FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Continuity of Mappings over the Union of Subspaces

Zbigniew Karno
Warsaw University
Białystok

Summary. Let X and Y be topological spaces and let X_1 and X_2 be subspaces of X. Let $f : X_1 \cup X_2 \to Y$ be a mapping defined on the union of X_1 and X_2 such that the restriction mappings $f_{|X_1}$ and $f_{|X_2}$ are continuous. It is well known that if X_1 and X_2 are both open (closed) subspaces of X, then f is continuous (see e.g. [6, p.106]).

The aim is to show, using Mizar System, the following theorem (see Section 5): *If X_1 and X_2 are weakly separated, then f is continuous* (compare also [15, p.358] for related results). This theorem generalizes the preceding one because if X_1 and X_2 are both open (closed), then these subspaces are weakly separated (see [5]). However, the following problem remains open.

Problem 1. Characterize the class of pairs of subspaces X_1 and X_2 of a topological space X such that $(*)$ for any topological space Y and for any mapping $f : X_1 \cup X_2 \to Y$, f is continuous if the restrictions $f_{|X_1}$ and $f_{|X_2}$ are continuous.

In some special case we have the following characterization: *X_1 and X_2 are separated iff X_1 misses X_2 and the condition $(*)$ is fulfilled.* In connection with this fact we hope that the following specification of the preceding problem has an affirmative answer.

Problem 2. Suppose the condition $(*)$ is fulfilled. Must X_1 and X_2 be weakly separated ?

Note that in the last section the concept of the union of two mappings is introduced and studied. In particular, all results presented above are reformulated using this notion. In the remaining sections we introduce concepts needed for the formulation and the proof of theorems on properties of continuous mappings, restriction mappings and modifications of the topology.

MML Identifier: `TMAP_1`.

The articles [13], [14], [2], [3], [1], [4], [11], [8], [10], [16], [7], [9], [12], and [5] provide the notation and terminology for this paper.

ISSN 0777–4028

1. Set-Theoretic Preliminaries

In the sequel A, B will denote non-empty sets. Next we state several propositions:

(1) For every function f from A into B and for every subset A_0 of A and for every subset B_0 of B holds $f\,^\circ\, A_0 \subseteq B_0$ if and only if $A_0 \subseteq f\,^{-1}\, B_0$.

(2) For every function f from A into B and for every non-empty subset A_0 of A and for every function f_0 from A_0 into B such that for every element c of A such that $c \in A_0$ holds $f(c) = f_0(c)$ holds $f \restriction A_0 = f_0$.

(3) For every function f from A into B and for every non-empty subset A_0 of A and for every element c of A such that $c \in A_0$ holds $f(c) = (f \restriction A_0)(c)$.

(4) For every function f from A into B and for every non-empty subset A_0 of A and for every subset C of A such that $C \subseteq A_0$ holds $f^\circ C = (f \restriction A_0)^\circ C$.

(5) For every function f from A into B and for every non-empty subset A_0 of A and for every subset D of B such that $f\,^{-1}\, D \subseteq A_0$ holds $f\,^{-1}\, D = (f \restriction A_0)\,^{-1}\, D$.

Let A, B be non-empty sets, and let A_1, A_2 be non-empty subsets of A, and let f_1 be a function from A_1 into B, and let f_2 be a function from A_2 into B. Let us assume that $f_1 \restriction (A_1 \cap A_2) = f_2 \restriction (A_1 \cap A_2)$. The functor $f_1 \cup f_2$ yielding a function from $A_1 \cup A_2$ into B is defined by:

(Def.1) $(f_1 \cup f_2) \restriction A_1 = f_1$ and $(f_1 \cup f_2) \restriction A_2 = f_2$.

The following proposition is true

(6) Let A, B be non-empty sets. Then for all non-empty subsets A_1, A_2 of A such that A_1 misses A_2 and for every function f_1 from A_1 into B and for every function f_2 from A_2 into B holds $f_1 \restriction (A_1 \cap A_2) = f_2 \restriction (A_1 \cap A_2)$ and $(f_1 \cup f_2) \restriction A_1 = f_1$ and $(f_1 \cup f_2) \restriction A_2 = f_2$.

We follow the rules: A, B are non-empty sets and A_1, A_2, A_3 are non-empty subsets of A. We now state four propositions:

(7) For every function g from $A_1 \cup A_2$ into B and for every function g_1 from A_1 into B and for every function g_2 from A_2 into B such that $g \restriction A_1 = g_1$ and $g \restriction A_2 = g_2$ holds $g = g_1 \cup g_2$.

(8) For every function f_1 from A_1 into B and for every function f_2 from A_2 into B such that $f_1 \restriction (A_1 \cap A_2) = f_2 \restriction (A_1 \cap A_2)$ holds $f_1 \cup f_2 = f_2 \cup f_1$.

(9) Let A_{12}, A_{23} be non-empty subsets of A. Suppose $A_{12} = A_1 \cup A_2$ and $A_{23} = A_2 \cup A_3$. Let f_1 be a function from A_1 into B. Let f_2 be a function from A_2 into B. Let f_3 be a function from A_3 into B. Suppose $f_1 \restriction (A_1 \cap A_2) = f_2 \restriction (A_1 \cap A_2)$ and $f_2 \restriction (A_2 \cap A_3) = f_3 \restriction (A_2 \cap A_3)$ and $f_1 \restriction (A_1 \cap A_3) = f_3 \restriction (A_1 \cap A_3)$. Then for every function f_{12} from A_{12} into B and for every function f_{23} from A_{23} into B such that $f_{12} = f_1 \cup f_2$ and $f_{23} = f_2 \cup f_3$ holds $f_{12} \cup f_3 = f_1 \cup f_{23}$.

(10) For every function f_1 from A_1 into B and for every function f_2 from A_2 into B such that $f_1 \restriction (A_1 \cap A_2) = f_2 \restriction (A_1 \cap A_2)$ holds A_1 is a subset

of A_2 if and only if $f_1 \cup f_2 = f_2$ but A_2 is a subset of A_1 if and only if $f_1 \cup f_2 = f_1$.

2. Selected Properties of Subspaces of Topological Spaces

In the sequel X is a topological space. Next we state four propositions:

(11) For every subspace X_0 of X holds the topological structure of X_0 is a strict subspace of X.

(12) For all topological spaces X_1, X_2 such that $X_1 =$ the topological structure of X_2 holds X_1 is a subspace of X if and only if X_2 is a subspace of X.

(13) For all topological spaces X_1, X_2 such that $X_2 =$ the topological structure of X_1 holds X_1 is a closed subspace of X if and only if X_2 is a closed subspace of X.

(14) For all topological spaces X_1, X_2 such that $X_2 =$ the topological structure of X_1 holds X_1 is an open subspace of X if and only if X_2 is an open subspace of X.

In the sequel X_1, X_2 will denote subspaces of X. Next we state several propositions:

(15) If X_1 is a subspace of X_2, then for every point x_1 of X_1 there exists a point x_2 of X_2 such that $x_2 = x_1$.

(16) For every point x of $X_1 \cup X_2$ holds there exists a point x_1 of X_1 such that $x_1 = x$ or there exists a point x_2 of X_2 such that $x_2 = x$.

(17) If X_1 meets X_2, then for every point x of $X_1 \cap X_2$ holds there exists a point x_1 of X_1 such that $x_1 = x$ and there exists a point x_2 of X_2 such that $x_2 = x$.

(18) For every point x of $X_1 \cup X_2$ and for every subset F_1 of X_1 and for every subset F_2 of X_2 such that F_1 is closed and $x \in F_1$ and F_2 is closed and $x \in F_2$ there exists a subset H of $X_1 \cup X_2$ such that H is closed and $x \in H$ and $H \subseteq F_1 \cup F_2$.

(19) For every point x of $X_1 \cup X_2$ and for every subset U_1 of X_1 and for every subset U_2 of X_2 such that U_1 is open and $x \in U_1$ and U_2 is open and $x \in U_2$ there exists a subset V of $X_1 \cup X_2$ such that V is open and $x \in V$ and $V \subseteq U_1 \cup U_2$.

(20) For every point x of $X_1 \cup X_2$ and for every point x_1 of X_1 and for every point x_2 of X_2 such that $x_1 = x$ and $x_2 = x$ and for every neighbourhood A_1 of x_1 and for every neighbourhood A_2 of x_2 there exists a subset V of $X_1 \cup X_2$ such that V is open and $x \in V$ and $V \subseteq A_1 \cup A_2$.

(21) For every point x of $X_1 \cup X_2$ and for every point x_1 of X_1 and for every point x_2 of X_2 such that $x_1 = x$ and $x_2 = x$ and for every neighbourhood A_1 of x_1 and for every neighbourhood A_2 of x_2 there exists a neighbourhood A of x such that $A \subseteq A_1 \cup A_2$.

In the sequel X_0, X_1, X_2, Y_1, Y_2 will be subspaces of X. One can prove the following propositions:

(22) If X_0 is a subspace of X_1, then X_0 meets X_1 and X_1 meets X_0.

(23) If X_0 is a subspace of X_1 but X_0 meets X_2 or X_2 meets X_0, then X_1 meets X_2 and X_2 meets X_1.

(24) If X_0 is a subspace of X_1 but X_1 misses X_2 or X_2 misses X_1, then X_0 misses X_2 and X_2 misses X_0.

(25) $X_0 \cup X_0 =$ the topological structure of X_0.

(26) $X_0 \cap X_0 =$ the topological structure of X_0.

(27) If Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2, then $Y_1 \cup Y_2$ is a subspace of $X_1 \cup X_2$.

(28) If Y_1 meets Y_2 and Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2, then $Y_1 \cap Y_2$ is a subspace of $X_1 \cap X_2$.

(29) If X_1 is a subspace of X_0 and X_2 is a subspace of X_0, then $X_1 \cup X_2$ is a subspace of X_0.

(30) If X_1 meets X_2 and X_1 is a subspace of X_0 and X_2 is a subspace of X_0, then $X_1 \cap X_2$ is a subspace of X_0.

(31) (i) If X_1 misses X_0 or X_0 misses X_1 but X_2 meets X_0 or X_0 meets X_2, then $(X_1 \cup X_2) \cap X_0 = X_2 \cap X_0$ and $X_0 \cap (X_1 \cup X_2) = X_0 \cap X_2$,

(ii) if X_1 meets X_0 or X_0 meets X_1 but X_2 misses X_0 or X_0 misses X_2, then $(X_1 \cup X_2) \cap X_0 = X_1 \cap X_0$ and $X_0 \cap (X_1 \cup X_2) = X_0 \cap X_1$.

(32) If X_1 meets X_2, then if X_1 is a subspace of X_0, then $X_1 \cap X_2$ is a subspace of $X_0 \cap X_2$ but if X_2 is a subspace of X_0, then $X_1 \cap X_2$ is a subspace of $X_1 \cap X_0$.

(33) If X_1 is a subspace of X_0 but X_0 misses X_2 or X_2 misses X_0, then $X_0 \cap (X_1 \cup X_2) =$ the topological structure of X_1 and $X_0 \cap (X_2 \cup X_1) =$ the topological structure of X_1.

(34) If X_1 meets X_2, then if X_1 is a subspace of X_0, then $X_0 \cap X_2$ meets X_1 and $X_2 \cap X_0$ meets X_1 but if X_2 is a subspace of X_0, then $X_1 \cap X_0$ meets X_2 and $X_0 \cap X_1$ meets X_2.

(35) If X_1 is a subspace of Y_1 and X_2 is a subspace of Y_2 but Y_1 misses Y_2 or $Y_1 \cap Y_2$ misses $X_1 \cup X_2$, then Y_1 misses X_2 and Y_2 misses X_1.

(36) Suppose X_1 is not a subspace of X_2 and X_2 is not a subspace of X_1 and $X_1 \cup X_2$ is a subspace of $Y_1 \cup Y_2$ and $Y_1 \cap (X_1 \cup X_2)$ is a subspace of X_1 and $Y_2 \cap (X_1 \cup X_2)$ is a subspace of X_2. Then Y_1 meets $X_1 \cup X_2$ and Y_2 meets $X_1 \cup X_2$.

(37) Suppose that

(i) X_1 meets X_2,

(ii) X_1 is not a subspace of X_2,

(iii) X_2 is not a subspace of X_1,

(iv) the topological structure of $X = Y_1 \cup Y_2 \cup X_0$,

(v) $Y_1 \cap (X_1 \cup X_2)$ is a subspace of X_1,

(vi) $Y_2 \cap (X_1 \cup X_2)$ is a subspace of X_2,
(vii) $X_0 \cap (X_1 \cup X_2)$ is a subspace of $X_1 \cap X_2$.
Then Y_1 meets $X_1 \cup X_2$ and Y_2 meets $X_1 \cup X_2$.

(38) Suppose that
(i) X_1 meets X_2,
(ii) X_1 is not a subspace of X_2,
(iii) X_2 is not a subspace of X_1,
(iv) $X_1 \cup X_2$ is not a subspace of $Y_1 \cup Y_2$,
(v) the topological structure of $X = Y_1 \cup Y_2 \cup X_0$,
(vi) $Y_1 \cap (X_1 \cup X_2)$ is a subspace of X_1,
(vii) $Y_2 \cap (X_1 \cup X_2)$ is a subspace of X_2,
(viii) $X_0 \cap (X_1 \cup X_2)$ is a subspace of $X_1 \cap X_2$.
Then $Y_1 \cup Y_2$ meets $X_1 \cup X_2$ and X_0 meets $X_1 \cup X_2$.

(39) $X_1 \cup X_2$ meets X_0 if and only if X_1 meets X_0 or X_2 meets X_0 but X_0 meets $X_1 \cup X_2$ if and only if X_0 meets X_1 or X_0 meets X_2.

(40) $X_1 \cup X_2$ misses X_0 if and only if X_1 misses X_0 and X_2 misses X_0 but X_0 misses $X_1 \cup X_2$ if and only if X_0 misses X_1 and X_0 misses X_2.

(41) If X_1 meets X_2, then if $X_1 \cap X_2$ meets X_0, then X_1 meets X_0 and X_2 meets X_0 but if X_0 meets $X_1 \cap X_2$, then X_0 meets X_1 and X_0 meets X_2.

(42) If X_1 meets X_2, then if X_1 misses X_0 or X_2 misses X_0, then $X_1 \cap X_2$ misses X_0 but if X_0 misses X_1 or X_0 misses X_2, then X_0 misses $X_1 \cap X_2$.

(43) For every closed subspace X_0 of X such that X_0 meets X_1 holds $X_0 \cap X_1$ is a closed subspace of X_1.

(44) For every open subspace X_0 of X such that X_0 meets X_1 holds $X_0 \cap X_1$ is an open subspace of X_1.

(45) For every closed subspace X_0 of X such that X_1 is a subspace of X_0 and X_0 misses X_2 holds X_1 is a closed subspace of $X_1 \cup X_2$ and X_1 is a closed subspace of $X_2 \cup X_1$.

(46) For every open subspace X_0 of X such that X_1 is a subspace of X_0 and X_0 misses X_2 holds X_1 is an open subspace of $X_1 \cup X_2$ and X_1 is an open subspace of $X_2 \cup X_1$.

3. Continuity of Mappings

We now define two new constructions. Let X, Y be topological spaces. A mapping from X into Y is a function from the carrier of X into the carrier of Y.

We say that f is continuous at x if and only if:

(Def.2) for every neighbourhood G of $f(x)$ there exists a neighbourhood H of x such that $f^\circ H \subseteq G$.

In the sequel X, Y denote topological spaces and f denotes a mapping from X into Y. One can prove the following propositions:

(47) For every point x of X holds f is continuous at x if and only if for every neighbourhood G of $f(x)$ holds $f^{-1}G$ is a neighbourhood of x.

(48) For every point x of X holds f is continuous at x if and only if for every subset G of Y such that G is open and $f(x) \in G$ there exists a subset H of X such that H is open and $x \in H$ and $f^{\circ} H \subseteq G$.

(49) f is continuous if and only if for every point x of X holds f is continuous at x.

(50) For all topological spaces X, Y, Z such that the carrier of Y = the carrier of Z and the topology of $Z \subseteq$ the topology of Y and for every mapping f from X into Y and for every mapping g from X into Z such that $f = g$ and for every point x of X such that f is continuous at x holds g is continuous at x.

(51) Let X, Y, Z be topological spaces. Then if the carrier of X = the carrier of Y and the topology of $Y \subseteq$ the topology of X, then for every mapping f from X into Z and for every mapping g from Y into Z such that $f = g$ and for every point x of X and for every point y of Y such that $x = y$ holds if g is continuous at y, then f is continuous at x.

Let X, Y, Z be topological spaces, and let f be a mapping from X into Y, and let g be a mapping from Y into Z. Then $g \cdot f$ is a mapping from X into Z.

We follow a convention: X, Y, Z are topological spaces, f is a mapping from X into Y, and g is a mapping from Y into Z. The following propositions are true:

(52) For every point x of X and for every point y of Y such that $y = f(x)$ holds if f is continuous at x and g is continuous at y, then $g \cdot f$ is continuous at x.

(53) For every point y of Y such that f is continuous and g is continuous at y and for every point x of X such that $x \in f^{-1}\{y\}$ holds $g \cdot f$ is continuous at x.

(54) For every point x of X such that f is continuous at x and g is continuous holds $g \cdot f$ is continuous at x.

Let X, Y be topological spaces. We introduce continuous mapping from X into Y as a synonym of continuous map from X into Y.

The following propositions are true:

(55) f is a continuous mapping from X into Y if and only if for every point x of X holds f is continuous at x.

(56) For all topological spaces X, Y, Z such that the carrier of Y = the carrier of Z and the topology of $Z \subseteq$ the topology of Y every continuous mapping from X into Y is a continuous mapping from X into Z.

(57) For all topological spaces X, Y, Z such that the carrier of X = the carrier of Y and the topology of $Y \subseteq$ the topology of X every continuous mapping from Y into Z is a continuous mapping from X into Z.

Let X, Y be topological spaces, and let X_0 be a subspace of X, and let f be a mapping from X into Y. The functor $f \restriction X_0$ yielding a mapping from X_0 into Y is defined by:

(Def.3) $f \restriction X_0 = f \restriction$ the carrier of X_0.

In the sequel X, Y will denote topological spaces, X_0 will denote a subspace of X, and f will denote a mapping from X into Y. The following propositions are true:

(58) For every point x of X such that $x \in$ the carrier of X_0 holds $f(x) = (f \restriction X_0)(x)$.

(59) For every mapping f_0 from X_0 into Y such that for every point x of X such that $x \in$ the carrier of X_0 holds $f(x) = f_0(x)$ holds $f \restriction X_0 = f_0$.

(60) If the topological structure of $X_0 =$ the topological structure of X, then $f = f \restriction X_0$.

(61) For every subset A of X such that $A \subseteq$ the carrier of X_0 holds $f\,^\circ\, A = (f \restriction X_0)\,^\circ\, A$.

(62) For every subset B of Y such that $f^{-1}\, B \subseteq$ the carrier of X_0 holds $f^{-1}\, B = (f \restriction X_0)^{-1}\, B$.

(63) For every mapping g from X_0 into Y there exists a mapping h from X into Y such that $h \restriction X_0 = g$.

In the sequel f is a mapping from X into Y and X_0 is a subspace of X. Next we state several propositions:

(64) For every point x of X and for every point x_0 of X_0 such that $x = x_0$ holds if f is continuous at x, then $f \restriction X_0$ is continuous at x_0.

(65) For every subset A of X and for every point x of X and for every point x_0 of X_0 such that $A \subseteq$ the carrier of X_0 and A is a neighbourhood of x and $x = x_0$ holds f is continuous at x if and only if $f \restriction X_0$ is continuous at x_0.

(66) For every subset A of X and for every point x of X and for every point x_0 of X_0 such that A is open and $x \in A$ and $A \subseteq$ the carrier of X_0 and $x = x_0$ holds f is continuous at x if and only if $f \restriction X_0$ is continuous at x_0.

(67) For every open subspace X_0 of X and for every point x of X and for every point x_0 of X_0 such that $x = x_0$ holds f is continuous at x if and only if $f \restriction X_0$ is continuous at x_0.

(68) For every continuous mapping f from X into Y and for every subspace X_0 of X holds $f \restriction X_0$ is a continuous mapping from X_0 into Y.

(69) For all topological spaces X, Y, Z and for every subspace X_0 of X and for every mapping f from X into Y and for every mapping g from Y into Z holds $(g \cdot f) \restriction X_0 = g \cdot (f \restriction X_0)$.

(70) For all topological spaces X, Y, Z and for every subspace X_0 of X and for every mapping g from Y into Z and for every mapping f from X into Y such that g is continuous and $f \restriction X_0$ is continuous holds $(g \cdot f) \restriction X_0$ is continuous.

(71) For all topological spaces X, Y, Z and for every subspace X_0 of X and for every continuous mapping g from Y into Z and for every mapping f from X into Y such that $f \upharpoonright X_0$ is a continuous mapping from X_0 into Y holds $(g \cdot f) \upharpoonright X_0$ is a continuous mapping from X_0 into Z.

Let X, Y be topological spaces, and let X_0, X_1 be subspaces of X, and let g be a mapping from X_0 into Y. Let us assume that X_1 is a subspace of X_0. The functor $g \upharpoonright X_1$ yielding a mapping from X_1 into Y is defined as follows:

(Def.4) $g \upharpoonright X_1 = g \upharpoonright$ the carrier of X_1.

For simplicity we follow a convention: X, Y denote topological spaces, X_0, X_1 denote subspaces of X, f denotes a mapping from X into Y, and g denotes a mapping from X_0 into Y. The following propositions are true:

(72) If X_1 is a subspace of X_0, then for every point x_0 of X_0 such that $x_0 \in$ the carrier of X_1 holds $g(x_0) = (g \upharpoonright X_1)(x_0)$.

(73) If X_1 is a subspace of X_0, then for every mapping g_1 from X_1 into Y such that for every point x_0 of X_0 such that $x_0 \in$ the carrier of X_1 holds $g(x_0) = g_1(x_0)$ holds $g \upharpoonright X_1 = g_1$.

(74) $g = g \upharpoonright X_0$.

(75) If X_1 is a subspace of X_0, then for every subset A of X_0 such that $A \subseteq$ the carrier of X_1 holds $g \circ A = (g \upharpoonright X_1) \circ A$.

(76) If X_1 is a subspace of X_0, then for every subset B of Y such that $g^{-1} B \subseteq$ the carrier of X_1 holds $g^{-1} B = (g \upharpoonright X_1)^{-1} B$.

(77) For every mapping g from X_0 into Y such that $g = f \upharpoonright X_0$ holds if X_1 is a subspace of X_0, then $g \upharpoonright X_1 = f \upharpoonright X_1$.

(78) If X_1 is a subspace of X_0, then $f \upharpoonright X_0 \upharpoonright X_1 = f \upharpoonright X_1$.

(79) For all subspaces X_0, X_1, X_2 of X such that X_1 is a subspace of X_0 and X_2 is a subspace of X_1 and for every mapping g from X_0 into Y holds $g \upharpoonright X_1 \upharpoonright X_2 = g \upharpoonright X_2$.

(80) For every mapping f from X into Y and for every mapping f_0 from X_1 into Y and for every mapping g from X_0 into Y such that $X_0 = X$ and $f = g$ holds $g \upharpoonright X_1 = f_0$ if and only if $f \upharpoonright X_1 = f_0$.

We follow the rules: X_0, X_1, X_2 are subspaces of X, f is a mapping from X into Y, and g is a mapping from X_0 into Y. One can prove the following propositions:

(81) For every point x_0 of X_0 and for every point x_1 of X_1 such that $x_0 = x_1$ holds if X_1 is a subspace of X_0 and g is continuous at x_0, then $g \upharpoonright X_1$ is continuous at x_1.

(82) If X_1 is a subspace of X_0, then for every point x_0 of X_0 and for every point x_1 of X_1 such that $x_0 = x_1$ holds if $f \upharpoonright X_0$ is continuous at x_0, then $f \upharpoonright X_1$ is continuous at x_1.

(83) If X_1 is a subspace of X_0, then for every subset A of X_0 and for every point x_0 of X_0 and for every point x_1 of X_1 such that $A \subseteq$ the carrier of

X_1 and A is a neighbourhood of x_0 and $x_0 = x_1$ holds g is continuous at x_0 if and only if $g \upharpoonright X_1$ is continuous at x_1.

(84) If X_1 is a subspace of X_0, then for every subset A of X_0 and for every point x_0 of X_0 and for every point x_1 of X_1 such that A is open and $x_0 \in A$ and $A \subseteq$ the carrier of X_1 and $x_0 = x_1$ holds g is continuous at x_0 if and only if $g \upharpoonright X_1$ is continuous at x_1.

(85) If X_1 is a subspace of X_0, then for every subset A of X and for every point x_0 of X_0 and for every point x_1 of X_1 such that A is open and $x_0 \in A$ and $A \subseteq$ the carrier of X_1 and $x_0 = x_1$ holds g is continuous at x_0 if and only if $g \upharpoonright X_1$ is continuous at x_1.

(86) If X_1 is an open subspace of X_0, then for every point x_0 of X_0 and for every point x_1 of X_1 such that $x_0 = x_1$ holds g is continuous at x_0 if and only if $g \upharpoonright X_1$ is continuous at x_1.

(87) If X_1 is an open subspace of X and X_1 is a subspace of X_0, then for every point x_0 of X_0 and for every point x_1 of X_1 such that $x_0 = x_1$ holds g is continuous at x_0 if and only if $g \upharpoonright X_1$ is continuous at x_1.

(88) If the topological structure of $X_1 = X_0$, then for every point x_0 of X_0 and for every point x_1 of X_1 such that $x_0 = x_1$ holds if $g \upharpoonright X_1$ is continuous at x_1, then g is continuous at x_0.

(89) For every continuous mapping g from X_0 into Y such that X_1 is a subspace of X_0 holds $g \upharpoonright X_1$ is a continuous mapping from X_1 into Y.

(90) If X_1 is a subspace of X_0 and X_2 is a subspace of X_1, then for every mapping g from X_0 into Y such that $g \upharpoonright X_1$ is a continuous mapping from X_1 into Y holds $g \upharpoonright X_2$ is a continuous mapping from X_2 into Y.

Let X be a topological space. The functor id_X yielding a mapping from X into X is defined as follows:

(Def.5) $\mathrm{id}_X = \mathrm{id}_{(\text{the carrier of } X)}$.

One can prove the following four propositions:

(91) For every point x of X holds $\mathrm{id}_X(x) = x$.

(92) For every mapping f from X into X such that for every point x of X holds $f(x) = x$ holds $f = \mathrm{id}_X$.

(93) For every mapping f from X into Y holds $f \cdot \mathrm{id}_X = f$ and $\mathrm{id}_Y \cdot f = f$.

(94) id_X is a continuous mapping from X into X.

We now define two new functors. Let X be a topological space, and let X_0 be a subspace of X. The functor $\overset{X_0}{\hookrightarrow}$ yielding a mapping from X_0 into X is defined by:

(Def.6) $\overset{X_0}{\hookrightarrow} = \mathrm{id}_X \upharpoonright X_0$.

We introduce the functor $X_0 \hookrightarrow X$ as a synonym of $\overset{X_0}{\hookrightarrow}$.

Next we state four propositions:

(95) For every subspace X_0 of X and for every point x of X such that $x \in$ the carrier of X_0 holds $(\overset{X_0}{\hookrightarrow})(x) = x$.

(96) For every subspace X_0 of X and for every mapping f_0 from X_0 into X such that for every point x of X such that $x \in$ the carrier of X_0 holds $x = f_0(x)$ holds $\overset{X_0}{\hookrightarrow} = f_0$.

(97) For every subspace X_0 of X and for every mapping f from X into Y holds $f \restriction X_0 = f \cdot (\overset{X_0}{\hookrightarrow})$.

(98) For every subspace X_0 of X holds $\overset{X_0}{\hookrightarrow}$ is a continuous mapping from X_0 into X.

4. A Modification of the Topology of Topological Spaces

In the sequel X will denote a topological space and H, G will denote subsets of X. Let us consider X, and let A be a subset of X. The A-extension of the topology of X yielding a family of subsets of X is defined as follows:

(Def.7) the A-extension of the topology of $X = \{H \cup G \cap A : H \in$ the topology of $X \wedge G \in$ the topology of $X\}$.

We now state several propositions:

(99) For every subset A of X holds the topology of $X \subseteq$ the A-extension of the topology of X.

(100) For every subset A of X holds $\{G \cap A : G \in$ the topology of $X\} \subseteq$ the A-extension of the topology of X, where G ranges over subsets of X.

(101) For every subset A of X and for all subsets C, D of X such that $C \in$ the topology of X and $D \in \{G \cap A : G \in$ the topology of $X\}$, where G ranges over subsets of X holds $C \cup D \in$ the A-extension of the topology of X and $C \cap D \in$ the A-extension of the topology of X.

(102) For every subset A of X holds $A \in$ the A-extension of the topology of X.

(103) For every subset A of X holds $A \in$ the topology of X if and only if the topology of $X =$ the A-extension of the topology of X.

Let X be a topological space, and let A be a subset of X. The X modified w.r.t. A yields a strict topological space and is defined by:

(Def.8) the X modified w.r.t. $A = \langle$the carrier of X, the A-extension of the topology of $X\rangle$.

In the sequel A will be a subset of X. The following three propositions are true:

(104) The carrier of the X modified w.r.t. $A =$ the carrier of X and the topology of the X modified w.r.t. $A =$ the A-extension of the topology of X.

(105) For every subset B of the X modified w.r.t. A such that $B = A$ holds B is open.

(106) A is open if and only if the topological structure of $X =$ the X modified w.r.t. A.

Let X be a topological space, and let A be a subset of X. The functor $\mathrm{modid}_{X,A}$ yields a mapping from X into the X modified w.r.t. A and is defined as follows:

(Def.9) $\mathrm{modid}_{X,A} = \mathrm{id}_{(\text{the carrier of } X)}$.

We now state several propositions:

(107) If A is open, then $\mathrm{modid}_{X,A} = \mathrm{id}_X$.

(108) For every point x of X such that $x \notin A$ holds $\mathrm{modid}_{X,A}$ is continuous at x.

(109) For every subspace X_0 of X such that (the carrier of X_0) $\cap A = \emptyset$ and for every point x_0 of X_0 holds $\mathrm{modid}_{X,A} \restriction X_0$ is continuous at x_0.

(110) For every subspace X_0 of X such that the carrier of $X_0 = A$ and for every point x_0 of X_0 holds $\mathrm{modid}_{X,A} \restriction X_0$ is continuous at x_0.

(111) For every subspace X_0 of X such that (the carrier of X_0) $\cap A = \emptyset$ holds $\mathrm{modid}_{X,A} \restriction X_0$ is a continuous mapping from X_0 into the X modified w.r.t. A.

(112) For every subspace X_0 of X such that the carrier of $X_0 = A$ holds $\mathrm{modid}_{X,A} \restriction X_0$ is a continuous mapping from X_0 into the X modified w.r.t. A.

(113) For every subset A of X holds A is open if and only if $\mathrm{modid}_{X,A}$ is a continuous mapping from X into the X modified w.r.t. A.

Let X be a topological space, and let X_0 be a subspace of X. The X modified w.r.t. X_0 yielding a strict topological space is defined as follows:

(Def.10) for every subset A of X such that $A =$ the carrier of X_0 holds the X modified w.r.t. $X_0 =$ the X modified w.r.t. A.

In the sequel X_0 will denote a subspace of X. The following three propositions are true:

(114) The carrier of the X modified w.r.t. $X_0 =$ the carrier of X and for every subset A of X such that $A =$ the carrier of X_0 holds the topology of the X modified w.r.t. $X_0 =$ the A-extension of the topology of X.

(115) For every subspace Y_0 of the X modified w.r.t. X_0 such that the carrier of $Y_0 =$ the carrier of X_0 holds Y_0 is an open subspace of the X modified w.r.t. X_0.

(116) X_0 is an open subspace of X if and only if the topological structure of $X =$ the X modified w.r.t. X_0.

Let X be a topological space, and let X_0 be a subspace of X. The functor modid_{X,X_0} yielding a mapping from X into the X modified w.r.t. X_0 is defined as follows:

(Def.11) for every subset A of X such that $A =$ the carrier of X_0 holds $\mathrm{modid}_{X,X_0} = \mathrm{modid}_{X,A}$.

We now state several propositions:

(117) If X_0 is an open subspace of X, then $\mathrm{modid}_{X,X_0} = \mathrm{id}_X$.

(118) For all subspaces X_0, X_1 of X such that X_0 misses X_1 and for every point x_1 of X_1 holds $\mathrm{modid}_{X,X_0} \restriction X_1$ is continuous at x_1.

(119) For every subspace X_0 of X and for every point x_0 of X_0 holds $\mathrm{modid}_{X,X_0} \restriction X_0$
is continuous at x_0.

(120) For all subspaces X_0, X_1 of X such that X_0 misses X_1 holds $\mathrm{modid}_{X,X_0} \restriction X_1$ is a continuous mapping from X_1 into the X modified w.r.t. X_0.

(121) For every subspace X_0 of X holds $\mathrm{modid}_{X,X_0} \restriction X_0$ is a continuous mapping from X_0 into the X modified w.r.t. X_0.

(122) For every subspace X_0 of X holds X_0 is an open subspace of X if and only if modid_{X,X_0} is a continuous mapping from X into the X modified w.r.t. X_0.

5. Continuity of Mappings over the Union of Subspaces

In the sequel X, Y denote topological spaces. We now state three propositions:

(123) For all subspaces X_1, X_2 of X and for every mapping g from $X_1 \cup X_2$ into Y and for every point x_1 of X_1 and for every point x_2 of X_2 and for every point x of $X_1 \cup X_2$ such that $x = x_1$ and $x = x_2$ holds g is continuous at x if and only if $g \restriction X_1$ is continuous at x_1 and $g \restriction X_2$ is continuous at x_2.

(124) Let f be a mapping from X into Y. Then for all subspaces X_1, X_2 of X and for every point x of $X_1 \cup X_2$ and for every point x_1 of X_1 and for every point x_2 of X_2 such that $x = x_1$ and $x = x_2$ holds $f \restriction (X_1 \cup X_2)$ is continuous at x if and only if $f \restriction X_1$ is continuous at x_1 and $f \restriction X_2$ is continuous at x_2.

(125) Let f be a mapping from X into Y. Then for all subspaces X_1, X_2 of X such that $X = X_1 \cup X_2$ and for every point x of X and for every point x_1 of X_1 and for every point x_2 of X_2 such that $x = x_1$ and $x = x_2$ holds f is continuous at x if and only if $f \restriction X_1$ is continuous at x_1 and $f \restriction X_2$ is continuous at x_2.

In the sequel X_1, X_2 will denote subspaces of X. One can prove the following propositions:

(126) If X_1 and X_2 are weakly separated, then for every mapping g from $X_1 \cup X_2$ into Y holds g is a continuous mapping from $X_1 \cup X_2$ into Y if and only if $g \restriction X_1$ is a continuous mapping from X_1 into Y and $g \restriction X_2$ is a continuous mapping from X_2 into Y.

(127) For all closed subspaces X_1, X_2 of X and for every mapping g from $X_1 \cup X_2$ into Y holds g is a continuous mapping from $X_1 \cup X_2$ into Y if and only if $g \restriction X_1$ is a continuous mapping from X_1 into Y and $g \restriction X_2$ is a continuous mapping from X_2 into Y.

(128) For all open subspaces X_1, X_2 of X and for every mapping g from $X_1 \cup X_2$ into Y holds g is a continuous mapping from $X_1 \cup X_2$ into Y if and only if $g \restriction X_1$ is a continuous mapping from X_1 into Y and $g \restriction X_2$ is a continuous mapping from X_2 into Y.

(129) If X_1 and X_2 are weakly separated, then for every mapping f from X into Y holds $f \restriction (X_1 \cup X_2)$ is a continuous mapping from $X_1 \cup X_2$ into Y if and only if $f \restriction X_1$ is a continuous mapping from X_1 into Y and $f \restriction X_2$ is a continuous mapping from X_2 into Y.

(130) For every mapping f from X into Y and for all closed subspaces X_1, X_2 of X holds $f \restriction (X_1 \cup X_2)$ is a continuous mapping from $X_1 \cup X_2$ into Y if and only if $f \restriction X_1$ is a continuous mapping from X_1 into Y and $f \restriction X_2$ is a continuous mapping from X_2 into Y.

(131) For every mapping f from X into Y and for all open subspaces X_1, X_2 of X holds $f \restriction (X_1 \cup X_2)$ is a continuous mapping from $X_1 \cup X_2$ into Y if and only if $f \restriction X_1$ is a continuous mapping from X_1 into Y and $f \restriction X_2$ is a continuous mapping from X_2 into Y.

(132) For every mapping f from X into Y and for all subspaces X_1, X_2 of X such that $X = X_1 \cup X_2$ and X_1 and X_2 are weakly separated holds f is a continuous mapping from X into Y if and only if $f \restriction X_1$ is a continuous mapping from X_1 into Y and $f \restriction X_2$ is a continuous mapping from X_2 into Y.

(133) For every mapping f from X into Y and for all closed subspaces X_1, X_2 of X such that $X = X_1 \cup X_2$ holds f is a continuous mapping from X into Y if and only if $f \restriction X_1$ is a continuous mapping from X_1 into Y and $f \restriction X_2$ is a continuous mapping from X_2 into Y.

(134) For every mapping f from X into Y and for all open subspaces X_1, X_2 of X such that $X = X_1 \cup X_2$ holds f is a continuous mapping from X into Y if and only if $f \restriction X_1$ is a continuous mapping from X_1 into Y and $f \restriction X_2$ is a continuous mapping from X_2 into Y.

(135) X_1 and X_2 are separated if and only if X_1 misses X_2 and for every topological space Y and for every mapping g from $X_1 \cup X_2$ into Y such that $g \restriction X_1$ is a continuous mapping from X_1 into Y and $g \restriction X_2$ is a continuous mapping from X_2 into Y holds g is a continuous mapping from $X_1 \cup X_2$ into Y.

(136) X_1 and X_2 are separated if and only if X_1 misses X_2 and for every topological space Y and for every mapping f from X into Y such that $f \restriction X_1$ is a continuous mapping from X_1 into Y and $f \restriction X_2$ is a continuous mapping from X_2 into Y holds $f \restriction (X_1 \cup X_2)$ is a continuous mapping from $X_1 \cup X_2$ into Y.

(137) For all subspaces X_1, X_2 of X such that $X = X_1 \cup X_2$ holds X_1 and X_2 are separated if and only if X_1 misses X_2 and for every topological space Y and for every mapping f from X into Y such that $f \restriction X_1$ is a continuous mapping from X_1 into Y and $f \restriction X_2$ is a continuous mapping

from X_2 into Y holds f is a continuous mapping from X into Y.

6. The Union of Continuous Mappings

Let X, Y be topological spaces, and let X_1, X_2 be subspaces of X, and let f_1 be a mapping from X_1 into Y, and let f_2 be a mapping from X_2 into Y. Let us assume that X_1 misses X_2 or $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$. The functor $f_1 \cup f_2$ yielding a mapping from $X_1 \cup X_2$ into Y is defined as follows:

(Def.12) $(f_1 \cup f_2) \restriction X_1 = f_1$ and $(f_1 \cup f_2) \restriction X_2 = f_2$.

In the sequel X, Y will denote topological spaces. We now state a number of propositions:

(138) For all subspaces X_1, X_2 of X and for every mapping g from $X_1 \cup X_2$ into Y holds $g = g \restriction X_1 \cup g \restriction X_2$.

(139) For all subspaces X_1, X_2 of X such that $X = X_1 \cup X_2$ and for every mapping g from X into Y holds $g = g \restriction X_1 \cup g \restriction X_2$.

(140) For all subspaces X_1, X_2 of X such that X_1 meets X_2 and for every mapping f_1 from X_1 into Y and for every mapping f_2 from X_2 into Y holds $(f_1 \cup f_2) \restriction X_1 = f_1$ and $(f_1 \cup f_2) \restriction X_2 = f_2$ if and only if $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$.

(141) For all subspaces X_1, X_2 of X and for every mapping f_1 from X_1 into Y and for every mapping f_2 from X_2 into Y such that $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$ holds X_1 is a subspace of X_2 if and only if $f_1 \cup f_2 = f_2$ but X_2 is a subspace of X_1 if and only if $f_1 \cup f_2 = f_1$.

(142) For all subspaces X_1, X_2 of X and for every mapping f_1 from X_1 into Y and for every mapping f_2 from X_2 into Y such that X_1 misses X_2 or $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$ holds $f_1 \cup f_2 = f_2 \cup f_1$.

(143) Let X_1, X_2, X_3 be subspaces of X. Let f_1 be a mapping from X_1 into Y. Let f_2 be a mapping from X_2 into Y. Let f_3 be a mapping from X_3 into Y. Suppose X_1 misses X_2 or $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$ but X_1 misses X_3 or $f_1 \restriction (X_1 \cap X_3) = f_3 \restriction (X_1 \cap X_3)$ but X_2 misses X_3 or $f_2 \restriction (X_2 \cap X_3) = f_3 \restriction (X_2 \cap X_3)$. Then $(f_1 \cup f_2) \cup f_3 = f_1 \cup (f_2 \cup f_3)$.

(144) For all subspaces X_1, X_2 of X such that X_1 meets X_2 and for every continuous mapping f_1 from X_1 into Y and for every continuous mapping f_2 from X_2 into Y such that $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$ holds if X_1 and X_2 are weakly separated, then $f_1 \cup f_2$ is a continuous mapping from $X_1 \cup X_2$ into Y.

(145) For all subspaces X_1, X_2 of X such that X_1 misses X_2 and for every continuous mapping f_1 from X_1 into Y and for every continuous mapping f_2 from X_2 into Y such that X_1 and X_2 are weakly separated holds $f_1 \cup f_2$ is a continuous mapping from $X_1 \cup X_2$ into Y.

(146) For all closed subspaces X_1, X_2 of X such that X_1 meets X_2 and for every continuous mapping f_1 from X_1 into Y and for every continuous

mapping f_2 from X_2 into Y such that $f_1 \upharpoonright (X_1 \cap X_2) = f_2 \upharpoonright (X_1 \cap X_2)$ holds $f_1 \cup f_2$ is a continuous mapping from $X_1 \cup X_2$ into Y.

(147) For all open subspaces X_1, X_2 of X such that X_1 meets X_2 and for every continuous mapping f_1 from X_1 into Y and for every continuous mapping f_2 from X_2 into Y such that $f_1 \upharpoonright (X_1 \cap X_2) = f_2 \upharpoonright (X_1 \cap X_2)$ holds $f_1 \cup f_2$ is a continuous mapping from $X_1 \cup X_2$ into Y.

(148) For all closed subspaces X_1, X_2 of X such that X_1 misses X_2 and for every continuous mapping f_1 from X_1 into Y and for every continuous mapping f_2 from X_2 into Y holds $f_1 \cup f_2$ is a continuous mapping from $X_1 \cup X_2$ into Y.

(149) For all open subspaces X_1, X_2 of X such that X_1 misses X_2 and for every continuous mapping f_1 from X_1 into Y and for every continuous mapping f_2 from X_2 into Y holds $f_1 \cup f_2$ is a continuous mapping from $X_1 \cup X_2$ into Y.

(150) For all subspaces X_1, X_2 of X holds X_1 and X_2 are separated if and only if X_1 misses X_2 and for every topological space Y and for every continuous mapping f_1 from X_1 into Y and for every continuous mapping f_2 from X_2 into Y holds $f_1 \cup f_2$ is a continuous mapping from $X_1 \cup X_2$ into Y.

(151) For all subspaces X_1, X_2 of X such that $X = X_1 \cup X_2$ and for every continuous mapping f_1 from X_1 into Y and for every continuous mapping f_2 from X_2 into Y such that $(f_1 \cup f_2) \upharpoonright X_1 = f_1$ and $(f_1 \cup f_2) \upharpoonright X_2 = f_2$ holds if X_1 and X_2 are weakly separated, then $f_1 \cup f_2$ is a continuous mapping from X into Y.

(152) For all closed subspaces X_1, X_2 of X and for every continuous mapping f_1 from X_1 into Y and for every continuous mapping f_2 from X_2 into Y such that $X = X_1 \cup X_2$ and $(f_1 \cup f_2) \upharpoonright X_1 = f_1$ and $(f_1 \cup f_2) \upharpoonright X_2 = f_2$ holds $f_1 \cup f_2$ is a continuous mapping from X into Y.

(153) For all open subspaces X_1, X_2 of X and for every continuous mapping f_1 from X_1 into Y and for every continuous mapping f_2 from X_2 into Y such that $X = X_1 \cup X_2$ and $(f_1 \cup f_2) \upharpoonright X_1 = f_1$ and $(f_1 \cup f_2) \upharpoonright X_2 = f_2$ holds $f_1 \cup f_2$ is a continuous mapping from X into Y.

Acknowledgments

I would like to express my thanks to Professors A. Trybulec and Cz. Byliński for their active interest in the publication of this article and for elucidating to me new advanced Mizar constructions.

References

[1] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[4] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.

[5] Zbigniew Karno. Separated and weakly separated subspaces of topological spaces. *Formalized Mathematics*, 2(**5**):665–674, 1991.

[6] Kazimierz Kuratowski. *Topology.* Volume I, PWN - Polish Scientific Publishers, Academic Press, Warsaw, New York and London, 1966.

[7] Beata Padlewska. Connected spaces. *Formalized Mathematics*, 1(**1**):239–244, 1990.

[8] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[9] Beata Padlewska. Locally connected spaces. *Formalized Mathematics*, 2(**1**):93–96, 1991.

[10] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[11] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[12] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.

[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[14] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[15] Eduard Čech. *Topological Spaces.* Academia, Publishing House of the Czechoslovak Academy of Sciences, Prague, 1966.

[16] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

Received May 22, 1992