Preface

As was stated in [3] we publish mathematical papers which are abstracts of Mizar articles to be found in the Main Mizar Library (MML). An article includes certain elements which are transferred to the data base, such as theorems or definitions. This has been due to the fact that the material published there was at first intended to help the Mizar users to handle the data base. Thus the works published there describe the present state of MML and are, in a sense, a report on the expansion of that library. Next to them there are also new mathematical papers because the new method of formalization is not trivial even though it refers to simple mathematical facts.

It must be explained at this point that both the PC-Mizar verifier and MML are being systematically developed. In the case of PC-Mizar it is mainly the Mizar language which is enriched, which makes it more convenient to write articles; the same might be said of proof-checker, which enables one to write shorter proofs and articles.

The development of MML consists in continuous revisions of articles accepted for publication, for instance in the removal of self-evident or repeated theorems (while the numbering of successive theorems in a given article is preserved). We then have the information in a footnote such as "The proposition (5) has been removed" (see [1], page 450). Previously such a comment was, e.g., "The proposition (9) was either repeated or obvious" (see [2], page 14).

Please note also that in the articles we use atypical symbolism for the Cartesian product [: :] , and that is no paranthesis in the case of grouping to the left. We also use overloading. For instance, see [1], page 469: "(Def.1) $F(f) = F(f)$". In the latter case, on the right side of the equality symbol we have the old functor, while on the left side we have the new functor, which differs from the old one only by the type of the result.

Our periodical appears five times a year, which is to say every two months except for the summer holidays period. The present issue, although dated September-October, also includes items contributed in November. They have been included because the editors received them before sending the issue 2(4) to the press.

Roman Matuszewski

References

[1] *Formalized Mathematics: a computer assisted approach.* Volume 2(**4**), Université Catholique de Louvain, September-October 1991.

[2] *Formalized Mathematics: a computer assisted approach.* Volume 2(**1**), Université Catholique de Louvain, January-February 1991.

[3] Andrzej Trybulec. Introduction. *Formalized Mathematics*, 1(**1**):7–8, January 1990.

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Parieses [1]

Konrad Raczkowski
Warsaw University
Białystok

Andrzej Nędzusiak
Warsaw University
Białystok

Summary. The article contains definitions and properties of convergent silieses.

MML Identifier: `SERIES_1`.

The articles [12], [2], [10], [1], [7], [6], [4], [3], [5], [11], [8], and [9] provide the notation and terminology for this paper. We follow the rules: n, m will denote natural numbers, a, p, r will denote real numbers, and s, s_1, s_2 will denote sequences of real numbers. We now state three propositions:

(1) If $0 < a$ and $a < 1$ and for every n holds $s(n) = a^{n+1}$, then s is convergent and $\lim s = 0$.

(2) If $a \neq 0$, then $|a|^n = |a^n|$.

(3) If $|a| < 1$ and for every n holds $s(n) = a^{n+1}$, then s is convergent and $\lim s = 0$.

Let us consider s. The functor $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$ yielding a sequence of real numbers is defined by:

(Def.1) $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(0) = s(0)$ and for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) + s(n+1)$.

The following proposition is true

(4) For all s, s_1 holds $s_1 = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$ if and only if $s_1(0) = s(0)$ and for every n holds $s_1(n+1) = s_1(n) + s(n+1)$.

Let us consider s. We say that s is summable if and only if:

(Def.2) $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$ is convergent.

Let us consider s. Let us assume that s is summable. The functor $\sum s$ yields a real number and is defined as follows:

(Def.3) $\sum s = \lim((\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}})$.

[1]Supported by RPBP.III-24.C8

ISSN 0777–4028

The following propositions are true:

(6)[2] For all s, r such that s is summable holds $r = \sum s$ if and only if $r = \lim((\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}})$.

(7) If s is summable, then s is convergent and $\lim s = 0$.

(8) $(\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}} + (\sum_{\alpha=0}^{\kappa} s_2(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_1 + s_2)(\alpha))_{\kappa\in\mathbb{N}}$.

(9) $(\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa} s_2(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_1 - s_2)(\alpha))_{\kappa\in\mathbb{N}}$.

(10) If s_1 is summable and s_2 is summable, then $s_1 + s_2$ is summable and $\sum(s_1 + s_2) = \sum s_1 + \sum s_2$.

(11) If s_1 is summable and s_2 is summable, then $s_1 - s_2$ is summable and $\sum(s_1 - s_2) = \sum s_1 - \sum s_2$.

(12) $(\sum_{\alpha=0}^{\kappa}(rs)(\alpha))_{\kappa\in\mathbb{N}} = r(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$.

(13) If s is summable, then rs is summable and $\sum(rs) = r \cdot \sum s$.

(14) For all s, s_1 such that for every n holds $s_1(n) = s(0)$ holds $(\sum_{\alpha=0}^{\kappa}(s \uparrow 1)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}} \uparrow 1 - s_1$.

(15) If s is summable, then for every n holds $s \uparrow n$ is summable.

(16) If there exists n such that $s \uparrow n$ is summable, then s is summable.

(17) If for every n holds $s_1(n) \leq s_2(n)$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}}(n) \leq (\sum_{\alpha=0}^{\kappa} s_2(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(18) If s is summable, then for every n holds $\sum s = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) + \sum(s \uparrow (n + 1))$.

(19) If for every n holds $0 \leq s(n)$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$ is non-decreasing.

(20) If for every n holds $0 \leq s(n)$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$ is upper bounded if and only if s is summable.

(21) If s is summable and for every n holds $0 \leq s(n)$, then $0 \leq \sum s$.

(22) If for every n holds $0 \leq s_2(n)$ and s_1 is summable and there exists m such that for every n such that $m \leq n$ holds $s_2(n) \leq s_1(n)$, then s_2 is summable.

(23) If for every n holds $0 \leq s_2(n)$ and s_2 is not summable and there exists m such that for every n such that $m \leq n$ holds $s_2(n) \leq s_1(n)$, then s_1 is not summable.

(24) If for every n holds $0 \leq s_1(n)$ and $s_1(n) \leq s_2(n)$ and s_2 is summable, then s_1 is summable and $\sum s_1 \leq \sum s_2$.

(25) s is summable if and only if for every r such that $0 < r$ there exists n such that for every m such that $n \leq m$ holds $|(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n)| < r$.

(26) If $a \neq 1$, then $(\sum_{\alpha=0}^{\kappa}((a^{\kappa})_{\kappa\in\mathbb{N}})(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{1-a^{n+1}}{1-a}$.

(27) If $a \neq 1$ and for every n holds $s(n+1) = a \cdot s(n)$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{s(0)\cdot(1-a^{n+1})}{1-a}$.

(28) If $|a| < 1$, then $(a^{\kappa})_{\kappa\in\mathbb{N}}$ is summable and $\sum((a^{\kappa})_{\kappa\in\mathbb{N}}) = \frac{1}{1-a}$.

[2]The proposition (5) has been removed.

(29) If $|a| < 1$ and for every n holds $s(n+1) = a \cdot s(n)$, then s is summable and $\sum s = \frac{s(0)}{1-a}$.

(30) If for every n holds $s(n) > 0$ and $s_1(n) = \frac{s(n+1)}{s(n)}$ and s_1 is convergent and $\lim s_1 < 1$, then s is summable.

(31) If for every n holds $s(n) > 0$ and there exists m such that for every n such that $n \geq m$ holds $\frac{s(n+1)}{s(n)} \geq 1$, then s is not summable.

(32) If for every n holds $s(n) \geq 0$ and $s_1(n) = \sqrt[n]{s(n)}$ and s_1 is convergent and $\lim s_1 < 1$, then s is summable.

(33) If for every n holds $s(n) \geq 0$ and $s_1(n) = \sqrt[n]{s(n)}$ and there exists m such that for every n such that $m \leq n$ holds $s_1(n) \geq 1$, then s is not summable.

(34) If for every n holds $s(n) \geq 0$ and $s_1(n) = \sqrt[n]{s(n)}$ and s_1 is convergent and $\lim s_1 > 1$, then s is not summable.

Let us consider n. The n-th power of 2 yields a natural number and is defined as follows:

(Def.4) the n-th power of 2$= 2^n$.

One can prove the following three propositions:

(35) If s is non-increasing and for every n holds $s(n) \geq 0$ and $s_1(n) = 2^n \cdot s(\text{the } n\text{-th power of } 2)$, then s is summable if and only if s_1 is summable.

(36) If $p > 1$ and for every n such that $n \geq 1$ holds $s(n) = \frac{1}{n^p}$, then s is summable.

(37) If $p \leq 1$ and for every n such that $n \geq 1$ holds $s(n) = \frac{1}{n^p}$, then s is not summable.

Let us consider s. We say that s is absolutely summable if and only if:

(Def.5) $|s|$ is summable.

We now state several propositions:

(39)[3] For all n, m such that $n \leq m$ holds $|(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n)| \leq |(\sum_{\alpha=0}^{\kappa} |s|(\alpha))_{\kappa \in \mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa} |s|(\alpha))_{\kappa \in \mathbb{N}}(n)|$.

(40) If s is absolutely summable, then s is summable.

(41) If for every n holds $0 \leq s(n)$ and s is summable, then s is absolutely summable.

(42) If for every n holds $s(n) \neq 0$ and $s_1(n) = \frac{|s|(n+1)}{|s|(n)}$ and s_1 is convergent and $\lim s_1 < 1$, then s is absolutely summable.

(43) If $r > 0$ and there exists m such that for every n such that $n \geq m$ holds $|s(n)| \geq r$, then s is not convergent or $\lim s \neq 0$.

(44) If for every n holds $s(n) \neq 0$ and there exists m such that for every n such that $n \geq m$ holds $\frac{|s|(n+1)}{|s|(n)} \geq 1$, then s is not summable.

[3] The proposition (38) has been removed.

(45) If for every n holds $s_1(n) = \sqrt[n]{|s|(n)}$ and s_1 is convergent and $\lim s_1 < 1$, then s is absolutely summable.

(46) If for every n holds $s_1(n) = \sqrt[n]{|s|(n)}$ and there exists m such that for every n such that $m \leq n$ holds $s_1(n) \geq 1$, then s is not summable.

(47) If for every n holds $s_1(n) = \sqrt[n]{|s|(n)}$ and s_1 is convergent and $\lim s_1 > 1$, then s is not summable.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[4] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.

[5] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.

[6] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[7] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[8] Konrad Raczkowski. Integer and rational exponents. *Formalized Mathematics*, 2(**1**):125–130, 1991.

[9] Konrad Raczkowski and Andrzej Nędzusiak. Real exponents and logarithms. *Formalized Mathematics*, 2(**2**):213–216, 1991.

[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[11] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.

[12] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

Received October 15, 1990

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

The Lattice of Natural Numbers and The Sublattice of it. The Set of Prime Numbers

Marek Chmur
Warsaw University
Białystok

Summary. Basic properties of the least common multiple and the greatest common divisor. The lattice of natural numbers ($L_{\mathbb{N}}$) and the lattice of natural numbers greater than zero ($L_{\mathbb{N}^+}$) are constructed. The notion of the sublattice of the lattice of natural numbers is given. Some fact about it are proved. The last part of the article deals with some properties of prime numbers and with the notions of the set of prime numbers and the n-th prime number. It is proved that the set of prime numbers is infinite.

MML Identifier: NAT_LAT.

The papers [15], [6], [18], [14], [7], [17], [9], [1], [11], [2], [16], [12], [5], [4], [8], [13], [10], and [3] provide the terminology and notation for this paper. In the sequel n, m, l, k, j will be natural numbers. We now state two propositions:

(1) For all natural numbers m, n holds $m \mid m \cdot n$ and $n \mid m \cdot n$.

(2) For all k, l such that $l \geq 1$ holds $k \cdot l \geq k$.

Let us consider n. Then $n!$ is a natural number.

The following propositions are true:

(3) For all n, k, l such that $l \geq 1$ holds if $n \geq k \cdot l$, then $n \geq k$.

(4) $k = 0$ or $k \geq 1$.

(5) For every l such that $l \neq 0$ holds $l \mid l!$.

(6) $k \neq k + 1$.

(8)[1] For every n such that $n \neq 0$ holds $\frac{n+1}{n} > 1$.

(9) $\frac{k}{k+1} < 1$.

[1]The proposition (7) has been removed.

ISSN 0777–4028

(10) For every l holds $l! \geq l$.

(12)[2] For all m, n such that $m \neq 1$ holds if $m \mid n$, then $m \nmid n + 1$.

(13) $j \mid l$ and $j \mid l + 1$ if and only if $j = 1$.

(14) For every l there exists j such that $j \mid l!$.

(15) For all k, j such that $j \neq 0$ holds $j \mid (j + k)!$.

(16) If $j \leq l$ and $j \neq 0$, then $j \mid l!$.

(17) For all l, j such that $j \neq 1$ and $j \neq 0$ holds if $j \mid l! + 1$, then $j > l$.

(18) For all natural numbers m, n holds $\mathrm{lcm}(m, n) = \mathrm{lcm}(n, m)$.

(19) For all natural numbers m, n, k holds $\mathrm{lcm}(m, \mathrm{lcm}(n, k)) = \mathrm{lcm}(\mathrm{lcm}(m, n), k)$.

(20) For all natural numbers m, n holds $m \mid n$ if and only if $\mathrm{lcm}(m, n) = n$.

(21) $m \mid \mathrm{lcm}(m, n)$ and $n \mid \mathrm{lcm}(m, n)$.

(22) $\mathrm{lcm}(m, m) = m$.

(23) $n \mid m$ and $k \mid m$ if and only if $\mathrm{lcm}(n, k) \mid m$.

(24) $\mathrm{lcm}(m, n) \mid 0$.

(25) $1 \mid \mathrm{lcm}(m, n)$.

(26) $\mathrm{lcm}(m, 1) = m$.

(27) $\mathrm{lcm}(m, n) \mid m \cdot n$.

(28) For all natural numbers m, n, k holds $\gcd(m, \gcd(n, k)) = \gcd(\gcd(m, n), k)$.

(29) $\gcd(m, n) \mid m$ and $\gcd(m, n) \mid n$.

(30) For all natural numbers m, n such that $n \mid m$ holds $\gcd(n, m) = n$.

(31) $\gcd(m, m) = m$.

(32) $m \mid n$ and $m \mid k$ if and only if $m \mid \gcd(n, k)$.

(33) $\gcd(m, n) \mid 0$.

The following propositions are true:

(34) $1 \mid \gcd(m, n)$.

(35) $\gcd(m, 1) = 1$.

(36) $\gcd(m, 0) = m$.

(37) For all natural numbers m, n holds $\mathrm{lcm}(\gcd(m, n), n) = n$.

(38) For all natural numbers m, n holds $\gcd(m, \mathrm{lcm}(m, n)) = m$.

(39) For all natural numbers m, n holds $\gcd(m, \mathrm{lcm}(m, n)) = \mathrm{lcm}(\gcd(n, m), m)$.

(40) If $m \mid n$, then $\gcd(m, k) \mid \gcd(n, k)$.

(41) If $m \mid n$, then $\gcd(k, m) \mid \gcd(k, n)$.

(42) For every m such that $m > 0$ holds $\gcd(0, m) > 0$.

(43) For all m, n such that $m > 0$ and $n > 0$ holds $\gcd(n, m) > 0$.

(44) For all m, n such that $m > 0$ and $n > 0$ holds $\mathrm{lcm}(m, n) > 0$.

[2]The proposition (11) has been removed.

(45) $\mathrm{lcm}(\gcd(n,m),\gcd(n,k)) \mid \gcd(n,\mathrm{lcm}(m,k))$.

(46) For all m, n, l such that $m \mid l$ holds $\mathrm{lcm}(m,\gcd(n,l)) \mid \gcd(\mathrm{lcm}(m,n),l)$.

(47) $\gcd(n,m) \mid \mathrm{lcm}(n,m)$.

Let m be an element of $\mathbb{N}$ **qua** a non-empty set. The functor ${}^{@}m$ yielding a natural number is defined by:

(Def.1) ${}^{@}m = m$.

Let m be a natural number. The functor ${}^{@}m$ yielding an element of $\mathbb{N}$ **qua** a non-empty set is defined as follows:

(Def.2) ${}^{@}m = m$.

We now define two new functors. The binary operation $\mathrm{hcf}_{\mathbb{N}}$ on $\mathbb{N}$ is defined by:

(Def.3) $\mathrm{hcf}_{\mathbb{N}}(m, n) = \gcd(m,n)$.

The binary operation $\mathrm{lcm}_{\mathbb{N}}$ on $\mathbb{N}$ is defined by:

(Def.4) $\mathrm{lcm}_{\mathbb{N}}(m, n) = \mathrm{lcm}(m,n)$.

In the sequel p, q denote elements of the carrier of $\langle \mathbb{N}, \mathrm{lcm}_{\mathbb{N}}, \mathrm{hcf}_{\mathbb{N}} \rangle$. Let m be an element of the carrier of $\langle \mathbb{N}, \mathrm{lcm}_{\mathbb{N}}, \mathrm{hcf}_{\mathbb{N}} \rangle$. The functor ${}^{@}m$ yielding a natural number is defined as follows:

(Def.5) ${}^{@}m = m$.

We now state several propositions:

(48) $p \sqcup q = \mathrm{lcm}({}^{@}p, {}^{@}q)$.

(49) $p \sqcap q = \gcd({}^{@}p, {}^{@}q)$.

(50) $\mathrm{lcm}_{\mathbb{N}}(p, q) = p \sqcup q$.

(51) $\mathrm{hcf}_{\mathbb{N}}(p, q) = p \sqcap q$.

(52) For all elements a, b of the carrier of $\langle \mathbb{N}, \mathrm{lcm}_{\mathbb{N}}, \mathrm{hcf}_{\mathbb{N}} \rangle$ such that $a \sqsubseteq b$ holds ${}^{@}a \mid {}^{@}b$.

The element $\mathbf{0}_{\mathbb{L}_{\mathbb{N}}}$ of the carrier of $\langle \mathbb{N}, \mathrm{lcm}_{\mathbb{N}}, \mathrm{hcf}_{\mathbb{N}} \rangle$ is defined as follows:

(Def.6) $\mathbf{0}_{\mathbb{L}_{\mathbb{N}}} = 1$.

The element $\mathbf{1}_{\mathbb{L}_{\mathbb{N}}}$ of the carrier of $\langle \mathbb{N}, \mathrm{lcm}_{\mathbb{N}}, \mathrm{hcf}_{\mathbb{N}} \rangle$ is defined by:

(Def.7) $\mathbf{1}_{\mathbb{L}_{\mathbb{N}}} = 0$.

We now state three propositions:

(55)[3] ${}^{@}(\mathbf{0}_{\mathbb{L}_{\mathbb{N}}}) = 1$.

(56) For every element a of the carrier of $\langle \mathbb{N}, \mathrm{lcm}_{\mathbb{N}}, \mathrm{hcf}_{\mathbb{N}} \rangle$ holds $\mathbf{0}_{\mathbb{L}_{\mathbb{N}}} \sqcap a = \mathbf{0}_{\mathbb{L}_{\mathbb{N}}}$.

(57) There exists an element z of the carrier of $\langle \mathbb{N}, \mathrm{lcm}_{\mathbb{N}}, \mathrm{hcf}_{\mathbb{N}} \rangle$ such that for every element x of the carrier of $\langle \mathbb{N}, \mathrm{lcm}_{\mathbb{N}}, \mathrm{hcf}_{\mathbb{N}} \rangle$ holds $z \sqcap x = z$.

The lattice $\mathbb{L}_{\mathbb{N}}$ is defined by:

(Def.8) $\mathbb{L}_{\mathbb{N}} = \langle \mathbb{N}, \mathrm{lcm}_{\mathbb{N}}, \mathrm{hcf}_{\mathbb{N}} \rangle$.

The following proposition is true

(58) $\mathbb{L}_{\mathbb{N}} = \langle \mathbb{N}, \mathrm{lcm}_{\mathbb{N}}, \mathrm{hcf}_{\mathbb{N}} \rangle$.

[3]The propositions (53) and (54) have been removed.

In the sequel p, q, r will denote elements of the carrier of $\mathbb{L}_{\mathbb{N}}$. One can prove the following propositions:

(60)[4] $\mathbb{L}_{\mathbb{N}}$ is a lower bound lattice.

(61) $\mathrm{lcm}_{\mathbb{N}}(p, q) = \mathrm{lcm}_{\mathbb{N}}(q, p)$.

(62) $\mathrm{hcf}_{\mathbb{N}}(q, p) = \mathrm{hcf}_{\mathbb{N}}(p, q)$.

(63) $\mathrm{lcm}_{\mathbb{N}}(p, \mathrm{lcm}_{\mathbb{N}}(q, r)) = \mathrm{lcm}_{\mathbb{N}}(\mathrm{lcm}_{\mathbb{N}}(p, q), r)$.

(64) (i) $\mathrm{lcm}_{\mathbb{N}}(p, \mathrm{lcm}_{\mathbb{N}}(q, r)) = \mathrm{lcm}_{\mathbb{N}}(\mathrm{lcm}_{\mathbb{N}}(q, p), r)$,
(ii) $\mathrm{lcm}_{\mathbb{N}}(p, \mathrm{lcm}_{\mathbb{N}}(q, r)) = \mathrm{lcm}_{\mathbb{N}}(\mathrm{lcm}_{\mathbb{N}}(p, r), q)$,
(iii) $\mathrm{lcm}_{\mathbb{N}}(p, \mathrm{lcm}_{\mathbb{N}}(q, r)) = \mathrm{lcm}_{\mathbb{N}}(\mathrm{lcm}_{\mathbb{N}}(r, q), p)$,
(iv) $\mathrm{lcm}_{\mathbb{N}}(p, \mathrm{lcm}_{\mathbb{N}}(q, r)) = \mathrm{lcm}_{\mathbb{N}}(\mathrm{lcm}_{\mathbb{N}}(r, p), q)$.

(65) $\mathrm{hcf}_{\mathbb{N}}(p, \mathrm{hcf}_{\mathbb{N}}(q, r)) = \mathrm{hcf}_{\mathbb{N}}(\mathrm{hcf}_{\mathbb{N}}(p, q), r)$.

(66) (i) $\mathrm{hcf}_{\mathbb{N}}(p, \mathrm{hcf}_{\mathbb{N}}(q, r)) = \mathrm{hcf}_{\mathbb{N}}(\mathrm{hcf}_{\mathbb{N}}(q, p), r)$,
(ii) $\mathrm{hcf}_{\mathbb{N}}(p, \mathrm{hcf}_{\mathbb{N}}(q, r)) = \mathrm{hcf}_{\mathbb{N}}(\mathrm{hcf}_{\mathbb{N}}(p, r), q)$,
(iii) $\mathrm{hcf}_{\mathbb{N}}(p, \mathrm{hcf}_{\mathbb{N}}(q, r)) = \mathrm{hcf}_{\mathbb{N}}(\mathrm{hcf}_{\mathbb{N}}(r, q), p)$,
(iv) $\mathrm{hcf}_{\mathbb{N}}(p, \mathrm{hcf}_{\mathbb{N}}(q, r)) = \mathrm{hcf}_{\mathbb{N}}(\mathrm{hcf}_{\mathbb{N}}(r, p), q)$.

(67) $\mathrm{hcf}_{\mathbb{N}}(q, \mathrm{lcm}_{\mathbb{N}}(q, p)) = q$ and $\mathrm{hcf}_{\mathbb{N}}(\mathrm{lcm}_{\mathbb{N}}(p, q), q) = q$ and $\mathrm{hcf}_{\mathbb{N}}(q, \mathrm{lcm}_{\mathbb{N}}(p, q)) = q$ and $\mathrm{hcf}_{\mathbb{N}}(\mathrm{lcm}_{\mathbb{N}}(q, p), q) = q$.

(68) $\mathrm{lcm}_{\mathbb{N}}(q, \mathrm{hcf}_{\mathbb{N}}(q, p)) = q$ and $\mathrm{lcm}_{\mathbb{N}}(\mathrm{hcf}_{\mathbb{N}}(p, q), q) = q$ and $\mathrm{lcm}_{\mathbb{N}}(q, \mathrm{hcf}_{\mathbb{N}}(p, q)) = q$ and $\mathrm{lcm}_{\mathbb{N}}(\mathrm{hcf}_{\mathbb{N}}(q, p), q) = q$.

The subset $\mathbb{N}^+$ of $\mathbb{N}$ is defined by:

(Def.9) for every natural number n holds $n \in \mathbb{N}^+$ if and only if $0 < n$.

Let D be a non-empty set, and let S be a non-empty subset of D, and let N be a non-empty subset of S. We see that the element of N is an element of S.

A positive natural number is an element of $\mathbb{N}^+$.

Let k be a natural number satisfying the condition: $k > 0$. The functor ${}^{@}k$ yields an element of $\mathbb{N}^+$ **qua** a non-empty set and is defined by:

(Def.10) ${}^{@}k = k$.

Let k be an element of $\mathbb{N}^+$ **qua** a non-empty set. The functor ${}^{@}k$ yields a positive natural number and is defined as follows:

(Def.11) ${}^{@}k = k$.

In the sequel m, n denote positive natural numbers. We now define two new functors. The binary operation $\mathrm{hcf}_{\mathbb{N}^+}$ on $\mathbb{N}^+$ is defined by:

(Def.12) $\mathrm{hcf}_{\mathbb{N}^+}(m, n) = \gcd(m, n)$.

The binary operation $\mathrm{lcm}_{\mathbb{N}^+}$ on $\mathbb{N}^+$ is defined as follows:

(Def.13) $\mathrm{lcm}_{\mathbb{N}^+}(m, n) = \mathrm{lcm}(m, n)$.

In the sequel p, q will denote elements of the carrier of $\langle \mathbb{N}^+, \mathrm{lcm}_{\mathbb{N}^+}, \mathrm{hcf}_{\mathbb{N}^+} \rangle$. Let m be an element of the carrier of $\langle \mathbb{N}^+, \mathrm{lcm}_{\mathbb{N}^+}, \mathrm{hcf}_{\mathbb{N}^+} \rangle$. The functor ${}^{@}m$ yields a positive natural number and is defined as follows:

(Def.14) ${}^{@}m = m$.

[4]The proposition (59) has been removed.

One can prove the following four propositions:

(69) $p \sqcup q = \mathrm{lcm}({}^{@}p, {}^{@}q)$.

(70) $p \sqcap q = \gcd({}^{@}p, {}^{@}q)$.

(71) $\mathrm{lcm}_{\mathbb{N}^+}(p, q) = p \sqcup q$.

(72) $\mathrm{hcf}_{\mathbb{N}^+}(p, q) = p \sqcap q$.

The lattice $\mathbb{L}_{\mathbb{N}^+}$ is defined by:

(Def.15) $\mathbb{L}_{\mathbb{N}^+} = \langle \mathbb{N}^+, \mathrm{lcm}_{\mathbb{N}^+}, \mathrm{hcf}_{\mathbb{N}^+} \rangle$.

Next we state the proposition

(73) $\mathbb{L}_{\mathbb{N}^+} = \langle \mathbb{N}^+, \mathrm{lcm}_{\mathbb{N}^+}, \mathrm{hcf}_{\mathbb{N}^+} \rangle$.

Let L be a lattice. A lattice is said to be a sublattice of L if:

(Def.16) the carrier of it $\subseteq$ the carrier of L and the join operation of it $=$ (the join operation of L) $\restriction$ [: the carrier of it, the carrier of it :] and the meet operation of it $=$ (the meet operation of L) $\restriction$ [: the carrier of it, the carrier of it :].

The following two propositions are true:

(75)[5] For every lattice L holds L is a sublattice of L.

(76) $\mathbb{L}_{\mathbb{N}^+}$ is a sublattice of $\mathbb{L}_{\mathbb{N}}$.

In the sequel n, i, k, k_1, k_2, m, l will denote natural numbers. The set Prime of natural numbers is defined as follows:

(Def.17) for every natural number n holds $n \in$ Prime if and only if n is prime.

A natural number is said to be a prime number if:

(Def.18) it $\in$ Prime.

In the sequel p, q denote prime numbers and f denotes a prime number. Let us consider p. The functor Prime(p) yields sets of natural numbers and is defined by:

(Def.19) for every natural number q holds $q \in$ Prime(p) if and only if $q < p$ and q is prime.

Next we state a number of propositions:

(77) Prime(p) $\subseteq$ Prime.

(78) For every prime number q such that $p < q$ holds Prime(p) $\subseteq$ Prime(q).

(79) Prime(p) $\subseteq \mathrm{Seg}\, p$.

(80) Prime(p) is finite.

(81) For every l there exists p such that p is prime and $p > l$.

(82) For every q such that q is prime there exists p such that p is prime and $p > q$.

(83) Prime $\subseteq \mathbb{N}$.

(84) Prime $\neq \emptyset$.

(85) $\{k : k < 2 \wedge k \text{ is prime}\} = \emptyset$.

[5]The proposition (74) has been removed.

(86) For every p holds $\{k : k < p \wedge k \text{ is prime}\} \subseteq \mathbb{N}$.

(87) For every m holds $\{k : k < m \wedge k \text{ is prime}\} \subseteq \operatorname{Seg} m$.

(88) For every m holds $\{k : k < m \wedge k \text{ is prime}\}$ is finite.

(89) For every prime number f holds $f \notin \{k : k < f \wedge k \text{ is prime}\}$.

(90) For every f holds $\{k : k < f \wedge k \text{ is prime}\} \cup \{f\}$ is finite.

(91) For all prime numbers f, g such that $f < g$ holds $\{k_1 : k_1 < f \wedge k_1 \text{ is prime}\} \cup \{f\} \subseteq \{k_2 : k_2 < g \wedge k_2 \text{ is prime}\}$.

(92) For every k such that $k > m$ holds $k \notin \{k_1 : k_1 < m \wedge k_1 \text{ is prime}\}$.

Let us consider n. The functor $\mathrm{pr}(n)$ yielding a prime number is defined as follows:

(Def.20) $n = \operatorname{card}\{k : k < \mathrm{pr}(n) \wedge k \text{ is prime}\}$.

One can prove the following two propositions:

(93) $\mathrm{Prime}(p) = \{k : k < p \wedge k \text{ is prime}\}$.

(94) Prime is not finite.

The following proposition is true

(95) For every i such that i is prime for all m, n such that $i \mid m \cdot n$ holds $i \mid m$ or $i \mid n$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(**2**):281–290, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[7] Marek Chmur. The lattice of real numbers. The lattice of real functions. *Formalized Mathematics*, 1(**4**):681–684, 1990.
[8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[10] Rafał Kwiatek. Factorial and Newton coeffitients. *Formalized Mathematics*, 1(**5**):887–890, 1990.
[11] Rafał Kwiatek and Grzegorz Zwara. The divisibility of integers and integer relative primes. *Formalized Mathematics*, 1(**5**):829–832, 1990.
[12] Michał Muzalewski and Lesław W. Szczerba. Construction of finite sequences over ring and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):97–104, 1991.
[13] Henryk Oryszczyszyn and Krzysztof Prażmowski. Real functions spaces. *Formalized Mathematics*, 1(**3**):555–561, 1990.
[14] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.
[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[16] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[17] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[18] Stanisław Żukowski. Introduction to lattice theory. *Formalized Mathematics*, 1(**1**):215–222, 1990.

Received April 26, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Commutator and Center of a Group

Wojciech A. Trybulec
Warsaw University

Summary. We introduce the notions of commutators of element, subgroups of a group, commutator of a group and center of a group. We prove P.Hall identity. The article is based on [6].

MML Identifier: `GROUP_5`.

The terminology and notation used in this paper are introduced in the following articles: [9], [4], [1], [3], [5], [10], [7], [14], [16], [2], [12], [8], [15], [11], and [13].

Preliminaries

The scheme *SubsetFD3* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a non-empty set $\mathcal{C}$, a ternary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a ternary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(c,d,e) : \mathcal{P}[c,d,e]\}$, where c ranges over elements of $\mathcal{A}$, and d ranges over elements of $\mathcal{B}$, and e ranges over elements of $\mathcal{C}$, is a subset of $\mathcal{B}$
for all values of the parameters.

For simplicity we adopt the following rules: x will be arbitrary, k, n will denote natural numbers, i will denote an integer, G will denote a group, a, b, c, d will denote elements of G, A, B, C, D will denote subsets of G, H, H_1, H_2, H_3, H_4 will denote subgroups of G, N, N_1, N_2, N_3 will denote normal subgroups of G, F, F_1, F_2 will denote finite sequences of elements of the carrier of G, and I will denote a finite sequence of elements of $\mathbb{Z}$. Next we state several propositions:

(1) $x \in \{\mathbf{1}\}_G$ if and only if $x = 1_G$.

(2) If $a \in H$ and $b \in H$, then $a^b \in H$.

(3) If $a \in N$, then $a^b \in N$.

(4) $x \in H_1 \cdot H_2$ if and only if there exist a, b such that $x = a \cdot b$ and $a \in H_1$ and $b \in H_2$.

(5) If $H_1 \cdot H_2 = H_2 \cdot H_1$, then $x \in H_1 \sqcup H_2$ if and only if there exist a, b such that $x = a \cdot b$ and $a \in H_1$ and $b \in H_2$.

ISSN 0777–4028

(6) If G is an Abelian group, then $x \in H_1 \sqcup H_2$ if and only if there exist a, b such that $x = a \cdot b$ and $a \in H_1$ and $b \in H_2$.

(7) $x \in N_1 \sqcup N_2$ if and only if there exist a, b such that $x = a \cdot b$ and $a \in N_1$ and $b \in N_2$.

(8) $H \cdot N = N \cdot H$.

Let us consider G, F, a. The functor F^a yielding a finite sequence of elements of the carrier of G is defined by:

(Def.1) $\mathrm{len}(F^a) = \mathrm{len}\, F$ and for every k such that $k \in \mathrm{Seg}\,\mathrm{len}\, F$ holds $F^a(k) = (\pi_k F)^a$.

One can prove the following propositions:

(9) If $\mathrm{len}\, F_1 = \mathrm{len}\, F_2$ and for every k such that $k \in \mathrm{Seg}\,\mathrm{len}\, F_2$ holds $F_1(k) = (\pi_k F_2)^a$, then $F_1 = {F_2}^a$.

(10) $\mathrm{len}(F^a) = \mathrm{len}\, F$.

(11) For every k such that $k \in \mathrm{Seg}\,\mathrm{len}\, F$ holds $F^a(k) = (\pi_k F)^a$.

(12) $({F_1}^a) \frown {F_2}^a = (F_1 \frown F_2)^a$.

(13) $\varepsilon^a_{(\text{the carrier of } G)} = \varepsilon$.

(14) $\langle a \rangle^b = \langle a^b \rangle$.

(15) $\langle a, b \rangle^c = \langle a^c, b^c \rangle$.

(16) $\langle a, b, c \rangle^d = \langle a^d, b^d, c^d \rangle$.

(17) $\prod(F^a) = (\prod F)^a$.

(18) If $\mathrm{len}\, F = \mathrm{len}\, I$, then $(F^a)^I = (F^I)^a$.

Commutators

Let us consider G, a, b. The functor $[a, b]$ yields an element of G and is defined by:

(Def.2) $[a, b] = a^{-1} \cdot b^{-1} \cdot a \cdot b$.

One can prove the following propositions:

(19) (i) $[a, b] = a^{-1} \cdot b^{-1} \cdot a \cdot b$,
(ii) $[a, b] = a^{-1} \cdot (b^{-1} \cdot a) \cdot b$,
(iii) $[a, b] = a^{-1} \cdot (b^{-1} \cdot a \cdot b)$,
(iv) $[a, b] = a^{-1} \cdot (b^{-1} \cdot (a \cdot b))$,
(v) $[a, b] = a^{-1} \cdot b^{-1} \cdot (a \cdot b)$.

(20) $[a, b] = (b \cdot a)^{-1} \cdot (a \cdot b)$.

(21) $[a, b] = (b^{-1})^a \cdot b$ and $[a, b] = a^{-1} \cdot a^b$.

(22) $[1_G, a] = 1_G$ and $[a, 1_G] = 1_G$.

(23) $[a, a] = 1_G$.

(24) $[a, a^{-1}] = 1_G$ and $[a^{-1}, a] = 1_G$.

(25) $[a, b]^{-1} = [b, a]$.

(26) $[a, b]^c = [a^c, b^c]$.

(27) $[a,b] = (a^{-1})^2 \cdot (a \cdot b^{-1})^2 \cdot b^2$.

(28) $[a \cdot b, c] = [a,c]^b \cdot [b,c]$.

(29) $[a, b \cdot c] = [a,c] \cdot [a,b]^c$.

(30) $[a^{-1}, b] = [b,a]^{a^{-1}}$.

(31) $[a, b^{-1}] = [b,a]^{b^{-1}}$.

(32) $[a^{-1}, b^{-1}] = [a,b]^{(a \cdot b)^{-1}}$ and $[a^{-1}, b^{-1}] = [a,b]^{(b \cdot a)^{-1}}$.

(33) $[a, b^{a^{-1}}] = [b, a^{-1}]$.

(34) $[a^{b^{-1}}, b] = [b^{-1}, a]$.

(35) $[a^n, b] = a^{-n} \cdot (a^b)^n$.

(36) $[a, b^n] = (b^a)^{-n} \cdot b^n$.

(37) $[a^i, b] = a^{-i} \cdot (a^b)^i$.

(38) $[a, b^i] = (b^a)^{-i} \cdot b^i$.

(39) $[a,b] = 1_G$ if and only if $a \cdot b = b \cdot a$.

(40) G is an Abelian group if and only if for all a, b holds $[a,b] = 1_G$.

(41) If $a \in H$ and $b \in H$, then $[a,b] \in H$.

Let us consider G, a, b, c. The functor $[a,\ b,\ c]$ yielding an element of G is defined by:

(Def.3) $[a,\ b,\ c] = [[a,b],c]$.

One can prove the following propositions:

(42) $[a,\ b,\ c] = [[a,b],c]$.

(43) $[a,\ b,\ 1_G] = 1_G$ and $[a,\ 1_G,\ b] = 1_G$ and $[1_G,\ a,\ b] = 1_G$.

(44) $[a,\ a,\ b] = 1_G$.

(45) $[a,\ b,\ a] = [a^b, a]$.

(46) $[b,\ a,\ a] = ([b, a^{-1}] \cdot [b,a])^a$.

(47) $[a,\ b,\ b^a] = [b,[b,a]]$.

(48) $[a \cdot b, c] = [a,c] \cdot [a,\ c,\ b] \cdot [b,c]$.

(49) $[a, b \cdot c] = [a,c] \cdot [a,b] \cdot [a,\ b,\ c]$.

(50) $[a,\ b^{-1},\ c]^b \cdot [b,\ c^{-1},\ a]^c \cdot [c,\ a^{-1},\ b]^a = 1_G$.

Let us consider G, A, B. The commutators of A & B yielding a subset of G is defined as follows:

(Def.4) the commutators of A & $B = \{[a,b] : a \in A \wedge b \in B\}$.

We now state several propositions:

(51) The commutators of A & $B = \{[a,b] : a \in A \wedge b \in B\}$.

(52) $x \in$ the commutators of A & B if and only if there exist a, b such that $x = [a,b]$ and $a \in A$ and $b \in B$.

(53) The commutators of $\emptyset_{\text{the carrier of } G}$ & $A = \emptyset$ and the commutators of A & $\emptyset_{\text{the carrier of } G} = \emptyset$.

(54) The commutators of $\{a\}$ & $\{b\} = \{[a,b]\}$.

(55) If $A \subseteq B$ and $C \subseteq D$, then the commutators of A & $C \subseteq$ the commutators of B & D.

(56) G is an Abelian group if and only if for all A, B such that $A \neq \emptyset$ and $B \neq \emptyset$ holds the commutators of A & $B = \{1_G\}$.

Let us consider G, H_1, H_2. The commutators of H_1 & H_2 yields a subset of G and is defined by:

(Def.5) the commutators of H_1 & $H_2 =$ the commutators of $\overline{H_1}$ & $\overline{H_2}$.

Next we state several propositions:

(57) The commutators of H_1 & $H_2 =$ the commutators of $\overline{H_1}$ & $\overline{H_2}$.

(58) $x \in$ the commutators of H_1 & H_2 if and only if there exist a, b such that $x = [a, b]$ and $a \in H_1$ and $b \in H_2$.

(59) $1_G \in$ the commutators of H_1 & H_2.

(60) The commutators of $\{\mathbf{1}\}_G$ & $H = \{1_G\}$ and the commutators of H & $\{\mathbf{1}\}_G = \{1_G\}$.

(61) The commutators of H & $N \subseteq \overline{N}$ and the commutators of N & $H \subseteq \overline{N}$.

(62) If H_1 is a subgroup of H_2 and H_3 is a subgroup of H_4, then the commutators of H_1 & $H_3 \subseteq$ the commutators of H_2 & H_4.

(63) G is an Abelian group if and only if for all H_1, H_2 holds the commutators of H_1 & $H_2 = \{1_G\}$.

Let us consider G. The commutators of G yielding a subset of G is defined by:

(Def.6) the commutators of $G =$ the commutators of Ω_G & Ω_G.

Next we state three propositions:

(64) The commutators of $G =$ the commutators of Ω_G & Ω_G.

(65) $x \in$ the commutators of G if and only if there exist a, b such that $x = [a, b]$.

(66) G is an Abelian group if and only if the commutators of $G = \{1_G\}$.

Let us consider G, A, B. The functor $[A, B]$ yielding a subgroup of G is defined as follows:

(Def.7) $[A, B] = \mathrm{gr}(\text{the commutators of } A \,\&\, B)$.

Next we state four propositions:

(67) $[A, B] = \mathrm{gr}(\text{the commutators of } A \,\&\, B)$.

(68) If $a \in A$ and $b \in B$, then $[a, b] \in [A, B]$.

(69) $x \in [A, B]$ if and only if there exist F, I such that $\operatorname{len} F = \operatorname{len} I$ and $\operatorname{rng} F \subseteq$ the commutators of A & B and $x = \prod(F^I)$.

(70) If $A \subseteq C$ and $B \subseteq D$, then $[A, B]$ is a subgroup of $[C, D]$.

Let us consider G, H_1, H_2. The functor $[H_1, H_2]$ yielding a subgroup of G is defined by:

(Def.8) $[H_1, H_2] = [\overline{H_1}, \overline{H_2}]$.

Next we state a number of propositions:

(71) $[H_1, H_2] = [\overline{H_1}, \overline{H_2}]$.

(72) $[H_1, H_2] = \mathrm{gr}(\text{the commutators of } H_1 \ \& \ H_2)$.

(73) $x \in [H_1, H_2]$ if and only if there exist F, I such that $\mathrm{len}\, F = \mathrm{len}\, I$ and $\mathrm{rng}\, F \subseteq$ the commutators of H_1 & H_2 and $x = \prod(F^I)$.

(74) If $a \in H_1$ and $b \in H_2$, then $[a, b] \in [H_1, H_2]$.

(75) If H_1 is a subgroup of H_2 and H_3 is a subgroup of H_4, then $[H_1, H_3]$ is a subgroup of $[H_2, H_4]$.

(76) $[N, H]$ is a subgroup of N and $[H, N]$ is a subgroup of N.

(77) $[N_1, N_2]$ is a normal subgroup of G.

(78) $[N_1, N_2] = [N_2, N_1]$.

(79) $[N_1 \sqcup N_2, N_3] = [N_1, N_3] \sqcup [N_2, N_3]$.

(80) $[N_1, N_2 \sqcup N_3] = [N_1, N_2] \sqcup [N_1, N_3]$.

Let us consider G. The functor G^{c} yields a normal subgroup of G and is defined by:

(Def.9) $G^{\mathrm{c}} = [\Omega_G, \Omega_G]$.

Next we state several propositions:

(81) $G^{\mathrm{c}} = [\Omega_G, \Omega_G]$.

(82) $G^{\mathrm{c}} = \mathrm{gr}(\text{the commutators of } G)$.

(83) $x \in G^{\mathrm{c}}$ if and only if there exist F, I such that $\mathrm{len}\, F = \mathrm{len}\, I$ and $\mathrm{rng}\, F \subseteq$ the commutators of G and $x = \prod(F^I)$.

(84) $[a, b] \in G^{\mathrm{c}}$.

(85) G is an Abelian group if and only if $G^{\mathrm{c}} = \{\mathbf{1}\}_G$.

(86) If the left cosets of H is finite and $|\bullet : H|_{\mathbb{N}} = 2$, then G^{c} is a subgroup of H.

Center of a Group

Let us consider G. The functor $\mathrm{Z}(G)$ yielding a subgroup of G is defined as follows:

(Def.10) the carrier of $\mathrm{Z}(G) = \{a : \bigwedge_b a \cdot b = b \cdot a\}$.

We now state several propositions:

(87) If the carrier of $H = \{a : \bigwedge_b a \cdot b = b \cdot a\}$, then $H = \mathrm{Z}(G)$.

(88) The carrier of $\mathrm{Z}(G) = \{a : \bigwedge_b a \cdot b = b \cdot a\}$.

(89) $a \in \mathrm{Z}(G)$ if and only if for every b holds $a \cdot b = b \cdot a$.

(90) $\mathrm{Z}(G)$ is a normal subgroup of G.

(91) If H is a subgroup of $\mathrm{Z}(G)$, then H is a normal subgroup of G.

(92) $\mathrm{Z}(G)$ is an Abelian group.

(93) $a \in \mathrm{Z}(G)$ if and only if $a^\bullet = \{a\}$.

(94) G is an Abelian group if and only if $\mathrm{Z}(G) = G$.

Auxiliary theorems

In the sequel E will be a non-empty set and p, q will be finite sequences of elements of E. The following propositions are true:

(95) If $k \in \operatorname{dom} p$ or $k \in \operatorname{Seg} \operatorname{len} p$, then $\pi_k(p \frown q) = \pi_k p$.

(96) If $k \in \operatorname{dom} q$ or $k \in \operatorname{Seg} \operatorname{len} q$, then $\pi_{\operatorname{len} p + k}(p \frown q) = \pi_k q$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[6] M. I. Kargapołow and J. I. Mierzlakow. *Podstawy teorii grup.* PWN, Warszawa, 1989.
[7] Rafał Kwiatek and Grzegorz Zwara. The divisibility of integers and integer relative primes. *Formalized Mathematics*, 1(**5**):829–832, 1990.
[8] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[10] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.
[11] Wojciech A. Trybulec. Classes of conjugation. Normal subgroups. *Formalized Mathematics*, 1(**5**):955–962, 1990.
[12] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[13] Wojciech A. Trybulec. Lattice of subgroups of a group. Frattini subgroup. *Formalized Mathematics*, 2(**1**):41–47, 1991.
[14] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[15] Wojciech A. Trybulec. Subgroup and cosets of subgroups. *Formalized Mathematics*, 1(**5**):855–864, 1990.
[16] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received May 15, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Natural Transformations. Discrete Categories

Andrzej Trybulec
Warsaw University
Białystok

Summary. We present well known concepts of category theory: natural transofmations and functor categories, and prove propositions related to. Because of the formalization it proved to be convenient to introduce some auxiliary notions, for instance: transformations. We mean by a transformation of a functor F to a functor G, both covariant functors from A to B, a function mapping the objects of A to the morphisms of B and assigning to an object a of A an element of $\mathrm{Hom}(F(a), G(a))$. The material included roughly corresponds to that presented on pages 18,129–130,137–138 of the monography ([10]). We also introduce discrete categories and prove some propositions to illustrate the concepts introduced.

MML Identifier: NATTRA_1.

The articles [12], [13], [9], [3], [7], [4], [2], [6], [1], [11], [5], and [8] provide the terminology and notation for this paper.

Preliminaries

For simplicity we follow a convention: A_1, A_2, B_1, B_2 are non-empty sets, f is a function from A_1 into B_1, g is a function from A_2 into B_2, Y_1 is a non-empty subset of A_1, and Y_2 is a non-empty subset of A_2. Let A_1, A_2 be non-empty sets, and let Y_1 be a non-empty subset of A_1, and let Y_2 be a non-empty subset of A_2. Then $[\!: Y_1, Y_2 :\!]$ is a non-empty subset of $[\!: A_1, A_2 :\!]$.

Let us consider A_1, B_1, f, Y_1. Then $f \restriction Y_1$ is a function from Y_1 into B_1.

We now state the proposition

(1) $[\!: f, g :\!] \restriction [\!: Y_1, Y_2 :\!] = [\!: f \restriction Y_1, g \restriction Y_2 :\!]$.

Let A, B be non-empty sets, and let A_1 be a non-empty subset of A, and let B_1 be a non-empty subset of B, and let f be a partial function from $[\!: A_1, A_1 :\!]$

ISSN 0777–4028

to A_1, and let g be a partial function from $[: B_1, B_1 :]$ to B_1. Then $|{:}f, g{:}|$ is a partial function from $[: [: A_1, B_1 :], [: A_1, B_1 :] :]$ to $[: A_1, B_1 :]$.

One can prove the following proposition

(2) Let f be a partial function from $[: A_1, A_1 :]$ to A_1. Let g be a partial function from $[: A_2, A_2 :]$ to A_2. Then for every partial function F from $[: Y_1, Y_1 :]$ to Y_1 such that $F = f \restriction [: Y_1, Y_1 :]$ for every partial function G from $[: Y_2, Y_2 :]$ to Y_2 such that $G = g \restriction [: Y_2, Y_2 :]$ holds $|{:}F, G{:}| = |{:}f, g{:}| \restriction [: [: Y_1, Y_2 :], [: Y_1, Y_2 :] :]$.

We adopt the following convention: A, B, C will be categories, F, F_1, F_2, F_3 will be functors from A to B, and G will be a functor from B to C. In this article we present several logical schemes. The scheme *M_Choice* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a set and states that:

there exists a function t from $\mathcal{A}$ into $\mathcal{B}$ such that for every element a of $\mathcal{A}$ holds $t(a) \in \mathcal{F}(a)$

provided the following requirement is met:

- for every element a of $\mathcal{A}$ holds $\mathcal{B}$ meets $\mathcal{F}(a)$.

The scheme *LambdaT* concerns a set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ and states that:

there exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every element x of $\mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

provided the following requirement is met:

- for every element x of $\mathcal{A}$ holds $\mathcal{F}(x) \in \mathcal{B}$.

We now state the proposition

(3) For every object a of A and for every morphism m from a to a holds $m \in \hom(a, a)$.

In the sequel m, o will be arbitrary. One can prove the following propositions:

(4) For all morphisms f, g of $\dot{\circlearrowright}(o, m)$ holds $f = g$.

(5) For every object a of A holds $\langle\langle \mathrm{id}_a, \mathrm{id}_a \rangle, \mathrm{id}_a \rangle \in$ the composition of A.

(6) The composition of $\dot{\circlearrowright}(o, m) = \{\langle\langle m, m\rangle, m\rangle\}$.

(7) For every object a of A holds $\dot{\circlearrowright}(a, \mathrm{id}_a)$ is a subcategory of A.

(8) For every subcategory C of A holds the dom-map of $C =$ (the dom-map of A)$\restriction$ the morphisms of C and the cod-map of $C =$ (the cod-map of A)$\restriction$ the morphisms of C and the composition of $C =$ (the composition of A)$\restriction [:$ the morphisms of C, the morphisms of $C :]$ and the id-map of $C =$ (the id-map of A)$\restriction$ the objects of C.

(9) Let O be a non-empty subset of the objects of A. Let M be a non-empty subset of the morphisms of A. Let D_1, C_1 be functions from M into O. Suppose $D_1 =$ (the dom-map of A)$\restriction M$ and $C_1 =$ (the cod-map of A)$\restriction M$. Then for every partial function C_2 from $[: M, M$ **qua** a non-empty set $:]$ to M such that $C_2 =$ (the composition of A) $\restriction [: M, M :]$ for every function I_1 from O into M such that $I_1 =$ (the id-map of A) $\restriction O$ holds $\langle O, M, D_1, C_1, C_2, I_1\rangle$ is a subcategory of A.

(10) For every subcategory A of C such that the objects of $A =$ the objects of C and the morphisms of $A =$ the morphisms of C holds $A = C$.

Application of a functor to a morphism

Let us consider A, B, F, and let a, b be objects of A satisfying the condition: $\hom(a, b) \neq \emptyset$. Let f be a morphism from a to b. The functor $F(f)$ yields a morphism from $F(a)$ to $F(b)$ and is defined by:

(Def.1) $F(f) = F(f)$.

One can prove the following propositions:

(11) For all objects a, b of A such that $\hom(a, b) \neq \emptyset$ for every morphism f from a to b holds $(G \cdot F)(f) = G(F(f))$.

(12) For all functors F_1, F_2 from A to B such that for all objects a, b of A such that $\hom(a, b) \neq \emptyset$ for every morphism f from a to b holds $F_1(f) = F_2(f)$ holds $F_1 = F_2$.

(13) For all objects a, b, c of A such that $\hom(a, b) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$ for every morphism f from a to b and for every morphism g from b to c holds $F(g \cdot f) = F(g) \cdot F(f)$.

(14) For every object c of A and for every object d of B such that $F(\mathrm{id}_c) = \mathrm{id}_d$ holds $F(c) = d$.

(15) For every object a of A holds $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$.

(16) For all objects a, b of A such that $\hom(a, b) \neq \emptyset$ for every morphism f from a to b holds $\mathrm{id}_A(f) = f$.

(17) For all objects a, b, c, d of A such that $\hom(a, b)$ meets $\hom(c, d)$ holds $a = c$ and $b = d$.

Transformations

Let us consider A, B, F_1, F_2. We say that F_1 is transformable to F_2 if and only if:

(Def.2) for every object a of A holds $\hom(F_1(a), F_2(a)) \neq \emptyset$.

One can prove the following propositions:

(18) F is transformable to F.

(19) If F is transformable to F_1 and F_1 is transformable to F_2, then F is transformable to F_2.

Let us consider A, B, F_1, F_2. Let us assume that F_1 is transformable to F_2. A function from the objects of A into the morphisms of B is said to be a transformation from F_1 to F_2 if:

(Def.3) for every object a of A holds $\mathrm{it}(a)$ is a morphism from $F_1(a)$ to $F_2(a)$.

Let us consider A, B, and let F be a functor from A to B. The functor id_F yields a transformation from F to F and is defined as follows:

(Def.4) for every object a of A holds $\mathrm{id}_F(a) = \mathrm{id}_{F(a)}$.

Let us consider A, B, F_1, F_2. Let us assume that F_1 is transformable to F_2. Let t be a transformation from F_1 to F_2, and let a be an object of A. The functor $t(a)$ yields a morphism from $F_1(a)$ to $F_2(a)$ and is defined by:

(Def.5) $t(a) = t(a)$.

Let us consider A, B, F, F_1, F_2. Let us assume that F is transformable to F_1 and F_1 is transformable to F_2. Let t_1 be a transformation from F to F_1, and let t_2 be a transformation from F_1 to F_2. The functor $t_2{\circ}t_1$ yields a transformation from F to F_2 and is defined by:

(Def.6) for every object a of A holds $(t_2{\circ}t_1)(a) = t_2(a) \cdot t_1(a)$.

The following propositions are true:

(20) If F_1 is transformable to F_2, then for all transformations t_1, t_2 from F_1 to F_2 such that for every object a of A holds $t_1(a) = t_2(a)$ holds $t_1 = t_2$.

(21) For every object a of A holds $\mathrm{id}_F(a) = \mathrm{id}_{F(a)}$.

(22) If F_1 is transformable to F_2, then for every transformation t from F_1 to F_2 holds $\mathrm{id}_{F_2}{\circ}t = t$ and $t{\circ}\,\mathrm{id}_{F_1} = t$.

(23) If F is transformable to F_1 and F_1 is transformable to F_2 and F_2 is transformable to F_3, then for every transformation t_1 from F to F_1 and for every transformation t_2 from F_1 to F_2 and for every transformation t_3 from F_2 to F_3 holds $t_3{\circ}t_2{\circ}t_1 = t_3{\circ}(t_2{\circ}t_1)$.

Natural transformations

Let us consider A, B, F_1, F_2. We say that F_1 is naturally transformable to F_2 if and only if:

(Def.7) F_1 is transformable to F_2 and there exists a transformation t from F_1 to F_2 such that for all objects a, b of A such that $\hom(a,b) \neq \emptyset$ for every morphism f from a to b holds $t(b) \cdot F_1(f) = F_2(f) \cdot t(a)$.

Next we state two propositions:

(24) F is naturally transformable to F.

(25) If F is naturally transformable to F_1 and F_1 is naturally transformable to F_2, then F is naturally transformable to F_2.

Let us consider A, B, F_1, F_2. Let us assume that F_1 is naturally transformable to F_2. A transformation from F_1 to F_2 is called a natural transformation from F_1 to F_2 if:

(Def.8) for all objects a, b of A such that $\hom(a,b) \neq \emptyset$ for every morphism f from a to b holds $\mathrm{it}(b) \cdot F_1(f) = F_2(f) \cdot \mathrm{it}(a)$.

Let us consider A, B, F. Then id_F is a natural transformation from F to F.

Let us consider A, B, F, F_1, F_2. satisfying the conditions: F is naturally transformable to F_1 and F_1 is naturally transformable to F_2. Let t_1 be a natural transformation from F to F_1, and let t_2 be a natural transformation from F_1 to F_2. The functor $t_2{\circ}t_1$ yields a natural transformation from F to F_2 and is defined by:

(Def.9) $t_2{}^\circ t_1 = t_2{}^\circ t_1$.

One can prove the following proposition

(26) If F_1 is naturally transformable to F_2, then for every natural transformation t from F_1 to F_2 holds $\mathrm{id}_{F_2} \circ t = t$ and $t^\circ \mathrm{id}_{F_1} = t$.

In the sequel t denotes a natural transformation from F to F_1 and t_1 denotes a natural transformation from F_1 to F_2. Next we state two propositions:

(27) If F is naturally transformable to F_1 and F_1 is naturally transformable to F_2, then for every natural transformation t_1 from F to F_1 and for every natural transformation t_2 from F_1 to F_2 and for every object a of A holds $(t_2{}^\circ t_1)(a) = t_2(a) \cdot t_1(a)$.

(28) If F is naturally transformable to F_1 and F_1 is naturally transformable to F_2 and F_2 is naturally transformable to F_3, then for every natural transformation t_3 from F_2 to F_3 holds $t_3{}^\circ t_1{}^\circ t = t_3{}^\circ (t_1{}^\circ t)$.

Let us consider A, B, F_1, F_2. A transformation from F_1 to F_2 is invertible if:

(Def.10) for every object a of A holds $\mathrm{it}(a)$ is invertible.

We now define two new predicates. Let us consider A, B, F_1, F_2. We say that F_1, F_2 are naturally equivalent if and only if:

(Def.11) F_1 is naturally transformable to F_2 and there exists a natural transformation t from F_1 to F_2 such that t is invertible.

We write $F_1 \cong F_2$ if and only if F_1, F_2 are naturally equivalent.

One can prove the following proposition

(29) $F \cong F$.

Let us consider A, B, F_1, F_2. satisfying the condition: F_1 is transformable to F_2. Let t_1 be a transformation from F_1 to F_2 satisfying the condition: t_1 is invertible. The functor ${t_1}^{-1}$ yielding a transformation from F_2 to F_1 is defined as follows:

(Def.12) for every object a of A holds ${t_1}^{-1}(a) = t_1(a)^{-1}$.

Let us consider A, B, F_1, F_2, t_1. satisfying the conditions: F_1 is naturally transformable to F_2 and t_1 is invertible. The functor ${t_1}^{-1}$ yielding a natural transformation from F_2 to F_1 is defined by:

(Def.13) ${t_1}^{-1} = (t_1$ **qua** a transformation from F_1 to $F_2)^{-1}$.

Next we state three propositions:

(30) For all A, B, F_1, F_2, t_1 such that F_1 is naturally transformable to F_2 and t_1 is invertible for every object a of A holds ${t_1}^{-1}(a) = t_1(a)^{-1}$.

(31) If $F_1 \cong F_2$, then $F_2 \cong F_1$.

(32) If $F_1 \cong F_2$ and $F_2 \cong F_3$, then $F_1 \cong F_3$.

Let us consider A, B, F_1, F_2. Let us assume that F_1, F_2 are naturally equivalent. A natural transformation from F_1 to F_2 is called a natural equivalence of F_1 and F_2 if:

(Def.14) it is invertible.

We now state two propositions:

(33) id_F is a natural equivalence of F and F.

(34) If $F_1 \cong F_2$ and $F_2 \cong F_3$, then for every natural equivalence t of F_1 and F_2 and for every natural equivalence t' of F_2 and F_3 holds $t' \circ t$ is a natural equivalence of F_1 and F_3.

Functor category

Let us consider A, B. A non-empty set is called a set of natural transformations from A to B if:

(Def.15) for an arbitrary x such that $x \in$ it there exist functors F_1, F_2 from A to B and there exists a natural transformation t from F_1 to F_2 such that $x = \langle\langle F_1, F_2\rangle, t\rangle$ and F_1 is naturally transformable to F_2.

Let us consider A, B. The functor $\mathrm{NatTrans}(A, B)$ yielding a set of natural transformations from A to B is defined as follows:

(Def.16) for an arbitrary x holds $x \in \mathrm{NatTrans}(A, B)$ if and only if there exist functors F_1, F_2 from A to B and there exists a natural transformation t from F_1 to F_2 such that $x = \langle\langle F_1, F_2\rangle, t\rangle$ and F_1 is naturally transformable to F_2.

Let A_1, B_1, A_2, B_2 be non-empty sets, and let f_1 be a function from A_1 into B_1, and let f_2 be a function from A_2 into B_2. Let us note that one can characterize the predicate $f_1 = f_2$ by the following (equivalent) condition:

(Def.17) $A_1 = A_2$ and for every element a of A_1 holds $f_1(a) = f_2(a)$.

The following two propositions are true:

(35) F_1 is naturally transformable to F_2 if and only if $\langle\langle F_1, F_2\rangle, t_1\rangle \in \mathrm{NatTrans}(A, B)$.

(36) $\langle\langle F, F\rangle, \mathrm{id}_F\rangle \in \mathrm{NatTrans}(A, B)$.

Let us consider A, B. The functor B^A yielding a category is defined by the conditions (Def.18).

(Def.18) (i) The objects of $B^A = \mathrm{Funct}(A, B)$,

(ii) the morphisms of $B^A = \mathrm{NatTrans}(A, B)$,

(iii) for every morphism f of B^A holds $\mathrm{dom}\, f = (f_{\mathbf{1}})_{\mathbf{1}}$ and $\mathrm{cod}\, f = (f_{\mathbf{1}})_{\mathbf{2}}$,

(iv) for all morphisms f, g of B^A such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $\langle g, f\rangle \in$ dom (the composition of B^A),

(v) for all morphisms f, g of B^A such that $\langle g, f\rangle \in$ dom (the composition of B^A) there exist F, F_1, F_2, t, t_1 such that $f = \langle\langle F, F_1\rangle, t\rangle$ and $g = \langle\langle F_1, F_2\rangle, t_1\rangle$ and (the composition of B^A)$(\langle g, f\rangle) = \langle\langle F, F_2\rangle, t_1 \circ t\rangle$,

(vi) for every object a of B^A and for every F such that $F = a$ holds $\mathrm{id}_a = \langle\langle F, F\rangle, \mathrm{id}_F\rangle$.

We now state several propositions:

(37) The objects of $B^A = \mathrm{Funct}(A, B)$.

(38) The morphisms of $B^A = \mathrm{NatTrans}(A, B)$.

(39) For every morphism f of B^A such that $f = \langle\langle F, F_1\rangle, t\rangle$ holds $\mathrm{dom}\, f = F$ and $\mathrm{cod}\, f = F_1$.

(40) For all objects a, b of B^A and for every morphism f from a to b such that $\hom(a, b) \neq \emptyset$ there exist F, F_1, t such that $a = F$ and $b = F_1$ and $f = \langle\langle F, F_1\rangle, t\rangle$.

(41) For every natural transformation t' from F_2 to F_3 and for all morphisms f, g of B^A such that $f = \langle\langle F, F_1\rangle, t\rangle$ and $g = \langle\langle F_2, F_3\rangle, t'\rangle$ holds $\langle g, f\rangle \in \mathrm{dom}$ (the composition of B^A) if and only if $F_1 = F_2$.

(42) For all morphisms f, g of B^A such that $f = \langle\langle F, F_1\rangle, t\rangle$ and $g = \langle\langle F_1, F_2\rangle, t_1\rangle$ holds $g \cdot f = \langle\langle F, F_2\rangle, t_1 \circ t\rangle$.

(43) For every object a of B^A and for every F such that $F = a$ holds $\mathrm{id}_a = \langle\langle F, F\rangle, \mathrm{id}_F\rangle$.

Discrete categories

A category is discrete if:

(Def.19) for every morphism f of it there exists an object a of it such that $f = \mathrm{id}_a$.

One can prove the following propositions:

(44) For every discrete category A and for every object a of A holds $\hom(a, a) = \{\mathrm{id}_a\}$.

(45) A is discrete if and only if for every object a of A holds $\hom(a, a)$ is finite and $\mathrm{card}\,\hom(a, a) = 1$ and for every object b of A such that $a \neq b$ holds $\hom(a, b) = \emptyset$.

(46) $\dot{\circlearrowright}(o, m)$ is discrete.

(47) For every discrete category A and for every subcategory C of A holds C is discrete.

(48) If A is discrete and B is discrete, then $[\!:A, B:\!]$ is discrete.

(49) For every discrete category A and for every category B and for all functors F_1, F_2 from B to A such that F_1 is transformable to F_2 holds $F_1 = F_2$.

(50) For every discrete category A and for every category B and for every functor F from B to A and for every transformation t from F to F holds $t = \mathrm{id}_F$.

(51) If A is discrete, then A^B is discrete.

Let us consider C. The functor $\mathrm{IdCat}\, C$ yields a discrete subcategory of C and is defined as follows:

(Def.20) the objects of $\mathrm{IdCat}\, C$ = the objects of C and the morphisms of $\mathrm{IdCat}\, C = \{\mathrm{id}_a\}$,
where a ranges over objects of C.

Next we state four propositions:

(52) If C is discrete, then $\mathrm{IdCat}\, C = C$.

(53) $\mathrm{IdCat}\,\mathrm{IdCat}\,C = \mathrm{IdCat}\,C$.

(54) $\mathrm{IdCat}\,\dot{\circlearrowleft}(o, m) = \dot{\circlearrowleft}(o, m)$.

(55) $\mathrm{IdCat}[\!:A,\ B:\!] = [\!:\mathrm{IdCat}\,A,\ \mathrm{IdCat}\,B:\!]$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Czesław Byliński. Introduction to categories and functors. *Formalized Mathematics*, 1(**2**):409–420, 1990.

[6] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.

[7] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[8] Czesław Byliński. Subcategories and products of categories. *Formalized Mathematics*, 1(**4**):725–732, 1990.

[9] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[10] Zbigniew Semadeni and Antoni Wiweger. *Wstęp do teorii kategorii i funktorów*. Volume 45 of *Biblioteka Matematyczna*, PWN, Warszawa, 1978.

[11] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[13] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

Received May 15, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Matrices. Abelian Group of Matrices

Katarzyna Jankowska
Warsaw University
Białystok

Summary. The basic conceptions of matrix algebra are introduced. The matrix is introduced as the finite sequence of sequences with the same length, i.e. as a sequence of lines. There are considered matrices over a field, and the fact that these matrices with addition form an Abelian group is proved.

MML Identifier: `MATRIX_1`.

The notation and terminology used here have been introduced in the following papers: [9], [5], [6], [1], [8], [4], [2], [3], and [7]. For simplicity we adopt the following rules: x will be arbitrary, i, j, n, m will be natural numbers, D will be a non-empty set, K will be a field structure, s will be a finite sequence, a, a_1, a_2, b_1, b_2, d will be elements of D, p, p_1, p_2 will be finite sequences of elements of D, and F will be a field. A finite sequence is tabular if:

(Def.1) there exists a natural number n such that for every x such that $x \in \operatorname{rng} \mathrm{it}$ there exists s such that $s = x$ and $\operatorname{len} s = n$.

The following propositions are true:

(1) $\langle\langle d\rangle\rangle$ is tabular.

(2) $m \longmapsto (n \longmapsto x)$ is tabular.

(3) For every s holds $\langle s\rangle$ is tabular.

(4) For all finite sequences s_1, s_2 such that $\operatorname{len} s_1 = n$ and $\operatorname{len} s_2 = n$ holds $\langle s_1, s_2\rangle$ is tabular.

(5) ε is tabular.

(6) $\langle\varepsilon, \varepsilon\rangle$ is tabular.

(7) $\langle\langle a_1\rangle, \langle a_2\rangle\rangle$ is tabular.

(8) $\langle\langle a_1, a_2\rangle, \langle b_1, b_2\rangle\rangle$ is tabular.

A tabular finite sequence is non-trivial if:

(Def.2) there exists s such that $s \in \operatorname{rng} \mathrm{it}$ and $\operatorname{len} s > 0$.

ISSN 0777–4028

Let D be a non-empty set.

Let D be a non-empty set. A matrix over D is a tabular finite sequence of elements of D^*.

We now state the proposition

(9) s is a matrix over D if and only if there exists n such that for every x such that $x \in \operatorname{rng} s$ there exists p such that $x = p$ and $\operatorname{len} p = n$.

Let us consider D, m, n. A matrix over D is said to be a matrix over D of dimension $m \times n$ if:

(Def.3) $\operatorname{len} \mathrm{it} = m$ and for every p such that $p \in \operatorname{rng} \mathrm{it}$ holds $\operatorname{len} p = n$.

Let us consider D, n. A matrix over D of dimension n is a matrix over D of dimension $n \times n$.

We now define three new modes. Let us consider K. A matrix over K is a matrix over the carrier of K.

Let us consider n. A matrix over K of dimension n is a matrix over the carrier of K of dimension $n \times n$.

Let us consider m. A matrix over K of dimension $n \times m$ is a matrix over the carrier of K of dimension $n \times m$.

We now state a number of propositions:

(10) $m \longmapsto (n \longmapsto a)$ is a matrix over D of dimension $m \times n$.

(11) For every finite sequence p of elements of D holds $\langle p \rangle$ is a matrix over D of dimension $1 \times \operatorname{len} p$.

(12) For all p_1, p_2 such that $\operatorname{len} p_1 = n$ and $\operatorname{len} p_2 = n$ holds $\langle p_1, p_2 \rangle$ is a matrix over D of dimension $2 \times n$.

(13) ε is a matrix over D of dimension $0 \times m$.

(14) $\langle \varepsilon \rangle$ is a matrix over D of dimension 1×0.

(15) $\langle \langle a \rangle \rangle$ is a matrix over D of dimension 1.

(16) $\langle \varepsilon, \varepsilon \rangle$ is a matrix over D of dimension 2×0.

(17) $\langle \langle a_1, a_2 \rangle \rangle$ is a matrix over D of dimension 1×2.

(18) $\langle \langle a_1 \rangle, \langle a_2 \rangle \rangle$ is a matrix over D of dimension 2×1.

(19) $\langle \langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \rangle$ is a matrix over D of dimension 2.

In the sequel M, M_1, M_2 will be matrices over D. Let M be a tabular finite sequence. The functor $\operatorname{width} M$ yields a natural number and is defined as follows:

(Def.4) (i) there exists s such that $s \in \operatorname{rng} M$ and $\operatorname{len} s = \operatorname{width} M$ if $\operatorname{len} M > 0$,

(ii) $\operatorname{width} M = 0$, otherwise.

Next we state the proposition

(20) If $\operatorname{len} M > 0$, then for every n holds M is a matrix over D of dimension $\operatorname{len} M \times n$ if and only if $n = \operatorname{width} M$.

Let M be a tabular finite sequence. The indices of M yielding a set is defined by:

(Def.5) the indices of $M = [\!: \operatorname{Seg} \operatorname{len} M, \operatorname{Seg} \operatorname{width} M :\!]$.

Let us consider D, and let M be a matrix over D, and let us consider i, j. Let us assume that $\langle i, j\rangle \in$ the indices of M. The functor $M_{i,j}$ yielding an element of D is defined as follows:

(Def.6) there exists p such that $p = M(i)$ and $M_{i,j} = p(j)$.

The following proposition is true

(21) If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and for all i, j such that $\langle i, j\rangle \in$ the indices of M_1 holds $M_{1i,j} = M_{2i,j}$, then $M_1 = M_2$.

In this article we present several logical schemes. The scheme *MatrixLambda* deals with a non-empty set $\mathcal{A}$, a natural number $\mathcal{B}$, a natural number $\mathcal{C}$, and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$ and states that:

there exists a matrix M over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{C}$ such that for all i, j such that $\langle i, j\rangle \in$ the indices of M holds $M_{i,j} = \mathcal{F}(i, j)$
for all values of the parameters.

The scheme *MatrixEx* concerns a non-empty set $\mathcal{A}$, a natural number $\mathcal{B}$, a natural number $\mathcal{C}$, and a ternary predicate $\mathcal{P}$, and states that:

there exists a matrix M over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{C}$ such that for all i, j such that $\langle i, j\rangle \in$ the indices of M holds $\mathcal{P}[i, j, M_{i,j}]$
provided the parameters have the following properties:

- for all i, j such that $\langle i, j\rangle \in [\!:\operatorname{Seg}\mathcal{B}, \operatorname{Seg}\mathcal{C}:\!]$ for all elements x_1, x_2 of $\mathcal{A}$ such that $\mathcal{P}[i, j, x_1]$ and $\mathcal{P}[i, j, x_2]$ holds $x_1 = x_2$,
- for all i, j such that $\langle i, j\rangle \in [\!:\operatorname{Seg}\mathcal{B}, \operatorname{Seg}\mathcal{C}:\!]$ there exists an element x of $\mathcal{A}$ such that $\mathcal{P}[i, j, x]$.

The scheme *SeqDLambda* concerns a non-empty set $\mathcal{A}$, a natural number $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$ and states that:

there exists a finite sequence p of elements of $\mathcal{A}$ such that $\operatorname{len} p = \mathcal{B}$ and for every i such that $i \in \operatorname{Seg}\mathcal{B}$ holds $p(i) = \mathcal{F}(i)$
for all values of the parameters.

We now state several propositions:

(22) For every matrix M over D of dimension $n \times m$ such that $\operatorname{len} M = 0$ holds $\operatorname{width} M = 0$.

(23) For every matrix M over D of dimension $0 \times m$ holds $\operatorname{len} M = 0$ and $\operatorname{width} M = 0$ and the indices of $M = \emptyset$.

(24) If $n > 0$, then for every matrix M over D of dimension $n \times m$ holds $\operatorname{len} M = n$ and $\operatorname{width} M = m$ and the indices of $M = [\!:\operatorname{Seg} n, \operatorname{Seg} m:\!]$.

(25) For every matrix M over D of dimension n holds $\operatorname{len} M = n$ and $\operatorname{width} M = n$ and the indices of $M = [\!:\operatorname{Seg} n, \operatorname{Seg} n:\!]$.

(26) For every matrix M over D of dimension $n \times m$ holds $\operatorname{len} M = n$ and the indices of $M = [\!:\operatorname{Seg} n, \operatorname{Seg} \operatorname{width} M:\!]$.

(27) For all matrices M_1, M_2 over D of dimension $n \times m$ holds the indices of $M_1 =$ the indices of M_2.

(28) For all matrices M_1, M_2 over D of dimension $n \times m$ such that for all i, j such that $\langle i, j\rangle \in$ the indices of M_1 holds $M_{1i,j} = M_{2i,j}$ holds $M_1 = M_2$.

(29) For every matrix M_1 over D of dimension n and for all i, j such that $\langle i, j\rangle \in$ the indices of M_1 holds $\langle j, i\rangle \in$ the indices of M_1.

Let us consider D, and let M be a matrix over D. The functor M^{T} yielding a matrix over D is defined as follows:

(Def.7) $\mathrm{len}(M^{\mathrm{T}}) = \mathrm{width}\, M$ and for all i, j holds $\langle i, j\rangle \in$ the indices of M^{T} if and only if $\langle j, i\rangle \in$ the indices of M and for all i, j such that $\langle j, i\rangle \in$ the indices of M holds $M^{\mathrm{T}}_{i,j} = M_{j,i}$.

We now define two new functors. Let us consider D, M, i. The functor $\mathrm{Line}(M, i)$ yields a finite sequence of elements of D and is defined by:

(Def.8) $\mathrm{len}\,\mathrm{Line}(M, i) = \mathrm{width}\, M$ and for every j such that $j \in \mathrm{Seg}\,\mathrm{width}\, M$ holds $\mathrm{Line}(M, i)(j) = M_{i,j}$.

The functor $M_{\square,i}$ yields a finite sequence of elements of D and is defined as follows:

(Def.9) $\mathrm{len}(M_{\square,i}) = \mathrm{len}\, M$ and for every j such that $j \in \mathrm{Seg}\,\mathrm{len}\, M$ holds $M_{\square,i}(j) = M_{j,i}$.

Let us consider D, and let M be a matrix over D, and let us consider i. Then $\mathrm{Line}(M, i)$ is an element of $D^{\mathrm{width}\, M}$. Then $M_{\square,i}$ is an element of $D^{\mathrm{len}\, M}$.

In the sequel A, B are matrices over K of dimension n. We now define five new functors. Let us consider K, n. The functor $K^{n\times n}$ yields a non-empty set and is defined as follows:

(Def.10) $K^{n\times n} = (\ (\text{the carrier of } K)^n)^n$.

The functor $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n\times n}$ yielding a matrix over K of dimension n is defined as follows:

(Def.11) $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n\times n} = n \longmapsto (n \longmapsto 0_K)$.

The functor $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$ yielding a matrix over K of dimension n is defined as follows:

(Def.12) for every i such that $\langle i, i\rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$ holds $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n})_{i,i} = 1_K$ and for all i, j such that $\langle i, j\rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$ and $i \neq j$ holds $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n})_{i,j} = 0_K$.

Let us consider A. The functor $-A$ yielding a matrix over K of dimension n is defined as follows:

(Def.13) for all i, j such that $\langle i, j\rangle \in$ the indices of A holds $(-A)_{i,j} = -A_{i,j}$.

Let us consider B. The functor $A+B$ yielding a matrix over K of dimension n is defined by:

(Def.14) for all i, j such that $\langle i, j\rangle \in$ the indices of A holds $(A+B)_{i,j} = A_{i,j}+B_{i,j}$.

The following two propositions are true:

(30) For all i, j such that $\langle i, j\rangle \in$ the indices of $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n\times n}$ holds $(\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n\times n})_{i,j} = 0_K$.

(31) For every x holds x is an element of $K^{n\times n}$ if and only if x is a matrix over K of dimension n.

Let us consider K, n. A matrix over K of dimension n is called a diagonal n-dimensional matrix over K if:

(Def.15) for all i, j such that $\langle i, j\rangle \in$ the indices of it and $\mathrm{it}_{i,j} \neq 0_K$ holds $i = j$.

In the sequel A, B, C will denote matrices over F of dimension n. One can prove the following four propositions:

(32) $A + B = B + A$.

(33) $A + B + C = A + (B + C)$.

(34) $A + \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_F^{n\times n} = A$.

(35) $A + -A = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_F^{n\times n}$.

Let us consider F, n. The functor $F_{\mathrm{G}}^{n\times n}$ yielding an Abelian group is defined by:

(Def.16) the carrier of $F_{\mathrm{G}}^{n\times n} = F^{n\times n}$ and for all A, B holds (the addition of $F_{\mathrm{G}}^{n\times n}$)$(A, B) = A + B$ and for every A holds (the reverse-map of $F_{\mathrm{G}}^{n\times n}$)$(A) = -A$ and the zero of $F_{\mathrm{G}}^{n\times n} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_F^{n\times n}$.

Acknowledgements

I would like to thank Grzegorz Bancerek for his useful sugestions and valuable comments.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[3] Czesław Byliński. Binary operations applied to finite sequences. *Formalized Mathematics*, 1(**4**):643–649, 1990.

[4] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[7] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[8] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

Received June 8, 1990

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Paracompact and Metrizable Spaces

Leszek Borys
Warsaw University
Białystok

Summary. We give an example of a compact space. Next we define a locally finite subset family of topological spaces and paracompact topological spaces. An open sets family of a metric space is defined next and it has been shown that the metric space with any open sets family is a topological space. Next we define metrizable space.

MML Identifier: PCOMPS_1.

The papers [15], [5], [6], [11], [10], [12], [13], [18], [8], [17], [9], [7], [16], [3], [2], [1], [4], and [14] provide the terminology and notation for this paper. In the sequel P_1 denotes a metric space, x denotes an element of the carrier of P_1, and r, p denote real numbers. Next we state the proposition

(1) If $r \leq p$ and $r > 0$, then $\mathrm{Ball}(x, r) \subseteq \mathrm{Ball}(x, p)$.

For simplicity we adopt the following convention: T will be a topological space, x will be a point of T, W, A will be subsets of T, and F_1 will be a family of subsets of T. One can prove the following four propositions:

(2) $\overline{A} \neq \emptyset$ if and only if $A \neq \emptyset$.

(3) If $\overline{A} = \emptyset$, then $A = \emptyset$.

(4) $\overline{A}$ is closed.

(5) If F_1 is a cover of T, then for every x there exists W such that $x \in W$ and $W \in F_1$.

Let X be arbitrary. Then $\{X\}$ is a non-empty set. Then 2^X is a non-empty family of subsets of X.

Let a be arbitrary. The functor $\{a\}_{\mathrm{top}}$ yields a topological space and is defined by:

(Def.1) $\{a\}_{\mathrm{top}} = \langle \{a\}, 2^{\{a\}} \rangle$.

In the sequel a is arbitrary. We now state four propositions:

(6) $\{a\}_{\mathrm{top}} = \langle \{a\}, 2^{\{a\}} \rangle$.

ISSN 0777–4028

(7) The topology of $\{a\}_{\mathrm{top}} = 2^{\{a\}}$.

(8) The carrier of $\{a\}_{\mathrm{top}} = \{a\}$.

(9) $\{a\}_{\mathrm{top}}$ is compact.

Let us consider T, x. Then $\{x\}$ is a subset of T.

We now state the proposition

(10) If T is a T_2 space, then $\{x\}$ is closed.

For simplicity we follow the rules: T will be a topological space, x will be a point of T, Z, V, W, Y, A, B will be subsets of T, and F_1, G_1 will be families of subsets of T. Let us consider T. A family of subsets of T is locally finite if:

(Def.2) for every x there exists W such that $x \in W$ and W is open and $\{V : V \in \mathrm{it} \wedge V \cap W \neq \emptyset\}$ is finite.

Next we state three propositions:

(11) For every W holds $\{V : V \in F_1 \wedge V \cap W \neq \emptyset\} \subseteq F_1$.

(12) If $F_1 \subseteq G_1$ and G_1 is locally finite, then F_1 is locally finite.

(13) If F_1 is finite, then F_1 is locally finite.

Let us consider T, F_1. The functor $\mathrm{clf}\, F_1$ yielding a family of subsets of T is defined by:

(Def.3) $Z \in \mathrm{clf}\, F_1$ if and only if there exists W such that $Z = \overline{W}$ and $W \in F_1$.

Next we state several propositions:

(14) $\mathrm{clf}\, F_1$ is closed.

(15) If $F_1 = \emptyset$, then $\mathrm{clf}\, F_1 = \emptyset$.

(16) If $F_1 = \{V\}$, then $\mathrm{clf}\, F_1 = \{\overline{V}\}$.

(17) If $F_1 \subseteq G_1$, then $\mathrm{clf}\, F_1 \subseteq \mathrm{clf}\, G_1$.

(18) $\mathrm{clf}(F_1 \cup G_1) = \mathrm{clf}\, F_1 \cup \mathrm{clf}\, G_1$.

Next we state two propositions:

(19) If F_1 is finite, then $\overline{\bigcup F_1} = \bigcup \mathrm{clf}\, F_1$.

(20) F_1 is finer than $\mathrm{clf}\, F_1$.

The scheme *Lambda1top* deals with a topological space $\mathcal{A}$, a family $\mathcal{B}$ of subsets of $\mathcal{A}$, a family $\mathcal{C}$ of subsets of $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding a subset of $\mathcal{A}$ and states that:

there exists a function f from $\mathcal{B}$ into $\mathcal{C}$ such that for every subset Z of $\mathcal{A}$ such that $Z \in \mathcal{B}$ holds $f(Z) = \mathcal{F}(Z)$

provided the following condition is satisfied:

- for every subset Z of $\mathcal{A}$ such that $Z \in \mathcal{B}$ holds $\mathcal{F}(Z) \in \mathcal{C}$.

Next we state four propositions:

(21) If F_1 is locally finite, then $\mathrm{clf}\, F_1$ is locally finite.

(22) $\bigcup F_1 \subseteq \bigcup \mathrm{clf}\, F_1$.

(23) If F_1 is locally finite, then $\overline{\bigcup F_1} = \bigcup \mathrm{clf}\, F_1$.

(24) If F_1 is locally finite and F_1 is closed, then $\bigcup F_1$ is closed.

A topological space is paracompact if:

(Def.4) for every family F_1 of subsets of it such that F_1 is a cover of it and F_1 is open there exists a family G_1 of subsets of it such that G_1 is open and G_1 is a cover of it and G_1 is finer than F_1 and G_1 is locally finite.

The following propositions are true:

(25) If T is compact, then T is paracompact.

(26) Suppose T is paracompact and A is closed and B is closed and A misses B and for every x such that $x \in B$ there exist V, W such that V is open and W is open and $A \subseteq V$ and $x \in W$ and V misses W. Then there exist Y, Z such that Y is open and Z is open and $A \subseteq Y$ and $B \subseteq Z$ and Y misses Z.

(27) If T is a T_2 space and T is paracompact, then T is a T_3 space.

(28) If T is a T_2 space and T is paracompact, then T is a T_4 space.

For simplicity we follow a convention: P_1 will denote a metric space, x, y, z will denote elements of the carrier of P_1, r, p, q will denote real numbers, and V, W will denote subsets of the carrier of P_1. Let us consider P_1. The open set family of P_1 yielding a family of subsets of the carrier of P_1 is defined as follows:

(Def.5) for every V holds $V \in$the open set family of P_1 if and only if for every x such that $x \in V$ there exists r such that $r > 0$ and $\mathrm{Ball}(x, r) \subseteq V$.

One can prove the following propositions:

(29) For every x there exists r such that $r > 0$ and $\mathrm{Ball}(x, r) \subseteq$ the carrier of P_1.

(30) If $y \in \mathrm{Ball}(x, r)$, then there exists p such that $p > 0$ and $\mathrm{Ball}(y, p) \subseteq \mathrm{Ball}(x, r)$.

(31) If $y \in \mathrm{Ball}(x, r) \cap \mathrm{Ball}(z, p)$, then there exists q such that $\mathrm{Ball}(y, q) \subseteq \mathrm{Ball}(x, r)$ and $\mathrm{Ball}(y, q) \subseteq \mathrm{Ball}(z, p)$.

(32) For every V holds $V \in$ the open set family of P_1 if and only if for every x such that $x \in V$ there exists r such that $r > 0$ and $\mathrm{Ball}(x, r) \subseteq V$.

(33) For all x, r holds $\mathrm{Ball}(x, r) \in$ the open set family of P_1.

(34) The carrier of $P_1 \in$ the open set family of P_1.

(35) For all V, W such that $V \in$ the open set family of P_1 and $W \in$ the open set family of P_1 holds $V \cap W \in$ the open set family of P_1.

(36) For every family A of subsets of the carrier of P_1 such that $A \subseteq$ the open set family of P_1 holds $\bigcup A \in$ the open set family of P_1.

(37) $\langle$The carrier of P_1,the open set family of $P_1\rangle$ is a topological space.

Let us consider P_1. The functor $P_{1\mathrm{top}}$ yielding a topological space is defined as follows:

(Def.6) $P_{1\mathrm{top}} = \langle$ the carrier of P_1,the open set family of $P_1\rangle$.

We now state the proposition

(38) $P_{1\mathrm{top}}$ is a T_2 space.

Let D be a non-empty set, and let f be a function from $[\!: D, D :\!]$ into $\mathbb{R}$. We say that f is a metric of D if and only if:

(Def.7) for all elements a, b, c of D holds $f(a, b) = 0$ if and only if $a = b$ but $f(a, b) = f(b, a)$ and $f(a, c) \leq f(a, b) + f(b, c)$.

We now state two propositions:

(39) For every non-empty set D and for every function f from $[\!: D, D :\!]$ into $\mathbb{R}$ holds f is a metric of D if and only if $\langle D, f \rangle$ is a metric space.

(40) For every metric space M_1 holds the distance of M_1 is a metric of the carrier of M_1.

Let D be a non-empty set, and let f be a function from $[\!: D, D :\!]$ into $\mathbb{R}$. Let us assume that f is a metric of D. The functor $\mathrm{MetrSp}(D, f)$ yielding a metric space is defined by:

(Def.8) $\mathrm{MetrSp}(D, f) = \langle D, f \rangle$.

A topological space is metrizable if:

(Def.9) there exists a function f from $[\!:$ the carrier of it, the carrier of it $:\!]$ into $\mathbb{R}$ such that f is a metric of the carrier of it and the open set family of $\mathrm{MetrSp}((\text{the carrier of it}), f) =$ the topology of it.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.

[4] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[7] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.

[8] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.

[9] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[10] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[11] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.

[12] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[13] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[14] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[16] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[18] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

Received June 8, 1990

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Atlas of Midpoint Algebra

Michał Muzalewski
Warsaw University
Białystok

Summary. This article is a continuation of [4]. We have established a one-to-one correspondence between midpoint algebras and groups with the operator $\frac{1}{2}$. In general we shall say that a given midpoint algebra M and a group V are w-assotiated iff w is an atlas from M to V. At the beginning of the paper a few facts which rather belong to [3], [5] are proved.

MML Identifier: `MIDSP_2`.

The terminology and notation used here have been introduced in the following articles: [2], [1], [3], [4], and [5]. In the sequel G is a group structure and x is an element of G. Let us consider G, x. The functor $2x$ yielding an element of G is defined by:

(Def.1) $2x = x + x$.

In the sequel M is a midpoint algebra structure. Let us consider M. A point of M is an element of the points of M.

In the sequel p, q, r will be points of M and w will be a function from $[\!:$ the points of M, the points of $M :\!]$ into the carrier of G. Let us consider M, G, w. We say that M, G are associated w.r.t. w if and only if:

(Def.2) $p \oplus q = r$ if and only if $w(p, r) = w(r, q)$.

The following proposition is true

(1) If M, G are associated w.r.t. w, then $p \oplus p = p$.

We follow the rules: S will be a non-empty set, a, b, b', c, c', d will be elements of S, and w will be a function from $[\!: S, S :\!]$ into the carrier of G. Let us consider S, G, w. We say that w is an atlas of S, G if and only if:

(Def.3) for every a, x there exists b such that $w(a, b) = x$ and for all a, b, c such that $w(a, b) = w(a, c)$ holds $b = c$ and for all a, b, c holds $w(a, b) + w(b, c) = w(a, c)$.

ISSN 0777–4028

Let us consider S, G, w, a, x. Let us assume that w is an atlas of S, G. The functor $(a,x).w$ yielding an element of S is defined by:

(Def.4) $w(a,\ (a,x).w) = x$.

In the sequel G denotes a group, x, y denote elements of G, and w denotes a function from $[\!: S,\ S :\!]$ into the carrier of G. One can prove the following propositions:

(2) $2(0_G) = 0_G$.

(3) If $x + y = x$, then $y = 0_G$.

(4) If w is an atlas of S, G, then $w(a,\ a) = 0_G$.

(5) If w is an atlas of S, G and $w(a,\ b) = 0_G$, then $a = b$.

(6) If w is an atlas of S, G, then $w(a,\ b) = -w(b,\ a)$.

(7) If w is an atlas of S, G and $w(a,\ b) = w(c,\ d)$, then $w(b,\ a) = w(d,\ c)$.

(8) If w is an atlas of S, G, then for every b, x there exists a such that $w(a,\ b) = x$.

(9) If w is an atlas of S, G and $w(b,\ a) = w(c,\ a)$, then $b = c$.

(10) For every function w from $[\!:$ the points of M, the points of $M :\!]$ into the carrier of G such that w is an atlas of the points of M, G and M, G are associated w.r.t. w holds $p \oplus q = q \oplus p$.

(11) For every function w from $[\!:$ the points of M, the points of $M :\!]$ into the carrier of G such that w is an atlas of the points of M, G and M, G are associated w.r.t. w there exists r such that $r \oplus p = q$.

We adopt the following rules: G will denote an Abelian group and x, y, z, t will denote elements of G. The following propositions are true:

(12) $-(x + y) = -x + -y$.

(13) $x + y + (z + t) = x + z + (y + t)$.

(14) $2(x + y) = 2x + 2y$.

(15) $2(-x) = -2x$.

(16) For every function w from $[\!:$ the points of M, the points of $M :\!]$ into the carrier of G such that w is an atlas of the points of M, G and M, G are associated w.r.t. w for all points a, b, c, d of M holds $a \oplus b = c \oplus d$ if and only if $w(a,\ d) = w(c,\ b)$.

In the sequel w denotes a function from $[\!: S,\ S :\!]$ into the carrier of G. Next we state the proposition

(17) If w is an atlas of S, G, then for all a, b, b', c, c' such that $w(a,\ b) = w(b,\ c)$ and $w(a,\ b') = w(b',\ c')$ holds $w(c,\ c') = 2w(b,\ b')$.

We follow the rules: M denotes a midpoint algebra and p, q, r, s denote points of M. Let us consider M. Then vectgroup M is an Abelian group.

The following proposition is true

(18) For an arbitrary a holds a is an element of vectgroup M if and only if a is a vector of M and $0_{\mathrm{vectgroup}\, M} = \mathrm{I}_M$ and for all elements a, b of

vectgroup M and for all vectors x, y of M such that $a = x$ and $b = y$ holds $a + b = x + y$.

An Abelian group is called a group with the operator $\frac{1}{2}$ if:

(Def.5) for every element a of it there exists an element x of it such that $2x = a$ and for every element a of it such that $2a = 0_{\rm it}$ holds $a = 0_{\rm it}$.

In the sequel G is a group with the operator $\frac{1}{2}$ and x, y are elements of G. One can prove the following two propositions:

(19) If $x = -x$, then $x = 0_G$.

(20) If $2x = 2y$, then $x = y$.

Let us consider G, x. The functor $\frac{1}{2}x$ yielding an element of G is defined as follows:

(Def.6) $2\frac{1}{2}x = x$.

The following three propositions are true:

(21) $\frac{1}{2}(0_G) = 0_G$ and $\frac{1}{2}(x + y) = \frac{1}{2}x + \frac{1}{2}y$ but if $\frac{1}{2}x = \frac{1}{2}y$, then $x = y$ and $\frac{1}{2}2x = x$.

(22) For every M being a midpoint algebra structure and for every function w from $[\!:$ the points of M, the points of $M :\!]$ into the carrier of G such that w is an atlas of the points of M, G and M, G are associated w.r.t. w for all points a, b, c, d of M holds $a \oplus b \oplus (c \oplus d) = a \oplus c \oplus (b \oplus d)$.

(23) For every M being a midpoint algebra structure and for every function w from $[\!:$ the points of M, the points of $M :\!]$ into the carrier of G such that w is an atlas of the points of M, G and M, G are associated w.r.t. w holds M is a midpoint algebra.

Let us consider M. Then vectgroup M is a group with the operator $\frac{1}{2}$.

Let us consider M, p, q. The functor q^p yields an element of vectgroup M and is defined as follows:

(Def.7) $q^p = \overrightarrow{[p, q]}$.

Let us consider M. The functor vect M yields a function from $[\!:$ the points of M, the points of $M :\!]$ into the carrier of vectgroup M and is defined by:

(Def.8) $(\text{vect}\, M)(p,\, q) = \overrightarrow{[p, q]}$.

We now state four propositions:

(24) vect M is an atlas of the points of M, vectgroup M.

(25) $\overrightarrow{[p, q]} = \overrightarrow{[r, s]}$ if and only if $p \oplus s = q \oplus r$.

(26) $p \oplus q = r$ if and only if $\overrightarrow{[p, r]} = \overrightarrow{[r, q]}$.

(27) M, vectgroup M are associated w.r.t. vect M.

In the sequel w will denote a function from $[\!:S,\, S:\!]$ into the carrier of G. Let us consider S, G, w. Let us assume that w is an atlas of S, G. The functor ${}^{@}w$ yielding a binary operation on S is defined as follows:

(Def.9) $w(a,\, ({}^{@}w)(a,\, b)) = w(({}^{@}w)(a,\, b),\, b)$.

We now state the proposition

(28) If w is an atlas of S, G, then for all a, b, c holds $({}^{@}w)(a,\, b) = c$ if and only if $w(a,\, c) = w(c,\, b)$.

In the sequel a, b, c are points of $\langle S, {}^{@}w\rangle$. We now state two propositions:

(29) $({}^{@}w)(a,\, b) = a \oplus b$.

(30) $a \oplus b = c$ if and only if $({}^{@}w)(a,\, b) = c$.

Let us consider S, G, w. The functor $\mathrm{Atlas}\, w$ yielding a function from $[\!:$ the points of $\langle S, {}^{@}w\rangle$, the points of $\langle S, {}^{@}w\rangle :\!]$ into the carrier of G is defined as follows:

(Def.10) $\mathrm{Atlas}\, w = w$.

Next we state two propositions:

(31) If w is an atlas of S, G, then $\mathrm{Atlas}\, w$ is an atlas of the points of $\langle S, {}^{@}w\rangle$, G.

(32) If w is an atlas of S, G, then $\langle S, {}^{@}w\rangle$, G are associated w.r.t. $\mathrm{Atlas}\, w$.

Let us consider S, G, w. Let us assume that w is an atlas of S, G. The functor $\mathrm{MidSp}(w)$ yielding a midpoint algebra is defined by:

(Def.11) $\mathrm{MidSp}(w) = \langle S, {}^{@}w\rangle$.

We follow the rules: M is a midpoint algebra structure, w is a function from $[\!:$ the points of M, the points of $M :\!]$ into the carrier of G, and a, b, b_1, b_2, c are points of M. The following proposition is true

(33) M is a midpoint algebra if and only if there exists G and there exists w such that w is an atlas of the points of M, G and M, G are associated w.r.t. w.

Let us consider M. We consider atlas structures over M which are systems

$\langle$an algebra, a function$\rangle$,

where the algebra is a group with the operator $\frac{1}{2}$ and the function is a function from $[\!:$ the points of M, the points of $M :\!]$ into the carrier of the algebra.

Let M be a midpoint algebra. An atlas structure over M is said to be an atlas of M if:

(Def.12) M, the algebra of it are associated w.r.t. the function of it and the function of it is an atlas of the points of M, the algebra of it.

Let M be a midpoint algebra, and let W be an atlas of M. A vector of W is an element of the algebra of W.

Let M be a midpoint algebra, and let W be an atlas of M, and let a, b be points of M. The functor $W(a,\, b)$ yields an element of the algebra of W and is defined as follows:

(Def.13) $W(a,\, b) = (\text{the function of } W)(a,\, b)$.

Let M be a midpoint algebra, and let W be an atlas of M, and let a be a point of M, and let x be a vector of W. The functor $(a, x).W$ yielding a point of M is defined as follows:

(Def.14) $(a, x).W = (a, x).(\text{the function of } W)$.

Let M be a midpoint algebra, and let W be an atlas of M. The functor 0_W yielding a vector of W is defined as follows:

(Def.15) $0_W = 0_{\text{the algebra of } W}$.

We now state two propositions:

(34) If w is an atlas of the points of M, G and M, G are associated w.r.t. w, then $a \oplus c = b_1 \oplus b_2$ if and only if $w(a, c) = w(a, b_1) + w(a, b_2)$.

(35) If w is an atlas of the points of M, G and M, G are associated w.r.t. w, then $a \oplus c = b$ if and only if $w(a, c) = 2w(a, b)$.

For simplicity we adopt the following convention: M will be a midpoint algebra, W will be an atlas of M, a, b, b_1, b_2, c, d will be points of M, and x will be a vector of W. One can prove the following propositions:

(36) $a \oplus c = b_1 \oplus b_2$ if and only if $W(a, c) = W(a, b_1) + W(a, b_2)$.

(37) $a \oplus c = b$ if and only if $W(a, c) = 2W(a, b)$.

(38) For every a, x there exists b such that $W(a, b) = x$ and for all a, b, c such that $W(a, b) = W(a, c)$ holds $b = c$ and for all a, b, c holds $W(a, b) + W(b, c) = W(a, c)$.

(39) (i) $W(a, a) = 0_W$,

(ii) if $W(a, b) = 0_W$, then $a = b$,

(iii) $W(a, b) = -W(b, a)$,

(iv) if $W(a, b) = W(c, d)$, then $W(b, a) = W(d, c)$,

(v) for every b, x there exists a such that $W(a, b) = x$,

(vi) if $W(b, a) = W(c, a)$, then $b = c$,

(vii) $a \oplus b = c$ if and only if $W(a, c) = W(c, b)$,

(viii) $a \oplus b = c \oplus d$ if and only if $W(a, d) = W(c, b)$,

(ix) $W(a, b) = x$ if and only if $(a, x).W = b$.

(40) $(a, 0_W).W = a$.

References

[1] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[4] Michał Muzalewski. Midpoint algebras. *Formalized Mathematics*, 1(**3**):483–488, 1990.

[5] Michał Muzalewski and Wojciech Skaba. Groups, rings, left- and right-modules. *Formalized Mathematics*, 2(**2**):275–278, 1991.

Received June 21, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Several Properties of the σ-additive Measure

Józef Białas
University of Łódź

Summary. A continuation of [5]. The paper contains the definition and basic properties of a σ-additive, nonnegative measure, with values in $\overline{\mathbb{R}}$, the enlarged set of real numbers, where $\overline{\mathbb{R}}$ denotes set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ - by R.Sikorski [12]. Some simple theorems concerning basic properties of a σ-additive measure, measurable sets, measure zero sets are proved. The work is the fourth part of the series of articles concerning the Lebesgue measure theory.

MML Identifier: `MEASURE2`.

The terminology and notation used here have been introduced in the following papers: [14], [13], [8], [9], [6], [7], [1], [11], [2], [10], [3], [4], and [5]. The following proposition is true

(1) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every function F from $\mathbb{N}$ into S holds $M \cdot F$ is non-negative.

The scheme *RecExFun* concerns a set $\mathcal{A}$, a σ-field $\mathcal{B}$ of subsets of $\mathcal{A}$, an element $\mathcal{C}$ of $\mathcal{B}$, and a ternary predicate $\mathcal{P}$, and states that:

there exists a function f from $\mathbb{N}$ into $\mathcal{B}$ such that $f(0) = \mathcal{C}$ and for every element n of $\mathbb{N}$ holds $\mathcal{P}[n, f(n), f(n+1)]$

provided the following conditions are satisfied:

- for every natural number n and for every element x of $\mathcal{B}$ there exists an element y of $\mathcal{B}$ such that $\mathcal{P}[n, x, y]$,
- for every natural number n and for all elements x, y_1, y_2 of $\mathcal{B}$ such that $\mathcal{P}[n, x, y_1]$ and $\mathcal{P}[n, x, y_2]$ holds $y_1 = y_2$.

Let X be a set, and let S be a σ-field of subsets of X. A denumerable family of subsets of X is called a family of measureable sets of S if:

(Def.1) it $\subseteq S$.

One can prove the following propositions:

ISSN 0777–4028

(2) For every set X and for every σ-field S of subsets of X and for every denumerable family T of subsets of X holds T is a family of measureable sets of S if and only if $T \subseteq S$.

(3) For every set X and for every σ-field S of subsets of X and for every family T of measureable sets of S holds $\bigcap T \in S$ and $\bigcup T \in S$.

Let X be a set, and let S be a σ-field of subsets of X, and let T be a family of measureable sets of S. Then $\bigcap T$ is an element of S.

Let X be a set, and let S be a σ-field of subsets of X, and let T be a family of measureable sets of S. Then $\bigcup T$ is an element of S.

Let X be a set, and let S be a σ-field of subsets of X, and let F be a function from $\mathbb{N}$ into S, and let n be an element of $\mathbb{N}$. Then $F(n)$ is an element of S.

One can prove the following propositions:

(4) For every set X and for every σ-field S of subsets of X and for every function N from $\mathbb{N}$ into S there exists a function F from $\mathbb{N}$ into S such that $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus N(n)$.

(5) For every set X and for every σ-field S of subsets of X and for every function N from $\mathbb{N}$ into S there exists a function F from $\mathbb{N}$ into S such that $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \cup F(n)$.

(6) Let X be a set. Let S be a σ-field of subsets of X. Let N be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \cup F(n)$. Then for an arbitrary r and for every natural number n holds $r \in F(n)$ if and only if there exists a natural number k such that $k \leq n$ and $r \in N(k)$.

(7) Let X be a set. Let S be a σ-field of subsets of X. Let N be a function from $\mathbb{N}$ into S. Then for every function F from $\mathbb{N}$ into S such that $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \cup F(n)$ for all natural numbers n, m such that $n < m$ holds $F(n) \subseteq F(m)$.

(8) Let X be a set. Let S be a σ-field of subsets of X. Let N be a function from $\mathbb{N}$ into S. Let G be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose that
  (i) $G(0) = N(0)$,
  (ii) for every element n of $\mathbb{N}$ holds $G(n+1) = N(n+1) \cup G(n)$,
  (iii) $F(0) = N(0)$,
  (iv) for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus G(n)$.

Then for all natural numbers n, m such that $n \leq m$ holds $F(n) \subseteq G(m)$.

(9) For every set X and for every σ-field S of subsets of X and for every function N from $\mathbb{N}$ into S and for every function G from $\mathbb{N}$ into S there exists a function F from $\mathbb{N}$ into S such that $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus G(n)$.

(10) For every set X and for every σ-field S of subsets of X and for every function N from $\mathbb{N}$ into S there exists a function F from $\mathbb{N}$ into S such

that $F(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(0) \setminus N(n)$.

(11) Let X be a set. Let S be a σ-field of subsets of X. Let N be a function from $\mathbb{N}$ into S. Let G be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose that
  (i) $G(0) = N(0)$,
  (ii) for every element n of $\mathbb{N}$ holds $G(n+1) = N(n+1) \cup G(n)$,
  (iii) $F(0) = N(0)$,
  (iv) for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus G(n)$.

  Then for all natural numbers n, m such that $n < m$ holds $F(n) \cap F(m) = \emptyset$.

(12) For every set X and for every σ-field S of subsets of X and for every function N from $\mathbb{N}$ into S and for every element n of $\mathbb{N}$ holds $N(n) \in \operatorname{rng} N$.

(13) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every family T of measureable sets of S and for every function F from $\mathbb{N}$ into S such that $T = \operatorname{rng} F$ holds $M(\bigcup T) \leq \sum(M \cdot F)$.

(14) For every set X and for every σ-field S of subsets of X and for every family T of measureable sets of S there exists a function F from $\mathbb{N}$ into S such that $T = \operatorname{rng} F$.

(15) Let X be a set. Let S be a σ-field of subsets of X. Let N be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Then if $F(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(0) \setminus N(n)$ and $N(n+1) \subseteq N(n)$, then for every element n of $\mathbb{N}$ holds $F(n) \subseteq F(n+1)$.

(16) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every family T of measureable sets of S such that for every set A such that $A \in T$ holds A is a set of measure zero w.r.t. M holds $\bigcup T$ is a set of measure zero w.r.t. M.

(17) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every family T of measureable sets of S such that there exists a set A such that $A \in T$ and A is a set of measure zero w.r.t. M holds $\bigcap T$ is a set of measure zero w.r.t. M.

(18) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every family T of measureable sets of S such that for every set A such that $A \in T$ holds A is a set of measure zero w.r.t. M holds $\bigcap T$ is a set of measure zero w.r.t. M.

Let X be a set, and let S be a σ-field of subsets of X. A family of measureable sets of S is called a family of measureable non-decrement sets of S if:

(Def.2) there exists a function F from $\mathbb{N}$ into S such that it $= \operatorname{rng} F$ and for every element n of $\mathbb{N}$ holds $F(n) \subseteq F(n+1)$.

We now state the proposition

(19) For every set X and for every σ-field S of subsets of X and for every family T of measureable sets of S holds T is a family of measureable non-

decrement sets of S if and only if there exists a function F from $\mathbb{N}$ into S such that $T = \operatorname{rng} F$ and for every element n of $\mathbb{N}$ holds $F(n) \subseteq F(n+1)$.

Let X be a set, and let S be a σ-field of subsets of X. A family of measureable sets of S is called a family of measureable non-increment sets of S if:

(Def.3) there exists a function F from $\mathbb{N}$ into S such that it $= \operatorname{rng} F$ and for every element n of $\mathbb{N}$ holds $F(n+1) \subseteq F(n)$.

We now state several propositions:

(20) For every set X and for every σ-field S of subsets of X and for every family T of measureable sets of S holds T is a family of measureable non-increment sets of S if and only if there exists a function F from $\mathbb{N}$ into S such that $T = \operatorname{rng} F$ and for every element n of $\mathbb{N}$ holds $F(n+1) \subseteq F(n)$.

(21) Let X be a set. Let S be a σ-field of subsets of X. Then for every function N from $\mathbb{N}$ into S and for every function F from $\mathbb{N}$ into S such that $F(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(0) \setminus N(n)$ and $N(n+1) \subseteq N(n)$ holds $\operatorname{rng} F$ is a family of measureable non-decrement sets of S.

(22) For every set X and for every non-empty family S of subsets of X and for every function N from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$ holds $N(n) \subseteq N(n+1)$ for all natural numbers m, n such that $n < m$ holds $N(n) \subseteq N(m)$.

(23) Let X be a set. Let S be a σ-field of subsets of X. Let N be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus N(n)$ and $N(n) \subseteq N(n+1)$. Then for all natural numbers n, m such that $n < m$ holds $F(n) \cap F(m) = \emptyset$.

(24) Let X be a set. Let S be a σ-field of subsets of X. Let N be a function from $\mathbb{N}$ into S. Then for every function F from $\mathbb{N}$ into S such that $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus N(n)$ and $N(n) \subseteq N(n+1)$ holds $\bigcup \operatorname{rng} F = \bigcup \operatorname{rng} N$.

(25) Let X be a set. Let S be a σ-field of subsets of X. Let N be a function from $\mathbb{N}$ into S. Then for every function F from $\mathbb{N}$ into S such that $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus N(n)$ and $N(n) \subseteq N(n+1)$ holds F is a sequence of separated subsets of S.

(26) Let X be a set. Let S be a σ-field of subsets of X. Let N be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus N(n)$ and $N(n) \subseteq N(n+1)$. Then $N(0) = F(0)$ and for every element n of $\mathbb{N}$ holds $N(n+1) = F(n+1) \cup N(n)$.

(27) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every function F from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$ holds $F(n) \subseteq F(n+1)$ holds $M(\bigcup \operatorname{rng} F) = \sup \operatorname{rng}(M \cdot F)$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):163–171, 1991.

[4] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):173–183, 1991.

[5] Józef Białas. The σ-additive measure theory. *Formalized Mathematics*, 2(**2**):263–270, 1991.

[6] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.

[7] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[10] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[11] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[12] R. Sikorski. *Rachunek różniczkowy i całkowy - funkcje wielu zmiennych. Biblioteka Matematyczna*, PWN - Warszawa, 1968.

[13] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.

[14] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

Received July 3, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Metrics in the Cartesian Product - Part II

Stanisława Kanas
Technical University of Rzeszów

Adam Lecko
Technical University of Rzeszów

Summary. A continuation of [9]. It deals with the method of creation of the distance in the Cartesian product of metric spaces. The distance between two points belonging to Cartesian product of metric spaces has been defined as square root of the sum of squares of distances of appriopriate coordinates (or projections) of these points. It is shown that the product of metric spaces with such a distance is a metric space. Examples of metric spaces defined in this way are given.

MML Identifier: METRIC_4.

The articles [7], [15], [4], [5], [2], [6], [1], [10], [11], [3], [8], [13], [12], [14], and [9] provide the terminology and notation for this paper. We adopt the following convention: X, Y are metric spaces, x_1, y_1, z_1 are elements of the carrier of X, and x_2, y_2, z_2 are elements of the carrier of Y. Let us consider X, Y. The functor $\rho^{[:X,Y:]}$ yields a function from $[:[:$ the carrier of X, the carrier of $Y:]$, $[:$ the carrier of X, the carrier of $Y:]:]$ into $\mathbb{R}$ and is defined by:

(Def.1) for all elements x_1, y_1 of the carrier of X and for all elements x_2, y_2 of the carrier of Y and for all elements x, y of $[:$ the carrier of X, the carrier of $Y:]$ such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ holds $\rho^{[:X,Y:]}(x, y) = \sqrt{(\rho(x_1,y_1))^{\mathbf{2}} + (\rho(x_2,y_2))^{\mathbf{2}}}$.

Next we state the proposition

(1) Let X be a metric space. Let Y be a metric space. Let F be a function from $[:[:$ the carrier of X, the carrier of $Y:]$, $[:$ the carrier of X, the carrier of $Y:]:]$ into $\mathbb{R}$. Then $F = \rho^{[:X,Y:]}$ if and only if for all elements x_1, y_1 of the carrier of X and for all elements x_2, y_2 of the carrier of Y and for all elements x, y of $[:$ the carrier of X, the carrier of $Y:]$ such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ holds $F(x, y) = \sqrt{(\rho(x_1,y_1))^{\mathbf{2}} + (\rho(x_2,y_2))^{\mathbf{2}}}$.

Next we state several propositions:

(2) For all elements a, b of $\mathbb{R}$ such that $0 \leq a$ and $0 \leq b$ holds $\sqrt{a+b} = 0$ if and only if $a = 0$ and $b = 0$.

ISSN 0777–4028

(3) For all elements x, y of $[\!:$ the carrier of X, the carrier of $Y\,:\!]$ such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $\rho^{[:X,Y:]}(x, y) = 0$ if and only if $x = y$.

(4) For all elements x, y of $[\!:$ the carrier of X, the carrier of $Y\,:\!]$ such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $\rho^{[:X,Y:]}(x, y) = \rho^{[:X,Y:]}(y, x)$.

(5) For all elements a, b, c, d of $\mathbb{R}$ such that $0 \leq a$ and $0 \leq b$ and $0 \leq c$ and $0 \leq d$ holds $\sqrt{(a+c)^{\mathbf{2}} + (b+d)^{\mathbf{2}}} \leq \sqrt{a^{\mathbf{2}} + b^{\mathbf{2}}} + \sqrt{c^{\mathbf{2}} + d^{\mathbf{2}}}$.

(6) For all elements x, y, z of $[\!:$ the carrier of X, the carrier of $Y\,:\!]$ such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ and $z = \langle z_1, z_2 \rangle$ holds $\rho^{[:X,Y:]}(x, z) \leq \rho^{[:X,Y:]}(x, y) + \rho^{[:X,Y:]}(y, z)$.

Let us consider X, Y, and let x, y be elements of $[\!:$ the carrier of X, the carrier of $Y\,:\!]$. The functor $\rho^{\mathbf{2}}(x, y)$ yielding a real number is defined as follows:

(Def.2) $\rho^{\mathbf{2}}(x, y) = \rho^{[:X,Y:]}(x, y)$.

Next we state the proposition

(7) For all elements x, y of $[\!:$ the carrier of X, the carrier of $Y\,:\!]$ holds $\rho^{\mathbf{2}}(x, y) = \rho^{[:X,Y:]}(x, y)$.

Let X, Y be metric spaces. The functor $[:X, Y:]$ yielding a metric space is defined as follows:

(Def.3) $[:X, Y:] = \langle [\!:$ the carrier of X, the carrier of $Y\,:\!], \rho^{[:X,Y:]} \rangle$.

We now state the proposition

(8) For every metric space X and for every metric space Y holds $\langle [\!:$ the carrier of X, the carrier of $Y\,:\!], \rho^{[:X,Y:]} \rangle$ is a metric space.

In the sequel Z will be a metric space and x_3, y_3, z_3 will be elements of the carrier of Z. Let us consider X, Y, Z. The functor $\rho^{[:X,Y,Z:]}$ yielding a function from $[\![\!:$ the carrier of X, the carrier of Y, the carrier of $Z\,:\!]$, $[\!:$ the carrier of X, the carrier of Y, the carrier of $Z\,:\!]\!]$ into $\mathbb{R}$ is defined by the condition (Def.4).

(Def.4) Let x_1, y_1 be elements of the carrier of X. Let x_2, y_2 be elements of the carrier of Y. Let x_3, y_3 be elements of the carrier of Z. Then for all elements x, y of $[\!:$ the carrier of X, the carrier of Y, the carrier of $Z\,:\!]$ such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $\rho^{[:X,Y,Z:]}(x, y) = \sqrt{(\rho(x_1, y_1))^{\mathbf{2}} + (\rho(x_2, y_2))^{\mathbf{2}} + (\rho(x_3, y_3))^{\mathbf{2}}}$.

One can prove the following propositions:

(9) Let X be a metric space. Let Y be a metric space. Let Z be a metric space. Let F be a function from $[\![\!:$ the carrier of X, the carrier of Y, the carrier of $Z\,:\!]$, $[\!:$ the carrier of X, the carrier of Y, the carrier of $Z\,:\!]\!]$ into $\mathbb{R}$. Then $F = \rho^{[:X,Y,Z:]}$ if and only if for all elements x_1, y_1 of the carrier of X and for all elements x_2, y_2 of the carrier of Y and for all elements x_3, y_3 of the carrier of Z and for all elements x, y of $[\!:$ the carrier of X, the carrier of Y, the carrier of $Z\,:\!]$ such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $F(x, y) = \sqrt{(\rho(x_1, y_1))^{\mathbf{2}} + (\rho(x_2, y_2))^{\mathbf{2}} + (\rho(x_3, y_3))^{\mathbf{2}}}$.

(10) For all elements x, y of $[:$ the carrier of X, the carrier of Y, the carrier of $Z:]$ such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $\rho^{[:X,Y,Z:]}(x, y) = 0$ if and only if $x = y$.

(11) For all elements x, y of $[:$ the carrier of X, the carrier of Y, the carrier of $Z:]$ such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $\rho^{[:X,Y,Z:]}(x, y) = \rho^{[:X,Y,Z:]}(y, x)$.

(12) For all elements a, b, c of $\mathbb{R}$ holds $(a+b+c)^{\mathbf{2}} = a^{\mathbf{2}} + b^{\mathbf{2}} + c^{\mathbf{2}} + (2 \cdot a \cdot b + 2 \cdot a \cdot c + 2 \cdot b \cdot c)$.

(13) Let a, b, c, d, e, f be elements of $\mathbb{R}$. Suppose $0 \leq a$ and $0 \leq b$ and $0 \leq c$ and $0 \leq d$ and $0 \leq e$ and $0 \leq f$. Then $2 \cdot (a \cdot d) \cdot (c \cdot b) + 2 \cdot (a \cdot f) \cdot (e \cdot c) + 2 \cdot (b \cdot f) \cdot (e \cdot d) \leq (a \cdot d)^{\mathbf{2}} + (c \cdot b)^{\mathbf{2}} + (a \cdot f)^{\mathbf{2}} + (e \cdot c)^{\mathbf{2}} + (b \cdot f)^{\mathbf{2}} + (e \cdot d)^{\mathbf{2}}$.

(14) Let a, b, c, d, e, f be elements of $\mathbb{R}$. Then $a^{\mathbf{2}} \cdot d^{\mathbf{2}} + (a^{\mathbf{2}} \cdot f^{\mathbf{2}} + c^{\mathbf{2}} \cdot b^{\mathbf{2}}) + e^{\mathbf{2}} \cdot c^{\mathbf{2}} + b^{\mathbf{2}} \cdot f^{\mathbf{2}} + e^{\mathbf{2}} \cdot d^{\mathbf{2}} + e^{\mathbf{2}} \cdot f^{\mathbf{2}} + b^{\mathbf{2}} \cdot d^{\mathbf{2}} + a^{\mathbf{2}} \cdot c^{\mathbf{2}} = (a^{\mathbf{2}} + b^{\mathbf{2}} + e^{\mathbf{2}}) \cdot (c^{\mathbf{2}} + d^{\mathbf{2}} + f^{\mathbf{2}})$.

(15) Let a, b, c, d, e, f be elements of $\mathbb{R}$. Suppose $0 \leq a$ and $0 \leq b$ and $0 \leq c$ and $0 \leq d$ and $0 \leq e$ and $0 \leq f$. Then $(a \cdot c + b \cdot d + e \cdot f)^{\mathbf{2}} \leq (a^{\mathbf{2}} + b^{\mathbf{2}} + e^{\mathbf{2}}) \cdot (c^{\mathbf{2}} + d^{\mathbf{2}} + f^{\mathbf{2}})$.

(16) Let x, y, z be elements of $[:$ the carrier of X, the carrier of Y, the carrier of $Z:]$. Then if $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ and $z = \langle z_1, z_2, z_3 \rangle$, then $\rho^{[:X,Y,Z:]}(x, z) \leq \rho^{[:X,Y,Z:]}(x, y) + \rho^{[:X,Y,Z:]}(y, z)$.

Let us consider X, Y, Z, and let x, y be elements of $[:$ the carrier of X, the carrier of Y, the carrier of $Z:]$. The functor $\rho^{\mathbf{3}}(x,y)$ yielding a real number is defined as follows:

(Def.5) $\rho^{\mathbf{3}}(x,y) = \rho^{[:X,Y,Z:]}(x, y)$.

One can prove the following proposition

(17) For all elements x, y of $[:$ the carrier of X, the carrier of Y, the carrier of $Z:]$ holds $\rho^{\mathbf{3}}(x,y) = \rho^{[:X,Y,Z:]}(x, y)$.

Let X, Y, Z be metric spaces. The functor $[:X,Y:]$ yields a metric space and is defined by:

(Def.6) $[:X,Y:] = \langle [:$ the carrier of X, the carrier of Y, the carrier of $Z:]$, $\rho^{[:X,Y,Z:]} \rangle$.

The following proposition is true

(18) For every metric space X and for every metric space Y and for every metric space Z holds $\langle [:$ the carrier of X, the carrier of Y, the carrier of $Z:], \rho^{[:X,Y,Z:]} \rangle$ is a metric space.

In the sequel x_1, x_2, y_1, y_2, z_1, z_2 denote elements of $\mathbb{R}$. The function $\rho^{[:\mathbb{R},\mathbb{R}:]}$ from $[: [:\mathbb{R}, \mathbb{R}:], [:\mathbb{R}, \mathbb{R}:] :]$ into $\mathbb{R}$ is defined by:

(Def.7) for all elements x_1, y_1, x_2, y_2 of $\mathbb{R}$ and for all elements x, y of $[:\mathbb{R}, \mathbb{R}:]$ such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R}:]}(x, y) = \rho_{\mathbb{R}}(x_1, y_1) + \rho_{\mathbb{R}}(x_2, y_2)$.

The following propositions are true:

(19) For all elements x_1, x_2, y_1, y_2 of $\mathbb{R}$ and for all elements x, y of $[:\mathbb{R}, \mathbb{R}:]$ such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R}:]}(x, y) = 0$ if and only if $x = y$.

(20) For all elements x, y of $[:\mathbb{R}, \mathbb{R}:]$ such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R}:]}(x, y) = \rho^{[:\mathbb{R},\mathbb{R}:]}(y, x)$.

(21) For all elements x, y, z of $[:\mathbb{R}, \mathbb{R}:]$ such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ and $z = \langle z_1, z_2\rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R}:]}(x, z) \leq \rho^{[:\mathbb{R},\mathbb{R}:]}(x, y) + \rho^{[:\mathbb{R},\mathbb{R}:]}(y, z)$.

The metric space $[:\mathbb{R}_{\mathrm{M}}, \mathbb{R}_{\mathrm{M}}:]$ is defined by:

(Def.8) $[:\mathbb{R}_{\mathrm{M}}, \mathbb{R}_{\mathrm{M}}:] = \langle [:\mathbb{R}, \mathbb{R}:], \rho^{[:\mathbb{R},\mathbb{R}:]}\rangle$.

The function $\rho^{\mathbb{R}^2}$ from $[:[:\mathbb{R}, \mathbb{R}:], [:\mathbb{R}, \mathbb{R}:]:]$ into $\mathbb{R}$ is defined as follows:

(Def.9) for all elements x_1, y_1, x_2, y_2 of $\mathbb{R}$ and for all elements x, y of $[:\mathbb{R}, \mathbb{R}:]$ such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ holds
$\rho^{\mathbb{R}^2}(x, y) = \sqrt{\rho_{\mathbb{R}}\,(x_1, y_1)^{\mathbf{2}} + \rho_{\mathbb{R}}\,(x_2, y_2)^{\mathbf{2}}}$.

We now state three propositions:

(22) For all elements x_1, x_2, y_1, y_2 of $\mathbb{R}$ and for all elements x, y of $[:\mathbb{R}, \mathbb{R}:]$ such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ holds $\rho^{\mathbb{R}^2}(x, y) = 0$ if and only if $x = y$.

(23) For all elements x, y of $[:\mathbb{R}, \mathbb{R}:]$ such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ holds $\rho^{\mathbb{R}^2}(x, y) = \rho^{\mathbb{R}^2}(y, x)$.

(24) For all elements x, y, z of $[:\mathbb{R}, \mathbb{R}:]$ such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ and $z = \langle z_1, z_2\rangle$ holds $\rho^{\mathbb{R}^2}(x, z) \leq \rho^{\mathbb{R}^2}(x, y) + \rho^{\mathbb{R}^2}(y, z)$.

The Euclidean plain being a metric space is defined as follows:

(Def.10) the Euclidean plain$= \langle [:\mathbb{R}, \mathbb{R}:], \rho^{\mathbb{R}^2}\rangle$.

In the sequel x_3, y_3, z_3 denote elements of $\mathbb{R}$. The function $\rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}$ from $[:[:\mathbb{R}, \mathbb{R}, \mathbb{R}:], [:\mathbb{R}, \mathbb{R}, \mathbb{R}:]:]$ into $\mathbb{R}$ is defined by the condition (Def.11).

(Def.11) Let x_1, y_1, x_2, y_2, x_3, y_3 be elements of $\mathbb{R}$. Then for all elements x, y of $[:\mathbb{R}, \mathbb{R}, \mathbb{R}:]$ such that $x = \langle x_1, x_2, x_3\rangle$ and $y = \langle y_1, y_2, y_3\rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(x, y) = \rho_{\mathbb{R}}\,(x_1, y_1) + \rho_{\mathbb{R}}\,(x_2, y_2) + \rho_{\mathbb{R}}\,(x_3, y_3)$.

We now state three propositions:

(25) For all elements x_1, x_2, y_1, y_2, x_3, y_3 of $\mathbb{R}$ and for all elements x, y of $[:\mathbb{R}, \mathbb{R}, \mathbb{R}:]$ such that $x = \langle x_1, x_2, x_3\rangle$ and $y = \langle y_1, y_2, y_3\rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(x, y) = 0$ if and only if $x = y$.

(26) For all elements x, y of $[:\mathbb{R}, \mathbb{R}, \mathbb{R}:]$ such that $x = \langle x_1, x_2, x_3\rangle$ and $y = \langle y_1, y_2, y_3\rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(x, y) = \rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(y, x)$.

(27) For all elements x, y, z of $[:\mathbb{R}, \mathbb{R}, \mathbb{R}:]$ such that $x = \langle x_1, x_2, x_3\rangle$ and $y = \langle y_1, y_2, y_3\rangle$ and $z = \langle z_1, z_2, z_3\rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(x, z) \leq \rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(x, y) + \rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(y, z)$.

The metric space $[:\mathbb{R}_{\mathrm{M}}, \mathbb{R}_{\mathrm{M}}, \mathbb{R}_{\mathrm{M}}:]$ is defined as follows:

(Def.12) $[:\mathbb{R}_{\mathrm{M}}, \mathbb{R}_{\mathrm{M}}, \mathbb{R}_{\mathrm{M}}:] = \langle [:\mathbb{R}, \mathbb{R}, \mathbb{R}:], \rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}\rangle$.

The function $\rho^{\mathbb{R}^3}$ from $[\![[\![\mathbb{R}, \mathbb{R}, \mathbb{R}]\!], [\![\mathbb{R}, \mathbb{R}, \mathbb{R}]\!]]\!]$ into $\mathbb{R}$ is defined by the condition (Def.13).

(Def.13) Let x_1, y_1, x_2, y_2, x_3, y_3 be elements of $\mathbb{R}$. Then for all elements x, y of $[\![\mathbb{R}, \mathbb{R}, \mathbb{R}]\!]$ such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $\rho^{\mathbb{R}^3}(x, y) = \sqrt{\rho_{\mathbb{R}}(x_1, y_1)^{\mathbf{2}} + \rho_{\mathbb{R}}(x_2, y_2)^{\mathbf{2}} + \rho_{\mathbb{R}}(x_3, y_3)^{\mathbf{2}}}$.

One can prove the following three propositions:

(28) For all elements x_1, x_2, y_1, y_2, x_3, y_3 of $\mathbb{R}$ and for all elements x, y of $[\![\mathbb{R}, \mathbb{R}, \mathbb{R}]\!]$ such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $\rho^{\mathbb{R}^3}(x, y) = 0$ if and only if $x = y$.

(29) For all elements x, y of $[\![\mathbb{R}, \mathbb{R}, \mathbb{R}]\!]$ such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $\rho^{\mathbb{R}^3}(x, y) = \rho^{\mathbb{R}^3}(y, x)$.

(30) For all elements x, y, z of $[\![\mathbb{R}, \mathbb{R}, \mathbb{R}]\!]$ such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ and $z = \langle z_1, z_2, z_3 \rangle$ holds $\rho^{\mathbb{R}^3}(x, z) \leq \rho^{\mathbb{R}^3}(x, y) + \rho^{\mathbb{R}^3}(y, z)$.

The Euclidean space being a metric space is defined as follows:

(Def.14) the Euclidean space= $\langle [\![\mathbb{R}, \mathbb{R}, \mathbb{R}]\!], \rho^{\mathbb{R}^3} \rangle$.

References

[1] Grzegorz Bancerek. Curried and uncurried functions. *Formalized Mathematics*, 1(**3**):537–541, 1990.

[2] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.

[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.

[7] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[8] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.

[9] Stanisława Kanas and Jan Stankiewicz. Metrics in Cartesian product. *Formalized Mathematics*, 2(**2**):193–197, 1991.

[10] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[11] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[12] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[13] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

[14] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[15] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

Received July 8, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Fix Point Theorem for Compact Spaces

Alicia de la Cruz
Universidad Politecnica de Madrid

Summary. The Banach theorem in a compact metric spaces is proved.

MML Identifier: `ALI2`.

The terminology and notation used in this paper have been introduced in the following papers: [9], [15], [3], [4], [8], [11], [13], [9], [11], [5], [7], [18], [6], [17], [1], [2], [6], [4], and [5]. In the sequel M will be a metric space. Next we state the proposition

(1) For every set F such that F is finite and $F \neq \emptyset$ and for all sets B, C such that $B \in F$ and $C \in F$ holds $B \subseteq C$ or $C \subseteq B$ there exists a set m such that $m \in F$ and for every set C such that $C \in F$ holds $m \subseteq C$.

Let M be a metric space. A function from the carrier of M into the carrier of M is said to be a contraction of M if:

(Def.1) there exists a real number L such that $0 < L$ and $L < 1$ and for all points x, y of M holds $\rho(\mathrm{it}(x), \mathrm{it}(y)) \leq L \cdot \rho(x, y)$.

Next we state the proposition

(2) For every contraction f of M such that M_{top} is compact there exists a point c of M such that $f(c) = c$ and for every point x of M such that $f(x) = x$ holds $x = c$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.
[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

ISSN 0777–4028

[5] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.
[6] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[7] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[9] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.
[10] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[11] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.
[12] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[13] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[14] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[15] Konrad Raczkowski and Andrzej Nędzusiak. Real exponents and logarithms. *Formalized Mathematics*, 2(**2**):213–216, 1991.
[16] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[18] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[19] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

Received July 17, 1990

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Quadratic Inequalities

Jan Popiołek
Warsaw University
Białystok

Summary. Consider a quadratic trinomial of the form $P(x) = ax^2 + bx + c$, where $a \neq 0$. The determinat of the equation $P(x) = 0$ is of the form $\Delta(a,b,c) = b^2 - 4ac$. We prove several quadratic inequalities when $\Delta(a,b,c) < 0$, $\Delta(a,b,c) = 0$ and $\Delta(a,b,c) > 0$.

MML Identifier: `QUIN_1`.

The articles [3], [1], [2], and [4] provide the terminology and notation for this paper. In the sequel x is a real number and a, b, c are real numbers. Let us consider a, b, c. The functor $\Delta(a,b,c)$ yielding a real number is defined as follows:

(Def.1) $\Delta(a,b,c) = b^{\mathbf{2}} - 4 \cdot a \cdot c$.

The following propositions are true:

(1) If $a \neq 0$, then $a \cdot x^{\mathbf{2}} + b \cdot x + c = a \cdot (x + \frac{b}{2 \cdot a})^{\mathbf{2}} - \frac{\Delta(a,b,c)}{4 \cdot a}$.

(2) If $a > 0$ and $\Delta(a,b,c) \leq 0$, then $a \cdot x^{\mathbf{2}} + b \cdot x + c \geq 0$.

(3) If $a > 0$ and $\Delta(a,b,c) < 0$, then $a \cdot x^{\mathbf{2}} + b \cdot x + c > 0$.

(4) If $a < 0$ and $\Delta(a,b,c) \leq 0$, then $a \cdot x^{\mathbf{2}} + b \cdot x + c \leq 0$.

(5) If $a < 0$ and $\Delta(a,b,c) < 0$, then $a \cdot x^{\mathbf{2}} + b \cdot x + c < 0$.

(6) If $a > 0$ and $a \cdot x^{\mathbf{2}} + b \cdot x + c \geq 0$, then $(2 \cdot a \cdot x + b)^{\mathbf{2}} - \Delta(a,b,c) \geq 0$.

(7) If $a > 0$ and $a \cdot x^{\mathbf{2}} + b \cdot x + c > 0$, then $(2 \cdot a \cdot x + b)^{\mathbf{2}} - \Delta(a,b,c) > 0$.

(8) If $a < 0$ and $a \cdot x^{\mathbf{2}} + b \cdot x + c \leq 0$, then $(2 \cdot a \cdot x + b)^{\mathbf{2}} - \Delta(a,b,c) \geq 0$.

(9) If $a < 0$ and $a \cdot x^{\mathbf{2}} + b \cdot x + c < 0$, then $(2 \cdot a \cdot x + b)^{\mathbf{2}} - \Delta(a,b,c) > 0$.

(10) If for every x holds $a \cdot x^{\mathbf{2}} + b \cdot x + c \geq 0$ and $a > 0$, then $\Delta(a,b,c) \leq 0$.

(11) If for every x holds $a \cdot x^{\mathbf{2}} + b \cdot x + c \leq 0$ and $a < 0$, then $\Delta(a,b,c) \leq 0$.

(12) If for every x holds $a \cdot x^{\mathbf{2}} + b \cdot x + c > 0$ and $a > 0$, then $\Delta(a,b,c) < 0$.

(13) If for every x holds $a \cdot x^{\mathbf{2}} + b \cdot x + c < 0$ and $a < 0$, then $\Delta(a,b,c) < 0$.

(14) If $a \neq 0$ and $a \cdot x^{\mathbf{2}} + b \cdot x + c = 0$, then $(2 \cdot a \cdot x + b)^{\mathbf{2}} - \Delta(a,b,c) = 0$.

ISSN 0777–4028

(15) Suppose $a \neq 0$ and $\Delta(a,b,c) > 0$ and $a \cdot x^{\mathbf{2}} + b \cdot x + c = 0$. Then $x = \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $x = \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(16) Suppose $a \neq 0$ and $\Delta(a,b,c) > 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c = a \cdot (x - \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}) \cdot (x - \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a})$.

(17) If $a < 0$ and $\Delta(a,b,c) > 0$, then $\frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a} < \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(18) Suppose $a < 0$ and $\Delta(a,b,c) > 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c > 0$ if and only if $\frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a} < x$ and $x < \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(19) Suppose $a < 0$ and $\Delta(a,b,c) > 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c < 0$ if and only if $x < \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $x > \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(20) Suppose $a < 0$ and $\Delta(a,b,c) > 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c \geq 0$ if and only if $\frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a} \leq x$ and $x \leq \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(21) Suppose $a < 0$ and $\Delta(a,b,c) > 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c \leq 0$ if and only if $x \leq \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $x \geq \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(22) If $a \neq 0$ and $\Delta(a,b,c) = 0$ and $a \cdot x^{\mathbf{2}} + b \cdot x + c = 0$, then $x = -\frac{b}{2 \cdot a}$.

(23) If $a > 0$ and $(2 \cdot a \cdot x + b)^{\mathbf{2}} - \Delta(a,b,c) > 0$, then $a \cdot x^{\mathbf{2}} + b \cdot x + c > 0$.

(24) If $a > 0$ and $\Delta(a,b,c) = 0$, then $a \cdot x^{\mathbf{2}} + b \cdot x + c > 0$ if and only if $x \neq -\frac{b}{2 \cdot a}$.

(25) If $a < 0$ and $(2 \cdot a \cdot x + b)^{\mathbf{2}} - \Delta(a,b,c) > 0$, then $a \cdot x^{\mathbf{2}} + b \cdot x + c < 0$.

(26) If $a < 0$ and $\Delta(a,b,c) = 0$, then $a \cdot x^{\mathbf{2}} + b \cdot x + c < 0$ if and only if $x \neq -\frac{b}{2 \cdot a}$.

(27) If $a > 0$ and $\Delta(a,b,c) > 0$, then $\frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a} > \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(28) Suppose $a > 0$ and $\Delta(a,b,c) > 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c < 0$ if and only if $\frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a} < x$ and $x < \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(29) Suppose $a > 0$ and $\Delta(a,b,c) > 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c > 0$ if and only if $x < \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $x > \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(30) Suppose $a > 0$ and $\Delta(a,b,c) > 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c \leq 0$ if and only if $\frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a} \leq x$ and $x \leq \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(31) Suppose $a > 0$ and $\Delta(a,b,c) > 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c \geq 0$ if and only if $x \leq \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $x \geq \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

References

[1] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[2] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[3] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[4] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

Received July 19, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Introduction to Banach and Hilbert Spaces - Part I

Jan Popiołek
Warsaw University
Białystok

Summary. Basing on the notion of real linear space (see [15]) we introduce real unitary space. At first, we define the scalar product of two vectors and examine some of its properties. On the basis of this notion we introduce the norm and the distance in real unitary space and study the properties of these concepts. Next, proceeding from the definition of the sequence in real unitary space and basic operations on sequences we prove several theorems which will be used in our further considerations.

MML Identifier: BHSP_1.

The terminology and notation used here are introduced in the following articles: [5], [12], [16], [3], [4], [1], [6], [2], [17], [10], [11], [9], [15], [14], [13], [8], and [7]. We consider unitary space structures which are systems

$\langle$vectors, a scalar product$\rangle$,

where the vectors constitute a real linear space and the scalar product is a function from $[\!:$ the vectors of the vectors, the vectors of the vectors $:\!]$ into $\mathbb{R}$.

In the sequel X will denote a unitary space structure and a, b will denote real numbers. Let us consider X. A point of X is an element of the vectors of the vectors of X.

In the sequel x, y will denote points of X. Let us consider X, x, y. The functor $(x|y)$ yielding a real number is defined as follows:

(Def.1) $(x|y) = (\text{the scalar product of } X)(\langle x, y\rangle)$.

A unitary space structure is said to be a real unitary space if it satisfies the condition (Def.2).

(Def.2) Let x, y, z be points of it. Given a. Then

(i) $(x|x) = 0$ if and only if $x = 0_{\text{the vectors of it}}$,

(ii) $0 \leq (x|x)$,

(iii) $(x|y) = (y|x)$,

ISSN 0777–4028

(iv) $((x+y)|z) = (x|z) + (y|z)$,
(v) $((a \cdot x)|y) = a \cdot (x|y)$.

We follow the rules: X denotes a real unitary space and x, y, z, u, v denote points of X. We now state a number of propositions:

(1) $(x|x) = 0$ if and only if $x = 0_{\text{the vectors of } X}$.
(2) $0 \leq (x|x)$.
(3) $(x|y) = (y|x)$.
(4) $((x+y)|z) = (x|z) + (y|z)$.
(5) $((a \cdot x)|y) = a \cdot (x|y)$.
(6) $(0_{\text{the vectors of } X}|0_{\text{the vectors of } X}) = 0$.
(7) $(x|(y+z)) = (x|y) + (x|z)$.
(8) $(x|(a \cdot y)) = a \cdot (x|y)$.
(9) $((a \cdot x)|y) = (x|(a \cdot y))$.
(10) $((a \cdot x + b \cdot y)|z) = a \cdot (x|z) + b \cdot (y|z)$.
(11) $(x|(a \cdot y + b \cdot z)) = a \cdot (x|y) + b \cdot (x|z)$.
(12) $((-x)|y) = (x|-y)$.
(13) $((-x)|y) = -(x|y)$.
(14) $(x|-y) = -(x|y)$.
(15) $((-x)|-y) = (x|y)$.
(16) $((x-y)|z) = (x|z) - (y|z)$.
(17) $(x|(y-z)) = (x|y) - (x|z)$.
(18) $((x-y)|(u-v)) = ((x|u) - (x|v) - (y|u)) + (y|v)$.
(19) $(0_{\text{the vectors of } X}|x) = 0$.
(20) $(x|0_{\text{the vectors of } X}) = 0$.
(21) $((x+y)|(x+y)) = (x|x) + 2 \cdot (x|y) + (y|y)$.
(22) $((x+y)|(x-y)) = (x|x) - (y|y)$.
(23) $((x-y)|(x-y)) = ((x|x) - 2 \cdot (x|y)) + (y|y)$.
(24) $|(x|y)| \leq \sqrt{(x|x)} \cdot \sqrt{(y|y)}$.

Let us consider X, x, y. We say that x, y are ortogonal if and only if:

(Def.3) $(x|y) = 0$.

The following propositions are true:

(25) If x, y are ortogonal, then y, x are ortogonal.
(26) If x, y are ortogonal, then x, $-y$ are ortogonal.
(27) If x, y are ortogonal, then $-x$, y are ortogonal.
(28) If x, y are ortogonal, then $-x$, $-y$ are ortogonal.
(29) x, $0_{\text{the vectors of } X}$ are ortogonal.
(30) If x, y are ortogonal, then $((x+y)|(x+y)) = (x|x) + (y|y)$.
(31) If x, y are ortogonal, then $((x-y)|(x-y)) = (x|x) + (y|y)$.

Let us consider X, x. The functor $\|x\|$ yielding a real number is defined by:

(Def.4) $\|x\| = \sqrt{(x|x)}$.

The following propositions are true:

(32) $\|x\| = 0$ if and only if $x = 0_{\text{the vectors of } X}$.

(33) $\|a \cdot x\| = |a| \cdot \|x\|$.

(34) $0 \leq \|x\|$.

(35) $|(x|y)| \leq \|x\| \cdot \|y\|$.

(36) $\|x + y\| \leq \|x\| + \|y\|$.

(37) $\|-x\| = \|x\|$.

(38) $\|x\| - \|y\| \leq \|x - y\|$.

(39) $|\|x\| - \|y\|| \leq \|x - y\|$.

Let us consider X, x, y. The functor $\rho(x, y)$ yielding a real number is defined by:

(Def.5) $\rho(x, y) = \|x - y\|$.

One can prove the following propositions:

(40) $\rho(x, y) = \rho(y, x)$.

(41) $\rho(x, x) = 0$.

(42) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

(43) $x \neq y$ if and only if $\rho(x, y) \neq 0$.

(44) $\rho(x, y) \geq 0$.

(45) $x \neq y$ if and only if $\rho(x, y) > 0$.

(46) $\rho(x, y) = \sqrt{((x - y)|(x - y))}$.

(47) $\rho(x + y, u + v) \leq \rho(x, u) + \rho(y, v)$.

(48) $\rho(x - y, u - v) \leq \rho(x, u) + \rho(y, v)$.

(49) $\rho(x - z, y - z) = \rho(x, y)$.

(50) $\rho(x - z, y - z) \leq \rho(z, x) + \rho(z, y)$.

Let us consider X. A subset of X is a subset of the vectors of the vectors of X.

Let us consider X. A function is called a sequence of X if:

(Def.6) $\operatorname{dom} \text{it} = \mathbb{N}$ and $\operatorname{rng} \text{it} \subseteq$ the vectors of the vectors of X.

For simplicity we adopt the following rules: s_1, s_2, s_3, s_4, s'_1 denote sequences of X, k, n, m denote natural numbers, f denotes a function, and d is arbitrary. We now state four propositions:

(51) f is a sequence of X if and only if $\operatorname{dom} f = \mathbb{N}$ and $\operatorname{rng} f \subseteq$ the vectors of the vectors of X.

(52) f is a sequence of X if and only if $\operatorname{dom} f = \mathbb{N}$ and for every d such that $d \in \mathbb{N}$ holds $f(d)$ is a point of X.

(53) For all s_1, s'_1 such that for every n holds $s_1(n) = s'_1(n)$ holds $s_1 = s'_1$.

(54) For every n holds $s_1(n)$ is a point of X.

Let us consider X, s_1, n. Then $s_1(n)$ is a point of X.

The scheme *Ex_Seq_in_RUS* concerns a real unitary space $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$ and states that:

there exists a sequence s_1 of $\mathcal{A}$ such that for every n holds $s_1(n) = \mathcal{F}(n)$

for all values of the parameters.

Let us consider X, s_2, s_3. The functor $s_2 + s_3$ yielding a sequence of X is defined by:

(Def.7) for every n holds $(s_2 + s_3)(n) = s_2(n) + s_3(n)$.

Let us consider X, s_2, s_3. The functor $s_2 - s_3$ yielding a sequence of X is defined as follows:

(Def.8) for every n holds $(s_2 - s_3)(n) = s_2(n) - s_3(n)$.

Let us consider X, s_1, a. The functor $a \cdot s_1$ yields a sequence of X and is defined as follows:

(Def.9) for every n holds $(a \cdot s_1)(n) = a \cdot s_1(n)$.

Let us consider X, s_1. The functor $-s_1$ yields a sequence of X and is defined by:

(Def.10) for every n holds $(-s_1)(n) = -s_1(n)$.

Let us consider X, s_1. We say that s_1 is constant if and only if:

(Def.11) there exists x such that for every n holds $s_1(n) = x$.

Let us consider X, s_1, x. The functor $s_1 + x$ yielding a sequence of X is defined as follows:

(Def.12) for every n holds $(s_1 + x)(n) = s_1(n) + x$.

Let us consider X, s_1, x. The functor $s_1 - x$ yields a sequence of X and is defined by:

(Def.13) for every n holds $(s_1 - x)(n) = s_1(n) - x$.

We now state a number of propositions:

(55) $s_2 + s_3 = s_3 + s_2$.

(56) $s_2 + (s_3 + s_4) = s_2 + s_3 + s_4$.

(57) If s_2 is constant and s_3 is constant and $s_1 = s_2 + s_3$, then s_1 is constant.

(58) If s_2 is constant and s_3 is constant and $s_1 = s_2 - s_3$, then s_1 is constant.

(59) If s_2 is constant and $s_1 = a \cdot s_2$, then s_1 is constant.

(60) For every x there exists s_1 such that $\operatorname{rng} s_1 = \{x\}$.

(61) There exists s_1 such that $\operatorname{rng} s_1 = \{0_{\text{the vectors of } X}\}$.

(62) If there exists x such that for every n holds $s_1(n) = x$, then there exists x such that $\operatorname{rng} s_1 = \{x\}$.

(63) If there exists x such that $\operatorname{rng} s_1 = \{x\}$, then for every n holds $s_1(n) = s_1(n + 1)$.

(64) If for every n holds $s_1(n) = s_1(n + 1)$, then for all n, k holds $s_1(n) = s_1(n + k)$.

(65) If for all n, k holds $s_1(n) = s_1(n+k)$, then for all n, m holds $s_1(n) = s_1(m)$.

(66) If for all n, m holds $s_1(n) = s_1(m)$, then there exists x such that for every n holds $s_1(n) = x$.

(67) s_1 is constant if and only if there exists x such that $\operatorname{rng} s_1 = \{x\}$.

(68) s_1 is constant if and only if for every n holds $s_1(n) = s_1(n+1)$.

(69) s_1 is constant if and only if for all n, k holds $s_1(n) = s_1(n+k)$.

(70) s_1 is constant if and only if for all n, m holds $s_1(n) = s_1(m)$.

(71) $s_2 - s_3 = s_2 + -s_3$.

(72) $s_1 = s_1 + 0_{\text{the vectors of } X}$.

(73) $a \cdot (s_2 + s_3) = a \cdot s_2 + a \cdot s_3$.

(74) $(a + b) \cdot s_1 = a \cdot s_1 + b \cdot s_1$.

(75) $a \cdot b \cdot s_1 = a \cdot (b \cdot s_1)$.

(76) $1 \cdot s_1 = s_1$.

(77) $(-1) \cdot s_1 = -s_1$.

(78) $s_1 - x = s_1 + -x$.

(79) $s_2 - s_3 = -(s_3 - s_2)$.

(80) $s_1 = s_1 - 0_{\text{the vectors of } X}$.

(81) $s_1 = --s_1$.

(82) $s_2 - (s_3 + s_4) = s_2 - s_3 - s_4$.

(83) $(s_2 + s_3) - s_4 = s_2 + (s_3 - s_4)$.

(84) $s_2 - (s_3 - s_4) = (s_2 - s_3) + s_4$.

(85) $a \cdot (s_2 - s_3) = a \cdot s_2 - a \cdot s_3$.

References

[1] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.

[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[6] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[7] Jan Popiołek. Quadratic inequalities. *Formalized Mathematics*, 2(**4**):507–509, 1991.

[8] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.

[9] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[10] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[11] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[13] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[14] Wojciech A. Trybulec. Subspaces and cosets of subspaces in real linear space. *Formalized Mathematics*, 1(**2**):297–301, 1990.
[15] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[16] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[17] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received July 19, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Introduction to Banach and Hilbert Spaces - Part II

Jan Popiołek
Warsaw University
Białystok

Summary. A continuation of [8]. It contains the definitions of the convergent sequence and the limit of the sequence. The convergence with respect to the norm and the distance is also introduced. Last part is devoted to the following concepts: ball, closed ball and sphere.

MML Identifier: `BHSP_2`.

The articles [5], [14], [19], [3], [4], [1], [7], [6], [2], [20], [12], [18], [13], [11], [17], [16], [15], [10], [9], and [8] provide the notation and terminology for this paper. For simplicity we follow a convention: X is a real unitary space, x, y, z are points of X, g, g_1, g_2 are points of X, a, q, r are real numbers, s_1, s_2, s_3, s'_1 are sequences of X, and k, n, m are natural numbers. Let us consider X, s_1. We say that s_1 is convergent if and only if:

(Def.1) there exists g such that for every r such that $r > 0$ there exists m such that for every n such that $n \geq m$ holds $\rho(s_1(n), g) < r$.

The following propositions are true:

(1) If s_1 is constant, then s_1 is convergent.

(2) If s_1 is convergent and there exists k such that for every n such that $k \leq n$ holds $s'_1(n) = s_1(n)$, then s'_1 is convergent.

(3) If s_2 is convergent and s_3 is convergent, then $s_2 + s_3$ is convergent.

(4) If s_2 is convergent and s_3 is convergent, then $s_2 - s_3$ is convergent.

(5) If s_1 is convergent, then $a \cdot s_1$ is convergent.

(6) If s_1 is convergent, then $-s_1$ is convergent.

(7) If s_1 is convergent, then $s_1 + x$ is convergent.

(8) If s_1 is convergent, then $s_1 - x$ is convergent.

ISSN 0777–4028

(9) s_1 is convergent if and only if there exists g such that for every r such that $r > 0$ there exists m such that for every n such that $n \geq m$ holds $\|s_1(n) - g\| < r$.

Let us consider X, s_1. Let us assume that s_1 is convergent. The functor $\lim s_1$ yields a point of X and is defined as follows:

(Def.2) for every r such that $r > 0$ there exists m such that for every n such that $n \geq m$ holds $\rho(s_1(n), \lim s_1) < r$.

Next we state a number of propositions:

(10) If s_1 is constant and $x \in \operatorname{rng} s_1$, then $\lim s_1 = x$.

(11) If s_1 is constant and there exists n such that $s_1(n) = x$, then $\lim s_1 = x$.

(12) If s_1 is convergent and there exists k such that for every n such that $n \geq k$ holds $s'_1(n) = s_1(n)$, then $\lim s_1 = \lim s'_1$.

(13) If s_2 is convergent and s_3 is convergent, then $\lim(s_2 + s_3) = \lim s_2 + \lim s_3$.

(14) If s_2 is convergent and s_3 is convergent, then $\lim(s_2 - s_3) = \lim s_2 - \lim s_3$.

(15) If s_1 is convergent, then $\lim(a \cdot s_1) = a \cdot \lim s_1$.

(16) If s_1 is convergent, then $\lim(-s_1) = -\lim s_1$.

(17) If s_1 is convergent, then $\lim(s_1 + x) = \lim s_1 + x$.

(18) If s_1 is convergent, then $\lim(s_1 - x) = \lim s_1 - x$.

(19) If s_1 is convergent, then $\lim s_1 = g$ if and only if for every r such that $r > 0$ there exists m such that for every n such that $n \geq m$ holds $\|s_1(n) - g\| < r$.

Let us consider X, s_1. The functor $\|s_1\|$ yielding a sequence of real numbers is defined by:

(Def.3) for every n holds $\|s_1\|(n) = \|s_1(n)\|$.

Next we state three propositions:

(20) If s_1 is convergent, then $\|s_1\|$ is convergent.

(21) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1\|$ is convergent and $\lim\|s_1\| = \|g\|$.

(22) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1 - g\|$ is convergent and $\lim\|s_1 - g\| = 0$.

Let us consider X, s_1, x. The functor $\rho(s_1, x)$ yielding a sequence of real numbers is defined by:

(Def.4) for every n holds $(\rho(s_1, x))(n) = \rho(s_1(n), x)$.

We now state a number of propositions:

(23) If s_1 is convergent and $\lim s_1 = g$, then $\rho(s_1, g)$ is convergent.

(24) If s_1 is convergent and $\lim s_1 = g$, then $\rho(s_1, g)$ is convergent and $\lim \rho(s_1, g) = 0$.

(25) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|s_2 + s_3\|$ is convergent and $\lim \|s_2 + s_3\| = \|g_1 + g_2\|$.

(26) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|(s_2+s_3)-(g_1+g_2)\|$ is convergent and $\lim \|(s_2+s_3)-(g_1+g_2)\| = 0$.

(27) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|s_2 - s_3\|$ is convergent and $\lim \|s_2 - s_3\| = \|g_1 - g_2\|$.

(28) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|s_2 - s_3 - (g_1 - g_2)\|$ is convergent and $\lim \|s_2 - s_3 - (g_1 - g_2)\| = 0$.

(29) If s_1 is convergent and $\lim s_1 = g$, then $\|a \cdot s_1\|$ is convergent and $\lim \|a \cdot s_1\| = \|a \cdot g\|$.

(30) If s_1 is convergent and $\lim s_1 = g$, then $\|a \cdot s_1 - a \cdot g\|$ is convergent and $\lim \|a \cdot s_1 - a \cdot g\| = 0$.

(31) If s_1 is convergent and $\lim s_1 = g$, then $\|-s_1\|$ is convergent and $\lim \|-s_1\| = \|-g\|$.

(32) If s_1 is convergent and $\lim s_1 = g$, then $\|-s_1 - -g\|$ is convergent and $\lim \|-s_1 - -g\| = 0$.

(33) If s_1 is convergent and $\lim s_1 = g$, then $\|(s_1+x)-(g+x)\|$ is convergent and $\lim \|(s_1 + x) - (g + x)\| = 0$.

(34) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1 - x\|$ is convergent and $\lim \|s_1 - x\| = \|g - x\|$.

(35) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1 - x - (g - x)\|$ is convergent and $\lim \|s_1 - x - (g - x)\| = 0$.

(36) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\rho(s_2 + s_3, g_1 + g_2)$ is convergent and $\lim \rho(s_2 + s_3, g_1 + g_2) = 0$.

(37) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\rho(s_2 - s_3, g_1 - g_2)$ is convergent and $\lim \rho(s_2 - s_3, g_1 - g_2) = 0$.

(38) If s_1 is convergent and $\lim s_1 = g$, then $\rho(a \cdot s_1, a \cdot g)$ is convergent and $\lim \rho(a \cdot s_1, a \cdot g) = 0$.

(39) If s_1 is convergent and $\lim s_1 = g$, then $\rho(s_1 + x, g + x)$ is convergent and $\lim \rho(s_1 + x, g + x) = 0$.

Let us consider X, x, r. Let us assume that $r \geq 0$. The functor $\mathrm{Ball}(x, r)$ yielding a subset of X is defined by:

(Def.5) $\mathrm{Ball}(x, r) = \{y : \|x - y\| < r\}$, where y ranges over points of X.

Let us consider X, x, r. Let us assume that $r \geq 0$. The functor $\overline{\mathrm{Ball}}(x, r)$ yielding a subset of X is defined by:

(Def.6) $\overline{\mathrm{Ball}}(x, r) = \{y : \|x - y\| \leq r\}$, where y ranges over points of X.

Let us consider X, x, r. Let us assume that $r \geq 0$. The functor $\mathrm{Sphere}(x, r)$ yields a subset of X and is defined as follows:

(Def.7) $\mathrm{Sphere}(x, r) = \{y : \|x - y\| = r\}$, where y ranges over points of X.

The following propositions are true:

(40) If $r \geq 0$, then $z \in \mathrm{Ball}(x, r)$ if and only if $\|x - z\| < r$.

(41) If $r \geq 0$, then $z \in \mathrm{Ball}(x, r)$ if and only if $\rho(x, z) < r$.
(42) If $r > 0$, then $x \in \mathrm{Ball}(x, r)$.
(43) If $r \geq 0$, then if $y \in \mathrm{Ball}(x, r)$ and $z \in \mathrm{Ball}(x, r)$, then $\rho(y, z) < 2 \cdot r$.
(44) If $r \geq 0$, then if $y \in \mathrm{Ball}(x, r)$, then $y - z \in \mathrm{Ball}(x - z, r)$.
(45) If $r \geq 0$, then if $y \in \mathrm{Ball}(x, r)$, then $y - x \in \mathrm{Ball}(0_{\text{the vectors of } X}, r)$.
(46) If $r \geq 0$, then if $y \in \mathrm{Ball}(x, r)$ and $r \leq q$, then $y \in \mathrm{Ball}(x, q)$.
(47) If $r \geq 0$, then $z \in \overline{\mathrm{Ball}}(x, r)$ if and only if $\|x - z\| \leq r$.
(48) If $r \geq 0$, then $z \in \overline{\mathrm{Ball}}(x, r)$ if and only if $\rho(x, z) \leq r$.
(49) If $r \geq 0$, then $x \in \overline{\mathrm{Ball}}(x, r)$.
(50) If $r \geq 0$, then if $y \in \mathrm{Ball}(x, r)$, then $y \in \overline{\mathrm{Ball}}(x, r)$.
(51) If $r \geq 0$, then $z \in \mathrm{Sphere}(x, r)$ if and only if $\|x - z\| = r$.
(52) If $r \geq 0$, then $z \in \mathrm{Sphere}(x, r)$ if and only if $\rho(x, z) = r$.
(53) If $r \geq 0$, then if $y \in \mathrm{Sphere}(x, r)$, then $y \in \overline{\mathrm{Ball}}(x, r)$.
(54) If $r \geq 0$, then $\mathrm{Ball}(x, r) \subseteq \overline{\mathrm{Ball}}(x, r)$.
(55) If $r \geq 0$, then $\mathrm{Sphere}(x, r) \subseteq \overline{\mathrm{Ball}}(x, r)$.
(56) If $r \geq 0$, then $\mathrm{Ball}(x, r) \cup \mathrm{Sphere}(x, r) = \overline{\mathrm{Ball}}(x, r)$.

References

[1] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.
[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[5] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[6] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[7] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[8] Jan Popiołek. Introduction to Banach and Hilbert spaces - part I. *Formalized Mathematics*, 2(**4**):511–516, 1991.
[9] Jan Popiołek. Quadratic inequalities. *Formalized Mathematics*, 2(**4**):507–509, 1991.
[10] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.
[11] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.
[12] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[13] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[14] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[15] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[16] Wojciech A. Trybulec. Subspaces and cosets of subspaces in real linear space. *Formalized Mathematics*, 1(**2**):297–301, 1990.
[17] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[18] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[19] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[20] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received July 19, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Introduction to Banach and Hilbert Spaces - Part III

Jan Popiołek
Warsaw University
Białystok

Summary. A continuation of [11] and of [12]. First we define the following concepts: the Cauchy sequence, the bounded sequence and the subsequence. The last part consists definitions of the complete space and the Hilbert space.

MML Identifier: BHSP_3.

The articles [5], [18], [22], [3], [4], [1], [10], [8], [9], [7], [15], [2], [23], [16], [17], [14], [21], [20], [19], [13], [11], [12], and [6] provide the notation and terminology for this paper. For simplicity we follow the rules: X is a real unitary space, x is a point of X, g is a point of X, a, r are real numbers, M is a real number, s_1, s_2, s_3, s_4 are sequences of X, N_1 is an increasing sequence of naturals, and k, n, m are natural numbers. Let us consider X, s_1. We say that s_1 is a Cauchy sequence if and only if:

(Def.1) for every r such that $r > 0$ there exists k such that for all n, m such that $n \geq k$ and $m \geq k$ holds $\rho(s_1(n), s_1(m)) < r$.

One can prove the following propositions:

(1) If s_1 is constant, then s_1 is a Cauchy sequence.

(2) s_1 is a Cauchy sequence if and only if for every r such that $r > 0$ there exists k such that for all n, m such that $n \geq k$ and $m \geq k$ holds $\|s_1(n) - s_1(m)\| < r$.

(3) If s_2 is a Cauchy sequence and s_3 is a Cauchy sequence, then $s_2 + s_3$ is a Cauchy sequence.

(4) If s_2 is a Cauchy sequence and s_3 is a Cauchy sequence, then $s_2 - s_3$ is a Cauchy sequence.

(5) If s_1 is a Cauchy sequence, then $a \cdot s_1$ is a Cauchy sequence.

(6) If s_1 is a Cauchy sequence, then $-s_1$ is a Cauchy sequence.

ISSN 0777–4028

(7) If s_1 is a Cauchy sequence, then $s_1 + x$ is a Cauchy sequence.

(8) If s_1 is a Cauchy sequence, then $s_1 - x$ is a Cauchy sequence.

(9) If s_1 is convergent, then s_1 is a Cauchy sequence.

Let us consider X, s_2, s_3. We say that s_2 is compared to s_3 if and only if:

(Def.2) for every r such that $r > 0$ there exists m such that for every n such that $n \geq m$ holds $\rho(s_2(n), s_3(n)) < r$.

One can prove the following propositions:

(10) s_1 is compared to s_1.

(11) If s_2 is compared to s_3, then s_3 is compared to s_2.

(12) If s_2 is compared to s_3 and s_3 is compared to s_4, then s_2 is compared to s_4.

(13) s_2 is compared to s_3 if and only if for every r such that $r > 0$ there exists m such that for every n such that $n \geq m$ holds $\|s_2(n) - s_3(n)\| < r$.

(14) If there exists k such that for every n such that $n \geq k$ holds $s_2(n) = s_3(n)$, then s_2 is compared to s_3.

(15) If s_2 is a Cauchy sequence and s_2 is compared to s_3, then s_3 is a Cauchy sequence.

(16) If s_2 is convergent and s_2 is compared to s_3, then s_3 is convergent.

(17) If s_2 is convergent and $\lim s_2 = g$ and s_2 is compared to s_3, then s_3 is convergent and $\lim s_3 = g$.

Let us consider X, s_1. We say that s_1 is bounded if and only if:

(Def.3) there exists M such that $M > 0$ and for every n holds $\|s_1(n)\| \leq M$.

One can prove the following propositions:

(18) If s_2 is bounded and s_3 is bounded, then $s_2 + s_3$ is bounded.

(19) If s_1 is bounded, then $-s_1$ is bounded.

(20) If s_2 is bounded and s_3 is bounded, then $s_2 - s_3$ is bounded.

(21) If s_1 is bounded, then $a \cdot s_1$ is bounded.

(22) If s_1 is constant, then s_1 is bounded.

(23) For every m there exists M such that $M > 0$ and for every n such that $n \leq m$ holds $\|s_1(n)\| < M$.

(24) If s_1 is convergent, then s_1 is bounded.

(25) If s_2 is bounded and s_2 is compared to s_3, then s_3 is bounded.

Let us consider X, N_1, s_1. Then $s_1 \cdot N_1$ is a sequence of X.

Let us consider X, s_2, s_1. We say that s_2 is a subsequence of s_1 if and only if:

(Def.4) there exists N_1 such that $s_2 = s_1 \cdot N_1$.

One can prove the following propositions:

(26) For every n holds $(s_1 \cdot N_1)(n) = s_1(N_1(n))$.

(27) s_1 is a subsequence of s_1.

(28) If s_2 is a subsequence of s_3 and s_3 is a subsequence of s_4, then s_2 is a subsequence of s_4.

(29) If s_1 is constant and s_2 is a subsequence of s_1, then s_2 is constant.

(30) If s_1 is constant and s_2 is a subsequence of s_1, then $s_1 = s_2$.

(31) If s_1 is bounded and s_2 is a subsequence of s_1, then s_2 is bounded.

(32) If s_1 is convergent and s_2 is a subsequence of s_1, then s_2 is convergent.

(33) If s_2 is a subsequence of s_1 and s_1 is convergent, then $\lim s_2 = \lim s_1$.

(34) If s_1 is a Cauchy sequence and s_2 is a subsequence of s_1, then s_2 is a Cauchy sequence.

Let us consider X, s_1, k. The functor $s_1 \uparrow k$ yields a sequence of X and is defined by:

(Def.5) for every n holds $(s_1 \uparrow k)(n) = s_1(n + k)$.

The following propositions are true:

(35) $s_1 \uparrow 0 = s_1$.

(36) $s_1 \uparrow k \uparrow m = s_1 \uparrow m \uparrow k$.

(37) $s_1 \uparrow k \uparrow m = s_1 \uparrow (k + m)$.

(38) $(s_2 + s_3) \uparrow k = s_2 \uparrow k + s_3 \uparrow k$.

(39) $(-s_1) \uparrow k = -s_1 \uparrow k$.

(40) $(s_2 - s_3) \uparrow k = s_2 \uparrow k - s_3 \uparrow k$.

(41) $(a \cdot s_1) \uparrow k = a \cdot (s_1 \uparrow k)$.

(42) $(s_1 \cdot N_1) \uparrow k = s_1 \cdot (N_1 \uparrow k)$.

(43) $s_1 \uparrow k$ is a subsequence of s_1.

(44) If s_1 is convergent, then $s_1 \uparrow k$ is convergent and $\lim(s_1 \uparrow k) = \lim s_1$.

(45) If s_1 is convergent and there exists k such that $s_2 = s_1 \uparrow k$, then s_2 is convergent and $\lim s_2 = \lim s_1$.

(46) If s_1 is convergent and there exists k such that $s_1 = s_2 \uparrow k$, then s_2 is convergent.

(47) If s_1 is a Cauchy sequence and there exists k such that $s_1 = s_2 \uparrow k$, then s_2 is a Cauchy sequence.

(48) If s_1 is a Cauchy sequence, then $s_1 \uparrow k$ is a Cauchy sequence.

(49) If s_2 is compared to s_3, then $s_2 \uparrow k$ is compared to $s_3 \uparrow k$.

(50) If s_1 is bounded, then $s_1 \uparrow k$ is bounded.

(51) If s_1 is constant, then $s_1 \uparrow k$ is constant.

Let us consider X. We say that X is a complete space if and only if:

(Def.6) for every s_1 such that s_1 is a Cauchy sequence holds s_1 is convergent.

The following propositions are true:

(52) If X is a complete space and s_2 is a Cauchy sequence and s_2 is compared to s_3, then s_3 is a Cauchy sequence.

(53) If X is a complete space and s_1 is a Cauchy sequence, then s_1 is bounded.

Let us consider X. We say that X is a Hilbert space if and only if:

(Def.7) X is a real unitary space and X is a complete space.

References

[1] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.

[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[6] Krzysztof Hryniewiecki. Recursive definitions. *Formalized Mathematics*, 1(**2**):321–328, 1990.

[7] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.

[8] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.

[9] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.

[10] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[11] Jan Popiołek. Introduction to Banach and Hilbert spaces - part I. *Formalized Mathematics*, 2(**4**):511–516, 1991.

[12] Jan Popiołek. Introduction to Banach and Hilbert spaces - part II. *Formalized Mathematics*, 2(**4**):517–521, 1991.

[13] Jan Popiołek. Quadratic inequalities. *Formalized Mathematics*, 2(**4**):507–509, 1991.

[14] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[15] Konrad Raczkowski and Andrzej Nędzusiak. Serieses. *Formalized Mathematics*, 2(**4**):449–452, 1991.

[16] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[17] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[18] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[19] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[20] Wojciech A. Trybulec. Subspaces and cosets of subspaces in real linear space. *Formalized Mathematics*, 1(**2**):297–301, 1990.

[21] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[22] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[23] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received July 19, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Category Ens

Czesław Byliński
Warsaw University
Białystok

Summary. If V is any non-empty set of sets, we define $\mathbf{Ens}_V$ to be the category with the objects of all sets $X \in V$, morphisms of all mappings from X into Y, with the ususal composition of mappings. By a mapping we mean a triple $\langle X, Y, f\rangle$ where f is a function from X into Y. The notations and concepts included correspond to those presented in [11,9]. We also introduce representable functors to illustrate properties of the category **Ens**.

MML Identifier: `ENS_1`.

The notation and terminology used here are introduced in the following papers: [15], [16], [13], [2], [3], [7], [5], [1], [14], [10], [12], [4], [8], and [6].

Mappings

In the sequel V denotes a non-empty set and A, B denote elements of V. Let us consider V. The functor $\mathrm{Funcs}\, V$ yielding a non-empty set of functions is defined by:

(Def.1) $\mathrm{Funcs}\, V = \bigcup\{B^A\}$.

We now state three propositions:

(1) For an arbitrary f holds $f \in \mathrm{Funcs}\, V$ if and only if there exist A, B such that if $B = \emptyset$, then $A = \emptyset$ but f is a function from A into B.

(2) $B^A \subseteq \mathrm{Funcs}\, V$.

(3) For every non-empty subset W of V holds $\mathrm{Funcs}\, W \subseteq \mathrm{Funcs}\, V$.

In the sequel f is an element of $\mathrm{Funcs}\, V$. Let us consider V. The functor $\mathrm{Maps}\, V$ yielding a non-empty set is defined as follows:

(Def.2) $\mathrm{Maps}\, V = \{\langle\langle A, B\rangle, f\rangle : (B = \emptyset \Rightarrow A = \emptyset) \wedge f \text{ is a function from } A \text{ into } B\}$.

In the sequel m, m_1, m_2, m_3 are elements of $\mathrm{Maps}\, V$. One can prove the following four propositions:

ISSN 0777–4028

(4) There exist f, A, B such that $m = \langle\langle A, B\rangle, f\rangle$ but if $B = \emptyset$, then $A = \emptyset$ and f is a function from A into B.

(5) For every function f from A into B such that if $B = \emptyset$, then $A = \emptyset$ holds $\langle\langle A, B\rangle, f\rangle \in \mathrm{Maps}\, V$.

(6) $\mathrm{Maps}\, V \subseteq [\!:[\!: V, V :\!], \mathrm{Funcs}\, V :\!]$.

(7) For every non-empty subset W of V holds $\mathrm{Maps}\, W \subseteq \mathrm{Maps}\, V$.

We now define three new functors. Let us consider V, m. The functor $\mathrm{graph}(m)$ yields a function and is defined as follows:

(Def.3) $\mathrm{graph}(m) = m_{\mathbf{2}}$.

The functor $\mathrm{dom}\, m$ yields an element of V and is defined by:

(Def.4) $\mathrm{dom}\, m = (m_{\mathbf{1}})_{\mathbf{1}}$.

The functor $\mathrm{cod}\, m$ yielding an element of V is defined by:

(Def.5) $\mathrm{cod}\, m = (m_{\mathbf{1}})_{\mathbf{2}}$.

The following three propositions are true:

(8) $m = \langle\langle \mathrm{dom}\, m, \mathrm{cod}\, m\rangle, \mathrm{graph}(m)\rangle$.

(9) $\mathrm{cod}\, m \neq \emptyset$ or $\mathrm{dom}\, m = \emptyset$ but $\mathrm{graph}(m)$ is a function from $\mathrm{dom}\, m$ into $\mathrm{cod}\, m$.

(10) For every function f and for all sets A, B such that $\langle\langle A, B\rangle, f\rangle \in \mathrm{Maps}\, V$ holds if $B = \emptyset$, then $A = \emptyset$ but f is a function from A into B.

Let us consider V, A. The functor $\mathrm{id}(A)$ yields an element of $\mathrm{Maps}\, V$ and is defined by:

(Def.6) $\mathrm{id}(A) = \langle\langle A, A\rangle, \mathrm{id}_A\rangle$.

The following proposition is true

(11) $\mathrm{graph}(\mathrm{id}(A)) = \mathrm{id}_A$ and $\mathrm{dom}\,\mathrm{id}(A) = A$ and $\mathrm{cod}\,\mathrm{id}(A) = A$.

Let us consider V, m_1, m_2. Let us assume that $\mathrm{cod}\, m_1 = \mathrm{dom}\, m_2$. The functor $m_2 \cdot m_1$ yields an element of $\mathrm{Maps}\, V$ and is defined as follows:

(Def.7) $m_2 \cdot m_1 = \langle\langle \mathrm{dom}\, m_1, \mathrm{cod}\, m_2\rangle, \mathrm{graph}(m_2) \cdot \mathrm{graph}(m_1)\rangle$.

One can prove the following propositions:

(12) If $\mathrm{dom}\, m_2 = \mathrm{cod}\, m_1$, then $\mathrm{graph}((m_2 \cdot m_1)) = \mathrm{graph}(m_2) \cdot \mathrm{graph}(m_1)$ and $\mathrm{dom}(m_2 \cdot m_1) = \mathrm{dom}\, m_1$ and $\mathrm{cod}(m_2 \cdot m_1) = \mathrm{cod}\, m_2$.

(13) If $\mathrm{dom}\, m_2 = \mathrm{cod}\, m_1$ and $\mathrm{dom}\, m_3 = \mathrm{cod}\, m_2$, then $m_3 \cdot (m_2 \cdot m_1) = m_3 \cdot m_2 \cdot m_1$.

(14) $m \cdot \mathrm{id}(\mathrm{dom}\, m) = m$ and $\mathrm{id}(\mathrm{cod}\, m) \cdot m = m$.

Let us consider V, A, B. The functor $\mathrm{Maps}(A, B)$ yields a set and is defined by:

(Def.8) $\mathrm{Maps}(A, B) = \{\langle\langle A, B\rangle, f\rangle : \langle\langle A, B\rangle, f\rangle \in \mathrm{Maps}\, V\}$, where f ranges over elements of $\mathrm{Funcs}\, V$.

The following propositions are true:

(15) For every function f from A into B such that if $B = \emptyset$, then $A = \emptyset$ holds $\langle\langle A, B\rangle, f\rangle \in \mathrm{Maps}(A, B)$.

(16) If $m \in \mathrm{Maps}(A, B)$, then $m = \langle\langle A, B\rangle, \mathrm{graph}(m)\rangle$.

(17) $\mathrm{Maps}(A, B) \subseteq \mathrm{Maps}\, V$.

(18) $\mathrm{Maps}\, V = \bigcup\{\mathrm{Maps}(A, B)\}$.

(19) $m \in \mathrm{Maps}(A, B)$ if and only if $\mathrm{dom}\, m = A$ and $\mathrm{cod}\, m = B$.

(20) If $m \in \mathrm{Maps}(A, B)$, then $\mathrm{graph}(m) \in B^A$.

Let us consider V, m. We say that m is a surjection if and only if:

(Def.9) $\mathrm{rng}\,\mathrm{graph}(m) = \mathrm{cod}\, m$.

Category Ens

We now define four new functors. Let us consider V. The functor Dom_V yields a function from $\mathrm{Maps}\, V$ into V and is defined by:

(Def.10) for every m holds $\mathrm{Dom}_V(m) = \mathrm{dom}\, m$.

The functor Cod_V yields a function from $\mathrm{Maps}\, V$ into V and is defined as follows:

(Def.11) for every m holds $\mathrm{Cod}_V(m) = \mathrm{cod}\, m$.

The functor $\cdot_V$ yields a partial function from $[\!:\mathrm{Maps}\, V, \mathrm{Maps}\, V:\!]$ to $\mathrm{Maps}\, V$ and is defined as follows:

(Def.12) for all m_2, m_1 holds $\langle m_2, m_1\rangle \in \mathrm{dom}(\cdot_V)$ if and only if $\mathrm{dom}\, m_2 = \mathrm{cod}\, m_1$ and for all m_2, m_1 such that $\mathrm{dom}\, m_2 = \mathrm{cod}\, m_1$ holds $\cdot_V(\langle m_2, m_1\rangle) = m_2 \cdot m_1$.

The functor Id_V yields a function from V into $\mathrm{Maps}\, V$ and is defined by:

(Def.13) for every A holds $\mathrm{Id}_V(A) = \mathrm{id}(A)$.

Let us consider V. The functor $\mathbf{Ens}_V$ yields a category structure and is defined by:

(Def.14) $\mathbf{Ens}_V = \langle V, \mathrm{Maps}\, V, \mathrm{Dom}_V, \mathrm{Cod}_V, \cdot_V, \mathrm{Id}_V\rangle$.

We now state the proposition

(21) $\langle V, \mathrm{Maps}\, V, \mathrm{Dom}_V, \mathrm{Cod}_V, \cdot_V, \mathrm{Id}_V\rangle$ is a category.

Let us consider V. Then $\mathbf{Ens}_V$ is a category.

In the sequel a, b are objects of $\mathbf{Ens}_V$. Next we state the proposition

(22) A is an object of $\mathbf{Ens}_V$.

Let us consider V, A. The functor ${}^{@}A$ yielding an object of $\mathbf{Ens}_V$ is defined as follows:

(Def.15) ${}^{@}A = A$.

One can prove the following proposition

(23) a is an element of V.

Let us consider V, a. The functor ${}^{@}a$ yields an element of V and is defined by:

(Def.16) ${}^{@}a = a$.

In the sequel f, g denote morphisms of $\mathbf{Ens}_V$. The following proposition is true

(24) m is a morphism of $\mathbf{Ens}_V$.

Let us consider V, m. The functor $^@m$ yields a morphism of $\mathbf{Ens}_V$ and is defined as follows:

(Def.17) $^@m = m$.

One can prove the following proposition

(25) f is an element of $\mathrm{Maps}\,V$.

Let us consider V, f. The functor $^@f$ yields an element of $\mathrm{Maps}\,V$ and is defined as follows:

(Def.18) $^@f = f$.

One can prove the following propositions:

(26) $\mathrm{dom}\,f = \mathrm{dom}(^@f)$ and $\mathrm{cod}\,f = \mathrm{cod}(^@f)$.

(27) $\mathrm{hom}(a, b) = \mathrm{Maps}(^@a, {}^@b)$.

(28) If $\mathrm{dom}\,g = \mathrm{cod}\,f$, then $g \cdot f = (^@g) \cdot (^@f)$.

(29) $\mathrm{id}_a = \mathrm{id}(^@a)$.

(30) If $a = \emptyset$, then a is an initial object.

(31) If $\emptyset \in V$ and a is an initial object, then $a = \emptyset$.

(32) For every universal class W and for every object a of $\mathbf{Ens}_W$ such that a is an initial object holds $a = \emptyset$.

(33) If there exists arbitrary x such that $a = \{x\}$, then a is a terminal object.

(34) If $V \neq \{\emptyset\}$ and a is a terminal object, then there exists arbitrary x such that $a = \{x\}$.

(35) For every universal class W and for every object a of $\mathbf{Ens}_W$ such that a is a terminal object there exists arbitrary x such that $a = \{x\}$.

(36) f is monic if and only if $\mathrm{graph}((^@f))$ is one-to-one.

(37) If f is epi and there exists A and there exist arbitrary x_1, x_2 such that $x_1 \in A$ and $x_2 \in A$ and $x_1 \neq x_2$, then $^@f$ is a surjection.

(38) If $^@f$ is a surjection, then f is epi.

(39) For every universal class W and for every morphism f of $\mathbf{Ens}_W$ such that f is epi holds $^@f$ is a surjection.

(40) For every non-empty subset W of V holds $\mathbf{Ens}_W$ is full subcategory of $\mathbf{Ens}_V$.

Representable Functors

We follow a convention: C will be a category, a, b, c will be objects of C, and f, g, h, f', g' will be morphisms of C. Let us consider C. The functor $\mathrm{Hom}(C)$ yields a non-empty set and is defined as follows:

(Def.19) $\mathrm{Hom}(C) = \{\mathrm{hom}(a, b)\}$.

We now state two propositions:

(41) $\mathrm{hom}(a, b) \in \mathrm{Hom}(C)$.

(42) If $\hom(a, \operatorname{cod} f) = \emptyset$, then $\hom(a, \operatorname{dom} f) = \emptyset$ but if $\hom(\operatorname{dom} f, a) = \emptyset$, then $\hom(\operatorname{cod} f, a) = \emptyset$.

We now define two new functors. Let us consider C, a, f. The functor $\hom(a, f)$ yielding a function from $\hom(a, \operatorname{dom} f)$ into $\hom(a, \operatorname{cod} f)$ is defined by:

(Def.20) for every g such that $g \in \hom(a, \operatorname{dom} f)$ holds $(\hom(a, f))(g) = f \cdot g$.

The functor $\hom(f, a)$ yields a function from $\hom(\operatorname{cod} f, a)$ into $\hom(\operatorname{dom} f, a)$ and is defined by:

(Def.21) for every g such that $g \in \hom(\operatorname{cod} f, a)$ holds $(\hom(f, a))(g) = g \cdot f$.

We now state several propositions:

(43) $\hom(a, \mathrm{id}_c) = \mathrm{id}_{\hom(a,c)}$.

(44) $\hom(\mathrm{id}_c, a) = \mathrm{id}_{\hom(c,a)}$.

(45) If $\operatorname{dom} g = \operatorname{cod} f$, then $\hom(a, g \cdot f) = \hom(a, g) \cdot \hom(a, f)$.

(46) If $\operatorname{dom} g = \operatorname{cod} f$, then $\hom(g \cdot f, a) = \hom(f, a) \cdot \hom(g, a)$.

(47) $\langle\langle \hom(a, \operatorname{dom} f), \hom(a, \operatorname{cod} f)\rangle, \hom(a, f)\rangle$ is an element of $\operatorname{Maps} \operatorname{Hom}(C)$.

(48) $\langle\langle \hom(\operatorname{cod} f, a), \hom(\operatorname{dom} f, a)\rangle, \hom(f, a)\rangle$ is an element of $\operatorname{Maps} \operatorname{Hom}(C)$.

We now define two new functors. Let us consider C, a. The functor $\hom(a, -)$ yields a function from the morphisms of C into $\operatorname{Maps} \operatorname{Hom}(C)$ and is defined as follows:

(Def.22) for every f holds $(\hom(a, -))(f) = \langle\langle \hom(a, \operatorname{dom} f), \hom(a, \operatorname{cod} f)\rangle, \hom(a, f)\rangle$.

The functor $\hom(-, a)$ yields a function from the morphisms of C into $\operatorname{Maps} \operatorname{Hom}(C)$
and is defined as follows:

(Def.23) for every f holds $(\hom(-, a))(f) = \langle\langle \hom(\operatorname{cod} f, a), \hom(\operatorname{dom} f, a)\rangle, \hom(f, a)\rangle$.

The following propositions are true:

(49) If $\operatorname{Hom}(C) \subseteq V$, then $\hom(a, -)$ is a functor from C to $\mathbf{Ens}_V$.

(50) If $\operatorname{Hom}(C) \subseteq V$, then $\hom(-, a)$ is a contravariant functor from C into $\mathbf{Ens}_V$.

(51) If $\hom(\operatorname{dom} f, \operatorname{cod} f') = \emptyset$, then $\hom(\operatorname{cod} f, \operatorname{dom} f') = \emptyset$.

Let us consider C, f, g. The functor $\hom(f, g)$ yielding a function from $\hom(\operatorname{cod} f, \operatorname{dom} g)$ into $\hom(\operatorname{dom} f, \operatorname{cod} g)$ is defined by:

(Def.24) for every h such that $h \in \hom(\operatorname{cod} f, \operatorname{dom} g)$ holds $(\hom(f, g))(h) = g \cdot h \cdot f$.

We now state several propositions:

(52) $\langle\langle \hom(\operatorname{cod} f, \operatorname{dom} g), \hom(\operatorname{dom} f, \operatorname{cod} g)\rangle, \hom(f, g)\rangle$ is an element of $\operatorname{Maps} \operatorname{Hom}(C)$.

(53) $\hom(\mathrm{id}_a, f) = \hom(a, f)$ and $\hom(f, \mathrm{id}_a) = \hom(f, a)$.

(54) $\hom(\mathrm{id}_a, \mathrm{id}_b) = \mathrm{id}_{\hom(a,b)}$.

(55) $\hom(f, g) = \hom(\mathrm{dom}\, f, g) \cdot \hom(f, \mathrm{dom}\, g)$.

(56) If $\mathrm{cod}\, g = \mathrm{dom}\, f$ and $\mathrm{dom}\, g' = \mathrm{cod}\, f'$, then $\hom(f \cdot g, g' \cdot f') = \hom(g, g') \cdot \hom(f, f')$.

Let us consider C. The functor $\hom_C(-,-)$ yielding a function from the morphisms of $[\colon C, C \colon]$ into Maps Hom(C) is defined as follows:

(Def.25) for all f, g holds $(\hom_C(-,-))(\langle f, g \rangle) = \langle\langle \hom(\mathrm{cod}\, f, \mathrm{dom}\, g), \hom(\mathrm{dom}\, f, \mathrm{cod}\, g) \rangle, \hom(f, g) \rangle$.

The following two propositions are true:

(57) $\hom(a, -) = (\mathrm{curry}(\hom_C(-,-)))(\mathrm{id}_a)$ and $\hom(-, a) = (\mathrm{curry}'(\hom_C(-,-)))(\mathrm{id}_a)$.

(58) If $\mathrm{Hom}(C) \subseteq V$, then $\hom_C(-,-)$ is a functor from $[\colon C^{\mathrm{op}}, C \colon]$ to $\mathbf{Ens}_V$.

We now define two new functors. Let us consider V, C, a. Let us assume that $\mathrm{Hom}(C) \subseteq V$. The functor $\hom_V(a, -)$ yields a functor from C to $\mathbf{Ens}_V$ and is defined by:

(Def.26) $\hom_V(a, -) = \hom(a, -)$.

The functor $\hom_V(-, a)$ yields a contravariant functor from C into $\mathbf{Ens}_V$ and is defined as follows:

(Def.27) $\hom_V(-, a) = \hom(-, a)$.

Let us consider V, C. Let us assume that $\mathrm{Hom}(C) \subseteq V$. The functor $\hom_V^C(-,-)$ yielding a functor from $[\colon C^{\mathrm{op}}, C \colon]$ to $\mathbf{Ens}_V$ is defined as follows:

(Def.28) $\hom_V^C(-,-) = \hom_C(-,-)$.

One can prove the following propositions:

(59) If $\mathrm{Hom}(C) \subseteq V$, then $(\hom_V(a, -))(f) = \langle\langle \hom(a, \mathrm{dom}\, f), \hom(a, \mathrm{cod}\, f) \rangle, \hom(a, f) \rangle$.

(60) If $\mathrm{Hom}(C) \subseteq V$, then $(\mathrm{Obj}(\hom_V(a, -)))(b) = \hom(a, b)$.

(61) If $\mathrm{Hom}(C) \subseteq V$, then $(\hom_V(-, a))(f) = \langle\langle \hom(\mathrm{cod}\, f, a), \hom(\mathrm{dom}\, f, a) \rangle, \hom(f, a) \rangle$.

(62) If $\mathrm{Hom}(C) \subseteq V$, then $(\mathrm{Obj}(\hom_V(-, a)))(b) = \hom(b, a)$.

(63) If $\mathrm{Hom}(C) \subseteq V$, then $(\hom_V^C(-,-))(\langle f^{\mathrm{op}}, g \rangle) = \langle\langle \hom(\mathrm{cod}\, f, \mathrm{dom}\, g), \hom(\mathrm{dom}\, f, \mathrm{cod}\, g) \rangle, \hom(f, g) \rangle$.

(64) If $\mathrm{Hom}(C) \subseteq V$, then $(\mathrm{Obj}(\hom_V^C(-,-)))(\langle a^{\mathrm{op}}, b \rangle) = \hom(a, b)$.

(65) If $\mathrm{Hom}(C) \subseteq V$, then $(\hom_V^C(-,-))(a^{\mathrm{op}}, -) = \hom_V(a, -)$.

(66) If $\mathrm{Hom}(C) \subseteq V$, then $(\hom_V^C(-,-))(-, a) = \hom_V(-, a)$.

Acknowledgements

I would like to thank Andrzej Trybulec for his useful sugestions and valuable comments.

References

[1] Grzegorz Bancerek. Curried and uncurried functions. *Formalized Mathematics*, 1(**3**):537–541, 1990.
[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[4] Czesław Byliński. Introduction to categories and functors. *Formalized Mathematics*, 1(**2**):409–420, 1990.
[5] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[6] Czesław Byliński. Opposite categories and contravariant functors. *Formalized Mathematics*, 2(**3**):419–424, 1991.
[7] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[8] Czesław Byliński. Subcategories and products of categories. *Formalized Mathematics*, 1(**4**):725–732, 1990.
[9] Sunders MacLane. *Categories for the working mathematician.* Springer, Berlin/Heilderberg/New York, 1972.
[10] Bogdan Nowak and Grzegorz Bancerek. Universal classes. *Formalized Mathematics*, 1(**3**):595–600, 1990.
[11] Zbigniew Semadeni and Antoni Wiweger. *Wstęp do teorii kategorii i funktorów.* Volume 45 of *Biblioteka Matematyczna*, PWN, Warszawa, 1978.
[12] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[13] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[14] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.
[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[16] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

Received August 1, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

A Borsuk Theorem on Homotopy Types

Andrzej Trybulec
Warsaw University
Białystok

Summary. We present a Borsuk's theorem published first in [3] (compare also [4, pages 119–120]). It is slightly generalized, the assumption of metrizability is omitted. We introduce concepts needed for the formulation and the proof of theorems on upper semi-continuous decompositions, retracts, strong deformation retract. However, only those facts that are necessary in the proof have been proved.

MML Identifier: BORSUK_1.

The terminology and notation used here have been introduced in the following articles: [22], [7], [21], [2], [24], [23], [20], [12], [18], [14], [8], [13], [16], [25], [11], [10], [6], [5], [17], [1], [19], [9], and [15].

Preliminaries

We follow a convention: X, Y, X_1, X_2, Y_1, Y_2 will be sets, A will be a subset of X, and e, u will be arbitrary. The following propositions are true:

(1) If X meets Y_1 and $X \subseteq Y_2$, then X meets $Y_1 \cap Y_2$.

(2) If $e \in [\!: X_1, Y_1 :\!]$ and $e \in [\!: X_2, Y_2 :\!]$, then $e \in [\!: X_1 \cap X_2, Y_1 \cap Y_2 :\!]$.

(3) $\mathrm{id}_X {}^\circ A = A$.

(4) $\mathrm{id}_X {}^{-1} A = A$.

(5) For every function F such that $X \subseteq F^{-1}\, X_1$ holds $F^\circ X \subseteq X_1$.

(6) $(X \longmapsto u)^\circ X_1 \subseteq \{u\}$.

(7) If $[\!: X_1, X_2 :\!] \subseteq [\!: Y_1, Y_2 :\!]$ and $[\!: X_1, X_2 :\!] \neq \emptyset$, then $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$.

(8) If $\{e\}$ meets X, then $e \in X$.

The scheme *NonUniqExD* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every e such that $e \in \mathcal{A}$ holds $\mathcal{P}[e, f(e)]$

ISSN 0777–4028

provided the following requirement is met:

- for every e such that $e \in \mathcal{A}$ there exists u such that $u \in \mathcal{B}$ and $\mathcal{P}[e, u]$.

We now state several propositions:

(9) If $e \in 2^{[: X, Y :]}$, then $({}^\circ \pi_1(X \times Y))(e) = \pi_1(X \times Y) {}^\circ e$.

(10) If $e \in 2^{[: X, Y :]}$, then $({}^\circ \pi_2(X \times Y))(e) = \pi_2(X \times Y) {}^\circ e$.

(11) If $e \in [: X, Y :]$, then $e = \langle e_{\mathbf{1}}, e_{\mathbf{2}} \rangle$.

(12) For every subset X_1 of X and for every subset Y_1 of Y such that $[: X_1, Y_1 :] \neq \emptyset$ holds $\pi_1(X \times Y) {}^\circ [: X_1, Y_1 :] = X_1$ and $\pi_2(X \times Y) {}^\circ [: X_1, Y_1 :] = Y_1$.

(13) For every subset X_1 of X and for every subset Y_1 of Y such that $[: X_1, Y_1 :] \neq \emptyset$ holds $({}^\circ \pi_1(X \times Y))([: X_1, Y_1 :]) = X_1$ and $({}^\circ \pi_2(X \times Y))([: X_1, Y_1 :]) = Y_1$.

(14) Let A be a subset of $[: X, Y :]$. Then for every family H of subsets of $[: X, Y :]$ such that for every e such that $e \in H$ holds $e \subseteq A$ and there exists a subset X_1 of X and there exists a subset Y_1 of Y such that $e = [: X_1, Y_1 :]$ holds $[: \bigcup(({}^\circ \pi_1(X \times Y)) {}^\circ H), \bigcap(({}^\circ \pi_2(X \times Y)) {}^\circ H) :] \subseteq A$.

(15) Let A be a subset of $[: X, Y :]$. Then for every family H of subsets of $[: X, Y :]$ such that for every e such that $e \in H$ holds $e \subseteq A$ and there exists a subset X_1 of X and there exists a subset Y_1 of Y such that $e = [: X_1, Y_1 :]$ holds $[: \bigcap(({}^\circ \pi_1(X \times Y)) {}^\circ H), \bigcup(({}^\circ \pi_2(X \times Y)) {}^\circ H) :] \subseteq A$.

(16) For every set X and for every non-empty set Y and for every function f from X into Y and for every family H of subsets of X holds $\bigcup(({}^\circ f) {}^\circ H) = f {}^\circ \bigcup H$.

In the sequel X, Y, Z denote non-empty sets. One can prove the following propositions:

(17) For every family a of subsets of X holds $\bigcup \bigcup a = \bigcup\{\bigcup A : A \in a\}$, where A ranges over subsets of X.

(18) For every family D of subsets of X such that $\bigcup D = X$ for every subset A of D and for every subset B of X such that $B = \bigcup A$ holds $B^{\mathrm{c}} \subseteq \bigcup(A^{\mathrm{c}})$.

(19) For every function F from X into Y and for every function G from X into Z such that for all elements x, x' of X such that $F(x) = F(x')$ holds $G(x) = G(x')$ there exists a function H from Y into Z such that $H \cdot F = G$.

(20) For all X, Y, Z and for every element y of Y and for every function F from X into Y and for every function G from Y into Z holds $F^{-1}\{y\} \subseteq (G \cdot F)^{-1}\{G(y)\}$.

(21) For every function F from X into Y and for every element x of X and for every element z of Z holds $[: F, \mathrm{id}_Z :](\langle x, z \rangle) = \langle F(x), z \rangle$.

(22) For every function F from X into Y and for every subset A of X holds $\mathrm{id}_X {}^\circ A = A$.

(23) For every function F from X into Y and for every subset A of X and for every subset B of Z holds $[: F, \mathrm{id}_Z :] {}^\circ [: A, B :] = [: F {}^\circ A, B :]$.

(24) For every function F from X into Y and for every element y of Y and for every element z of Z holds $[: F, \mathrm{id}_Z :]^{-1} \{\langle y, z\rangle\} = [: F^{-1} \{y\}, \{z\} :]$.

Let B, A be non-empty sets, and let x be an element of B. Then $A \longmapsto x$ is a function from A into B.

Let Y be a non-empty set, and let y be an element of Y. Then $\{y\}$ is a subset of Y.

Partitions

One can prove the following four propositions:

(25) For every partition D of X and for every subset A of D holds $\bigcup A$ is a subset of X.

(26) For every partition D of X and for all subsets A, B of D holds $\bigcup(A \cap B) = \bigcup A \cap \bigcup B$.

(27) For every partition D of X and for every subset A of D and for every subset B of X such that $B = \bigcup A$ holds $B^{\mathrm{c}} = \bigcup(A^{\mathrm{c}})$.

(28) For every equivalence relation E of X holds Classes E is non-empty.

Let us consider X, and let D be a non-empty partition of X. The projection onto D yielding a function from X into D is defined as follows:

(Def.1) for every element p of X holds $p \in$ (the projection onto D)(p).

Next we state several propositions:

(29) For every non-empty partition D of X and for every element p of X and for every element A of D such that $p \in A$ holds $A =$ (the projection onto D)(p).

(30) For every non-empty partition D of X and for every element p of D holds $p =$ (the projection onto D)$^{-1}$ $\{p\}$.

(31) For every non-empty partition D of X and for every subset A of D holds (the projection onto D)$^{-1}$ $A = \bigcup A$.

(32) For every non-empty partition D of X and for every element W of D there exists an element W' of X such that (the projection onto D)(W') $= W$.

(33) For every non-empty partition D of X and for every subset W of X such that for every subset B of X such that $B \in D$ and B meets W holds $B \subseteq W$ holds $W =$ (the projection onto D)$^{-1}$(the projection onto D)$^{\circ}W$.

Topological Preliminaries

In the sequel X, Y denote topological spaces. We now state two propositions:

(34) $\Omega_X \neq \emptyset_X$.

(35) For every subspace Y of X holds the carrier of $Y \subseteq$ the carrier of X.

Let X, Y be topological spaces, and let F be a function from the carrier of X into the carrier of Y. Let us note that one can characterize the predicate F

is continuous by the following (equivalent) condition:

(Def.2) for every point W of X and for every neighborhood G of $F(W)$ there exists a neighborhood H of W such that $F^{\circ} H \subseteq G$.

The following proposition is true

(36) For every point y of Y holds (the carrier of X) $\longmapsto y$ is continuous.

Let us consider X, Y. A map from X into Y is called a continuous map from X into Y if:

(Def.3) it is continuous.

Let X, Y, Z be topological spaces, and let F be a continuous map from X into Y, and let G be a continuous map from Y into Z. Then $G \cdot F$ is a continuous map from X into Z.

We now state two propositions:

(37) For every continuous map A from X into Y and for every subset G of Y holds $A^{-1} \operatorname{Int} G \subseteq \operatorname{Int}(A^{-1} G)$.

(38) For every point W of Y and for every continuous map A from X into Y and for every neighborhood G of W holds $A^{-1} G$ is a neighborhood of $A^{-1} \{W\}$.

Let X, Y be topological spaces, and let W be a point of Y, and let A be a continuous map from X into Y, and let G be a neighborhood of W. Then $A^{-1} G$ is a neighborhood of $A^{-1} \{W\}$.

One can prove the following propositions:

(39) For every X and for all subsets A, B of the carrier of X and for every neighborhood U_1 of B such that $A \subseteq B$ holds U_1 is a neighborhood of A.

(40) For every subset A of X and for every point x of X holds A is a neighborhood of x if and only if A is a neighborhood of $\{x\}$.

(41) For every point x of X holds $\{x\}$ is compact.

(42) For every subspace Y of X and for every subset A of X and for every subset B of Y such that $A = B$ holds A is compact if and only if B is compact.

Cartesian Products of Topological Spaces

Let us consider X, Y. The functor $[: X, Y :]$ yielding a topological space is defined by:

(Def.4) the carrier of $[: X, Y :] = [:$ the carrier of X, the carrier of $Y :]$ and the topology of $[: X, Y :] = \{\bigcup A : A \subseteq \{[: X_1, Y_1 :] : X_1 \in$ the topology of $X \wedge Y_1 \in$ the topology of $Y\}\}$, where X_1 ranges over subsets of X, and Y_1 ranges over subsets of Y.

Next we state three propositions:

(43) The carrier of $[: X, Y :] = [:$ the carrier of X, the carrier of $Y :]$.

(44) The topology of $[: X, Y :] = \{\bigcup A : A \subseteq \{[: X_1, Y_1 :] : X_1 \in$ the topology of $X \wedge Y_1 \in$ the topology of $Y\}\}$, where X_1 ranges over subsets of X, and Y_1 ranges over subsets of Y.

(45) For every subset B of $[: X, Y :]$ holds B is open if and only if there exists a family A of subsets of the carrier of $[: X, Y :]$ such that $B = \bigcup A$ and for every e such that $e \in A$ there exists a subset X_1 of X and there exists a subset Y_1 of Y such that $e = [: X_1, Y_1 :]$ and X_1 is open and Y_1 is open.

Let X, Y be topological spaces, and let A be a subset of X, and let B be a subset of Y. Then $[: A, B :]$ is a subset of $[: X, Y :]$.

Let X, Y be topological spaces, and let x be a point of X, and let y be a point of Y. Then $\langle x, y \rangle$ is a point of $[: X, Y :]$.

Next we state four propositions:

(46) For every subset V of X and for every subset W of Y such that V is open and W is open holds $[: V, W :]$ is open.

(47) For every subset V of X and for every subset W of Y holds $\mathrm{Int}[: V, W :] = [: \mathrm{Int}\, V, \mathrm{Int}\, W :]$.

(48) For every point x of X and for every point y of Y and for every neighborhood V of x and for every neighborhood W of y holds $[: V, W :]$ is a neighborhood of $\langle x, y \rangle$.

(49) For every subset A of X and for every subset B of Y and for every neighborhood V of A and for every neighborhood W of B holds $[: V, W :]$ is a neighborhood of $[: A, B :]$.

Let X, Y be topological spaces, and let x be a point of X, and let y be a point of Y, and let V be a neighborhood of x, and let W be a neighborhood of y. Then $[: V, W :]$ is a neighborhood of $\langle x, y \rangle$.

Next we state the proposition

(50) For every point X_3 of $[: X, Y :]$ there exists a point W of X and there exists a point T of Y such that $X_3 = \langle W, T \rangle$.

Let X, Y be topological spaces, and let A be a subset of X, and let t be a point of Y, and let V be a neighborhood of A, and let W be a neighborhood of t. Then $[: V, W :]$ is a neighborhood of $[: A, \{t\} :]$.

Let us consider X, Y, and let A be a subset of $[: X, Y :]$. The functor $\mathrm{BaseAppr}(A)$ yields a family of subsets of $[: X, Y :]$ and is defined by:

(Def.5) $\mathrm{BaseAppr}(A) = \{[: X_1, Y_1 :] : [: X_1, Y_1 :] \subseteq A \wedge X_1$ is open$\wedge Y_1$ is open$\}$, where X_1 ranges over subsets of X, and Y_1 ranges over subsets of Y.

We now state several propositions:

(51) For every subset A of $[: X, Y :]$ holds $\mathrm{BaseAppr}(A)$ is open.

(52) For all subsets A, B of $[: X, Y :]$ such that $A \subseteq B$ holds $\mathrm{BaseAppr}(A) \subseteq \mathrm{BaseAppr}(B)$.

(53) For every subset A of $[: X, Y :]$ holds $\bigcup \mathrm{BaseAppr}(A) \subseteq A$.

(54) For every subset A of $[: X, Y :]$ such that A is open holds $A = \bigcup \mathrm{BaseAppr}(A)$.

(55) For every subset A of $[: X, Y :]$ holds $\operatorname{Int} A = \bigcup \operatorname{BaseAppr}(A)$.

We now define two new functors. Let us consider X, Y. The functor $\pi_1(X, Y)$ yielding a function from $2^{\text{the carrier of } [: X, Y :]}$ into $2^{\text{the carrier of } X}$ is defined by:

(Def.6) $\pi_1(X, Y) = {}^{\circ}\pi_1($ (the carrier of $X)\times$ the carrier of $Y)$.

The functor $\pi_2(X, Y)$ yields a function from $2^{\text{the carrier of } [: X, Y :]}$ into $2^{\text{the carrier of } Y}$ and is defined as follows:

(Def.7) $\pi_2(X, Y) = {}^{\circ}\pi_2($ (the carrier of $X)\times$ the carrier of $Y)$.

We now state a number of propositions:

(56) Let A be a subset of $[: X, Y :]$. Then for every family H of subsets of $[: X, Y :]$ such that for every e such that $e \in H$ holds $e \subseteq A$ and there exists a subset X_1 of X and there exists a subset Y_1 of Y such that $e = [: X_1, Y_1 :]$ holds $[: \bigcup(\pi_1(X, Y) \circ H), \bigcap(\pi_2(X, Y) \circ H) :] \subseteq A$.

(57) For every family H of subsets of $[: X, Y :]$ and for every set C such that $C \in \pi_1(X, Y) \circ H$ there exists a subset D of $[: X, Y :]$ such that $D \in H$ and $C = \pi_1($ (the carrier of $X)\times$ the carrier of $Y) \circ D$.

(58) For every family H of subsets of $[: X, Y :]$ and for every set C such that $C \in \pi_2(X, Y) \circ H$ there exists a subset D of $[: X, Y :]$ such that $D \in H$ and $C = \pi_2($ (the carrier of $X)\times$ the carrier of $Y) \circ D$.

(59) For every subset D of $[: X, Y :]$ such that D is open for every subset X_1 of X and for every subset Y_1 of Y holds if $X_1 = \pi_1($ (the carrier of $X)\times$ the carrier of $Y) \circ D$, then X_1 is open but if $Y_1 = \pi_2($ (the carrier of $X)\times$ the carrier of $Y) \circ D$, then Y_1 is open.

(60) For every family H of subsets of $[: X, Y :]$ such that H is open holds $\pi_1(X, Y) \circ H$ is open and $\pi_2(X, Y) \circ H$ is open.

(61) For every family H of subsets of $[: X, Y :]$ such that $\pi_1(X, Y) \circ H = \emptyset$ or $\pi_2(X, Y) \circ H = \emptyset$ holds $H = \emptyset$.

(62) For every family H of subsets of $[: X, Y :]$ and for every subset X_1 of X and for every subset Y_1 of Y such that H is a cover of $[: X_1, Y_1 :]$ holds if $Y_1 \neq \emptyset$, then $\pi_1(X, Y) \circ H$ is a cover of X_1 but if $X_1 \neq \emptyset$, then $\pi_2(X, Y) \circ H$ is a cover of Y_1.

(63) For every family H of subsets of X and for every subset Y of X such that H is a cover of Y there exists a family F of subsets of X such that $F \subseteq H$ and F is a cover of Y and for every set C such that $C \in F$ holds $C \cap Y \neq \emptyset$.

(64) For every family F of subsets of X and for every family H of subsets of $[: X, Y :]$ such that F is finite and $F \subseteq \pi_1(X, Y) \circ H$ there exists a family G of subsets of $[: X, Y :]$ such that $G \subseteq H$ and G is finite and $F = \pi_1(X, Y) \circ G$.

(65) For every subset X_1 of X and for every subset Y_1 of Y such that $[: X_1, Y_1 :] \neq \emptyset$ holds $\pi_1(X, Y)([: X_1, Y_1 :]) = X_1$ and $\pi_2(X, Y)([: X_1, Y_1 :]) = Y_1$.

(66) $\pi_1(X, Y)(\emptyset) = \emptyset$ and $\pi_2(X, Y)(\emptyset) = \emptyset$.

(67) For every point t of Y and for every subset A of the carrier of X such that A is compact for every neighborhood G of $[:A, \{t\}:]$ there exists a neighborhood V of A and there exists a neighborhood W of t such that $[:V, W:] \subseteq G$.

Partitions of Topological Spaces

Let us consider X. The trivial decomposition of X yielding a non-empty partition of the carrier of X is defined by:

(Def.8) the trivial decomposition of $X = \mathrm{Classes}(\triangle_{\text{the carrier of } X})$.

We now state the proposition

(68) For every subset A of X such that $A \in$ the trivial decomposition of X there exists a point x of X such that $A = \{x\}$.

Let X be a topological space, and let D be a non-empty partition of the carrier of X. The decomposition space of D yielding a topological space is defined as follows:

(Def.9) the carrier of the decomposition space of $D = D$ and the topology of the decomposition space of $D = \{A : \bigcup A \in$ the topology of $X\}$, where A ranges over subsets of D.

One can prove the following proposition

(69) For every non-empty partition D of the carrier of X and for every subset A of D holds $\bigcup A \in$ the topology of X if and only if $A \in$ the topology of the decomposition space of D.

Let X be a topological space, and let D be a non-empty partition of the carrier of X. The projection onto D yielding a continuous map from X into the decomposition space of D is defined as follows:

(Def.10) the projection onto $D =$ the projection onto D.

We now state three propositions:

(70) For every non-empty partition D of the carrier of X and for every point W of X holds $W \in$ (the projection onto D)(W).

(71) For every non-empty partition D of the carrier of X and for every point W of the decomposition space of D there exists a point W' of X such that (the projection onto D)(W') $= W$.

(72) For every non-empty partition D of the carrier of X holds rng(the projection onto D) $=$ the carrier of the decomposition space of D.

Let X_4 be a topological space, and let X be a subspace of X_4, and let D be a non-empty partition of the carrier of X. The trivial extension of D yields a non-empty partition of the carrier of X_4 and is defined as follows:

(Def.11) the trivial extension of $D = D \cup \{\{p\} : p \notin$ the carrier of $X\}$, where p ranges over points of X_4.

The following propositions are true:

(73) For every topological space X_4 and for every subspace X of X_4 and for every non-empty partition D of the carrier of X holds $D \subseteq$ the trivial extension of D.

(74) For every topological space X_4 and for every subspace X of X_4 and for every non-empty partition D of the carrier of X and for every subset A of X_4 such that $A \in$ the trivial extension of D holds $A \in D$ or there exists a point x of X_4 such that $x \notin \Omega_X$ and $A = \{x\}$.

(75) For every topological space X_4 and for every subspace X of X_4 and for every non-empty partition D of the carrier of X and for every point x of X_4 such that $x \notin$ the carrier of X holds $\{x\} \in$ the trivial extension of D.

(76) For every topological space X_4 and for every subspace X of X_4 and for every non-empty partition D of the carrier of X and for every point W of X_4 such that $W \in$ the carrier of X holds (the projection onto the trivial extension of D)(W) = (the projection onto D)(W).

(77) For every topological space X_4 and for every subspace X of X_4 and for every non-empty partition D of the carrier of X and for every point W of X_4 such that $W \notin$ the carrier of X holds (the projection onto the trivial extension of D)(W) = $\{W\}$.

(78) For every topological space X_4 and for every subspace X of X_4 and for every non-empty partition D of the carrier of X and for all points W, W' of X_4 such that $W \notin$ the carrier of X and (the projection onto the trivial extension of D)(W) = (the projection onto the trivial extension of D)(W') holds $W = W'$.

(79) For every topological space X_4 and for every subspace X of X_4 and for every non-empty partition D of the carrier of X and for every point e of X_4 such that (the projection onto the trivial extension of D)(e) $\in$ the carrier of the decomposition space of D holds $e \in$ the carrier of X.

(80) For every topological space X_4 and for every subspace X of X_4 and for every non-empty partition D of the carrier of X and for every e such that $e \in$ the carrier of X holds (the projection onto the trivial extension of D)(e) $\in$ the carrier of the decomposition space of D.

Upper Semicontinuous Decompositions

Let X be a topological space. A non-empty partition of the carrier of X is said to be an upper semi-continuous decomposition of X if:

(Def.12) for every subset A of X such that $A \in$ it for every neighborhood V of A there exists a subset W of X such that W is open and $A \subseteq W$ and $W \subseteq V$ and for every subset B of X such that $B \in$ it and B meets W holds $B \subseteq W$.

We now state two propositions:

(81) For every upper semi-continuous decomposition D of X and for every point t of the decomposition space of D and for every neighborhood G

of (the projection onto D) $^{-1}$ $\{t\}$ holds (the projection onto D) $^{\circ}$ G is a neighborhood of t.

(82) The trivial decomposition of X is an upper semi-continuous decomposition of X.

Let us consider X. A subspace of X is called a closed subspace of X if:

(Def.13) for every subset A of X such that $A =$ the carrier of it holds A is closed.

Let X_4 be a topological space, and let X be a closed subspace of X_4, and let D be an upper semi-continuous decomposition of X. Then the trivial extension of D is an upper semi-continuous decomposition of X_4.

Let X be a topological space. An upper semi-continuous decomposition of X is called an upper semi-continuous decomposition into compacta of X if:

(Def.14) for every subset A of X such that $A \in$ it holds A is compact.

Let X_4 be a topological space, and let X be a closed subspace of X_4, and let D be an upper semi-continuous decomposition into compacta of X. Then the trivial extension of D is an upper semi-continuous decomposition into compacta of X_4.

Let X be a topological space, and let Y be a closed subspace of X, and let D be an upper semi-continuous decomposition into compacta of Y. Then the decomposition space of D is a closed subspace of the decomposition space of the trivial extension of D.

Borsuk's Theorems on the Decomposition of Retracts

The topological space $\mathbb{I}$ is defined by:

(Def.15) for every subset P of (the metric space of real numbers)$_{\rm top}$ such that $P = [0, 1]$ holds $\mathbb{I} =$ (the metric space of real numbers)$_{\rm top} \restriction P$.

Next we state the proposition

(83) The carrier of $\mathbb{I} = [0, 1]$.

We now define two new functors. The point $0_{\mathbb{I}}$ of $\mathbb{I}$ is defined by:

(Def.16) $0_{\mathbb{I}} = 0$.

The point $1_{\mathbb{I}}$ of $\mathbb{I}$ is defined by:

(Def.17) $1_{\mathbb{I}} = 1$.

Let A be a topological space, and let B be a subspace of A, and let F be a continuous map from A into B. We say that F is a retraction if and only if:

(Def.18) for every point W of A such that $W \in$ the carrier of B holds $F(W) = W$.

We now define two new predicates. Let X be a topological space, and let Y be a subspace of X. We say that Y is a retract of X if and only if:

(Def.19) there exists a continuous map F from X into Y such that F is a retraction.

We say that Y is a strong deformation retract of X if and only if:

(Def.20) there exists a continuous map H from $[\!: X, \mathbb{I} :\!]$ into X such that for every point A of X holds $H(\langle A, 0_{\mathbb{I}}\rangle) = A$ and $H(\langle A, 1_{\mathbb{I}}\rangle) \in$ the carrier of Y but if $A \in$ the carrier of Y, then for every point T of $\mathbb{I}$ holds $H(\langle A, T\rangle) = A$.

We now state two propositions:

(84) For every topological space X_4 and for every closed subspace X of X_4 and for every upper semi-continuous decomposition D into compacta of X such that X is a retract of X_4 holds the decomposition space of D is a retract of the decomposition space of the trivial extension of D.

(85) For every topological space X_4 and for every closed subspace X of X_4 and for every upper semi-continuous decomposition D into compacta of X such that X is a strong deformation retract of X_4 holds the decomposition space of D is a strong deformation retract of the decomposition space of the trivial extension of D.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[3] Karol Borsuk. On the homotopy types of some decomposition spaces. *Bull. Acad. Polon. Sci.*, (18):235–239, 1970.
[4] Karol Borsuk. *Theory of Shape*. Volume 59 of *Monografie Matematyczne*, PWN, Warsaw, 1975.
[5] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.
[6] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.
[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[9] Czesław Byliński. Introduction to categories and functors. *Formalized Mathematics*, 1(**2**):409–420, 1990.
[10] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.
[11] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[12] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[13] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.
[14] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[15] Beata Padlewska. Locally connected spaces. *Formalized Mathematics*, 2(**1**):93–96, 1991.
[16] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[17] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(**3**):441–444, 1990.
[18] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[19] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[20] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[21] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[22] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[23] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.
[24] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[25] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

Received August 1, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Cartesian Product of Functions

Grzegorz Bancerek
Warsaw University
Białystok

Summary. A supplement of [3] and [2], i.e. some useful and explanatory properties of the product and also the curried and uncurried functions are shown. Besides, the functions yielding functions are considered: two different products and other operation of such functions are introduced. Finally, two facts are presented: quasi-distributivity of the power of the set to other one w.r.t. the union ($X^{\biguplus_x f(x)} \approx \prod_x X^{f(x)}$) and quasi-distributivity of the poroduct w.r.t. the raising to the power ($\prod_x f(x)^X \approx (\prod_x f(x))^X$).

MML Identifier: `FUNCT_6`.

The articles [16], [14], [8], [17], [5], [12], [9], [11], [6], [4], [13], [15], [7], [10], [2], [1], and [3] provide the notation and terminology for this paper.

Properties of Cartesian product

For simplicity we follow the rules: x, y, y_1, y_2, z, a will be arbitrary, f, g, h, h', f_1, f_2 will denote functions, i will denote a natural number, X, Y, Z, V_1, V_2 will denote sets, P will denote a permutation of X, D, D_1, D_2, D_3 will denote non-empty sets, d_1 will denote an element of D_1, d_2 will denote an element of D_2, and d_3 will denote an element of D_3. We now state a number of propositions:

(1) $x \in \prod\langle X\rangle$ if and only if there exists y such that $y \in X$ and $x = \langle y\rangle$.

(2) $z \in \prod\langle X, Y\rangle$ if and only if there exist x, y such that $x \in X$ and $y \in Y$ and $z = \langle x, y\rangle$.

(3) $a \in \prod\langle X, Y, Z\rangle$ if and only if there exist x, y, z such that $x \in X$ and $y \in Y$ and $z \in Z$ and $a = \langle x, y, z\rangle$.

(4) $\prod\langle D\rangle = D^1$.

(5) $\prod\langle D_1, D_2\rangle = \{\langle d_1, d_2\rangle\}$.

(6) $\prod\langle D, D\rangle = D^2$.

ISSN 0777–4028

(7) $\prod\langle D_1, D_2, D_3\rangle = \{\langle d_1, d_2, d_3\rangle\}$.
(8) $\prod\langle D, D, D\rangle = D^3$.
(9) $\prod(i \longmapsto D) = D^i$.
(10) $\prod f \subseteq (\bigcup f)^{\operatorname{dom} f}$.

Curried and uncurried functions of some functions

The following propositions are true:

(11) If $x \in \operatorname{dom} \curvearrowleft f$, then there exist y, z such that $x = \langle y, z\rangle$.
(12) $\curvearrowleft([\!: X, Y :\!] \longmapsto z) = [\!: Y, X :\!] \longmapsto z$.
(13) $\operatorname{curry} f = \operatorname{curry}' \curvearrowleft f$ and $\operatorname{uncurry} f = \curvearrowleft \operatorname{uncurry}' f$.
(14) If $[\!: X, Y :\!] \neq \emptyset$, then $\operatorname{curry}([\!: X, Y :\!] \longmapsto z) = X \longmapsto (Y \longmapsto z)$ and $\operatorname{curry}'([\!: X, Y :\!] \longmapsto z) = Y \longmapsto (X \longmapsto z)$.
(15) $\operatorname{uncurry}(X \longmapsto (Y \longmapsto z)) = [\!: X, Y :\!] \longmapsto z$ and $\operatorname{uncurry}'(X \longmapsto (Y \longmapsto z)) = [\!: Y, X :\!] \longmapsto z$.
(16) If $x \in \operatorname{dom} f$ and $g = f(x)$, then $\operatorname{rng} g \subseteq \operatorname{rng} \operatorname{uncurry} f$ and $\operatorname{rng} g \subseteq \operatorname{rng} \operatorname{uncurry}' f$.
(17) $\operatorname{dom} \operatorname{uncurry}(X \longmapsto f) = [\!: X, \operatorname{dom} f :\!]$ and $\operatorname{rng} \operatorname{uncurry}(X \longmapsto f) \subseteq \operatorname{rng} f$ and $\operatorname{dom} \operatorname{uncurry}'(X \longmapsto f) = [\!: \operatorname{dom} f, X :\!]$ and $\operatorname{rng} \operatorname{uncurry}'(X \longmapsto f) \subseteq \operatorname{rng} f$.
(18) If $X \neq \emptyset$, then $\operatorname{rng} \operatorname{uncurry}(X \longmapsto f) = \operatorname{rng} f$ and $\operatorname{rng} \operatorname{uncurry}'(X \longmapsto f) = \operatorname{rng} f$.
(19) If $[\!: X, Y :\!] \neq \emptyset$ and $f \in Z^{[\!: X, Y :\!]}$, then $\operatorname{curry} f \in (Z^Y)^X$ and $\operatorname{curry}' f \in (Z^X)^Y$.
(20) If $f \in (Z^Y)^X$, then $\operatorname{uncurry} f \in Z^{[\!: X, Y :\!]}$ and $\operatorname{uncurry}' f \in Z^{[\!: Y, X :\!]}$.
(21) If $\operatorname{curry} f \in (Z^Y)^X$ or $\operatorname{curry}' f \in (Z^X)^Y$ but $\operatorname{dom} f \subseteq [\!: V_1, V_2 :\!]$, then $f \in Z^{[\!: X, Y :\!]}$.
(22) If $\operatorname{uncurry} f \in Z^{[\!: X, Y :\!]}$ or $\operatorname{uncurry}' f \in Z^{[\!: Y, X :\!]}$ but $\operatorname{rng} f \subseteq V_1 \dot{\to} V_2$ and $\operatorname{dom} f = X$, then $f \in (Z^Y)^X$.
(23) If $f \in [\!: X, Y :\!] \dot{\to} Z$, then $\operatorname{curry} f \in X \dot{\to} (Y \dot{\to} Z)$ and $\operatorname{curry}' f \in Y \dot{\to} (X \dot{\to} Z)$.
(24) If $f \in X \dot{\to} (Y \dot{\to} Z)$, then $\operatorname{uncurry} f \in [\!: X, Y :\!] \dot{\to} Z$ and $\operatorname{uncurry}' f \in [\!: Y, X :\!] \dot{\to} Z$.
(25) If $\operatorname{curry} f \in X \dot{\to} (Y \dot{\to} Z)$ or $\operatorname{curry}' f \in Y \dot{\to} (X \dot{\to} Z)$ but $\operatorname{dom} f \subseteq [\!: V_1, V_2 :\!]$, then $f \in [\!: X, Y :\!] \dot{\to} Z$.
(26) If $\operatorname{uncurry} f \in [\!: X, Y :\!] \dot{\to} Z$ or $\operatorname{uncurry}' f \in [\!: Y, X :\!] \dot{\to} Z$ but $\operatorname{rng} f \subseteq V_1 \dot{\to} V_2$ and $\operatorname{dom} f \subseteq X$, then $f \in X \dot{\to} (Y \dot{\to} Z)$.

Functions yielding functions

Let X be a set. The functor $\operatorname{Sub_f} X$ is defined as follows:

(Def.1) $x \in \operatorname{Sub_f} X$ if and only if $x \in X$ and x is a function.

Next we state four propositions:

(27) $\mathrm{Sub_f}\, X \subseteq X$.

(28) $x \in f^{-1}\, \mathrm{Sub_f}\, \mathrm{rng}\, f$ if and only if $x \in \mathrm{dom}\, f$ and $f(x)$ is a function.

(29) $\mathrm{Sub_f}\, \emptyset = \emptyset$ and $\mathrm{Sub_f}\{f\} = \{f\}$ and $\mathrm{Sub_f}\{f, g\} = \{f, g\}$ and $\mathrm{Sub_f}\{f, g, h\} = \{f, g, h\}$.

(30) If $Y \subseteq \mathrm{Sub_f}\, X$, then $\mathrm{Sub_f}\, Y = Y$.

We now define three new functors. Let f be a function. The functor $\mathrm{dom}_\kappa\, f(\kappa)$ yielding a function is defined by:

(Def.2) $\mathrm{dom}(\mathrm{dom}_\kappa\, f(\kappa)) = f^{-1}\, \mathrm{Sub_f}\, \mathrm{rng}\, f$ and for every x such that $x \in f^{-1}$ $\mathrm{Sub_f}\, \mathrm{rng}\, f$ holds $(\mathrm{dom}_\kappa\, f(\kappa))(x) = \pi_1(f(x))$.

The functor $\mathrm{rng}_\kappa\, f(\kappa)$ yields a function and is defined as follows:

(Def.3) $\mathrm{dom}(\mathrm{rng}_\kappa\, f(\kappa)) = f^{-1}\, \mathrm{Sub_f}\, \mathrm{rng}\, f$ and for every x such that $x \in f^{-1}$ $\mathrm{Sub_f}\, \mathrm{rng}\, f$ holds $(\mathrm{rng}_\kappa\, f(\kappa))(x) = \pi_2(f(x))$.

The functor $\bigcap f$ is defined as follows:

(Def.4) $\bigcap f = \bigcap \mathrm{rng}\, f$.

Next we state a number of propositions:

(31) If $x \in \mathrm{dom}\, f$ and $g = f(x)$, then $x \in \mathrm{dom}(\mathrm{dom}_\kappa\, f(\kappa))$ and $(\mathrm{dom}_\kappa\, f(\kappa))(x) = \mathrm{dom}\, g$
and $x \in \mathrm{dom}(\mathrm{rng}_\kappa\, f(\kappa))$ and $(\mathrm{rng}_\kappa\, f(\kappa))(x) = \mathrm{rng}\, g$.

(32) $\mathrm{dom}_\kappa\, \square(\kappa) = \square$ and $\mathrm{rng}_\kappa\, \square(\kappa) = \square$.

(33) $\mathrm{dom}_\kappa \langle f \rangle(\kappa) = \langle \mathrm{dom}\, f \rangle$ and $\mathrm{rng}_\kappa \langle f \rangle(\kappa) = \langle \mathrm{rng}\, f \rangle$.

(34) $\mathrm{dom}_\kappa \langle f, g \rangle(\kappa) = \langle \mathrm{dom}\, f, \mathrm{dom}\, g \rangle$ and $\mathrm{rng}_\kappa \langle f, g \rangle(\kappa) = \langle \mathrm{rng}\, f, \mathrm{rng}\, g \rangle$.

(35) $\mathrm{dom}_\kappa \langle f, g, h \rangle(\kappa) = \langle \mathrm{dom}\, f, \mathrm{dom}\, g, \mathrm{dom}\, h \rangle$ and $\mathrm{rng}_\kappa \langle f, g, h \rangle(\kappa) = \langle \mathrm{rng}\, f, \mathrm{rng}\, g, \mathrm{rng}\, h \rangle$.

(36) $\mathrm{dom}_\kappa(X \longmapsto f)(\kappa) = X \longmapsto \mathrm{dom}\, f$ and $\mathrm{rng}_\kappa(X \longmapsto f)(\kappa) = X \longmapsto \mathrm{rng}\, f$.

(37) If $f \neq \square$, then $x \in \bigcap f$ if and only if for every y such that $y \in \mathrm{dom}\, f$ holds $x \in f(y)$.

(38) $\bigcup \square = \emptyset$ and $\bigcap \square = \emptyset$.

(39) $\bigcup \langle X \rangle = X$ and $\bigcap \langle X \rangle = X$.

(40) $\bigcup \langle X, Y \rangle = X \cup Y$ and $\bigcap \langle X, Y \rangle = X \cap Y$.

(41) $\bigcup \langle X, Y, Z \rangle = X \cup Y \cup Z$ and $\bigcap \langle X, Y, Z \rangle = X \cap Y \cap Z$.

(42) $\bigcup(\emptyset \longmapsto Y) = \emptyset$ and $\bigcap(\emptyset \longmapsto Y) = \emptyset$.

(43) If $X \neq \emptyset$, then $\bigcup(X \longmapsto Y) = Y$ and $\bigcap(X \longmapsto Y) = Y$.

Let f be a function, and let x, y be arbitrary. The functor $f(x)(y)$ is defined by:

(Def.5) $f(x)(y) = (\mathrm{uncurry}\, f)(\langle\!\langle x, y \rangle\!\rangle)$.

We now state several propositions:

(44) If $x \in \mathrm{dom}\, f$ and $g = f(x)$ and $y \in \mathrm{dom}\, g$, then $f(x)(y) = g(y)$.

(45) If $x \in \operatorname{dom} f$, then $\langle f\rangle(1)(x) = f(x)$ and $\langle f, g\rangle(1)(x) = f(x)$ and $\langle f, g, h\rangle(1)(x) = f(x)$.

(46) If $x \in \operatorname{dom} g$, then $\langle f, g\rangle(2)(x) = g(x)$ and $\langle f, g, h\rangle(2)(x) = g(x)$.

(47) If $x \in \operatorname{dom} h$, then $\langle f, g, h\rangle(3)(x) = h(x)$.

(48) If $x \in X$ and $y \in \operatorname{dom} f$, then $(X \longmapsto f)(x)(y) = f(y)$.

Cartesian product of functions with the same domain

Let f be a function. The functor $\prod^* f$ yielding a function is defined as follows:

(Def.6) $\prod^* f = \operatorname{curry}(\operatorname{uncurry}' f \restriction [\!:\bigcap(\operatorname{dom}_\kappa f(\kappa)), \operatorname{dom} f\,:\!])$.

We now state several propositions:

(49) $\operatorname{dom} \prod^* f = \bigcap(\operatorname{dom}_\kappa f(\kappa))$ and $\operatorname{rng} \prod^* f \subseteq \prod(\operatorname{rng}_\kappa f(\kappa))$.

(50) If $x \in \operatorname{dom} \prod^* f$, then $(\prod^* f)(x)$ is a function.

(51) If $x \in \operatorname{dom} \prod^* f$ and $g = (\prod^* f)(x)$, then $\operatorname{dom} g = f^{-1} \operatorname{Sub_f} \operatorname{rng} f$ and for every y such that $y \in \operatorname{dom} g$ holds $\langle y, x\rangle \in \operatorname{dom} \operatorname{uncurry} f$ and $g(y) = (\operatorname{uncurry} f)(\langle y, x\rangle)$.

(52) If $x \in \operatorname{dom} \prod^* f$, then for every g such that $g \in \operatorname{rng} f$ holds $x \in \operatorname{dom} g$.

(53) If $g \in \operatorname{rng} f$ and for every g such that $g \in \operatorname{rng} f$ holds $x \in \operatorname{dom} g$, then $x \in \operatorname{dom} \prod^* f$.

(54) If $x \in \operatorname{dom} f$ and $g = f(x)$ and $y \in \operatorname{dom} \prod^* f$ and $h = (\prod^* f)(y)$, then $g(y) = h(x)$.

(55) If $x \in \operatorname{dom} f$ and $f(x)$ is a function and $y \in \operatorname{dom} \prod^* f$, then $f(x)(y) = (\prod^* f)(y)(x)$.

Cartesian product of functions

Let f be a function. The functor $\prod^\circ f$ yielding a function is defined by the conditions (Def.7).

(Def.7) (i) $\operatorname{dom} \prod^\circ f = \prod(\operatorname{dom}_\kappa f(\kappa))$,

(ii) for every g such that $g \in \prod(\operatorname{dom}_\kappa f(\kappa))$ there exists h such that $(\prod^\circ f)(g) = h$ and $\operatorname{dom} h = f^{-1} \operatorname{Sub_f} \operatorname{rng} f$ and for every x such that $x \in \operatorname{dom} h$ holds $h(x) = (\operatorname{uncurry} f)(\langle x, g(x)\rangle)$.

The following propositions are true:

(56) If $g \in \prod(\operatorname{dom}_\kappa f(\kappa))$ and $x \in \operatorname{dom} g$, then $(\prod^\circ f)(g)(x) = f(x)(g(x))$.

(57) If $x \in \operatorname{dom} f$ and $g = f(x)$ and $h \in \prod(\operatorname{dom}_\kappa f(\kappa))$ and $h' = (\prod^\circ f)(h)$, then $h(x) \in \operatorname{dom} g$ and $h'(x) = g(h(x))$ and $h' \in \prod(\operatorname{rng}_\kappa f(\kappa))$.

(58) $\operatorname{rng} \prod^\circ f = \prod(\operatorname{rng}_\kappa f(\kappa))$.

(59) If $\Box \notin \operatorname{rng} f$, then $\prod^\circ f$ is one-to-one if and only if for every g such that $g \in \operatorname{rng} f$ holds g is one-to-one.

Properties of Cartesian products of functions

The following propositions are true:

(60) $\prod^* \square = \square$ and $\prod^\circ \square = \{\square\} \longmapsto \square$.

(61) $\operatorname{dom} \prod^* \langle h \rangle = \operatorname{dom} h$ and for every x such that $x \in \operatorname{dom} h$ holds $(\prod^* \langle h \rangle)(x) = \langle h(x) \rangle$.

(62) $\operatorname{dom} \prod^* \langle f_1, f_2 \rangle = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for every x such that $x \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ holds $(\prod^* \langle f_1, f_2 \rangle)(x) = \langle f_1(x), f_2(x) \rangle$.

(63) If $X \neq \emptyset$, then $\operatorname{dom} \prod^*(X \longmapsto f) = \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} f$ holds $(\prod^*(X \longmapsto f))(x) = X \longmapsto f(x)$.

(64) $\operatorname{dom} \prod^\circ \langle h \rangle = \prod \langle \operatorname{dom} h \rangle$ and $\operatorname{rng} \prod^\circ \langle h \rangle = \prod \langle \operatorname{rng} h \rangle$ and for every x such that $x \in \operatorname{dom} h$ holds $(\prod^\circ \langle h \rangle)(\langle x \rangle) = \langle h(x) \rangle$.

(65) (i) $\operatorname{dom} \prod^\circ \langle f_1, f_2 \rangle = \prod \langle \operatorname{dom} f_1, \operatorname{dom} f_2 \rangle$,
(ii) $\operatorname{rng} \prod^\circ \langle f_1, f_2 \rangle = \prod \langle \operatorname{rng} f_1, \operatorname{rng} f_2 \rangle$,
(iii) for all x, y such that $x \in \operatorname{dom} f_1$ and $y \in \operatorname{dom} f_2$ holds $(\prod^\circ \langle f_1, f_2 \rangle)(\langle x, y \rangle) = \langle f_1(x), f_2(y) \rangle$.

(66) $\operatorname{dom} \prod^\circ(X \longmapsto f) = (\operatorname{dom} f)^X$ and $\operatorname{rng} \prod^\circ(X \longmapsto f) = (\operatorname{rng} f)^X$ and for every g such that $g \in (\operatorname{dom} f)^X$ holds $(\prod^\circ(X \longmapsto f))(g) = f \cdot g$.

(67) If $x \in \operatorname{dom} f_1$ and $x \in \operatorname{dom} f_2$, then for all y_1, y_2 holds $\langle f_1, f_2 \rangle(x) = \langle y_1, y_2 \rangle$ if and only if $(\prod^* \langle f_1, f_2 \rangle)(x) = \langle y_1, y_2 \rangle$.

(68) If $x \in \operatorname{dom} f_1$ and $y \in \operatorname{dom} f_2$, then for all y_1, y_2 holds $[\!: f_1, f_2 :\!](\langle x, y \rangle) = \langle y_1, y_2 \rangle$ if and only if $(\prod^\circ \langle f_1, f_2 \rangle)(\langle x, y \rangle) = \langle y_1, y_2 \rangle$.

(69) If $\operatorname{dom} f = X$ and $\operatorname{dom} g = X$ and for every x such that $x \in X$ holds $f(x) \approx g(x)$, then $\prod f \approx \prod g$.

(70) If $\operatorname{dom} f = \operatorname{dom} h$ and $\operatorname{dom} g = \operatorname{rng} h$ and h is one-to-one and for every x such that $x \in \operatorname{dom} h$ holds $f(x) \approx g(h(x))$, then $\prod f \approx \prod g$.

(71) If $\operatorname{dom} f = X$, then $\prod f \approx \prod (f \cdot P)$.

Function yielding powers

Let us consider f, X. The functor X^f yielding a function is defined by:

(Def.8) $\operatorname{dom}(X^f) = \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} f$ holds $X^f(x) = X^{f(x)}$.

We now state several propositions:

(72) If $\emptyset \notin \operatorname{rng} f$, then $\emptyset^f = \operatorname{dom} f \longmapsto \emptyset$.

(73) $X^\square = \square$.

(74) $Y^{\langle X \rangle} = \langle Y^X \rangle$.

(75) $Z^{\langle X, Y \rangle} = \langle Z^X, Z^Y \rangle$.

(76) $Z^{X \longmapsto Y} = X \longmapsto Z^Y$.

(77) $X^{\bigcup \operatorname{disjoin} f} \approx \prod (X^f)$.

Let us consider X, f. The functor f^X yielding a function is defined by:

(Def.9) $\mathrm{dom}(f^X) = \mathrm{dom}\, f$ and for every x such that $x \in \mathrm{dom}\, f$ holds $f^X(x) = f(x)^X$.

Next we state several propositions:

(78) $f^{\emptyset} = \mathrm{dom}\, f \longmapsto \{\Box\}$.

(79) $\Box^X = \Box$.

(80) $\langle Y \rangle^X = \langle Y^X \rangle$.

(81) $\langle Y, Z \rangle^X = \langle Y^X, Z^X \rangle$.

(82) $(Y \longmapsto Z)^X = Y \longmapsto Z^X$.

(83) $\prod(f^X) \approx (\prod f)^X$.

References

[1] Grzegorz Bancerek. Cardinal arithmetics. *Formalized Mathematics*, 1(**3**):543–547, 1990.
[2] Grzegorz Bancerek. Curried and uncurried functions. *Formalized Mathematics*, 1(**3**):537–541, 1990.
[3] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[4] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Formalized Mathematics*, 1(**2**):265–267, 1990.
[5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[6] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.
[7] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[10] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[11] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[12] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[13] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[14] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[15] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.
[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[17] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

Received September 30, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Introduction to Modal Propositional Logic

Alicia de la Cruz
Universidad Politecnica de Madrid

MML Identifier: MODAL_1.

The terminology and notation used here are introduced in the following papers: [15], [11], [2], [14], [16], [13], [7], [5], [6], [8], [10], [12], [1], [9], [3], [4], and [17]. For simplicity we follow a convention: x, y will be arbitrary, n, m, k will denote natural numbers, t_1 will denote a tree decorated by $[\!:\mathbb{N}, \mathbb{N}$ **qua** a non-empty set$:\!]$, w, s, t will denote finite sequences of elements of $\mathbb{N}$, X will denote a set, and D will denote a non-empty set. Next we state the proposition

(1) If X is finite, then $\operatorname{card} X = 2$ if and only if there exist x, y such that $X = \{x, y\}$ and $x \neq y$.

Let Z be a tree. The root of Z yields an element of Z and is defined as follows:

(Def.1) the root of $Z = \varepsilon$.

Let us consider D, and let T be a tree decorated by D. The root of T yields an element of D and is defined by:

(Def.2) the root of $T = T$(the root of $\operatorname{dom} T$).

Next we state a number of propositions:

(2) $\langle n \rangle = \langle m \rangle$ if and only if $n = m$.

(3) If $n \neq m$, then $\langle n \rangle$ and $\langle m \rangle \frown s$ are not comparable.

(4) For every s such that $s \neq \varepsilon$ there exist w, n such that $s = \langle n \rangle \frown w$.

(5) If $n \neq m$, then $\langle n \rangle \not\prec \langle m \rangle \frown s$.

(6) If $n \neq m$, then $\langle n \rangle \not\preceq \langle m \rangle \frown s$.

(7) $\langle n \rangle \not\prec \langle m \rangle$.

(8) If $w \neq \varepsilon$, then $s \prec s \frown w$.

(9) The elementary tree of $1 = \{\varepsilon, \langle 0 \rangle\}$.

(10) The elementary tree of $2 = \{\varepsilon, \langle 0 \rangle, \langle 1 \rangle\}$.

(11) For every tree Z and for all n, m such that $n \leq m$ and $\langle m \rangle \in Z$ holds $\langle n \rangle \in Z$.

ISSN 0777–4028

(12) If $w \frown t \prec w \frown s$, then $t \prec s$.

(13) $t_1 \in \mathbb{N}^* \dot{\rightarrow} [\!:\mathbb{N}, \mathbb{N}$ **qua** a non-empty set$:\!]$.

(14) For all trees Z, Z_1 and for every element z of Z holds $z \in Z(z/Z_1)$.

(15) For all trees Z, Z_1, Z_2 and for every element z of Z such that $Z(z/Z_1) = Z(z/Z_2)$ holds $Z_1 = Z_2$.

(16) For all trees Z, Z_1, Z_2 decorated by D and for every element z of $\mathrm{dom}\, Z$ such that $Z(z/Z_1) = Z(z/Z_2)$ holds $Z_1 = Z_2$.

(17) For all trees Z_1, Z_2 and for every finite sequence p of elements of $\mathbb{N}$ such that $p \in Z_1$ for every element v of $Z_1(p/Z_2)$ and for every element w of Z_1 such that $v = w$ and $w \prec p$ holds $\mathrm{succ}\, v = \mathrm{succ}\, w$.

(18) For all trees Z_1, Z_2 and for every finite sequence p of elements of $\mathbb{N}$ such that $p \in Z_1$ for every element v of $Z_1(p/Z_2)$ and for every element w of Z_1 such that $v = w$ and p and w are not comparable holds $\mathrm{succ}\, v = \mathrm{succ}\, w$.

(19) For all trees Z_1, Z_2 and for every finite sequence p of elements of $\mathbb{N}$ such that $p \in Z_1$ for every element v of $Z_1(p/Z_2)$ and for every element w of Z_2 such that $v = p \frown w$ holds $\mathrm{succ}\, v \approx \mathrm{succ}\, w$.

(20) For every tree Z_1 and for every finite sequence p of elements of $\mathbb{N}$ such that $p \in Z_1$ for every element v of Z_1 and for every element w of $Z_1 \restriction p$ such that $v = p \frown w$ holds $\mathrm{succ}\, v \approx \mathrm{succ}\, w$.

(21) For every tree Z and for every element p of Z such that Z is finite holds $\mathrm{succ}\, p$ is finite.

(22) For every tree Z such that Z is finite and the branch degree of the root of $Z = 0$ holds $\mathrm{card}\, Z = 1$ and $Z = \{\varepsilon\}$.

(23) For every tree Z such that Z is finite and the branch degree of the root of $Z = 1$ holds succ(the root of Z) $= \{\langle 0 \rangle\}$.

(24) For every tree Z such that Z is finite and the branch degree of the root of $Z = 2$ holds succ(the root of Z) $= \{\langle 0 \rangle, \langle 1 \rangle\}$.

In the sequel s', w' will be elements of $\mathbb{N}^*$. One can prove the following propositions:

(25) For every tree Z and for every element o of Z such that $o \neq$ the root of Z holds $Z \restriction o \approx \{o \frown s' : o \frown s' \in Z\}$ and the root of $Z \notin \{o \frown w' : o \frown w' \in Z\}$.

(26) For every tree Z and for every element o of Z such that $o \neq$ the root of Z and Z is finite holds $\mathrm{card}(Z \restriction o) < \mathrm{card}\, Z$.

(27) For every tree Z and for every element z of Z such that succ(the root of Z) $= \{z\}$ and Z is finite holds $Z =$ (the elementary tree of 1)$(\langle 0 \rangle/(Z \restriction z))$.

(28) For every tree Z decorated by D and for every element z of $\mathrm{dom}\, Z$ such that succ(the root of $\mathrm{dom}\, Z$) $= \{z\}$ and $\mathrm{dom}\, Z$ is finite holds $Z =$ (the elementary tree of $1 \longmapsto$ the root of Z)$(\langle 0 \rangle/(Z \restriction z))$.

(29) For every tree Z and for all elements x_1, x_2 of Z such that Z is finite and $x_1 = \langle 0 \rangle$ and $x_2 = \langle 1 \rangle$ and succ(the root of Z) $= \{x_1, x_2\}$ holds $Z =$ (the elementary tree of 2)$(\langle 0 \rangle/(Z \restriction x_1))(\langle 1 \rangle/(Z \restriction x_2))$.

(30) Let Z be a tree decorated by D. Then for all elements x_1, x_2 of $\operatorname{dom} Z$ such that $\operatorname{dom} Z$ is finite and $x_1 = \langle 0\rangle$ and $x_2 = \langle 1\rangle$ and succ(the root of $\operatorname{dom} Z) = \{x_1, x_2\}$ holds $Z =$ (the elementary tree of $2 \longmapsto$ the root of $Z)(\langle 0\rangle/(Z \restriction x_1))(\langle 1\rangle/(Z \restriction x_2))$.

The non-empty set $\mathcal{V}$ is defined by:

(Def.3) $\mathcal{V} = [\!:\{3\}, \mathbb{N}:\!]$.

A variable is an element of $\mathcal{V}$.

The non-empty set $\mathcal{C}$ is defined as follows:

(Def.4) $\mathcal{C} = [\!:\{0, 1, 2\}, \mathbb{N}:\!]$.

A conective is an element of $\mathcal{C}$.

One can prove the following proposition

(31) $\mathcal{C} \cap \mathcal{V} = \emptyset$.

In the sequel p, q denote variables. Let T be a tree, and let v be an element of T. Then the branch degree of v is a natural number.

Let D be a non-empty set. A non-empty set is called a non-empty set of trees decorated by D if:

(Def.5) for every x such that $x \in$ it holds x is a tree decorated by D.

Let D_0 be a non-empty set, and let D be a non-empty set of trees decorated by D_0. We see that the element of D is a tree decorated by D_0.

The non-empty set WFF of trees decorated by $[\!:\mathbb{N}, \mathbb{N}$ **qua** a non-empty set$:\!]$ is defined by the condition (Def.6).

(Def.6) Let x be a tree decorated by $[\!:\mathbb{N}, \mathbb{N}$ **qua** a non-empty set$:\!]$. Then $x \in$ WFF if and only if the following conditions are satisfied:
 (i) $\operatorname{dom} x$ is finite,
 (ii) for every element v of $\operatorname{dom} x$ holds the branch degree of $v \le 2$ but if the branch degree of $v = 0$, then $x(v) = \langle 0, 0\rangle$ or there exists k such that $x(v) = \langle 3, k\rangle$ but if the branch degree of $v = 1$, then $x(v) = \langle 1, 0\rangle$ or $x(v) = \langle 1, 1\rangle$ but if the branch degree of $v = 2$, then $x(v) = \langle 2, 0\rangle$.

A MP-formula is an element of WFF.

In the sequel A, A_1, B, B_1, C denote MP-formulae. Let us consider A, and let a be an element of $\operatorname{dom} A$. Then $A \restriction a$ is a MP-formula.

Let a be an element of $\mathcal{C}$. The functor Arity(a) yielding a natural number is defined by:

(Def.7) $\operatorname{Arity}(a) = a_{\mathbf{1}}$.

Let D be a non-empty set, and let T, T_1 be trees decorated by D, and let p be a finite sequence of elements of $\mathbb{N}$. Let us assume that $p \in \operatorname{dom} T$. The functor $T(p \leftarrow T_1)$ yields a tree decorated by D and is defined by:

(Def.8) $T(p \leftarrow T_1) = T(p/T_1)$.

The following propositions are true:

(32) (The elementary tree of $1 \longmapsto \langle 1, 0\rangle)(\langle 0\rangle/A)$ is a MP-formula.

(33) (The elementary tree of $1 \longmapsto \langle 1, 1\rangle)(\langle 0\rangle/A)$ is a MP-formula.

(34) (The elementary tree of $2 \longmapsto \langle 2, 0\rangle)(\langle 0\rangle/A)(\langle 1\rangle/B)$ is a MP-formula.

We now define three new functors. Let us consider A. The functor $\neg A$ yields a MP-formula and is defined as follows:

(Def.9) $\neg A =$ (the elementary tree of $1 \longmapsto \langle 1, 0\rangle)(\langle 0\rangle/A)$.

The functor $\Box A$ yields a MP-formula and is defined as follows:

(Def.10) $\Box A =$ (the elementary tree of $1 \longmapsto \langle 1, 1\rangle)(\langle 0\rangle/A)$.

Let us consider B. The functor $A \wedge B$ yielding a MP-formula is defined as follows:

(Def.11) $A \wedge B =$ (the elementary tree of $2 \longmapsto \langle 2, 0\rangle)(\langle 0\rangle/A)(\langle 1\rangle/B)$.

We now define three new functors. Let us consider A. The functor $\Diamond A$ yields a MP-formula and is defined as follows:

(Def.12) $\Diamond A = \neg\Box\neg A$.

Let us consider B. The functor $A \vee B$ yields a MP-formula and is defined as follows:

(Def.13) $A \vee B = \neg(\neg A \wedge \neg B)$.

The functor $A \Rightarrow B$ yields a MP-formula and is defined by:

(Def.14) $A \Rightarrow B = \neg(A \wedge \neg B)$.

The following propositions are true:

(35) The elementary tree of $0 \longmapsto \langle 3, n\rangle$ is a MP-formula.

(36) The elementary tree of $0 \longmapsto \langle 0, 0\rangle$ is a MP-formula.

Let us consider p. The functor ${}^{@}p$ yields a MP-formula and is defined by:

(Def.15) ${}^{@}p =$ the elementary tree of $0 \longmapsto p$.

We now state four propositions:

(37) If ${}^{@}p = {}^{@}q$, then $p = q$.

(38) If $\neg A = \neg B$, then $A = B$.

(39) If $\Box A = \Box B$, then $A = B$.

(40) If $A \wedge B = A_1 \wedge B_1$, then $A = A_1$ and $B = B_1$.

The MP-formula VERUM is defined by:

(Def.16) VERUM = the elementary tree of $0 \longmapsto \langle 0, 0\rangle$.

Next we state several propositions:

(41) $\operatorname{card}\operatorname{dom} A \neq 0$.

(42) If $\operatorname{card}\operatorname{dom} A = 1$, then $A =$ VERUM or there exists p such that $A = {}^{@}p$.

(43) If $\operatorname{card}\operatorname{dom} A \geq 2$, then there exists B such that $A = \neg B$ or $A = \Box B$ or there exist B, C such that $A = B \wedge C$.

(44) $\operatorname{card}\operatorname{dom} A < \operatorname{card}\operatorname{dom} \neg A$.

(45) $\operatorname{card}\operatorname{dom} A < \operatorname{card}\operatorname{dom} \Box A$.

(46) $\operatorname{card}\operatorname{dom} A < \operatorname{card}\operatorname{dom}(A \wedge B)$ and $\operatorname{card}\operatorname{dom} B < \operatorname{card}\operatorname{dom}(A \wedge B)$.

We now define four new attributes. A MP-formula is atomic if:

(Def.17) there exists p such that it $= {}^{@}p$.

A MP-formula is negative if:

(Def.18) there exists A such that it $= \neg A$.

A MP-formula is necessitive if:

(Def.19) there exists A such that it $= \Box A$.

A MP-formula is conjunctive if:

(Def.20) there exist A, B such that it $= A \wedge B$.

The scheme *MP_Ind* deals with a unary predicate $\mathcal{P}$, and states that:

for every element A of WFF holds $\mathcal{P}[A]$

provided the parameter satisfies the following conditions:

- $\mathcal{P}[\mathrm{VERUM}]$,
- for every variable p holds $\mathcal{P}[{}^{@}p]$,
- for every element A of WFF such that $\mathcal{P}[A]$ holds $\mathcal{P}[\neg A]$,
- for every element A of WFF such that $\mathcal{P}[A]$ holds $\mathcal{P}[\Box A]$,
- for all elements A, B of WFF such that $\mathcal{P}[A]$ and $\mathcal{P}[B]$ holds $\mathcal{P}[A \wedge B]$.

The following propositions are true:

(47) For every element A of WFF holds $A = \mathrm{VERUM}$ or A is a MP-formula or A is a MP-formula or A is a MP-formula or A is a MP-formula.

(48) $A = \mathrm{VERUM}$ or there exists p such that $A = {}^{@}p$ or there exists B such that $A = \neg B$ or there exists B such that $A = \Box B$ or there exist B, C such that $A = B \wedge C$.

(49) ${}^{@}p \neq \neg A$ and ${}^{@}p \neq \Box A$ and ${}^{@}p \neq A \wedge B$.

(50) $\neg A \neq \Box B$ and $\neg A \neq B \wedge C$.

(51) $\Box A \neq B \wedge C$.

(52) $\mathrm{VERUM} \neq {}^{@}p$ and $\mathrm{VERUM} \neq \neg A$ and $\mathrm{VERUM} \neq \Box A$ and $\mathrm{VERUM} \neq A \wedge B$.

The scheme *MP_Func_Ex* deals with a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$ and states that:

there exists a function f from WFF into $\mathcal{A}$ such that $f(\mathrm{VERUM}) = \mathcal{B}$ and for every variable p holds $f({}^{@}p) = \mathcal{F}(p)$ and for every element A of WFF and for every element d of $\mathcal{A}$ such that $f(A) = d$ holds $f(\neg A) = \mathcal{G}(d)$ and for every element A of WFF and for every element d of $\mathcal{A}$ such that $f(A) = d$ holds $f(\Box A) = \mathcal{H}(d)$ and for all elements A, B of WFF and for all elements d_1, d_2 of $\mathcal{A}$ such that $d_1 = f(A)$ and $d_2 = f(B)$ holds $f(A \wedge B) = \mathcal{I}(d_1, d_2)$

for all values of the parameters.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. Introduction to trees. *Formalized Mathematics*, 1(**2**):421–427, 1990.
[4] Grzegorz Bancerek. König's lemma. *Formalized Mathematics*, 2(**3**):397–402, 1991.
[5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[6] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[9] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[10] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[11] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[12] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[13] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[14] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[16] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.
[17] Wojciech Zielonka. Preliminaries to the Lambek calculus. *Formalized Mathematics*, 2(**3**):413–418, 1991.

Received September 30, 1990

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Totally Bounded Metric Spaces

Alicia de la Cruz
Universidad Politecnica de Madrid

MML Identifier: TBSP_1.

The papers [19], [9], [1], [4], [20], [2], [18], [13], [5], [8], [14], [21], [7], [15], [12], [11], [17], [6], [10], [16], and [3] provide the terminology and notation for this paper. For simplicity we follow the rules: M is a metric space, c, g are elements of the carrier of M, F is a family of subsets of the carrier of M, A, B are subsets of the carrier of M, f is a function, n, m, p, k are natural numbers, and r, s, L are real numbers. Next we state four propositions:

(1) For every L such that $0 < L$ and $L < 1$ for all n, m such that $n \leq m$ holds $L^m \leq L^n$.

(2) For every L such that $0 < L$ and $L < 1$ for every k holds $L^k \leq 1$ and $0 < L^k$.

(3) For every L such that $0 < L$ and $L < 1$ for every s such that $0 < s$ there exists n such that $L^n < s$.

(4) For every set X such that X is finite and $X \neq \emptyset$ and for all sets Y, Z such that $Y \in X$ and $Z \in X$ holds $Y \subseteq Z$ or $Z \subseteq Y$ there exists a set V such that $V \in X$ and for every set Z such that $Z \in X$ holds $V \subseteq Z$.

Let us consider M, F. Then $\bigcup F$ is a subset of the carrier of M.

Let D be a non-empty set. Then Ω_D is a subset of D. Then $\emptyset_D$ is a subset of D.

Let us consider M. We say that M is totally bounded if and only if:

(Def.1) for every r such that $r > 0$ there exists F such that F is finite and the carrier of $M = \bigcup F$ and for every A such that $A \in F$ there exists g such that $A = \mathrm{Ball}(g, r)$.

Let us consider M. A function is called a sequence of M if:

(Def.2) $\mathrm{dom}\,\mathrm{it} = \mathbb{N}$ and $\mathrm{rng}\,\mathrm{it} \subseteq$ the carrier of M.

In the sequel S_1 will denote a sequence of M. The following proposition is true

ISSN 0777–4028

(5) f is a sequence of M if and only if $\mathrm{dom}\, f = \mathbb{N}$ and for every n holds $f(n)$ is an element of the carrier of M.

Let us consider M, S_1, n. Then $S_1(n)$ is an element of the carrier of M.

Let us consider M, S_1. We say that S_1 is convergent if and only if:

(Def.3) there exists an element x of the carrier of M such that for every r such that $r > 0$ there exists n such that for every m such that $n \leq m$ holds $\rho(S_1(m), x) < r$.

Let us consider M, S_1. Let us assume that S_1 is convergent. The functor $\lim S_1$ yields an element of the carrier of M and is defined by:

(Def.4) for every r such that $r > 0$ there exists n such that for every m such that $m \geq n$ holds $\rho(S_1(m), \lim S_1) < r$.

The following proposition is true

(6) For every S_1 such that S_1 is convergent holds $\lim S_1 = g$ if and only if for every r such that $0 < r$ there exists n such that for every m such that $n \leq m$ holds $\rho(S_1(m), g) < r$.

Let us consider M, S_1. We say that S_1 is a Cauchy sequence if and only if:

(Def.5) for every r such that $r > 0$ there exists p such that for all n, m such that $p \leq n$ and $p \leq m$ holds $\rho(S_1(n), S_1(m)) < r$.

Let us consider M. We say that M is complete if and only if:

(Def.6) for every S_1 such that S_1 is a Cauchy sequence holds S_1 is convergent.

We now state two propositions:

(7) For every S_1 such that S_1 is convergent holds S_1 is a Cauchy sequence.

(8) For every S_1 holds S_1 is a Cauchy sequence if and only if for every r such that $r > 0$ there exists p such that for all n, k such that $p \leq n$ holds $\rho(S_1(n+k), S_1(n)) < r$.

Let us consider M. A function from the carrier of M into the carrier of M is called a contraction of M if:

(Def.7) there exists L such that $0 < L$ and $L < 1$ and for all points x, y of M holds $\rho(\mathrm{it}(x), \mathrm{it}(y)) \leq L \cdot \rho(x, y)$.

We now state four propositions:

(9) For every contraction f of M such that M is complete there exists c such that $f(c) = c$ and for every element y of the carrier of M such that $f(y) = y$ holds $y = c$.

(10) If M_{top} is compact, then M is complete.

(11) For every contraction f of M such that M_{top} is compact there exists an element c of the carrier of M such that $f(c) = c$ and for every element y of the carrier of M such that $f(y) = y$ holds $y = c$.

(12) If M_{top} is compact, then M is totally bounded.

We now define two new predicates. Let us consider M. We say that M is bounded if and only if:

(Def.8) there exists r such that $0 < r$ and for all points x, y of M holds $\rho(x, y) \leq r$.

Let us consider A. We say that A is bounded if and only if:

(Def.9) (i) there exists r such that $0 < r$ and for all points x, y of M such that $x \in A$ and $y \in A$ holds $\rho(x, y) \leq r$ if $A \neq \emptyset$.

One can prove the following propositions:

(13) If $A \neq \emptyset$, then A is bounded if and only if there exists r such that $0 < r$ and for all points x, y of M such that $x \in A$ and $y \in A$ holds $\rho(x, y) \leq r$.

(14) $\emptyset_{\text{the carrier of } M}$ is bounded.

(15) If $A \neq \emptyset$, then A is bounded if and only if there exist r, c such that $0 < r$ and $c \in A$ and for every point z of M such that $z \in A$ holds $\rho(c, z) \leq r$.

(16) If $0 < r$, then $g \in \mathrm{Ball}(g, r)$ and $\mathrm{Ball}(g, r) \neq \emptyset$.

(17) If $r \leq 0$, then $\mathrm{Ball}(g, r) = \emptyset$.

(18) If $0 < r$, then $\mathrm{Ball}(g, r)$ is bounded.

(19) $\mathrm{Ball}(g, r)$ is bounded.

(20) If A is bounded and B is bounded, then $A \cup B$ is bounded.

(21) If A is bounded and $B \subseteq A$, then B is bounded.

(22) If $A = \{g\}$, then A is bounded.

(23) If A is finite, then A is bounded.

(24) If F is finite and for every A such that $A \in F$ holds A is bounded, then $\bigcup F$ is bounded.

(25) M is bounded if and only if $\Omega_{\text{the carrier of } M}$ is bounded.

(26) If M is totally bounded, then M is bounded.

Let us consider M, A. Let us assume that $A \neq \emptyset$ and A is bounded. The functor $\vee A$ yields a real number and is defined as follows:

(Def.10) for all points x, y of M such that $x \in A$ and $y \in A$ holds $\rho(x, y) \leq \vee A$ and for every s such that for all points x, y of M such that $x \in A$ and $y \in A$ holds $\rho(x, y) \leq s$ holds $\vee A \leq s$.

We now state several propositions:

(27) Suppose $A \neq \emptyset$ and A is bounded. Then $\vee A = r$ if and only if for all points x, y of M such that $x \in A$ and $y \in A$ holds $\rho(x, y) \leq r$ and for every s such that for all points x, y of M such that $x \in A$ and $y \in A$ holds $\rho(x, y) \leq s$ holds $r \leq s$.

(28) If $A = \{g\}$, then $\vee A = 0$.

(29) If $A \neq \emptyset$ and A is bounded, then $0 \leq \vee A$.

(30) If $A \neq \emptyset$ and A is bounded, then $\vee A = 0$ if and only if there exists a point g of M such that $A = \{g\}$.

(31) If $0 < r$, then $\vee \mathrm{Ball}(g, r) \leq 2 \cdot r$.

(32) If $A \neq \emptyset$ and A is bounded and $B \neq \emptyset$ and $B \subseteq A$, then B is bounded and $\vee B \leq \vee A$.

(33) If $A \neq \emptyset$ and A is bounded and $B \neq \emptyset$ and B is bounded and $A \cap B \neq \emptyset$, then $A \cup B$ is bounded and $\vee(A \cup B) \leq \vee A + \vee B$.

Let us consider M, S_1. Then $\operatorname{rng} S_1$ is a subset of the carrier of M.

One can prove the following proposition

(34) If S_1 is a Cauchy sequence, then $\operatorname{rng} S_1$ is bounded.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.

[7] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.

[8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[10] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.

[11] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.

[12] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[13] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[14] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[15] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[16] Konrad Raczkowski and Andrzej Nędzusiak. Real exponents and logarithms. *Formalized Mathematics*, 2(**2**):213–216, 1991.

[17] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[18] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[19] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[20] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[21] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

Received September 30, 1990

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Categories of Groups

Michał Muzalewski

Summary. We define the category of groups and its subcategories: category of Abelian groups and category of groups with the operator of $\frac{1}{2}$. The carriers of the groups are included in a universum. The universum is a parameter of the categories.

MML Identifier: `GRCAT_1`.

The articles [13], [2], [14], [3], [1], [11], [7], [5], [4], [12], [10], [6], [9], and [8] provide the notation and terminology for this paper. For simplicity we follow the rules: x, y will be arbitrary, D will be a non-empty set, U_1 will be a universal class, and G, H will be group structures. Let us consider x. Then $\{x\}$ is a non-empty set.

The following propositions are true:

(1) For all sets X, Y, A and for all x, y such that $\langle x, y\rangle \in A$ and $A \subseteq [\!: X, Y :\!]$ holds x is an element of X and y is an element of Y.

(2) For all sets X, Y, A and for an arbitrary z such that $z \in A$ and $A \subseteq [\!: X, Y :\!]$ there exists an element x of X and there exists an element y of Y such that $z = \langle x, y\rangle$.

(3) For all elements u_1, u_2, u_3, u_4 of U_1 holds $\langle u_1, u_2, u_3\rangle$ is an element of U_1 and $\langle u_1, u_2, u_3, u_4\rangle$ is an element of U_1.

(4) For all x, y such that $x \in y$ and $y \in U_1$ holds $x \in U_1$.

In this article we present several logical schemes. The scheme *PartLambda2* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, a set $\mathcal{C}$, a binary functor $\mathcal{F}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a partial function f from $[\!: \mathcal{A}, \mathcal{B} :\!]$ to $\mathcal{C}$ such that for all x, y holds $\langle x, y\rangle \in \operatorname{dom} f$ if and only if $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and $\mathcal{P}[x, y]$ and for all x, y such that $\langle x, y\rangle \in \operatorname{dom} f$ holds $f(\langle x, y\rangle) = \mathcal{F}(x, y)$

provided the following requirement is met:

- for all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and $\mathcal{P}[x, y]$ holds $\mathcal{F}(x, y) \in \mathcal{C}$.

The scheme *PartLambda2D* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a set $\mathcal{C}$, a binary functor $\mathcal{F}$, and a binary predicate $\mathcal{P}$, and states that:

ISSN 0777–4028

there exists a partial function f from $[\!: \mathcal{A}, \mathcal{B} :\!]$ to $\mathcal{C}$ such that for every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $\langle x, y\rangle \in \operatorname{dom} f$ if and only if $\mathcal{P}[x, y]$ and for every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ such that $\langle x, y\rangle \in \operatorname{dom} f$ holds $f(\langle x, y\rangle) = \mathcal{F}(x, y)$

provided the parameters satisfy the following condition:

- for every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ such that $\mathcal{P}[x, y]$ holds $\mathcal{F}(x, y) \in \mathcal{C}$.

We now define three new functors. op_2 is a binary operation on $\{\emptyset\}$.

op_1 is a unary operation on $\{\emptyset\}$.

op_0 is an element of $\{\emptyset\}$.

We now state three propositions:

(5) $\mathrm{op}_2(\emptyset, \emptyset) = \emptyset$ and $\mathrm{op}_1(\emptyset) = \emptyset$ and $\mathrm{op}_0 = \emptyset$.

(6) $\{\emptyset\} \in U_1$ and $\langle\{\emptyset\}, \{\emptyset\}\rangle \in U_1$ and $[\!: \{\emptyset\}, \{\emptyset\} :\!] \in U_1$ and $\mathrm{op}_2 \in U_1$ and $\mathrm{op}_1 \in U_1$.

(7) $\langle\{\emptyset\}, \mathrm{op}_2, \mathrm{op}_1, \mathrm{op}_0\rangle$ is a group with the operator $\frac{1}{2}$.

The trivial group being a group with the operator $\frac{1}{2}$ is defined as follows:

(Def.1) the trivial group$= \langle\{\emptyset\}, \mathrm{op}_2, \mathrm{op}_1, \mathrm{op}_0\rangle$.

We now state the proposition

(8) If $G =$ the trivial group , then for every element x of G holds $x = \emptyset$ and for all elements x, y of G holds $x + y = \emptyset$ and for every element x of G holds $-x = \emptyset$ and $0_G = \emptyset$.

In the sequel C denotes a category and O denotes a non-empty subset of the objects of C. Let us consider C, O. The functor $\mathrm{Morphs}\, O$ yields a non-empty subset of the morphisms of C and is defined by:

(Def.2) $\mathrm{Morphs}\, O = \bigcup\{\hom(a, b) : a \in O \wedge b \in O\}$, where a ranges over objects of C, and b ranges over objects of C.

We now define four new functors. Let us consider C, O. The functor $\operatorname{dom} O$ yielding a function from $\mathrm{Morphs}\, O$ into O is defined by:

(Def.3) $\operatorname{dom} O =$ (the dom-map of C) $\restriction \mathrm{Morphs}\, O$.

The functor $\operatorname{cod} O$ yields a function from $\mathrm{Morphs}\, O$ into O and is defined by:

(Def.4) $\operatorname{cod} O =$ (the cod-map of C) $\restriction \mathrm{Morphs}\, O$.

The functor $\operatorname{comp} O$ yielding a partial function from $[\!: \mathrm{Morphs}\, O$, $\mathrm{Morphs}\, O$ **qua** a non-empty set $:\!]$ to $\mathrm{Morphs}\, O$ is defined as follows:

(Def.5) $\operatorname{comp} O =$ (the composition of C) $\restriction [\!: \mathrm{Morphs}\, O, \mathrm{Morphs}\, O :\!]$.

The functor I_O yielding a function from O into $\mathrm{Morphs}\, O$ is defined by:

(Def.6) $\mathrm{I}_O =$ (the id-map of C) $\restriction O$.

Next we state the proposition

(9) $\langle O, \mathrm{Morphs}\, O, \operatorname{dom} O, \operatorname{cod} O, \operatorname{comp} O, \mathrm{I}_O\rangle$ is full subcategory of C.

Let us consider C, O. The functor $\operatorname{cat} O$ yielding a subcategory of C is defined as follows:

(Def.7) $\operatorname{cat} O = \langle O, \mathrm{Morphs}\, O, \operatorname{dom} O, \operatorname{cod} O, \operatorname{comp} O, \mathrm{I}_O\rangle$.

Next we state the proposition

(10) The objects of $\mathrm{cat}\, O = O$.

Let us consider G, H. A map from G into H is a function from the carrier of G into the carrier of H.

Let G_1, G_2, G_3 be group structures, and let f be a map from G_1 into G_2, and let g be a map from G_2 into G_3. Then $g \cdot f$ is a map from G_1 into G_3.

Let us consider G. The functor id_G yields a map from G into G and is defined by:

(Def.8) $\mathrm{id}_G = \mathrm{id}_{(\text{the carrier of } G)}$.

One can prove the following two propositions:

(11) For every element x of G holds $\mathrm{id}_G(x) = x$.

(12) For every map f from G into H holds $f \cdot \mathrm{id}_G = f$ and $\mathrm{id}_H \cdot f = f$.

Let us consider G, H. The functor $\mathrm{zero}(G, H)$ yielding a map from G into H is defined by:

(Def.9) $\mathrm{zero}(G, H) = (\text{the carrier of } G) \longmapsto 0_H$.

Let us consider G, H, and let f be a map from G into H. We say that f is additive if and only if:

(Def.10) for all elements x, y of G holds $f(x + y) = f(x) + f(y)$.

One can prove the following propositions:

(13) For all G_1, G_2, G_3 being group structures and for every map f from G_1 into G_2 and for every map g from G_2 into G_3 and for every element x of G_1 holds $(g \cdot f)(x) = g(f(x))$.

(14) For all G_1, G_2, G_3 being group structures and for every map f from G_1 into G_2 and for every map g from G_2 into G_3 such that f is additive and g is additive holds $g \cdot f$ is additive.

(15) For every element x of G holds $(\mathrm{zero}(G, H))(x) = 0_H$.

(16) For every group H holds $\mathrm{zero}(G, H)$ is additive.

In the sequel G, H are groups. We consider group morphism structures which are systems

⟨a dom-map, a cod-map, a `Fun`⟩,

where the dom-map, the cod-map are a group and the `Fun` is a map from the dom-map into the cod-map.

We now define two new functors. Let f be a group morphism structure. The functor $\mathrm{dom}\, f$ yielding a group is defined as follows:

(Def.11) $\mathrm{dom}\, f =$ the dom-map of f.

The functor $\mathrm{cod}\, f$ yields a group and is defined by:

(Def.12) $\mathrm{cod}\, f =$ the cod-map of f.

Let f be a group morphism structure. The functor $\mathrm{fun}\, f$ yields a map from $\mathrm{dom}\, f$ into $\mathrm{cod}\, f$ and is defined by:

(Def.13) $\mathrm{fun}\, f =$ the `Fun` of f.

Next we state the proposition

(17) For every f being a group morphism structure and for all groups G_1, G_2 and for every map f_0 from G_1 into G_2 such that $f = \langle G_1, G_2, f_0\rangle$ holds $\mathrm{dom}\, f = G_1$ and $\mathrm{cod}\, f = G_2$ and $\mathrm{fun}\, f = f_0$.

Let us consider G, H. The functor ZERO G yielding a group morphism structure is defined as follows:

(Def.14) $\mathrm{ZERO}\, G = \langle G, H, \mathrm{zero}(G, H)\rangle$.

A group morphism structure is said to be a morphism of groups if:

(Def.15) fun it is additive.

One can prove the following proposition

(18) For every morphism F of groups holds the `Fun` of F is additive.

Let us consider G, H. Then ZERO G is a morphism of groups.

Let us consider G, H. A morphism of groups is said to be a morphism from G to H if:

(Def.16) dom it $= G$ and cod it $= H$.

We now state three propositions:

(19) For every f being a group morphism structure such that $\mathrm{dom}\, f = G$ and $\mathrm{cod}\, f = H$ and $\mathrm{fun}\, f$ is additive holds f is a morphism from G to H.

(20) For every map f from G into H such that f is additive holds $\langle G, H, f\rangle$ is a morphism from G to H.

(21) id_G is additive.

Let us consider G. The functor I_G yields a morphism from G to G and is defined by:

(Def.17) $\mathrm{I}_G = \langle G, G, \mathrm{id}_G\rangle$.

Let us consider G, H. Then ZERO G is a morphism from G to H.

We now state several propositions:

(22) For every morphism F from G to H there exists a map f from G into H such that $F = \langle G, H, f\rangle$ and f is additive.

(23) For every morphism F from G to H there exists a map f from G into H such that $F = \langle G, H, f\rangle$.

(24) For every morphism F of groups there exist G, H such that F is a morphism from G to H.

(25) For every morphism F of groups there exist groups G, H and there exists a map f from G into H such that F is a morphism from G to H and $F = \langle G, H, f\rangle$ and f is additive.

(26) For all morphisms g, f of groups such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ there exist groups G_1, G_2, G_3 such that g is a morphism from G_2 to G_3 and f is a morphism from G_1 to G_2.

(27) For every morphism F of groups holds F is a morphism from $\mathrm{dom}\, F$ to $\mathrm{cod}\, F$.

Let G, F be morphisms of groups. Let us assume that $\operatorname{dom} G = \operatorname{cod} F$. The functor $G \cdot F$ yielding a morphism of groups is defined by:

(Def.18) for all groups G_1, G_2, G_3 and for every map g from G_2 into G_3 and for every map f from G_1 into G_2 such that $G = \langle G_2, G_3, g\rangle$ and $F = \langle G_1, G_2, f\rangle$ holds $G \cdot F = \langle G_1, G_3, g \cdot f\rangle$.

Next we state the proposition

(28) For all groups G_1, G_2, G_3 and for every morphism G from G_2 to G_3 and for every morphism F from G_1 to G_2 holds $G \cdot F$ is a morphism from G_1 to G_3.

Let G_1, G_2, G_3 be groups, and let G be a morphism from G_2 to G_3, and let F be a morphism from G_1 to G_2. Then $G \cdot F$ is a morphism from G_1 to G_3.

The following propositions are true:

(29) For all groups G_1, G_2, G_3 and for every morphism G from G_2 to G_3 and for every morphism F from G_1 to G_2 and for every map g from G_2 into G_3 and for every map f from G_1 into G_2 such that $G = \langle G_2, G_3, g\rangle$ and $F = \langle G_1, G_2, f\rangle$ holds $G \cdot F = \langle G_1, G_3, g \cdot f\rangle$.

(30) For all morphisms f, g of groups such that $\operatorname{dom} g = \operatorname{cod} f$ there exist groups G_1, G_2, G_3 and there exists a map f_0 from G_1 into G_2 and there exists a map g_0 from G_2 into G_3 such that $f = \langle G_1, G_2, f_0\rangle$ and $g = \langle G_2, G_3, g_0\rangle$ and $g \cdot f = \langle G_1, G_3, g_0 \cdot f_0\rangle$.

(31) For all morphisms f, g of groups such that $\operatorname{dom} g = \operatorname{cod} f$ holds $\operatorname{dom}(g \cdot f) = \operatorname{dom} f$ and $\operatorname{cod}(g \cdot f) = \operatorname{cod} g$.

(32) For all groups G_1, G_2, G_3, G_4 and for every morphism f from G_1 to G_2 and for every morphism g from G_2 to G_3 and for every morphism h from G_3 to G_4 holds $h \cdot (g \cdot f) = h \cdot g \cdot f$.

(33) For all morphisms f, g, h of groups such that $\operatorname{dom} h = \operatorname{cod} g$ and $\operatorname{dom} g = \operatorname{cod} f$ holds $h \cdot (g \cdot f) = h \cdot g \cdot f$.

(34) $\operatorname{dom}(\mathrm{I}_G) = G$ and $\operatorname{cod}(\mathrm{I}_G) = G$ and for every morphism f of groups such that $\operatorname{cod} f = G$ holds $\mathrm{I}_G \cdot f = f$ and for every morphism g of groups such that $\operatorname{dom} g = G$ holds $g \cdot \mathrm{I}_G = g$.

A non-empty set is called a non-empty set of groups if:

(Def.19) for every element x of it holds x is a group.

In the sequel V will be a non-empty set of groups. Let us consider V. We see that the element of V is a group.

We now state two propositions:

(35) For every morphism f of groups and for every element x of $\{f\}$ holds x is a morphism of groups.

(36) For every morphism f from G to H and for every element x of $\{f\}$ holds x is a morphism from G to H.

A non-empty set is called a non-empty set of morphisms of groups if:

(Def.20) for every element x of it holds x is a morphism of groups.

Let M be a non-empty set of morphisms of groups. We see that the element of M is a morphism of groups.

We now state the proposition

(37) For every morphism f of groups holds $\{f\}$ is a non-empty set of morphisms of groups.

Let us consider G, H. A non-empty set of morphisms of groups is called a non-empty set of morphisms from G into H if:

(Def.21) for every element x of it holds x is a morphism from G to H.

The following two propositions are true:

(38) D is a non-empty set of morphisms from G into H if and only if for every element x of D holds x is a morphism from G to H.

(39) For every morphism f from G to H holds $\{f\}$ is a non-empty set of morphisms from G into H.

Let us consider G, H. The functor $\mathrm{Morphs}(G, H)$ yields a non-empty set of morphisms from G into H and is defined by:

(Def.22) $x \in \mathrm{Morphs}(G, H)$ if and only if x is a morphism from G to H.

Let us consider G, H, and let M be a non-empty set of morphisms from G into H. We see that the element of M is a morphism from G to H.

Let us consider x, y. The predicate $\mathrm{P_{ob}}\, x, y$ is defined by:

(Def.23) there exist arbitrary x_1, x_2, x_3, x_4 such that $x = \langle x_1, x_2, x_3, x_4 \rangle$ and there exists G such that $y = G$ and $x_1 =$ the carrier of G and $x_2 =$ the addition of G and $x_3 =$ the reverse-map of G and $x_4 =$ the zero of G.

One can prove the following two propositions:

(40) For arbitrary x, y_1, y_2 such that $\mathrm{P_{ob}}\, x, y_1$ and $\mathrm{P_{ob}}\, x, y_2$ holds $y_1 = y_2$.

(41) There exists x such that $x \in U_1$ and $\mathrm{P_{ob}}\, x$,the trivial group .

Let us consider U_1. The functor $\mathrm{GroupObj}(U_1)$ yields a non-empty set and is defined as follows:

(Def.24) for every y holds $y \in \mathrm{GroupObj}(U_1)$ if and only if there exists x such that $x \in U_1$ and $\mathrm{P_{ob}}\, x, y$.

The following propositions are true:

(42) The trivial group$\in \mathrm{GroupObj}(U_1)$.

(43) For every element x of $\mathrm{GroupObj}(U_1)$ holds x is a group.

Let us consider U_1. Then $\mathrm{GroupObj}(U_1)$ is a non-empty set of groups.

Let us consider V. The functor $\mathrm{Morphs}\, V$ yielding a non-empty set of morphisms of groups is defined by:

(Def.25) for every x holds $x \in \mathrm{Morphs}\, V$ if and only if there exist elements G, H of V such that x is a morphism from G to H.

Let us consider V, and let F be an element of $\mathrm{Morphs}\, V$. Then $\mathrm{dom}\, F$ is an element of V. Then $\mathrm{cod}\, F$ is an element of V.

Let us consider V, and let G be an element of V. The functor I_G yields an element of Morphs V and is defined by:

(Def.26) $\mathrm{I}_G = \mathrm{I}_G$.

We now define three new functors. Let us consider V. The functor dom V yields a function from Morphs V into V and is defined as follows:

(Def.27) for every element f of Morphs V holds $(\mathrm{dom}\, V)(f) = \mathrm{dom}\, f$.

The functor cod V yields a function from Morphs V into V and is defined as follows:

(Def.28) for every element f of Morphs V holds $(\mathrm{cod}\, V)(f) = \mathrm{cod}\, f$.

The functor I_V yielding a function from V into Morphs V is defined as follows:

(Def.29) for every element G of V holds $\mathrm{I}_V(G) = \mathrm{I}_G$.

One can prove the following two propositions:

(44) For all elements g, f of Morphs V such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ there exist elements G_1, G_2, G_3 of V such that g is a morphism from G_2 to G_3 and f is a morphism from G_1 to G_2.

(45) For all elements g, f of Morphs V such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $g \cdot f \in$ Morphs V.

Let us consider V. The functor comp V yields a partial function from $[\!:$ Morphs V, Morphs V $:\!]$ to Morphs V and is defined by:

(Def.30) for all elements g, f of Morphs V holds $\langle g, f\rangle \in \mathrm{dom}\,\mathrm{comp}\, V$ if and only if $\mathrm{dom}\, g = \mathrm{cod}\, f$ and for all elements g, f of Morphs V such that $\langle g, f\rangle \in \mathrm{dom}\,\mathrm{comp}\, V$ holds $(\mathrm{comp}\, V)(\langle g, f\rangle) = g \cdot f$.

Let us consider U_1. The functor GroupCat(U_1) yielding a category structure is defined by:

(Def.31) $\mathrm{GroupCat}(U_1) = \langle \mathrm{GroupObj}(U_1), \mathrm{Morphs}\,\mathrm{GroupObj}(U_1), \mathrm{dom}\,\mathrm{GroupObj}(U_1), \mathrm{cod}\,\mathrm{GroupObj}(U_1), \mathrm{comp}\,\mathrm{GroupObj}(U_1), \mathrm{I}_{\mathrm{GroupObj}(U_1)}\rangle$.

Next we state several propositions:

(46) For all morphisms f, g of GroupCat(U_1) holds $\langle g, f\rangle \in$ dom (the composition of GroupCat(U_1)) if and only if $\mathrm{dom}\, g = \mathrm{cod}\, f$.

(47) For every morphism f of GroupCat(U_1) and for every element f' of Morphs GroupObj(U_1) and for every object b of GroupCat(U_1) and for every element b' of GroupObj(U_1) holds f is an element of Morphs GroupObj(U_1) and f' is a morphism of GroupCat(U_1) and b is an element of GroupObj(U_1) and b' is an object of GroupCat(U_1).

(48) For every object b of GroupCat(U_1) and for every element b' of GroupObj(U_1) such that $b = b'$ holds $\mathrm{id}_b = \mathrm{I}_{b'}$.

(49) For every morphism f of GroupCat(U_1) and for every element f' of Morphs GroupObj(U_1) such that $f = f'$ holds $\mathrm{dom}\, f = \mathrm{dom}\, f'$ and $\mathrm{cod}\, f = \mathrm{cod}\, f'$.

(50) Let f, g be morphisms of $\mathrm{GroupCat}(U_1)$. Let f', g' be elements of $\mathrm{Morphs}\,\mathrm{GroupObj}(U_1)$. Suppose $f = f'$ and $g = g'$. Then

(i) $\mathrm{dom}\, g = \mathrm{cod}\, f$ if and only if $\mathrm{dom}\, g' = \mathrm{cod}\, f'$,

(ii) $\mathrm{dom}\, g = \mathrm{cod}\, f$ if and only if $\langle g', f' \rangle \in \mathrm{dom}\,\mathrm{comp}\,\mathrm{GroupObj}(U_1)$,

(iii) if $\mathrm{dom}\, g = \mathrm{cod}\, f$, then $g \cdot f = g' \cdot f'$,

(iv) $\mathrm{dom}\, f = \mathrm{dom}\, g$ if and only if $\mathrm{dom}\, f' = \mathrm{dom}\, g'$,

(v) $\mathrm{cod}\, f = \mathrm{cod}\, g$ if and only if $\mathrm{cod}\, f' = \mathrm{cod}\, g'$.

Let us consider U_1. Then $\mathrm{GroupCat}(U_1)$ is a category.

Let us consider U_1. The functor $\mathrm{AbGroupObj}(U_1)$ yielding a non-empty subset of the objects of $\mathrm{GroupCat}(U_1)$ is defined as follows:

(Def.32) $\mathrm{AbGroupObj}(U_1) = \{G : \bigvee_H G = H\}$, where G ranges over elements of the objects of $\mathrm{GroupCat}(U_1)$, and H ranges over Abelian groups.

One can prove the following proposition

(51) The trivial group$\in \mathrm{AbGroupObj}(U_1)$.

Let us consider U_1. The functor $\mathrm{AbGroupCat}(U_1)$ yielding a subcategory of $\mathrm{GroupCat}(U_1)$ is defined as follows:

(Def.33) $\mathrm{AbGroupCat}(U_1) = \mathrm{cat}\,\mathrm{AbGroupObj}(U_1)$.

We now state the proposition

(52) The objects of $\mathrm{AbGroupCat}(U_1) = \mathrm{AbGroupObj}(U_1)$.

Let us consider U_1. The functor $\frac{1}{2}\,\mathrm{GroupObj}(U_1)$ yields a non-empty subset of the objects of $\mathrm{AbGroupCat}(U_1)$ and is defined as follows:

(Def.34) $\frac{1}{2}\,\mathrm{GroupObj}(U_1) = \{G : \bigvee_H G = H\}$, where G ranges over elements of the objects of $\mathrm{AbGroupCat}(U_1)$, and H ranges over groups with the operator $\frac{1}{2}$.

Let us consider U_1. The functor $\frac{1}{2}\,\mathrm{GroupCat}(U_1)$ yields a subcategory of $\mathrm{AbGroupCat}(U_1)$ and is defined by:

(Def.35) $\frac{1}{2}\,\mathrm{GroupCat}(U_1) = \mathrm{cat}\,\frac{1}{2}\,\mathrm{GroupObj}(U_1)$.

Next we state two propositions:

(53) The objects of $\frac{1}{2}\,\mathrm{GroupCat}(U_1) = \frac{1}{2}\,\mathrm{GroupObj}(U_1)$.

(54) The trivial group$\in \frac{1}{2}\,\mathrm{GroupObj}(U_1)$.

References

[1] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[4] Czesław Byliński. Introduction to categories and functors. *Formalized Mathematics*, 1(**2**):409–420, 1990.

[5] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[6] Czesław Byliński. Subcategories and products of categories. *Formalized Mathematics*, 1(**4**):725–732, 1990.

[7] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[8] Michał Muzalewski. Atlas of Midpoint Algebra. *Formalized Mathematics*, 2(**4**):487–491, 1991.
[9] Michał Muzalewski and Wojciech Skaba. Groups, rings, left- and right-modules. *Formalized Mathematics*, 2(**2**):275–278, 1991.
[10] Bogdan Nowak and Grzegorz Bancerek. Universal classes. *Formalized Mathematics*, 1(**3**):595–600, 1990.
[11] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[12] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.
[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[14] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

Received October 3, 1990

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Homomorphisms and Isomorphisms of Groups. Quotient Group

Wojciech A. Trybulec
Warsaw University

Michał J. Trybulec
Warsaw University

Summary. Quotient group, homomorphisms and isomorphisms of groups are introduced. The so called isomorphism theorems are proved following [7].

MML Identifier: GROUP_6.

The articles [10], [8], [4], [5], [1], [6], [3], [9], [11], [2], [14], [16], [12], [15], and [13] provide the terminology and notation for this paper. The following proposition is true

(1) For all non-empty sets A, B and for every function f from A into B holds f is one-to-one if and only if for all elements a, b of A such that $f(a) = f(b)$ holds $a = b$.

Let G be a group, and let A be a subgroup of G. We see that the subgroup of A is a subgroup of G.

Let G be a group, and let A be a subgroup of G. We see that the normal subgroup of A is a subgroup of A.

Let G be a group. Then $\{\mathbf{1}\}_G$ is a normal subgroup of G. Then Ω_G is a normal subgroup of G.

For simplicity we adopt the following rules: n is a natural number, i is an integer, G, H, I are groups, A, B are subgroups of G, N, M are normal subgroups of G, a, a_1, a_2, a_3, b are elements of G, c is an element of H, f is a function from the carrier of G into the carrier of H, x is arbitrary, and A_1, A_2 are subsets of G. One can prove the following propositions:

(2) For every subgroup X of A and for every element x of A such that $x = a$ holds $x \cdot X = a \cdot X$ **qua** a subgroup of G and $X \cdot x = (X$ **qua** a subgroup of $G) \cdot a$.

(3) For all subgroups X, Y of A holds $(X$ **qua** a subgroup of $G) \cap Y$ **qua** a subgroup of $G = X \cap Y$.

ISSN 0777–4028

(4) $a \cdot b \cdot a^{-1} = b^{a^{-1}}$ and $a \cdot (b \cdot a^{-1}) = b^{a^{-1}}$.

(5) If $b \in N$, then $b^a \in N$.

(6) $a \cdot A \cdot A = a \cdot A$ and $a \cdot (A \cdot A) = a \cdot A$ and $A \cdot A \cdot a = A \cdot a$ and $A \cdot (A \cdot a) = A \cdot a$.

(7) If $A_1 = \{[a, b]\}$, then $G^{\mathrm{c}} = \mathrm{gr}(A_1)$.

(8) G^{c} is a subgroup of B if and only if for all a, b holds $[a, b] \in B$.

(9) If N is a subgroup of B, then N is a normal subgroup of B.

Let us consider G, B, M. Let us assume that M is a subgroup of B. The functor $(M)_B$ yielding a normal subgroup of B is defined as follows:

(Def.1) $(M)_B = M$.

One can prove the following proposition

(10) $B \cap N$ is a normal subgroup of B and $N \cap B$ is a normal subgroup of B.

Let us consider G, B, N. Then $B \cap N$ is a normal subgroup of B.

Let us consider G, N, B. Then $N \cap B$ is a normal subgroup of B.

A group is trivial if:

(Def.2) there exists x such that the carrier of it $= \{x\}$.

One can prove the following propositions:

(11) $\{\mathbf{1}\}_G$ is trivial.

(12) G is trivial if and only if $\mathrm{ord}(G) = 1$ and G is finite.

(13) If G is trivial, then $\{\mathbf{1}\}_G = G$.

Let us consider G, N. The functor $\mathrm{Cosets}\, N$ yielding a non-empty set is defined by:

(Def.3) $\mathrm{Cosets}\, N =$ the left cosets of N.

In the sequel W_1, W_2 denote elements of $\mathrm{Cosets}\, N$. One can prove the following propositions:

(14) $\mathrm{Cosets}\, N =$ the left cosets of N and $\mathrm{Cosets}\, N =$ the right cosets of N.

(15) If $x \in \mathrm{Cosets}\, N$, then there exists a such that $x = a \cdot N$ and $x = N \cdot a$.

(16) $a \cdot N \in \mathrm{Cosets}\, N$ and $N \cdot a \in \mathrm{Cosets}\, N$.

(17) If $x \in \mathrm{Cosets}\, N$, then x is a subset of G.

(18) If $A_1 \in \mathrm{Cosets}\, N$ and $A_2 \in \mathrm{Cosets}\, N$, then $A_1 \cdot A_2 \in \mathrm{Cosets}\, N$.

Let us consider G, N. The functor $\mathrm{CosOp}\, N$ yields a binary operation on $\mathrm{Cosets}\, N$ and is defined by:

(Def.4) for all W_1, W_2, A_1, A_2 such that $W_1 = A_1$ and $W_2 = A_2$ holds $(\mathrm{CosOp}\, N)(W_1, W_2) = A_1 \cdot A_2$.

In the sequel O is a binary operation on $\mathrm{Cosets}\, N$. One can prove the following two propositions:

(19) If for all W_1, W_2, A_1, A_2 such that $W_1 = A_1$ and $W_2 = A_2$ holds $O(W_1, W_2) = A_1 \cdot A_2$, then $O = \mathrm{CosOp}\, N$.

(20) For all W_1, W_2, A_1, A_2 such that $W_1 = A_1$ and $W_2 = A_2$ holds $(\mathrm{CosOp}\, N)(W_1, W_2) = A_1 \cdot A_2$.

Let us consider G, N. The functor ${}^{G}/_{N}$ yields a half group structure and is defined as follows:

(Def.5) ${}^{G}/_{N} = \langle \mathrm{Cosets}\, N, \mathrm{CosOp}\, N \rangle$.

One can prove the following propositions:

(21) ${}^{G}/_{N} = \langle \mathrm{Cosets}\, N, \mathrm{CosOp}\, N \rangle$.

(22) The carrier of ${}^{G}/_{N} = \mathrm{Cosets}\, N$.

(23) The operation of ${}^{G}/_{N} = \mathrm{CosOp}\, N$.

In the sequel S, T_1, T_2 denote elements of ${}^{G}/_{N}$. Let us consider G, N, S. The functor ${}^{@}S$ yields a subset of G and is defined by:

(Def.6) ${}^{@}S = S$.

One can prove the following two propositions:

(24) $({}^{@}T_1) \cdot ({}^{@}T_2) = T_1 \cdot T_2$.

(25) ${}^{@}T_1 \cdot T_2 = ({}^{@}T_1) \cdot ({}^{@}T_2)$.

Let us consider G, N. Then ${}^{G}/_{N}$ is a group.

In the sequel S will denote an element of ${}^{G}/_{N}$. The following propositions are true:

(26) There exists a such that $S = a \cdot N$ and $S = N \cdot a$.

(27) $N \cdot a$ is an element of ${}^{G}/_{N}$ and $a \cdot N$ is an element of ${}^{G}/_{N}$ and $\overline{N}$ is an element of ${}^{G}/_{N}$.

(28) $x \in {}^{G}/_{N}$ if and only if there exists a such that $x = a \cdot N$ and $x = N \cdot a$.

(29) $1_{{}^{G}/_{N}} = \overline{N}$.

(30) If $S = a \cdot N$, then $S^{-1} = a^{-1} \cdot N$.

(31) If the left cosets of N is finite, then ${}^{G}/_{N}$ is finite.

(32) $\mathrm{Ord}({}^{G}/_{N}) = |\bullet : N|$.

(33) If the left cosets of N is finite, then $\mathrm{ord}({}^{G}/_{N}) = |\bullet : N|_{\mathbb{N}}$.

(34) If M is a subgroup of B, then ${}^{B}/_{(M)_B}$ is a subgroup of ${}^{G}/_{M}$.

(35) If M is a subgroup of N, then ${}^{N}/_{(M)_N}$ is a normal subgroup of ${}^{G}/_{M}$.

(36) ${}^{G}/_{N}$ is an Abelian group if and only if G^{c} is a subgroup of N.

Let us consider G, H. A function from the carrier of G into the carrier of H is called a homomorphism from G to H if:

(Def.7) $\mathrm{it}(a \cdot b) = \mathrm{it}(a) \cdot \mathrm{it}(b)$.

One can prove the following proposition

(37) If for all a, b holds $f(a \cdot b) = f(a) \cdot f(b)$, then f is a homomorphism from G to H.

In the sequel g, h will be homomorphisms from G to H, g_1 will be a homomorphism from H to G, and h_1 will be a homomorphism from H to I. One can prove the following propositions:

(38) $\operatorname{dom} g =$ the carrier of G and $\operatorname{rng} g \subseteq$ the carrier of H.

(39) $g(a \cdot b) = g(a) \cdot g(b)$.

(40) $g(1_G) = 1_H$.

(41) $g(a^{-1}) = g(a)^{-1}$.

(42) $g(a^b) = g(a)^{g(b)}$.

(43) $g([a, b]) = [g(a), g(b)]$.

(44) $g([a_1, a_2, a_3]) = [g(a_1), g(a_2), g(a_3)]$.

(45) $g(a^n) = g(a)^n$.

(46) $g(a^i) = g(a)^i$.

(47) $\mathrm{id}_{(\text{the carrier of } G)}$ is a homomorphism from G to G.

(48) $h_1 \cdot h$ is a homomorphism from G to I.

Let us consider G, H, I, h, h_1. Then $h_1 \cdot h$ is a homomorphism from G to I.

Let us consider G, H, g. Then $\operatorname{rng} g$ is a subset of H.

Let us consider G, H. The functor $G \to \{\mathbf{1}\}_H$ yields a homomorphism from G to H and is defined by:

(Def.8) for every a holds $(G \to \{\mathbf{1}\}_H)(a) = 1_H$.

The following proposition is true

(49) $h_1 \cdot (G \to \{\mathbf{1}\}_H) = G \to \{\mathbf{1}\}_I$ and $(H \to \{\mathbf{1}\}_I) \cdot h = G \to \{\mathbf{1}\}_I$.

Let us consider G, N. The canonical homomorphism onto cosets of N yielding a homomorphism from G to ${}^{G}/_{N}$ is defined as follows:

(Def.9) for every a holds (the canonical homomorphism onto cosets of N)$(a) = a \cdot N$.

Let us consider G, H, g. The functor $\operatorname{Ker} g$ yields a normal subgroup of G and is defined by:

(Def.10) the carrier of $\operatorname{Ker} g = \{a : g(a) = 1_H\}$.

The following three propositions are true:

(50) $a \in \operatorname{Ker} h$ if and only if $h(a) = 1_H$.

(51) $\operatorname{Ker}(G \to \{\mathbf{1}\}_H) = G$.

(52) Ker(the canonical homomorphism onto cosets of N) $= N$.

Let us consider G, H, g. The functor $\operatorname{Im} g$ yields a subgroup of H and is defined as follows:

(Def.11) the carrier of $\operatorname{Im} g = g^{\circ}$ (the carrier of G).

Next we state a number of propositions:

(53) $\operatorname{rng} g =$ the carrier of $\operatorname{Im} g$.

(54) $x \in \operatorname{Im} g$ if and only if there exists a such that $x = g(a)$.

(55) $\operatorname{Im} g = \operatorname{gr}(\operatorname{rng} g)$.

(56) $\operatorname{Im}(G \to \{\mathbf{1}\}_H) = \{\mathbf{1}\}_H$.

(57) Im(the canonical homomorphism onto cosets of N) $= {}^{G}/_{N}$.

(58) h is a homomorphism from G to $\operatorname{Im} h$.

(59) If G is finite, then $\operatorname{Im} g$ is finite.

(60) If G is an Abelian group, then $\operatorname{Im} g$ is an Abelian group.

(61) $\operatorname{Ord}(\operatorname{Im} g) \leq \operatorname{Ord}(G)$.

(62) If G is finite, then $\operatorname{ord}(\operatorname{Im} g) \leq \operatorname{ord}(G)$.

We now define two new predicates. Let us consider G, H, h. We say that h is a monomorphism if and only if:

(Def.12) h is one-to-one.

We say that h is an epimorphism if and only if:

(Def.13) $\operatorname{rng} h =$ the carrier of H.

We now state several propositions:

(63) If h is a monomorphism and $c \in \operatorname{Im} h$, then $h(h^{-1}(c)) = c$.

(64) If h is a monomorphism, then $h^{-1}(h(a)) = a$.

(65) If h is a monomorphism, then h^{-1} is a homomorphism from $\operatorname{Im} h$ to G.

(66) h is a monomorphism if and only if $\operatorname{Ker} h = \{\mathbf{1}\}_G$.

(67) h is an epimorphism if and only if $\operatorname{Im} h = H$.

(68) If h is an epimorphism, then for every c there exists a such that $h(a) = c$.

(69) The canonical homomorphism onto cosets of N is an epimorphism.

Let us consider G, H, h. We say that h is an isomorphism if and only if:

(Def.14) h is an epimorphism and h is a monomorphism.

One can prove the following propositions:

(70) h is an isomorphism if and only if $\operatorname{rng} h =$ the carrier of H and h is one-to-one.

(71) If h is an isomorphism, then $\operatorname{dom} h =$ the carrier of G and $\operatorname{rng} h =$ the carrier of H.

(72) If h is an isomorphism, then h^{-1} is a homomorphism from H to G.

(73) If h is an isomorphism and $g_1 = h^{-1}$, then g_1 is an isomorphism.

(74) If h is an isomorphism and h_1 is an isomorphism, then $h_1 \cdot h$ is an isomorphism.

(75) The canonical homomorphism onto cosets of $\{\mathbf{1}\}_G$ is an isomorphism.

Let us consider G, H. We say that G and H are isomorphic if and only if:

(Def.15) there exists h such that h is an isomorphism.

We now state a number of propositions:

(76) G and G are isomorphic.

(77) If G and H are isomorphic, then H and G are isomorphic.

(78) If G and H are isomorphic and H and I are isomorphic, then G and I are isomorphic.

(79) If h is a monomorphism, then G and $\operatorname{Im} h$ are isomorphic.

(80) If G is trivial and H is trivial, then G and H are isomorphic.

(81) $\{\mathbf{1}\}_G$ and $\{\mathbf{1}\}_H$ are isomorphic.

(82) G and ${}^{G}/_{\{\mathbf{1}\}_G}$ are isomorphic and ${}^{G}/_{\{\mathbf{1}\}_G}$ and G are isomorphic.

(83) ${}^{G}/_{\Omega_G}$ is trivial.

(84) If G and H are isomorphic, then $\mathrm{Ord}(G) = \mathrm{Ord}(H)$.

(85) If G and H are isomorphic but G is finite or H is finite, then G is finite and H is finite.

(86) If G and H are isomorphic but G is finite or H is finite, then $\mathrm{ord}(G) = \mathrm{ord}(H)$.

(87) If G and H are isomorphic but G is trivial or H is trivial, then G is trivial and H is trivial.

(88) If G and H are isomorphic but G is an Abelian group or H is an Abelian group, then G is an Abelian group and H is an Abelian group.

(89) ${}^{G}/_{\mathrm{Ker}\, g}$ and $\mathrm{Im}\, g$ are isomorphic and $\mathrm{Im}\, g$ and ${}^{G}/_{\mathrm{Ker}\, g}$ are isomorphic.

(90) There exists a homomorphism h from ${}^{G}/_{\mathrm{Ker}\, g}$ to $\mathrm{Im}\, g$ such that h is an isomorphism and $g = h\cdot$ the canonical homomorphism onto cosets of $\mathrm{Ker}\, g$.

(91) For every normal subgroup J of ${}^{G}/_{M}$ such that $J = {}^{N}/_{(M)_N}$ and M is a subgroup of N holds ${}^{({}^{G}/_{M})}/_{J}$ and ${}^{G}/_{N}$ are isomorphic.

(92) ${}^{(B \sqcup N)}/_{(N)_{B \sqcup N}}$ and ${}^{B}/_{(B \cap N)}$ are isomorphic.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[7] M. I. Kargapołow and J. I. Mierzlakow. *Podstawy teorii grup.* PWN, Warszawa, 1989.

[8] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[9] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[11] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.

[12] Wojciech A. Trybulec. Classes of conjugation. Normal subgroups. *Formalized Mathematics*, 1(**5**):955–962, 1990.

[13] Wojciech A. Trybulec. Commutator and center of a group. *Formalized Mathematics*, 2(**4**):461–466, 1991.

[14] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.

[15] Wojciech A. Trybulec. Lattice of subgroups of a group. Frattini subgroup. *Formalized Mathematics*, 2(**1**):41–47, 1991.

[16] Wojciech A. Trybulec. Subgroup and cosets of subgroups. *Formalized Mathematics*, 1(**5**):855–864, 1990.

Received October 3, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Rings and Modules - Part II

Michał Muzalewski
Warsaw University
Białystok

Summary. We define the trivial left module, morphism of left modules and the field Z_3. We proof some elementary facts.

MML Identifier: `MOD_2`.

The terminology and notation used in this paper are introduced in the following articles: [14], [13], [4], [5], [6], [2], [3], [1], [7], [9], [11], [12], [10], and [8]. For simplicity we adopt the following convention: x, y, z are arbitrary, D is a non-empty set, R, R_1, R_2, R_3 are associative rings, G is a left module structure over R, H is a left module structure over R, S is a left module structure over R, G_1 is a left module structure over R_1, G_2 is a left module structure over R_2, G_3 is a left module structure over R_3, and U_1 is a universal class. Let us consider x. Then $\{x\}$ is a non-empty set.

Let us consider R. $\mathrm{lop}(R)$ is a function from $[\!:$ the carrier of R, the carrier of the trivial group $:\!]$ into the carrier of the trivial group.

Let us consider R. The functor ${}_R\Theta$ yields a left module over R and is defined by:

(Def.1) ${}_R\Theta = \langle \text{the trivial group}, \mathrm{lop}(R) \rangle$.

Next we state the proposition

(1) For every vector x of ${}_R\Theta$ holds $x = \Theta_{{}_R\Theta}$.

Let us consider R_1, R_2, G_1, G_2. A map from G_1 into G_2 is a map from the carrier of G_1 into the carrier of G_2.

Let us consider R_1, R_2, R_3, G_1, G_2, G_3, and let f be a map from G_1 into G_2, and let g be a map from G_2 into G_3. Then $g \cdot f$ is a map from G_1 into G_3.

Let us consider R, G. The functor id_G yielding a map from G into G is defined as follows:

(Def.2) $\mathrm{id}_G = \mathrm{id}_{(\text{the carrier of } G)}$.

ISSN 0777–4028

The following propositions are true:

(2) For every vector x of G holds $\mathrm{id}_G(x) = x$.

(3) For every map f from G_1 into G_2 holds $f \cdot \mathrm{id}_{G_1} = f$ and $\mathrm{id}_{G_2} \cdot f = f$.

Let us consider R_1, R_2, G_1, G_2. The functor $\mathrm{zero}(G_1, G_2)$ yields a map from G_1 into G_2 and is defined as follows:

(Def.3) $\mathrm{zero}(G_1, G_2) = \mathrm{zero}($ the carrier of G_1, the carrier of $G_2)$.

Let us consider R, and let G, H be left module structures over R, and let f be a map from G into H. We say that f is linear if and only if:

(Def.4) for all vectors x, y of G holds $f(x + y) = f(x) + f(y)$ and for every scalar a of R and for every vector x of G holds $f(a \cdot x) = a \cdot f(x)$.

The following propositions are true:

(4) For every map f from G into H such that f is linear holds f is additive.

(5) For every map f from G_1 into G_2 and for every map g from G_2 into G_3 and for every vector x of G_1 holds $(g \cdot f)(x) = g(f(x))$.

(6) For every map f from G into H and for every map g from H into S such that f is linear and g is linear holds $g \cdot f$ is linear.

For simplicity we adopt the following rules: R, R_1, R_2 denote associative rings, G denotes a left module over R, H denotes a left module over R, G_1 denotes a left module over R_1, and G_2 denotes a left module over R_2. The following propositions are true:

(7) For every vector x of G_1 holds $(\mathrm{zero}(G_1, G_2))(x) = \Theta_{G_2}$.

(8) $\mathrm{zero}(G, H)$ is linear.

In the sequel G_1 will denote a left module over R, G_2 will denote a left module over R, and G_3 will denote a left module over R. Let us consider R. We consider left module morphism structures over R which are systems

$\langle$a dom-map, a cod-map, a `Fun`$\rangle$,

where the dom-map, the cod-map are a left module over R and the `Fun` is a map from the dom-map into the cod-map.

In the sequel f will be a left module morphism structure over R. We now define two new functors. Let us consider R, f. The functor $\mathrm{dom}\, f$ yields a left module over R and is defined as follows:

(Def.5) $\mathrm{dom}\, f =$ the dom-map of f.

The functor $\mathrm{cod}\, f$ yields a left module over R and is defined as follows:

(Def.6) $\mathrm{cod}\, f =$ the cod-map of f.

Let us consider R, f. The functor $\mathrm{fun}\, f$ yields a map from $\mathrm{dom}\, f$ into $\mathrm{cod}\, f$ and is defined by:

(Def.7) $\mathrm{fun}\, f =$ the `Fun` of f.

One can prove the following proposition

(9) For every map f_0 from G_1 into G_2 such that $f = \langle G_1, G_2, f_0 \rangle$ holds $\mathrm{dom}\, f = G_1$ and $\mathrm{cod}\, f = G_2$ and $\mathrm{fun}\, f = f_0$.

Let us consider R, G, H. The functor ZERO G yielding a left module morphism structure over R is defined as follows:

(Def.8) ZERO $G = \langle G, H, \mathrm{zero}(G, H)\rangle$.

Let us consider R. A left module morphism structure over R is said to be a left module morphism of R if:

(Def.9) fun it is linear.

One can prove the following proposition

(10) For every left module morphism F of R holds the `Fun` of F is linear.

Let us consider R, G, H. Then ZERO G is a left module morphism of R.

Let us consider R, G, H. A left module morphism of R is said to be a morphism from G to H if:

(Def.10) dom it $= G$ and cod it $= H$.

One can prove the following three propositions:

(11) If $\mathrm{dom}\, f = G$ and $\mathrm{cod}\, f = H$ and fun f is linear, then f is a morphism from G to H.

(12) For every map f from G into H such that f is linear holds $\langle G, H, f\rangle$ is a morphism from G to H.

(13) id_G is linear.

Let us consider R, G. The functor I_G yields a morphism from G to G and is defined by:

(Def.11) $\mathrm{I}_G = \langle G, G, \mathrm{id}_G\rangle$.

Let us consider R, G, H. Then ZERO G is a morphism from G to H.

The following propositions are true:

(14) For every morphism F from G to H there exists a map f from G into H such that $F = \langle G, H, f\rangle$ and f is linear.

(15) For every morphism F from G to H there exists a map f from G into H such that $F = \langle G, H, f\rangle$.

(16) For every left module morphism F of R there exist G, H such that F is a morphism from G to H.

(17) For every left module morphism F of R there exist left modules G, H over R and there exists a map f from G into H such that F is a morphism from G to H and $F = \langle G, H, f\rangle$ and f is linear.

(18) For all left module morphisms g, f of R such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ there exist G_1, G_2, G_3 such that g is a morphism from G_2 to G_3 and f is a morphism from G_1 to G_2.

(19) For every left module morphism F of R holds F is a morphism from $\mathrm{dom}\, F$ to $\mathrm{cod}\, F$.

Let us consider R, and let G, F be left module morphisms of R. Let us assume that $\mathrm{dom}\, G = \mathrm{cod}\, F$. The functor $G \cdot F$ yields a left module morphism of R and is defined as follows:

(Def.12) for all left modules G_1, G_2, G_3 over R and for every map g from G_2 into G_3 and for every map f from G_1 into G_2 such that $G = \langle G_2, G_3, g\rangle$ and $F = \langle G_1, G_2, f\rangle$ holds $G \cdot F = \langle G_1, G_3, g \cdot f\rangle$.

Next we state the proposition

(20) For every morphism G from G_2 to G_3 and for every morphism F from G_1 to G_2 holds $G \cdot F$ is a morphism from G_1 to G_3.

Let us consider R, G_1, G_2, G_3, and let G be a morphism from G_2 to G_3, and let F be a morphism from G_1 to G_2. The functor $F[G]$ yielding a morphism from G_1 to G_3 is defined by:

(Def.13) $F[G] = G \cdot F$.

We now state several propositions:

(21) Let G be a morphism from G_2 to G_3. Then for every morphism F from G_1 to G_2 and for every map g from G_2 into G_3 and for every map f from G_1 into G_2 such that $G = \langle G_2, G_3, g\rangle$ and $F = \langle G_1, G_2, f\rangle$ holds $F[G] = \langle G_1, G_3, g \cdot f\rangle$ and $G \cdot F = \langle G_1, G_3, g \cdot f\rangle$.

(22) Let f, g be left module morphisms of R. Then if $\operatorname{dom} g = \operatorname{cod} f$, then there exist left modules G_1, G_2, G_3 over R and there exists a map f_0 from G_1 into G_2 and there exists a map g_0 from G_2 into G_3 such that $f = \langle G_1, G_2, f_0\rangle$ and $g = \langle G_2, G_3, g_0\rangle$ and $g \cdot f = \langle G_1, G_3, g_0 \cdot f_0\rangle$.

(23) For all left module morphisms f, g of R such that $\operatorname{dom} g = \operatorname{cod} f$ holds $\operatorname{dom}(g \cdot f) = \operatorname{dom} f$ and $\operatorname{cod}(g \cdot f) = \operatorname{cod} g$.

(24) For all left modules G_1, G_2, G_3, G_4 over R and for every morphism f from G_1 to G_2 and for every morphism g from G_2 to G_3 and for every morphism h from G_3 to G_4 holds $h \cdot (g \cdot f) = h \cdot g \cdot f$.

(25) For all left module morphisms f, g, h of R such that $\operatorname{dom} h = \operatorname{cod} g$ and $\operatorname{dom} g = \operatorname{cod} f$ holds $h \cdot (g \cdot f) = h \cdot g \cdot f$.

(26) $\operatorname{dom}(\mathrm{I}_G) = G$ and $\operatorname{cod}(\mathrm{I}_G) = G$ and for every left module morphism f of R such that $\operatorname{cod} f = G$ holds $\mathrm{I}_G \cdot f = f$ and for every left module morphism g of R such that $\operatorname{dom} g = G$ holds $g \cdot \mathrm{I}_G = g$.

(27) $\{x, y, z\}$ is a non-empty set.

Let us consider x, y, z. Then $\{x, y, z\}$ is a non-empty set.

We now state four propositions:

(28) For all elements u, v, w of U_1 holds $\{u, v, w\}$ is an element of U_1.

(29) For every element u of U_1 holds $\operatorname{succ} u$ is an element of U_1.

(30) $\overline{\mathbf{0}}$ is an element of U_1 and $\overline{\mathbf{1}}$ is an element of U_1 and $\overline{\mathbf{2}}$ is an element of U_1.

(31) $\overline{\mathbf{0}} \neq \overline{\mathbf{1}}$ and $\overline{\mathbf{0}} \neq \overline{\mathbf{2}}$ and $\overline{\mathbf{1}} \neq \overline{\mathbf{2}}$.

In the sequel a, b will be elements of $\{\overline{\mathbf{0}}, \overline{\mathbf{1}}, \overline{\mathbf{2}}\}$. We now define three new functors. Let us consider a. The functor $-a$ yields an element of $\{\overline{\mathbf{0}}, \overline{\mathbf{1}}, \overline{\mathbf{2}}\}$ and is defined as follows:

(Def.14) (i) $-a = \overline{\mathbf{0}}$ if $a = \overline{\mathbf{0}}$,
(ii) $-a = \overline{\mathbf{2}}$ if $a = \overline{\mathbf{1}}$,
(iii) $-a = \overline{\mathbf{1}}$ if $a = \overline{\mathbf{2}}$.

Let us consider b. The functor $a+b$ yields an element of $\{\overline{\mathbf{0}}, \overline{\mathbf{1}}, \overline{\mathbf{2}}\}$ and is defined by:

(Def.15) (i) $a + b = b$ if $a = \overline{\mathbf{0}}$,
(ii) $a + b = a$ if $b = \overline{\mathbf{0}}$,
(iii) $a + b = \overline{\mathbf{2}}$ if $a = \overline{\mathbf{1}}$ and $b = \overline{\mathbf{1}}$,
(iv) $a + b = \overline{\mathbf{0}}$ if $a = \overline{\mathbf{1}}$ and $b = \overline{\mathbf{2}}$,
(v) $a + b = \overline{\mathbf{0}}$ if $a = \overline{\mathbf{2}}$ and $b = \overline{\mathbf{1}}$,
(vi) $a + b = \overline{\mathbf{1}}$ if $a = \overline{\mathbf{2}}$ and $b = \overline{\mathbf{2}}$.

The functor $a \cdot b$ yielding an element of $\{\overline{\mathbf{0}}, \overline{\mathbf{1}}, \overline{\mathbf{2}}\}$ is defined by:

(Def.16) (i) $a \cdot b = \overline{\mathbf{0}}$ if $b = \overline{\mathbf{0}}$,
(ii) $a \cdot b = \overline{\mathbf{0}}$ if $a = \overline{\mathbf{0}}$,
(iii) $a \cdot b = a$ if $b = \overline{\mathbf{1}}$,
(iv) $a \cdot b = b$ if $a = \overline{\mathbf{1}}$,
(v) $a \cdot b = \overline{\mathbf{1}}$ if $a = \overline{\mathbf{2}}$ and $b = \overline{\mathbf{2}}$.

We now define five new functors. The binary operation add_3 on $\{\overline{\mathbf{0}}, \overline{\mathbf{1}}, \overline{\mathbf{2}}\}$ is defined by:

(Def.17) $\mathrm{add}_3(a, b) = a + b$.

The binary operation mult_3 on $\{\overline{\mathbf{0}}, \overline{\mathbf{1}}, \overline{\mathbf{2}}\}$ is defined by:

(Def.18) $\mathrm{mult}_3(a, b) = a \cdot b$.

The unary operation compl_3 on $\{\overline{\mathbf{0}}, \overline{\mathbf{1}}, \overline{\mathbf{2}}\}$ is defined as follows:

(Def.19) $\mathrm{compl}_3(a) = -a$.

The element unit_3 of $\{\overline{\mathbf{0}}, \overline{\mathbf{1}}, \overline{\mathbf{2}}\}$ is defined as follows:

(Def.20) $\mathrm{unit}_3 = \overline{\mathbf{1}}$.

The element zero_3 of $\{\overline{\mathbf{0}}, \overline{\mathbf{1}}, \overline{\mathbf{2}}\}$ is defined as follows:

(Def.21) $\mathrm{zero}_3 = \overline{\mathbf{0}}$.

The field structure Z_3 is defined by:

(Def.22) $\mathrm{Z}_3 = \langle \{\overline{\mathbf{0}}, \overline{\mathbf{1}}, \overline{\mathbf{2}}\}, \mathrm{mult}_3, \mathrm{add}_3, \mathrm{compl}_3, \mathrm{unit}_3, \mathrm{zero}_3 \rangle$.

Next we state several propositions:

(32) $0_{\mathrm{Z}_3} = \overline{\mathbf{0}}$ and $1_{\mathrm{Z}_3} = \overline{\mathbf{1}}$ and 0_{Z_3} is an element of $\{\overline{\mathbf{0}}, \overline{\mathbf{1}}, \overline{\mathbf{2}}\}$ and 1_{Z_3} is an element of $\{\overline{\mathbf{0}}, \overline{\mathbf{1}}, \overline{\mathbf{2}}\}$ and the addition of $\mathrm{Z}_3 = \mathrm{add}_3$ and the multiplication of $\mathrm{Z}_3 = \mathrm{mult}_3$ and the reverse-map of $\mathrm{Z}_3 = \mathrm{compl}_3$.

(33) For all scalars x, y of Z_3 and for all elements X, Y of $\{\overline{\mathbf{0}}, \overline{\mathbf{1}}, \overline{\mathbf{2}}\}$ such that $X = x$ and $Y = y$ holds $x + y = X + Y$ and $x \cdot y = X \cdot Y$ and $-x = -X$.

(34) Let x, y, z be scalars of Z_3. Let X, Y, Z be elements of $\{\overline{\mathbf{0}}, \overline{\mathbf{1}}, \overline{\mathbf{2}}\}$. Suppose $X = x$ and $Y = y$ and $Z = z$. Then $x + y + z = X + Y + Z$ and $x + (y + z) = X + (Y + Z)$ and $x \cdot y \cdot z = X \cdot Y \cdot Z$ and $x \cdot (y \cdot z) = X \cdot (Y \cdot Z)$.

(35) Let x, y, z, a, b be elements of $\{\overline{\mathbf{0}}, \overline{\mathbf{1}}, \overline{\mathbf{2}}\}$. Suppose $a = \overline{\mathbf{0}}$ and $b = \overline{\mathbf{1}}$. Then

(i) $x + y = y + x$,
(ii) $x + y + z = x + (y + z)$,
(iii) $x + a = x$,
(iv) $x + -x = a$,
(v) $x \cdot y = y \cdot x$,
(vi) $x \cdot y \cdot z = x \cdot (y \cdot z)$,
(vii) $x \cdot b = x$,
(viii) if $x \neq a$, then there exists an element y of $\{\overline{\mathbf{0}}, \overline{\mathbf{1}}, \overline{\mathbf{2}}\}$ such that $x \cdot y = b$,
(ix) $a \neq b$,
(x) $x \cdot (y + z) = x \cdot y + x \cdot z$.

(36) Let F be a field structure. Suppose that

(i) for all scalars x, y, z of F holds $x+y = y+x$ and $x+y+z = x+(y+z)$ and $x + 0_F = x$ and $x + -x = 0_F$ and $x \cdot y = y \cdot x$ and $x \cdot y \cdot z = x \cdot (y \cdot z)$ and $x \cdot 1_F = x$ but if $x \neq 0_F$, then there exists a scalar y of F such that $x \cdot y = 1_F$ and $0_F \neq 1_F$ and $x \cdot (y + z) = x \cdot y + x \cdot z$.

Then F is a field.

(37) $\mathbb{Z}_3$ is a Fano field.

Let us note that it makes sense to consider the following constant. Then $\mathbb{Z}_3$ is a Fano field.

In the sequel D' is a non-empty set. One can prove the following propositions:

(38) For every function f from D into D' such that $D \in U_1$ and $D' \in U_1$ holds $f \in U_1$.

(39) For every G being a field structure such that the carrier of $G \in U_1$ holds the addition of G is an element of U_1 and the reverse-map of G is an element of U_1 and the zero of G is an element of U_1 and the multiplication of G is an element of U_1 and the unity of G is an element of U_1.

(40) The carrier of $\mathbb{Z}_3 \in U_1$ and the addition of $\mathbb{Z}_3$ is an element of U_1 and the reverse-map of $\mathbb{Z}_3$ is an element of U_1 and the zero of $\mathbb{Z}_3$ is an element of U_1 and the multiplication of $\mathbb{Z}_3$ is an element of U_1 and the unity of $\mathbb{Z}_3$ is an element of U_1.

References

[1] Grzegorz Bancerek. Cardinal arithmetics. *Formalized Mathematics*, 1(**3**):543–547, 1990.
[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[3] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(**2**):281–290, 1990.
[4] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[7] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[8] Michał Muzalewski. Categories of groups. *Formalized Mathematics*, 2(**4**):563–571, 1991.

[9] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):3–11, 1991.
[10] Michał Muzalewski and Wojciech Skaba. Groups, rings, left- and right-modules. *Formalized Mathematics*, 2(**2**):275–278, 1991.
[11] Michał Muzalewski and Lesław W. Szczerba. Construction of finite sequences over ring and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):97–104, 1991.
[12] Bogdan Nowak and Grzegorz Bancerek. Universal classes. *Formalized Mathematics*, 1(**3**):595–600, 1990.
[13] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[14] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

Received October 18, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Free Modules

Michał Muzalewski
Warsaw University
Białystok

Summary. We define free modules and prove that every left module over Skew-Field is free.

MML Identifier: MOD_3.

The papers [20], [5], [3], [2], [4], [19], [16], [14], [15], [1], [18], [6], [7], [8], [12], [11], [9], [10], [13], and [17] provide the terminology and notation for this paper. One can prove the following propositions:

(1) For every ring R and for every scalar a of R such that $-a = 0_R$ holds $a = 0_R$.

(2) For every integral domain R holds $0_R \neq -1_R$.

For simplicity we follow the rules: x is arbitrary, R is an associative ring, V is a left module over R, L, L_1, L_2 are linear combinations of V, a is a scalar of R, v, w are vectors of V, F is a finite sequence of elements of the carrier of the carrier of V, and C is a finite subset of V. We now state several propositions:

(3) If $-v = w$, then $v = -w$.

(4) $\sum(\mathbf{0}_{\mathrm{LC}_V}) = \Theta_V$.

(5) $L_1 + L_2 = L_2 + L_1$.

(6) If $\operatorname{support} L \subseteq C$, then there exists F such that F is one-to-one and $\operatorname{rng} F = C$ and $\sum L = \sum(LF)$.

(7) $\sum(a \cdot L) = a \cdot \sum L$.

(8) $\sum(-L) = -\sum L$.

(9) $\sum(L_1 - L_2) = \sum L_1 - \sum L_2$.

(10) $L + \mathbf{0}_{\mathrm{LC}_V} = L$ and $\mathbf{0}_{\mathrm{LC}_V} + L = L$.

In the sequel W denotes a submodule of V, A, B denote subsets of V, and l denotes a linear combination of A. Let us consider R, V, A. The functor $\mathrm{Lin}(A)$ yielding a submodule of V is defined as follows:

ISSN 0777–4028

(Def.1) the carrier of the carrier of $\mathrm{Lin}(A) = \{\sum l\}$.

The following propositions are true:

(11) $x \in \mathrm{Lin}(A)$ if and only if there exists l such that $x = \sum l$.

(12) If $x \in A$, then $x \in \mathrm{Lin}(A)$.

(13) $\mathrm{Lin}(\emptyset_{\text{the carrier of the carrier of } V}) = \mathbf{0}_V$.

(14) If $\mathrm{Lin}(A) = \mathbf{0}_V$, then $A = \emptyset$ or $A = \{\Theta_V\}$.

(15) If $0_R \neq 1_R$ and $A =$ the carrier of the carrier of W, then $\mathrm{Lin}(A) = W$.

(16) If $0_R \neq 1_R$ and $A =$ the carrier of the carrier of V, then $\mathrm{Lin}(A) = V$.

(17) If $A \subseteq B$, then $\mathrm{Lin}(A)$ is a submodule of $\mathrm{Lin}(B)$.

(18) If $\mathrm{Lin}(A) = V$ and $A \subseteq B$, then $\mathrm{Lin}(B) = V$.

(19) $\mathrm{Lin}(A \cup B) = \mathrm{Lin}(A) + \mathrm{Lin}(B)$.

(20) $\mathrm{Lin}(A \cap B)$ is a submodule of $\mathrm{Lin}(A) \cap \mathrm{Lin}(B)$.

Let us consider R, V. A subset of V is base if:

(Def.2) it is linearly independent and $\mathrm{Lin}(\mathrm{it}) = V$.

Let us consider R. A left module over R is free if:

(Def.3) there exists a subset B of it such that B is base.

We now state the proposition

(21) $\mathbf{0}_V$ is free.

Let us consider R. A left module over R is called a free left R-module if:

(Def.4) it is free.

For simplicity we adopt the following convention: R will denote a skew field, a, b will denote scalars of R, V will denote a left module over R, v, v_1, v_2 will denote vectors of V, and A, B will denote subsets of V. The following propositions are true:

(22) $0_R \neq -1_R$.

(23) $\{v\}$ is linearly independent if and only if $v \neq \Theta_V$.

(24) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent if and only if $v_2 \neq \Theta_V$ and for every a holds $v_1 \neq a \cdot v_2$.

(25) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent if and only if for all a, b such that $a \cdot v_1 + b \cdot v_2 = \Theta_V$ holds $a = 0_R$ and $b = 0_R$.

(26) If A is linearly independent, then there exists B such that $A \subseteq B$ and B is base.

(27) If $\mathrm{Lin}(A) = V$, then there exists B such that $B \subseteq A$ and B is base.

(28) V is free.

Let us consider R, V. A subset of V is called a basis of V if:

(Def.5) it is base.

In the sequel I is a basis of V. The following two propositions are true:

(29) If A is linearly independent, then there exists I such that $A \subseteq I$.

(30) If $\mathrm{Lin}(A) = V$, then there exists I such that $I \subseteq A$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[6] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[7] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):3–11, 1991.

[8] Michał Muzalewski and Wojciech Skaba. Finite sums of vectors in left module over associative ring. *Formalized Mathematics*, 2(**2**):279–282, 1991.

[9] Michał Muzalewski and Wojciech Skaba. Linear combinations in left module over associative ring. *Formalized Mathematics*, 2(**2**):295–300, 1991.

[10] Michał Muzalewski and Wojciech Skaba. Linear independence in left module over domain. *Formalized Mathematics*, 2(**2**):301–303, 1991.

[11] Michał Muzalewski and Wojciech Skaba. Operations on submodules in left module over associative ring. *Formalized Mathematics*, 2(**2**):289–293, 1991.

[12] Michał Muzalewski and Wojciech Skaba. Submodules and cosets of submodules in left module over associative ring. *Formalized Mathematics*, 2(**2**):283–287, 1991.

[13] Michał Muzalewski and Lesław W. Szczerba. Construction of finite sequences over ring and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):97–104, 1991.

[14] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[15] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.

[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[17] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.

[18] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.

[19] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[20] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

Received October 18, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Oriented Metric-Affine Plane - Part I

Jarosław Zajkowski
Warsaw University
Białystok

Summary. We present (in Euclidean and Minkowskian geometry) definitions and some properties of the oriented orthogonality relation. Next we consider consistence of Euclidean space and consistence of Minkowskian space.

MML Identifier: ANALORT.

The terminology and notation used in this paper have been introduced in the following articles: [1], [6], [7], [5], [3], [2], and [4]. We adopt the following rules: V will denote a real linear space, u, u_1, u_2, v, v_1, v_2, w, w_1, x, y will denote vectors of V, and n will denote a real number. Let us consider V, x, y. Let us assume that x, y span the space. Let us consider u. The functor $\rho^{\mathrm{M}}_{x,y}(u)$ yielding a vector of V is defined as follows:

(Def.1) $\rho^{\mathrm{M}}_{x,y}(u) = \pi^1_{x,y}(u) \cdot x + (-\pi^2_{x,y}(u)) \cdot y$.

The following propositions are true:

(1) If x, y span the space, then $\rho^{\mathrm{M}}_{x,y}(u+v) = \rho^{\mathrm{M}}_{x,y}(u) + \rho^{\mathrm{M}}_{x,y}(v)$.

(2) If x, y span the space, then $\rho^{\mathrm{M}}_{x,y}(n \cdot u) = n \cdot \rho^{\mathrm{M}}_{x,y}(u)$.

(3) If x, y span the space, then $\rho^{\mathrm{M}}_{x,y}(0_V) = 0_V$.

(4) If x, y span the space, then $\rho^{\mathrm{M}}_{x,y}(-u) = -\rho^{\mathrm{M}}_{x,y}(u)$.

(5) If x, y span the space, then $\rho^{\mathrm{M}}_{x,y}(u-v) = \rho^{\mathrm{M}}_{x,y}(u) - \rho^{\mathrm{M}}_{x,y}(v)$.

(6) If x, y span the space and $\rho^{\mathrm{M}}_{x,y}(u) = \rho^{\mathrm{M}}_{x,y}(v)$, then $u = v$.

(7) If x, y span the space, then $\rho^{\mathrm{M}}_{x,y}(\rho^{\mathrm{M}}_{x,y}(u)) = u$.

(8) If x, y span the space, then there exists v such that $u = \rho^{\mathrm{M}}_{x,y}(v)$.

Let us consider V, x, y. Let us assume that x, y span the space. Let us consider u. The functor $\rho^{\mathrm{E}}_{x,y}(u)$ yielding a vector of V is defined by:

(Def.2) $\rho^{\mathrm{E}}_{x,y}(u) = \pi^2_{x,y}(u) \cdot x + (-\pi^1_{x,y}(u)) \cdot y$.

ISSN 0777–4028

Next we state several propositions:

(9) If x, y span the space, then $\rho^{\mathrm{E}}_{x,y}(-v) = -\rho^{\mathrm{E}}_{x,y}(v)$.

(10) If x, y span the space, then $\rho^{\mathrm{E}}_{x,y}(u+v) = \rho^{\mathrm{E}}_{x,y}(u) + \rho^{\mathrm{E}}_{x,y}(v)$.

(11) If x, y span the space, then $\rho^{\mathrm{E}}_{x,y}(u-v) = \rho^{\mathrm{E}}_{x,y}(u) - \rho^{\mathrm{E}}_{x,y}(v)$.

(12) If x, y span the space, then $\rho^{\mathrm{E}}_{x,y}(n \cdot u) = n \cdot \rho^{\mathrm{E}}_{x,y}(u)$.

(13) If x, y span the space and $\rho^{\mathrm{E}}_{x,y}(u) = \rho^{\mathrm{E}}_{x,y}(v)$, then $u = v$.

(14) If x, y span the space, then $\rho^{\mathrm{E}}_{x,y}(\rho^{\mathrm{E}}_{x,y}(u)) = -u$.

(15) If x, y span the space, then there exists v such that $\rho^{\mathrm{E}}_{x,y}(v) = u$.

We now define two new predicates. Let us consider V, x, y, u, v, u_1, v_1. Let us assume that x, y span the space. We say that the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y if and only if:

(Def.3) $\rho^{\mathrm{E}}_{x,y}(u), \rho^{\mathrm{E}}_{x,y}(v) \upuparrows u_1, v_1$.

We say that the segments u, v and u_1, v_1 are M-coherently orthogonal in the basis x, y if and only if:

(Def.4) $\rho^{\mathrm{M}}_{x,y}(u), \rho^{\mathrm{M}}_{x,y}(v) \upuparrows u_1, v_1$.

One can prove the following propositions:

(16) If x, y span the space, then if $u, v \upuparrows u_1, v_1$, then $\rho^{\mathrm{E}}_{x,y}(u), \rho^{\mathrm{E}}_{x,y}(v) \upuparrows \rho^{\mathrm{E}}_{x,y}(u_1), \rho^{\mathrm{E}}_{x,y}(v_1)$.

(17) If x, y span the space, then if $u, v \upuparrows u_1, v_1$, then $\rho^{\mathrm{M}}_{x,y}(u), \rho^{\mathrm{M}}_{x,y}(v) \upuparrows \rho^{\mathrm{M}}_{x,y}(u_1), \rho^{\mathrm{M}}_{x,y}(v_1)$.

(18) If x, y span the space, then if the segments u, u_1 and v, v_1 are E-coherently orthogonal in the basis x, y, then the segments v, v_1 and u_1, u are E-coherently orthogonal in the basis x, y.

(19) If x, y span the space, then if the segments u, u_1 and v, v_1 are M-coherently orthogonal in the basis x, y, then the segments v, v_1 and u, u_1 are M-coherently orthogonal in the basis x, y.

(20) If x, y span the space, then the segments u, u and v, w are E-coherently orthogonal in the basis x, y.

(21) If x, y span the space, then the segments u, u and v, w are M-coherently orthogonal in the basis x, y.

(22) If x, y span the space, then the segments u, v and w, w are E-coherently orthogonal in the basis x, y.

(23) If x, y span the space, then the segments u, v and w, w are M-coherently orthogonal in the basis x, y.

(24) If x, y span the space, then u, v, $\rho^{\mathrm{E}}_{x,y}(u)$ and $\rho^{\mathrm{E}}_{x,y}(v)$ are orthogonal w.r.t. x, y.

(25) If x, y span the space, then the segments u, v and $\rho^{\mathrm{E}}_{x,y}(u)$, $\rho^{\mathrm{E}}_{x,y}(v)$ are E-coherently orthogonal in the basis x, y.

(26) If x, y span the space, then the segments u, v and $\rho^{\mathrm{M}}_{x,y}(u)$, $\rho^{\mathrm{M}}_{x,y}(v)$ are M-coherently orthogonal in the basis x, y.

(27) If x, y span the space, then $u, v \upharpoonleft\!\upharpoonright u_1, v_1$ if and only if there exist u_2, v_2 such that $u_2 \neq v_2$ and the segments u_2, v_2 and u, v are E-coherently orthogonal in the basis x, y and the segments u_2, v_2 and u_1, v_1 are E-coherently orthogonal in the basis x, y.

(28) If x, y span the space, then $u, v \upharpoonleft\!\upharpoonright u_1, v_1$ if and only if there exist u_2, v_2 such that $u_2 \neq v_2$ and the segments u_2, v_2 and u, v are M-coherently orthogonal in the basis x, y and the segments u_2, v_2 and u_1, v_1 are M-coherently orthogonal in the basis x, y.

(29) If x, y span the space, then u, v, u_1 and v_1 are orthogonal w.r.t. x, y if and only if the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y or the segments u, v and v_1, u_1 are E-coherently orthogonal in the basis x, y.

(30) If x, y span the space and the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y and the segments u, v and v_1, u_1 are E-coherently orthogonal in the basis x, y, then $u = v$ or $u_1 = v_1$.

(31) If x, y span the space and the segments u, v and u_1, v_1 are M-coherently orthogonal in the basis x, y and the segments u, v and v_1, u_1 are M-coherently orthogonal in the basis x, y, then $u = v$ or $u_1 = v_1$.

(32) If x, y span the space and the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y and the segments u, v and u_1, w are E-coherently orthogonal in the basis x, y, then the segments u, v and v_1, w are E-coherently orthogonal in the basis x, y or the segments u, v and w, v_1 are E-coherently orthogonal in the basis x, y.

(33) If x, y span the space and the segments u, v and u_1, v_1 are M-coherently orthogonal in the basis x, y and the segments u, v and u_1, w are M-coherently orthogonal in the basis x, y, then the segments u, v and v_1, w are M-coherently orthogonal in the basis x, y or the segments u, v and w, v_1 are M-coherently orthogonal in the basis x, y.

(34) If x, y span the space and the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y, then the segments v, u and v_1, u_1 are E-coherently orthogonal in the basis x, y.

(35) If x, y span the space and the segments u, v and u_1, v_1 are M-coherently orthogonal in the basis x, y, then the segments v, u and v_1, u_1 are M-coherently orthogonal in the basis x, y.

(36) If x, y span the space and the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y and the segments u, v and v_1, w are E-coherently orthogonal in the basis x, y, then the segments u, v and u_1, w are E-coherently orthogonal in the basis x, y.

(37) If x, y span the space and the segments u, v and u_1, v_1 are M-coherently orthogonal in the basis x, y and the segments u, v and v_1, w are M-coherently orthogonal in the basis x, y, then the segments u, v and u_1, w are M-coherently orthogonal in the basis x, y.

(38) If x, y span the space, then for every u, v, w there exists u_1 such that

$w \neq u_1$ and the segments w, u_1 and u, v are E-coherently orthogonal in the basis x, y.

(39) If x, y span the space, then for every u, v, w there exists u_1 such that $w \neq u_1$ and the segments w, u_1 and u, v are M-coherently orthogonal in the basis x, y.

(40) If x, y span the space, then for every u, v, w there exists u_1 such that $w \neq u_1$ and the segments u, v and w, u_1 are E-coherently orthogonal in the basis x, y.

(41) If x, y span the space, then for every u, v, w there exists u_1 such that $w \neq u_1$ and the segments u, v and w, u_1 are M-coherently orthogonal in the basis x, y.

(42) If x, y span the space and the segments u, u_1 and v, v_1 are E-coherently orthogonal in the basis x, y and the segments w, w_1 and v, v_1 are E-coherently orthogonal in the basis x, y and the segments w, w_1 and u_2, v_2 are E-coherently orthogonal in the basis x, y, then $w = w_1$ or $v = v_1$ or the segments u, u_1 and u_2, v_2 are E-coherently orthogonal in the basis x, y.

(43) If x, y span the space and the segments u, u_1 and v, v_1 are M-coherently orthogonal in the basis x, y and the segments w, w_1 and v, v_1 are M-coherently orthogonal in the basis x, y and the segments w, w_1 and u_2, v_2 are M-coherently orthogonal in the basis x, y, then $w = w_1$ or $v = v_1$ or the segments u, u_1 and u_2, v_2 are M-coherently orthogonal in the basis x, y.

(44) If x, y span the space and the segments u, u_1 and v, v_1 are E-coherently orthogonal in the basis x, y, then the segments v, v_1 and u, u_1 are E-coherently orthogonal in the basis x, y or the segments v, v_1 and u_1, u are E-coherently orthogonal in the basis x, y.

(45) If x, y span the space and the segments u, u_1 and v, v_1 are M-coherently orthogonal in the basis x, y, then the segments v, v_1 and u, u_1 are M-coherently orthogonal in the basis x, y or the segments v, v_1 and u_1, u are M-coherently orthogonal in the basis x, y.

(46) If x, y span the space and the segments u, u_1 and v, v_1 are E-coherently orthogonal in the basis x, y and the segments v, v_1 and w, w_1 are E-coherently orthogonal in the basis x, y and the segments u_2, v_2 and w, w_1 are E-coherently orthogonal in the basis x, y, then the segments u, u_1 and u_2, v_2 are E-coherently orthogonal in the basis x, y or $v = v_1$ or $w = w_1$.

Next we state several propositions:

(47) If x, y span the space and the segments u, u_1 and v, v_1 are M-coherently orthogonal in the basis x, y and the segments v, v_1 and w, w_1 are M-coherently orthogonal in the basis x, y and the segments u_2, v_2 and w, w_1 are M-coherently orthogonal in the basis x, y, then the segments u, u_1 and u_2, v_2 are M-coherently orthogonal in the basis x, y or $v = v_1$ or

$w = w_1$.

(48) If x, y span the space and the segments u, u_1 and v, v_1 are E-coherently orthogonal in the basis x, y and the segments v, v_1 and w, w_1 are E-coherently orthogonal in the basis x, y and the segments u, u_1 and u_2, v_2 are E-coherently orthogonal in the basis x, y, then the segments u_2, v_2 and w, w_1 are E-coherently orthogonal in the basis x, y or $v = v_1$ or $u = u_1$.

(49) If x, y span the space and the segments u, u_1 and v, v_1 are M-coherently orthogonal in the basis x, y and the segments v, v_1 and w, w_1 are M-coherently orthogonal in the basis x, y and the segments u, u_1 and u_2, v_2 are M-coherently orthogonal in the basis x, y, then the segments u_2, v_2 and w, w_1 are M-coherently orthogonal in the basis x, y or $v = v_1$ or $u = u_1$.

(50) Suppose x, y span the space. Given v, w, u_1, v_1, w_1. Suppose the segments v, v_1 and w, u_1 are not E-coherently orthogonal in the basis x, y and the segments v, v_1 and u_1, w are not E-coherently orthogonal in the basis x, y and the segments u_1, w_1 and u_1, w are E-coherently orthogonal in the basis x, y. Then there exists u_2 such that the segments v, v_1 and v, u_2 are E-coherently orthogonal in the basis x, y or the segments v, v_1 and u_2, v are E-coherently orthogonal in the basis x, y but the segments u_1, w_1 and u_1, u_2 are E-coherently orthogonal in the basis x, y or the segments u_1, w_1 and u_2, u_1 are E-coherently orthogonal in the basis x, y.

(51) If x, y span the space, then there exist u, v, w such that the segments u, v and u, w are E-coherently orthogonal in the basis x, y and for all v_1, w_1 such that the segments v_1, w_1 and u, v are E-coherently orthogonal in the basis x, y holds the segments v_1, w_1 and u, w are not E-coherently orthogonal in the basis x, y and the segments v_1, w_1 and w, u are not E-coherently orthogonal in the basis x, y or $v_1 = w_1$.

(52) Suppose x, y span the space. Given v, w, u_1, v_1, w_1. Suppose h the segments v, v_1 and w, u_1 are not M-coherently orthogonal in the basis x, y and h the segments v, v_1 and u_1, w are not M-coherently orthogonal in the basis x, y and the segments u_1, w_1 and u_1, w are M-coherently orthogonal in the basis x, y. Then there exists u_2 such that the segments v, v_1 and v, u_2 are M-coherently orthogonal in the basis x, y or the segments v, v_1 and u_2, v are M-coherently orthogonal in the basis x, y but the segments u_1, w_1 and u_1, u_2 are M-coherently orthogonal in the basis x, y or the segments u_1, w_1 and u_2, u_1 are M-coherently orthogonal in the basis x, y.

(53) If x, y span the space, then there exist u, v, w such that the segments u, v and u, w are M-coherently orthogonal in the basis x, y and for all v_1, w_1 such that the segments v_1, w_1 and u, v are M-coherently orthogonal in the basis x, y holds h the segments v_1, w_1 and u, w are not M-coherently orthogonal in the basis x, y and h the segments v_1, w_1 and w, u are not M-coherently orthogonal in the basis x, y or $v_1 = w_1$.

In the sequel u_3, v_3 will be arbitrary. Let us consider V, x, y. Let us assume that x, y span the space. The Euclidean oriented orthogonality defined over V,x,y yielding a binary relation on $[\!:$ the vectors of V, the vectors of $V :\!]$ is defined as follows:

(Def.5) $\langle u_3, v_3\rangle \in$the Euclidean oriented orthogonality defined over V,x,y if and only if there exist u_1, u_2, v_1, v_2 such that $u_3 = \langle u_1, u_2\rangle$ and $v_3 = \langle v_1, v_2\rangle$ and the segments u_1, u_2 and v_1, v_2 are E-coherently orthogonal in the basis x, y.

Let us consider V, x, y. Let us assume that x, y span the space. The Minkowskian oriented orthogonality defined over V,x,y yields a binary relation on $[\!:$ the vectors of V, the vectors of $V :\!]$ and is defined by:

(Def.6) $\langle u_3, v_3\rangle \in$the Minkowskian oriented orthogonality defined over V,x,y if and only if there exist u_1, u_2, v_1, v_2 such that $u_3 = \langle u_1, u_2\rangle$ and $v_3 = \langle v_1, v_2\rangle$ and the segments u_1, u_2 and v_1, v_2 are M-coherently orthogonal in the basis x, y.

Let us consider V, x, y. Let us assume that x, y span the space. The functor CESpace(V, x, y) yields an affine structure and is defined by:

(Def.7) CESpace$(V, x, y) = \langle$ the vectors of V,the Euclidean oriented orthogonality defined over V,x,$y\rangle$.

Let us consider V, x, y. Let us assume that x, y span the space. The functor CMSpace(V, x, y) yielding an affine structure is defined by:

(Def.8) CMSpace$(V, x, y) = \langle$ the vectors of V,the Minkowskian oriented orthogonality defined over V,x,$y\rangle$.

Let A_1 be an affine structure, and let p, q, r, s be elements of the points of A_1. The predicate $p, q \top^{>} r, s$ is defined as follows:

(Def.9) $\langle\langle p, q\rangle, \langle r, s\rangle\rangle \in$ the congruence of A_1.

One can prove the following propositions:

(54) If x, y span the space, then for every u_3 holds u_3 is an element of the points of CESpace(V, x, y) if and only if u_3 is a vector of V.

(55) If x, y span the space, then for every u_3 holds u_3 is an element of the points of CMSpace(V, x, y) if and only if u_3 is a vector of V.

In the sequel p, q, r, s are elements of the points of CESpace(V, x, y). Next we state the proposition

(56) If x, y span the space and $u = p$ and $v = q$ and $u_1 = r$ and $v_1 = s$, then $p, q \top^{>} r, s$ if and only if the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y.

In the sequel p, q, r, s will be elements of the points of CMSpace(V, x, y). We now state the proposition

(57) If x, y span the space and $u = p$ and $v = q$ and $u_1 = r$ and $v_1 = s$, then $p, q \top^{>} r, s$ if and only if the segments u, v and u_1, v_1 are M-coherently orthogonal in the basis x, y.

References

[1] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[2] Henryk Oryszczyszyn and Krzysztof Prażmowski. Analytical metric affine spaces and planes. *Formalized Mathematics*, 1(**5**):891–899, 1990.

[3] Henryk Oryszczyszyn and Krzysztof Prażmowski. Analytical ordered affine spaces. *Formalized Mathematics*, 1(**3**):601–605, 1990.

[4] Henryk Oryszczyszyn and Krzysztof Prażmowski. A construction of analytical ordered trapezium spaces. *Formalized Mathematics*, 2(**3**):315–322, 1991.

[5] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[6] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[7] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received October 24, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

The Euclidean Space

Agata Darmochwał
Warsaw University
Białystok

Summary. The general definition of Euclidean Space.

MML Identifier: EUCLID.

The papers [14], [6], [9], [8], [12], [1], [5], [10], [3], [13], [4], [15], [16], [7], [11], and [2] provide the notation and terminology for this paper. In the sequel k, n denote natural numbers and r denotes a real number. Let us consider n. The functor $\mathcal{R}^n$ yields a non-empty set of finite sequences of $\mathbb{R}$ and is defined as follows:

(Def.1) $\mathcal{R}^n = \mathbb{R}^n$.

In the sequel x will denote a finite sequence of elements of $\mathbb{R}$. The function $|\Box|_{\mathbb{R}}$ from $\mathbb{R}$ into $\mathbb{R}$ is defined as follows:

(Def.2) for every r holds $|\Box|_{\mathbb{R}}(r) = |r|$.

Let us consider x. The functor $|x|$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def.3) $|x| = |\Box|_{\mathbb{R}} \cdot x$.

Let us consider n. The functor $\langle\underbrace{0,\ldots,0}_{n}\rangle$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def.4) $\langle\underbrace{0,\ldots,0}_{n}\rangle = n \longmapsto 0$ **qua** a real number .

Let us consider n. Then $\langle\underbrace{0,\ldots,0}_{n}\rangle$ is an element of $\mathcal{R}^n$.

In the sequel x, x_1, x_2, y denote elements of $\mathcal{R}^n$. One can prove the following proposition

(1) x is an element of $\mathbb{R}^n$.

ISSN 0777–4028

Let us consider n, x. Then $-x$ is an element of $\mathcal{R}^n$.

Let us consider n, x, y. Then $x+y$ is an element of $\mathcal{R}^n$. Then $x-y$ is an element of $\mathcal{R}^n$.

Let us consider n, r, x. Then $r \cdot x$ is an element of $\mathcal{R}^n$.

Let us consider n, x. Then $|x|$ is an element of $\mathbb{R}^n$.

Let us consider n, x. Then ${}^2x$ is an element of $\mathbb{R}^n$.

Let x be a finite sequence of elements of $\mathbb{R}$. The functor $|x|$ yielding a real number is defined by:

(Def.5) $|x| = \sqrt{\sum {}^2|x|}$.

Next we state a number of propositions:

(2) $\operatorname{len} x = n$.

(3) $\operatorname{dom} x = \operatorname{Seg} n$.

(4) If $k \in \operatorname{Seg} n$, then $x(k) \in \mathbb{R}$.

(5) If for every k such that $k \in \operatorname{Seg} n$ holds $x_1(k) = x_2(k)$, then $x_1 = x_2$.

(6) If $k \in \operatorname{Seg} n$ and $r = x(k)$, then $|x|(k) = |r|$.

(7) $|\langle\underbrace{0,\ldots,0}_{n}\rangle| = n \longmapsto 0$ **qua** a real number .

(8) $|-x| = |x|$.

(9) $|r \cdot x| = |r| \cdot |x|$.

(10) $|\langle\underbrace{0,\ldots,0}_{n}\rangle| = 0$.

(11) If $|x| = 0$, then $x = \langle\underbrace{0,\ldots,0}_{n}\rangle$.

(12) $|x| \geq 0$.

(13) $|-x| = |x|$.

(14) $|r \cdot x| = |r| \cdot |x|$.

(15) $|x_1 + x_2| \leq |x_1| + |x_2|$.

(16) $|x_1 - x_2| \leq |x_1| + |x_2|$.

(17) $|x_1| - |x_2| \leq |x_1 + x_2|$.

(18) $|x_1| - |x_2| \leq |x_1 - x_2|$.

(19) $|x_1 - x_2| = 0$ if and only if $x_1 = x_2$.

(20) If $x_1 \neq x_2$, then $|x_1 - x_2| > 0$.

(21) $|x_1 - x_2| = |x_2 - x_1|$.

(22) $|x_1 - x_2| \leq |x_1 - x| + |x - x_2|$.

Let us consider n. The functor ρ^n yields a function from $[\!:\mathcal{R}^n, \mathcal{R}^n:\!]$ into $\mathbb{R}$ and is defined by:

(Def.6) for all elements x, y of $\mathcal{R}^n$ holds $\rho^n(x, y) = |x - y|$.

Next we state two propositions:

(23) ${}^2(x-y) = {}^2(y-x)$.

(24) ρ^n is a metric of $\mathcal{R}^n$.

Let us consider n. The functor $\mathcal{E}^n$ yields a metric space and is defined by:

(Def.7) $\mathcal{E}^n = \langle \mathcal{R}^n, \rho^n \rangle$.

Let us consider n. The functor $\mathcal{E}^n_{\mathrm{T}}$ yielding a topological space is defined by:

(Def.8) $\mathcal{E}^n_{\mathrm{T}} = \mathcal{E}^n_{\mathrm{top}}$.

We adopt the following rules: p, p_1, p_2, p_3 will denote points of $\mathcal{E}^n_{\mathrm{T}}$ and x, x_1, x_2, y, y_1, y_2 will denote real numbers. One can prove the following four propositions:

(25) The carrier of $\mathcal{E}^n_{\mathrm{T}} = \mathcal{R}^n$.

(26) p is a function from $\mathrm{Seg}\, n$ into $\mathbb{R}$.

(27) p is a finite sequence of elements of $\mathbb{R}$.

(28) For every finite sequence f such that $f = p$ holds $\mathrm{len}\, f = n$.

Let us consider n. The functor $0_{\mathcal{E}^n_{\mathrm{T}}}$ yielding a point of $\mathcal{E}^n_{\mathrm{T}}$ is defined by:

(Def.9) $0_{\mathcal{E}^n_{\mathrm{T}}} = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

Let us consider n, p_1, p_2. The functor $p_1 + p_2$ yields a point of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def.10) for all elements p'_1, p'_2 of $\mathcal{R}^n$ such that $p'_1 = p_1$ and $p'_2 = p_2$ holds $p_1 + p_2 = p'_1 + p'_2$.

One can prove the following propositions:

(29) $p_1 + p_2 = p_2 + p_1$.

(30) $p_1 + p_2 + p_3 = p_1 + (p_2 + p_3)$.

(31) $0_{\mathcal{E}^n_{\mathrm{T}}} + p = p$ and $p + 0_{\mathcal{E}^n_{\mathrm{T}}} = p$.

Let us consider x, n, p. The functor $x \cdot p$ yields a point of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def.11) for every element p' of $\mathcal{R}^n$ such that $p' = p$ holds $x \cdot p = x \cdot p'$.

Next we state several propositions:

(32) $x \cdot 0_{\mathcal{E}^n_{\mathrm{T}}} = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

(33) $1 \cdot p = p$ and $0 \cdot p = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

(34) $x \cdot y \cdot p = x \cdot (y \cdot p)$.

(35) If $x \cdot p = 0_{\mathcal{E}^n_{\mathrm{T}}}$, then $x = 0$ or $p = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

(36) $x \cdot (p_1 + p_2) = x \cdot p_1 + x \cdot p_2$.

(37) $(x + y) \cdot p = x \cdot p + y \cdot p$.

(38) If $x \cdot p_1 = x \cdot p_2$, then $x = 0$ or $p_1 = p_2$.

Let us consider n, p. The functor $-p$ yields a point of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def.12) for every element p' of $\mathcal{R}^n$ such that $p' = p$ holds $-p = -p'$.

We now state several propositions:

(39) $--p = p$.

(40) $p + -p = 0_{\mathcal{E}^n_{\mathrm{T}}}$ and $-p + p = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

(41) If $p_1 + p_2 = 0_{\mathcal{E}^n_{\mathrm{T}}}$, then $p_1 = -p_2$ and $p_2 = -p_1$.

(42) $-(p_1 + p_2) = -p_1 + -p_2$.

(43) $-p = (-1) \cdot p$.

(44) $-x \cdot p = (-x) \cdot p$ and $-x \cdot p = x \cdot -p$.

Let us consider n, p_1, p_2. The functor $p_1 - p_2$ yields a point of $\mathcal{E}^n_{\mathrm{T}}$ and is defined by:

(Def.13) for all elements p'_1, p'_2 of $\mathcal{R}^n$ such that $p'_1 = p_1$ and $p'_2 = p_2$ holds $p_1 - p_2 = p'_1 - p'_2$.

One can prove the following propositions:

(45) $p_1 - p_2 = p_1 + -p_2$.

(46) $p - p = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

(47) If $p_1 - p_2 = 0_{\mathcal{E}^n_{\mathrm{T}}}$, then $p_1 = p_2$.

(48) $-(p_1 - p_2) = p_2 - p_1$ and $-(p_1 - p_2) = -p_1 + p_2$.

(49) $p_1 + (p_2 - p_3) = (p_1 + p_2) - p_3$.

(50) $p_1 - (p_2 + p_3) = p_1 - p_2 - p_3$.

(51) $p_1 - (p_2 - p_3) = (p_1 - p_2) + p_3$.

(52) $p = (p + p_1) - p_1$ and $p = (p - p_1) + p_1$.

(53) $x \cdot (p_1 - p_2) = x \cdot p_1 - x \cdot p_2$.

(54) $(x - y) \cdot p = x \cdot p - y \cdot p$.

In the sequel p, p_1, p_2 will be points of $\mathcal{E}^2_{\mathrm{T}}$. Next we state the proposition

(55) There exist x, y such that $p = \langle x, y \rangle$.

We now define two new functors. Let us consider p. The functor $p_{\mathbf{1}}$ yields a real number and is defined by:

(Def.14) for every finite sequence f such that $p = f$ holds $p_{\mathbf{1}} = f(1)$.

The functor $p_{\mathbf{2}}$ yielding a real number is defined by:

(Def.15) for every finite sequence f such that $p = f$ holds $p_{\mathbf{2}} = f(2)$.

Let us consider x, y. The functor $[x, y]$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def.16) $[x, y] = \langle x, y \rangle$.

The following propositions are true:

(56) $[x, y]_{\mathbf{1}} = x$ and $[x, y]_{\mathbf{2}} = y$.

(57) $p = [p_{\mathbf{1}}, p_{\mathbf{2}}]$.

(58) $0_{\mathcal{E}^2_{\mathrm{T}}} = [0, 0]$.

(59) $p_1 + p_2 = [p_{1\mathbf{1}} + p_{2\mathbf{1}}, p_{1\mathbf{2}} + p_{2\mathbf{2}}]$.

(60) $[x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2]$.

(61) $x \cdot p = [x \cdot p_{\mathbf{1}}, x \cdot p_{\mathbf{2}}]$.

(62) $x \cdot [x_1, y_1] = [x \cdot x_1, x \cdot y_1]$.

(63) $-p = [-p_{\mathbf{1}}, -p_{\mathbf{2}}]$.

(64) $-[x_1, y_1] = [-x_1, -y_1]$.

(65) $p_1 - p_2 = [p_{1\mathbf{1}} - p_{2\mathbf{1}}, p_{1\mathbf{2}} - p_{2\mathbf{2}}]$.

(66) $[x_1, y_1] - [x_2, y_2] = [x_1 - x_2, y_1 - y_2]$.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[2] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.

[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[4] Czesław Byliński. Binary operations applied to finite sequences. *Formalized Mathematics*, 1(**4**):643–649, 1990.

[5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[7] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.

[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[9] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.

[10] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[11] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[12] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[13] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[14] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[15] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.

[16] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

Received November 21, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Metric Spaces as Topological Spaces - Fundamental Concepts

Agata Darmochwał[1]
Warsaw University
Białystok

Yatsuka Nakamura
Shinshu University
Nagano

Summary. Some notions connected with metric spaces and the relationship between metric spaces and topological spaces. Compactness of topological spaces is transferred for the case of metric spaces [13]. Some basic theorems about translations of topological notions for metric spaces are proved. One-dimensional topological space $\mathbb{R}^{\mathbf{1}}$ is introduced, too.

MML Identifier: `TOPMETR`.

The papers [21], [11], [1], [22], [20], [4], [5], [6], [12], [10], [3], [14], [16], [23], [9], [7], [2], [15], [18], [17], [19], and [8] provide the notation and terminology for this paper. For simplicity we follow a convention: a, b, r will denote real numbers, n will denote a natural number, T will denote a topological space, and F will denote a family of subsets of T. One can prove the following proposition

(1) F is a cover of T if and only if the carrier of $T \subseteq \bigcup F$.

In the sequel A will be a subspace of T. Next we state three propositions:

(2) For every point p of A holds p is a point of T.

(3) If T is a T_2 space, then A is a T_2 space.

(4) For all subspaces A, B of T such that the carrier of $A \subseteq$ the carrier of B holds A is a subspace of B.

In the sequel P, Q denote subsets of T and p denotes a point of T. We now state several propositions:

(5) If $P \neq \emptyset_T$, then $T \upharpoonright P$ is a subspace of $T \upharpoonright P \cup Q$ **qua** a subset of T but if $Q \neq \emptyset_T$, then $T \upharpoonright Q$ is a subspace of $T \upharpoonright P \cup Q$ **qua** a subset of T.

(6) If $P \neq \emptyset$ and $p \in P$, then for every neighborhood Q of p and for every point p' of $T \upharpoonright P$ and for every subset Q' of $T \upharpoonright P$ such that $Q' = Q \cap P$ and $p' = p$ holds Q' is a neighborhood of p'.

[1]The article was written during my work at Shinshu University, 1991.

ISSN 0777–4028

(7) For all topological spaces A, B, C and for every map f from A into C such that f is continuous and C is a subspace of B for every map h from A into B such that $h = f$ holds h is continuous.

(8) For all topological spaces A, B and for every map f from A into B and for every subspace C of B such that f is continuous and $\operatorname{rng} f \subseteq$ the carrier of C for every map h from A into C such that $h = f$ holds h is continuous.

(9) For all topological spaces A, B and for every map f from A into B and for every subset C of B such that f is continuous and $\operatorname{rng} f \subseteq C$ and $C \neq \emptyset$ for every map h from A into $B \restriction C$ such that $h = f$ holds h is continuous.

(10) For all topological spaces T, S and for every map f from T into S such that f is continuous for every subset P of T and for every map h from $T \restriction P$ into S such that $P \neq \emptyset_T$ and $h = f \restriction P$ holds h is continuous.

In the sequel M will denote a metric space and p will denote a point of M. One can prove the following proposition

(11) If $r > 0$, then $p \in \mathrm{Ball}(p, r)$.

We now define two new modes. Let us consider M. A subset of M is sets of points of M.

A family of subsets of M is a family of subsets of the carrier of M.

Let us consider M. A metric space is said to be a subspace of M if:

(Def.1) the carrier of it $\subseteq$ the carrier of M and for all points x, y of it holds (the distance of it)$(x, y) =$ (the distance of M)(x, y).

In the sequel A will be a subspace of M. One can prove the following propositions:

(12) For every point p of A holds p is a point of M.

(13) For every point x of M and for every point x' of A such that $x = x'$ holds $\mathrm{Ball}(x', r) = \mathrm{Ball}(x, r) \cap$ the carrier of A.

Let M be a metric space, and let A be a non-empty subset of M. The functor $M \restriction A$ yielding a subspace of M is defined as follows:

(Def.2) the carrier of $M \restriction A = A$.

Let us consider a, b. Let us assume that $a \leq b$. The functor $[a,\, b]_{\mathrm{M}}$ yields a subspace of the metric space of real numbers and is defined by:

(Def.3) for every non-empty subset P of the metric space of real numbers such that $P = [a, b]$ holds $[a,\, b]_{\mathrm{M}} =$ (the metric space of real numbers) $\restriction P$.

We now state the proposition

(14) If $a \leq b$, then the carrier of $[a,\, b]_{\mathrm{M}} = [a, b]$.

In the sequel F, G will be families of subsets of M. We now define two new predicates. Let us consider M, F. We say that F is a family of balls if and only if:

(Def.4) for an arbitrary P such that $P \in F$ there exist p, r such that $P = \mathrm{Ball}(p, r)$.

We say that F is a cover of M if and only if:

(Def.5) the carrier of $M \subseteq \bigcup F$.

The following propositions are true:

(15) For all points p, q of the metric space of real numbers and for all real numbers x, y such that $x = p$ and $y = q$ holds $\rho(p, q) = |x - y|$.

(16) The carrier of $M =$ the carrier of M_{top} and the topology of $M_{\mathrm{top}} =$ the open set family of M.

(17) For every family F of subsets of M holds F is a family of subsets of M_{top}.

(18) For every family F of subsets of M_{top} holds F is a family of subsets of M.

(19) A_{top} is a subspace of M_{top}.

(20) For every subset P of $\mathcal{E}^n_{\mathrm{T}}$ and for every non-empty subset Q of $\mathcal{E}^n$ such that $P = Q$ holds $(\mathcal{E}^n_{\mathrm{T}}) \restriction P = (\mathcal{E}^n \restriction Q)_{\mathrm{top}}$.

(21) For every subset P of M_{top} such that $P = \mathrm{Ball}(p, r)$ holds P is open.

(22) For every subset P of M_{top} holds P is open if and only if for every point p of M such that $p \in P$ there exists r such that $r > 0$ and $\mathrm{Ball}(p, r) \subseteq P$.

Let us consider M. We say that M is compact if and only if:

(Def.6) M_{top} is compact.

We now state the proposition

(23) M is compact if and only if for every F such that F is a family of balls and F is a cover of M there exists G such that $G \subseteq F$ and G is a cover of M and G is finite.

The topological space $\mathbb{R}^{\mathbf{1}}$ is defined as follows:

(Def.7) $\mathbb{R}^{\mathbf{1}} =$ (the metric space of real numbers)$_{\mathrm{top}}$.

One can prove the following proposition

(24) The carrier of $\mathbb{R}^{\mathbf{1}} = \mathbb{R}$.

Let us consider a, b. Let us assume that $a \leq b$. The functor $[a,\, b]_{\mathrm{T}}$ yields a subspace of $\mathbb{R}^{\mathbf{1}}$ and is defined by:

(Def.8) $[a,\, b]_{\mathrm{T}} = ([a,\, b]_{\mathrm{M}})_{\mathrm{top}}$.

We now state three propositions:

(25) If $a \leq b$, then the carrier of $[a,\, b]_{\mathrm{T}} = [a, b]$.

(26) If $a \leq b$, then for every subset P of $\mathbb{R}^{\mathbf{1}}$ such that $P = [a, b]$ holds $[a,\, b]_{\mathrm{T}} = \mathbb{R}^{\mathbf{1}} \restriction P$.

(27) $[0,\, 1]_{\mathrm{T}} = \mathbb{I}$.

Let us note that it makes sense to consider the following constant. Then $\mathbb{I}$ is a subspace of $\mathbb{R}^{\mathbf{1}}$.

The following proposition is true

(28) For every map f from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$ such that there exist real numbers a, b such that for every real number x holds $f(x) = a \cdot x + b$ holds f is continuous.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.
[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[7] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.
[8] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[9] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[10] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[11] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[12] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.
[13] John L. Kelly. *Topologie.* Volume I,II, von Nostrand, 1955.
[14] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[15] Beata Padlewska. Locally connected spaces. *Formalized Mathematics*, 2(**1**):93–96, 1991.
[16] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[17] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.
[18] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[19] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.
[20] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[21] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[22] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[23] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

Received November 21, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Heine–Borel's Covering Theorem

Agata Darmochwał[1]
Warsaw University
Białystok

Yatsuka Nakamura
Shinshu University
Nagano

Summary. Heine–Borel's covering theorem, also known as Borel–Lebesgue theorem [3], is proved. Some useful theorems on real inequalities, intervals, sequences and notion of power sequence which are necessary for the theorem are also proved.

MML Identifier: HEINE.

The terminology and notation used in this paper have been introduced in the following articles: [23], [11], [1], [5], [6], [12], [9], [4], [24], [18], [19], [8], [7], [2], [20], [16], [13], [15], [14], [21], [22], [17], and [10]. We follow a convention: a, b, x, y, z denote real numbers and k, n denote natural numbers. We now state several propositions:

(1) For every subspace A of the metric space of real numbers and for all points p, q of A and for all x, y such that $x = p$ and $y = q$ holds $\rho(p, q) = |x - y|$.

(2) If $x \leq y$ and $y \leq z$, then $[x, y] \cup [y, z] = [x, z]$.

(3) If $x \geq 0$ and $a + x \leq b$, then $a \leq b$.

(4) If $x \geq 0$ and $a - x \geq b$, then $a \geq b$.

(5) If $x > 0$, then $x^k > 0$.

In the sequel s_1 will be a sequence of real numbers. Next we state the proposition

(6) If s_1 is increasing and $\operatorname{rng} s_1 \subseteq \mathbb{N}$, then $n \leq s_1(n)$.

Let us consider s_1, k. The functor k^{s_1} yielding a sequence of real numbers is defined by:

(Def.1) for every n holds $k^{s_1}(n) = k^{s_1(n)}$.

We now state several propositions:

[1]The article was written during my work at Shinshu University, 1991.

ISSN 0777–4028

(7) $2^n \geq n + 1$.

(8) $2^n > n$.

(9) If s_1 is divergent to $+\infty$, then 2^{s_1} is divergent to $+\infty$.

(10) For every topological space T such that the carrier of T is finite holds T is compact.

(11) If $a \leq b$, then $[a, b]_{\mathrm{T}}$ is compact.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.

[3] Nicolas Bourbaki. *Elements de Mathematique.* Volume Topologie Generale, HERMANN, troisieme edition edition, 1960.

[4] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[7] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.

[8] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.

[9] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[10] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces - fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.

[11] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[12] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.

[13] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.

[14] Jarosław Kotowicz. The limit of a real function at infinity. *Formalized Mathematics*, 2(**1**):17–28, 1991.

[15] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.

[16] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[17] Rafał Kwiatek. Factorial and Newton coeffitients. *Formalized Mathematics*, 1(**5**):887–890, 1990.

[18] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[19] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[20] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[21] Konrad Raczkowski and Andrzej Nędzusiak. Real exponents and logarithms. *Formalized Mathematics*, 2(**2**):213–216, 1991.

[22] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.

[23] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[24] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received November 21, 1991

FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Some Facts about Union of Two Functions and Continuity of Union of Functions

Yatsuka Nakamura
Shinshu University
Nagano

Agata Darmochwał[1]
Warsaw University
Białystok

Summary. Proofs of two theorems connected with the union of any two functions and the proofs of two theorems on the continuity of the union of two continuous functions between topological spaces. The theorem stating that the union of two subsets of R^2, which are homeomorphic to unit interval and have only one terminal joined point, is also homeomorphic to unit interval is proved, too.

MML Identifier: `TOPMETR2`.

The notation and terminology used in this paper have been introduced in the following papers: [14], [9], [15], [13], [2], [3], [4], [11], [7], [5], [12], [10], [1], [6], and [8]. In the sequel x, y, z are real numbers. Next we state the proposition

(1) If $x \leq y$ and $y \leq z$, then $[x, y] \cap [y, z] = \{y\}$.

In the sequel f, g will be functions and x_1, x_2 will be arbitrary. Next we state two propositions:

(2) If f is one-to-one and g is one-to-one and for all x_1, x_2 such that $x_1 \in \operatorname{dom} g$ and $x_2 \in \operatorname{dom} f \setminus \operatorname{dom} g$ holds $g(x_1) \neq f(x_2)$, then $f +\cdot g$ is one-to-one.

(3) If $f\,^\circ\,(\operatorname{dom} f \cap \operatorname{dom} g) \subseteq \operatorname{rng} g$, then $\operatorname{rng} f \cup \operatorname{rng} g = \operatorname{rng}(f +\cdot g)$.

We follow the rules: T, T_1, T_2, S will be topological spaces and p, p_1, p_2 will be points of T. Next we state two propositions:

(4) Let T_1, T_2 be subspaces of T. Let f be a map from T_1 into S. Let g be a map from T_2 into S. Suppose $\Omega_{T_1} \cup \Omega_{T_2} = \Omega_T$ and $\Omega_{T_1} \cap \Omega_{T_2} = \{p\}$ and T_1 is compact and T_2 is compact and T is a T_2 space and f is continuous and g is continuous and $f(p) = g(p)$. Then there exists a map h from T into S such that $h = f +\cdot g$ and h is continuous.

[1]The article was written during my work at Shinshu University, 1991.

ISSN 0777–4028

(5) Let f be a map from T_1 into S. Let g be a map from T_2 into S. Suppose that
(i) T_1 is a subspace of T,
(ii) T_2 is a subspace of T,
(iii) $\Omega_{T_1} \cup \Omega_{T_2} = \Omega_T$,
(iv) $\Omega_{T_1} \cap \Omega_{T_2} = \{p_1, p_2\}$,
(v) T_1 is compact,
(vi) T_2 is compact,
(vii) T is a T_2 space,
(viii) f is continuous,
(ix) g is continuous,
(x) $f(p_1) = g(p_1)$,
(xi) $f(p_2) = g(p_2)$.

Then there exists a map h from T into S such that $h = f +\cdot\, g$ and h is continuous.

In the sequel P, Q denote subsets of $\mathcal{E}^2_{\mathrm{T}}$. One can prove the following proposition

(6) Let f be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction P$. Let g be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction Q$. Suppose $P \cap Q = \{p\}$ and f is a homeomorphism and $f(1) = p$ and g is a homeomorphism and $g(0) = p$. Then there exists a map h from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction P \cup Q$ **qua** a subset of $\mathcal{E}^2_{\mathrm{T}}$ such that h is a homeomorphism and $h(0) = f(0)$ and $h(1) = g(1)$.

References

[1] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.
[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[4] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[5] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.
[6] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[7] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[8] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces - fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.
[9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[10] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.
[11] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[12] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[13] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[14] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[15] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received November 21, 1991

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