).
(32) \( X \cup Y = Y \cup X \).
(33) \( X \cap Y = Y \cap X \).
(34) \( (X \cup Y) \cup Z = X \cup (Y \cup Z) \).
(35) \( (X \cap Y) \cap Z = X \cap (Y \cap Z) \).
(36) \( X \cap (X \cup Y) = X \) and \( (X \cup Y) \cap X = X \) and \( X \cap (Y \cup X) = X \) and \( (Y \cup X) \cap X = X \).
(37) \( X \cup X \cap Y = X \) and \( X \cap Y \cup X = X \) and \( X \cup Y \cap X = X \) and \( Y \cap X \cup X = X \).
(38) \( X \cap (Y \cup Z) = X \cap Y \cup X \cap Z \) and \( (Y \cup Z) \cap X = Y \cap X \cup Z \cap X \).
(39) \( X \cup Y \cap Z = (X \cup Y) \cap (X \cup Z) \) and \( Y \cap Z \cup X = (Y \cap X) \cap (Z \cup X) \).
(40) If \( X \cap Y \cup X \cap Z = X \), then \( X \subseteq Y \cup Z \).
(41) If \( (X \cup Y) \cap (X \cup Z) = X \), then \( Y \cap Z \subseteq X \).
(42) \( X \cap Y \cup Y \cap Z \cup Z \cap X = (X \cup Y) \cap (Y \cup Z) \cap (Z \cup X) \).
(43) If \( X \cup Y \subseteq Z \), then \( X \subseteq Z \) and \( Y \subseteq Z \).
(44) If \( X \subseteq Y \cap Z \), then \( X \subseteq Y \) and \( X \subseteq Z \).
(45) \( (X \cup Y) \cup Z = X \cup Z \cup (Y \cup Z) \) and \( X \cup (Y \cup Z) = (X \cup Y) \cup (X \cup Z) \).
(46) \( (X \cap Y) \cap Z = X \cap Z \cap (Y \cap Z) \) and \( X \cap (Y \cap Z) = (X \cap Y) \cap (X \cap Z) \).
(47) \( X \cup (X \cup Y) = X \cup Y \) and \( X \cup Y \cup Y = X \cup Y \).
(48) \( X \cap (X \cap Y) = X \cap Y \) and \( X \cap Y \cap Y = X \cap Y \).

3. The Empty Many Sorted Set

Next we state several propositions:

(49) \( \emptyset_I \subseteq X \).
(50) If \( X \subseteq \emptyset_I \), then \( X = \emptyset_I \).
(51) If \( X \subseteq Y \) and \( X \subseteq Z \) and \( Y \cap Z = \emptyset_I \), then \( X = \emptyset_I \).
(52) If \( X \subseteq Y \) and \( Y \cap Z = \emptyset_I \), then \( X \cap Z = \emptyset_I \).
(53) \( X \cup \emptyset_I = X \) and \( \emptyset_I \cup X = X \).
(54) If \( X \cup Y = \emptyset_I \), then \( X = \emptyset_I \) and \( Y = \emptyset_I \).
(55) \( X \cap \emptyset_I = \emptyset_I \) and \( \emptyset_I \cap X = \emptyset_I \).
(56) If \( X \subseteq Y \cup Z \) and \( X \cap Z = \emptyset_I \), then \( X \subseteq Y \).
(57) If \( Y \subseteq X \) and \( X \cap Y = \emptyset_I \), then \( Y = \emptyset_I \).
4. The Difference and the Symmetric Difference

We now state a number of propositions:

(58) \( X \setminus Y = \emptyset_I \) iff \( X \subseteq Y \).

(59) If \( X \subseteq Y \), then \( X \setminus Z \subseteq Y \setminus Z \).

(60) If \( X \subseteq Y \), then \( Z \setminus Y \subseteq Z \setminus X \).

(61) If \( X \subseteq Y \) and \( Z \subseteq V \), then \( X \setminus V \subseteq Y \setminus Z \).

(62) \( X \setminus Y \subseteq X \).

(63) If \( X \subseteq Y \setminus X \), then \( X = \emptyset_I \).

(64) \( X \setminus X = \emptyset_I \).

(65) \( X \setminus \emptyset_I = X \).

(66) \( \emptyset_I \setminus X = \emptyset_I \).

(67) \( X \setminus (X \cup Y) = \emptyset_I \) and \( X \setminus (Y \cup X) = \emptyset_I \).

(68) \( X \cap (Y \setminus Z) = X \cap Y \setminus Z \).

(69) \( (X \setminus Y) \cap Y = \emptyset_I \) and \( Y \cap (X \setminus Y) = \emptyset_I \).

(70) \( X \setminus (Y \setminus Z) = (X \setminus Y) \cup X \cap Z \).

(71) \( (X \setminus Y) \cup X \cap Y = X \) and \( X \cap Y \cup (X \setminus Y) = X \).

(72) If \( X \subseteq Y \), then \( Y = X \setminus (Y \setminus X) \) and \( Y = (Y \setminus X) \cup X \).

(73) \( X \cup (Y \setminus X) = X \cup Y \) and \( (Y \setminus X) \cup X = Y \cup X \).

(74) \( X \setminus (X \setminus Y) = X \cap Y \).

(75) \( X \setminus Y \cap Z = (X \setminus Y) \cup (X \setminus Z) \).

(76) \( X \setminus X \cap Y = X \setminus Y \) and \( X \setminus Y \cap X = X \setminus Y \).

(77) \( X \cap Y = \emptyset_I \) iff \( X \setminus Y = X \).

(78) \( (X \cup Y) \setminus Z = (X \setminus Z) \cup (Y \setminus Z) \).

(79) \( X \setminus Y \cap Z = X \setminus (Y \cup Z) \).

(80) \( X \cap Y \setminus Z = (X \setminus Z) \cap (Y \setminus Z) \).

(81) \( (X \cup Y) \setminus Y = X \setminus Y \).

(82) If \( X \subseteq Y \cup Z \), then \( X \setminus Y \subseteq Z \) and \( X \setminus Z \subseteq Y \).

(83) \( (X \cup Y) \setminus X \cap Y = (X \setminus Y) \cup (Y \setminus X) \).

(84) \( X \setminus Y \cap Y = X \setminus Y \).

(85) \( X \setminus (Y \cup Z) = (X \setminus Y) \cap (X \setminus Z) \).

(86) If \( X \setminus Y = Y \setminus X \), then \( X = Y \).

(87) \( X \cap (Y \setminus Z) = X \cap Y \setminus X \cap Z \) and \( (Y \setminus Z) \cap X = Y \cap X \setminus Z \cap X \).

(88) If \( X \setminus Y \subseteq Z \), then \( X \subseteq Y \cup Z \).

(89) \( X \setminus Y \subseteq X \setminus Y \).

(90) \( X \setminus Y = (X \setminus Y) \cup (Y \setminus X) \).

(91) \( X \setminus \emptyset_I = X \) and \( \emptyset_I \setminus X = X \).

(92) \( X \setminus X = \emptyset_I \).
\[ (93) \quad X \div Y = Y \div X. \]
\[ (94) \quad X \cup Y = (X \div Y) \cup X \cap Y. \]
\[ (95) \quad X \div Y = (X \cup Y) \setminus X \cap Y. \]
\[ (96) \quad (X \div Y) \setminus Z = (X \setminus (Y \cup Z)) \cup (Y \setminus (X \cup Z)). \]
\[ (97) \quad X \setminus (Y \div Z) = (X \setminus (Y \cup Z)) \cup X \cap Y \cap Z. \]
\[ (98) \quad (X \div Y) \div Z = X \div (Y \div Z). \]
\[ (99) \quad \text{If } X \setminus Y \subseteq Z \text{ and } Y \setminus X \subseteq Z, \text{ then } X \div Y \subseteq Z. \]
\[ (100) \quad X \cup Y = X \div (Y \setminus X). \]
\[ (101) \quad X \cap Y = X \div (X \setminus Y). \]
\[ (102) \quad X \setminus Y = X \div X \cap Y. \]
\[ (103) \quad Y \setminus X = X \div (X \cup Y). \]
\[ (104) \quad X \cup Y = X \div Y \div X \cap Y. \]
\[ (105) \quad X \cap Y = X \div Y \div (X \cup Y). \]

5. Meeting and Overlapping

The following propositions are true:

\[ (106) \quad \text{If } X \text{ overlaps } Y \text{ or } X \text{ overlaps } Z, \text{ then } X \text{ overlaps } Y \cup Z. \]
\[ (107) \quad \text{If } X \text{ overlaps } Y, \text{ then } Y \text{ overlaps } X. \]
\[ (108) \quad \text{If } X \text{ overlaps } Y \text{ and } Y \subseteq Z, \text{ then } X \text{ overlaps } Z. \]
\[ (109) \quad \text{If } X \text{ overlaps } Y \text{ and } X \subseteq Z, \text{ then } Z \text{ overlaps } Y. \]
\[ (110) \quad \text{If } X \subseteq Y \text{ and } Z \subseteq V \text{ and } X \text{ overlaps } Z, \text{ then } Y \text{ overlaps } V. \]
\[ (111) \quad \text{If } X \text{ overlaps } Y \cap Z, \text{ then } X \text{ overlaps } Y \text{ and } X \text{ overlaps } Z. \]
\[ (112) \quad \text{If } X \text{ overlaps } Z \text{ and } X \subseteq V, \text{ then } X \text{ overlaps } Z \cap V. \]
\[ (113) \quad \text{If } X \text{ overlaps } Y \setminus Z, \text{ then } X \text{ overlaps } Y. \]
\[ (114) \quad \text{If } Y \text{ does not overlap } Z, \text{ then } X \cap Y \text{ does not overlap } X \cap Z \text{ and } Y \cap X \text{ does not overlap } Z \cap X. \]
\[ (115) \quad \text{If } X \text{ overlaps } Y \setminus Z, \text{ then } Y \text{ overlaps } X \setminus Z. \]
\[ (116) \quad \text{If } X \text{ meets } Y \text{ and } Y \subseteq Z, \text{ then } X \text{ meets } Z. \]
\[ (117) \quad \text{If } X \text{ meets } Y, \text{ then } Y \text{ meets } X. \]
\[ (118) \quad Y \text{ misses } X \setminus Y. \]
\[ (119) \quad X \cap Y \text{ misses } X \setminus Y. \]
\[ (120) \quad X \cap Y \text{ misses } X \div Y. \]
\[ (121) \quad \text{If } X \text{ misses } Y, \text{ then } X \cap Y = \emptyset. \]
\[ (122) \quad \text{If } X \neq \emptyset, \text{ then } X \text{ meets } X. \]
\[ (123) \quad \text{If } X \subseteq Y \text{ and } X \subseteq Z \text{ and } Y \text{ misses } Z, \text{ then } X = \emptyset. \]
\[ (124) \quad \text{If } Z \cup V = X \cup Y \text{ and } X \text{ misses } Z \text{ and } Y \text{ misses } V, \text{ then } X = V \text{ and } Y = Z. \]
(125) If $Z \cup V = X \cup Y$ and $Y$ misses $Z$ and $X$ miss $V$, then $X = Z$ and $Y = V$.
(126) If $X$ misses $Y$, then $X \setminus Y = X$ and $Y \setminus X = Y$.
(127) If $X$ misses $Y$, then $(X \cup Y) \setminus Y = X$ and $(X \cup Y) \setminus X = Y$.
(128) If $X \setminus Y = X$, then $X$ misses $Y$ and $Y$ misses $X$.
(129) $X \setminus Y$ misses $Y \setminus X$.

6. The Second Inclusion

Let us consider $I$, $X$, $Y$. The predicate $X \subseteq Y$ is defined as follows:

(Def.14) For every $x$ such that $x \in X$ holds $x \in Y$.

The following three propositions are true:

(130) If $X \subseteq Y$, then $X \subseteq Y$.
(131) $X \subseteq X$.
(132) If $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$.

7. Non Empty and Non-empty Many Sorted Sets

The following propositions are true:

(133) $\emptyset \in \emptyset$.
(134) For every many sorted set $X$ of \emptyset holds $X = \emptyset$.

We follow a convention: $I$ will be a non empty set and $x$, $X$, $Y$, $Z$ will be many sorted sets of $I$.

The following propositions are true:

(135) If $X$ overlaps $Y$, then $X$ meets $Y$.
(136) It is not true that there exists $x$ such that $x \in \emptyset_I$.
(137) If $x \in X$ and $x \in Y$, then $X \cap Y \neq \emptyset_I$.
(138) $X$ does not overlap $\emptyset_I$ and $\emptyset_I$ does not overlap $X$.
(139) If $X \cap Y = \emptyset_I$, then $X$ does not overlap $Y$.
(140) If $X$ overlaps $X$, then $X \neq \emptyset_I$.

Let $I$ be a set. A many sorted set of $I$ is empty yielding if:

(Def.15) For every $i$ such that $i \in I$ holds it$(i)$ is empty.

A many sorted set of $I$ is non empty set yielding if:

(Def.16) For every $i$ such that $i \in I$ holds it$(i)$ is non empty.

Let $I$ be a non empty set. Observe that every many sorted set of $I$ which is non-empty is also non empty and every many sorted set of $I$ which is empty is also non-empty.

One can prove the following propositions:
(141) $X$ is empty iff $X = \emptyset_I$.
(142) If $Y$ is empty and $X \subseteq Y$, then $X$ is empty.
(143) If $X$ is non-empty and $X \subseteq Y$, then $Y$ is non-empty.
(144) If $X$ is non-empty and $X \subseteq Y$, then $X \subseteq Y$.
(145) If $X$ is non-empty and $X \subseteq Y$, then $Y$ is non-empty.

In the sequel $X$ denotes a non-empty many sorted set of $I$.

The following propositions are true:

(146) There exists $x$ such that $x \in X$.
(147) If for every $x$ holds $x \in X$ iff $x \in Y$, then $X = Y$.
(148) If for every $x$ holds $x \in X$ iff $x \in Y$ and $x \in Z$, then $X = Y \cap Z$.

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