Preface

As was stated in [3] we publish mathematical papers which are abstracts of Mizar articles to be found in the Main Mizar Library (MML). An article includes certain elements which are transferred to the data base, such as theorems or definitions. This has been due to the fact that the material published there was at first intended to help the Mizar users to handle the data base. Thus the works published there describe the present state of MML and are, in a sense, a report on the expansion of that library. Next to them there are also new mathematical papers because the new method of formalization is not trivial even though it refers to simple mathematical facts.

It must be explained at this point that both the PC-Mizar verifier and MML are being systematically developed. In the case of PC-Mizar it is mainly the Mizar language which is enriched, which makes it more convenient to write articles; the same might be said of proof-checker, which enables one to write shorter proofs and articles.

The development of MML consists in continuous revisions of articles accepted for publication, for instance in the removal of self-evident or repeated theorems (while the numbering of successive theorems in a given article is preserved). We then have the information in a footnote such as "The proposition (5) has been removed" (see [1], page 450). Previously such a comment was, e.g., "The proposition (9) was either repeated or obvious" (see [2], page 14).

Please note also that in the articles we use atypical symbolism for the Cartesian product [: :] , and that is no paranthesis in the case of grouping to the left. In the present issue we have changed the format of certain operation. In [4] the functor represented in Mizar by "$\emptyset$.X" (the empty set treated as the finite subset of X) was unfortunately TEXed as 0_X. Now we corrected this and it is TEXed as $\emptyset_X$.

Our periodical appears five times a year, which is to say every two months except for the summer holidays period. The present issue, although dated November-December, also includes items contributed after. They have been included because the editors received them before sending the issue 2(5) to the press.

Roman Matuszewski

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FORMALIZED MATHEMATICS
Vol.2,No.5, November–December 1991
Université Catholique de Louvain

The Topological Space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, Line Segments and Special Polygonal Arcs

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Summary. The notions of arc and line segment are introduced in two-dimensional topological real space $\mathcal{E}^2_{\mathrm{T}}$. Some basic theorems for these notions are proved. Using line segments, the notion of special polygonal arc is defined. It has been shown that any special polygonal arc is homeomorphic to unit interval $\mathbb{I}$. The notion of unit square $\square_{\mathcal{E}^2_{\mathrm{T}}}$ has been also introduced and some facts about it have been proved.

MML Identifier: `TOPREAL1`.

The articles [22], [21], [13], [1], [24], [20], [6], [7], [18], [4], [8], [15], [23], [17], [25], [11], [16], [9], [19], [2], [5], [14], [3], [10], and [12] provide the notation and terminology for this paper. In the sequel l_1 will denote a real number and i, j, n will denote natural numbers. The scheme *Fraenkel_Alt* concerns a non-empty set $\mathcal{A}$, and two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

$\{v : \mathcal{P}[v] \lor \mathcal{Q}[v]\} = \{v_1 : \mathcal{P}[v_1]\} \cup \{v_2 : \mathcal{Q}[v_2]\}$, where v_2 ranges over elements of $\mathcal{A}$, and v_1 ranges over elements of $\mathcal{A}$, and v ranges over elements of $\mathcal{A}$ for all values of the parameters.

In the sequel d_1, d_2, d_3 will be arbitrary. We now state the proposition

(2)[2] $\langle d_1, d_2, d_3\rangle$ is one-to-one if and only if $d_1 \neq d_2$ and $d_2 \neq d_3$ and $d_1 \neq d_3$.

In the sequel D denotes a non-empty set and p denotes a finite sequence of elements of D. Let us consider D, p, n. The functor $p \restriction n$ yielding a finite sequence of elements of D is defined by:

(Def.1) $p \restriction n = p \restriction \mathrm{Seg}\, n$.

One can prove the following proposition

[1]The article was written during my work at Shinshu University, 1991.
[2]The proposition (1) has been removed.

ISSN 0777–4028

(3) If $n \leq \operatorname{len} p$, then $\operatorname{len}(p \restriction n) = n$.

Let us consider T. A finite sequence of elements of T is a finite sequence of elements of the carrier of T.

We adopt the following convention: p, p_1, p_2, q, q_1, q_2 will be points of $\mathcal{E}^2_{\mathrm{T}}$ and P, Q, P_1, P_2 will be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Let us consider p_1, p_2, P. We say that P is an arc from p_1 to p_2 if and only if:

(Def.2) $P \neq \emptyset$ and there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction P$ such that f is a homeomorphism and $f(0) = p_1$ and $f(1) = p_2$.

One can prove the following two propositions:

(4) If P is an arc from p_1 to p_2, then $p_1 \in P$ and $p_2 \in P$.

(5) If P is an arc from p_1 to p_2 and Q is an arc from p_2 to q_1 and $P \cap Q = \{p_2\}$, then $P \cup Q$ is an arc from p_1 to q_1.

The subset $\square_{\mathcal{E}^2}$ of $\mathcal{E}^2_{\mathrm{T}}$ is defined by the condition (Def.3).

(Def.3) $\square_{\mathcal{E}^2} = \{p : p_{\mathbf{1}} = 0 \wedge p_{\mathbf{2}} \leq 1 \wedge p_{\mathbf{2}} \geq 0 \ \vee\ p_{\mathbf{1}} \leq 1 \wedge p_{\mathbf{1}} \geq 0 \wedge p_{\mathbf{2}} = 1 \ \vee\ p_{\mathbf{1}} \leq 1 \wedge p_{\mathbf{1}} \geq 0 \wedge p_{\mathbf{2}} = 0 \ \vee\ p_{\mathbf{1}} = 1 \wedge p_{\mathbf{2}} \leq 1 \wedge p_{\mathbf{2}} \geq 0\}$.

Let us consider p_1, p_2. The functor $\mathcal{L}(p_1, p_2)$ yielding a non-empty subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def.4) $\mathcal{L}(p_1, p_2) = \{p : \bigvee_{l_1} [0 \leq l_1 \wedge l_1 \leq 1 \wedge p = (1 - l_1) \cdot p_1 + l_1 \cdot p_2]\}$.

Next we state a number of propositions:

(6) $p_1 \in \mathcal{L}(p_1, p_2)$ and $p_2 \in \mathcal{L}(p_1, p_2)$.

(7) $\mathcal{L}(p, p) = \{p\}$.

(8) $\mathcal{L}(p_1, p_2) = \mathcal{L}(p_2, p_1)$.

(9) If $p_{1\mathbf{1}} \leq p_{2\mathbf{1}}$ and $p \in \mathcal{L}(p_1, p_2)$, then $p_{1\mathbf{1}} \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq p_{2\mathbf{1}}$.

(10) If $p_{1\mathbf{2}} \leq p_{2\mathbf{2}}$ and $p \in \mathcal{L}(p_1, p_2)$, then $p_{1\mathbf{2}} \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq p_{2\mathbf{2}}$.

(11) If $p \in \mathcal{L}(p_1, p_2)$, then $\mathcal{L}(p_1, p_2) = \mathcal{L}(p_1, p) \cup \mathcal{L}(p, p_2)$.

(12) If $q_1 \in \mathcal{L}(p_1, p_2)$ and $q_2 \in \mathcal{L}(p_1, p_2)$, then $\mathcal{L}(q_1, q_2) \subseteq \mathcal{L}(p_1, p_2)$.

(13) If $p \in \mathcal{L}(p_1, p_2)$ and $q \in \mathcal{L}(p_1, p_2)$, then $\mathcal{L}(p_1, p_2) = \mathcal{L}(p_1, p) \cup \mathcal{L}(p, q) \cup \mathcal{L}(q, p_2)$.

(14) If $p \in \mathcal{L}(p_1, p_2)$, then $\mathcal{L}(p_1, p) \cap \mathcal{L}(p, p_2) = \{p\}$.

(15) If $p_1 \neq p_2$, then $\mathcal{L}(p_1, p_2)$ is an arc from p_1 to p_2.

(16) If P is an arc from p_1 to p_2 and $P \cap \mathcal{L}(p_2, q_1) = \{p_2\}$, then $P \cup \mathcal{L}(p_2, q_1)$ is an arc from p_1 to q_1.

(17) If P is an arc from p_2 to p_1 and $\mathcal{L}(q_1, p_2) \cap P = \{p_2\}$, then $\mathcal{L}(q_1, p_2) \cup P$ is an arc from q_1 to p_1.

(18) If $p_1 \neq p_2$ or $p_2 \neq q_1$ but $\mathcal{L}(p_1, p_2) \cap \mathcal{L}(p_2, q_1) = \{p_2\}$, then $\mathcal{L}(p_1, p_2) \cup \mathcal{L}(p_2, q_1)$ is an arc from p_1 to q_1.

(19) (i) $\mathcal{L}([0,0],[0,1]) = \{p_1 : p_{1\mathbf{1}} = 0 \wedge p_{1\mathbf{2}} \leq 1 \wedge p_{1\mathbf{2}} \geq 0\}$,

(ii) $\mathcal{L}([0,1],[1,1]) = \{p_2 : p_{2\mathbf{1}} \leq 1 \wedge p_{2\mathbf{1}} \geq 0 \wedge p_{2\mathbf{2}} = 1\}$,

(iii) $\mathcal{L}([0,0],[1,0]) = \{q_1 : q_{1\mathbf{1}} \leq 1 \wedge q_{1\mathbf{1}} \geq 0 \wedge q_{1\mathbf{2}} = 0\}$,

(iv) $\mathcal{L}([1,0],[1,1]) = \{q_2 : q_{2\mathbf{1}} = 1 \wedge q_{2\mathbf{2}} \leq 1 \wedge q_{2\mathbf{2}} \geq 0\}$.

(20) $\square_{\mathcal{E}^2} = \mathcal{L}([0,0],[0,1]) \cup \mathcal{L}([0,1],[1,1]) \cup (\mathcal{L}([0,0],[1,0]) \cup \mathcal{L}([1,0],[1,1]))$.

(21) $\mathcal{L}([0,0],[0,1]) \cap \mathcal{L}([0,1],[1,1]) = \{[0,1]\}$.

(22) $\mathcal{L}([0,0],[1,0]) \cap \mathcal{L}([1,0],[1,1]) = \{[1,0]\}$.

(23) $\mathcal{L}([0,0],[0,1]) \cap \mathcal{L}([0,0],[1,0]) = \{[0,0]\}$.

(24) $\mathcal{L}([0,1],[1,1]) \cap \mathcal{L}([1,0],[1,1]) = \{[1,1]\}$.

(25) $\mathcal{L}([0,0],[1,0]) \cap \mathcal{L}([0,1],[1,1]) = \emptyset$.

(26) $\mathcal{L}([0,0],[0,1]) \cap \mathcal{L}([1,0],[1,1]) = \emptyset$.

In the sequel f, f_1, f_2, h will be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Let us consider f, i, j. The functor $\mathcal{L}(f,i,j)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def.5) (i) for all p_1, p_2 such that $p_1 = f(i)$ and $p_2 = f(j)$ holds $\mathcal{L}(f,i,j) = \mathcal{L}(p_1,p_2)$ if $i \in \mathrm{Seg}\,\mathrm{len}\,f$ and $j \in \mathrm{Seg}\,\mathrm{len}\,f$,

(ii) $\mathcal{L}(f,i,j) = \emptyset$, otherwise.

The following proposition is true

(27) If $i \in \mathrm{Seg}\,\mathrm{len}\,f$ and $j \in \mathrm{Seg}\,\mathrm{len}\,f$, then $f(i) \in \mathcal{L}(f,i,j)$ and $f(j) \in \mathcal{L}(f,i,j)$.

Let us consider f. The functor $\widetilde{\mathcal{L}}(f)$ yields a subset of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def.6) $\widetilde{\mathcal{L}}(f) = \bigcup\{\mathcal{L}(f,i,i+1) : 1 \leq i \wedge i \leq \mathrm{len}\,f - 1\}$.

One can prove the following propositions:

(28) $\mathrm{len}\,f = 0$ or $\mathrm{len}\,f = 1$ if and only if $\widetilde{\mathcal{L}}(f) = \emptyset$.

(29) If $\mathrm{len}\,f \geq 2$, then $\widetilde{\mathcal{L}}(f) \neq \emptyset$.

Let us consider f. We say that f is a special sequence if and only if the conditions (Def.7) is satisfied.

(Def.7) (i) f is one-to-one,

(ii) $\mathrm{len}\,f \geq 3$,

(iii) for every i such that $1 \leq i$ and $i \leq \mathrm{len}\,f-2$ holds $\mathcal{L}(f,i,i+1) \cap \mathcal{L}(f,i+1,i+2) = \{f(i+1)\}$,

(iv) for all i, j such that $i-j > 1$ or $j-i > 1$ holds $\mathcal{L}(f,i,i+1) \cap \mathcal{L}(f,j,j+1) = \emptyset$,

(v) for all i, p_1, p_2 such that $1 \leq i$ and $i \leq \mathrm{len}\,f - 1$ and $p_1 = f(i)$ and $p_2 = f(i+1)$ holds $p_{1\mathbf{1}} = p_{2\mathbf{1}}$ or $p_{1\mathbf{2}} = p_{2\mathbf{2}}$.

The following propositions are true:

(30) There exist f_1, f_2 such that f_1 is a special sequence and f_2 is a special sequence and $\square_{\mathcal{E}^2} = \widetilde{\mathcal{L}}(f_1) \cup \widetilde{\mathcal{L}}(f_2)$ and $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) = \{[0,0],[1,1]\}$ and $f_1(1) = [0,0]$ and $f_1(\mathrm{len}\,f_1) = [1,1]$ and $f_2(1) = [0,0]$ and $f_2(\mathrm{len}\,f_2) = [1,1]$.

(31) If h is a special sequence and $P = \widetilde{\mathcal{L}}(h)$, then for all p_1, p_2 such that $p_1 = h(1)$ and $p_2 = h(\mathrm{len}\,h)$ holds P is an arc from p_1 to p_2.

Let us consider P. We say that P is a special polygonal arc if and only if:

(Def.8) there exists f such that f is a special sequence and $P = \widetilde{\mathcal{L}}(f)$.

The following propositions are true:

(32) If P is a special polygonal arc, then $P \neq \emptyset$.

(33) If f is a special sequence, then $\widetilde{\mathcal{L}}(f)$ is a special polygonal arc.

(34) There exist P_1, P_2 such that P_1 is a special polygonal arc and P_2 is a special polygonal arc and $\square_{\mathcal{E}^2} = P_1 \cup P_2$ and $P_1 \cap P_2 = \{[0,0],[1,1]\}$.

(35) If P is a special polygonal arc, then there exist p_1, p_2 such that P is an arc from p_1 to p_2.

(36) If P is a special polygonal arc, then there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \upharpoonright P$ such that f is a homeomorphism.

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Received November 21, 1991

FORMALIZED MATHEMATICS
Vol.2, No.5, November–December 1991
Université Catholique de Louvain

Cyclic Groups and Some of Their Properties - Part I

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Summary. Some properties of finite groups are proved. The notion of cyclic group is defined next, some cyclic groups are given, for example the group of integers with addition operations. Chosen properties of cyclic groups are proved next.

MML Identifier: `GR_CY_1`.

The articles [19], [7], [12], [8], [13], [2], [3], [16], [6], [5], [18], [1], [11], [4], [15], [28], [17], [21], [14], [23], [27], [22], [25], [26], [24], [20], [10], and [9] provide the notation and terminology for this paper. For simplicity we adopt the following rules: i_1 denotes an element of $\mathbb{Z}$, j_1 denotes an integer, p, s, k, n, l, m denote natural numbers, x is arbitrary, G denotes a group, a, b denote elements of G, and I denotes a finite sequence of elements of $\mathbb{Z}$. We now state several propositions:

(1) For every n such that $n > 0$ holds $m \bmod n = (n \cdot k + m) \bmod n$.

(2) For every n such that $n > 0$ holds $(p+s) \bmod n = ((p \bmod n)+s) \bmod n$.

(3) For every n such that $n > 0$ holds $(p+s) \bmod n = (p+(s \bmod n)) \bmod n$.

(4) For every k such that $k < n$ holds $k \bmod n = k$.

(5) For every n such that $n > 0$ holds $n \bmod n = 0$.

(6) For every n such that $n > 0$ holds $0 = 0 \bmod n$.

(7) If $k + l = m$, then $l \leq m$.

(8) For all k, l, m such that $l = m$ and $m = k + l$ holds $k = 0$.

Let us consider n satisfying the condition: $n > 0$. The functor $\mathbb{Z}_n$ yields a non-empty subset of $\mathbb{N}$ and is defined by:

(Def.1) $\mathbb{Z}_n = \{p : p < n\}$.

We now state several propositions:

ISSN 0777–4028

(9) For every n such that $n > 0$ holds if $x \in \mathbb{Z}_n$, then x is a natural number.

(10) For every n such that $n > 0$ holds $s \in \mathbb{Z}_n$ if and only if $s < n$.

(11) For every n such that $n > 0$ holds $\mathbb{Z}_n \subseteq \mathbb{N}$.

(12) For every n such that $n > 0$ holds $0 \in \mathbb{Z}_n$.

(13) $\mathbb{Z}_1 = \{0\}$.

The binary operation $+_{\mathbb{Z}}$ on $\mathbb{Z}$ is defined by:

(Def.2) for all elements i_1, i_2 of $\mathbb{Z}$ holds $(+_{\mathbb{Z}})(i_1, i_2) = +_{\mathbb{R}}(i_1, i_2)$.

The following propositions are true:

(14) For all integers i_1, i_2 holds $(+_{\mathbb{Z}})(i_1, i_2) = i_1 + i_2$.

(15) For every i_1 such that $i_1 = 0$ holds i_1 is a unity w.r.t. $+_{\mathbb{Z}}$.

(16) $\mathbf{1}_{+_{\mathbb{Z}}} = 0$.

(17) $+_{\mathbb{Z}}$ has a unity.

(18) $+_{\mathbb{Z}}$ is commutative.

(19) $+_{\mathbb{Z}}$ is associative.

Let F be a finite sequence of elements of $\mathbb{Z}$. The functor $\sum F$ yields an integer and is defined by:

(Def.3) $\sum F = +_{\mathbb{Z}} \circledast F$.

Next we state several propositions:

(20) $\sum(I \frown \langle i_1 \rangle) = \sum I + {}^{@}i_1$.

(21) $\sum\langle i_1 \rangle = i_1$.

(22) $\sum(\varepsilon_{\mathbb{Z}}) = 0$.

(23) For all non-empty sets D, D_1 holds $\varepsilon_D = \varepsilon_{D_1}$.

(24) For every finite sequence I of elements of $\mathbb{Z}$ holds $\prod((\operatorname{len} I \longmapsto a)^I) = a^{\sum I}$.

Let G be a group, and let a be an element of G. Then $\{a\}$ is a subset of G.

We now state several propositions:

(25) $b \in \operatorname{gr}(\{a\})$ if and only if there exists j_1 such that $b = a^{j_1}$.

(26) If G is finite, then a is not of order 0.

(27) If G is finite, then $\operatorname{ord}(a) = \operatorname{ord}(\operatorname{gr}(\{a\}))$.

(28) If G is finite, then $\operatorname{ord}(a) \mid \operatorname{ord}(G)$.

(29) If G is finite, then $a^{\operatorname{ord}(G)} = 1_G$.

(30) If G is finite, then $(a^n)^{-1} = a^{\operatorname{ord}(G) - (n \bmod \operatorname{ord}(G))}$.

(31) For every strict group G such that $\operatorname{ord}(G) > 1$ there exists an element a of G such that $a \neq 1_G$.

(32) For every strict group G such that G is finite and $\operatorname{ord}(G) = p$ and p is prime and for every strict subgroup H of G holds $H = \{\mathbf{1}\}_G$ or $H = G$.

(33) $\langle \mathbb{Z}, +_{\mathbb{Z}} \rangle$ is a group.

The group $\mathbb{Z}^+$ is defined as follows:

(Def.4) $\mathbb{Z}^+ = \langle \mathbb{Z}, +_{\mathbb{Z}} \rangle$.

Let D be a non-empty set, and let D_1 be a non-empty subset of D, and let D_2 be a non-empty subset of D_1. We see that the element of D_2 is an element of D_1.

Let us consider n satisfying the condition: $n > 0$. The functor $+_n$ yielding a binary operation on $\mathbb{Z}_n$ is defined by:

(Def.5) for all elements k, l of $\mathbb{Z}_n$ holds $+_n(k, l) = (k + l) \bmod n$.

Next we state the proposition

(34) For every n such that $n > 0$ holds $\langle \mathbb{Z}_n, +_n \rangle$ is a group.

Let us consider n satisfying the condition: $n > 0$. The functor $\mathbb{Z}_n^+$ yields a strict group and is defined by:

(Def.6) $\mathbb{Z}_n^+ = \langle \mathbb{Z}_n, +_n \rangle$.

Next we state two propositions:

(35) $1_{\mathbb{Z}^+} = 0$.

(36) For every n such that $n > 0$ holds $1_{\mathbb{Z}_n^+} = 0$.

Let h be an element of $\mathbb{Z}^+$. The functor ${}^{@}h$ yields an integer and is defined as follows:

(Def.7) ${}^{@}h = h$.

Let h be an integer. The functor ${}^{@}h$ yielding an element of $\mathbb{Z}^+$ is defined as follows:

(Def.8) ${}^{@}h = h$.

The following proposition is true

(37) For every element h of $\mathbb{Z}^+$ holds $h^{-1} = -{}^{@}h$.

In the sequel G_1 will denote a subgroup of $\mathbb{Z}^+$ and h will denote an element of $\mathbb{Z}^+$. Next we state two propositions:

(38) For every h such that $h = 1$ and for every k holds $h^k = k$.

(39) For all h, j_1 such that $h = 1$ holds $j_1 = h^{j_1}$.

A strict group is said to be a cyclic group if:

(Def.9) there exists an element a of it such that it $= \mathrm{gr}(\{a\})$.

One can prove the following propositions:

(40) $\{\mathbf{1}\}_G$ is a cyclic group.

(41) For every strict group G holds G is a cyclic group if and only if there exists an element a of G such that for every element b of G there exists j_1 such that $b = a^{j_1}$.

(42) For every strict group G such that G is finite holds G is a cyclic group if and only if there exists an element a of G such that for every element b of G there exists n such that $b = a^n$.

(43) For every strict group G such that G is finite holds G is a cyclic group if and only if there exists an element a of G such that $\mathrm{ord}(a) = \mathrm{ord}(G)$.

(44) For every strict subgroup H of G such that G is finite and G is a cyclic group and H is a subgroup of G holds H is a cyclic group.

(45) For every strict group G such that G is finite and $\mathrm{ord}(G) = p$ and p is prime holds G is a cyclic group.

(46) For every n such that $n > 0$ there exists an element g of $\mathbb{Z}_n^+$ such that for every element b of $\mathbb{Z}_n^+$ there exists j_1 such that $b = g^{j_1}$.

(47) If G is a cyclic group, then G is an Abelian group.

(48) $\mathbb{Z}^+$ is a cyclic group.

(49) For every n such that $n > 0$ holds $\mathbb{Z}_n^+$ is a cyclic group.

(50) $\mathbb{Z}^+$ is an Abelian group.

(51) For every n such that $n > 0$ holds $\mathbb{Z}_n^+$ is an Abelian group.

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Received November 22, 1991

FORMALIZED MATHEMATICS
Vol.2,No.5, November–December 1991
Université Catholique de Louvain

Isomorphisms of Categories

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Summary. We continue the development of the category theory basically following [12] (compare also [11]). We define the concept of isomorphic categories and prove basic facts related, e.g. that the Cartesian product of categories is associative up to the isomorphism. We introduce the composition of a functor and a transformation, and of transformation and a functor, and afterwards we define again those concepts for natural transformations. Let us observe, that we have to duplicate those concepts because of the permissiveness: if a functor F is not naturally transformable to G, then natural transformation from F to G has no fixed meaning, hence we cannot claim that the composition of it with a functor as a transformation results in a natural transformation. We define also the so called horizontal composition of transformations ([12], p.140, exercise **4.2**,**5**(C)) and prove *interchange law* ([11], p.44). We conclude with the definition of equivalent categories.

MML Identifier: `ISOCAT_1`.

The articles [16], [17], [4], [5], [3], [7], [1], [2], [10], [13], [8], [14], [6], [9], and [15] provide the notation and terminology for this paper. We adopt the following convention: A, B, C, D will denote categories, F, F_1, F_2 will denote functors from A to B, and G will denote a functor from B to C. One can prove the following propositions:

(1) For all functions F, G such that F is one-to-one and G is one-to-one holds $[\!: F, G :\!]$ is one-to-one.

(2) $\operatorname{rng} \pi_1(A \times B) =$ the morphisms of A and $\operatorname{rng} \pi_2(B \times A) =$ the morphisms of A.

(3) For every morphism f of A such that f is invertible holds $F(f)$ is invertible.

(4) For every functor F from A to B and for every functor G from B to A holds $F \cdot \mathrm{id}_A = F$ and $\mathrm{id}_A \cdot G = G$.

ISSN 0777–4028

(5) For all objects a, b of A such that $\hom(a,b) \neq \emptyset$ and for every morphism f from a to b and for every functor F from A to B and for every functor G from B to C holds $(G \cdot F)(f) = G(F(f))$.

(6) For all objects a, b, c of A such that $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$ and for every morphism f from a to b and for every morphism g from b to c and for every functor F from A to B holds $F(g \cdot f) = F(g) \cdot F(f)$.

(7) For all functors F_1, F_2 from A to B such that F_1 is transformable to F_2 and for every transformation t from F_1 to F_2 and for every object a of A holds $t(a) \in \hom(F_1(a), F_2(a))$.

(8) For all functors F_1, F_2 from A to B and for all functors G_1, G_2 from B to C such that F_1 is transformable to F_2 and G_1 is transformable to G_2 holds $G_1 \cdot F_1$ is transformable to $G_2 \cdot F_2$.

(9) For all functors F_1, F_2 from A to B such that F_1 is transformable to F_2 and for every transformation t from F_1 to F_2 such that t is invertible and for every object a of A holds $F_1(a)$ and $F_2(a)$ are isomorphic.

Let us consider C, D. Let us observe that the mode below can be characterized by another conditions, which are equivalent to the formulas previously defining them. In accordance the mode Let us note that one can characterize the mode functor from C to D, by the following (equivalent) condition:

(Def.1) (i) for every object c of C there exists an object d of D such that $\mathrm{it}(\mathrm{id}_c) = \mathrm{id}_d$,

(ii) for every morphism f of C holds $\mathrm{it}(\mathrm{id}_{\mathrm{dom}\, f}) = \mathrm{id}_{\mathrm{dom}\,\mathrm{it}(f)}$ and $\mathrm{it}(\mathrm{id}_{\mathrm{cod}\, f}) = \mathrm{id}_{\mathrm{cod}\,\mathrm{it}(f)}$,

(iii) for all morphisms f, g of C such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $\mathrm{it}(g \cdot f) = \mathrm{it}(g) \cdot \mathrm{it}(f)$.

Let us consider A. Then id_A is a functor from A to A. Let us consider B, C, and let F be a functor from A to B, and let G be a functor from B to C. Then $G \cdot F$ is a functor from A to C.

In the sequel o, m are arbitrary. We now state three propositions:

(10) If F is an isomorphism, then for every morphism g of B there exists a morphism f of A such that $F(f) = g$.

(11) If F is an isomorphism, then for every object b of B there exists an object a of A such that $F(a) = b$.

(12) If F is one-to-one, then $\mathrm{Obj}\, F$ is one-to-one.

Let us consider A, B, F. Let us assume that F is an isomorphism. The functor F^{-1} yields a functor from B to A and is defined by:

(Def.2) $F^{-1} = F^{-1}$.

Let us consider A, B, F. Let us note that one can characterize the predicate F is an isomorphism by the following (equivalent) condition:

(Def.3) F is one-to-one and $\mathrm{rng}\, F =$ the morphisms of B.

Next we state several propositions:

(13) If F is an isomorphism, then F^{-1} is an isomorphism.

(14) If F is an isomorphism, then $(\operatorname{Obj} F)^{-1} = \operatorname{Obj}(F^{-1})$.

(15) If F is an isomorphism, then $(F^{-1})^{-1} = F$.

(16) If F is an isomorphism, then $F \cdot F^{-1} = \mathrm{id}_B$ and $F^{-1} \cdot F = \mathrm{id}_A$.

(17) If F is an isomorphism and G is an isomorphism, then $G \cdot F$ is an isomorphism.

In the sequel t_1 denotes a natural transformation from F_1 to F_2 and t_2 denotes a natural transformation from F to F_2. We now define two new predicates. Let us consider A, B. We say that A and B are isomorphic if and only if:

(Def.4) there exists a functor F from A to B such that F is an isomorphism.

We write $A \cong B$ if A and B are isomorphic.

The following propositions are true:

(18) $A \cong A$.

(19) If $A \cong B$, then $B \cong A$.

(20) If $A \cong B$ and $B \cong C$, then $A \cong C$.

(21) $[\!: \dot{\circlearrowleft}(o, m),\ A :\!] \cong A$.

(22) $[\!: A,\ B :\!] \cong [\!: B,\ A :\!]$.

(23) $[\!: [\!: A,\ B :\!],\ C :\!] \cong [\!: A,\ [\!: B,\ C :\!] :\!]$.

(24) If $A \cong B$ and $C \cong D$, then $[\!: A,\ C :\!] \cong [\!: B,\ D :\!]$.

Let us consider A, B, C, and let F_1, F_2 be functors from A to B satisfying the condition: F_1 is transformable to F_2. Let t be a transformation from F_1 to F_2, and let G be a functor from B to C. The functor $G \cdot t$ yields a transformation from $G \cdot F_1$ to $G \cdot F_2$ and is defined as follows:

(Def.5) $G \cdot t = G \cdot t$.

Let us consider A, B, C, and let G_1, G_2 be functors from B to C satisfying the condition: G_1 is transformable to G_2. Let F be a functor from A to B, and let t be a transformation from G_1 to G_2. The functor $t \cdot F$ yielding a transformation from $G_1 \cdot F$ to $G_2 \cdot F$ is defined by:

(Def.6) $t \cdot F = t \cdot \operatorname{Obj} F$.

We now state three propositions:

(25) For all functors G_1, G_2 from B to C such that G_1 is transformable to G_2 and for every functor F from A to B and for every transformation t from G_1 to G_2 and for every object a of A holds $(t \cdot F)(a) = t(F(a))$.

(26) For all functors F_1, F_2 from A to B such that F_1 is transformable to F_2 and for every transformation t from F_1 to F_2 and for every functor G from B to C and for every object a of A holds $(G \cdot t)(a) = G(t(a))$.

(27) For all functors F_1, F_2 from A to B and for all functors G_1, G_2 from B to C such that F_1 is naturally transformable to F_2 and G_1 is naturally transformable to G_2 holds $G_1 \cdot F_1$ is naturally transformable to $G_2 \cdot F_2$.

Let us consider A, B, C, and let F_1, F_2 be functors from A to B satisfying the condition: F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2, and let G be a functor from B to C. The functor $G \cdot t$ yielding a natural transformation from $G \cdot F_1$ to $G \cdot F_2$ is defined by:

(Def.7) $G \cdot t = G \cdot t$.

Next we state the proposition

(28) For all functors F_1, F_2 from A to B such that F_1 is naturally transformable to F_2 and for every natural transformation t from F_1 to F_2 and for every functor G from B to C and for every object a of A holds $(G \cdot t)(a) = G(t(a))$.

Let us consider A, B, C, and let G_1, G_2 be functors from B to C satisfying the condition: G_1 is naturally transformable to G_2. Let F be a functor from A to B, and let t be a natural transformation from G_1 to G_2. The functor $t \cdot F$ yields a natural transformation from $G_1 \cdot F$ to $G_2 \cdot F$ and is defined as follows:

(Def.8) $t \cdot F = t \cdot F$.

The following proposition is true

(29) For all functors G_1, G_2 from B to C such that G_1 is naturally transformable to G_2 and for every functor F from A to B and for every natural transformation t from G_1 to G_2 and for every object a of A holds $(t \cdot F)(a) = t(F(a))$.

For simplicity we follow the rules: F, F_1, F_2, F_3 are functors from A to B, G, G_1, G_2, G_3 are functors from B to C, H, H_1, H_2 are functors from C to D, s is a natural transformation from F_1 to F_2, s' is a natural transformation from F_2 to F_3, t is a natural transformation from G_1 to G_2, t' is a natural transformation from G_2 to G_3, and u is a natural transformation from H_1 to H_2. We now state a number of propositions:

(30) If F_1 is naturally transformable to F_2, then for every object a of A holds $\hom(F_1(a), F_2(a)) \neq \emptyset$.

(31) If F_1 is naturally transformable to F_2, then for all natural transformations t_1, t_2 from F_1 to F_2 such that for every object a of A holds $t_1(a) = t_2(a)$ holds $t_1 = t_2$.

(32) If F_1 is naturally transformable to F_2 and F_2 is naturally transformable to F_3, then $G \cdot (s' \circ s) = G \cdot s' \circ G \cdot s$.

(33) If G_1 is naturally transformable to G_2 and G_2 is naturally transformable to G_3, then $(t' \circ t) \cdot F = t' \cdot F \circ t \cdot F$.

(34) If H_1 is naturally transformable to H_2, then $(u \cdot G) \cdot F = u \cdot (G \cdot F)$.

(35) If G_1 is naturally transformable to G_2, then $(H \cdot t) \cdot F = H \cdot (t \cdot F)$.

(36) If F_1 is naturally transformable to F_2, then $(H \cdot G) \cdot s = H \cdot (G \cdot s)$.

(37) $\mathrm{id}_G \cdot F = \mathrm{id}_{(G \cdot F)}$.

(38) $G \cdot \mathrm{id}_F = \mathrm{id}_{(G \cdot F)}$.

(39) If G_1 is naturally transformable to G_2, then $t \cdot \mathrm{id}_B = t$.

(40) If F_1 is naturally transformable to F_2, then $\mathrm{id}_B \cdot s = s$.

Let us consider A, B, C, F_1, F_2, G_1, G_2, s, t. The functor $t\,s$ yields a natural transformation from $G_1 \cdot F_1$ to $G_2 \cdot F_2$ and is defined as follows:

(Def.9) $t\,s = t \cdot F_2 \circ G_1 \cdot s$.

We now state several propositions:

(41) If F_1 is naturally transformable to F_2 and G_1 is naturally transformable to G_2, then $t\,s = G_2 \cdot s \circ t \cdot F_1$.

(42) If F_1 is naturally transformable to F_2, then $\mathrm{id}_{(\mathrm{id}_B)}\,s = s$.

(43) If G_1 is naturally transformable to G_2, then $t\,\mathrm{id}_{(\mathrm{id}_B)} = t$.

(44) If F_1 is naturally transformable to F_2 and G_1 is naturally transformable to G_2 and H_1 is naturally transformable to H_2, then $u\,(t\,s) = (u\,t)\,s$.

(45) If G_1 is naturally transformable to G_2, then $t \cdot F = t\,\mathrm{id}_F$.

(46) If F_1 is naturally transformable to F_2, then $G \cdot s = \mathrm{id}_G\,s$.

(47) If F_1 is naturally transformable to F_2 and F_2 is naturally transformable to F_3 and G_1 is naturally transformable to G_2 and G_2 is naturally transformable to G_3, then $(t' \circ t)\,(s' \circ s) = t'\,s' \circ t\,s$.

(48) For every functor F from A to B and for every functor G from C to D and for all functors I, J from B to C such that $I \cong J$ holds $G \cdot I \cong G \cdot J$ and $I \cdot F \cong J \cdot F$.

(49) For every functor F from A to B and for every functor G from B to A and for every functor I from A to A such that $I \cong \mathrm{id}_A$ holds $F \cdot I \cong F$ and $I \cdot G \cong G$.

We now define two new predicates. Let A, B be categories. We say that A is equivalent with B if and only if:

(Def.10) there exists a functor F from A to B and there exists a functor G from B to A such that $G \cdot F \cong \mathrm{id}_A$ and $F \cdot G \cong \mathrm{id}_B$.

A and B are equivalent stands for A is equivalent with B.

We now state four propositions:

(50) If $A \cong B$, then A is equivalent with B.

(51) A is equivalent with A.

(52) If A and B are equivalent, then B and A are equivalent.

(53) If A and B are equivalent and B and C are equivalent, then A and C are equivalent.

Let us consider A, B. Let us assume that A and B are equivalent. A functor from A to B is called an equivalence of A and B if:

(Def.11) there exists a functor G from B to A such that $G \cdot \mathrm{it} \cong \mathrm{id}_A$ and $\mathrm{it} \cdot G \cong \mathrm{id}_B$.

Next we state several propositions:

(54) id_A is an equivalence of A and A.

(55) If A and B are equivalent and B and C are equivalent, then for every equivalence F of A and B and for every equivalence G of B and C holds $G \cdot F$ is an equivalence of A and C.

(56) If A and B are equivalent, then for every equivalence F of A and B there exists an equivalence G of B and A such that $G \cdot F \cong \mathrm{id}_A$ and $F \cdot G \cong \mathrm{id}_B$.

(57) For every functor F from A to B and for every functor G from B to A such that $G \cdot F \cong \mathrm{id}_A$ holds F is faithful.

(58) If A and B are equivalent, then for every equivalence F of A and B holds F is full and F is faithful and for every object b of B there exists an object a of A such that b and $F(a)$ are isomorphic.

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Received November 22, 1991

FORMALIZED MATHEMATICS
Vol.2,No.5, November–December 1991
Université Catholique de Louvain

Similarity of Formulae

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Summary. The main objective of the paper is to define the concept of the similarity of formulas. We mean by similar formulas the two formulas that differs only in the names of bound variables. Some authors (compare [16]) call such formulas *congruent*. We use the word *similar* following [14,12,15]. The concept is unjustfully neglected in many logical handbooks. It is intuitively quite clear, however the exact definition is not simple. As far as we know, only W.A.Pogorzelski and T.Prucnal [15] define it in the precise way. We follow basically the Pogorzelski's definition (compare [14]). We define renumaration of bound variables and we say that two formulas are similar if after renumaration are equal. Therefore we need a rule of chosing bound variables independent of the original choice. Quite obvious solution is to use consecutively variables $x_{k+1}, x_{k+2}, \ldots$, where k is the maximal index of free variable occurring in the formula. Therefore after the renumaration we get the new formula in which different quantifiers bind different variables. It is the reason that the result of renumaration applied to a formula φ we call φ *with variables separated*.

MML Identifier: `CQC_SIM1`.

The notation and terminology used in this paper are introduced in the following articles: [23], [27], [20], [24], [19], [13], [5], [6], [18], [3], [10], [26], [21], [11], [2], [25], [22], [8], [17], [1], [9], [4], and [7]. One can prove the following four propositions:

(1) For arbitrary x, y and for every function f holds $(f +\cdot (\{x\} \longmapsto y)) \circ \{x\} = \{y\}$.

(2) For all sets K, L and for arbitrary x, y and for every function f holds $(f +\cdot (L \longmapsto y)) \circ K \subseteq f \circ K \cup \{y\}$.

(3) For arbitrary x, y and for every function g and for every set A holds $(g +\cdot (\{x\} \longmapsto y)) \circ (A \setminus \{x\}) = g \circ (A \setminus \{x\})$.

(4) For arbitrary x, y and for every function g and for every set A such that $y \notin g \circ (A \setminus \{x\})$ holds $(g +\cdot (\{x\} \longmapsto y)) \circ (A \setminus \{x\}) = (g +\cdot (\{x\} \longmapsto y)) \circ A \setminus \{y\}$.

ISSN 0777–4028

For simplicity we follow the rules: p, q, r, s denote elements of CQC-WFF, x denotes an element of BoundVar, i, k, l, m, n denote elements of $\mathbb{N}$, l_1 denotes a variables list of k, and P denotes a k-ary predicate symbol. The following propositions are true:

(5) If p is atomic, then there exist k, P, l_1 such that $p = P[l_1]$.

(6) If p is negative, then there exists q such that $p = \neg q$.

(7) If p is conjunctive, then there exist q, r such that $p = q \wedge r$.

(8) If p is universal, then there exist x, q such that $p = \forall_x q$.

(9) For every non-empty set D and for every finite sequence l of elements of D holds $\operatorname{rng} l = \{l(i) : 1 \leq i \wedge i \leq \operatorname{len} l\}$.

In this article we present several logical schemes. The scheme *NUBFuncExD* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every element e of $\mathcal{A}$ holds $\mathcal{P}[e, f(e)]$

provided the parameters satisfy the following condition:

- for every element e of $\mathcal{A}$ there exists an element u of $\mathcal{B}$ such that $\mathcal{P}[e, u]$.

The scheme *NUBFuncEx2D* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a non-empty set $\mathcal{C}$, and a ternary predicate $\mathcal{P}$, and states that:

there exists a function f from $[\![\mathcal{A}, \mathcal{B}]\!]$ into $\mathcal{C}$ such that for every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $\mathcal{P}[x, y, f(\langle x, y \rangle)]$

provided the parameters meet the following condition:

- for every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ there exists an element u of $\mathcal{C}$ such that $\mathcal{P}[x, y, u]$.

The scheme *QC_Func_ExN* deals with a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a ternary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$ and states that:

there exists a function F from WFF into $\mathcal{A}$ such that for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1, p)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2, p)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(d_1, p)$

for all values of the parameters.

The scheme *CQCF2_Func_Ex* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\mathcal{B}^{\mathcal{A}}$, a ternary functor $\mathcal{F}$ yielding an element of $\mathcal{B}^{\mathcal{A}}$, a binary functor $\mathcal{G}$ yielding an element of $\mathcal{B}^{\mathcal{A}}$, a 4-ary functor $\mathcal{H}$ yielding an element of $\mathcal{B}^{\mathcal{A}}$, and a ternary functor $\mathcal{I}$ yielding an element of $\mathcal{B}^{\mathcal{A}}$ and states that:

there exists a function F from CQC-WFF into $\mathcal{B}^{\mathcal{A}}$ such that $F(\mathrm{VERUM}) = \mathcal{C}$ and for every k and for every variables list l of k and for every k-ary predicate symbol P holds $F(P[l]) = \mathcal{F}(k, P, l)$ and for all r, s, x and for all functions f,

g from $\mathcal{A}$ into $\mathcal{B}$ such that $f = F(r)$ and $g = F(s)$ holds $F(\neg r) = \mathcal{G}(f, r)$ and $F(r \wedge s) = \mathcal{H}(f, g, r, s)$ and $F(\forall_x r) = \mathcal{I}(x, f, r)$
for all values of the parameters.

The scheme *CQCF2_FUniq* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a function $\mathcal{C}$ from CQC-WFF into $\mathcal{B}^{\mathcal{A}}$, a function $\mathcal{D}$ from CQC-WFF into $\mathcal{B}^{\mathcal{A}}$, a function $\mathcal{E}$ from $\mathcal{A}$ into $\mathcal{B}$, a ternary functor $\mathcal{F}$ yielding a function from $\mathcal{A}$ into $\mathcal{B}$, a binary functor $\mathcal{G}$ yielding a function from $\mathcal{A}$ into $\mathcal{B}$, a 4-ary functor $\mathcal{H}$ yielding a function from $\mathcal{A}$ into $\mathcal{B}$, and a ternary functor $\mathcal{I}$ yielding a function from $\mathcal{A}$ into $\mathcal{B}$ and states that:

$\mathcal{C} = \mathcal{D}$

provided the parameters meet the following requirements:

- $\mathcal{C}(\mathrm{VERUM}) = \mathcal{E}$,
- for all k, l_1, P holds $\mathcal{C}(P[l_1]) = \mathcal{F}(k, P, l_1)$,
- Given r, s, x. Then for all functions f, g from $\mathcal{A}$ into $\mathcal{B}$ such that $f = \mathcal{C}(r)$ and $g = \mathcal{C}(s)$ holds $\mathcal{C}(\neg r) = \mathcal{G}(f, r)$ and $\mathcal{C}(r \wedge s) = \mathcal{H}(f, g, r, s)$ and $\mathcal{C}(\forall_x r) = \mathcal{I}(x, f, r)$,
- $\mathcal{D}(\mathrm{VERUM}) = \mathcal{E}$,
- for all k, l_1, P holds $\mathcal{D}(P[l_1]) = \mathcal{F}(k, P, l_1)$,
- Given r, s, x. Then for all functions f, g from $\mathcal{A}$ into $\mathcal{B}$ such that $f = \mathcal{D}(r)$ and $g = \mathcal{D}(s)$ holds $\mathcal{D}(\neg r) = \mathcal{G}(f, r)$ and $\mathcal{D}(r \wedge s) = \mathcal{H}(f, g, r, s)$ and $\mathcal{D}(\forall_x r) = \mathcal{I}(x, f, r)$.

We now state four propositions:

(10) p is a subformula of $\neg p$.

(11) p is a subformula of $p \wedge q$ and q is a subformula of $p \wedge q$.

(12) p is a subformula of $\forall_x p$.

(13) For every variables list l of k and for every i such that $1 \leq i$ and $i \leq \operatorname{len} l$ holds $l(i) \in \mathrm{BoundVar}$.

Let D be a non-empty set, and let f be a function from D into CQC-WFF. The functor $\mathrm{NEG}(f)$ yielding an element of CQC-WFF^D is defined as follows:

(Def.1) for every element a of D and for every element p of CQC-WFF such that $p = f(a)$ holds $(\mathrm{NEG}(f))(a) = \neg p$.

In the sequel f, h will denote elements of $\mathrm{BoundVar}^{\mathrm{BoundVar}}$ and K will denote a finite subset of BoundVar. Let f, g be functions from $[\!: \mathbb{N}, \mathrm{BoundVar}^{\mathrm{BoundVar}} :\!]$ into CQC-WFF, and let n be a natural number. The functor $\mathrm{CON}(f, g, n)$ yields an element of $\text{CQC-WFF}^{[\!: \mathbb{N}, \mathrm{BoundVar}^{\mathrm{BoundVar}} :\!]}$ and is defined by:

(Def.2) for all k, h, p, q such that $p = f(\langle k, h \rangle)$ and $q = g(\langle k + n, h \rangle)$ holds $(\mathrm{CON}(f, g, n))(\langle k, h \rangle) = p \wedge q$.

Let f be a function from $[\!: \mathbb{N}, \mathrm{BoundVar}^{\mathrm{BoundVar}} :\!]$ into CQC-WFF, and let x be a bound variable. The functor $\mathrm{UNIV}(x, f)$ yielding an element of $\text{CQC-WFF}^{[\!: \mathbb{N}, \mathrm{BoundVar}^{\mathrm{BoundVar}} :\!]}$ is defined by:

(Def.3) for all k, h, p such that $p = f(\langle k + 1, h +\cdot (\{x\} \longmapsto \mathrm{x}_k) \rangle)$ holds $(\mathrm{UNIV}(x, f))(\langle k, h \rangle) = \forall_{\mathrm{x}_k} p$.

Let us consider k, and let l be a variables list of k, and let f be an element of $\mathrm{BoundVar}^{\mathrm{BoundVar}}$. Then $f \cdot l$ is a variables list of k.

Let us consider k, and let P be a k-ary predicate symbol, and let l be a variables list of k. The functor $\mathrm{ATOM}(P, l)$ yields an element of

$\text{CQC-WFF}^{[\![\mathbb{N}, \mathrm{BoundVar}^{\mathrm{BoundVar}}]\!]}$

and is defined as follows:

(Def.4) for all n, h holds $(\mathrm{ATOM}(P, l))(\langle n, h\rangle) = P[h \cdot l]$.

Let us consider p. The number of quantifiers in p yields an element of $\mathbb{N}$ and is defined by the condition (Def.5).

(Def.5) There exists a function F from CQC-WFF into $\mathbb{N}$ such that the number of quantifiers in $p = F(p)$ and for all r, s, x, k and for every variables list l of k and for every k-ary predicate symbol P and for all elements r', s' of $\mathbb{N}$ such that $r' = F(r)$ and $s' = F(s)$ holds $F(\mathrm{VERUM}) = 0$ and $F(P[l]) = 0$ and $F(\neg r) = r'$ and $F(r \wedge s) = r' + s'$ and $F(\forall_x r) = r' + 1$.

Let f be a function from CQC-WFF into

$\text{CQC-WFF}^{[\![\mathbb{N}, \mathrm{BoundVar}^{\mathrm{BoundVar}}]\!]}$,

and let x be an element of CQC-WFF. Then $f(x)$ is an element of $\text{CQC-WFF}^{[\![\mathbb{N}, \mathrm{BoundVar}^{\mathrm{BoundVar}}]\!]}$

The function Renum from CQC-WFF into $\text{CQC-WFF}^{[\![\mathbb{N}, \mathrm{BoundVar}^{\mathrm{BoundVar}}]\!]}$ is defined by the conditions (Def.6).

(Def.6) (i) $\mathrm{Renum}(\mathrm{VERUM}) = [\![\mathbb{N}, \mathrm{BoundVar}^{\mathrm{BoundVar}}]\!] \longmapsto \mathrm{VERUM}$,

(ii) for every k and for every variables list l of k and for every k-ary predicate symbol P holds $\mathrm{Renum}(P[l]) = \mathrm{ATOM}(P, l)$,

(iii) for all r, s, x and for all functions f, g from $[\![\mathbb{N}, \mathrm{BoundVar}^{\mathrm{BoundVar}}]\!]$ into CQC-WFF such that $f = \mathrm{Renum}(r)$ and $g = \mathrm{Renum}(s)$ holds $\mathrm{Renum}(\neg r) = \mathrm{NEG}(f)$ and $\mathrm{Renum}(r \wedge s) = \mathrm{CON}(f, g,$ the number of quantifiers in $r)$ and $\mathrm{Renum}(\forall_x r) = \mathrm{UNIV}(x, f)$.

Let us consider p, k, f. The functor $\mathrm{Renum}_{k,f}(p)$ yields an element of CQC-WFF and is defined by:

(Def.7) $\mathrm{Renum}_{k,f}(p) = \mathrm{Renum}(p)(\langle k, f\rangle)$.

Next we state several propositions:

(14) The number of quantifiers in VERUM $= 0$.

(15) The number of quantifiers in $P[l_1] = 0$.

(16) The number of quantifiers in $\neg p =$ the number of quantifiers in p.

(17) The number of quantifiers in $p \wedge q =$ (the number of quantifiers in p) + (the number of quantifiers in q).

(18) The number of quantifiers in $\forall_x p =$ (the number of quantifiers in p) + 1.

Let A be a non-empty subset of $\mathbb{N}$. The functor $\min A$ yields a natural number and is defined by:

(Def.8) $\min A \in A$ and for every k such that $k \in A$ holds $\min A \leq k$.

We now state two propositions:

(19) For all non-empty subsets A, B of $\mathbb{N}$ such that $A \subseteq B$ holds $\min B \leq \min A$.

(20) For every element p of WFF holds $\mathrm{snb}(p)$ is finite.

The scheme *MaxFinDomElem* concerns a non-empty set $\mathcal{A}$, a set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

there exists an element x of $\mathcal{A}$ such that $x \in \mathcal{B}$ and for every element y of $\mathcal{A}$ such that $y \in \mathcal{B}$ holds $\mathcal{P}[x, y]$

provided the parameters meet the following requirements:

- $\mathcal{B}$ is finite and $\mathcal{B} \neq \emptyset$ and $\mathcal{B} \subseteq \mathcal{A}$,
- for all elements x, y of $\mathcal{A}$ holds $\mathcal{P}[x, y]$ or $\mathcal{P}[y, x]$,
- for all elements x, y, z of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$.

Let us consider p. The functor $\mathrm{NBI}(p)$ yielding a non-empty subset of $\mathbb{N}$ is defined as follows:

(Def.9) $\mathrm{NBI}(p) = \{k : \bigwedge_i [k \leq i \Rightarrow \mathrm{x}_i \notin \mathrm{snb}(p)]\}$.

Let us consider p. The functor $|\bullet : p|_{\mathbb{N}}$ yielding a natural number is defined as follows:

(Def.10) $|\bullet : p|_{\mathbb{N}} = \min \mathrm{NBI}(p)$.

Next we state several propositions:

(21) $|\bullet : p|_{\mathbb{N}} = 0$ if and only if p is closed.

(22) If $\mathrm{x}_i \in \mathrm{snb}(p)$, then $i < |\bullet : p|_{\mathbb{N}}$.

(23) $|\bullet : \mathrm{VERUM}|_{\mathbb{N}} = 0$.

(24) $|\bullet : \neg p|_{\mathbb{N}} = |\bullet : p|_{\mathbb{N}}$.

(25) $|\bullet : p|_{\mathbb{N}} \leq |\bullet : p \wedge q|_{\mathbb{N}}$ and $|\bullet : q|_{\mathbb{N}} \leq |\bullet : p \wedge q|_{\mathbb{N}}$.

Let C be a non-empty set, and let D be a non-empty subset of C. Then id_D is an element of D^D.

Let us consider p. The functor p with variables separated yielding an element of CQC-WFF is defined as follows:

(Def.11) p with variables separated $= \mathrm{Renum}_{|\bullet : p|_{\mathbb{N}}, \mathrm{id}_{\mathrm{BoundVar}}}(p)$.

The following proposition is true

(26) VERUM with variables separated $=$ VERUM.

The scheme *CQCInd* deals with a unary predicate $\mathcal{P}$, and states that:

for every r holds $\mathcal{P}[r]$

provided the following requirements are met:

- $\mathcal{P}[\mathrm{VERUM}]$,
- for every k and for every variables list l of k and for every k-ary predicate symbol P holds $\mathcal{P}[P[l]]$,
- for every r such that $\mathcal{P}[r]$ holds $\mathcal{P}[\neg r]$,
- for all r, s such that $\mathcal{P}[r]$ and $\mathcal{P}[s]$ holds $\mathcal{P}[r \wedge s]$,
- for all r, x such that $\mathcal{P}[r]$ holds $\mathcal{P}[\forall_x r]$.

We now state four propositions:

(27) $P[l_1]$ with variables separated $= P[l_1]$.

(28) If p is atomic, then p with variables separated $= p$.

(29) $\neg p$ with variables separated $= \neg(p$ with variables separated).

(30) If p is negative and $q = \mathrm{Arg}(p)$, then p with variables separated $= \neg(q$ with variables separated).

Let us consider p, and let X be a subset of $[\!:$ CQC-WFF, $\mathbb{N}$, Fin BoundVar, BoundVar$^{\mathrm{BoundVar}} :\!]$. We say that X is closed w.r.t. p if and only if the conditions (Def.12) is satisfied.

(Def.12) (i) $\langle p, |\bullet : p|_{\mathbb{N}}, \emptyset_{\mathrm{BoundVar}}, \mathrm{id}_{\mathrm{BoundVar}}\rangle \in X$,

(ii) for all q, k, K, f such that $\langle \neg q, k, K, f\rangle \in X$ holds $\langle q, k, K, f\rangle \in X$,

(iii) for all q, r, k, K, f such that $\langle q \wedge r, k, K, f\rangle \in X$ holds $\langle q, k, K, f\rangle \in X$ and $\langle r, k +$ the number of quantifiers in $q, K, f\rangle \in X$,

(iv) for all q, x, k, K, f such that $\langle \forall_x q, k, K, f\rangle \in X$ holds $\langle q, k+1, K \cup \{x\}, f +\cdot (\{x\} \longmapsto \mathrm{x}_k)\rangle \in X$.

Let D be a non-empty set, and let x be an element of D. Then $\{x\}$ is an element of Fin D.

Let us consider p. The functor $\mathbf{Quadruples}_p$ yields a subset of $[\!:$ CQC-WFF, $\mathbb{N}$, Fin BoundVar, BoundVar$^{\mathrm{BoundVar}} :\!]$ and is defined by:

(Def.13) $\mathbf{Quadruples}_p$ is closed w.r.t. p and for every subset D of $[\!:$ CQC-WFF, $\mathbb{N}$, Fin BoundVar, BoundVar$^{\mathrm{BoundVar}} :\!]$ such that D is closed w.r.t. p holds $\mathbf{Quadruples}_p \subseteq D$.

One can prove the following propositions:

(31) $\langle p, |\bullet : p|_{\mathbb{N}}, \emptyset_{\mathrm{BoundVar}}, \mathrm{id}_{\mathrm{BoundVar}}\rangle \in \mathbf{Quadruples}_p$.

(32) For all q, k, K, f such that $\langle \neg q, k, K, f\rangle \in \mathbf{Quadruples}_p$ holds $\langle q, k, K, f\rangle \in \mathbf{Quadruples}_p$.

(33) For all q, r, k, K, f such that $\langle q \wedge r, k, K, f\rangle \in \mathbf{Quadruples}_p$ holds $\langle q, k, K, f\rangle \in \mathbf{Quadruples}_p$ and $\langle r, k +$ the number of quantifiers in $q, K, f\rangle \in \mathbf{Quadruples}_p$.

(34) For all q, x, k, K, f such that $\langle \forall_x q, k, K, f\rangle \in \mathbf{Quadruples}_p$ holds $\langle q, k+1, K \cup \{x\}, f +\cdot (\{x\} \longmapsto \mathrm{x}_k)\rangle \in \mathbf{Quadruples}_p$.

(35) Suppose $\langle q, k, K, f\rangle \in \mathbf{Quadruples}_p$. Then

(i) $\langle q, k, K, f\rangle = \langle p, |\bullet : p|_{\mathbb{N}}, \emptyset_{\mathrm{BoundVar}}, \mathrm{id}_{\mathrm{BoundVar}}\rangle$, or

(ii) $\langle \neg q, k, K, f\rangle \in \mathbf{Quadruples}_p$, or

(iii) there exists r such that $\langle q \wedge r, k, K, f\rangle \in \mathbf{Quadruples}_p$, or

(iv) there exist r, l such that $k = l +$ the number of quantifiers in r and $\langle r \wedge q, l, K, f\rangle \in \mathbf{Quadruples}_p$, or

(v) there exist x, l, h such that $l + 1 = k$ and $h +\cdot (\{x\} \longmapsto \mathrm{x}_l) = f$ but $\langle \forall_x q, l, K, h\rangle \in \mathbf{Quadruples}_p$ or $\langle \forall_x q, l, K \setminus \{x\}, h\rangle \in \mathbf{Quadruples}_p$.

The scheme *Sep_regression* deals with an element $\mathcal{A}$ of CQC-WFF, and a 4-ary predicate $\mathcal{P}$, and states that:

for all q, k, K, f such that $\langle q, k, K, f\rangle \in \mathbf{Quadruples}_{\mathcal{A}}$ holds $\mathcal{P}[q, k, K, f]$ provided the following conditions are met:

- $\mathcal{P}[\mathcal{A}, |\bullet : \mathcal{A}|_{\mathbb{N}}, \emptyset_{\mathrm{BoundVar}}, \mathrm{id}_{\mathrm{BoundVar}}]$,
- for all q, k, K, f such that $\langle \neg q, k, K, f\rangle \in \mathbf{Quadruples}_{\mathcal{A}}$ and $\mathcal{P}[\neg q, k, K, f]$ holds $\mathcal{P}[q, k, K, f]$,
- for all q, r, k, K, f such that $\langle q \wedge r, k, K, f\rangle \in \mathbf{Quadruples}_{\mathcal{A}}$ and $\mathcal{P}[q \wedge r, k, K, f]$ holds $\mathcal{P}[q, k, K, f]$ and $\mathcal{P}[r, k + \text{the number of quantifiers in } q, K, f]$,
- for all q, x, k, K, f such that $\langle \forall_x q, k, K, f\rangle \in \mathbf{Quadruples}_{\mathcal{A}}$ and $\mathcal{P}[\forall_x q, k, K, f]$ holds $\mathcal{P}[q, k+1, K \cup \{x\}, f +\cdot (\{x\} \longmapsto \mathrm{x}_k)]$.

We now state a number of propositions:

(36) For all q, k, K, f such that $\langle q, k, K, f\rangle \in \mathbf{Quadruples}_p$ holds q is a subformula of p.

(37) $\mathbf{Quadruples}_{\mathrm{VERUM}} = \{\langle \mathrm{VERUM}, 0, \emptyset_{\mathrm{BoundVar}}, \mathrm{id}_{\mathrm{BoundVar}}\rangle\}$.

(38) For every k and for every variables list l of k and for every k-ary predicate symbol P holds
$\mathbf{Quadruples}_{P[l]} = \{\langle P[l], |\bullet : P[l]|_{\mathbb{N}}, \emptyset_{\mathrm{BoundVar}}, \mathrm{id}_{\mathrm{BoundVar}}\rangle\}$.

(39) For all q, k, K, f such that $\langle q, k, K, f\rangle \in \mathbf{Quadruples}_p$ holds $\mathrm{snb}(q) \subseteq \mathrm{snb}(p) \cup K$.

(40) If $\langle q, m, K, f\rangle \in \mathbf{Quadruples}_p$ and $\mathrm{x}_i \in f^\circ K$, then $i < m$.

(41) If $\langle q, m, K, f\rangle \in \mathbf{Quadruples}_p$, then $\mathrm{x}_m \notin f^\circ K$.

(42) If $\langle q, m, K, f\rangle \in \mathbf{Quadruples}_p$ and $\mathrm{x}_i \in f^\circ \mathrm{snb}(p)$, then $i < m$.

(43) If $\langle q, m, K, f\rangle \in \mathbf{Quadruples}_p$ and $\mathrm{x}_i \in f^\circ \mathrm{snb}(q)$, then $i < m$.

(44) If $\langle q, m, K, f\rangle \in \mathbf{Quadruples}_p$, then $\mathrm{x}_m \notin f^\circ \mathrm{snb}(q)$.

(45) $\mathrm{snb}(p) = \mathrm{snb}(p\,\text{with variables separated})$.

(46) $|\bullet : p|_{\mathbb{N}} = |\bullet : p\,\text{with variables separated}\,|_{\mathbb{N}}$.

Let us consider p, q. We say that p and q are similar if and only if:

(Def.14) $p\,$with variables separated $= q\,$with variables separated.

One can prove the following propositions:

(47) p and p are similar.

(48) If p and q are similar, then q and p are similar.

(49) If p and q are similar and q and r are similar, then p and r are similar.

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Received November 22, 1991

FORMALIZED MATHEMATICS
Vol.2,No.5, November–December 1991
Université Catholique de Louvain

Category of Rings

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Summary. We define the category of non-associative rings. The carriers of the rings are included in a universum. The universum is a parameter of the category.

MML Identifier: `RINGCAT1`.

The papers [14], [2], [15], [3], [1], [12], [7], [8], [5], [4], [13], [11], [6], [10], and [9] provide the terminology and notation for this paper. For simplicity we follow a convention: x, y will be arbitrary, D will be a non-empty set, U_1 will be a universal class, and G, H will be field structures. Let us consider G, H. A map from G into H is a function from the carrier of G into the carrier of H.

Let G_1, G_2, G_3 be field structures, and let f be a map from G_1 into G_2, and let g be a map from G_2 into G_3. Then $g \cdot f$ is a map from G_1 into G_3.

Let us consider G. The functor id_G yields a map from G into G and is defined by:

(Def.1) $\mathrm{id}_G = \mathrm{id}_{(\text{the carrier of } G)}$.

The following propositions are true:

(1) For every scalar x of G holds $\mathrm{id}_G(x) = x$.

(2) For every map f from G into H holds $f \cdot \mathrm{id}_G = f$ and $\mathrm{id}_H \cdot f = f$.

Let us consider G, H. A map from G into H is linear if:

(Def.2) for all scalars x, y of G holds $\mathrm{it}(x+y) = \mathrm{it}(x) + \mathrm{it}(y)$ and for all scalars x, y of G holds $\mathrm{it}(x \cdot y) = \mathrm{it}(x) \cdot \mathrm{it}(y)$ and $\mathrm{it}(1_G) = 1_H$.

We now state the proposition

(3) For all G_1, G_2, G_3 being field structures and for every map f from G_1 into G_2 and for every map g from G_2 into G_3 such that f is linear and g is linear holds $g \cdot f$ is linear.

ISSN 0777–4028

We consider ring morphisms structures which are systems

⟨a dom-map, a cod-map, a `Fun`⟩,

where the dom-map, the cod-map are a ring and the `Fun` is a map from the dom-map into the cod-map.

We now define three new functors. Let us consider f. The functor $\mathrm{dom}\, f$ yields a ring and is defined by:

(Def.3) $\mathrm{dom}\, f =$ the dom-map of f.

The functor $\mathrm{cod}\, f$ yields a ring and is defined by:

(Def.4) $\mathrm{cod}\, f =$ the cod-map of f.

The functor $\mathrm{fun}\, f$ yields a map from the dom-map of f into the cod-map of f and is defined by:

(Def.5) $\mathrm{fun}\, f =$ the `Fun` of f.

In the sequel G, H, G_1, G_2, G_3, G_4 will denote rings. A ring morphisms structure is called a morphism of rings if:

(Def.6) fun it is linear.

Let us consider G. The functor I_G yields a strict morphism of rings and is defined as follows:

(Def.7) $\mathrm{I}_G = \langle G, G, \mathrm{id}_G \rangle$.

Let us consider G, H. The predicate $G \leq H$ is defined as follows:

(Def.8) there exists a morphism F of rings such that $\mathrm{dom}\, F = G$ and $\mathrm{cod}\, F = H$.

We now state the proposition

(4) $G \leq G$.

Let us consider G, H. Let us assume that $G \leq H$. A strict morphism of rings is said to be a morphism from G to H if:

(Def.9) dom it $= G$ and cod it $= H$.

Let us consider G. Then I_G is a strict morphism from G to G.

We now state three propositions:

(5) For all morphisms g, f of rings such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ there exist G_1, G_2, G_3 such that $G_1 \leq G_2$ and $G_2 \leq G_3$ and the ring morphisms structure of g is a morphism from G_2 to G_3 and the ring morphisms structure of f is a morphism from G_1 to G_2.

(6) For every strict morphism F of rings holds F is a morphism from $\mathrm{dom}\, F$ to $\mathrm{cod}\, F$ and $\mathrm{dom}\, F \leq \mathrm{cod}\, F$.

(7) For every strict morphism F of rings there exist G, H and there exists a map f from G into H such that F is a morphism from G to H and $F = \langle G, H, f \rangle$ and f is linear.

Let G, F be morphisms of rings. Let us assume that $\mathrm{dom}\, G = \mathrm{cod}\, F$. The functor $G \cdot F$ yields a strict morphism of rings and is defined by:

(Def.10) for all G_1, G_2, G_3 and for every map g from G_2 into G_3 and for every map f from G_1 into G_2 such that the ring morphisms structure of $G = \langle G_2, G_3, g\rangle$ and the ring morphisms structure of $F = \langle G_1, G_2, f\rangle$ holds $G \cdot F = \langle G_1, G_3, g \cdot f\rangle$.

We now state two propositions:

(8) If $G_1 \leq G_2$ and $G_2 \leq G_3$, then $G_1 \leq G_3$.

(9) For every morphism G from G_2 to G_3 and for every morphism F from G_1 to G_2 such that $G_1 \leq G_2$ and $G_2 \leq G_3$ holds $G \cdot F$ is a morphism from G_1 to G_3.

Let us consider G_1, G_2, G_3, and let G be a morphism from G_2 to G_3, and let F be a morphism from G_1 to G_2. Let us assume that $G_1 \leq G_2$ and $G_2 \leq G_3$. The functor $F[G]$ yields a strict morphism from G_1 to G_3 and is defined as follows:

(Def.11) $F[G] = G \cdot F$.

The following propositions are true:

(10) For all strict morphisms f, g of rings such that $\operatorname{dom} g = \operatorname{cod} f$ there exist G_1, G_2, G_3 and there exists a map f_0 from G_1 into G_2 and there exists a map g_0 from G_2 into G_3 such that $f = \langle G_1, G_2, f_0\rangle$ and $g = \langle G_2, G_3, g_0\rangle$ and $g \cdot f = \langle G_1, G_3, g_0 \cdot f_0\rangle$.

(11) For all strict morphisms f, g of rings such that $\operatorname{dom} g = \operatorname{cod} f$ holds $\operatorname{dom}(g \cdot f) = \operatorname{dom} f$ and $\operatorname{cod}(g \cdot f) = \operatorname{cod} g$.

(12) For every morphism f from G_1 to G_2 and for every morphism g from G_2 to G_3 and for every morphism h from G_3 to G_4 such that $G_1 \leq G_2$ and $G_2 \leq G_3$ and $G_3 \leq G_4$ holds $h \cdot (g \cdot f) = (h \cdot g) \cdot f$.

(13) For all strict morphisms f, g, h of rings such that $\operatorname{dom} h = \operatorname{cod} g$ and $\operatorname{dom} g = \operatorname{cod} f$ holds $h \cdot (g \cdot f) = (h \cdot g) \cdot f$.

(14) $\operatorname{dom}(\mathrm{I}_G) = G$ and $\operatorname{cod}(\mathrm{I}_G) = G$ and for every strict morphism f of rings such that $\operatorname{cod} f = G$ holds $\mathrm{I}_G \cdot f = f$ and for every strict morphism g of rings such that $\operatorname{dom} g = G$ holds $g \cdot \mathrm{I}_G = g$.

A non-empty set is said to be a non-empty set of rings if:

(Def.12) for every element x of it holds x is a strict ring.

In the sequel V denotes a non-empty set of rings. Let us consider V. We see that the element of V is a ring.

One can prove the following two propositions:

(15) For every strict morphism f of rings and for every element x of $\{f\}$ holds x is a strict morphism of rings.

(16) For every morphism f from G to H and for every element x of $\{f\}$ holds x is a morphism from G to H.

A non-empty set is said to be a non-empty set of morphisms of rings if:

(Def.13) for every element x of it holds x is a strict morphism of rings.

Let M be a non-empty set of morphisms of rings. We see that the element of M is a morphism of rings.

Next we state the proposition

(17) For every strict morphism f of rings holds $\{f\}$ is a non-empty set of morphisms of rings.

Let us consider G, H. A non-empty set of morphisms of rings is called a non-empty set of morphisms from G into H if:

(Def.14) for every element x of it holds x is a morphism from G to H.

The following two propositions are true:

(18) D is a non-empty set of morphisms from G into H if and only if for every element x of D holds x is a morphism from G to H.

(19) For every morphism f from G to H holds $\{f\}$ is a non-empty set of morphisms from G into H.

Let us consider G, H. Let us assume that $G \leq H$. The functor $\mathrm{Morphs}(G, H)$ yielding a non-empty set of morphisms from G into H is defined by:

(Def.15) $x \in \mathrm{Morphs}(G, H)$ if and only if x is a morphism from G to H.

Let us consider G, H, and let M be a non-empty set of morphisms from G into H. We see that the element of M is a morphism from G to H.

Let us consider x, y. The predicate $\mathrm{P_{ob}}\, x, y$ is defined by the condition (Def.16).

(Def.16) There exist arbitrary $x_1, x_2, x_3, x_4, x_5, x_6$ such that $x = \langle\langle x_1, x_2, x_3, x_4\rangle, x_5, x_6\rangle$ and there exists a strict ring G such that $y = G$ and $x_1 =$ the carrier of G and $x_2 =$ the addition of G and $x_3 =$ the reverse-map of G and $x_4 =$ the zero of G and $x_5 =$ the multiplication of G and $x_6 =$ the unity of G.

We now state two propositions:

(20) For arbitrary x, y_1, y_2 such that $\mathrm{P_{ob}}\, x, y_1$ and $\mathrm{P_{ob}}\, x, y_2$ holds $y_1 = y_2$.

(21) There exists x such that $x \in U_1$ and $\mathrm{P_{ob}}\, x, \mathrm{Z}_3$.

Let us consider U_1. The functor $\mathrm{RingObj}(U_1)$ yielding a non-empty set is defined as follows:

(Def.17) for every y holds $y \in \mathrm{RingObj}(U_1)$ if and only if there exists x such that $x \in U_1$ and $\mathrm{P_{ob}}\, x, y$.

We now state two propositions:

(22) $\mathrm{Z}_3 \in \mathrm{RingObj}(U_1)$.

(23) For every element x of $\mathrm{RingObj}(U_1)$ holds x is a strict ring.

Let us consider U_1. Then $\mathrm{RingObj}(U_1)$ is a non-empty set of rings.

Let us consider V. The functor $\mathrm{Morphs}\, V$ yielding a non-empty set of morphisms of rings is defined as follows:

(Def.18) $x \in \mathrm{Morphs}\, V$ if and only if there exist elements G, H of V such that $G \leq H$ and x is a morphism from G to H.

Let us consider V, and let F be an element of $\operatorname{Morphs} V$. Then $\operatorname{dom} F$ is an element of V. Then $\operatorname{cod} F$ is an element of V.

Let us consider V, and let G be an element of V. The functor I_G yields a strict element of $\operatorname{Morphs} V$ and is defined by:

(Def.19) $\mathrm{I}_G = \mathrm{I}_G$.

We now define three new functors. Let us consider V. The functor $\operatorname{dom} V$ yields a function from $\operatorname{Morphs} V$ into V and is defined as follows:

(Def.20) for every element f of $\operatorname{Morphs} V$ holds $(\operatorname{dom} V)(f) = \operatorname{dom} f$.

The functor $\operatorname{cod} V$ yielding a function from $\operatorname{Morphs} V$ into V is defined as follows:

(Def.21) for every element f of $\operatorname{Morphs} V$ holds $(\operatorname{cod} V)(f) = \operatorname{cod} f$.

The functor I_V yields a function from V into $\operatorname{Morphs} V$ and is defined by:

(Def.22) for every element G of V holds $\mathrm{I}_V(G) = \mathrm{I}_G$.

We now state two propositions:

(24) For all elements g, f of $\operatorname{Morphs} V$ such that $\operatorname{dom} g = \operatorname{cod} f$ there exist elements G_1, G_2, G_3 of V such that $G_1 \leq G_2$ and $G_2 \leq G_3$ and g is a morphism from G_2 to G_3 and f is a morphism from G_1 to G_2.

(25) For all elements g, f of $\operatorname{Morphs} V$ such that $\operatorname{dom} g = \operatorname{cod} f$ holds $g \cdot f \in \operatorname{Morphs} V$.

Let us consider V. The functor $\operatorname{comp} V$ yielding a partial function from $[\!:\operatorname{Morphs} V, \operatorname{Morphs} V:\!]$ to $\operatorname{Morphs} V$ is defined as follows:

(Def.23) for all elements g, f of $\operatorname{Morphs} V$ holds $\langle g, f\rangle \in \operatorname{dom} \operatorname{comp} V$ if and only if $\operatorname{dom} g = \operatorname{cod} f$ and for all elements g, f of $\operatorname{Morphs} V$ such that $\langle g, f\rangle \in \operatorname{dom} \operatorname{comp} V$ holds $(\operatorname{comp} V)(\langle g, f\rangle) = g \cdot f$.

Let us consider U_1. The functor $\operatorname{RingCat}(U_1)$ yielding a strict category structure is defined by:

(Def.24) $\operatorname{RingCat}(U_1) = \langle \operatorname{RingObj}(U_1), \operatorname{Morphs} \operatorname{RingObj}(U_1), \operatorname{dom} \operatorname{RingObj}(U_1), \operatorname{cod} \operatorname{RingObj}(U_1), \operatorname{comp} \operatorname{RingObj}(U_1), \mathrm{I}_{\operatorname{RingObj}(U_1)}\rangle$.

The following propositions are true:

(26) For all morphisms f, g of $\operatorname{RingCat}(U_1)$ holds $\langle g, f\rangle \in \operatorname{dom}$ (the composition of $\operatorname{RingCat}(U_1)$) if and only if $\operatorname{dom} g = \operatorname{cod} f$.

(27) For every morphism f of $\operatorname{RingCat}(U_1)$ and for every element f' of $\operatorname{Morphs} \operatorname{RingObj}(U_1)$ and for every object b of $\operatorname{RingCat}(U_1)$ and for every element b' of $\operatorname{RingObj}(U_1)$ holds f is a strict element of $\operatorname{Morphs} \operatorname{RingObj}(U_1)$ and f' is a morphism of $\operatorname{RingCat}(U_1)$ and b is a strict element of $\operatorname{RingObj}(U_1)$ and b' is an object of $\operatorname{RingCat}(U_1)$.

(28) For every object b of $\operatorname{RingCat}(U_1)$ and for every element b' of $\operatorname{RingObj}(U_1)$ such that $b = b'$ holds $\mathrm{id}_b = \mathrm{I}_{b'}$.

(29) For every morphism f of $\mathrm{RingCat}(U_1)$ and for every element f' of $\mathrm{Morphs}\,\mathrm{RingObj}(U_1)$ such that $f = f'$ holds $\mathrm{dom}\, f = \mathrm{dom}\, f'$ and $\mathrm{cod}\, f = \mathrm{cod}\, f'$.

(30) Let f, g be morphisms of $\mathrm{RingCat}(U_1)$. Let f', g' be elements of $\mathrm{Morphs}\,\mathrm{RingObj}(U_1)$. Suppose $f = f'$ and $g = g'$. Then

(i) $\mathrm{dom}\, g = \mathrm{cod}\, f$ if and only if $\mathrm{dom}\, g' = \mathrm{cod}\, f'$,

(ii) $\mathrm{dom}\, g = \mathrm{cod}\, f$ if and only if $\langle g', f' \rangle \in \mathrm{dom}\,\mathrm{comp}\,\mathrm{RingObj}(U_1)$,

(iii) if $\mathrm{dom}\, g = \mathrm{cod}\, f$, then $g \cdot f = g' \cdot f'$,

(iv) $\mathrm{dom}\, f = \mathrm{dom}\, g$ if and only if $\mathrm{dom}\, f' = \mathrm{dom}\, g'$,

(v) $\mathrm{cod}\, f = \mathrm{cod}\, g$ if and only if $\mathrm{cod}\, f' = \mathrm{cod}\, g'$.

Let us consider U_1. Then $\mathrm{RingCat}(U_1)$ is a strict category.

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Received December 5, 1991

FORMALIZED MATHEMATICS
Vol.2,No.5, November–December 1991
Université Catholique de Louvain

Category of Left Modules

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Summary. We define the category of left modules over an associative ring. The carriers of the modules are included in a universum. The universum is a parameter of the category.

MML Identifier: `MODCAT_1`.

The papers [12], [1], [2], [4], [5], [7], [3], [11], [10], [9], [6], and [8] provide the terminology and notation for this paper. For simplicity we adopt the following convention: x, y are arbitrary, D is a non-empty set, U_1 is a universal class, R is an associative ring, and G, H are left modules over R. Let us consider R. A non-empty set is said to be a non-empty set of left-modules of R if:

(Def.1) for every element x of it holds x is a strict left module over R.

In the sequel V is a non-empty set of left-modules of R. Let us consider R, V. We see that the element of V is a left module over R.

We now state two propositions:

(1) For every left module morphism f of R and for every element x of $\{f\}$ holds x is a left module morphism of R.

(2) For every strict morphism f from G to H and for every element x of $\{f\}$ holds x is a strict morphism from G to H.

Let us consider R. A non-empty set is said to be a non-empty set of morphisms of left-modules of R if:

(Def.2) for every element x of it holds x is a strict left module morphism of R.

Let us consider R, and let M be a non-empty set of morphisms of left-modules of R. We see that the element of M is a left module morphism of R.

Next we state the proposition

(3) For every strict left module morphism f of R holds $\{f\}$ is a non-empty set of morphisms of left-modules of R.

ISSN 0777–4028

Let us consider R, G, H. A non-empty set of morphisms of left-modules of R is called a non-empty set of morphisms of left-modules from G into H if:

(Def.3) for every element x of it holds x is a strict morphism from G to H.

The following two propositions are true:

(4) D is a non-empty set of morphisms of left-modules from G into H if and only if for every element x of D holds x is a strict morphism from G to H.

(5) For every strict morphism f from G to H holds $\{f\}$ is a non-empty set of morphisms of left-modules from G into H.

Let us consider R, G, H. The functor $\mathrm{Morphs}(G, H)$ yields a non-empty set of morphisms of left-modules from G into H and is defined as follows:

(Def.4) $x \in \mathrm{Morphs}(G, H)$ if and only if x is a strict morphism from G to H.

Let us consider R, G, H, and let M be a non-empty set of morphisms of left-modules from G into H. We see that the element of M is a morphism from G to H.

Let us consider x, y, R. The predicate $\mathrm{P_{ob}}\, x, y, R$ is defined by:

(Def.5) there exist arbitrary x_1, x_2 such that $x = \langle x_1, x_2 \rangle$ and there exists a strict left module G over R such that $y = G$ and $x_1 =$ the carrier of G and $x_2 =$ the left multiplication of G.

One can prove the following propositions:

(6) For arbitrary x, y_1, y_2 such that $\mathrm{P_{ob}}\, x, y_1, R$ and $\mathrm{P_{ob}}\, x, y_2, R$ holds $y_1 = y_2$.

(7) For every U_1 there exists x such that $x \in \{\langle G, f \rangle\}$, where G ranges over elements of $\mathrm{GroupObj}(U_1)$, and f ranges over elements of $\{\emptyset\}^{[\text{: the carrier of } R, \{\emptyset\} \text{ :}]}$ and $\mathrm{P_{ob}}\, x, {}_R\Theta, R$.

Let us consider U_1, R. The functor $\mathrm{LModObj}(U_1, R)$ yielding a non-empty set is defined as follows:

(Def.6) for every y holds $y \in \mathrm{LModObj}(U_1, R)$ if and only if there exists x such that $x \in \{\langle G, f \rangle\}$, where G ranges over elements of $\mathrm{GroupObj}(U_1)$, and f ranges over elements of $\{\emptyset\}^{[\text{: the carrier of } R, \{\emptyset\} \text{ :}]}$ and $\mathrm{P_{ob}}\, x, y, R$.

One can prove the following two propositions:

(8) ${}_R\Theta \in \mathrm{LModObj}(U_1, R)$.

(9) For every element x of $\mathrm{LModObj}(U_1, R)$ holds x is a strict left module over R.

Let us consider U_1, R. Then $\mathrm{LModObj}(U_1, R)$ is a non-empty set of left-modules of R.

Let us consider R, V. The functor $\mathrm{Morphs}\, V$ yields a non-empty set of morphisms of left-modules of R and is defined as follows:

(Def.7) for every x holds $x \in \mathrm{Morphs}\, V$ if and only if there exist strict elements G, H of V such that x is a strict morphism from G to H.

We now define two new functors. Let us consider R, V, and let F be an element of $\mathrm{Morphs}\,V$. The functor $\mathrm{dom}'\,F$ yields an element of V and is defined as follows:

(Def.8) $\mathrm{dom}'\,F = \mathrm{dom}\,F$.

The functor $\mathrm{cod}'\,F$ yields an element of V and is defined by:

(Def.9) $\mathrm{cod}'\,F = \mathrm{cod}\,F$.

Let us consider R, V, and let G be an element of V. The functor I_G yielding a strict element of $\mathrm{Morphs}\,V$ is defined as follows:

(Def.10) $\mathrm{I}_G = \mathrm{I}_G$.

We now define three new functors. Let us consider R, V. The functor $\mathrm{dom}\,V$ yields a function from $\mathrm{Morphs}\,V$ into V and is defined by:

(Def.11) for every element f of $\mathrm{Morphs}\,V$ holds $(\mathrm{dom}\,V)(f) = \mathrm{dom}'\,f$.

The functor $\mathrm{cod}\,V$ yields a function from $\mathrm{Morphs}\,V$ into V and is defined as follows:

(Def.12) for every element f of $\mathrm{Morphs}\,V$ holds $(\mathrm{cod}\,V)(f) = \mathrm{cod}'\,f$.

The functor I_V yields a function from V into $\mathrm{Morphs}\,V$ and is defined by:

(Def.13) for every element G of V holds $\mathrm{I}_V(G) = \mathrm{I}_G$.

One can prove the following three propositions:

(10) For all elements g, f of $\mathrm{Morphs}\,V$ such that $\mathrm{dom}'\,g = \mathrm{cod}'\,f$ there exist strict elements G_1, G_2, G_3 of V such that g is a morphism from G_2 to G_3 and f is a morphism from G_1 to G_2.

(11) For all elements g, f of $\mathrm{Morphs}\,V$ such that $\mathrm{dom}'\,g = \mathrm{cod}'\,f$ holds $g \cdot f \in \mathrm{Morphs}\,V$.

(12) For all elements g, f of $\mathrm{Morphs}\,V$ such that $\mathrm{dom}\,g = \mathrm{cod}\,f$ holds $g \cdot f \in \mathrm{Morphs}\,V$.

Let us consider R, V. The functor $\mathrm{comp}\,V$ yields a partial function from $[\!:\mathrm{Morphs}\,V, \mathrm{Morphs}\,V:\!]$ to $\mathrm{Morphs}\,V$ and is defined by:

(Def.14) for all elements g, f of $\mathrm{Morphs}\,V$ holds $\langle g, f\rangle \in \mathrm{dom}\,\mathrm{comp}\,V$ if and only if $\mathrm{dom}'\,g = \mathrm{cod}'\,f$ and for all elements g, f of $\mathrm{Morphs}\,V$ such that $\langle g, f\rangle \in \mathrm{dom}\,\mathrm{comp}\,V$ holds $(\mathrm{comp}\,V)(\langle g, f\rangle) = g \cdot f$.

The following proposition is true

(13) For all elements g, f of $\mathrm{Morphs}\,V$ holds $\langle g, f\rangle \in \mathrm{dom}\,\mathrm{comp}\,V$ if and only if $\mathrm{dom}\,g = \mathrm{cod}\,f$.

Let us consider U_1, R. The functor $\mathrm{LModCat}(U_1, R)$ yields a strict category structure and is defined by:

(Def.15) $\mathrm{LModCat}(U_1, R) = \langle \mathrm{LModObj}(U_1, R), \mathrm{Morphs}\,\mathrm{LModObj}(U_1, R), \mathrm{dom}\,\mathrm{LModObj}(U_1, R), \mathrm{cod}\,\mathrm{LModObj}(U_1, R), \mathrm{comp}\,\mathrm{LModObj}(U_1, R), \mathrm{I}_{\mathrm{LModObj}(U_1, R)}\rangle$.

One can prove the following propositions:

(14) For all morphisms f, g of $\mathrm{LModCat}(U_1, R)$ holds $\langle g, f\rangle \in \mathrm{dom}\,(\text{the composition of } \mathrm{LModCat}(U_1, R))$ if and only if $\mathrm{dom}\,g = \mathrm{cod}\,f$.

(15) Let f be a morphism of $\mathrm{LModCat}(U_1, R)$. Then for every element f' of $\mathrm{Morphs}\,\mathrm{LModObj}(U_1, R)$ and for every object b of $\mathrm{LModCat}(U_1, R)$ and for every element b' of $\mathrm{LModObj}(U_1, R)$ holds f is a strict element of $\mathrm{Morphs}\,\mathrm{LModObj}(U_1, R)$ and f' is a morphism of $\mathrm{LModCat}(U_1, R)$ and b is a strict element of $\mathrm{LModObj}(U_1, R)$ and b' is an object of $\mathrm{LModCat}(U_1, R)$.

(16) For every object b of $\mathrm{LModCat}(U_1, R)$ and for every element b' of $\mathrm{LModObj}(U_1, R)$ such that $b = b'$ holds $\mathrm{id}_b = \mathrm{I}_{b'}$.

(17) For every morphism f of $\mathrm{LModCat}(U_1, R)$ and for every element f' of $\mathrm{Morphs}\,\mathrm{LModObj}(U_1, R)$ such that $f = f'$ holds $\mathrm{dom}\, f = \mathrm{dom}\, f'$ and $\mathrm{cod}\, f = \mathrm{cod}\, f'$.

(18) Let f, g be morphisms of $\mathrm{LModCat}(U_1, R)$. Let f', g' be elements of $\mathrm{Morphs}\,\mathrm{LModObj}(U_1, R)$. Suppose $f = f'$ and $g = g'$. Then

(i) $\mathrm{dom}\, g = \mathrm{cod}\, f$ if and only if $\mathrm{dom}\, g' = \mathrm{cod}\, f'$,

(ii) $\mathrm{dom}\, g = \mathrm{cod}\, f$ if and only if $\langle g', f' \rangle \in \mathrm{dom}\,\mathrm{comp}\,\mathrm{LModObj}(U_1, R)$,

(iii) if $\mathrm{dom}\, g = \mathrm{cod}\, f$, then $g \cdot f = g' \cdot f'$,

(iv) $\mathrm{dom}\, f = \mathrm{dom}\, g$ if and only if $\mathrm{dom}\, f' = \mathrm{dom}\, g'$,

(v) $\mathrm{cod}\, f = \mathrm{cod}\, g$ if and only if $\mathrm{cod}\, f' = \mathrm{cod}\, g'$.

Let us consider U_1, R. Then $\mathrm{LModCat}(U_1, R)$ is a strict category.

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Received December 12, 1991

FORMALIZED MATHEMATICS
Vol.2,No.5, November–December 1991
Université Catholique de Louvain

Real Function One-Side Differantiability

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Summary. We define real function one-side differantiability and one-side continuity. Main properties of one-side differentiability function are proved. Connections between one-side differential and differential real function at the point are demonstrated.

MML Identifier: FDIFF_3.

The terminology and notation used in this paper have been introduced in the following papers: [17], [2], [4], [1], [11], [5], [7], [14], [18], [3], [8], [9], [10], [16], [15], [12], [13], and [6]. For simplicity we follow the rules: h, h_1, h_2 are real sequences convergent to 0, c is a constant real sequence, f, f_1, f_2 are partial functions from $\mathbb{R}$ to $\mathbb{R}$, x_0, r, r_1, g, g_1, g_2 are real numbers, n is a natural number, and a is a sequence of real numbers. The following propositions are true:

(1) If there exists r such that $r > 0$ and $[x_0 - r, x_0] \subseteq \operatorname{dom} f$, then there exist h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom} f$ and for every n holds $h(n) < 0$.

(2) If there exists r such that $r > 0$ and $[x_0, x_0 + r] \subseteq \operatorname{dom} f$, then there exist h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom} f$ and for every n holds $h(n) > 0$.

(3) Suppose For all h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h+c) \subseteq \operatorname{dom} f$ and for every n holds $h(n) < 0$ holds $h^{-1}\,(f \cdot (h+c) - f \cdot c)$ is convergent and $\{x_0\} \subseteq \operatorname{dom} f$. Given h_1, h_2, c. Suppose $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h_1 + c) \subseteq \operatorname{dom} f$ and for every n holds $h_1(n) < 0$ and $\operatorname{rng}(h_2 + c) \subseteq \operatorname{dom} f$ and for every n holds $h_2(n) < 0$. Then $\lim({h_1}^{-1}\,(f \cdot (h_1 + c) - f \cdot c)) = \lim({h_2}^{-1}\,(f \cdot (h_2 + c) - f \cdot c))$.

(4) Suppose For all h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h+c) \subseteq \operatorname{dom} f$ and for every n holds $h(n) > 0$ holds $h^{-1}\,(f \cdot (h+c) - f \cdot c)$ is convergent and $\{x_0\} \subseteq \operatorname{dom} f$. Given h_1, h_2, c. Suppose $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h_1 + c) \subseteq$

ISSN 0777–4028

dom f and $\operatorname{rng}(h_2 + c) \subseteq \operatorname{dom} f$ and for every n holds $h_1(n) > 0$ and for every n holds $h_2(n) > 0$. Then $\lim({h_1}^{-1} (f \cdot (h_1 + c) - f \cdot c)) = \lim({h_2}^{-1} (f \cdot (h_2 + c) - f \cdot c))$.

We now define four new predicates. Let us consider f, x_0. We say that f is left continous in x_0 if and only if:

(Def.1) $x_0 \in \operatorname{dom} f$ and for every a such that $\operatorname{rng} a \subseteq]-\infty, x_0[\cap \operatorname{dom} f$ and a is convergent and $\lim a = x_0$ holds $f \cdot a$ is convergent and $f(x_0) = \lim(f \cdot a)$.

We say that f is right continous in x_0 if and only if:

(Def.2) $x_0 \in \operatorname{dom} f$ and for every a such that $\operatorname{rng} a \subseteq]x_0, +\infty[\cap \operatorname{dom} f$ and a is convergent and $\lim a = x_0$ holds $f \cdot a$ is convergent and $f(x_0) = \lim(f \cdot a)$.

We say that f is right differentiable in x_0 if and only if the conditions (Def.3) is satisfied.

(Def.3) (i) There exists r such that $r > 0$ and $[x_0, x_0 + r] \subseteq \operatorname{dom} f$,

(ii) for all h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom} f$ and for every n holds $h(n) > 0$ holds $h^{-1} (f \cdot (h + c) - f \cdot c)$ is convergent.

We say that f is left differentiable in x_0 if and only if the conditions (Def.4) is satisfied.

(Def.4) (i) There exists r such that $r > 0$ and $[x_0 - r, x_0] \subseteq \operatorname{dom} f$,

(ii) for all h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom} f$ and for every n holds $h(n) < 0$ holds $h^{-1} (f \cdot (h + c) - f \cdot c)$ is convergent.

One can prove the following propositions:

(5) If f is left differentiable in x_0, then f is left continous in x_0.

(6) Suppose f is left continous in x_0 and $f(x_0) \neq g_2$ and there exists r such that $r > 0$ and $[x_0 - r, x_0] \subseteq \operatorname{dom} f$. Then there exists r_1 such that $r_1 > 0$ and $[x_0 - r_1, x_0] \subseteq \operatorname{dom} f$ and for every g such that $g \in [x_0 - r_1, x_0]$ holds $f(g) \neq g_2$.

(7) If f is right differentiable in x_0, then f is right continous in x_0.

(8) Suppose f is right continous in x_0 and $f(x_0) \neq g_2$ and there exists r such that $r > 0$ and $[x_0, x_0 + r] \subseteq \operatorname{dom} f$. Then there exists r_1 such that $r_1 > 0$ and $[x_0, x_0 + r_1] \subseteq \operatorname{dom} f$ and for every g such that $g \in [x_0, x_0 + r_1]$ holds $f(g) \neq g_2$.

Let us consider x_0, f. Let us assume that f is left differentiable in x_0. The functor $f'_{-}(x_0)$ yielding a real number is defined by:

(Def.5) for all h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom} f$ and for every n holds $h(n) < 0$ holds $f'_{-}(x_0) = \lim(h^{-1} (f \cdot (h + c) - f \cdot c))$.

Let us consider x_0, f. Let us assume that f is right differentiable in x_0. The functor $f'_{+}(x_0)$ yields a real number and is defined by:

(Def.6) for all h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom} f$ and for every n holds $h(n) > 0$ holds $f'_{+}(x_0) = \lim(h^{-1} (f \cdot (h + c) - f \cdot c))$.

The following propositions are true:

(9) f is left differentiable in x_0 and $f'_{-}(x_0) = g$ if and only if the following conditions are satisfied:

(i) there exists r such that $0 < r$ and $[x_0 - r, x_0] \subseteq \operatorname{dom} f$,

(ii) for all h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom} f$ and for every n holds $h(n) < 0$ holds $h^{-1} (f \cdot (h + c) - f \cdot c)$ is convergent and $\lim(h^{-1} (f \cdot (h + c) - f \cdot c)) = g$.

(10) If f_1 is left differentiable in x_0 and f_2 is left differentiable in x_0, then $f_1 + f_2$ is left differentiable in x_0 and $(f_1 + f_2)'_{-}(x_0) = {f_1}'_{-}(x_0) + {f_2}'_{-}(x_0)$.

(11) If f_1 is left differentiable in x_0 and f_2 is left differentiable in x_0, then $f_1 - f_2$ is left differentiable in x_0 and $(f_1 - f_2)'_{-}(x_0) = {f_1}'_{-}(x_0) - {f_2}'_{-}(x_0)$.

(12) If f_1 is left differentiable in x_0 and f_2 is left differentiable in x_0, then $f_1 f_2$ is left differentiable in x_0 and $(f_1 f_2)'_{-}(x_0) = {f_1}'_{-}(x_0) \cdot f_2(x_0) + {f_2}'_{-}(x_0) \cdot f_1(x_0)$.

(13) If f_1 is left differentiable in x_0 and f_2 is left differentiable in x_0 and $f_2(x_0) \neq 0$, then $\frac{f_1}{f_2}$ is left differentiable in x_0 and $(\frac{f_1}{f_2})'_{-}(x_0) = \frac{{f_1}'_{-}(x_0) \cdot f_2(x_0) - {f_2}'_{-}(x_0) \cdot f_1(x_0)}{f_2(x_0)^{\mathbf{2}}}$.

(14) If f is left differentiable in x_0 and $f(x_0) \neq 0$, then $\frac{1}{f}$ is left differentiable in x_0 and $(\frac{1}{f})'_{-}(x_0) = -\frac{f'_{-}(x_0)}{f(x_0)^{\mathbf{2}}}$.

(15) f is right differentiable in x_0 and $f'_{+}(x_0) = g_1$ if and only if the following conditions are satisfied:

(i) there exists r such that $r > 0$ and $[x_0, x_0 + r] \subseteq \operatorname{dom} f$,

(ii) for all h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom} f$ and for every n holds $h(n) > 0$ holds $h^{-1} (f \cdot (h + c) - f \cdot c)$ is convergent and $\lim(h^{-1} (f \cdot (h + c) - f \cdot c)) = g_1$.

(16) If f_1 is right differentiable in x_0 and f_2 is right differentiable in x_0, then $f_1 + f_2$ is right differentiable in x_0 and $(f_1 + f_2)'_{+}(x_0) = {f_1}'_{+}(x_0) + {f_2}'_{+}(x_0)$.

(17) If f_1 is right differentiable in x_0 and f_2 is right differentiable in x_0, then $f_1 - f_2$ is right differentiable in x_0 and $(f_1 - f_2)'_{+}(x_0) = {f_1}'_{+}(x_0) - {f_2}'_{+}(x_0)$.

(18) If f_1 is right differentiable in x_0 and f_2 is right differentiable in x_0, then $f_1 f_2$ is right differentiable in x_0 and $(f_1 f_2)'_{+}(x_0) = {f_1}'_{+}(x_0) \cdot f_2(x_0) + {f_2}'_{+}(x_0) \cdot f_1(x_0)$.

(19) If f_1 is right differentiable in x_0 and f_2 is right differentiable in x_0 and $f_2(x_0) \neq 0$, then $\frac{f_1}{f_2}$ is right differentiable in x_0 and $(\frac{f_1}{f_2})'_{+}(x_0) = \frac{{f_1}'_{+}(x_0) \cdot f_2(x_0) - {f_2}'_{+}(x_0) \cdot f_1(x_0)}{f_2(x_0)^{\mathbf{2}}}$.

(20) If f is right differentiable in x_0 and $f(x_0) \neq 0$, then $\frac{1}{f}$ is right differentiable in x_0 and $(\frac{1}{f})'_{+}(x_0) = -\frac{f'_{+}(x_0)}{f(x_0)^{\mathbf{2}}}$.

(21) If f is right differentiable in x_0 and f is left differentiable in x_0 and $f'_{+}(x_0) = f'_{-}(x_0)$, then f is differentiable in x_0 and $f'(x_0) = f'_{+}(x_0)$ and $f'(x_0) = f'_{-}(x_0)$.

(22) If f is differentiable in x_0, then f is right differentiable in x_0 and f is left differentiable in x_0 and $f'(x_0) = f'_+(x_0)$ and $f'(x_0) = f'_-(x_0)$.

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Received December 12, 1991

FORMALIZED MATHEMATICS
Vol.2,No.5, November–December 1991
Université Catholique de Louvain

Sequences in Metric Spaces

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Summary. Sequences in metric spaces are defined. The article contains definitions of bounded, convergent, Cauchy sequences. The subsequences are introduced too. Some theorems concerning sequences are proved.

MML Identifier: `METRIC_6`.

The terminology and notation used in this paper have been introduced in the following articles: [11], [14], [4], [5], [3], [6], [13], [12], [7], [10], [8], [9], [1], and [2]. For simplicity we follow a convention: X will be a metric space, x, y, z will be elements of the carrier of X, V will be a subset of the carrier of X, A will be a non-empty set, a will be an element of A, G will be a function from $[\!: A, A :\!]$ into $\mathbb{R}$, k, n, m will be natural numbers, and r will be a real number. The following propositions are true:

(1) $|\rho(x,z) - \rho(y,z)| \leq \rho(x,y)$.

(2) If G is a metric of A, then for all elements a, b of A holds $0 \leq G(a, b)$.

Let us consider A, G. We say that G is not a pseudo metric if and only if:

(Def.1) for all elements a, b of A holds $G(a, b) = 0$ if and only if $a = b$.

Let us consider A, G. We say that G is symmetric if and only if:

(Def.2) for all elements a, b of A holds $G(a, b) = G(b, a)$.

Let us consider A, G. We say that G satisfies triangle inequality if and only if:

(Def.3) for all elements a, b, c of A holds $G(a, c) \leq G(a, b) + G(b, c)$.

Next we state three propositions:

(3) G is a metric of A if and only if G is not a pseudo metric and G is symmetric and G satisfies triangle inequality.

(4) For every strict metric space X holds the distance of X is not a pseudo metric and the distance of X is symmetric and the distance of X satisfies triangle inequality.

ISSN 0777–4028

(5) G is a metric of A if and only if G is not a pseudo metric and for all elements a, b, c of A holds $G(b, c) \leq G(a, b) + G(a, c)$.

Let us consider A, G. Let us assume that G is a metric of A. The functor $\widetilde{G}_A$ yielding a function from $[: A, A :]$ into $\mathbb{R}$ is defined as follows:

(Def.4) for all elements a, b of A holds $\widetilde{G}_A(a, b) = \frac{G(a, b)}{1+G(a, b)}$.

The following proposition is true

(6) If G is a metric of A, then $\widetilde{G}_A$ is a metric of A.

Let X be a metric space. A sequence of elements of X is defined by:

(Def.5) it is a function from $\mathbb{N}$ into the carrier of X.

Let X be a metric space. We see that the sequence of elements of X is a function from $\mathbb{N}$ into the carrier of X.

Next we state the proposition

(7) For every function F from $\mathbb{N}$ into the carrier of X holds F is a sequence of elements of X.

We follow the rules: S, S_1, T denote sequences of elements of X, N_1 denotes an increasing sequence of naturals, and F denotes a function from $\mathbb{N}$ into the carrier of X. The following propositions are true:

(8) F is a sequence of elements of X if and only if for every a such that $a \in \mathbb{N}$ holds $F(a)$ is an element of the carrier of X.

(9) For all S, T such that for every n holds $S(n) = T(n)$ holds $S = T$.

(10) For every x there exists S such that $\operatorname{rng} S = \{x\}$.

(11) If there exists x such that for every n holds $S(n) = x$, then there exists x such that $\operatorname{rng} S = \{x\}$.

Let us consider X, S. We say that S is constant if and only if:

(Def.6) there exists x such that for every n holds $S(n) = x$.

The following proposition is true

(13)[1] S is constant if and only if there exists x such that $\operatorname{rng} S = \{x\}$.

Let us consider X, S. We say that S is convergent if and only if:

(Def.7) there exists x such that for every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\rho(S(n), x) < r$.

Let us consider X, S, x. We say that S is convergent to x if and only if:

(Def.8) for every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\rho(S(n), x) < r$.

Let us consider X, S. We say that S satisfies the Cauchy condition if and only if:

(Def.9) for every r such that $0 < r$ there exists m such that for all n, k such that $m \leq n$ and $m \leq k$ holds $\rho(S(n), S(k)) < r$.

Let us consider X, V. We say that V is bounded if and only if:

[1]The proposition (12) has been removed.

(Def.10) there exist r, x such that $0 < r$ and $V \subseteq \mathrm{Ball}(x, r)$.

Let us consider X, S. We say that S is bounded if and only if:

(Def.11) there exist r, x such that $0 < r$ and $\mathrm{rng}\, S \subseteq \mathrm{Ball}(x, r)$.

Let us consider X, V, S. We say that V contains almost all sequence S if and only if:

(Def.12) there exists m such that for every n such that $m \leq n$ holds $S(n) \in V$.

Let us consider X, s_1, s_2. We say that s_1 is a subsequence of s_2 if and only if:

(Def.13) there exists N_1 such that $s_1 = s_2 \cdot N_1$.

Next we state the proposition

(16)[2] S is convergent to x if and only if for every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\rho(S(n), x) < r$.

We now state three propositions:

(20)[3] S is bounded if and only if there exist r, x such that $0 < r$ and for every n holds $S(n) \in \mathrm{Ball}(x, r)$.

(21) If S is convergent to x, then S is convergent.

(22) If S is convergent, then there exists x such that S is convergent to x.

Let us consider X, S, x. The functor $\rho(S, x)$ yields a sequence of real numbers and is defined as follows:

(Def.14) for every n holds $(\rho(S, x))(n) = \rho(S(n), x)$.

Next we state the proposition

(23) $\rho(S, x)$ is a sequence of real numbers if and only if for every n holds $(\rho(S, x))(n) = \rho(S(n), x)$.

Let us consider X, S, T. The functor $\rho(S, T)$ yields a sequence of real numbers and is defined by:

(Def.15) for every n holds $(\rho(S, T))(n) = \rho(S(n), T(n))$.

Next we state the proposition

(24) $\rho(S, T)$ is a sequence of real numbers if and only if for every n holds $(\rho(S, T))(n) = \rho(S(n), T(n))$.

Let us consider X, S. Let us assume that S is convergent. The functor $\lim S$ yields an element of the carrier of X and is defined as follows:

(Def.16) for every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\rho(S(n), \lim S) < r$.

One can prove the following propositions:

(25) If S is convergent, then $\lim S = x$ if and only if for every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\rho(S(n), x) < r$.

[2]The propositions (14) and (15) have been removed.

[3]The propositions (17)–(19) have been removed.

(26) If S is convergent to x, then $\lim S = x$.

(27) S is convergent to x if and only if S is convergent and $\lim S = x$.

(28) If S is convergent, then there exists x such that S is convergent to x and $\lim S = x$.

(29) S is convergent to x if and only if $\rho(S, x)$ is convergent and $\lim \rho(S, x) = 0$.

(30) If S is convergent to x, then for every r such that $0 < r$ holds $\mathrm{Ball}(x, r)$ contains almost all sequence S.

(31) If for every r such that $0 < r$ holds $\mathrm{Ball}(x, r)$ contains almost all sequence S, then for every V such that $x \in V$ and $V \in$ the open set family of X holds V contains almost all sequence S.

(32) If for every V such that $x \in V$ and $V \in$ the open set family of X holds V contains almost all sequence S, then S is convergent to x.

(33) S is convergent to x if and only if for every r such that $0 < r$ holds $\mathrm{Ball}(x, r)$ contains almost all sequence S.

(34) S is convergent to x if and only if for every V such that $x \in V$ and $V \in$ the open set family of X holds V contains almost all sequence S.

(35) For every r such that $0 < r$ holds $\mathrm{Ball}(x, r)$ contains almost all sequence S if and only if for every V such that $x \in V$ and $V \in$ the open set family of X holds V contains almost all sequence S.

(36) If S is convergent and T is convergent, then $\rho(\lim S, \lim T) = \lim \rho(S, T)$.

(37) If S is convergent to x and S is convergent to y, then $x = y$.

(38) If S is constant, then S is convergent.

(39) If S is convergent to x and S_1 is a subsequence of S, then S_1 is convergent to x.

(40) If S satisfies the Cauchy condition and S_1 is a subsequence of S, then S_1 satisfies the Cauchy condition.

(41) If S is convergent, then S satisfies the Cauchy condition.

(42) If S is constant, then S satisfies the Cauchy condition.

(43) If S is convergent, then S is bounded.

(44) If S satisfies the Cauchy condition, then S is bounded.

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Received December 12, 1991

FORMALIZED MATHEMATICS
Vol.2,No.5, November–December 1991
Université Catholique de Louvain

The Topological Space $\mathcal{E}^2_{\rm T}$. Simple Closed Curves

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Summary. Continuation of [13]. The fact that the unit square is compact is shown in the beginning of the article. Next the notion of simple closed curve is introduced. It is proved that any simple closed curve can be divided into two independent parts which are homeomorphic to unit interval $\mathbb{I}$.

MML Identifier: `TOPREAL2`.

The notation and terminology used here have been introduced in the following articles: [22], [21], [14], [1], [24], [20], [6], [7], [18], [4], [8], [23], [17], [25], [11], [16], [9], [19], [2], [5], [15], [3], [10], [12], and [13]. We follow the rules: p_1, p_2, q_1, q_2 will denote points of $\mathcal{E}^2_{\rm T}$ and P, Q, P_1, P_2 will denote subsets of $\mathcal{E}^2_{\rm T}$. The following propositions are true:

(1) If $p_1 \neq p_2$ and $p_1 \in \square_{\mathcal{E}^2}$ and $p_2 \in \square_{\mathcal{E}^2}$, then there exist P_1, P_2 such that P_1 is an arc from p_1 to p_2 and P_2 is an arc from p_1 to p_2 and $\square_{\mathcal{E}^2} = P_1 \cup P_2$ and $P_1 \cap P_2 = \{p_1, p_2\}$.

(2) $\square_{\mathcal{E}^2}$ is compact.

(3) For every map f from $(\mathcal{E}^2_{\rm T}) \restriction Q$ into $(\mathcal{E}^2_{\rm T}) \restriction P$ such that f is a homeomorphism and Q is an arc from q_1 to q_2 and $P \neq \emptyset$ and for all p_1, p_2 such that $p_1 = f(q_1)$ and $p_2 = f(q_2)$ holds P is an arc from p_1 to p_2.

Let us consider P. We say that P is a simple closed curve if and only if:

(Def.1) $P \neq \emptyset$ and there exists a map f from $(\mathcal{E}^2_{\rm T}) \restriction \square_{\mathcal{E}^2}$ into $(\mathcal{E}^2_{\rm T}) \restriction P$ such that f is a homeomorphism.

Next we state two propositions:

(4) If P is a simple closed curve, then there exist p_1, p_2 such that $p_1 \neq p_2$ and $p_1 \in P$ and $p_2 \in P$.

(5) P is a simple closed curve if and only if the following conditions are satisfied:

ISSN 0777–4028

(i) there exist p_1, p_2 such that $p_1 \neq p_2$ and $p_1 \in P$ and $p_2 \in P$,
(ii) for all p_1, p_2 such that $p_1 \neq p_2$ and $p_1 \in P$ and $p_2 \in P$ there exist P_1, P_2 such that P_1 is an arc from p_1 to p_2 and P_2 is an arc from p_1 to p_2 and $P = P_1 \cup P_2$ and $P_1 \cap P_2 = \{p_1, p_2\}$.

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Received December 30, 1991

FORMALIZED MATHEMATICS
Vol.2,No.5, November–December 1991
Université Catholique de Louvain

Separated and Weakly Separated Subspaces of Topological Spaces

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Summary. A new concept of weakly separated subsets and subspaces of topological spaces is described in Mizar formalizm. Based on [1], in comparison with the notion of separated subsets (subspaces), some properties of such subsets (subspaces) are discussed. Some necessary facts concerning closed subspaces, open subspaces and the union and the meet of two subspaces are also introduced. To present the main theorems we first formulate basic definitions. Let X be a topological space. Two subsets A_1 and A_2 of X are called *weakly separated* if $A_1 \setminus A_2$ and $A_2 \setminus A_1$ are separated. Two subspaces X_1 and X_2 of X are called *weakly separated* if their carriers are weakly separated. The following theorem contains a useful characterization of weakly separated subsets in the special case when $A_1 \cup A_2$ is equal to the carrier of X. *A_1 and A_2 are weakly separated iff there are such subsets of X, C_1 and C_2 closed (open) and C open (closed, respectively), that $A_1 \cup A_2 = C_1 \cup C_2 \cup C$, $C_1 \subset A_1$, $C_2 \subset A_2$ and $C \subset A_1 \cap A_2$.* Next theorem divided into two parts contains similar characterization of weakly separated subspaces in the special case when the union of X_1 and X_2 is equal to X. *If X_1 meets X_2, then X_1 and X_2 are weakly separated iff either X_1 is a subspace of X_2 or X_2 is a subspace of X_1 or there are such open (closed) subspaces Y_1 and Y_2 of X that Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2 and either X is equal to the union of Y_1 and Y_2 or there is a(n) closed (open, respectively) subspace Y of X being a subspace of the meet of X_1 and X_2 and with the property that X is the union of all Y_1, Y_2 and Y. If X_1 misses X_2, then X_1 and X_2 are weakly separated iff X_1 and X_2 are open (closed) subspaces of X.* Moreover, the following simple characterization of separated subspaces by means of weakly separated ones is obtained. *X_1 and X_2 are separated iff there are weakly separated subspaces Y_1 and Y_2 of X such that X_1 is a subspace of Y_1, X_2 is a subspace of Y_2 and either Y_1 misses Y_2 or the meet of Y_1 and Y_2 misses the union of X_1 and X_2.*

MML Identifier: `TSEP_1`.

The papers [6], [7], [4], [3], [8], [2], and [5] provide the notation and terminology for this paper.

ISSN 0777–4028

1. Some properties of subspaces of topological spaces

In the sequel X is a topological space. We now state a number of propositions:

(1) For every subspace X_0 of X holds the carrier of X_0 is a subset of X.

(2) X is a subspace of X.

(3) For every strict topological space X holds $X \restriction \Omega_X = X$.

(4) For all subspaces X_1, X_2 of X holds the carrier of $X_1 \subseteq$ the carrier of X_2 if and only if X_1 is a subspace of X_2.

(5) For all strict subspaces X_1, X_2 of X holds the carrier of X_1 = the carrier of X_2 if and only if $X_1 = X_2$.

(6) For all strict subspaces X_1, X_2 of X holds X_1 is a subspace of X_2 and X_2 is a subspace of X_1 if and only if $X_1 = X_2$.

(7) For every subspace X_1 of X and for every subspace X_2 of X_1 holds X_2 is a subspace of X.

(8) For every subspace X_0 of X and for all subsets C, A of X and for every subset B of X_0 such that C is closed and $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$ holds B is closed if and only if A is closed.

(9) For every subspace X_0 of X and for all subsets C, A of X and for every subset B of X_0 such that C is open and $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$ holds B is open if and only if A is open.

(10) For every non-empty subset A_0 of X there exists a strict subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(11) For every subspace X_0 of X and for every subset A of X such that $A =$ the carrier of X_0 holds X_0 is a closed subspace of X if and only if A is closed.

(12) For every closed subspace X_0 of X and for every subset A of X and for every subset B of X_0 such that $A = B$ holds B is closed if and only if A is closed.

(13) For every closed subspace X_1 of X and for every closed subspace X_2 of X_1 holds X_2 is a closed subspace of X.

(14) For every closed subspace X_1 of X and for every subspace X_2 of X such that the carrier of $X_1 \subseteq$ the carrier of X_2 holds X_1 is a closed subspace of X_2.

(15) For every non-empty subset A_0 of X such that A_0 is closed there exists a strict closed subspace X_0 of X such that $A_0 =$ the carrier of X_0.

Let X be a topological space. A subspace of X is said to be an open subspace of X if:

(Def.1) for every subset A of X such that $A =$ the carrier of it holds A is open.

The following propositions are true:

(16) For every subspace X_0 of X and for every subset A of X such that $A =$ the carrier of X_0 holds X_0 is an open subspace of X if and only if A is open.

(17) For every open subspace X_0 of X and for every subset A of X and for every subset B of X_0 such that $A = B$ holds B is open if and only if A is open.

(18) For every open subspace X_1 of X and for every open subspace X_2 of X_1 holds X_2 is an open subspace of X.

(19) For every open subspace X_1 of X and for every subspace X_2 of X such that the carrier of $X_1 \subseteq$ the carrier of X_2 holds X_1 is an open subspace of X_2.

(20) For every non-empty subset A_0 of X such that A_0 is open there exists a strict open subspace X_0 of X such that $A_0 =$ the carrier of X_0.

2. Operations on subspaces of topological spaces

In the sequel X denotes a topological space. Let us consider X, and let X_1, X_2 be subspaces of X. The functor $X_1 \cup X_2$ yielding a strict subspace of X is defined by:

(Def.2) the carrier of $X_1 \cup X_2 =$ (the carrier of X_1) $\cup$ (the carrier of X_2).

In the sequel X_1, X_2, X_3 will denote subspaces of X. One can prove the following propositions:

(21) $X_1 \cup X_2 = X_2 \cup X_1$ and $(X_1 \cup X_2) \cup X_3 = X_1 \cup (X_2 \cup X_3)$.

(22) X_1 is a subspace of $X_1 \cup X_2$ and X_2 is a subspace of $X_1 \cup X_2$.

(23) For all strict subspaces X_1, X_2 of X holds X_1 is a subspace of X_2 if and only if $X_1 \cup X_2 = X_2$ but X_2 is a subspace of X_1 if and only if $X_1 \cup X_2 = X_1$.

(24) For all closed subspaces X_1, X_2 of X holds $X_1 \cup X_2$ is a closed subspace of X.

(25) For all open subspaces X_1, X_2 of X holds $X_1 \cup X_2$ is an open subspace of X.

We now define two new predicates. Let us consider X, and let X_1, X_2 be subspaces of X. We say that X_1 misses X_2 if and only if:

(Def.3) (the carrier of X_1) $\cap$ (the carrier of X_2) $= \emptyset$.

We say that X_1 meets X_2 if and only if:

(Def.4) (the carrier of X_1) $\cap$ (the carrier of X_2) $\neq \emptyset$.

The following three propositions are true:

(26) X_1 misses X_2 if and only if X_1 does not meet X_2.

(27) X_1 misses X_2 if and only if X_2 misses X_1 but X_1 meets X_2 if and only if X_2 meets X_1.

(28) For all subsets A_1, A_2 of X such that A_1 = the carrier of X_1 and A_2 = the carrier of X_2 holds X_1 misses X_2 if and only if A_1 misses A_2 but X_1 meets X_2 if and only if A_1 meets A_2.

Let us consider X, and let X_1, X_2 be subspaces of X. Let us assume that X_1 meets X_2. The functor $X_1 \cap X_2$ yielding a strict subspace of X is defined by:

(Def.5) the carrier of $X_1 \cap X_2$ = (the carrier of X_1) $\cap$ (the carrier of X_2).

In the sequel X_1, X_2, X_3 will denote subspaces of X. We now state several propositions:

(29) If X_1 meets X_2 or X_2 meets X_1, then $X_1 \cap X_2 = X_2 \cap X_1$ but if X_1 meets X_2 and $X_1 \cap X_2$ meets X_3 or X_2 meets X_3 and X_1 meets $X_2 \cap X_3$, then $(X_1 \cap X_2) \cap X_3 = X_1 \cap (X_2 \cap X_3)$.

(30) If X_1 meets X_2, then $X_1 \cap X_2$ is a subspace of X_1 and $X_1 \cap X_2$ is a subspace of X_2.

(31) For all strict subspaces X_1, X_2 of X such that X_1 meets X_2 holds X_1 is a subspace of X_2 if and only if $X_1 \cap X_2 = X_1$ but X_2 is a subspace of X_1 if and only if $X_1 \cap X_2 = X_2$.

(32) For all closed subspaces X_1, X_2 of X such that X_1 meets X_2 holds $X_1 \cap X_2$ is a closed subspace of X.

(33) For all open subspaces X_1, X_2 of X such that X_1 meets X_2 holds $X_1 \cap X_2$ is an open subspace of X.

(34) If X_1 meets X_2, then $X_1 \cap X_2$ is a subspace of $X_1 \cup X_2$.

(35) For every subspace Y of X such that X_1 meets Y or Y meets X_1 but X_2 meets Y or Y meets X_2 holds $(X_1 \cup X_2) \cap Y = X_1 \cap Y \cup X_2 \cap Y$ and $Y \cap (X_1 \cup X_2) = Y \cap X_1 \cup Y \cap X_2$.

(36) For every subspace Y of X such that X_1 meets X_2 holds $X_1 \cap X_2 \cup Y = (X_1 \cup Y) \cap (X_2 \cup Y)$ and $Y \cup X_1 \cap X_2 = (Y \cup X_1) \cap (Y \cup X_2)$.

3. Separated and weakly separated subsets of topological spaces

Let X be a topological space, and let A_1, A_2 be subsets of X. Let us note that one can characterize the predicate A_1 and A_2 are separated by the following (equivalent) condition:

(Def.6) $\overline{A_1} \cap A_2 = \emptyset$ and $A_1 \cap \overline{A_2} = \emptyset$.

In the sequel X is a topological space and A_1, A_2 are subsets of X. We now state a number of propositions:

(37) If A_1 and A_2 are separated, then A_1 misses A_2.

(38) If A_1 is closed and A_2 is closed, then A_1 misses A_2 if and only if A_1 and A_2 are separated.

(39) If $A_1 \cup A_2$ is closed and A_1 and A_2 are separated, then A_1 is closed and A_2 is closed.

(40) If A_1 misses A_2, then if A_1 is open, then A_1 misses $\overline{A_2}$ but if A_2 is open, then $\overline{A_1}$ misses A_2.

(41) If A_1 is open and A_2 is open, then A_1 misses A_2 if and only if A_1 and A_2 are separated.

(42) If $A_1 \cup A_2$ is open and A_1 and A_2 are separated, then A_1 is open and A_2 is open.

(43) For every subset C of X such that A_1 and A_2 are separated holds $A_1 \cap C$ and $A_2 \cap C$ are separated and $C \cap A_1$ and $C \cap A_2$ are separated.

(44) For every subset B of X holds if A_1 and B are separated or A_2 and B are separated, then $A_1 \cap A_2$ and B are separated but if B and A_1 are separated or B and A_2 are separated, then B and $A_1 \cap A_2$ are separated.

(45) For every subset B of X holds A_1 and B are separated and A_2 and B are separated if and only if $A_1 \cup A_2$ and B are separated but B and A_1 are separated and B and A_2 are separated if and only if B and $A_1 \cup A_2$ are separated.

(46) A_1 and A_2 are separated if and only if there exist subsets C_1, C_2 of X such that $A_1 \subseteq C_1$ and $A_2 \subseteq C_2$ and C_1 misses A_2 and C_2 misses A_1 and C_1 is closed and C_2 is closed.

(47) A_1 and A_2 are separated if and only if there exist subsets C_1, C_2 of X such that $A_1 \subseteq C_1$ and $A_2 \subseteq C_2$ and $C_1 \cap C_2$ misses $A_1 \cup A_2$ and C_1 is closed and C_2 is closed.

(48) A_1 and A_2 are separated if and only if there exist subsets C_1, C_2 of X such that $A_1 \subseteq C_1$ and $A_2 \subseteq C_2$ and C_1 misses A_2 and C_2 misses A_1 and C_1 is open and C_2 is open.

(49) A_1 and A_2 are separated if and only if there exist subsets C_1, C_2 of X such that $A_1 \subseteq C_1$ and $A_2 \subseteq C_2$ and $C_1 \cap C_2$ misses $A_1 \cup A_2$ and C_1 is open and C_2 is open.

Let X be a topological space, and let A_1, A_2 be subsets of X. We say that A_1 and A_2 are weakly separated if and only if:

(Def.7) $A_1 \setminus A_2$ and $A_2 \setminus A_1$ are separated.

In the sequel X will be a topological space and A_1, A_2 will be subsets of X. We now state a number of propositions:

(50) If A_1 and A_2 are weakly separated, then A_2 and A_1 are weakly separated.

(51) A_1 misses A_2 and A_1 and A_2 are weakly separated if and only if A_1 and A_2 are separated.

(52) If $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$, then A_1 and A_2 are weakly separated.

(53) If A_1 is closed and A_2 is closed, then A_1 and A_2 are weakly separated.

(54) If A_1 is open and A_2 is open, then A_1 and A_2 are weakly separated.

(55) For every subset C of X such that A_1 and A_2 are weakly separated holds $A_1 \cup C$ and $A_2 \cup C$ are weakly separated and $C \cup A_1$ and $C \cup A_2$ are weakly separated.

(56) For all subsets B_1, B_2 of X such that $B_1 \subseteq A_2$ and $B_2 \subseteq A_1$ holds if A_1 and A_2 are weakly separated, then $A_1 \cup B_1$ and $A_2 \cup B_2$ are weakly separated and $B_1 \cup A_1$ and $B_2 \cup A_2$ are weakly separated.

(57) For every subset B of X holds if A_1 and B are weakly separated and A_2 and B are weakly separated, then $A_1 \cap A_2$ and B are weakly separated but if B and A_1 are weakly separated and B and A_2 are weakly separated, then B and $A_1 \cap A_2$ are weakly separated.

(58) For every subset B of X holds if A_1 and B are weakly separated and A_2 and B are weakly separated, then $A_1 \cup A_2$ and B are weakly separated but if B and A_1 are weakly separated and B and A_2 are weakly separated, then B and $A_1 \cup A_2$ are weakly separated.

(59) A_1 and A_2 are weakly separated if and only if there exist subsets C_1, C_2, C of X such that $C_1 \cap (A_1 \cup A_2) \subseteq A_1$ and $C_2 \cap (A_1 \cup A_2) \subseteq A_2$ and $C \cap (A_1 \cup A_2) \subseteq A_1 \cap A_2$ and the carrier of $X = C_1 \cup C_2 \cup C$ and C_1 is closed and C_2 is closed and C is open.

(60) Suppose A_1 and A_2 are weakly separated and $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$. Then there exist non-empty subsets C_1, C_2 of X such that C_1 is closed and C_2 is closed and $C_1 \cap (A_1 \cup A_2) \subseteq A_1$ and $C_2 \cap (A_1 \cup A_2) \subseteq A_2$ but $A_1 \cup A_2 \subseteq C_1 \cup C_2$ or there exists a non-empty subset C of X such that C is open and $C \cap (A_1 \cup A_2) \subseteq A_1 \cap A_2$ and the carrier of $X = C_1 \cup C_2 \cup C$.

(61) If $A_1 \cup A_2 =$ the carrier of X, then A_1 and A_2 are weakly separated if and only if there exist subsets C_1, C_2, C of X such that $A_1 \cup A_2 = C_1 \cup C_2 \cup C$ and $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ and $C \subseteq A_1 \cap A_2$ and C_1 is closed and C_2 is closed and C is open.

(62) Suppose $A_1 \cup A_2 =$ the carrier of X and A_1 and A_2 are weakly separated and $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$. Then there exist non-empty subsets C_1, C_2 of X such that C_1 is closed and C_2 is closed and $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ but $A_1 \cup A_2 = C_1 \cup C_2$ or there exists a non-empty subset C of X such that $A_1 \cup A_2 = C_1 \cup C_2 \cup C$ and $C \subseteq A_1 \cap A_2$ and C is open.

(63) A_1 and A_2 are weakly separated if and only if there exist subsets C_1, C_2, C of X such that $C_1 \cap (A_1 \cup A_2) \subseteq A_1$ and $C_2 \cap (A_1 \cup A_2) \subseteq A_2$ and $C \cap (A_1 \cup A_2) \subseteq A_1 \cap A_2$ and the carrier of $X = C_1 \cup C_2 \cup C$ and C_1 is open and C_2 is open and C is closed.

(64) Suppose A_1 and A_2 are weakly separated and $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$. Then there exist non-empty subsets C_1, C_2 of X such that C_1 is open and C_2 is open and $C_1 \cap (A_1 \cup A_2) \subseteq A_1$ and $C_2 \cap (A_1 \cup A_2) \subseteq A_2$ but $A_1 \cup A_2 \subseteq C_1 \cup C_2$ or there exists a non-empty subset C of X such that C is closed and $C \cap (A_1 \cup A_2) \subseteq A_1 \cap A_2$ and the carrier of $X = C_1 \cup C_2 \cup C$.

(65) If $A_1 \cup A_2 =$ the carrier of X, then A_1 and A_2 are weakly separated if and only if there exist subsets C_1, C_2, C of X such that $A_1 \cup A_2 = C_1 \cup C_2 \cup C$ and $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ and $C \subseteq A_1 \cap A_2$ and C_1 is open and C_2 is open and C is closed.

(66) Suppose $A_1 \cup A_2 =$ the carrier of X and A_1 and A_2 are weakly separated

and $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$. Then there exist non-empty subsets C_1, C_2 of X such that C_1 is open and C_2 is open and $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ but $A_1 \cup A_2 = C_1 \cup C_2$ or there exists a non-empty subset C of X such that $A_1 \cup A_2 = C_1 \cup C_2 \cup C$ and $C \subseteq A_1 \cap A_2$ and C is closed.

(67) A_1 and A_2 are separated if and only if there exist subsets B_1, B_2 of X such that B_1 and B_2 are weakly separated and $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$ and $B_1 \cap B_2$ misses $A_1 \cup A_2$.

4. Separated and weakly separated subspaces of topological spaces

In the sequel X is a topological space. Let us consider X, and let X_1, X_2 be subspaces of X. We say that X_1 and X_2 are separated if and only if:

(Def.8) for all subsets A_1, A_2 of X such that $A_1 =$ the carrier of X_1 and $A_2 =$ the carrier of X_2 holds A_1 and A_2 are separated.

In the sequel X_1, X_2 will denote subspaces of X. One can prove the following propositions:

(68) If X_1 and X_2 are separated, then X_1 misses X_2.

(69) If X_1 and X_2 are separated, then X_2 and X_1 are separated.

(70) For all closed subspaces X_1, X_2 of X holds X_1 misses X_2 if and only if X_1 and X_2 are separated.

(71) If $X = X_1 \cup X_2$ and X_1 and X_2 are separated, then X_1 is a closed subspace of X and X_2 is a closed subspace of X.

(72) If $X_1 \cup X_2$ is a closed subspace of X and X_1 and X_2 are separated, then X_1 is a closed subspace of X and X_2 is a closed subspace of X.

(73) For all open subspaces X_1, X_2 of X holds X_1 misses X_2 if and only if X_1 and X_2 are separated.

(74) If $X = X_1 \cup X_2$ and X_1 and X_2 are separated, then X_1 is an open subspace of X and X_2 is an open subspace of X.

(75) If $X_1 \cup X_2$ is an open subspace of X and X_1 and X_2 are separated, then X_1 is an open subspace of X and X_2 is an open subspace of X.

(76) For all subspaces Y, X_1, X_2 of X such that X_1 meets Y and X_2 meets Y holds if X_1 and X_2 are separated, then $X_1 \cap Y$ and $X_2 \cap Y$ are separated and $Y \cap X_1$ and $Y \cap X_2$ are separated.

(77) For all subspaces Y_1, Y_2 of X such that Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2 holds if X_1 and X_2 are separated, then Y_1 and Y_2 are separated.

(78) For every subspace Y of X such that X_1 meets X_2 holds if X_1 and Y are separated or X_2 and Y are separated, then $X_1 \cap X_2$ and Y are separated but if Y and X_1 are separated or Y and X_2 are separated, then Y and $X_1 \cap X_2$ are separated.

(79) For every subspace Y of X holds X_1 and Y are separated and X_2 and Y are separated if and only if $X_1 \cup X_2$ and Y are separated but Y and X_1 are separated and Y and X_2 are separated if and only if Y and $X_1 \cup X_2$ are separated.

(80) X_1 and X_2 are separated if and only if there exist closed subspaces Y_1, Y_2 of X such that X_1 is a subspace of Y_1 and X_2 is a subspace of Y_2 and Y_1 misses X_2 and Y_2 misses X_1.

(81) X_1 and X_2 are separated if and only if there exist closed subspaces Y_1, Y_2 of X such that X_1 is a subspace of Y_1 and X_2 is a subspace of Y_2 but Y_1 misses Y_2 or $Y_1 \cap Y_2$ misses $X_1 \cup X_2$.

(82) X_1 and X_2 are separated if and only if there exist open subspaces Y_1, Y_2 of X such that X_1 is a subspace of Y_1 and X_2 is a subspace of Y_2 and Y_1 misses X_2 and Y_2 misses X_1.

(83) X_1 and X_2 are separated if and only if there exist open subspaces Y_1, Y_2 of X such that X_1 is a subspace of Y_1 and X_2 is a subspace of Y_2 but Y_1 misses Y_2 or $Y_1 \cap Y_2$ misses $X_1 \cup X_2$.

Let X be a topological space, and let X_1, X_2 be subspaces of X. We say that X_1 and X_2 are weakly separated if and only if:

(Def.9) for all subsets A_1, A_2 of X such that $A_1 =$ the carrier of X_1 and $A_2 =$ the carrier of X_2 holds A_1 and A_2 are weakly separated.

In the sequel X_1, X_2 will denote subspaces of X. The following propositions are true:

(84) If X_1 and X_2 are weakly separated, then X_2 and X_1 are weakly separated.

(85) X_1 misses X_2 and X_1 and X_2 are weakly separated if and only if X_1 and X_2 are separated.

(86) If X_1 is a subspace of X_2 or X_2 is a subspace of X_1, then X_1 and X_2 are weakly separated.

(87) For all closed subspaces X_1, X_2 of X holds X_1 and X_2 are weakly separated.

(88) For all open subspaces X_1, X_2 of X holds X_1 and X_2 are weakly separated.

(89) For every subspace Y of X such that X_1 and X_2 are weakly separated holds $X_1 \cup Y$ and $X_2 \cup Y$ are weakly separated and $Y \cup X_1$ and $Y \cup X_2$ are weakly separated.

(90) For all subspaces Y_1, Y_2 of X such that Y_1 is a subspace of X_2 and Y_2 is a subspace of X_1 holds if X_1 and X_2 are weakly separated, then $X_1 \cup Y_1$ and $X_2 \cup Y_2$ are weakly separated and $Y_1 \cup X_1$ and $Y_2 \cup X_2$ are weakly separated.

(91) For all subspaces Y, X_1, X_2 of X such that X_1 meets X_2 holds if X_1 and Y are weakly separated and X_2 and Y are weakly separated, then $X_1 \cap X_2$

and Y are weakly separated but if Y and X_1 are weakly separated and Y and X_2 are weakly separated, then Y and $X_1 \cap X_2$ are weakly separated.

(92) For every subspace Y of X holds if X_1 and Y are weakly separated and X_2 and Y are weakly separated, then $X_1 \cup X_2$ and Y are weakly separated but if Y and X_1 are weakly separated and Y and X_2 are weakly separated, then Y and $X_1 \cup X_2$ are weakly separated.

(93) Let X be a strict topological space. Let X_1, X_2 be subspaces of X. Suppose X_1 meets X_2. Then X_1 and X_2 are weakly separated if and only if X_1 is a subspace of X_2 or X_2 is a subspace of X_1 or there exist closed subspaces Y_1, Y_2 of X such that $Y_1 \cap (X_1 \cup X_2)$ is a subspace of X_1 and $Y_2 \cap (X_1 \cup X_2)$ is a subspace of X_2 but $X_1 \cup X_2$ is a subspace of $Y_1 \cup Y_2$ or there exists an open subspace Y of X such that $X = Y_1 \cup Y_2 \cup Y$ and $Y \cap (X_1 \cup X_2)$ is a subspace of $X_1 \cap X_2$.

(94) Suppose $X = X_1 \cup X_2$ and X_1 meets X_2. Then X_1 and X_2 are weakly separated if and only if X_1 is a subspace of X_2 or X_2 is a subspace of X_1 or there exist closed subspaces Y_1, Y_2 of X such that Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2 but $X = Y_1 \cup Y_2$ or there exists an open subspace Y of X such that $X = Y_1 \cup Y_2 \cup Y$ and Y is a subspace of $X_1 \cap X_2$.

(95) If $X = X_1 \cup X_2$ and X_1 misses X_2, then X_1 and X_2 are weakly separated if and only if X_1 is a closed subspace of X and X_2 is a closed subspace of X.

(96) Let X be a strict topological space. Let X_1, X_2 be subspaces of X. Suppose X_1 meets X_2. Then X_1 and X_2 are weakly separated if and only if X_1 is a subspace of X_2 or X_2 is a subspace of X_1 or there exist open subspaces Y_1, Y_2 of X such that $Y_1 \cap (X_1 \cup X_2)$ is a subspace of X_1 and $Y_2 \cap (X_1 \cup X_2)$ is a subspace of X_2 but $X_1 \cup X_2$ is a subspace of $Y_1 \cup Y_2$ or there exists a closed subspace Y of X such that $X = Y_1 \cup Y_2 \cup Y$ and $Y \cap (X_1 \cup X_2)$ is a subspace of $X_1 \cap X_2$.

(97) Suppose $X = X_1 \cup X_2$ and X_1 meets X_2. Then X_1 and X_2 are weakly separated if and only if X_1 is a subspace of X_2 or X_2 is a subspace of X_1 or there exist open subspaces Y_1, Y_2 of X such that Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2 but $X = Y_1 \cup Y_2$ or there exists a closed subspace Y of X such that $X = Y_1 \cup Y_2 \cup Y$ and Y is a subspace of $X_1 \cap X_2$.

(98) If $X = X_1 \cup X_2$ and X_1 misses X_2, then X_1 and X_2 are weakly separated if and only if X_1 is an open subspace of X and X_2 is an open subspace of X.

(99) X_1 and X_2 are separated if and only if there exist subspaces Y_1, Y_2 of X such that Y_1 and Y_2 are weakly separated and X_1 is a subspace of Y_1 and X_2 is a subspace of Y_2 but Y_1 misses Y_2 or $Y_1 \cap Y_2$ misses $X_1 \cup X_2$.

Acknowledgements

I would like to express my gratitude to Professors A. Trybulec and C. Byliński for their valuable advice. I am also extremely grateful to W.A. Trybulec and to all participants of the Mizar Summer School (1991) for acquainting me with the Mizar system.

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Received January 8, 1992

FORMALIZED MATHEMATICS
Vol.2,No.5, November–December 1991
Université Catholique de Louvain

The de l'Hospital Theorem

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Summary. List of theorems concerning the de l'Hospital Theorem. We discuss the case when both functions have the zero value at a point and when the quotient of their differentials is convergent at this point.

MML Identifier: `L'HOSPIT`.

The papers [21], [4], [1], [2], [17], [15], [6], [9], [16], [3], [5], [12], [13], [20], [14], [18], [19], [8], [11], [7], and [10] provide the terminology and notation for this paper. We adopt the following rules: f, g will be partial functions from $\mathbb{R}$ to $\mathbb{R}$, r, r_1, r_2, g_1, g_2, x_0, t will be real numbers, and a will be a sequence of real numbers. Next we state a number of propositions:

(1) If f is continuous in x_0 and for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}\, f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}\, f$, then f is convergent in x_0.

(2) f is right convergent in x_0 and $\lim_{x_0^+} f = t$ if and only if the following conditions are satisfied:

(i) for every r such that $x_0 < r$ there exists t such that $t < r$ and $x_0 < t$ and $t \in \mathrm{dom}\, f$,

(ii) for every a such that a is convergent and $\lim a = x_0$ and $\mathrm{rng}\, a \subseteq \mathrm{dom}\, f \cap]x_0, +\infty[$ holds $f \cdot a$ is convergent and $\lim(f \cdot a) = t$.

(3) f is left convergent in x_0 and $\lim_{x_0^-} f = t$ if and only if the following conditions are satisfied:

(i) for every r such that $r < x_0$ there exists t such that $r < t$ and $t < x_0$ and $t \in \mathrm{dom}\, f$,

(ii) for every a such that a is convergent and $\lim a = x_0$ and $\mathrm{rng}\, a \subseteq \mathrm{dom}\, f \cap]-\infty, x_0[$ holds $f \cdot a$ is convergent and $\lim(f \cdot a) = t$.

(4) Suppose There exists a neighbourhood N of x_0 such that $N \setminus \{x_0\} \subseteq \mathrm{dom}\, f$. Then for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}\, f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}\, f$.

ISSN 0777–4028

(5) Given a neighbourhood N of x_0 such that
(i) f is differentiable on N,
(ii) g is differentiable on N,
(iii) $N \setminus \{x_0\} \subseteq \mathrm{dom}(\frac{f}{g})$,
(iv) $N \subseteq \mathrm{dom}(\frac{f'_{\restriction N}}{g'_{\restriction N}})$,
(v) $f(x_0) = 0$,
(vi) $g(x_0) = 0$,
(vii) $\frac{f'_{\restriction N}}{g'_{\restriction N}}$ is divergent to $+\infty$ in x_0.

Then $\frac{f}{g}$ is divergent to $+\infty$ in x_0.

(6) Given a neighbourhood N of x_0 such that
(i) f is differentiable on N,
(ii) g is differentiable on N,
(iii) $N \setminus \{x_0\} \subseteq \mathrm{dom}(\frac{f}{g})$,
(iv) $N \subseteq \mathrm{dom}(\frac{f'_{\restriction N}}{g'_{\restriction N}})$,
(v) $f(x_0) = 0$,
(vi) $g(x_0) = 0$,
(vii) $\frac{f'_{\restriction N}}{g'_{\restriction N}}$ is divergent to $-\infty$ in x_0.

Then $\frac{f}{g}$ is divergent to $-\infty$ in x_0.

(7) Given r such that
(i) $r > 0$,
(ii) f is differentiable on $]x_0, x_0 + r[$,
(iii) g is differentiable on $]x_0, x_0 + r[$,
(iv) $]x_0, x_0 + r[\subseteq \mathrm{dom}(\frac{f}{g})$,
(v) $[x_0, x_0 + r] \subseteq \mathrm{dom}(\frac{f'_{\restriction]x_0,x_0+r[}}{g'_{\restriction]x_0,x_0+r[}})$,
(vi) $f(x_0) = 0$,
(vii) $g(x_0) = 0$,
(viii) f is continuous in x_0,
(ix) g is continuous in x_0,
(x) $\frac{f'_{\restriction]x_0,x_0+r[}}{g'_{\restriction]x_0,x_0+r[}}$ is right convergent in x_0.

Then $\frac{f}{g}$ is right convergent in x_0 and there exists r such that $r > 0$ and $\lim_{x_0^+}(\frac{f}{g}) = \lim_{x_0^+}(\frac{f'_{\restriction]x_0,x_0+r[}}{g'_{\restriction]x_0,x_0+r[}})$.

(8) Given r such that
(i) $r > 0$,
(ii) f is differentiable on $]x_0 - r, x_0[$,
(iii) g is differentiable on $]x_0 - r, x_0[$,
(iv) $]x_0 - r, x_0[\subseteq \mathrm{dom}(\frac{f}{g})$,

(v) $[x_0 - r, x_0] \subseteq \mathrm{dom}(\frac{f'_{\restriction]x_0-r,x_0[}}{g'_{\restriction]x_0-r,x_0[}})$,
(vi) $f(x_0) = 0$,
(vii) $g(x_0) = 0$,
(viii) f is continuous in x_0,
(ix) g is continuous in x_0,
(x) $\frac{f'_{\restriction]x_0-r,x_0[}}{g'_{\restriction]x_0-r,x_0[}}$ is left convergent in x_0.

Then $\frac{f}{g}$ is left convergent in x_0 and there exists r such that $r > 0$ and $\lim_{x_0^-}(\frac{f}{g}) = \lim_{x_0^-}(\frac{f'_{\restriction]x_0-r,x_0[}}{g'_{\restriction]x_0-r,x_0[}})$.

(9) Given a neighbourhood N of x_0 such that
(i) f is differentiable on N,
(ii) g is differentiable on N,
(iii) $N \setminus \{x_0\} \subseteq \mathrm{dom}(\frac{f}{g})$,
(iv) $N \subseteq \mathrm{dom}(\frac{f'_{\restriction N}}{g'_{\restriction N}})$,
(v) $f(x_0) = 0$,
(vi) $g(x_0) = 0$,
(vii) $\frac{f'_{\restriction N}}{g'_{\restriction N}}$ is convergent in x_0.

Then $\frac{f}{g}$ is convergent in x_0 and there exists a neighbourhood N of x_0 such that $\lim_{x_0}(\frac{f}{g}) = \lim_{x_0}(\frac{f'_{\restriction N}}{g'_{\restriction N}})$.

(10) Given a neighbourhood N of x_0 such that
(i) f is differentiable on N,
(ii) g is differentiable on N,
(iii) $N \setminus \{x_0\} \subseteq \mathrm{dom}(\frac{f}{g})$,
(iv) $N \subseteq \mathrm{dom}(\frac{f'_{\restriction N}}{g'_{\restriction N}})$,
(v) $f(x_0) = 0$,
(vi) $g(x_0) = 0$,
(vii) $\frac{f'_{\restriction N}}{g'_{\restriction N}}$ is continuous in x_0.

Then $\frac{f}{g}$ is convergent in x_0 and $\lim_{x_0}(\frac{f}{g}) = \frac{f'(x_0)}{g'(x_0)}$.

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Received February 20, 1992

FORMALIZED MATHEMATICS
Vol.2,No.5, November–December 1991
Université Catholique de Louvain

Comma Category

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Summary. Comma category of two functors is introduced.

MML Identifier: COMMACAT.

The terminology and notation used in this paper have been introduced in the following articles: [9], [10], [1], [5], [2], [7], [4], [3], [6], and [8]. We now define four new functors. Let x be arbitrary. The functor $x_{\mathbf{1},\mathbf{1}}$ is defined by:

(Def.1) $x_{\mathbf{1},\mathbf{1}} = (x_{\mathbf{1}})_{\mathbf{1}}$.

The functor $x_{\mathbf{1},\mathbf{2}}$ is defined as follows:

(Def.2) $x_{\mathbf{1},\mathbf{2}} = (x_{\mathbf{1}})_{\mathbf{2}}$.

The functor $x_{\mathbf{2},\mathbf{1}}$ is defined by:

(Def.3) $x_{\mathbf{2},\mathbf{1}} = (x_{\mathbf{2}})_{\mathbf{1}}$.

The functor $x_{\mathbf{2},\mathbf{2}}$ is defined as follows:

(Def.4) $x_{\mathbf{2},\mathbf{2}} = (x_{\mathbf{2}})_{\mathbf{2}}$.

In the sequel x, x_1, x_2, y, y_1, y_2 are arbitrary. One can prove the following proposition

(1) $\langle\langle x_1, x_2\rangle, y\rangle_{\mathbf{1},\mathbf{1}} = x_1$ and $\langle\langle x_1, x_2\rangle, y\rangle_{\mathbf{1},\mathbf{2}} = x_2$ and $\langle x, \langle y_1, y_2\rangle\rangle_{\mathbf{2},\mathbf{1}} = y_1$ and $\langle x, \langle y_1, y_2\rangle\rangle_{\mathbf{2},\mathbf{2}} = y_2$.

Let D_1, D_2, D_3 be non-empty sets, and let x be an element of $[\![\, [\![\, D_1, D_2 \,]\!], D_3 \,]\!]$. Then $x_{\mathbf{1},\mathbf{1}}$ is an element of D_1. Then $x_{\mathbf{1},\mathbf{2}}$ is an element of D_2.

Let D_1, D_2, D_3 be non-empty sets, and let x be an element of $[\![\, D_1, [\![\, D_2, D_3 \,]\!] \,]\!]$. Then $x_{\mathbf{2},\mathbf{1}}$ is an element of D_2. Then $x_{\mathbf{2},\mathbf{2}}$ is an element of D_3.

For simplicity we follow a convention: C, D, E are categories, c is an object of C, d is an object of D, x is arbitrary, f is a morphism of E, g is a morphism of C, h is a morphism of D, F is a functor from C to E, and G is a functor from D to E. Let us consider C, D, E, and let F be a functor from C to E, and let G be a functor from D to E. Let us assume that there exist c_1, d_1, f_1 such

ISSN 0777–4028

that $f_1 \in \hom(F(c_1), G(d_1))$. The functor $\mathrm{Obj}_{(F,G)}$ yields a non-empty subset of [: [: the objects of C, the objects of D :], the morphisms of E :] and is defined as follows:

(Def.5) $\mathrm{Obj}_{(F,G)} = \{\langle\langle c, d\rangle, f\rangle : f \in \hom(F(c), G(d))\}$.

In the sequel o, o_1, o_2 will denote elements of $\mathrm{Obj}_{(F,G)}$. The following proposition is true

(2) Suppose there exist c, d, f such that $f \in \hom(F(c), G(d))$. Then $o = \langle\langle o_{\mathbf{1},\mathbf{1}}, o_{\mathbf{1},\mathbf{2}}\rangle, o_{\mathbf{2}}\rangle$ and $o_{\mathbf{2}} \in \hom(F(o_{\mathbf{1},\mathbf{1}}), G(o_{\mathbf{1},\mathbf{2}}))$ and $\mathrm{dom}(o_{\mathbf{2}}) = F(o_{\mathbf{1},\mathbf{1}})$ and $\mathrm{cod}(o_{\mathbf{2}}) = G(o_{\mathbf{1},\mathbf{2}})$.

Let us consider C, D, E, F, G. Let us assume that there exist c_1, d_1, f_1 such that $f_1 \in \hom(F(c_1), G(d_1))$. The functor $\mathrm{Morph}_{(F,G)}$ yielding a non-empty subset of [: [: $\mathrm{Obj}_{(F,G)}$, $\mathrm{Obj}_{(F,G)}$ **qua** a non-empty set :], [: the morphisms of C, the morphisms of D :] :] is defined by:

(Def.6) $\mathrm{Morph}_{(F,G)} = \{\langle\langle o_1, o_2\rangle, \langle g, h\rangle\rangle : \mathrm{dom}\, g = {o_1}_{\mathbf{1},\mathbf{1}} \wedge \mathrm{cod}\, g = {o_2}_{\mathbf{1},\mathbf{1}} \wedge \mathrm{dom}\, h = {o_1}_{\mathbf{1},\mathbf{2}} \wedge \mathrm{cod}\, h = {o_2}_{\mathbf{1},\mathbf{2}} \wedge {o_2}_{\mathbf{2}} \cdot F(g) = G(h) \cdot {o_1}_{\mathbf{2}}\}$.

In the sequel k, k_1, k_2, k' denote elements of $\mathrm{Morph}_{(F,G)}$. Let us consider C, D, E, F, G, k. Then $k_{\mathbf{1},\mathbf{1}}$ is an element of $\mathrm{Obj}_{(F,G)}$. Then $k_{\mathbf{1},\mathbf{2}}$ is an element of $\mathrm{Obj}_{(F,G)}$. Then $k_{\mathbf{2},\mathbf{1}}$ is a morphism of C. Then $k_{\mathbf{2},\mathbf{2}}$ is a morphism of D.

The following proposition is true

(3) Suppose There exist c, d, f such that $f \in \hom(F(c), G(d))$. Then

(i) $k = \langle\langle k_{\mathbf{1},\mathbf{1}}, k_{\mathbf{1},\mathbf{2}}\rangle, \langle k_{\mathbf{2},\mathbf{1}}, k_{\mathbf{2},\mathbf{2}}\rangle\rangle$,

(ii) $\mathrm{dom}(k_{\mathbf{2},\mathbf{1}}) = (k_{\mathbf{1},\mathbf{1}})_{\mathbf{1},\mathbf{1}}$,

(iii) $\mathrm{cod}(k_{\mathbf{2},\mathbf{1}}) = (k_{\mathbf{1},\mathbf{2}})_{\mathbf{1},\mathbf{1}}$,

(iv) $\mathrm{dom}(k_{\mathbf{2},\mathbf{2}}) = (k_{\mathbf{1},\mathbf{1}})_{\mathbf{1},\mathbf{2}}$,

(v) $\mathrm{cod}(k_{\mathbf{2},\mathbf{2}}) = (k_{\mathbf{1},\mathbf{2}})_{\mathbf{1},\mathbf{2}}$,

(vi) $(k_{\mathbf{1},\mathbf{2}})_{\mathbf{2}} \cdot F(k_{\mathbf{2},\mathbf{1}}) = G(k_{\mathbf{2},\mathbf{2}}) \cdot (k_{\mathbf{1},\mathbf{1}})_{\mathbf{2}}$.

Let us consider C, D, E, F, G, k_1, k_2. Let us assume that there exist c_1, d_1, f_1 such that $f_1 \in \hom(F(c_1), G(d_1))$. Let us assume that ${k_1}_{\mathbf{1},\mathbf{2}} = {k_2}_{\mathbf{1},\mathbf{1}}$. The functor $k_2 \cdot k_1$ yielding an element of $\mathrm{Morph}_{(F,G)}$ is defined as follows:

(Def.7) $k_2 \cdot k_1 = \langle\langle {k_1}_{\mathbf{1},\mathbf{1}}, {k_2}_{\mathbf{1},\mathbf{2}}\rangle, \langle {k_2}_{\mathbf{2},\mathbf{1}} \cdot {k_1}_{\mathbf{2},\mathbf{1}}, {k_2}_{\mathbf{2},\mathbf{2}} \cdot {k_1}_{\mathbf{2},\mathbf{2}}\rangle\rangle$.

Let us consider C, D, E, F, G. The functor $\circ_{(F,G)}$ yields a partial function from [: $\mathrm{Morph}_{(F,G)}$, $\mathrm{Morph}_{(F,G)}$:] to $\mathrm{Morph}_{(F,G)}$ and is defined by:

(Def.8) $\mathrm{dom}(\circ_{(F,G)}) = \{\langle k_1, k_2\rangle : {k_1}_{\mathbf{1},\mathbf{1}} = {k_2}_{\mathbf{1},\mathbf{2}}\}$ and for all k, k' such that $\langle k, k'\rangle \in \mathrm{dom}(\circ_{(F,G)})$ holds $\circ_{(F,G)}(\langle k, k'\rangle) = k \cdot k'$.

Let us consider C, D, E, F, G. Let us assume that there exist c_1, d_1, f_1 such that $f_1 \in \hom(F(c_1), G(d_1))$. The functor (F, G) yielding a strict category is defined by the conditions (Def.9).

(Def.9) (i) The objects of $(F, G) = \mathrm{Obj}_{(F,G)}$,

(ii) the morphisms of $(F, G) = \mathrm{Morph}_{(F,G)}$,

(iii) for every k holds (the dom-map of (F, G))$(k) = k_{\mathbf{1},\mathbf{1}}$,

(iv) for every k holds (the cod-map of (F, G))$(k) = k_{\mathbf{1},\mathbf{2}}$,

(v) for every o holds (the id-map of (F, G))$(o) = \langle\langle o, o\rangle, \langle \mathrm{id}_{(o_{\mathbf{1},\mathbf{1}})}, \mathrm{id}_{(o_{\mathbf{1},\mathbf{2}})}\rangle\rangle$,

(vi) the composition of $(F,G) = \circ_{(F,G)}$.

We now state two propositions:

(4) The objects of $\dot{\circlearrowleft}(x,y) = \{x\}$ and the morphisms of $\dot{\circlearrowleft}(x,y) = \{y\}$.

(5) For all objects a, b of $\dot{\circlearrowleft}(x,y)$ holds $\hom(a,b) = \{y\}$.

Let us consider C, c. The functor $\dot{\circlearrowleft}(c)$ yielding a strict subcategory of C is defined as follows:

(Def.10) $\dot{\circlearrowleft}(c) = \dot{\circlearrowleft}(c, \mathrm{id}_c)$.

We now define two new functors. Let us consider C, c. The functor (c,C) yields a strict category and is defined by:

(Def.11) $(c,C) = (\overset{\dot{\circlearrowleft}(c)}{\hookrightarrow}, \mathrm{id}_C)$.

The functor (C,c) yields a strict category and is defined as follows:

(Def.12) $(C,c) = (\mathrm{id}_C, \overset{\dot{\circlearrowleft}(c)}{\hookrightarrow})$.

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Received February 20, 1992

FORMALIZED MATHEMATICS
Vol.2, No.5, November–December 1991
Université Catholique de Louvain

Context-Free Grammar - Part 1

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Summary. The concept of context-free grammar and of derivability in grammar are introduced. Moreover, the language (set of finite sequences of symbols) generated by grammar and some grammars are defined. The notion convenient to prove facts on language generated by grammar with exchange of symbols on grammar of union and concatenation of languages is included.

MML Identifier: LANG1.

The notation and terminology used here have been introduced in the following papers: [9], [7], [1], [8], [10], [11], [4], [2], [6], [5], and [3]. We consider context-free grammars which are systems

⟨symbols, a initial symbol, rules⟩,

where the symbols constitute a non-empty set, the initial symbol is an element of the symbols, and the rules constitute a relation between the symbols and (the symbols)*.

We now define two new modes. Let G be a context-free grammar. A symbol of G is an element of the symbols of G.

A string of G is an element of (the symbols of G)*.

Let D be a non-empty set, and let p, q be elements of D^*. Then $p \frown q$ is an element of D^*.

Let D be a non-empty set. Then ε_D is an element of D^*. Let d be an element of D. Then $\langle d \rangle$ is an element of D^*. Let e be an element of D. Then $\langle d, e \rangle$ is an element of D^*.

In the sequel G will denote a context-free grammar, s will denote a symbol of G, and n, m will denote strings of G. Let us consider G, s, n. The predicate $s \Rightarrow n$ is defined as follows:

(Def.1) $\langle s, n \rangle \in$ the rules of G.

We now define two new functors. Let us consider G. The terminals of G yields a set and is defined as follows:

ISSN 0777–4028

(Def.2) the terminals of $G = \{s : \neg \bigvee_n s \Rightarrow n\}$.

The nonterminals of G yielding a set is defined as follows:

(Def.3) the nonterminals of $G = \{s : \bigvee_n s \Rightarrow n\}$.

Next we state the proposition

(1) (The terminals of G) $\cup$ (the nonterminals of G) = the symbols of G.

Let us consider G, n, m. The predicate $n \Rightarrow m$ is defined by:

(Def.4) there exist strings n_1, n_2, n_3 of G and there exists s such that $n = n_1 \frown \langle s \rangle \frown n_2$ and $m = n_1 \frown n_3 \frown n_2$ and $s \Rightarrow n_3$.

In the sequel n_1, n_2, n_3 denote strings of G. One can prove the following four propositions:

(2) If $s \Rightarrow n$, then $n_1 \frown \langle s \rangle \frown n_2 \Rightarrow n_1 \frown n \frown n_2$.

(3) If $s \Rightarrow n$, then $\langle s \rangle \Rightarrow n$.

(4) If $\langle s \rangle \Rightarrow n$, then $s \Rightarrow n$.

(5) If $n_1 \Rightarrow n_2$, then $n \frown n_1 \Rightarrow n \frown n_2$ and $n_1 \frown n \Rightarrow n_2 \frown n$.

Let us consider G, n, m. The predicate $n \Rightarrow_* m$ is defined by the condition (Def.5).

(Def.5) There exists a finite sequence p such that $\operatorname{len} p \geq 1$ and $p(1) = n$ and $p(\operatorname{len} p) = m$ and for every natural number i such that $i \geq 1$ and $i < \operatorname{len} p$ there exist strings a, b of G such that $p(i) = a$ and $p(i+1) = b$ and $a \Rightarrow b$.

The following three propositions are true:

(6) $n \Rightarrow_* n$.

(7) If $n \Rightarrow m$, then $n \Rightarrow_* m$.

(8) If $n_2 \Rightarrow_* n_1$ and $n_3 \Rightarrow_* n_2$, then $n_3 \Rightarrow_* n_1$.

Let us consider G. The language generated by G yielding a set is defined by:

(Def.6) the language generated by $G = \{a : \operatorname{rng} a \subseteq$ the terminals of $G \wedge \langle$the initial symbol of $G\rangle \Rightarrow_* a\}$, where a ranges over elements of (the symbols of $G)^*$.

Next we state the proposition

(9) $n \in$ the language generated by G if and only if $\operatorname{rng} n \subseteq$ the terminals of G and $\langle$the initial symbol of $G\rangle \Rightarrow_* n$.

Let a be arbitrary. Then $\{a\}$ is a non-empty set. Let b be arbitrary. Then $\{a, b\}$ is a non-empty set.

Let D, E be non-empty sets, and let a be an element of $[\!: D, E :\!]$. Then $\{a\}$ is a relation between D and E. Let b be an element of $[\!: D, E :\!]$. Then $\{a, b\}$ is a relation between D and E.

We now define three new functors. Let a be arbitrary. The functor $\{a \Rightarrow \varepsilon\}$ yielding a strict context-free grammar is defined by:

(Def.7) the symbols of $\{a \Rightarrow \varepsilon\} = \{a\}$ and the rules of $\{a \Rightarrow \varepsilon\} = \{\langle a, \varepsilon \rangle\}$.

Let b be arbitrary. The functor $\{a \Rightarrow b\}$ yielding a strict context-free grammar is defined as follows:

(Def.8) the symbols of $\{a \Rightarrow b\} = \{a, b\}$ and the initial symbol of $\{a \Rightarrow b\} = a$ and the rules of $\{a \Rightarrow b\} = \{\langle a, \langle b\rangle\rangle\}$.

The functor $\left\{\begin{array}{c} a \Rightarrow ba \\ a \Rightarrow \varepsilon \end{array}\right\}$ yields a strict context-free grammar and is defined by:

(Def.9) the symbols of $\left\{\begin{array}{c} a \Rightarrow ba \\ a \Rightarrow \varepsilon \end{array}\right\} = \{a, b\}$ and the initial symbol of $\left\{\begin{array}{c} a \Rightarrow ba \\ a \Rightarrow \varepsilon \end{array}\right\} =$ a and the rules of $\left\{\begin{array}{c} a \Rightarrow ba \\ a \Rightarrow \varepsilon \end{array}\right\} = \{\langle a, \langle b, a\rangle\rangle, \langle a, \varepsilon\rangle\}$.

Let D be a non-empty set. The total grammar over D yields a strict context-free grammar and is defined as follows:

(Def.10) the symbols of the total grammar over $D = D \cup \{D\}$ and the initial symbol of the total grammar over $D = D$ and the rules of the total grammar over $D = \{\langle D, \langle d, D\rangle\rangle : d = d\} \cup \{\langle D, \varepsilon\rangle\}$, where d ranges over elements of D.

In the sequel a, b are arbitrary and D denotes a non-empty set. Next we state several propositions:

(10) The terminals of $\{a \Rightarrow \varepsilon\} = \emptyset$.

(11) The language generated by $\{a \Rightarrow \varepsilon\} = \{\varepsilon\}$.

(12) If $a \neq b$, then the terminals of $\{a \Rightarrow b\} = \{b\}$.

(13) If $a \neq b$, then the language generated by $\{a \Rightarrow b\} = \{\langle b\rangle\}$.

(14) If $a \neq b$, then the terminals of $\left\{\begin{array}{c} a \Rightarrow ba \\ a \Rightarrow \varepsilon \end{array}\right\} = \{b\}$.

(15) If $a \neq b$, then the language generated by $\left\{\begin{array}{c} a \Rightarrow ba \\ a \Rightarrow \varepsilon \end{array}\right\} = \{b\}^*$.

(16) The terminals of the total grammar over $D = D$.

(17) The language generated by the total grammar over $D = D^*$.

We now define two new attributes. A context-free grammar is efective if:

(Def.11) the language generated by it is non-empty and the initial symbol of it $\in$ the nonterminals of it and for every symbol s of it such that $s \in$ the terminals of it there exists a string p of it such that $p \in$ the language generated by it and $s \in \operatorname{rng} p$.

A context-free grammar is finite if:

(Def.12) the rules of it is finite.

Let G be an efective context-free grammar. Then the nonterminals of G is a non-empty subset of the symbols of G.

Let X be a set, and let Y be a non-empty set, and let f be a function from X into Y. Then graph f is a relation between X and Y.

Let X, Y be non-empty sets, and let p be a finite sequence of elements of X, and let f be a function from X into Y. Then $f \cdot p$ is an element of Y^*.

Let X, Y be non-empty sets, and let f be a function from X into Y. The functor f^* yielding a function from X^* into Y^* is defined as follows:

(Def.13) for every element p of X^* holds $f^*(p) = f \cdot p$.

Let R be a binary relation. The functor R^* yielding a binary relation is defined by the condition (Def.14).

(Def.14) Let x, y be arbitrary. Then $\langle x, y\rangle \in R^*$ if and only if the following conditions are satisfied:
- (i) $x \in \operatorname{field} R$,
- (ii) $y \in \operatorname{field} R$,
- (iii) there exists a finite sequence p such that $\operatorname{len} p \geq 1$ and $p(1) = x$ and $p(\operatorname{len} p) = y$ and for every natural number i such that $i \geq 1$ and $i < \operatorname{len} p$ holds $\langle p(i), p(i+1)\rangle \in R$.

In the sequel R is a binary relation. We now state the proposition

(18) $R \subseteq R^*$.

Let X be a non-empty set, and let R be a binary relation on X. Then R^* is a binary relation on X.

Let G be a context-free grammar, and let X be a non-empty set, and let f be a function from the symbols of G into X. The functor $G(f)$ yielding a strict context-free grammar is defined by:

(Def.15) $G(f) = \langle X, f(\text{the initial symbol of } G), (\operatorname{graph} f)^\smile \cdot \text{the rules of } G \cdot \operatorname{graph}(f^*)\rangle$.

The following proposition is true

(19) For all non-empty sets D_1, D_2 such that $D_1 \subseteq D_2$ holds ${D_1}^* \subseteq {D_2}^*$.

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Received February 21, 1992

FORMALIZED MATHEMATICS
Vol.2,No.5, November–December 1991
Université Catholique de Louvain

Completeness of the σ-Additive Measure. Measure Theory

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Summary. Definitions and basic properties of a σ-additive, non-negative measure, with values in $\overline{\mathbb{R}}$, the enlarged set of real numbers, where $\overline{\mathbb{R}}$ denotes set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ - by [13]. The article includs the text being a continuation of the paper [5]. Some theorems concerning basic properties of a σ-additive measure and completeness of the measure are proved.

MML Identifier: MEASURE3.

The papers [15], [14], [9], [10], [7], [8], [1], [12], [2], [11], [3], [4], [6], and [5] provide the terminology and notation for this paper. One can prove the following four propositions:

(1) For every *Real number* x such that $-\infty < x$ and $x < +\infty$ holds x is a real number.

(2) For every *Real number* x such that $x \neq -\infty$ and $x \neq +\infty$ holds x is a real number.

(3) For all functions F_1, F_2 from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F_1 is non-negative and F_2 is non-negative holds if for every natural number n holds $(\operatorname{Ser} F_1)(n) \leq (\operatorname{Ser} F_2)(n)$, then $\sum F_1 \leq \sum F_2$.

(4) For all functions F_1, F_2 from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F_1 is non-negative and F_2 is non-negative holds if for every natural number n holds $(\operatorname{Ser} F_1)(n) = (\operatorname{Ser} F_2)(n)$, then $\sum F_1 = \sum F_2$.

Let X be a set, and let S be a σ-field of subsets of X. A denumerable family of subsets of X is called a subfamily of S if:

(Def.1) it $\subseteq S$.

Let X be a set, and let S be a σ-field of subsets of X, and let F be a function from $\mathbb{N}$ into S. Then $\operatorname{rng} F$ is a subfamily of S.

ISSN 0777–4028

Let X be a set, and let S be a σ-field of subsets of X, and let A be a subfamily of S. Then $\bigcup A$ is an element of S.

Let X be a set, and let S be a σ-field of subsets of X, and let A be a subfamily of S. Then $\bigcap A$ is an element of S.

One can prove the following propositions:

(5) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every function F from $\mathbb{N}$ into S and for every element A of S such that $\bigcap \operatorname{rng} F \subseteq A$ and for every element n of $\mathbb{N}$ holds $A \subseteq F(n)$ holds $M(A) = M(\bigcap \operatorname{rng} F)$.

(6) Let X be a set. Let S be a σ-field of subsets of X. Let G be a function from $\mathbb{N}$ into S. Then for every function F from $\mathbb{N}$ into S such that $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$ holds $\bigcup \operatorname{rng} G = F(0) \setminus \bigcap \operatorname{rng} F$.

(7) Let X be a set. Let S be a σ-field of subsets of X. Let G be a function from $\mathbb{N}$ into S. Then for every function F from $\mathbb{N}$ into S such that $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$ holds $\bigcap \operatorname{rng} F = F(0) \setminus \bigcup \operatorname{rng} G$.

(8) Let X be a set. Let S be a σ-field of subsets of X. Let M be a σ-measure on S. Let G be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose $M(F(0)) < +\infty$ and $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$. Then $M(\bigcap \operatorname{rng} F) = M(F(0)) - M(\bigcup \operatorname{rng} G)$.

(9) Let X be a set. Let S be a σ-field of subsets of X. Let M be a σ-measure on S. Let G be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose $M(F(0)) < +\infty$ and $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$. Then $M(\bigcup \operatorname{rng} G) = M(F(0)) - M(\bigcap \operatorname{rng} F)$.

(10) Let X be a set. Let S be a σ-field of subsets of X. Let M be a σ-measure on S. Let G be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose $M(F(0)) < +\infty$ and $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$. Then $M(\bigcap \operatorname{rng} F) = M(F(0)) - \sup \operatorname{rng}(M \cdot G)$.

(11) Let X be a set. Let S be a σ-field of subsets of X. Let M be a σ-measure on S. Let G be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose $M(F(0)) < +\infty$ and $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n + 1) = F(0) \setminus F(n)$ and $F(n + 1) \subseteq F(n)$. Then $\sup \operatorname{rng}(M \cdot G)$ is a real number and $M(F(0))$ is a real number and $\inf \operatorname{rng}(M \cdot F)$ is a real number.

(12) Let X be a set. Let S be a σ-field of subsets of X. Let M be a σ-measure on S. Let G be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose $M(F(0)) < +\infty$ and $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$. Then $\sup \operatorname{rng}(M \cdot G) = M(F(0)) - \inf \operatorname{rng}(M \cdot F)$.

(13) Let X be a set. Let S be a σ-field of subsets of X. Let M be a σ-measure on S. Let G be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose $M(F(0)) < +\infty$ and $G(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(0) \setminus F(n)$ and $F(n+1) \subseteq F(n)$. Then $\inf \operatorname{rng}(M \cdot F) = M(F(0)) - \sup \operatorname{rng}(M \cdot G)$.

(14) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every function F from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$ holds $F(n+1) \subseteq F(n)$ and $M(F(0)) < +\infty$ holds $M(\bigcap \operatorname{rng} F) = \inf \operatorname{rng}(M \cdot F)$.

(15) For every set X and for every σ-field S of subsets of X and for every measure M on S and for every family T of measureable sets of S and for every sequence F of separated subsets of S such that $T = \operatorname{rng} F$ holds $\sum(M \cdot F) \leq M(\bigcup T)$.

(16) For every set X and for every σ-field S of subsets of X and for every measure M on S and for every sequence F of separated subsets of S holds $\sum(M \cdot F) \leq M(\bigcup \operatorname{rng} F)$.

(17) For every set X and for every σ-field S of subsets of X and for every measure M on S such that for every sequence F of separated subsets of S holds $M(\bigcup \operatorname{rng} F) \leq \sum(M \cdot F)$ holds M is a σ-measure on S.

Let X be a set, and let S be a σ-field of subsets of X, and let M be a σ-measure on S. We say that M is complete on S if and only if:

(Def.2) for every subset A of X and for every set B such that $B \in S$ holds if $A \subseteq B$ and $M(B) = 0_{\overline{\mathbb{R}}}$, then $A \in S$.

Let X be a set, and let S be a σ-field of subsets of X, and let M be a σ-measure on S. A subset of X is called a set with measure zero w.r.t. M if:

(Def.3) there exists a set B such that $B \in S$ and it $\subseteq B$ and $M(B) = 0_{\overline{\mathbb{R}}}$.

Let X be a set, and let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor $\mathrm{COM}(S, M)$ yielding a non-empty family of subsets of X is defined as follows:

(Def.4) for an arbitrary A holds $A \in \mathrm{COM}(S, M)$ if and only if there exists a set B such that $B \in S$ and there exists a set C with measure zero w.r.t. M such that $A = B \cup C$.

Let X be a set, and let S be a σ-field of subsets of X, and let M be a σ-measure on S, and let A be an element of $\mathrm{COM}(S, M)$. The functor $\mathrm{MeasPart} A$ yields a non-empty family of subsets of X and is defined as follows:

(Def.5) for an arbitrary B holds $B \in \mathrm{MeasPart} A$ if and only if $B \in S$ and $B \subseteq A$ and $A \setminus B$ is a set with measure zero w.r.t. M.

Let X be a set, and let S be a σ-field of subsets of X, and let M be a σ-measure on S, and let F be a function from $\mathbb{N}$ into $\mathrm{COM}(S, M)$, and let n be a natural number. Then $F(n)$ is an element of $\mathrm{COM}(S, M)$.

We now state four propositions:

(18) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every function F from $\mathbb{N}$ into $\mathrm{COM}(S, M)$ there exists a function G from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$ holds $G(n) \in \mathrm{MeasPart}F(n)$.

(19) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every function F from $\mathbb{N}$ into $\mathrm{COM}(S, M)$ and for every function G from $\mathbb{N}$ into S there exists a function H from $\mathbb{N}$ into 2^X such that for every element n of $\mathbb{N}$ holds $H(n) = F(n) \setminus G(n)$.

(20) Let X be a set. Then for every σ-field S of subsets of X and for every σ-measure M on S and for every function F from $\mathbb{N}$ into 2^X such that for every element n of $\mathbb{N}$ holds $F(n)$ is a set with measure zero w.r.t. M there exists a function G from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$ holds $F(n) \subseteq G(n)$ and $M(G(n)) = 0_{\overline{\mathbb{R}}}$.

(21) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every non-empty family D of subsets of X such that for an arbitrary A holds $A \in D$ if and only if there exists a set B such that $B \in S$ and there exists a set C with measure zero w.r.t. M such that $A = B \cup C$ holds D is a σ-field of subsets of X.

Let X be a set, and let S be a σ-field of subsets of X, and let M be a σ-measure on S. Then $\mathrm{COM}(S, M)$ is a σ-field of subsets of X.

Next we state the proposition

(22) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for all sets B_1, B_2 such that $B_1 \in S$ and $B_2 \in S$ and for all sets C_1, C_2 with measure zero w.r.t. M such that $B_1 \cup C_1 = B_2 \cup C_2$ holds $M(B_1) = M(B_2)$.

Let X be a set, and let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor $\mathrm{COM}(M)$ yields a σ-measure on $\mathrm{COM}(S, M)$ and is defined by:

(Def.6) for every set B such that $B \in S$ and for every set C with measure zero w.r.t. M holds $(\mathrm{COM}(M))(B \cup C) = M(B)$.

The following proposition is true

(23) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S holds $\mathrm{COM}(M)$ is complete on $\mathrm{COM}(S, M)$.

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Received February 22, 1992

FORMALIZED MATHEMATICS
Vol.2,No.5, November–December 1991
Université Catholique de Louvain

Series in Banach and Hilbert Spaces

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Summary. In [20] the series of real numbers were investigated. The introduction to Banach and Hilbert Spaces ([12,13,14]), enables us to arrive at the concept of series in Hilbert Space. We start with the notions: partial sums of series, sum and n-th sum of series, convergent series (summable series), absolutely convergent series. We prove some basic theorems: the necessary condition for a series to converge, Weierstrass' test, d'Alembert's test, Cauchy's test.

MML Identifier: `BHSP_4`.

The notation and terminology used here have been introduced in the following articles: [5], [23], [28], [3], [4], [1], [10], [8], [9], [7], [20], [2], [29], [21], [22], [17], [27], [26], [24], [16], [12], [13], [15], [6], [11], [14], [25], [18], and [19]. For simplicity we adopt the following convention: X denotes a real unitary space, a, b, r denote real numbers, s_1, s_2, s_3 denote sequences of X, R_1, R_2, R_3 denote sequences of real numbers, and k, n, m denote natural numbers. The scheme *Rec_Func_Ex_RUS* deals with a real unitary space $\mathcal{A}$, a point $\mathcal{B}$ of $\mathcal{A}$, and a binary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$ and states that:

there exists a function f from $\mathbb{N}$ into the vectors of the vectors of $\mathcal{A}$ such that $f(0) = \mathcal{B}$ and for every element n of $\mathbb{N}$ and for every point x of $\mathcal{A}$ such that $x = f(n)$ holds $f(n+1) = \mathcal{F}(n, x)$

for all values of the parameters.

Let us consider X, s_1. The functor $(\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}}$ yields a sequence of X and is defined as follows:

(Def.1) $(\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}}(0) = s_1(0)$ and for every n holds $(\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}}(n) + s_1(n+1)$.

Next we state several propositions:

(1) $(\sum_{\alpha=0}^{\kappa} s_2(\alpha))_{\kappa\in\mathbb{N}} + (\sum_{\alpha=0}^{\kappa} s_3(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_2 + s_3)(\alpha))_{\kappa\in\mathbb{N}}$.

(2) $(\sum_{\alpha=0}^{\kappa} s_2(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa} s_3(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_2 - s_3)(\alpha))_{\kappa\in\mathbb{N}}$.

(3) $(\sum_{\alpha=0}^{\kappa}(a \cdot s_1)(\alpha))_{\kappa\in\mathbb{N}} = a \cdot (\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}}$.

ISSN 0777–4028

(4) $(\sum_{\alpha=0}^{\kappa}(-s_1)(\alpha))_{\kappa\in\mathbb{N}} = -(\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}}$.

(5) $a\cdot(\sum_{\alpha=0}^{\kappa} s_2(\alpha))_{\kappa\in\mathbb{N}} + b\cdot(\sum_{\alpha=0}^{\kappa} s_3(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(a\cdot s_2+b\cdot s_3)(\alpha))_{\kappa\in\mathbb{N}}$.

Let us consider X, s_1. We say that s_1 is summable if and only if:

(Def.2) $(\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}}$ is convergent.

Let us consider X, s_1. Let us assume that s_1 is summable. The functor $\sum s_1$ yielding a point of X is defined by:

(Def.3) $\sum s_1 = \lim((\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}})$.

Next we state several propositions:

(6) If s_2 is summable and s_3 is summable, then $s_2 + s_3$ is summable and $\sum(s_2 + s_3) = \sum s_2 + \sum s_3$.

(7) If s_2 is summable and s_3 is summable, then $s_2 - s_3$ is summable and $\sum(s_2 - s_3) = \sum s_2 - \sum s_3$.

(8) If s_1 is summable, then $a\cdot s_1$ is summable and $\sum(a\cdot s_1) = a\cdot\sum s_1$.

(9) If s_1 is summable, then s_1 is convergent and $\lim s_1 = 0_{\text{the vectors of } X}$.

(10) If X is a Hilbert space, then s_1 is summable if and only if for every r such that $r > 0$ there exists k such that for all n, m such that $n \geq k$ and $m \geq k$ holds $\|(\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}}(m)\| < r$.

(11) If s_1 is summable, then $(\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}}$ is bounded.

(12) For all s_1, s_2 such that for every n holds $s_2(n) = s_1(0)$ holds $(\sum_{\alpha=0}^{\kappa}(s_1\uparrow 1)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}}\uparrow 1 - s_2$.

(13) If s_1 is summable, then for every k holds $s_1\uparrow k$ is summable.

(14) If there exists k such that $s_1\uparrow k$ is summable, then s_1 is summable.

Let us consider X, s_1, n. The functor $\sum_{\kappa=0}^{n} s_1(\kappa)$ yielding a point of X is defined by:

(Def.4) $\sum_{\kappa=0}^{n} s_1(\kappa) = (\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}}(n)$.

We now state several propositions:

(15) $\sum_{\kappa=0}^{n} s_1(\kappa) = (\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(16) $\sum_{\kappa=0}^{0} s_1(\kappa) = s_1(0)$.

(17) $\sum_{\kappa=0}^{1} s_1(\kappa) = \sum_{\kappa=0}^{0} s_1(\kappa) + s_1(1)$.

(18) $\sum_{\kappa=0}^{1} s_1(\kappa) = s_1(0) + s_1(1)$.

(19) $\sum_{\kappa=0}^{n+1} s_1(\kappa) = \sum_{\kappa=0}^{n} s_1(\kappa) + s_1(n+1)$.

(20) $s_1(n+1) = \sum_{\kappa=0}^{n+1} s_1(\kappa) - \sum_{\kappa=0}^{n} s_1(\kappa)$.

(21) $s_1(1) = \sum_{\kappa=0}^{1} s_1(\kappa) - \sum_{\kappa=0}^{0} s_1(\kappa)$.

Let us consider X, s_1, n, m. The functor $\sum_{\kappa=n+1}^{m} s_1(\kappa)$ yielding a point of X is defined by:

(Def.5) $\sum_{\kappa=n+1}^{m} s_1(\kappa) = \sum_{\kappa=0}^{n} s_1(\kappa) - \sum_{\kappa=0}^{m} s_1(\kappa)$.

The following propositions are true:

(22) $\sum_{\kappa=n+1}^{m} s_1(\kappa) = \sum_{\kappa=0}^{n} s_1(\kappa) - \sum_{\kappa=0}^{m} s_1(\kappa)$.

(23) $\sum_{\kappa=1+1}^{0} s_1(\kappa) = s_1(1)$.

(24) $\sum_{\kappa=n+1+1}^{n} s_1(\kappa) = s_1(n+1)$.

(25) If X is a Hilbert space, then s_1 is summable if and only if for every r such that $r > 0$ there exists k such that for all n, m such that $n \geq k$ and $m \geq k$ holds $\|\sum_{\kappa=0}^{n} s_1(\kappa) - \sum_{\kappa=0}^{m} s_1(\kappa)\| < r$.

(26) If X is a Hilbert space, then s_1 is summable if and only if for every r such that $r > 0$ there exists k such that for all n, m such that $n \geq k$ and $m \geq k$ holds $\|\sum_{\kappa=n+1}^{m} s_1(\kappa)\| < r$.

Let us consider R_1, n. The functor $\sum_{\kappa=0}^{n} R_1(\kappa)$ yields a real number and is defined by:

(Def.6) $\sum_{\kappa=0}^{n} R_1(\kappa) = (\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}(n)$.

Let us consider R_1, n, m. The functor $\sum_{\kappa=n+1}^{m} R_1(\kappa)$ yielding a real number is defined by:

(Def.7) $\sum_{\kappa=n+1}^{m} R_1(\kappa) = \sum_{\kappa=0}^{n} R_1(\kappa) - \sum_{\kappa=0}^{m} R_1(\kappa)$.

Let us consider X, s_1. We say that s_1 is absolutely summable if and only if:

(Def.8) $\|s_1\|$ is summable.

The following propositions are true:

(27) If s_2 is absolutely summable and s_3 is absolutely summable, then s_2+s_3 is absolutely summable.

(28) If s_1 is absolutely summable, then $a \cdot s_1$ is absolutely summable.

(29) If for every n holds $\|s_1\|(n) \leq R_1(n)$ and R_1 is summable, then s_1 is absolutely summable.

(30) If for every n holds $s_1(n) \neq 0_{\text{the vectors of } X}$ and $R_1(n) = \frac{\|s_1(n+1)\|}{\|s_1(n)\|}$ and R_1 is convergent and $\lim R_1 < 1$, then s_1 is absolutely summable.

(31) If $r > 0$ and there exists m such that for every n such that $n \geq m$ holds $\|s_1(n)\| \geq r$, then s_1 is not convergent or $\lim s_1 \neq 0_{\text{the vectors of } X}$.

(32) If for every n holds $s_1(n) \neq 0_{\text{the vectors of } X}$ and there exists m such that for every n such that $n \geq m$ holds $\frac{\|s_1(n+1)\|}{\|s_1(n)\|} \geq 1$, then s_1 is not summable.

(33) If for every n holds $s_1(n) \neq 0_{\text{the vectors of } X}$ and for every n holds $R_1(n) = \frac{\|s_1(n+1)\|}{\|s_1(n)\|}$ and R_1 is convergent and $\lim R_1 > 1$, then s_1 is not summable.

(34) If for every n holds $R_1(n) = \sqrt[n]{\|s_1(n)\|}$ and R_1 is convergent and $\lim R_1 < 1$, then s_1 is absolutely summable.

(35) If for every n holds $R_1(n) = \sqrt[n]{\|s_1\|(n)}$ and there exists m such that for every n such that $n \geq m$ holds $R_1(n) \geq 1$, then s_1 is not summable.

(36) If for every n holds $R_1(n) = \sqrt[n]{\|s_1\|(n)}$ and R_1 is convergent and $\lim R_1 > 1$, then s_1 is not summable.

(37) $(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}$ is non-decreasing.

(38) For every n holds $(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}(n) \geq 0$.

(39) For every n holds $\|(\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}}(n)\| \leq (\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(40) For every n holds $\|\sum_{\kappa=0}^{n} s_1(\kappa)\| \leq \sum_{\kappa=0}^{n}\|s_1\|(\kappa)$.

(41) For all n, m holds $\|(\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}}(n)\| \leq |(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}(n)|$.

(42) For all n, m holds $\|\sum_{\kappa=0}^{m} s_1(\kappa) - \sum_{\kappa=0}^{n} s_1(\kappa)\| \leq |\sum_{\kappa=0}^{m}\|s_1\|(\kappa) - \sum_{\kappa=0}^{n}\|s_1\|(\kappa)|$.

(43) For all n, m holds $\|\sum_{\kappa=m+1}^{n} s_1(\kappa)\| \leq |\sum_{\kappa=m+1}^{n}\|s_1\|(\kappa)|$.

(44) If X is a Hilbert space, then if s_1 is absolutely summable, then s_1 is summable.

Let us consider X, s_1, R_1. The functor $R_1 \cdot s_1$ yielding a sequence of X is defined as follows:

(Def.9) for every n holds $(R_1 \cdot s_1)(n) = R_1(n) \cdot s_1(n)$.

One can prove the following propositions:

(45) $R_1 \cdot (s_2 + s_3) = R_1 \cdot s_2 + R_1 \cdot s_3$.

(46) $(R_2 + R_3) \cdot s_1 = R_2 \cdot s_1 + R_3 \cdot s_1$.

(47) $(R_2\, R_3) \cdot s_1 = R_2 \cdot (R_3 \cdot s_1)$.

(48) $(a\, R_1) \cdot s_1 = a \cdot (R_1 \cdot s_1)$.

(49) $R_1 \cdot -s_1 = (-R_1) \cdot s_1$.

(50) If R_1 is convergent and s_1 is convergent, then $R_1 \cdot s_1$ is convergent.

(51) If R_1 is bounded and s_1 is bounded, then $R_1 \cdot s_1$ is bounded.

(52) If R_1 is convergent and s_1 is convergent, then $R_1 \cdot s_1$ is convergent and $\lim(R_1 \cdot s_1) = \lim R_1 \cdot \lim s_1$.

Let us consider R_1. We say that R_1 is a Cauchy sequence if and only if:

(Def.10) for every r such that $r > 0$ there exists k such that for all n, m such that $n \geq k$ and $m \geq k$ holds $|R_1(n) - R_1(m)| < r$.

One can prove the following propositions:

(53) If X is a Hilbert space, then if s_1 is a Cauchy sequence and R_1 is a Cauchy sequence, then $R_1 \cdot s_1$ is a Cauchy sequence.

(54) For every n holds $(\sum_{\alpha=0}^{\kappa}((R_1 - R_1 \uparrow 1) \cdot (\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}})(\alpha))_{\kappa\in\mathbb{N}}(n) = (\sum_{\alpha=0}^{\kappa}(R_1 \cdot s_1)(\alpha))_{\kappa\in\mathbb{N}}(n+1) - (R_1 \cdot (\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}})(n+1)$.

(55) For every n holds $(\sum_{\alpha=0}^{\kappa}(R_1 \cdot s_1)(\alpha))_{\kappa\in\mathbb{N}}(n+1) = (R_1 \cdot (\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}})(n+1) - (\sum_{\alpha=0}^{\kappa}((R_1 \uparrow 1 - R_1) \cdot (\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}})(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(56) For every n holds $\sum_{\kappa=0}^{n+1}(R_1 \cdot s_1)(\kappa) = (R_1 \cdot (\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}})(n+1) - \sum_{\kappa=0}^{n}((R_1 \uparrow 1 - R_1) \cdot (\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}})(\kappa)$.

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Received April 1, 1992

FORMALIZED MATHEMATICS
Vol.2,No.5, November–December 1991
Université Catholique de Louvain

Products and Coproducts in Categories

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Summary. A product and coproduct in categories are introduced. The concepts included corresponds to that presented in [7].

MML Identifier: CAT_3.

The papers [9], [1], [2], [8], [4], [6], [3], and [5] provide the notation and terminology for this paper.

1. Indexed families

For simplicity we adopt the following rules: I will be a set, x, x_1, x_2, y, y_1, y_2 will be arbitrary, A will be a non-empty set, C, D will be categories, a, b, c, d will be objects of C, and f, g, h, k, p_1, p_2, q_1, q_2, i_1, i_2, j_1, j_2 will be morphisms of C. Let us consider I, x, A, and let F be a function from I into A. Let us assume that $x \in I$. The functor F_x yielding an element of A is defined as follows:

(Def.1) $F_x = F(x)$.

The scheme *LambdaIdx* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$ and states that:

there exists a function F from $\mathcal{A}$ into $\mathcal{B}$ such that for every x such that $x \in \mathcal{A}$ holds $F_x = \mathcal{F}(x)$

for all values of the parameters.

The following proposition is true

(1) For all functions F_1, F_2 from I into A such that for every x such that $x \in I$ holds $F_{1x} = F_{2x}$ holds $F_1 = F_2$.

ISSN 0777–4028

The scheme *FuncIdx_correctn* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$ and states that:
(i) there exists a function F from $\mathcal{A}$ into $\mathcal{B}$ such that for every x such that $x \in \mathcal{A}$ holds $F_x = \mathcal{F}(x)$,
(ii) for all functions F_1, F_2 from $\mathcal{A}$ into $\mathcal{B}$ such that for every x such that $x \in \mathcal{A}$ holds $F_{1x} = \mathcal{F}(x)$ and for every x such that $x \in \mathcal{A}$ holds $F_{2x} = \mathcal{F}(x)$ holds $F_1 = F_2$
for all values of the parameters.

Let us consider A, I, and let a be an element of A. Then $I \longmapsto a$ is a function from I into A.

The following proposition is true

(2) For every element a of A such that $x \in I$ holds $(I \longmapsto a)_x = a$.

Let us consider x_1, x_2, y_1, y_2. The functor $[x_1 \longmapsto y_1, x_2 \longmapsto y_2]$ yields a function and is defined as follows:

(Def.2) $[x_1 \longmapsto y_1, x_2 \longmapsto y_2] = (\{x_1\} \longmapsto y_1) +\cdot (\{x_2\} \longmapsto y_2)$.

The following propositions are true:

(3) $\mathrm{dom}[x_1 \longmapsto y_1, x_2 \longmapsto y_2] = \{x_1, x_2\}$ and $\mathrm{rng}[x_1 \longmapsto y_1, x_2 \longmapsto y_2] \subseteq \{y_1, y_2\}$.

(4) If $x_1 \neq x_2$, then $[x_1 \longmapsto y_1, x_2 \longmapsto y_2](x_1) = y_1$ and $[x_1 \longmapsto y_1, x_2 \longmapsto y_2](x_2) = y_2$.

(5) If $x_1 \neq x_2$, then $\mathrm{rng}[x_1 \longmapsto y_1, x_2 \longmapsto y_2] = \{y_1, y_2\}$.

(6) $[x_1 \longmapsto y, x_2 \longmapsto y] = \{x_1, x_2\} \longmapsto y$.

Let us consider A, x_1, x_2, and let y_1, y_2 be elements of A. Then $[x_1 \longmapsto y_1, x_2 \longmapsto y_2]$ is a function from $\{x_1, x_2\}$ into A.

The following proposition is true

(7) If $x_1 \neq x_2$, then for all elements y_1, y_2 of A holds $[x_1 \longmapsto y_1, x_2 \longmapsto y_2]_{x_1} = y_1$ and $[x_1 \longmapsto y_1, x_2 \longmapsto y_2]_{x_2} = y_2$.

2. Indexed families of morphisms

We now define two new functors. Let us consider C, I, and let F be a function from I into the morphisms of C. The functor $\mathrm{dom}_\kappa F(\kappa)$ yielding a function from I into the objects of C is defined as follows:

(Def.3) for every x such that $x \in I$ holds $(\mathrm{dom}_\kappa F(\kappa))_x = \mathrm{dom}(F_x)$.

The functor $\mathrm{cod}_\kappa F(\kappa)$ yielding a function from I into the objects of C is defined by:

(Def.4) for every x such that $x \in I$ holds $(\mathrm{cod}_\kappa F(\kappa))_x = \mathrm{cod}(F_x)$.

We now state four propositions:

(8) $\mathrm{dom}_\kappa(I \longmapsto f)(\kappa) = I \longmapsto \mathrm{dom}\, f$.

(9) $\mathrm{cod}_\kappa(I \longmapsto f)(\kappa) = I \longmapsto \mathrm{cod}\, f$.

(10) $\mathrm{dom}_\kappa[x_1 \longmapsto p_1, x_2 \longmapsto p_2](\kappa) = [x_1 \longmapsto \mathrm{dom}\, p_1, x_2 \longmapsto \mathrm{dom}\, p_2]$.

(11) $\operatorname{cod}_\kappa[x_1 \longmapsto p_1, x_2 \longmapsto p_2](\kappa) = [x_1 \longmapsto \operatorname{cod} p_1, x_2 \longmapsto \operatorname{cod} p_2]$.

Let us consider C, I, and let F be a function from I into the morphisms of C. The functor $F^{\rm op}$ yields a function from I into the morphisms of $C^{\rm op}$ and is defined as follows:

(Def.5) for every x such that $x \in I$ holds $(F^{\rm op})_x = (F_x)^{\rm op}$.

Next we state three propositions:

(12) $(I \longmapsto f)^{\rm op} = I \longmapsto f^{\rm op}$.

(13) If $x_1 \neq x_2$, then $[x_1 \longmapsto p_1, x_2 \longmapsto p_2]^{\rm op} = [x_1 \longmapsto {p_1}^{\rm op}, x_2 \longmapsto {p_2}^{\rm op}]$.

(14) For every function F from I into the morphisms of C holds $(F^{\rm op})^{\rm op} = F$.

Let us consider C, I, and let F be a function from I into the morphisms of $C^{\rm op}$. The functor ${}^{\rm op}F$ yielding a function from I into the morphisms of C is defined by:

(Def.6) for every x such that $x \in I$ holds $({}^{\rm op}F)_x = {}^{\rm op}(F_x)$.

The following propositions are true:

(15) For every morphism f of $C^{\rm op}$ holds ${}^{\rm op}(I \longmapsto f) = I \longmapsto {}^{\rm op}f$.

(16) If $x_1 \neq x_2$, then for all morphisms p_1, p_2 of $C^{\rm op}$ holds ${}^{\rm op}[x_1 \longmapsto p_1, x_2 \longmapsto p_2] = [x_1 \longmapsto {}^{\rm op}p_1, x_2 \longmapsto {}^{\rm op}p_2]$.

(17) For every function F from I into the morphisms of C holds ${}^{\rm op}(F^{\rm op}) = F$.

We now define two new functors. Let us consider C, I, and let F be a function from I into the morphisms of C, and let us consider f. The functor $F \cdot f$ yields a function from I into the morphisms of C and is defined as follows:

(Def.7) for every x such that $x \in I$ holds $(F \cdot f)_x = F_x \cdot f$.

The functor $f \cdot F$ yielding a function from I into the morphisms of C is defined by:

(Def.8) for every x such that $x \in I$ holds $(f \cdot F)_x = f \cdot F_x$.

The following four propositions are true:

(18) If $x_1 \neq x_2$, then $[x_1 \longmapsto p_1, x_2 \longmapsto p_2] \cdot f = [x_1 \longmapsto p_1 \cdot f, x_2 \longmapsto p_2 \cdot f]$.

(19) If $x_1 \neq x_2$, then $f \cdot [x_1 \longmapsto p_1, x_2 \longmapsto p_2] = [x_1 \longmapsto f \cdot p_1, x_2 \longmapsto f \cdot p_2]$.

(20) For every function F from I into the morphisms of C such that $\operatorname{dom}_\kappa F(\kappa) = I \longmapsto \operatorname{cod} f$ holds $\operatorname{dom}_\kappa F \cdot f(\kappa) = I \longmapsto \operatorname{dom} f$ and $\operatorname{cod}_\kappa F \cdot f(\kappa) = \operatorname{cod}_\kappa F(\kappa)$.

(21) For every function F from I into the morphisms of C such that $\operatorname{cod}_\kappa F(\kappa) = I \longmapsto \operatorname{dom} f$ holds $\operatorname{dom}_\kappa f \cdot F(\kappa) = \operatorname{dom}_\kappa F(\kappa)$ and $\operatorname{cod}_\kappa f \cdot F(\kappa) = I \longmapsto \operatorname{cod} f$.

Let us consider C, I, and let F, G be functions from I into the morphisms of C. The functor $F \cdot G$ yields a function from I into the morphisms of C and is defined by:

(Def.9) for every x such that $x \in I$ holds $(F \cdot G)_x = F_x \cdot G_x$.

We now state four propositions:

(22) For all functions F, G from I into the morphisms of C such that $\mathrm{dom}_\kappa F(\kappa) = \mathrm{cod}_\kappa G(\kappa)$ holds $\mathrm{dom}_\kappa F \cdot G(\kappa) = \mathrm{dom}_\kappa G(\kappa)$ and $\mathrm{cod}_\kappa F \cdot G(\kappa) = \mathrm{cod}_\kappa F(\kappa)$.

(23) If $x_1 \neq x_2$, then $[x_1 \longmapsto p_1, x_2 \longmapsto p_2] \cdot [x_1 \longmapsto q_1, x_2 \longmapsto q_2] = [x_1 \longmapsto p_1 \cdot q_1, x_2 \longmapsto p_2 \cdot q_2]$.

(24) For every function F from I into the morphisms of C holds $F \cdot f = F \cdot (I \longmapsto f)$.

(25) For every function F from I into the morphisms of C holds $f \cdot F = (I \longmapsto f) \cdot F$.

3. Retractions and coretractions

We now define two new attributes. Let us consider C. A morphism of C is retraction if:

(Def.10) there exists g such that $\mathrm{cod}\, g = \mathrm{dom}\, \mathrm{it}$ and $\mathrm{it} \cdot g = \mathrm{id}_{\mathrm{cod}\, \mathrm{it}}$.

A morphism of C is coretraction if:

(Def.11) there exists g such that $\mathrm{dom}\, g = \mathrm{cod}\, \mathrm{it}$ and $g \cdot \mathrm{it} = \mathrm{id}_{\mathrm{dom}\, \mathrm{it}}$.

The following propositions are true:

(26) If f is retraction, then f is epi.

(27) If f is coretraction, then f is monic.

(28) If f is retraction and g is retraction and $\mathrm{dom}\, g = \mathrm{cod}\, f$, then $g \cdot f$ is retraction.

(29) If f is coretraction and g is coretraction and $\mathrm{dom}\, g = \mathrm{cod}\, f$, then $g \cdot f$ is coretraction.

(30) If $\mathrm{dom}\, g = \mathrm{cod}\, f$ and $g \cdot f$ is retraction, then g is retraction.

(31) If $\mathrm{dom}\, g = \mathrm{cod}\, f$ and $g \cdot f$ is coretraction, then f is coretraction.

(32) If f is retraction and f is monic, then f is invertible.

(33) If f is coretraction and f is epi, then f is invertible.

(34) f is invertible if and only if f is retraction and f is coretraction.

(35) For every functor T from C to D such that f is retraction holds $T(f)$ is retraction.

(36) For every functor T from C to D such that f is coretraction holds $T(f)$ is coretraction.

(37) f is retraction if and only if f^{op} is coretraction.

(38) f is coretraction if and only if f^{op} is retraction.

4. Morphisms determined by a terminal object

Let us consider C, a, b. Let us assume that b is a terminal object. $|_b a$ is a morphism from a to b.

Next we state three propositions:

(39) If b is a terminal object, then $\mathrm{dom}|_b a = a$ and $\mathrm{cod}|_b a = b$.

(40) If b is a terminal object and $\mathrm{dom}\, f = a$ and $\mathrm{cod}\, f = b$, then $|_b a = f$.

(41) For every morphism f from a to b such that b is a terminal object holds $|_b a = f$.

5. Morphisms determined by an iniatial object

Let us consider C, a, b. Let us assume that a is an initial object. $|^a b$ is a morphism from a to b.

Next we state three propositions:

(42) If a is an initial object, then $\mathrm{dom}|^a b = a$ and $\mathrm{cod}|^a b = b$.

(43) If a is an initial object and $\mathrm{dom}\, f = a$ and $\mathrm{cod}\, f = b$, then $|^a b = f$.

(44) For every morphism f from a to b such that a is an initial object holds $|^a b = f$.

6. Products

Let us consider C, a, I. A function from I into the morphisms of C is said to be a projections family from a onto I if:

(Def.12) $\mathrm{dom}_\kappa\, \mathrm{it}(\kappa) = I \longmapsto a$.

We now state several propositions:

(45) For every projections family F from a onto I such that $x \in I$ holds $\mathrm{dom}(F_x) = a$.

(46) For every function F from $\emptyset$ into the morphisms of C holds F is a projections family from a onto $\emptyset$.

(47) If $\mathrm{dom}\, f = a$, then $\{y\} \longmapsto f$ is a projections family from a onto $\{y\}$.

(48) If $\mathrm{dom}\, p_1 = a$ and $\mathrm{dom}\, p_2 = a$, then $[x_1 \longmapsto p_1, x_2 \longmapsto p_2]$ is a projections family from a onto $\{x_1, x_2\}$.

(49) For every projections family F from a onto $\emptyset$ holds $F = \square$.

(50) For every projections family F from a onto I such that $\mathrm{cod}\, f = a$ holds $F \cdot f$ is a projections family from $\mathrm{dom}\, f$ onto I.

(51) For every function F from I into the morphisms of C and for every projections family G from a onto I such that $\mathrm{dom}_\kappa\, F(\kappa) = \mathrm{cod}_\kappa\, G(\kappa)$ holds $F \cdot G$ is a projections family from a onto I.

(52) For every projections family F from $\mathrm{cod}\, f$ onto I holds $f^{\mathrm{op}} \cdot F^{\mathrm{op}} = (F \cdot f)^{\mathrm{op}}$.

Let us consider C, a, I, and let F be a function from I into the morphisms of C. We say that a is a product w.r.t. F if and only if the conditions (Def.13) is satisfied.

(Def.13) (i) F is a projections family from a onto I,

(ii) for every b and for every projections family F' from b onto I such that $\mathrm{cod}_\kappa F(\kappa) = \mathrm{cod}_\kappa F'(\kappa)$ there exists h such that $h \in \mathrm{hom}(b, a)$ and for every k such that $k \in \mathrm{hom}(b, a)$ holds for every x such that $x \in I$ holds $F_x \cdot k = F'_x$ if and only if $h = k$.

One can prove the following propositions:

(53) For every projections family F from c onto I and for every projections family F' from d onto I such that c is a product w.r.t. F and d is a product w.r.t. F' and $\mathrm{cod}_\kappa F(\kappa) = \mathrm{cod}_\kappa F'(\kappa)$ holds c and d are isomorphic.

(54) For every projections family F from c onto I such that c is a product w.r.t. F and for all x_1, x_2 such that $x_1 \in I$ and $x_2 \in I$ holds $\mathrm{hom}(\mathrm{cod}(F_{x_1}), \mathrm{cod}(F_{x_2})) \neq \emptyset$ and for every x such that $x \in I$ holds F_x is retraction.

(55) For every function F from $\emptyset$ into the morphisms of C holds a is a product w.r.t. F if and only if a is a terminal object.

(56) For every projections family F from a onto I such that a is a product w.r.t. F and $\mathrm{dom}\, f = b$ and $\mathrm{cod}\, f = a$ and f is invertible holds b is a product w.r.t. $F \cdot f$.

(57) a is a product w.r.t. $\{y\} \longmapsto \mathrm{id}_a$.

(58) For every projections family F from a onto I such that a is a product w.r.t. F and for every x such that $x \in I$ holds $\mathrm{cod}(F_x)$ is a terminal object holds a is a terminal object.

Let us consider C, c, p_1, p_2. We say that c is a product w.r.t. p_1 and p_2 if and only if the conditions (Def.14) is satisfied.

(Def.14) (i) $\mathrm{dom}\, p_1 = c$,

(ii) $\mathrm{dom}\, p_2 = c$,

(iii) for all d, f, g such that $f \in \mathrm{hom}(d, \mathrm{cod}\, p_1)$ and $g \in \mathrm{hom}(d, \mathrm{cod}\, p_2)$ there exists h such that $h \in \mathrm{hom}(d, c)$ and for every k such that $k \in \mathrm{hom}(d, c)$ holds $p_1 \cdot k = f$ and $p_2 \cdot k = g$ if and only if $h = k$.

The following propositions are true:

(59) If $x_1 \neq x_2$, then c is a product w.r.t. p_1 and p_2 if and only if c is a product w.r.t. $[x_1 \longmapsto p_1, x_2 \longmapsto p_2]$.

(60) Suppose $\mathrm{hom}(c, a) \neq \emptyset$ and $\mathrm{hom}(c, b) \neq \emptyset$. Let p_1 be a morphism from c to a. Let p_2 be a morphism from c to b. Then c is a product w.r.t. p_1 and p_2 if and only if for every d such that $\mathrm{hom}(d, a) \neq \emptyset$ and $\mathrm{hom}(d, b) \neq \emptyset$ holds $\mathrm{hom}(d, c) \neq \emptyset$ and for every morphism f from d to a and for every morphism g from d to b there exists a morphism h from d to c such that for every morphism k from d to c holds $p_1 \cdot k = f$ and $p_2 \cdot k = g$ if and only if $h = k$.

(61) If c is a product w.r.t. p_1 and p_2 and d is a product w.r.t. q_1 and q_2 and $\mathrm{cod}\, p_1 = \mathrm{cod}\, q_1$ and $\mathrm{cod}\, p_2 = \mathrm{cod}\, q_2$, then c and d are isomorphic.

(62) If c is a product w.r.t. p_1 and p_2 and $\hom(\operatorname{cod} p_1, \operatorname{cod} p_2) \neq \emptyset$ and $\hom(\operatorname{cod} p_2, \operatorname{cod} p_1) \neq \emptyset$, then p_1 is retraction and p_2 is retraction.

(63) If c is a product w.r.t. p_1 and p_2 and $h \in \hom(c, c)$ and $p_1 \cdot h = p_1$ and $p_2 \cdot h = p_2$, then $h = \mathrm{id}_c$.

(64) If c is a product w.r.t. p_1 and p_2 and $\operatorname{dom} f = d$ and $\operatorname{cod} f = c$ and f is invertible, then d is a product w.r.t. $p_1 \cdot f$ and $p_2 \cdot f$.

(65) If c is a product w.r.t. p_1 and p_2 and $\operatorname{cod} p_2$ is a terminal object, then c and $\operatorname{cod} p_1$ are isomorphic.

(66) If c is a product w.r.t. p_1 and p_2 and $\operatorname{cod} p_1$ is a terminal object, then c and $\operatorname{cod} p_2$ are isomorphic.

7. Coproducts

Let us consider C, c, I. A function from I into the morphisms of C is said to be a injections family into c on I if:

(Def.15) $\operatorname{cod}_\kappa \operatorname{it}(\kappa) = I \longmapsto c$.

We now state a number of propositions:

(67) For every injections family F into c on I such that $x \in I$ holds $\operatorname{cod}(F_x) = c$.

(68) For every function F from $\emptyset$ into the morphisms of C holds F is a injections family into a on $\emptyset$.

(69) If $\operatorname{cod} f = a$, then $\{y\} \longmapsto f$ is a injections family into a on $\{y\}$.

(70) If $\operatorname{cod} p_1 = c$ and $\operatorname{cod} p_2 = c$, then $[x_1 \longmapsto p_1, x_2 \longmapsto p_2]$ is a injections family into c on $\{x_1, x_2\}$.

(71) For every injections family F into c on $\emptyset$ holds $F = \square$.

(72) For every injections family F into b on I such that $\operatorname{dom} f = b$ holds $f \cdot F$ is a injections family into $\operatorname{cod} f$ on I.

(73) For every injections family F into b on I and for every function G from I into the morphisms of C such that $\operatorname{dom}_\kappa F(\kappa) = \operatorname{cod}_\kappa G(\kappa)$ holds $F \cdot G$ is a injections family into b on I.

(74) For every function F from I into the morphisms of C holds F is a projections family from c onto I if and only if F^{op} is a injections family into c^{op} on I.

(75) For every function F from I into the morphisms of C^{op} and for every object c of C^{op} holds F is a injections family into c on I if and only if ${}^{\mathrm{op}}F$ is a projections family from ${}^{\mathrm{op}}c$ onto I.

(76) For every injections family F into $\operatorname{dom} f$ on I holds $F^{\mathrm{op}} \cdot f^{\mathrm{op}} = (f \cdot F)^{\mathrm{op}}$.

Let us consider C, c, I, and let F be a function from I into the morphisms of C. We say that c is a coproduct w.r.t. F if and only if the conditions (Def.16) is satisfied.

(Def.16) (i) F is a injections family into c on I,

(ii) for every d and for every injections family F' into d on I such that $\mathrm{dom}_\kappa F(\kappa) = \mathrm{dom}_\kappa F'(\kappa)$ there exists h such that $h \in \hom(c,d)$ and for every k such that $k \in \hom(c,d)$ holds for every x such that $x \in I$ holds $k \cdot F_x = F'_x$ if and only if $h = k$.

One can prove the following propositions:

(77) For every function F from I into the morphisms of C holds c is a product w.r.t. F if and only if c^{op} is a coproduct w.r.t. F^{op}.

(78) For every injections family F into c on I and for every injections family F' into d on I such that c is a coproduct w.r.t. F and d is a coproduct w.r.t. F' and $\mathrm{dom}_\kappa F(\kappa) = \mathrm{dom}_\kappa F'(\kappa)$ holds c and d are isomorphic.

(79) For every injections family F into c on I such that c is a coproduct w.r.t. F and for all x_1, x_2 such that $x_1 \in I$ and $x_2 \in I$ holds $\hom(\mathrm{dom}(F_{x_1}), \mathrm{dom}(F_{x_2})) \neq \emptyset$ and for every x such that $x \in I$ holds F_x is coretraction.

(80) For every injections family F into a on I such that a is a coproduct w.r.t. F and $\mathrm{dom}\, f = a$ and $\mathrm{cod}\, f = b$ and f is invertible holds b is a coproduct w.r.t. $f \cdot F$.

(81) For every injections family F into a on $\emptyset$ holds a is a coproduct w.r.t. F if and only if a is an initial object.

(82) a is a coproduct w.r.t. $\{y\} \longmapsto \mathrm{id}_a$.

(83) For every injections family F into a on I such that a is a coproduct w.r.t. F and for every x such that $x \in I$ holds $\mathrm{dom}(F_x)$ is an initial object holds a is an initial object.

Let us consider C, c, i_1, i_2. We say that c is a coproduct w.r.t. i_1 and i_2 if and only if the conditions (Def.17) is satisfied.

(Def.17) (i) $\mathrm{cod}\, i_1 = c$,

(ii) $\mathrm{cod}\, i_2 = c$,

(iii) for all d, f, g such that $f \in \hom(\mathrm{dom}\, i_1, d)$ and $g \in \hom(\mathrm{dom}\, i_2, d)$ there exists h such that $h \in \hom(c,d)$ and for every k such that $k \in \hom(c,d)$ holds $k \cdot i_1 = f$ and $k \cdot i_2 = g$ if and only if $h = k$.

We now state several propositions:

(84) c is a product w.r.t. p_1 and p_2 if and only if c^{op} is a coproduct w.r.t. ${p_1}^{\mathrm{op}}$ and ${p_2}^{\mathrm{op}}$.

(85) If $x_1 \neq x_2$, then c is a coproduct w.r.t. i_1 and i_2 if and only if c is a coproduct w.r.t. $[x_1 \longmapsto i_1, x_2 \longmapsto i_2]$.

(86) If c is a coproduct w.r.t. i_1 and i_2 and d is a coproduct w.r.t. j_1 and j_2 and $\mathrm{dom}\, i_1 = \mathrm{dom}\, j_1$ and $\mathrm{dom}\, i_2 = \mathrm{dom}\, j_2$, then c and d are isomorphic.

(87) Suppose $\hom(a,c) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$. Let i_1 be a morphism from a to c. Let i_2 be a morphism from b to c. Then c is a coproduct w.r.t. i_1 and i_2 if and only if for every d such that $\hom(a,d) \neq \emptyset$ and $\hom(b,d) \neq \emptyset$ holds $\hom(c,d) \neq \emptyset$ and for every morphism f from a to d and for every

morphism g from b to d there exists a morphism h from c to d such that for every morphism k from c to d holds $k \cdot i_1 = f$ and $k \cdot i_2 = g$ if and only if $h = k$.

(88) If c is a coproduct w.r.t. i_1 and i_2 and $\hom(\operatorname{dom} i_1, \operatorname{dom} i_2) \neq \emptyset$ and $\hom(\operatorname{dom} i_2, \operatorname{dom} i_1) \neq \emptyset$, then i_1 is coretraction and i_2 is coretraction.

(89) If c is a coproduct w.r.t. i_1 and i_2 and $h \in \hom(c, c)$ and $h \cdot i_1 = i_1$ and $h \cdot i_2 = i_2$, then $h = \mathrm{id}_c$.

(90) If c is a coproduct w.r.t. i_1 and i_2 and $\operatorname{dom} f = c$ and $\operatorname{cod} f = d$ and f is invertible, then d is a coproduct w.r.t. $f \cdot i_1$ and $f \cdot i_2$.

(91) If c is a coproduct w.r.t. i_1 and i_2 and $\operatorname{dom} i_2$ is an initial object, then $\operatorname{dom} i_1$ and c are isomorphic.

(92) If c is a coproduct w.r.t. i_1 and i_2 and $\operatorname{dom} i_1$ is an initial object, then $\operatorname{dom} i_2$ and c are isomorphic.

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Received May 11, 1992

FORMALIZED MATHEMATICS
Vol.2,No.5, November–December 1991
Université Catholique de Louvain

Transpose Matrices and Groups of Permutations

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Summary. Some facts concerning matrices with dimention 2×2 are shown. Upper and lower triangular matrices, and operation of deleting rows and columns in a matrix are introduced. Besides, we deal with sets of permutations and the fact that all permutations of finite set constitute a finite group is proved. Some proofs are based on [11] and [14].

MML Identifier: `MATRIX_2`.

The articles [17], [7], [8], [3], [15], [2], [1], [19], [18], [21], [20], [4], [13], [16], [9], [6], [12], [10], and [5] provide the notation and terminology for this paper.

1. Some examples of matrices

For simplicity we follow a convention: x, x_1, x_2, y_1, y_2 are arbitrary, i, j, k, n, m are natural numbers, D is a non-empty set, K is a field, s is a finite sequence, and a, b, c, d are elements of D. The scheme *SeqDEx* concerns a non-empty set $\mathcal{A}$, a natural number $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a finite sequence p of elements of $\mathcal{A}$ such that $\operatorname{dom} p = \operatorname{Seg} \mathcal{B}$ and for every k such that $k \in \operatorname{Seg} \mathcal{B}$ holds $\mathcal{P}[k, p(k)]$
provided the following requirement is met:

- for every k such that $k \in \operatorname{Seg} \mathcal{B}$ there exists an element x of $\mathcal{A}$ such that $\mathcal{P}[k, x]$.

Let us consider D, a, b. Then $\langle a, b \rangle$ is a finite sequence of elements of D.

Let us consider n, m, and let a be arbitrary. The functor $\begin{pmatrix} a & \ldots & a \\ \vdots & \ddots & \vdots \\ a & \ldots & a \end{pmatrix}^{n \times m}$ yielding a tabular finite sequence is defined as follows:

ISSN 0777–4028

(Def.1) $$\begin{pmatrix} a & \dots & a \\ \vdots & \ddots & \vdots \\ a & \dots & a \end{pmatrix}^{n \times m} = n \longmapsto (m \longmapsto a).$$

Let us consider D, n, m, d. Then $\begin{pmatrix} d & \dots & d \\ \vdots & \ddots & \vdots \\ d & \dots & d \end{pmatrix}^{n \times m}$ is a matrix over D of dimension $n \times m$.

Next we state the proposition

(1) If $\langle i, j \rangle \in$ the indices of $\begin{pmatrix} a & \dots & a \\ \vdots & \ddots & \vdots \\ a & \dots & a \end{pmatrix}^{n \times m}$, then $(\begin{pmatrix} a & \dots & a \\ \vdots & \ddots & \vdots \\ a & \dots & a \end{pmatrix}^{n \times m})_{i,j} = a$.

In the sequel a', b' are elements of the carrier of K. Next we state the proposition

(2) $$\begin{pmatrix} a' & \dots & a' \\ \vdots & \ddots & \vdots \\ a' & \dots & a' \end{pmatrix}^{n \times n} + \begin{pmatrix} b' & \dots & b' \\ \vdots & \ddots & \vdots \\ b' & \dots & b' \end{pmatrix}^{n \times n} = \begin{pmatrix} a'+b' & \dots & a'+b' \\ \vdots & \ddots & \vdots \\ a'+b' & \dots & a'+b' \end{pmatrix}^{n \times n}.$$

Let a, b, c, d be arbitrary. The functor $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ yielding a tabular finite sequence is defined as follows:

(Def.2) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \langle \langle a, b \rangle, \langle c, d \rangle \rangle.$$

The following two propositions are true:

(3) $\operatorname{len} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = 2$ and $\operatorname{width} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = 2$ and the indices of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = [\!: \operatorname{Seg} 2, \operatorname{Seg} 2 :\!]$.

(4) $\langle 1, 1 \rangle \in$ the indices of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$ and $\langle 1, 2 \rangle \in$ the indices of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$ and $\langle 2, 1 \rangle \in$ the indices of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$ and $\langle 2, 2 \rangle \in$ the indices of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$.

Let us consider D, and let a be an element of D. Then $\langle a \rangle$ is an element of D^1.

Let us consider D, and let us consider n, and let p be an element of D^n. Then $\langle p \rangle$ is a matrix over D of dimension $1 \times n$.

One can prove the following proposition

(5) $\langle 1, 1 \rangle \in$ the indices of $\langle\langle a \rangle\rangle$ and $\langle\langle a \rangle\rangle_{1,1} = a$.

Let us consider D, and let a, b, c, d be elements of D. Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix over D of dimension 2.

Next we state the proposition

(6) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{1,1} = a$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{1,2} = b$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2,1} = c$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2,2} = d$.

Let us consider n, and let K be a field. A matrix over K of dimension n is said to be an upper triangular matrix over K of dimension n if:

(Def.3) for all i, j such that $\langle i, j \rangle \in$ the indices of it holds if $i > j$, then $\mathrm{it}_{i,j} = 0_K$.

Let us consider n, K. A matrix over K of dimension n is said to be a lower triangular matrix over K of dimension n if:

(Def.4) for all i, j such that $\langle i, j \rangle \in$ the indices of it holds if $i < j$, then $\mathrm{it}_{i,j} = 0_K$.

The following proposition is true

(7) For every matrix M over D such that $\operatorname{len} M = n$ holds M is a matrix over D of dimension $n \times \operatorname{width} M$.

2. Deleting of rows and columns in a matrix

Let us consider i, and let p be a finite sequence. Let us assume that $i \in \operatorname{dom} p$. The functor $p_{\restriction i}$ yielding a finite sequence is defined by:

(Def.5) $p_{\restriction i} = p \cdot \operatorname{Sgm}(\operatorname{Seg} \operatorname{len} p \setminus \{i\})$.

We now state three propositions:

(8) For every finite sequence p such that $\operatorname{len} p > 0$ and for every i such that $i \in \operatorname{dom} p$ there exists m such that $\operatorname{len} p = m + 1$ and $\operatorname{len}(p_{\restriction i}) = m$.

(9) For every finite sequence p of elements of D and for every i such that $i \in \operatorname{dom} p$ holds $p_{\restriction i}$ is a finite sequence of elements of D.

(10) For every matrix M over K of dimension $n \times m$ and for every k such that $k \in \operatorname{Seg} n$ holds $M(k) = \operatorname{Line}(M, k)$.

Let us consider i, and let us consider K, and let M be a matrix over K. Let us assume that $i \in \operatorname{Seg} \operatorname{width} M$. The deleting of i-column in M yielding a matrix over K is defined as follows:

(Def.6) len(the deleting of i-column in M) $= \operatorname{len} M$ and for every k such that $k \in \operatorname{Seg} \operatorname{len} M$ holds (the deleting of i-column in M)$(k) = \operatorname{Line}(M, k)_{\restriction i}$.

The following propositions are true:

(11) For all matrices M_1, M_2 over D holds $M_1 = M_2$ if and only if ${M_1}^{\mathrm{T}} = {M_2}^{\mathrm{T}}$ and $\operatorname{len} M_1 = \operatorname{len} M_2$.

(12) For every matrix M over D such that $\operatorname{width} M > 0$ holds $\operatorname{len}(M^{\mathrm{T}}) = \operatorname{width} M$ and $\operatorname{width}(M^{\mathrm{T}}) = \operatorname{len} M$.

(13) For all matrices M_1, M_2 over D such that $\operatorname{width} M_1 > 0$ and $\operatorname{width} M_2 > 0$ holds $M_1 = M_2$ if and only if ${M_1}^{\mathrm{T}} = {M_2}^{\mathrm{T}}$ and $\operatorname{width}({M_1}^{\mathrm{T}}) = \operatorname{width}({M_2}^{\mathrm{T}})$.

(14) For all matrices M_1, M_2 over D such that $\operatorname{width} M_1 > 0$ and $\operatorname{width} M_2 > 0$ holds $M_1 = M_2$ if and only if ${M_1}^{\mathrm{T}} = {M_2}^{\mathrm{T}}$ and $\operatorname{width} M_1 = \operatorname{width} M_2$.

(15) For every matrix M over D such that $\operatorname{len} M > 0$ and $\operatorname{width} M > 0$ holds $(M^{\mathrm{T}})^{\mathrm{T}} = M$.

(16) For every matrix M over D and for every i such that $i \in \operatorname{Seg} \operatorname{len} M$ holds $\operatorname{Line}(M, i) = (M^{\mathrm{T}})_{\square, i}$.

(17) For every matrix M over D and for every j such that $j \in \operatorname{Seg} \operatorname{width} M$ holds $\operatorname{Line}(M^{\mathrm{T}}, j) = M_{\square, j}$.

(18) For every matrix M over D and for every i such that $i \in \operatorname{Seg} \operatorname{len} M$ holds $M(i) = \operatorname{Line}(M, i)$.

Let us consider i, and let us consider K, and let M be a matrix over K. Let us assume that $i \in \operatorname{Seg} \operatorname{len} M$ and $\operatorname{width} M > 0$. The deleting of i-row in M yields a matrix over K and is defined by:

(Def.7) (i) the deleting of i-row in $M = \varepsilon$ if $\operatorname{len} M = 1$,

(ii) width(the deleting of i-row in M) $= \operatorname{width} M$ and for every k such that $k \in \operatorname{Seg} \operatorname{width} M$ holds (the deleting of i-row in $M)_{\square, k} = (M_{\square, k})_{\restriction i}$, otherwise.

Let us consider i, j, and let us consider n, and let us consider K, and let M be a matrix over K of dimension n. The deleting of i-row and j-column in M yields a matrix over K and is defined as follows:

(Def.8) (i) the deleting of i-row and j-column in $M = \varepsilon$ if $n = 1$,

(ii) the deleting of i-row and j-column in $M =$ the deleting of j-column in the deleting of i-row in M, otherwise.

3. Sets of permutations

Let us consider n, and let q, p be permutations of $\operatorname{Seg} n$. Then $p \cdot q$ is a permutation of $\operatorname{Seg} n$.

A set is permutational if:

(Def.9) there exists n such that for every x such that $x \in$ it holds x is a permutation of $\operatorname{Seg} n$.

Let P be a permutational non-empty set. The functor $\operatorname{len} P$ yielding a natural number is defined as follows:

(Def.10) there exists s such that $s \in P$ and $\operatorname{len} P = \operatorname{len} s$.

Let P be a permutational non-empty set. We see that the element of P is a permutation of $\operatorname{Seg} \operatorname{len} P$.

One can prove the following proposition

(19) For every n there exists a permutational non-empty set P such that $\operatorname{len} P = n$.

Let us consider n. The permutations of n-element set constitute a permutational non-empty set defined as follows:

(Def.11) $x \in$ the permutations of n-element set if and only if x is a permutation of $\operatorname{Seg} n$.

The following propositions are true:

(20) len(the permutations of n-element set) $= n$.

(21) The permutations of 1-element set $= \{\mathrm{id}_1\}$.

Let us consider n, and let p be an element of the permutations of n-element set. The functor $\operatorname{len} p$ yields a natural number and is defined as follows:

(Def.12) there exists a finite sequence s such that $s = p$ and $\operatorname{len} p = \operatorname{len} s$.

We now state the proposition

(22) For every element p of the permutations of n-element set holds $\operatorname{len} p = n$.

4. Group of permutations

In the sequel p, q denote elements of the permutations of n-element set. Let us consider n. The functor A_n yielding a strict half group structure is defined by:

(Def.13) the carrier of $A_n =$ the permutations of n-element set and for all elements q, p of the permutations of n-element set holds (the operation of $A_n)(q, p) = p \cdot q$.

One can prove the following propositions:

(23) id_n is an element of A_n.

(24) $p \cdot \mathrm{id}_n = p$ and $\mathrm{id}_n \cdot p = p$.

(25) $p \cdot p^{-1} = \mathrm{id}_n$ and $p^{-1} \cdot p = \mathrm{id}_n$.

(26) p^{-1} is an element of A_n.

(27) p is an element of A_n if and only if p is an element of the permutations of n-element set.

Let us consider n. A permutation of n element set is an element of the permutations of n-element set.

Then A_n is a strict group.

We now state the proposition

(28) $\mathrm{id}_n = 1_{A_n}$.

Let us consider n, and let p be a permutation of $\operatorname{Seg} n$. We say that p is a transposition if and only if:

(Def.14) there exist i, j such that $i \in \operatorname{dom} p$ and $j \in \operatorname{dom} p$ and $i \neq j$ and $p(i) = j$ and $p(j) = i$ and for every k such that $k \neq i$ and $k \neq j$ and $k \in \operatorname{dom} p$ holds $p(k) = k$.

We now define two new predicates. Let us consider n, and let p be a permutation of $\operatorname{Seg} n$. We say that p is even if and only if:

(Def.15) there exists a finite sequence l of elements of the carrier of A_n such that $\operatorname{len} l \bmod 2 = 0$ and $p = \prod l$ and for every i such that $i \in \operatorname{dom} l$ there exists q such that $l(i) = q$ and q is a transposition.

p is odd stands for p is not even.

We now state the proposition

(29) $\mathrm{id}_{\operatorname{Seg} n}$ is even.

Let us consider K, n, and let x be an element of the carrier of K, and let p be an element of the permutations of n-element set. The functor $(-1)^{\operatorname{sgn}(p)}x$ yields an element of the carrier of K and is defined by:

(Def.16) (i) $(-1)^{\operatorname{sgn}(p)}x = x$ if p is even,

(ii) $(-1)^{\operatorname{sgn}(p)}x = -x$, otherwise.

Let X be a set. Let us assume that X is finite. The functor Ω^{f}_X yields an element of $\operatorname{Fin} X$ and is defined as follows:

(Def.17) $\Omega^{\mathrm{f}}_X = X$.

We now state the proposition

(30) The permutations of n-element set is finite.

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Received May 20, 1992

FORMALIZED MATHEMATICS
Vol.2, No.5, November–December 1991
Université Catholique de Louvain

Complete Lattices

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Summary. In the first section the lattice of subsets of distinct set is introduced. The join and meet operations are, respectively, union and intersection of sets, and the ordering relation is inclusion. It is shown that this lattice is Boolean, i.e. distributive and complimentary. The socond section introduced the poset generated in a distinct lattice by its ordering relation. Besides, it is proved that posets which have l.u.b.'s and g.l.b.'s for every two elements generate lattices with the same ordering relations. In the last section the concept of complete lattice is introduced and discussed. Finally, the fact that the function f from subsets of distinct set yielding elements of this set is a infinite union of some complete lattice, if f yields an element a for singleton $\{a\}$ and $f(f^\circ X) = f(\bigsqcup X)$ for every subset X, is proved. Some concepts and proofs are based on [6] and [7].

MML Identifier: LATTICE3.

The notation and terminology used here are introduced in the following articles: [10], [8], [13], [4], [5], [3], [17], [14], [15], [1], [9], [2], [16], [11], and [12].

1. Boolean lattice of subsets

Let X be a non-empty set, and let x, y be elements of X. Then $\{x,y\}$ is a non-empty subset of X.

Let X be a set, and let x, y be elements of 2^X. Then $x \cup y$ is a subset of X. Then $x \cap y$ is a subset of X.

Let X be a set. The lattice of subsets of X yields a strict lattice structure and is defined by:

(Def.1) the carrier of the lattice of subsets of $X = 2^X$ and for all elements Y, Z of 2^X holds (the join operation of the lattice of subsets of X)(Y, Z) $= Y \cup Z$ and (the meet operation of the lattice of subsets of X)(Y, Z) $= Y \cap Z$.

ISSN 0777–4028

In the sequel X will denote a set and x, y will denote elements of the lattice of subsets of X. The following propositions are true:

(1) $x \sqcup y = x \cup y$ and $x \sqcap y = x \cap y$.

(2) $x \sqsubseteq y$ if and only if $x \subseteq y$.

Let us consider X. Then the lattice of subsets of X is a strict lattice.

In the sequel x will denote an element of the lattice of subsets of X. The following propositions are true:

(3) The lattice of subsets of X is a lower bound lattice and $\perp_{\text{the lattice of subsets of } X} = \emptyset$.

(4) The lattice of subsets of X is an upper bound lattice and $\top_{\text{the lattice of subsets of } X} = X$.

Let us consider X. Then the lattice of subsets of X is a strict Boolean lattice.

Next we state the proposition

(5) For every element x of the lattice of subsets of X holds $x^{\mathrm{c}} = X \setminus x$.

2. Correspondence between lattices and posets

Let L be a lattice. Then $\mathrm{LattRel}(L)$ is an order in the carrier of L.

Let L be a lattice. The functor $\mathrm{Poset}(L)$ yields a strict poset and is defined as follows:

(Def.2) $\mathrm{Poset}(L) = \langle \text{the carrier of } L, \mathrm{LattRel}(L) \rangle$.

Next we state the proposition

(6) For all lattices L_1, L_2 such that $\mathrm{Poset}(L_1) = \mathrm{Poset}(L_2)$ holds the lattice structure of $L_1 =$ the lattice structure of L_2.

Let L be a lattice, and let p be an element of L. The functor $p^{\cdot}$ yields an element of $\mathrm{Poset}(L)$ and is defined as follows:

(Def.3) $p^{\cdot} = p$.

Let L be a lattice, and let p be an element of $\mathrm{Poset}(L)$. The functor ${}^{\cdot}p$ yielding an element of L is defined as follows:

(Def.4) ${}^{\cdot}p = p$.

In the sequel L is a lattice, p, q are elements of L, and p' is an element of $\mathrm{Poset}(L)$. We now state the proposition

(7) $p \sqsubseteq q$ if and only if $p^{\cdot} \leq q^{\cdot}$.

Let X be a set, and let O be an order in X. Then $O^{\smile}$ is an order in X.

Let A be a poset. The functor $A^{\smile}$ yields a strict poset and is defined as follows:

(Def.5) $A^{\smile} = \langle \text{the carrier of } A, (\text{the order of } A)^{\smile} \rangle$.

In the sequel A will be a poset and a, b, c will be elements of A. One can prove the following proposition

(8) $(A^{\smile})^{\smile} =$ the poset of A.

Let A be a poset, and let a be an element of A. The functor $a^{\smile}$ yielding an element of $A^{\smile}$ is defined as follows:

(Def.6) $a^{\smile} = a$.

Let A be a poset, and let a be an element of $A^{\smile}$. The functor $\frown a$ yielding an element of A is defined by:

(Def.7) $\frown a = a$.

One can prove the following proposition

(9) $a \leq b$ if and only if $b^{\smile} \leq a^{\smile}$.

We now define four new predicates. Let A be a poset, and let X be a set, and let a be an element of A. The predicate $a \leq X$ is defined as follows:

(Def.8) for every element b of A such that $b \in X$ holds $a \leq b$.

We write $X \geq a$ if $a \leq X$. The predicate $X \leq a$ is defined by:

(Def.9) for every element b of A such that $b \in X$ holds $b \leq a$.

We write $a \geq X$ if and only if $X \leq a$.

We now define two new attributes. A poset has l.u.b.'s if:

(Def.10) for every elements x, y of it there exists an element z of it such that $x \leq z$ and $y \leq z$ and for every element z' of it such that $x \leq z'$ and $y \leq z'$ holds $z \leq z'$.

A poset has g.l.b.'s if:

(Def.11) for every elements x, y of it there exists an element z of it such that $z \leq x$ and $z \leq y$ and for every element z' of it such that $z' \leq x$ and $z' \leq y$ holds $z' \leq z$.

We now state two propositions:

(10) A has l.u.b.'s if and only if $A^{\smile}$ has g.l.b.'s.

(11) For every lattice L holds Poset(L) has l.u.b.'s and g.l.b.'s.

A poset is complete if:

(Def.12) for every set X there exists an element a of it such that $X \leq a$ and for every element b of it such that $X \leq b$ holds $a \leq b$.

Next we state the proposition

(12) If A is complete, then A has l.u.b.'s and g.l.b.'s.

Let A be a poset satisfying the condition: A has l.u.b.'s. Let a, b be elements of A. The functor $a \sqcup b$ yielding an element of A is defined as follows:

(Def.13) $a \leq a \sqcup b$ and $b \leq a \sqcup b$ and for every element c of A such that $a \leq c$ and $b \leq c$ holds $a \sqcup b \leq c$.

Let A be a poset satisfying the condition: A has g.l.b.'s. Let a, b be elements of A. The functor $a \sqcap b$ yields an element of A and is defined as follows:

(Def.14) $a \sqcap b \leq a$ and $a \sqcap b \leq b$ and for every element c of A such that $c \leq a$ and $c \leq b$ holds $c \leq a \sqcap b$.

For simplicity we follow a convention: V denotes a poset with l.u.b.'s, u_1, u_2, u_3 denote elements of V, N denotes a poset with g.l.b.'s, n_1, n_2, n_3 denote elements of N, K denotes a poset with l.u.b.'s and g.l.b.'s, and k_1, k_2 denote elements of K. The following propositions are true:

(13) $u_1 \sqcup u_2 = u_2 \sqcup u_1$.

(14) $(u_1 \sqcup u_2) \sqcup u_3 = u_1 \sqcup (u_2 \sqcup u_3)$.

(15) $n_1 \sqcap n_2 = n_2 \sqcap n_1$.

(16) $(n_1 \sqcap n_2) \sqcap n_3 = n_1 \sqcap (n_2 \sqcap n_3)$.

(17) $k_1 \sqcap k_2 \sqcup k_2 = k_2$.

(18) $k_1 \sqcap (k_1 \sqcup k_2) = k_1$.

(19) For every A being a poset with l.u.b.'s and g.l.b.'s there exists a strict lattice L such that the poset of $A = \text{Poset}(L)$.

Let us consider A satisfying the condition: A has l.u.b.'s and g.l.b.'s. The functor $\mathbb{L}_A$ yields a strict lattice and is defined as follows:

(Def.15) the poset of $A = \text{Poset}(\mathbb{L}_A)$.

The following proposition is true

(20) $\text{LattRel}(L)^\smile = \text{LattRel}(L^\circ)$ and $\text{Poset}(L)^\smile = \text{Poset}(L^\circ)$.

3. Complete lattices

Let L be a lattice structure. A subset of L is a subset of the carrier of L.

We now define four new predicates. Let L be a lattice structure, and let p be an element of L, and let X be a set. The predicate $p \sqsubseteq X$ is defined by:

(Def.16) for every element q of L such that $q \in X$ holds $p \sqsubseteq q$.

We write $X \sqsupseteq p$ if $p \sqsubseteq X$. The predicate $X \sqsubseteq p$ is defined by:

(Def.17) for every element q of L such that $q \in X$ holds $q \sqsubseteq p$.

We write $p \sqsupseteq X$ if $X \sqsubseteq p$.

We now state two propositions:

(21) For every lattice L and for all elements p, q, r of L holds $p \sqsubseteq \{q, r\}$ if and only if $p \sqsubseteq q \sqcap r$.

(22) For every lattice L and for all elements p, q, r of L holds $p \sqsupseteq \{q, r\}$ if and only if $q \sqcup r \sqsubseteq p$.

We now define three new attributes. A lattice structure is complete if:

(Def.18) for every set X there exists an element p of it such that $X \sqsubseteq p$ and for every element r of it such that $X \sqsubseteq r$ holds $p \sqsubseteq r$.

A lattice structure is $\bigsqcup$-distributive if it satisfies the condition (Def.19).

(Def.19) Given X. Let a, b, c be elements of it. Then if $X \sqsubseteq a$ and for every element d of it such that $X \sqsubseteq d$ holds $a \sqsubseteq d$ and $\{b \sqcap a' : a' \in X\} \sqsubseteq c$, where a' ranges over elements of it and for every element d of it such that $\{b \sqcap a' : a' \in X\} \sqsubseteq d$, where a' ranges over elements of it holds $c \sqsubseteq d$, then $b \sqcap a \sqsubseteq c$.

A lattice structure is $\sqcap$-distributive if it satisfies the condition (Def.20).

(Def.20) Given X. Let a, b, c be elements of it. Then if $X \sqsupseteq a$ and for every element d of it such that $X \sqsupseteq d$ holds $d \sqsubseteq a$ and $\{b \sqcup a' : a' \in X\} \sqsupseteq c$, where a' ranges over elements of it and for every element d of it such that $\{b \sqcup a' : a' \in X\} \sqsupseteq d$, where a' ranges over elements of it holds $d \sqsubseteq c$, then $c \sqsubseteq b \sqcup a$.

We now state several propositions:

(23) For every Boolean lattice B and for every element a of B holds $X \sqsubseteq a$ if and only if $\{b^{\rm c} : b \in X\} \sqsupseteq a^{\rm c}$, where b ranges over elements of B.

(24) For every Boolean lattice B and for every element a of B holds $X \sqsupseteq a$ if and only if $\{b^{\rm c} : b \in X\} \sqsubseteq a^{\rm c}$, where b ranges over elements of B.

(25) The lattice of subsets of X is complete.

(26) The lattice of subsets of X is $\bigsqcup$-distributive.

(27) The lattice of subsets of X is $\bigsqcap$-distributive.

Next we state four propositions:

(28) $p \sqsubseteq X$ if and only if $p^{\cdot} \leq X$.

(29) $p' \leq X$ if and only if ${}^{\cdot}p' \sqsubseteq X$.

(30) $X \sqsubseteq p$ if and only if $X \leq p^{\cdot}$.

(31) $X \leq p'$ if and only if $X \sqsubseteq {}^{\cdot}p'$.

Let A be a complete poset. Then $\mathbb{L}_A$ is a complete strict lattice.

Let L be a lattice structure satisfying the condition: L is a complete lattice. Let X be a set. The functor $\bigsqcup_L X$ yields an element of L and is defined by:

(Def.21) $X \sqsubseteq \bigsqcup_L X$ and for every element r of L such that $X \sqsubseteq r$ holds $\bigsqcup_L X \sqsubseteq r$.

Let L be a lattice structure, and let X be a set. The functor $\bigsqcap_L X$ yielding an element of L is defined as follows:

(Def.22) $\bigsqcap_L X = \bigsqcup_L \{p : p \sqsubseteq X\}$, where p ranges over elements of L.

We now define two new functors. Let L be a lattice structure, and let X be a subset of L. We introduce the functor $\bigsqcup X$ as a synonym of $\bigsqcup_L X$. We introduce the functor $\bigsqcap X$ as a synonym of $\bigsqcap_L X$.

We adopt the following rules: C denotes a complete lattice, a, b, c denote elements of C, and X, Y denote sets. Next we state a number of propositions:

(32) $\bigsqcup_C \{a \sqcap b : b \in X\} \sqsubseteq a \sqcap \bigsqcup_C X$.

(33) C is $\bigsqcup$-distributive if and only if for all X, a holds $a \sqcap \bigsqcup_C X \sqsubseteq \bigsqcup_C \{a \sqcap b : b \in X\}$.

(34) $a = \bigsqcap_C X$ if and only if $a \sqsubseteq X$ and for every b such that $b \sqsubseteq X$ holds $b \sqsubseteq a$.

(35) $a \sqcup \bigsqcap_C X \sqsubseteq \bigsqcap_C \{a \sqcup b : b \in X\}$.

(36) C is $\bigsqcap$-distributive if and only if for all X, a holds $\bigsqcap_C \{a \sqcup b : b \in X\} \sqsubseteq a \sqcup \bigsqcap_C X$.

(37) $\bigsqcup_C X = \bigsqcap_C\{a : a \sqsupseteq X\}$.
(38) If $a \in X$, then $a \sqsubseteq \bigsqcup_C X$ and $\bigsqcap_C X \sqsubseteq a$.
(39) If $X \sqsubseteq a$, then $\bigsqcup_C X \sqsubseteq a$.
(40) If $a \sqsubseteq X$, then $a \sqsubseteq \bigsqcap_C X$.
(41) If $a \in X$ and $X \sqsubseteq a$, then $\bigsqcup_C X = a$.
(42) If $a \in X$ and $a \sqsubseteq X$, then $\bigsqcap_C X = a$.
(43) $\bigsqcup\{a\} = a$ and $\bigsqcap\{a\} = a$.
(44) $a \sqcup b = \bigsqcup\{a, b\}$ and $a \sqcap b = \bigsqcap\{a, b\}$.
(45) $a = \bigsqcup_C\{b : b \sqsubseteq a\}$ and $a = \bigsqcap_C\{c : a \sqsubseteq c\}$.
(46) If $X \subseteq Y$, then $\bigsqcup_C X \sqsubseteq \bigsqcup_C Y$ and $\bigsqcap_C Y \sqsubseteq \bigsqcap_C X$.
(47) $\bigsqcup_C X = \bigsqcup_C\{a : \bigvee_b[a \sqsubseteq b \wedge b \in X]\}$ and $\bigsqcap_C X = \bigsqcap_C\{b : \bigvee_a[a \sqsubseteq b \wedge a \in X]\}$.
(48) If for every a such that $a \in X$ there exists b such that $a \sqsubseteq b$ and $b \in Y$, then $\bigsqcup_C X \sqsubseteq \bigsqcup_C Y$.
(49) If $X \subseteq 2^{\text{the carrier of } C}$, then $\bigsqcup_C \bigcup X = \bigsqcup_C\{\bigsqcup Y : Y \in X\}$, where Y ranges over subsets of C.
(50) C is a lower bound lattice and $\bot_C = \bigsqcup_C \emptyset$.
(51) C is an upper bound lattice and $\top_C = \bigsqcup_C$ (the carrier of C).
(52) If C is an implicative lattice, then $a \Rightarrow b = \bigsqcup_C\{c : a \sqcap c \sqsubseteq b\}$.
(53) If C is an implicative lattice, then C is $\bigsqcup$-distributive.
(54) For every complete $\bigsqcup$-distributive lattice D and for every element a of D holds $a \sqcap \bigsqcup_D X = \bigsqcup_D\{a \sqcap b_1 : b_1 \in X\}$, where b_1 ranges over elements of D and $\bigsqcup_D X \sqcap a = \bigsqcup_D\{b_2 \sqcap a : b_2 \in X\}$, where b_2 ranges over elements of D.
(55) For every complete $\bigsqcap$-distributive lattice D and for every element a of D holds $a \sqcup \bigsqcap_D X = \bigsqcap_D\{a \sqcup b_1 : b_1 \in X\}$, where b_1 ranges over elements of D and $\bigsqcap_D X \sqcup a = \bigsqcap_D\{b_2 \sqcup a : b_2 \in X\}$, where b_2 ranges over elements of D.

In this article we present several logical schemes. The scheme *SingleFraenkel* deals with a constant $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{A} : \mathcal{P}[a]\} = \{\mathcal{A}\}$, where a ranges over elements of $\mathcal{B}$

provided the parameters meet the following requirement:

- there exists an element a of $\mathcal{B}$ such that $\mathcal{P}[a]$.

The scheme *FuncFraenkel* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, a function $\mathcal{C}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{C} \circ \{\mathcal{F}(x) : \mathcal{P}[x]\} = \{\mathcal{C}(\mathcal{F}(x)) : \mathcal{P}[x]\}$, where x ranges over elements of $\mathcal{A}$, and x ranges over elements of $\mathcal{A}$

provided the parameters satisfy the following condition:

- $\mathcal{B} \subseteq \operatorname{dom} \mathcal{C}$.

The following proposition is true

(56) Let D be a non-empty set. Let f be a function from 2^D into D. Then if for every element a of D holds $f(\{a\}) = a$ and for every subset X of 2^D holds $f(f^\circ X) = f(\bigcup X)$, then there exists a complete strict lattice L such that the carrier of $L = D$ and for every subset X of L holds $\bigsqcup X = f(X)$.

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Received May 13, 1992

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