Preface

We offer to our Readers Volume 2 of mathematical papers which are abstracts of Mizar articles to be found in the Main Mizar Library (MML). They are usually published in the order in which they have been approved for MML. A careful Reader may note that our publication has several peculiarities due to two facts. First, it is an endeavour to make a machine translation into English. Secondly, changes in the PC Mizar system and continually updated MML influence the quality of the texts published and the topical value of the papers. Hence, first, the standard of English is not always satisfactory. Secondly, the quality of the papers is very closely related to what is actually taking place in MML. Originally, obvious theorems (relative to the power of Checker) were not identified. As the system PC Mizar was developing, some theorems became obvious. It is likewise with repeated theorems (which accounts for the footnotes in the text of the type "The proposition (k) was either repeated or obvious"). Those theorems can be classed in two groups. The first includes accidental repetitions: the author did not know that such a theorem was already included in MML and proved it again. There were few such cases. The other includes some 500 eliminated theorems because they were so-called definitional theorems. The authors wrote those theorems because previously it had not been possible directly to refer to definitions.

The Readers are also requested to note that in the present issue we have changed the formats of certain operations. The operation symbol of the removal of n initial terms from a sequence has been changed from $^\frown$ into $\uparrow$ (see [2] and [1]). Likewise, the operation symbol of the multiplication of real functions $\diamond$ has been removed (see [1]).

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FORMALIZED MATHEMATICS
Vol.2,No.1, January–February 1991
Université Catholique de Louvain

Construction of Rings and Left-, Right-, and Bi-Modules over a Ring

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Summary. Definitions of some classes of rings and left-, right-, and bi-modules over a ring and some elementary theorems on rings and skew fields.

MML Identifier: VECTSP_2.

The articles [9], [8], [11], [3], [1], [10], [7], [4], [2], [5], and [6] provide the notation and terminology for this paper. In the sequel F_1 will denote a field structure. Let us consider F_1. A scalar of F_1 is an element of the carrier of F_1.

In the sequel x, y will denote scalars of F_1. Let us consider F_1, x, y. The functor $x - y$ yields a scalar of F_1 and is defined as follows:

(Def.1) $x - y = x + (-y)$.

In the sequel F denotes a field. A field structure is called a ring if:

(Def.2) Let x, y, z be scalars of it . Then
- (i) $x + y = y + x$,
- (ii) $(x + y) + z = x + (y + z)$,
- (iii) $x + 0_{\rm it} = x$,
- (iv) $x + (-x) = 0_{\rm it}$,
- (v) $x \cdot (1_{\rm it}) = x$,
- (vi) $(1_{\rm it}) \cdot x = x$,
- (vii) $x \cdot (y + z) = x \cdot y + x \cdot z$,
- (viii) $(y + z) \cdot x = y \cdot x + z \cdot x$.

The following proposition is true

[1]Supported by RPBP.III-24.C6.

ISSN 0777–4028

(1) The following conditions are equivalent:

(i) for all scalars x, y, z of F_1 holds $x+y = y+x$ and $(x+y)+z = x+(y+z)$ and $x + 0_{F_1} = x$ and $x + (-x) = 0_{F_1}$ and $x \cdot (1_{F_1}) = x$ and $(1_{F_1}) \cdot x = x$ and $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$,

(ii) F_1 is a ring.

In the sequel R is a ring and x, y, z are scalars of R. Next we state several propositions:

(2) $x + y = y + x$.

(3) $(x + y) + z = x + (y + z)$.

(4) $x + 0_R = x$.

(5) $x + (-x) = 0_R$.

(6) $x \cdot (1_R) = x$ and $(1_R) \cdot x = x$.

(7) $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$.

A ring is called an associative ring if:

(Def.3) for all scalars x, y, z of it holds $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

The following proposition is true

(8) For all scalars x, y, z of R holds $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ if and only if R is an associative ring.

In the sequel R will denote an associative ring and x, y, z will denote scalars of R. One can prove the following proposition

(9) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

An associative ring is said to be a commutative ring if:

(Def.4) for all scalars x, y of it holds $x \cdot y = y \cdot x$.

One can prove the following proposition

(10) If for all scalars x, y of R holds $x \cdot y = y \cdot x$, then R is a commutative ring.

In the sequel R will denote a commutative ring and x, y will denote scalars of R. The following proposition is true

(11) $x \cdot y = y \cdot x$.

A commutative ring is said to be an integral domain if:

(Def.5) $0_{\mathrm{it}} \neq 1_{\mathrm{it}}$ and for all scalars x, y of it such that $x \cdot y = 0_{\mathrm{it}}$ holds $x = 0_{\mathrm{it}}$ or $y = 0_{\mathrm{it}}$.

We now state two propositions:

(12) If $0_R \neq 1_R$ and for all x, y such that $x \cdot y = 0_R$ holds $x = 0_R$ or $y = 0_R$, then R is an integral domain.

(13) F is an integral domain.

In the sequel R denotes an integral domain and x, y denote scalars of R. The following propositions are true:

(14) $0_R \neq 1_R$.

(15) If $x \cdot y = 0_R$, then $x = 0_R$ or $y = 0_R$.

An associative ring is called a skew field if:

(Def.6) for every scalar x of it holds if $x \neq 0_{\rm it}$, then there exists a scalar y of it such that $x \cdot y = 1_{\rm it}$ but $0_{\rm it} \neq 1_{\rm it}$.

In the sequel R denotes an associative ring. The following proposition is true

(16) If for every scalar x of R holds if $x \neq 0_R$, then there exists a scalar y of R such that $x \cdot y = 1_R$ but $0_R \neq 1_R$, then R is a skew field.

In the sequel S_1 will denote a skew field and x, y will denote scalars of S_1. The following propositions are true:

(17) If $x \neq 0_{S_1}$, then there exists y such that $x \cdot y = 1_{S_1}$.

(18) $0_{S_1} \neq 1_{S_1}$.

(19) F is a skew field.

We see that the field is a skew field.

In the sequel R is a ring and x, y, z are scalars of R. Next we state a number of propositions:

(20) $x - y = x + (-y)$.

(21) $-0_R = 0_R$.

(22) $x + y = z$ if and only if $x = z - y$ but $x + y = z$ if and only if $y = z - x$.

(23) $x - 0_R = x$ and $0_R - x = -x$.

(24) If $x + y = x + z$, then $y = z$ but if $x + y = z + y$, then $x = z$.

(25) $-(x + y) = (-x) + (-y)$.

(26) $x \cdot 0_R = 0_R$ and $0_R \cdot x = 0_R$.

(27) $-(-x) = x$.

(28) $(-x) \cdot y = -x \cdot y$.

(29) $x \cdot (-y) = -x \cdot y$.

(30) $(-x) \cdot (-y) = x \cdot y$.

(31) $x \cdot (y - z) = x \cdot y - x \cdot z$.

(32) $(x - y) \cdot z = x \cdot z - y \cdot z$.

(33) $(x + y) - z = x + (y - z)$.

(34) $x = 0_R$ if and only if $-x = 0_R$.

(35) $x - (y + z) = (x - y) - z$.

(36) $x - (y - z) = (x - y) + z$.

(37) $x - x = 0_R$ and $(-x) + x = 0_R$.

(38) For every x, y there exists z such that $x = y + z$ and $x = z + y$.

In the sequel S_1 denotes a skew field and x, y, z denote scalars of S_1. We now state four propositions:

(39) If $x \cdot y = 1_{S_1}$, then $x \neq 0_{S_1}$ and $y \neq 0_{S_1}$.

(40) If $x \neq 0_{S_1}$, then there exists y such that $y \cdot x = 1_{S_1}$.

(41) If $x \cdot y = 1_{S_1}$, then $y \cdot x = 1_{S_1}$.

(42) If $x \cdot y = x \cdot z$ and $x \neq 0_{S_1}$, then $y = z$.

Let us consider S_1, x. Let us assume that $x \neq 0_{S_1}$. The functor x^{-1} yielding a scalar of S_1 is defined by:

(Def.7) $x \cdot (x^{-1}) = 1_{S_1}$.

Let us consider S_1, x, y. Let us assume that $y \neq 0_{S_1}$. The functor $\frac{x}{y}$ yielding a scalar of S_1 is defined by:

(Def.8) $\frac{x}{y} = x \cdot y^{-1}$.

One can prove the following propositions:

(43) If $x \neq 0_{S_1}$, then $x \cdot x^{-1} = 1_{S_1}$ and $x^{-1} \cdot x = 1_{S_1}$.

(44) If $y \neq 0_{S_1}$, then $\frac{x}{y} = x \cdot y^{-1}$.

(45) If $x \cdot y = 1_{S_1}$, then $x = y^{-1}$ and $y = x^{-1}$.

(46) If $x \neq 0_{S_1}$ and $y \neq 0_{S_1}$, then $x^{-1} \cdot y^{-1} = (y \cdot x)^{-1}$.

(47) If $x \cdot y = 0_{S_1}$, then $x = 0_{S_1}$ or $y = 0_{S_1}$.

(48) If $x \neq 0_{S_1}$, then $x^{-1} \neq 0_{S_1}$.

(49) If $x \neq 0_{S_1}$, then $(x^{-1})^{-1} = x$.

(50) If $x \neq 0_{S_1}$, then $\frac{1_{S_1}}{x} = x^{-1}$ and $\frac{1_{S_1}}{x^{-1}} = x$.

(51) If $x \neq 0_{S_1}$, then $x \cdot \frac{1_{S_1}}{x} = 1_{S_1}$ and $\frac{1_{S_1}}{x} \cdot x = 1_{S_1}$.

(52) If $x \neq 0_{S_1}$, then $\frac{x}{x} = 1_{S_1}$.

(53) If $y \neq 0_{S_1}$ and $z \neq 0_{S_1}$, then $\frac{x}{y} = \frac{x \cdot z}{y \cdot z}$.

(54) If $y \neq 0_{S_1}$, then $-\frac{x}{y} = \frac{-x}{y}$ and $\frac{x}{-y} = -\frac{x}{y}$.

(55) If $z \neq 0_{S_1}$, then $\frac{x}{z} + \frac{y}{z} = \frac{x+y}{z}$ and $\frac{x}{z} - \frac{y}{z} = \frac{x-y}{z}$.

(56) If $y \neq 0_{S_1}$ and $z \neq 0_{S_1}$, then $\frac{x}{\frac{y}{z}} = \frac{x \cdot z}{y}$.

(57) If $y \neq 0_{S_1}$, then $\frac{x}{y} \cdot y = x$.

Let us consider F_1. We consider left module structures over F_1 which are systems

$\langle$a carrier, a left multiplication$\rangle$,

where the carrier is an Abelian group and the left multiplication is a function from $[\![$ the carrier of F_1, the carrier of the carrier $]\!]$ into the carrier of the carrier.

In the sequel L_1 denotes a left module structure over F_1. We now define two new modes. Let us consider F_1, L_1. A scalar of L_1 is a scalar of F_1.

A vector of L_1 is an element of the carrier of L_1.

Let us consider F_1. We consider right module structures over F_1 which are systems

$\langle$a carrier, a right multiplication$\rangle$,

where the carrier is an Abelian group and the right multiplication is a function from $[\![$ the carrier of the carrier, the carrier of F_1 $]\!]$ into the carrier of the carrier.

In the sequel R_1 will denote a right module structure over F_1. We now define two new modes. Let us consider F_1, R_1. A scalar of R_1 is a scalar of F_1.

A vector of R_1 is an element of the carrier of R_1.

Let us consider F_1. We consider bimodule structures over F_1 which are systems

⟨a carrier, a left multiplication, a right multiplication⟩,

where the carrier is an Abelian group, the left multiplication is a function from $[\!:$ the carrier of F_1, the carrier of the carrier $:\!]$ into the carrier of the carrier, and the right multiplication is a function from $[\!:$ the carrier of the carrier, the carrier of F_1 $:\!]$ into the carrier of the carrier.

In the sequel B_1 will denote a bimodule structure over F_1. We now define two new modes. Let us consider F_1, B_1. A scalar of B_1 is a scalar of F_1.

A vector of B_1 is an element of the carrier of B_1.

In the sequel R is a ring. Let us consider R. The functor $\mathrm{AbGr}(R)$ yields an Abelian group and is defined by:

(Def.9) $\mathrm{AbGr}(R) = \langle$ the carrier of R, the addition of R, the reverse-map of R, the zero of $R\rangle$.

Next we state the proposition

(58) $\mathrm{AbGr}(R) = \langle$ the carrier of R, the addition of R, the reverse-map of R, the zero of $R\rangle$.

Let us consider R. The functor $\mathrm{LeftModMult}(R)$ yielding a function from $[\!:$ the carrier of R, the carrier of $\mathrm{AbGr}(R)$ $:\!]$ into the carrier of $\mathrm{AbGr}(R)$ is defined as follows:

(Def.10) $\mathrm{LeftModMult}(R) =$ the multiplication of R.

Next we state the proposition

(59) $\mathrm{LeftModMult}(R) =$ the multiplication of R.

Let us consider R. The functor $\mathrm{LeftMod}(R)$ yielding a left module structure over R is defined as follows:

(Def.11) $\mathrm{LeftMod}(R) = \langle \mathrm{AbGr}(R), \mathrm{LeftModMult}(R)\rangle$.

We now state the proposition

(60) $\mathrm{LeftMod}(R) = \langle \mathrm{AbGr}(R), \mathrm{LeftModMult}(R)\rangle$.

In the sequel V will be a left module structure over R. Let us consider R, V, and let x be a scalar of R, and let v be a vector of V. The functor $x \cdot v$ yielding a vector of V is defined as follows:

(Def.12) for every scalar x' of V such that $x' = x$ holds $x \cdot v =$ (the left multiplication of V)(x', v).

The following proposition is true

(62)[2] For every V being a left module structure over R and for every scalar x of R and for every vector v of V and for every scalar x' of V such that $x' = x$ holds $x \cdot v =$ (the left multiplication of V)(x', v).

Let us consider R. The functor $\mathrm{RightModMult}(R)$ yields a function from $[\!:$ the carrier of $\mathrm{AbGr}(R)$, the carrier of R $:\!]$ into the carrier of $\mathrm{AbGr}(R)$ and is defined as follows:

[2]The proposition (61) was either repeated or obvious.

(Def.13) RightModMult(R) = the multiplication of R.

We now state the proposition

(63) RightModMult(R) = the multiplication of R.

Let us consider R. The functor RightMod(R) yielding a right module structure over R is defined as follows:

(Def.14) $\mathrm{RightMod}(R) = \langle \mathrm{AbGr}(R), \mathrm{RightModMult}(R)\rangle$.

We now state the proposition

(64) $\mathrm{RightMod}(R) = \langle \mathrm{AbGr}(R), \mathrm{RightModMult}(R)\rangle$.

In the sequel V will denote a right module structure over R. Let us consider R, V, and let x be a scalar of R, and let v be a vector of V. The functor $v \cdot x$ yielding a vector of V is defined as follows:

(Def.15) for every scalar x' of V such that $x' = x$ holds $v \cdot x =$ (the right multiplication of V)(v, x').

We now state the proposition

(66)[3] For every V being a right module structure over R and for every scalar x of R and for every vector v of V and for every scalar x' of V such that $x' = x$ holds $v \cdot x =$ (the right multiplication of V)(v, x').

Let us consider R. The functor BiMod(R) yielding a bimodule structure over R is defined as follows:

(Def.16) $\mathrm{BiMod}(R) = \langle \mathrm{AbGr}(R), \mathrm{LeftModMult}(R), \mathrm{RightModMult}(R)\rangle$.

The following proposition is true

(67) $\mathrm{BiMod}(R) = \langle \mathrm{AbGr}(R), \mathrm{LeftModMult}(R), \mathrm{RightModMult}(R)\rangle$.

In the sequel V is a bimodule structure over R. Let us consider R, V, and let x be a scalar of R, and let v be a vector of V. The functor $x \cdot v$ yields a vector of V and is defined as follows:

(Def.17) for every scalar x' of V such that $x' = x$ holds $x \cdot v =$ (the left multiplication of V)(x', v).

One can prove the following proposition

(69)[4] For every V being a bimodule structure over R and for every scalar x of R and for every vector v of V and for every scalar x' of V such that $x' = x$ holds $x \cdot v =$ (the left multiplication of V)(x', v).

Let us consider R, V, and let x be a scalar of R, and let v be a vector of V. The functor $v \cdot x$ yields a vector of V and is defined by:

(Def.18) for every scalar x' of V such that $x' = x$ holds $v \cdot x =$ (the right multiplication of V)(v, x').

The following proposition is true

[3]The proposition (65) was either repeated or obvious.

[4]The proposition (68) was either repeated or obvious.

(70) For every V being a bimodule structure over R and for every scalar x of R and for every vector v of V and for every scalar x' of V such that $x' = x$ holds $v \cdot x =$ (the right multiplication of V)(v, x').

In the sequel R will denote an associative ring. Next we state the proposition

(71) Let x, y be scalars of R. Let v, w be vectors of LeftMod(R). Then $x \cdot (v+w) = x \cdot v + x \cdot w$ and $(x+y) \cdot v = x \cdot v + y \cdot v$ and $(x \cdot y) \cdot v = x \cdot (y \cdot v)$ and $(1_R) \cdot v = v$.

Let us consider R. A left module structure over R is called a left module over R if:

(Def.19) Let x, y be scalars of R. Let v, w be vectors of it . Then $x \cdot (v + w) = x \cdot v + x \cdot w$ and $(x + y) \cdot v = x \cdot v + y \cdot v$ and $(x \cdot y) \cdot v = x \cdot (y \cdot v)$ and $(1_R) \cdot v = v$.

We now state the proposition

(72) Let V be a left module structure over R. Then the following conditions are equivalent:
 (i) for all scalars x, y of R and for all vectors v, w of V holds $x \cdot (v+w) = x \cdot v + x \cdot w$ and $(x + y) \cdot v = x \cdot v + y \cdot v$ and $(x \cdot y) \cdot v = x \cdot (y \cdot v)$ and $(1_R) \cdot v = v$,
 (ii) V is a left module over R.

Let us consider R. Then LeftMod(R) is a left module over R.

For simplicity we adopt the following rules: R is an associative ring, x, y are scalars of R, L_2 is a left module over R, and v, w are vectors of L_2. We now state several propositions:

(73) $x \cdot (v + w) = x \cdot v + x \cdot w$.

(74) $(x + y) \cdot v = x \cdot v + y \cdot v$.

(75) $(x \cdot y) \cdot v = x \cdot (y \cdot v)$.

(76) $(1_R) \cdot v = v$.

(77) Let x, y be scalars of R. Let v, w be vectors of RightMod(R). Then $(v+w) \cdot x = v \cdot x + w \cdot x$ and $v \cdot (x+y) = v \cdot x + v \cdot y$ and $v \cdot (y \cdot x) = (v \cdot y) \cdot x$ and $v \cdot (1_R) = v$.

Let us consider R. A right module structure over R is said to be a right module over R if:

(Def.20) Let x, y be scalars of R. Let v, w be vectors of it . Then $(v + w) \cdot x = v \cdot x + w \cdot x$ and $v \cdot (x + y) = v \cdot x + v \cdot y$ and $v \cdot (y \cdot x) = (v \cdot y) \cdot x$ and $v \cdot (1_R) = v$.

The following proposition is true

(78) Let V be a right module structure over R. Then the following conditions are equivalent:
 (i) for all scalars x, y of R and for all vectors v, w of V holds $(v+w) \cdot x = v \cdot x + w \cdot x$ and $v \cdot (x + y) = v \cdot x + v \cdot y$ and $v \cdot (y \cdot x) = (v \cdot y) \cdot x$ and $v \cdot (1_R) = v$,
 (ii) V is a right module over R.

Let us consider R. Then RightMod(R) is a right module over R.

For simplicity we follow the rules: R is an associative ring, x, y are scalars of R, R_2 is a right module over R, and v, w are vectors of R_2. We now state four propositions:

(79) $(v + w) \cdot x = v \cdot x + w \cdot x$.

(80) $v \cdot (x + y) = v \cdot x + v \cdot y$.

(81) $v \cdot (y \cdot x) = (v \cdot y) \cdot x$.

(82) $v \cdot (1_R) = v$.

Let us consider R. A bimodule structure over R is said to be a bimodule over R if:

(Def.21) Let x, y be scalars of R. Let v, w be vectors of it . Then

(i) $x \cdot (v + w) = x \cdot v + x \cdot w$,

(ii) $(x + y) \cdot v = x \cdot v + y \cdot v$,

(iii) $(x \cdot y) \cdot v = x \cdot (y \cdot v)$,

(iv) $(1_R) \cdot v = v$,

(v) $(v + w) \cdot x = v \cdot x + w \cdot x$,

(vi) $v \cdot (x + y) = v \cdot x + v \cdot y$,

(vii) $v \cdot (y \cdot x) = (v \cdot y) \cdot x$,

(viii) $v \cdot (1_R) = v$,

(ix) $x \cdot (v \cdot y) = (x \cdot v) \cdot y$.

Next we state two propositions:

(83) Let V be a bimodule structure over R. Then the following conditions are equivalent:

(i) for all scalars x, y of R and for all vectors v, w of V holds $x \cdot (v + w) = x \cdot v + x \cdot w$ and $(x + y) \cdot v = x \cdot v + y \cdot v$ and $(x \cdot y) \cdot v = x \cdot (y \cdot v)$ and $(1_R) \cdot v = v$ and $(v + w) \cdot x = v \cdot x + w \cdot x$ and $v \cdot (x + y) = v \cdot x + v \cdot y$ and $v \cdot (y \cdot x) = (v \cdot y) \cdot x$ and $v \cdot (1_R) = v$ and $x \cdot (v \cdot y) = (x \cdot v) \cdot y$,

(ii) V is a bimodule over R.

(84) BiMod(R) is a bimodule over R.

Let us consider R. Then BiMod(R) is a bimodule over R.

For simplicity we follow the rules: R will be an associative ring, x, y will be scalars of R, R_2 will be a bimodule over R, and v, w will be vectors of R_2. The following propositions are true:

(85) $x \cdot (v + w) = x \cdot v + x \cdot w$.

(86) $(x + y) \cdot v = x \cdot v + y \cdot v$.

(87) $(x \cdot y) \cdot v = x \cdot (y \cdot v)$.

(88) $(1_R) \cdot v = v$.

(89) $(v + w) \cdot x = v \cdot x + w \cdot x$.

(90) $v \cdot (x + y) = v \cdot x + v \cdot y$.

(91) $v \cdot (y \cdot x) = (v \cdot y) \cdot x$.

(92) $v \cdot (1_R) = v$.

(93) $x \cdot (v \cdot y) = (x \cdot v) \cdot y$.

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Received June 20, 1990

FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

Desargues Theorem In Projective 3-Space

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Summary. Proof of the Desargues theorem in Fanoian projective at least 3-dimensional space.

MML Identifier: `PROJDES1`.

The notation and terminology used in this paper are introduced in the following papers: [5], [1], [2], [3], and [4]. We follow a convention: F_1 will be an at least 3-dimensional projective space defined in terms of collinearity and a, a', b, b', c, c', d, d', o, p, q, r, s, t, u, x will be elements of the points of F_1. One can prove the following propositions:

(1) If a, b and c are collinear, then b, c and a are collinear and c, a and b are collinear and b, a and c are collinear and a, c and b are collinear and c, b and a are collinear.

(2) If $a \neq b$ and a, b and c are collinear and a, b and d are collinear, then a, c and d are collinear.

(3) If $p \neq q$ and a, b and p are collinear and a, b and q are collinear and p, q and r are collinear, then a, b and r are collinear.

(4) If $p \neq q$, then there exists r such that p, q and r are not collinear.

(5) There exist q, r such that p, q and r are not collinear.

(6) If a, b and c are not collinear and a, b and b' are collinear and $a \neq b'$, then a, b' and c are not collinear.

(7) If a, b and c are not collinear and a, b and d are collinear and a, c and d are collinear, then $a = d$.

(8) If o, a and d are not collinear and o, d and d' are collinear and a, d and s are collinear and $d \neq d'$ and a', d' and s are collinear and o, a and a' are collinear and $o \neq a'$, then $s \neq d$.

[1]Supported by RPBP.III-24.C6.

ISSN 0777–4028

Let us consider F_1, a, b, c, d. We say that a, b, c, d are coplanar if and only if:

(Def.1) there exists an element x of the points of F_1 such that a, b and x are collinear and c, d and x are collinear.

One can prove the following propositions:

(10)[2] If a, b and c are collinear or b, c and d are collinear or c, d and a are collinear or d, a and b are collinear, then a, b, c, d are coplanar.

(11) Suppose a, b, c, d are coplanar. Then b, c, d, a are coplanar and c, d, a, b are coplanar and d, a, b, c are coplanar and b, a, c, d are coplanar and c, b, d, a are coplanar and d, c, a, b are coplanar and a, d, b, c are coplanar and a, c, d, b are coplanar and b, d, a, c are coplanar and c, a, b, d are coplanar and d, b, c, a are coplanar and c, a, d, b are coplanar and d, b, a, c are coplanar and a, c, b, d are coplanar and b, d, c, a are coplanar and a, b, d, c are coplanar and a, d, c, b are coplanar and b, c, a, d are coplanar and b, a, d, c are coplanar and c, b, a, d are coplanar and c, d, b, a are coplanar and d, a, c, b are coplanar and d, c, b, a are coplanar.

(12) If a, b and c are not collinear and a, b, c, p are coplanar and a, b, c, q are coplanar and a, b, c, r are coplanar and a, b, c, s are coplanar, then p, q, r, s are coplanar.

(13) If p, q and r are not collinear and a, b, c, p are coplanar and a, b, c, r are coplanar and a, b, c, q are coplanar and p, q, r, s are coplanar, then a, b, c, s are coplanar.

(14) If $p \neq q$ and p, q and r are collinear and a, b, c, p are coplanar and a, b, c, q are coplanar, then a, b, c, r are coplanar.

(15) If a, b and c are not collinear and a, b, c, p are coplanar and a, b, c, q are coplanar and a, b, c, r are coplanar and a, b, c, s are coplanar, then there exists x such that p, q and x are collinear and r, s and x are collinear.

(16) There exist a, b, c, d such that a, b, c, d are not coplanar.

(17) If p, q and r are not collinear, then there exists s such that p, q, r, s are not coplanar.

(18) If $a = b$ or $a = c$ or $b = c$ or $a = d$ or $b = d$ or $d = c$, then a, b, c, d are coplanar.

(19) If a, b, c, o are not coplanar and o, a and a' are collinear and $a \neq a'$, then a, b, c, a' are not coplanar.

(20) Suppose that

(i) a, b and c are not collinear,

(ii) a', b' and c' are not collinear,

(iii) a, b, c, p are coplanar,

(iv) a, b, c, q are coplanar,

(v) a, b, c, r are coplanar,

[2] The proposition (9) was either repeated or obvious.

(vi) a', b', c', p are coplanar,
(vii) a', b', c', q are coplanar,
(viii) a', b', c', r are coplanar,
(ix) a, b, c, a' are not coplanar.
Then p, q and r are collinear.

(21) Suppose that
(i) $a \neq a'$,
(ii) o, a and a' are collinear,
(iii) a, b, c, o are not coplanar,
(iv) a', b' and c' are not collinear,
(v) a, b and p are collinear,
(vi) a', b' and p are collinear,
(vii) b, c and q are collinear,
(viii) b', c' and q are collinear,
(ix) a, c and r are collinear,
(x) a', c' and r are collinear.
Then p, q and r are collinear.

(22) If a, b, c, d are not coplanar and a, b, c, o are coplanar and a, b and o are not collinear, then a, b, d, o are not coplanar.

(23) If a, b, c, o are not coplanar and o, a and a' are collinear and o, b and b' are collinear and o, c and c' are collinear and $o \neq a'$ and $o \neq b'$ and $o \neq c'$, then a', b' and c' are not collinear and a', b', c', o are not coplanar.

(24) Suppose that
(i) a, b, c, o are coplanar,
(ii) a, b, c, d are not coplanar,
(iii) a, b, d, o are not coplanar,
(iv) b, c, d, o are not coplanar,
(v) a, c, d, o are not coplanar,
(vi) o, d and d' are collinear,
(vii) o, a and a' are collinear,
(viii) o, b and b' are collinear,
(ix) o, c and c' are collinear,
(x) a, d and s are collinear,
(xi) a', d' and s are collinear,
(xii) b, d and t are collinear,
(xiii) b', d' and t are collinear,
(xiv) c, d and u are collinear,
(xv) $o \neq a'$,
(xvi) $o \neq d'$,
(xvii) $d \neq d'$,
(xviii) $o \neq b'$.
Then s, t and u are not collinear.

Let us consider F_1, o, a, b, c. We say that o, a, b, and c constitute a quadrangle if and only if:

(Def.2) a, b and c are not collinear and o, a and b are not collinear and o, b and c are not collinear and o, c and a are not collinear.

The following propositions are true:

(26)[3] Suppose that
(i) o, a and b are not collinear,
(ii) o, b and c are not collinear,
(iii) o, a and c are not collinear,
(iv) o, a and a' are collinear,
(v) o, b and b' are collinear,
(vi) o, c and c' are collinear,
(vii) a, b and p are collinear,
(viii) a', b' and p are collinear,
(ix) $a \neq a'$,
(x) b, c and r are collinear,
(xi) b', c' and r are collinear,
(xii) a, c and q are collinear,
(xiii) $b \neq b'$,
(xiv) a', c' and q are collinear,
(xv) $o \neq a'$,
(xvi) $o \neq b'$,
(xvii) $o \neq c'$.

Then r, q and p are collinear.

(27) For every at least 3-dimensional projective space C_1 defined in terms of collinearity holds C_1 is a Desarguesian at least 3-dimensional projective space defined in terms of collinearity.

We see that the at least 3-dimensional projective space defined in terms of collinearity is a Desarguesian at least 3-dimensional projective space defined in terms of collinearity.

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Received August 13, 1990

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FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

The Limit of a Real Function at Infinity

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Summary. We introduce the halflines (*open* and *closed*), real sequences divergent to infinity (*plus* and *minus*) and the proper and improper limit of a real function at infinity. We prove basic properties of halflines, sequeces divergent to infinity and the limit of function at infinity.

MML Identifier: `LIMFUNC1`.

The articles [14], [4], [1], [2], [12], [10], [5], [6], [11], [15], [3], [7], [8], [13], and [9] provide the terminology and notation for this paper. For simplicity we follow a convention: r, r_1, r_2, g, g_1, g_2 are real numbers, X is a subset of $\mathbb{R}$, n, m, k are natural numbers, s_1, s_2, s_3 are sequences of real numbers, and f, f_1, f_2 are partial functions from $\mathbb{R}$ to $\mathbb{R}$. Let us consider n, m. Then $\max(n, m)$ is a natural number.

We now state four propositions:

(1) If $0 \leq r_1$ and $r_1 < r_2$ and $0 < g_1$ and $g_1 \leq g_2$, then $r_1 \cdot g_1 < r_2 \cdot g_2$.

(2) If $r \neq 0$, then $(-r)^{-1} = -r^{-1}$.

(3) If $r_1 < r_2$ and $r_2 < 0$ and $0 < g$, then $\frac{g}{r_2} < \frac{g}{r_1}$.

(4) If $r < 0$, then $r^{-1} < 0$.

Let us consider r. We introduce the functor $]-\infty, r[$ as a synonym of $\mathrm{HL}(r)$.

We now define three new functors. Let us consider r. The functor $]-\infty, r]$ yielding a subset of $\mathbb{R}$ is defined as follows:

(Def.1) $]-\infty, r] = \{g : g \leq r\}$.

The functor $[r, +\infty[$ yields a subset of $\mathbb{R}$ and is defined as follows:

(Def.2) $[r, +\infty[= \{g : r \leq g\}$.

The functor $]r, +\infty[$ yielding a subset of $\mathbb{R}$ is defined by:

(Def.3) $]r, +\infty[= \{g : r < g\}$.

[1]Supported by RPBP.III-24.C8

ISSN 0777–4028

One can prove the following propositions:

(5) $X =]-\infty, r]$ if and only if $X = \{g : g \leq r\}$.

(6) $X = [r, +\infty[$ if and only if $X = \{g : r \leq g\}$.

(7) $X =]r, +\infty[$ if and only if $X = \{g : r < g\}$.

(8) If $r_1 \leq r_2$, then $]r_2, +\infty[\subseteq]r_1, +\infty[$.

(9) If $r_1 \leq r_2$, then $[r_2, +\infty[\subseteq [r_1, +\infty[$.

(10) $]r, +\infty[\subseteq [r, +\infty[$.

(11) $]r, g[\subseteq]r, +\infty[$.

(12) $[r, g] \subseteq [r, +\infty[$.

(13) If $r_1 \leq r_2$, then $]-\infty, r_1[\subseteq]-\infty, r_2[$.

(14) If $r_1 \leq r_2$, then $]-\infty, r_1] \subseteq]-\infty, r_2]$.

(15) $]-\infty, r[\subseteq]-\infty, r]$.

(16) $]g, r[\subseteq]-\infty, r[$.

(17) $[g, r] \subseteq]-\infty, r]$.

(18) $]-\infty, r[\cap]g, +\infty[=]g, r[$.

(19) $]-\infty, r] \cap [g, +\infty[= [g, r]$.

(20) If $r \leq r_1$, then $]r_1, r_2[\subseteq]r, +\infty[$ and $[r_1, r_2] \subseteq [r, +\infty[$.

(21) If $r < r_1$, then $[r_1, r_2] \subseteq]r, +\infty[$.

(22) If $r_2 \leq r$, then $]r_1, r_2[\subseteq]-\infty, r[$ and $[r_1, r_2] \subseteq]-\infty, r]$.

(23) If $r_2 < r$, then $[r_1, r_2] \subseteq]-\infty, r[$.

(24) $\mathbb{R} \setminus]r, +\infty[=]-\infty, r]$ and $\mathbb{R} \setminus [r, +\infty[=]-\infty, r[$ and $\mathbb{R} \setminus]-\infty, r[= [r, +\infty[$ and $\mathbb{R} \setminus]-\infty, r] =]r, +\infty[$.

(25) $\mathbb{R} \setminus]r_1, r_2[=]-\infty, r_1] \cup [r_2, +\infty[$ and $\mathbb{R} \setminus [r_1, r_2] =]-\infty, r_1[\cup]r_2, +\infty[$.

(26) If s_1 is non-decreasing, then s_1 is lower bounded but if s_1 is non-increasing, then s_1 is upper bounded.

(27) If s_1 is non-zero and s_1 is convergent and $\lim s_1 = 0$ and s_1 is non-decreasing, then for every n holds $s_1(n) < 0$.

(28) If s_1 is non-zero and s_1 is convergent and $\lim s_1 = 0$ and s_1 is non-increasing, then for every n holds $0 < s_1(n)$.

(29) If s_1 is convergent and $0 < \lim s_1$, then there exists n such that for every m such that $n \leq m$ holds $0 < s_1(m)$.

(30) If s_1 is convergent and $0 < \lim s_1$, then there exists n such that for every m such that $n \leq m$ holds $\frac{\lim s_1}{2} < s_1(m)$.

We now define two new predicates. Let us consider s_1. We say that s_1 is divergent to $+\infty$ if and only if:

(Def.4) for every r there exists n such that for every m such that $n \leq m$ holds $r < s_1(m)$.

We say that s_1 is divergent to $-\infty$ if and only if:

(Def.5) for every r there exists n such that for every m such that $n \leq m$ holds $s_1(m) < r$.

Next we state a number of propositions:

(33)[2] If s_1 is divergent to $+\infty$ or s_1 is divergent to $-\infty$, then there exists n such that for every m such that $n \leq m$ holds $s_1 \uparrow m$ is non-zero.

(34) If $s_1 \uparrow k$ is divergent to $+\infty$, then s_1 is divergent to $+\infty$ but if $s_1 \uparrow k$ is divergent to $-\infty$, then s_1 is divergent to $-\infty$.

(35) If s_2 is divergent to $+\infty$ and s_3 is divergent to $+\infty$, then $s_2 + s_3$ is divergent to $+\infty$.

(36) If s_2 is divergent to $+\infty$ and s_3 is lower bounded, then s_2+s_3 is divergent to $+\infty$.

(37) If s_2 is divergent to $+\infty$ and s_3 is divergent to $+\infty$, then s_2s_3 is divergent to $+\infty$.

(38) If s_2 is divergent to $-\infty$ and s_3 is divergent to $-\infty$, then $s_2 + s_3$ is divergent to $-\infty$.

(39) If s_2 is divergent to $-\infty$ and s_3 is upper bounded, then $s_2 + s_3$ is divergent to $-\infty$.

(40) If s_1 is divergent to $+\infty$ and $r > 0$, then rs_1 is divergent to $+\infty$ but if s_1 is divergent to $+\infty$ and $r < 0$, then rs_1 is divergent to $-\infty$ but if s_1 is divergent to $+\infty$ and $r = 0$, then $\operatorname{rng}(rs_1) = \{0\}$ and rs_1 is constant.

(41) If s_1 is divergent to $-\infty$ and $r > 0$, then rs_1 is divergent to $-\infty$ but if s_1 is divergent to $-\infty$ and $r < 0$, then rs_1 is divergent to $+\infty$ but if s_1 is divergent to $-\infty$ and $r = 0$, then $\operatorname{rng}(rs_1) = \{0\}$ and rs_1 is constant.

(42) If s_1 is divergent to $+\infty$, then $-s_1$ is divergent to $-\infty$ but if s_1 is divergent to $-\infty$, then $-s_1$ is divergent to $+\infty$.

(43) If s_1 is lower bounded and s_2 is divergent to $-\infty$, then s_1-s_2 is divergent to $+\infty$.

(44) If s_1 is upper bounded and s_2 is divergent to $+\infty$, then $s_1 - s_2$ is divergent to $-\infty$.

(45) If s_1 is divergent to $+\infty$ and s_2 is convergent, then $s_1 + s_2$ is divergent to $+\infty$.

(46) If s_1 is divergent to $-\infty$ and s_2 is convergent, then $s_1 + s_2$ is divergent to $-\infty$.

(47) If for every n holds $s_1(n) = n$, then s_1 is divergent to $+\infty$.

(48) If for every n holds $s_1(n) = -n$, then s_1 is divergent to $-\infty$.

(49) If s_2 is divergent to $+\infty$ and there exists r such that $r > 0$ and for every n holds $s_3(n) \geq r$, then s_2s_3 is divergent to $+\infty$.

(50) If s_2 is divergent to $-\infty$ and there exists r such that $0 < r$ and for every n holds $s_3(n) \geq r$, then s_2s_3 is divergent to $-\infty$.

(51) If s_2 is divergent to $-\infty$ and s_3 is divergent to $-\infty$, then s_2s_3 is divergent to $+\infty$.

[2]The propositions (31)–(32) were either repeated or obvious.

(52) If s_1 is divergent to $+\infty$ or s_1 is divergent to $-\infty$, then $|s_1|$ is divergent to $+\infty$.

(53) If s_1 is divergent to $+\infty$ and s_2 is a subsequence of s_1, then s_2 is divergent to $+\infty$.

(54) If s_1 is divergent to $-\infty$ and s_2 is a subsequence of s_1, then s_2 is divergent to $-\infty$.

(55) If s_2 is divergent to $+\infty$ and s_3 is convergent and $0 < \lim s_3$, then $s_2 s_3$ is divergent to $+\infty$.

(56) If s_1 is non-decreasing and s_1 is not upper bounded, then s_1 is divergent to $+\infty$.

(57) If s_1 is non-increasing and s_1 is not lower bounded, then s_1 is divergent to $-\infty$.

(58) If s_1 is increasing and s_1 is not upper bounded, then s_1 is divergent to $+\infty$.

(59) If s_1 is decreasing and s_1 is not lower bounded, then s_1 is divergent to $-\infty$.

(60) If s_1 is monotone, then s_1 is convergent or s_1 is divergent to $+\infty$ or s_1 is divergent to $-\infty$.

(61) If s_1 is divergent to $+\infty$ or s_1 is divergent to $-\infty$ but s_1 is non-zero, then ${s_1}^{-1}$ is convergent and $\lim {s_1}^{-1} = 0$.

Next we state several propositions:

(62) If s_1 is non-zero and s_1 is convergent and $\lim s_1 = 0$ and there exists k such that for every n such that $k \leq n$ holds $0 < s_1(n)$, then ${s_1}^{-1}$ is divergent to $+\infty$.

(63) If s_1 is non-zero and s_1 is convergent and $\lim s_1 = 0$ and there exists k such that for every n such that $k \leq n$ holds $s_1(n) < 0$, then ${s_1}^{-1}$ is divergent to $-\infty$.

(64) If s_1 is non-zero and s_1 is convergent and $\lim s_1 = 0$ and s_1 is non-decreasing, then ${s_1}^{-1}$ is divergent to $-\infty$.

(65) If s_1 is non-zero and s_1 is convergent and $\lim s_1 = 0$ and s_1 is non-increasing, then ${s_1}^{-1}$ is divergent to $+\infty$.

(66) If s_1 is non-zero and s_1 is convergent and $\lim s_1 = 0$ and s_1 is increasing, then ${s_1}^{-1}$ is divergent to $-\infty$.

(67) If s_1 is non-zero and s_1 is convergent and $\lim s_1 = 0$ and s_1 is decreasing, then ${s_1}^{-1}$ is divergent to $+\infty$.

(68) If s_2 is bounded but s_3 is divergent to $+\infty$ or s_3 is divergent to $-\infty$ and s_3 is non-zero, then $\frac{s_2}{s_3}$ is convergent and $\lim \frac{s_2}{s_3} = 0$.

(69) If s_1 is divergent to $+\infty$ and for every n holds $s_1(n) \leq s_2(n)$, then s_2 is divergent to $+\infty$.

(70) If s_1 is divergent to $-\infty$ and for every n holds $s_2(n) \leq s_1(n)$, then s_2 is divergent to $-\infty$.

We now define several new predicates. Let us consider f. We say that f is convergent in $+\infty$ if and only if:

(Def.6) for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and there exists g such that for every s_1 such that s_1 is divergent to $+\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

We say that f is divergent in $+\infty$ to $+\infty$ if and only if:

(Def.7) for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is divergent to $+\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is divergent to $+\infty$.

We say that f is divergent in $+\infty$ to $-\infty$ if and only if:

(Def.8) for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is divergent to $+\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is divergent to $-\infty$.

We say that f is convergent in $-\infty$ if and only if:

(Def.9) for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and there exists g such that for every s_1 such that s_1 is divergent to $-\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

We say that f is divergent in $-\infty$ to $+\infty$ if and only if:

(Def.10) for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is divergent to $-\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is divergent to $+\infty$.

We say that f is divergent in $-\infty$ to $-\infty$ if and only if:

(Def.11) for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is divergent to $-\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is divergent to $-\infty$.

We now state a number of propositions:

(77)[3] f is convergent in $+\infty$ if and only if for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and there exists g such that for every g_1 such that $0 < g_1$ there exists r such that for every r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < g_1$.

(78) f is convergent in $-\infty$ if and only if for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and there exists g such that for every g_1 such that $0 < g_1$ there exists r such that for every r_1 such that $r_1 < r$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < g_1$.

(79) f is divergent in $+\infty$ to $+\infty$ if and only if for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and for every g there exists r such that for every r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} f$ holds $g < f(r_1)$.

(80) f is divergent in $+\infty$ to $-\infty$ if and only if for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and for every g there exists r such that for every r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} f$ holds $f(r_1) < g$.

[3]The propositions (71)–(76) were either repeated or obvious.

(81) f is divergent in $-\infty$ to $+\infty$ if and only if for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and for every g there exists r such that for every r_1 such that $r_1 < r$ and $r_1 \in \operatorname{dom} f$ holds $g < f(r_1)$.

(82) f is divergent in $-\infty$ to $-\infty$ if and only if for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and for every g there exists r such that for every r_1 such that $r_1 < r$ and $r_1 \in \operatorname{dom} f$ holds $f(r_1) < g$.

(83) If f_1 is divergent in $+\infty$ to $+\infty$ and f_2 is divergent in $+\infty$ to $+\infty$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, then $f_1 + f_2$ is divergent in $+\infty$ to $+\infty$ and $f_1 f_2$ is divergent in $+\infty$ to $+\infty$.

(84) If f_1 is divergent in $+\infty$ to $-\infty$ and f_2 is divergent in $+\infty$ to $-\infty$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, then $f_1 + f_2$ is divergent in $+\infty$ to $-\infty$ and $f_1 f_2$ is divergent in $+\infty$ to $+\infty$.

(85) If f_1 is divergent in $-\infty$ to $+\infty$ and f_2 is divergent in $-\infty$ to $+\infty$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, then $f_1 + f_2$ is divergent in $-\infty$ to $+\infty$ and $f_1 f_2$ is divergent in $-\infty$ to $+\infty$.

(86) If f_1 is divergent in $-\infty$ to $-\infty$ and f_2 is divergent in $-\infty$ to $-\infty$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, then $f_1 + f_2$ is divergent in $-\infty$ to $-\infty$ and $f_1 f_2$ is divergent in $-\infty$ to $+\infty$.

(87) If f_1 is divergent in $+\infty$ to $+\infty$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom}(f_1 + f_2)$ and there exists r such that f_2 is lower bounded on $]r, +\infty[$, then $f_1 + f_2$ is divergent in $+\infty$ to $+\infty$.

(88) If f_1 is divergent in $+\infty$ to $+\infty$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom}(f_1 f_2)$ and there exist r, r_1 such that $0 < r$ and for every g such that $g \in \operatorname{dom} f_2 \cap]r_1, +\infty[$ holds $r \leq f_2(g)$, then $f_1 f_2$ is divergent in $+\infty$ to $+\infty$.

(89) If f_1 is divergent in $-\infty$ to $+\infty$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom}(f_1 + f_2)$ and there exists r such that f_2 is lower bounded on $]-\infty, r[$, then $f_1 + f_2$ is divergent in $-\infty$ to $+\infty$.

(90) If f_1 is divergent in $-\infty$ to $+\infty$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom}(f_1 f_2)$ and there exist r, r_1 such that $0 < r$ and for every g such that $g \in \operatorname{dom} f_2 \cap]-\infty, r_1[$ holds $r \leq f_2(g)$, then $f_1 f_2$ is divergent in $-\infty$ to $+\infty$.

(91) If f is divergent in $+\infty$ to $+\infty$ and $r > 0$, then rf is divergent in $+\infty$ to $+\infty$ but if f is divergent in $+\infty$ to $+\infty$ and $r < 0$, then rf is divergent in $+\infty$ to $-\infty$ but if f is divergent in $+\infty$ to $-\infty$ and $r > 0$, then rf is divergent in $+\infty$ to $-\infty$ but if f is divergent in $+\infty$ to $-\infty$ and $r < 0$, then rf is divergent in $+\infty$ to $+\infty$.

(92) If f is divergent in $-\infty$ to $+\infty$ and $r > 0$, then rf is divergent in $-\infty$ to $+\infty$ but if f is divergent in $-\infty$ to $+\infty$ and $r < 0$, then rf is divergent in $-\infty$ to $-\infty$ but if f is divergent in $-\infty$ to $-\infty$ and $r > 0$, then rf is divergent in $-\infty$ to $-\infty$ but if f is divergent in $-\infty$ to $-\infty$ and $r < 0$, then rf is divergent in $-\infty$ to $+\infty$.

(93) If f is divergent in $+\infty$ to $+\infty$ or f is divergent in $+\infty$ to $-\infty$, then

$|f|$ is divergent in $+\infty$ to $+\infty$.

(94) If f is divergent in $-\infty$ to $+\infty$ or f is divergent in $-\infty$ to $-\infty$, then $|f|$ is divergent in $-\infty$ to $+\infty$.

(95) If there exists r such that f is non-decreasing on $]r, +\infty[$ and f is not upper bounded on $]r, +\infty[$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$, then f is divergent in $+\infty$ to $+\infty$.

(96) If there exists r such that f is increasing on $]r, +\infty[$ and f is not upper bounded on $]r, +\infty[$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$, then f is divergent in $+\infty$ to $+\infty$.

(97) If there exists r such that f is non-increasing on $]r, +\infty[$ and f is not lower bounded on $]r, +\infty[$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$, then f is divergent in $+\infty$ to $-\infty$.

(98) If there exists r such that f is decreasing on $]r, +\infty[$ and f is not lower bounded on $]r, +\infty[$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$, then f is divergent in $+\infty$ to $-\infty$.

(99) If there exists r such that f is non-increasing on $]-\infty, r[$ and f is not upper bounded on $]-\infty, r[$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$, then f is divergent in $-\infty$ to $+\infty$.

(100) If there exists r such that f is decreasing on $]-\infty, r[$ and f is not upper bounded on $]-\infty, r[$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$, then f is divergent in $-\infty$ to $+\infty$.

(101) If there exists r such that f is non-decreasing on $]-\infty, r[$ and f is not lower bounded on $]-\infty, r[$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$, then f is divergent in $-\infty$ to $-\infty$.

The following propositions are true:

(102) If there exists r such that f is increasing on $]-\infty, r[$ and f is not lower bounded on $]-\infty, r[$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$, then f is divergent in $-\infty$ to $-\infty$.

(103) Suppose f_1 is divergent in $+\infty$ to $+\infty$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and there exists r such that $\operatorname{dom} f \cap]r, +\infty[\subseteq \operatorname{dom} f_1 \cap]r, +\infty[$ and for every g such that $g \in \operatorname{dom} f \cap]r, +\infty[$ holds $f_1(g) \leq f(g)$. Then f is divergent in $+\infty$ to $+\infty$.

(104) Suppose f_1 is divergent in $+\infty$ to $-\infty$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and there exists r such that $\operatorname{dom} f \cap]r, +\infty[\subseteq \operatorname{dom} f_1 \cap]r, +\infty[$ and for every g such that $g \in \operatorname{dom} f \cap]r, +\infty[$ holds $f(g) \leq f_1(g)$. Then f is divergent in $+\infty$ to $-\infty$.

(105) Suppose f_1 is divergent in $-\infty$ to $+\infty$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and there exists r such that $\operatorname{dom} f \cap]-\infty, r[\subseteq \operatorname{dom} f_1 \cap]-\infty, r[$ and for every g such that $g \in \operatorname{dom} f \cap]-\infty, r[$ holds $f_1(g) \leq f(g)$. Then f is divergent in $-\infty$ to $+\infty$.

(106) Suppose f_1 is divergent in $-\infty$ to $-\infty$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and there exists r such that $\operatorname{dom} f \cap]-\infty, r[\subseteq$

$\mathrm{dom}\, f_1 \cap]-\infty, r[$ and for every g such that $g \in \mathrm{dom}\, f \cap]-\infty, r[$ holds $f(g) \leq f_1(g)$. Then f is divergent in $-\infty$ to $-\infty$.

(107) If f_1 is divergent in $+\infty$ to $+\infty$ and there exists r such that $]r, +\infty[\subseteq \mathrm{dom}\, f \cap \mathrm{dom}\, f_1$ and for every g such that $g \in]r, +\infty[$ holds $f_1(g) \leq f(g)$, then f is divergent in $+\infty$ to $+\infty$.

(108) If f_1 is divergent in $+\infty$ to $-\infty$ and there exists r such that $]r, +\infty[\subseteq \mathrm{dom}\, f \cap \mathrm{dom}\, f_1$ and for every g such that $g \in]r, +\infty[$ holds $f(g) \leq f_1(g)$, then f is divergent in $+\infty$ to $-\infty$.

(109) If f_1 is divergent in $-\infty$ to $+\infty$ and there exists r such that $]-\infty, r[\subseteq \mathrm{dom}\, f \cap \mathrm{dom}\, f_1$ and for every g such that $g \in]-\infty, r[$ holds $f_1(g) \leq f(g)$, then f is divergent in $-\infty$ to $+\infty$.

(110) If f_1 is divergent in $-\infty$ to $-\infty$ and there exists r such that $]-\infty, r[\subseteq \mathrm{dom}\, f \cap \mathrm{dom}\, f_1$ and for every g such that $g \in]-\infty, r[$ holds $f(g) \leq f_1(g)$, then f is divergent in $-\infty$ to $-\infty$.

Let us consider f. Let us assume that f is convergent in $+\infty$. The functor $\lim_{+\infty} f$ yielding a real number is defined by:

(Def.12) for every s_1 such that s_1 is divergent to $+\infty$ and $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = \lim_{+\infty} f$.

Let us consider f. Let us assume that f is convergent in $-\infty$. The functor $\lim_{-\infty} f$ yields a real number and is defined by:

(Def.13) for every s_1 such that s_1 is divergent to $-\infty$ and $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = \lim_{-\infty} f$.

Next we state a number of propositions:

(111) If f is convergent in $+\infty$, then $\lim_{+\infty} f = g$ if and only if for every s_1 such that s_1 is divergent to $+\infty$ and $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

(112) If f is convergent in $-\infty$, then $\lim_{-\infty} f = g$ if and only if for every s_1 such that s_1 is divergent to $-\infty$ and $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

(113) If f is convergent in $-\infty$, then $\lim_{-\infty} f = g$ if and only if for every g_1 such that $0 < g_1$ there exists r such that for every r_1 such that $r_1 < r$ and $r_1 \in \mathrm{dom}\, f$ holds $|f(r_1) - g| < g_1$.

(114) If f is convergent in $+\infty$, then $\lim_{+\infty} f = g$ if and only if for every g_1 such that $0 < g_1$ there exists r such that for every r_1 such that $r < r_1$ and $r_1 \in \mathrm{dom}\, f$ holds $|f(r_1) - g| < g_1$.

(115) If f is convergent in $+\infty$, then rf is convergent in $+\infty$ and $\lim_{+\infty}(rf) = r \cdot (\lim_{+\infty} f)$.

(116) If f is convergent in $+\infty$, then $-f$ is convergent in $+\infty$ and $\lim_{+\infty}(-f) = -\lim_{+\infty} f$.

(117) If f_1 is convergent in $+\infty$ and f_2 is convergent in $+\infty$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_1 + f_2)$, then $f_1 + f_2$ is convergent in $+\infty$ and $\lim_{+\infty}(f_1 + f_2) = \lim_{+\infty} f_1 + \lim_{+\infty} f_2$.

(118) If f_1 is convergent in $+\infty$ and f_2 is convergent in $+\infty$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_1 - f_2)$, then $f_1 - f_2$ is convergent in $+\infty$ and $\lim_{+\infty}(f_1 - f_2) = \lim_{+\infty} f_1 - \lim_{+\infty} f_2$.

(119) If f is convergent in $+\infty$ and $f^{-1}\{0\} = \emptyset$ and $\lim_{+\infty} f \neq 0$, then $\frac{1}{f}$ is convergent in $+\infty$ and $\lim_{+\infty} \frac{1}{f} = (\lim_{+\infty} f)^{-1}$.

(120) If f is convergent in $+\infty$, then $|f|$ is convergent in $+\infty$ and $\lim_{+\infty} |f| = |\lim_{+\infty} f|$.

(121) If f is convergent in $+\infty$ and $\lim_{+\infty} f \neq 0$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}\, f$ and $f(g) \neq 0$, then $\frac{1}{f}$ is convergent in $+\infty$ and $\lim_{+\infty} \frac{1}{f} = (\lim_{+\infty} f)^{-1}$.

(122) If f_1 is convergent in $+\infty$ and f_2 is convergent in $+\infty$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_1 f_2)$, then $f_1 f_2$ is convergent in $+\infty$ and $\lim_{+\infty}(f_1 f_2) = (\lim_{+\infty} f_1) \cdot (\lim_{+\infty} f_2)$.

(123) If f_1 is convergent in $+\infty$ and f_2 is convergent in $+\infty$ and $\lim_{+\infty} f_2 \neq 0$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}\, \frac{f_1}{f_2}$, then $\frac{f_1}{f_2}$ is convergent in $+\infty$ and $\lim_{+\infty} \frac{f_1}{f_2} = \frac{\lim_{+\infty} f_1}{\lim_{+\infty} f_2}$.

(124) If f is convergent in $-\infty$, then rf is convergent in $-\infty$ and $\lim_{-\infty}(rf) = r \cdot (\lim_{-\infty} f)$.

(125) If f is convergent in $-\infty$, then $-f$ is convergent in $-\infty$ and $\lim_{-\infty}(-f) = -\lim_{-\infty} f$.

(126) If f_1 is convergent in $-\infty$ and f_2 is convergent in $-\infty$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_1 + f_2)$, then $f_1 + f_2$ is convergent in $-\infty$ and $\lim_{-\infty}(f_1 + f_2) = \lim_{-\infty} f_1 + \lim_{-\infty} f_2$.

(127) If f_1 is convergent in $-\infty$ and f_2 is convergent in $-\infty$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_1 - f_2)$, then $f_1 - f_2$ is convergent in $-\infty$ and $\lim_{-\infty}(f_1 - f_2) = \lim_{-\infty} f_1 - \lim_{-\infty} f_2$.

(128) If f is convergent in $-\infty$ and $f^{-1}\{0\} = \emptyset$ and $\lim_{-\infty} f \neq 0$, then $\frac{1}{f}$ is convergent in $-\infty$ and $\lim_{-\infty} \frac{1}{f} = (\lim_{-\infty} f)^{-1}$.

(129) If f is convergent in $-\infty$, then $|f|$ is convergent in $-\infty$ and $\lim_{-\infty} |f| = |\lim_{-\infty} f|$.

(130) If f is convergent in $-\infty$ and $\lim_{-\infty} f \neq 0$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}\, f$ and $f(g) \neq 0$, then $\frac{1}{f}$ is convergent in $-\infty$ and $\lim_{-\infty} \frac{1}{f} = (\lim_{-\infty} f)^{-1}$.

(131) If f_1 is convergent in $-\infty$ and f_2 is convergent in $-\infty$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_1 f_2)$, then $f_1 f_2$ is convergent in $-\infty$ and $\lim_{-\infty}(f_1 f_2) = (\lim_{-\infty} f_1) \cdot (\lim_{-\infty} f_2)$.

(132) If f_1 is convergent in $-\infty$ and f_2 is convergent in $-\infty$ and $\lim_{-\infty} f_2 \neq 0$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}\, \frac{f_1}{f_2}$, then $\frac{f_1}{f_2}$ is convergent in $-\infty$ and $\lim_{-\infty} \frac{f_1}{f_2} = \frac{\lim_{-\infty} f_1}{\lim_{-\infty} f_2}$.

(133) If f_1 is convergent in $+\infty$ and $\lim_{+\infty} f_1 = 0$ and for every r there exists

g such that $r < g$ and $g \in \mathrm{dom}(f_1 f_2)$ and there exists r such that f_2 is bounded on $]r, +\infty[$, then $f_1 f_2$ is convergent in $+\infty$ and $\lim_{+\infty}(f_1 f_2) = 0$.

(134) If f_1 is convergent in $-\infty$ and $\lim_{-\infty} f_1 = 0$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_1 f_2)$ and there exists r such that f_2 is bounded on $]-\infty, r[$, then $f_1 f_2$ is convergent in $-\infty$ and $\lim_{-\infty}(f_1 f_2) = 0$.

(135) Suppose that
(i) f_1 is convergent in $+\infty$,
(ii) f_2 is convergent in $+\infty$,
(iii) $\lim_{+\infty} f_1 = \lim_{+\infty} f_2$,
(iv) for every r there exists g such that $r < g$ and $g \in \mathrm{dom}\, f$,
(v) there exists r such that $\mathrm{dom}\, f_1 \cap\,]r, +\infty[\ \subseteq \mathrm{dom}\, f_2 \cap\,]r, +\infty[$ and $\mathrm{dom}\, f \cap]r, +\infty[\ \subseteq \mathrm{dom}\, f_1 \cap]r, +\infty[$ or $\mathrm{dom}\, f_2 \cap]r, +\infty[\ \subseteq \mathrm{dom}\, f_1 \cap]r, +\infty[$ and $\mathrm{dom}\, f \cap\,]r, +\infty[\ \subseteq \mathrm{dom}\, f_2 \cap\,]r, +\infty[$ but for every g such that $g \in \mathrm{dom}\, f \cap\,]r, +\infty[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$.
Then f is convergent in $+\infty$ and $\lim_{+\infty} f = \lim_{+\infty} f_1$.

(136) Suppose f_1 is convergent in $+\infty$ and f_2 is convergent in $+\infty$ and $\lim_{+\infty} f_1 = \lim_{+\infty} f_2$ and there exists r such that $]r, +\infty[\ \subseteq (\mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2) \cap \mathrm{dom}\, f$ and for every g such that $g \in\,]r, +\infty[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$. Then f is convergent in $+\infty$ and $\lim_{+\infty} f = \lim_{+\infty} f_1$.

(137) Suppose that
(i) f_1 is convergent in $-\infty$,
(ii) f_2 is convergent in $-\infty$,
(iii) $\lim_{-\infty} f_1 = \lim_{-\infty} f_2$,
(iv) for every r there exists g such that $g < r$ and $g \in \mathrm{dom}\, f$,
(v) there exists r such that $\mathrm{dom}\, f_1 \cap\,]-\infty, r[\ \subseteq \mathrm{dom}\, f_2 \cap\,]-\infty, r[$ and $\mathrm{dom}\, f \cap]-\infty, r[\ \subseteq \mathrm{dom}\, f_1 \cap]-\infty, r[$ or $\mathrm{dom}\, f_2 \cap]-\infty, r[\ \subseteq \mathrm{dom}\, f_1 \cap]-\infty, r[$ and $\mathrm{dom}\, f \cap\,]-\infty, r[\ \subseteq \mathrm{dom}\, f_2 \cap\,]-\infty, r[$ but for every g such that $g \in \mathrm{dom}\, f \cap\,]-\infty, r[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$.
Then f is convergent in $-\infty$ and $\lim_{-\infty} f = \lim_{-\infty} f_1$.

(138) Suppose f_1 is convergent in $-\infty$ and f_2 is convergent in $-\infty$ and $\lim_{-\infty} f_1 = \lim_{-\infty} f_2$ and there exists r such that $]-\infty, r[\ \subseteq (\mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2) \cap \mathrm{dom}\, f$ and for every g such that $g \in\,]-\infty, r[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$. Then f is convergent in $-\infty$ and $\lim_{-\infty} f = \lim_{-\infty} f_1$.

(139) Suppose that
(i) f_1 is convergent in $+\infty$,
(ii) f_2 is convergent in $+\infty$,
(iii) there exists r such that $\mathrm{dom}\, f_1 \cap\,]r, +\infty[\ \subseteq \mathrm{dom}\, f_2 \cap\,]r, +\infty[$ and for every g such that $g \in \mathrm{dom}\, f_1 \cap\,]r, +\infty[$ holds $f_1(g) \leq f_2(g)$ or $\mathrm{dom}\, f_2 \cap\,]r, +\infty[\ \subseteq \mathrm{dom}\, f_1 \cap]r, +\infty[$ and for every g such that $g \in \mathrm{dom}\, f_2 \cap]r, +\infty[$ holds $f_1(g) \leq f_2(g)$.
Then $\lim_{+\infty} f_1 \leq \lim_{+\infty} f_2$.

(140) Suppose that
(i) f_1 is convergent in $-\infty$,
(ii) f_2 is convergent in $-\infty$,

(iii) there exists r such that $\mathrm{dom}\, f_1 \cap]-\infty, r[\subseteq \mathrm{dom}\, f_2 \cap]-\infty, r[$ and for every g such that $g \in \mathrm{dom}\, f_1 \cap]-\infty, r[$ holds $f_1(g) \leq f_2(g)$ or $\mathrm{dom}\, f_2 \cap]-\infty, r[\subseteq \mathrm{dom}\, f_1 \cap]-\infty, r[$ and for every g such that $g \in \mathrm{dom}\, f_2 \cap]-\infty, r[$ holds $f_1(g) \leq f_2(g)$.

Then $\lim_{-\infty} f_1 \leq \lim_{-\infty} f_2$.

(141) If f is divergent in $+\infty$ to $+\infty$ or f is divergent in $+\infty$ to $-\infty$ but for every r there exists g such that $r < g$ and $g \in \mathrm{dom}\, f$ and $f(g) \neq 0$, then $\frac{1}{f}$ is convergent in $+\infty$ and $\lim_{+\infty} \frac{1}{f} = 0$.

We now state several propositions:

(142) If f is divergent in $-\infty$ to $+\infty$ or f is divergent in $-\infty$ to $-\infty$ but for every r there exists g such that $g < r$ and $g \in \mathrm{dom}\, f$ and $f(g) \neq 0$, then $\frac{1}{f}$ is convergent in $-\infty$ and $\lim_{-\infty} \frac{1}{f} = 0$.

(143) If f is convergent in $+\infty$ and $\lim_{+\infty} f = 0$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}\, f$ and $f(g) \neq 0$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f \cap]r, +\infty[$ holds $0 \leq f(g)$, then $\frac{1}{f}$ is divergent in $+\infty$ to $+\infty$.

(144) If f is convergent in $+\infty$ and $\lim_{+\infty} f = 0$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}\, f$ and $f(g) \neq 0$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f \cap]r, +\infty[$ holds $f(g) \leq 0$, then $\frac{1}{f}$ is divergent in $+\infty$ to $-\infty$.

(145) If f is convergent in $-\infty$ and $\lim_{-\infty} f = 0$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}\, f$ and $f(g) \neq 0$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f \cap]-\infty, r[$ holds $0 \leq f(g)$, then $\frac{1}{f}$ is divergent in $-\infty$ to $+\infty$.

(146) If f is convergent in $-\infty$ and $\lim_{-\infty} f = 0$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}\, f$ and $f(g) \neq 0$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f \cap]-\infty, r[$ holds $f(g) \leq 0$, then $\frac{1}{f}$ is divergent in $-\infty$ to $-\infty$.

(147) If f is convergent in $+\infty$ and $\lim_{+\infty} f = 0$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f \cap]r, +\infty[$ holds $0 < f(g)$, then $\frac{1}{f}$ is divergent in $+\infty$ to $+\infty$.

(148) If f is convergent in $+\infty$ and $\lim_{+\infty} f = 0$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f \cap]r, +\infty[$ holds $f(g) < 0$, then $\frac{1}{f}$ is divergent in $+\infty$ to $-\infty$.

(149) If f is convergent in $-\infty$ and $\lim_{-\infty} f = 0$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f \cap]-\infty, r[$ holds $0 < f(g)$, then $\frac{1}{f}$ is divergent in $-\infty$ to $+\infty$.

(150) If f is convergent in $-\infty$ and $\lim_{-\infty} f = 0$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f \cap]-\infty, r[$ holds $f(g) < 0$, then $\frac{1}{f}$ is divergent in $-\infty$ to $-\infty$.

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Received August 20, 1990

FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

One-Side Limits of a Real Function at a Point

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Summary. We introduce the left-side and the right-side limit of a real function at a point. We prove a few properties of the operations on the proper and improper one-side limits and show that Cauchy and Heine characterizations of one-side limit are equivalent.

MML Identifier: `LIMFUNC2`.

The articles [15], [4], [1], [2], [13], [11], [5], [7], [12], [14], [3], [8], [9], [10], and [6] provide the terminology and notation for this paper. For simplicity we adopt the following convention: r, r_1, r_2, g, g_1, x_0 will be real numbers, n, k will be natural numbers, s_1 will be a sequence of real numbers, and f, f_1, f_2 will be partial functions from $\mathbb{R}$ to $\mathbb{R}$. We now state several propositions:

(1) If s_1 is convergent and $r < \lim s_1$, then there exists n such that for every k such that $n \leq k$ holds $r < s_1(k)$.

(2) If s_1 is convergent and $\lim s_1 < r$, then there exists n such that for every k such that $n \leq k$ holds $s_1(k) < r$.

(3) If $0 < r_2$ and $]r_1 - r_2, r_1[\subseteq \operatorname{dom} f$, then for every r such that $r < r_1$ there exists g such that $r < g$ and $g < r_1$ and $g \in \operatorname{dom} f$.

(4) If $0 < r_2$ and $]r_1, r_1 + r_2[\subseteq \operatorname{dom} f$, then for every r such that $r_1 < r$ there exists g such that $g < r$ and $r_1 < g$ and $g \in \operatorname{dom} f$.

(5) If for every n holds $x_0 - \frac{1}{n+1} < s_1(n)$ and $s_1(n) < x_0$ and $s_1(n) \in \operatorname{dom} f$, then s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]-\infty, x_0[$.

(6) If for every n holds $x_0 < s_1(n)$ and $s_1(n) < x_0 + \frac{1}{n+1}$ and $s_1(n) \in \operatorname{dom} f$, then s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]x_0, +\infty[$.

[1]Supported by RPBP.III-24.C8

ISSN 0777–4028

We now define several new predicates. Let us consider f, x_0. We say that f is left convergent in x_0 if and only if:

(Def.1) (i) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$,

(ii) there exists g such that for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]-\infty, x_0[$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

We say that f is left divergent to $+\infty$ in x_0 if and only if:

(Def.2) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]-\infty, x_0[$ holds $f \cdot s_1$ is divergent to $+\infty$.

We say that f is left divergent to $-\infty$ in x_0 if and only if:

(Def.3) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]-\infty, x_0[$ holds $f \cdot s_1$ is divergent to $-\infty$.

We say that f is right convergent in x_0 if and only if:

(Def.4) (i) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$,

(ii) there exists g such that for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]x_0, +\infty[$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

We say that f is right divergent to $+\infty$ in x_0 if and only if:

(Def.5) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]x_0, +\infty[$ holds $f \cdot s_1$ is divergent to $+\infty$.

We say that f is right divergent to $-\infty$ in x_0 if and only if:

(Def.6) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]x_0, +\infty[$ holds $f \cdot s_1$ is divergent to $-\infty$.

We now state a number of propositions:

(7) f is left convergent in x_0 if and only if the following conditions are satisfied:

(i) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$,

(ii) there exists g such that for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]-\infty, x_0[$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

(8) f is left divergent to $+\infty$ in x_0 if and only if for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]-\infty, x_0[$ holds $f \cdot s_1$ is divergent to $+\infty$.

(9) f is left divergent to $-\infty$ in x_0 if and only if for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]-\infty, x_0[$ holds $f \cdot s_1$ is divergent to $-\infty$.

(10) f is right convergent in x_0 if and only if the following conditions are satisfied:

(i) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$,

(ii) there exists g such that for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]x_0, +\infty[$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

(11) f is right divergent to $+\infty$ in x_0 if and only if for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]x_0, +\infty[$ holds $f \cdot s_1$ is divergent to $+\infty$.

(12) f is right divergent to $-\infty$ in x_0 if and only if for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]x_0, +\infty[$ holds $f \cdot s_1$ is divergent to $-\infty$.

(13) f is left convergent in x_0 if and only if the following conditions are satisfied:

(i) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$,

(ii) there exists g such that for every g_1 such that $0 < g_1$ there exists r such that $r < x_0$ and for every r_1 such that $r < r_1$ and $r_1 < x_0$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < g_1$.

(14) f is left divergent to $+\infty$ in x_0 if and only if the following conditions are satisfied:

(i) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$,

(ii) for every g_1 there exists r such that $r < x_0$ and for every r_1 such that $r < r_1$ and $r_1 < x_0$ and $r_1 \in \operatorname{dom} f$ holds $g_1 < f(r_1)$.

(15) f is left divergent to $-\infty$ in x_0 if and only if the following conditions are satisfied:

(i) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$,

(ii) for every g_1 there exists r such that $r < x_0$ and for every r_1 such that $r < r_1$ and $r_1 < x_0$ and $r_1 \in \operatorname{dom} f$ holds $f(r_1) < g_1$.

(16) f is right convergent in x_0 if and only if the following conditions are satisfied:

(i) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$,

(ii) there exists g such that for every g_1 such that $0 < g_1$ there exists r such that $x_0 < r$ and for every r_1 such that $r_1 < r$ and $x_0 < r_1$ and $r_1 \in \operatorname{dom} f$

holds $|f(r_1) - g| < g_1$.

(17) f is right divergent to $+\infty$ in x_0 if and only if the following conditions are satisfied:
(i) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$,
(ii) for every g_1 there exists r such that $x_0 < r$ and for every r_1 such that $r_1 < r$ and $x_0 < r_1$ and $r_1 \in \operatorname{dom} f$ holds $g_1 < f(r_1)$.

(18) f is right divergent to $-\infty$ in x_0 if and only if the following conditions are satisfied:
(i) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$,
(ii) for every g_1 there exists r such that $x_0 < r$ and for every r_1 such that $r_1 < r$ and $x_0 < r_1$ and $r_1 \in \operatorname{dom} f$ holds $f(r_1) < g_1$.

(19) If f_1 is left divergent to $+\infty$ in x_0 and f_2 is left divergent to $+\infty$ in x_0 and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, then $f_1 + f_2$ is left divergent to $+\infty$ in x_0 and $f_1 f_2$ is left divergent to $+\infty$ in x_0.

(20) If f_1 is left divergent to $-\infty$ in x_0 and f_2 is left divergent to $-\infty$ in x_0 and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, then $f_1 + f_2$ is left divergent to $-\infty$ in x_0 and $f_1 f_2$ is left divergent to $+\infty$ in x_0.

(21) If f_1 is right divergent to $+\infty$ in x_0 and f_2 is right divergent to $+\infty$ in x_0 and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, then $f_1 + f_2$ is right divergent to $+\infty$ in x_0 and $f_1 f_2$ is right divergent to $+\infty$ in x_0.

(22) If f_1 is right divergent to $-\infty$ in x_0 and f_2 is right divergent to $-\infty$ in x_0 and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, then $f_1 + f_2$ is right divergent to $-\infty$ in x_0 and $f_1 f_2$ is right divergent to $+\infty$ in x_0.

(23) If f_1 is left divergent to $+\infty$ in x_0 and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom}(f_1 + f_2)$ and there exists r such that $0 < r$ and f_2 is lower bounded on $]x_0 - r, x_0[$, then $f_1 + f_2$ is left divergent to $+\infty$ in x_0.

(24) Suppose that
(i) f_1 is left divergent to $+\infty$ in x_0,
(ii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom}(f_1 f_2)$,
(iii) there exist r, r_1 such that $0 < r$ and $0 < r_1$ and for every g such that $g \in \operatorname{dom} f_2 \cap]x_0 - r, x_0[$ holds $r_1 \leq f_2(g)$.
Then $f_1 f_2$ is left divergent to $+\infty$ in x_0.

(25) If f_1 is right divergent to $+\infty$ in x_0 and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom}(f_1 + f_2)$ and there exists r such that $0 < r$ and f_2 is lower bounded on $]x_0, x_0 + r[$, then

$f_1 + f_2$ is right divergent to $+\infty$ in x_0.

(26) Suppose that

(i) f_1 is right divergent to $+\infty$ in x_0,

(ii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_1 f_2)$,

(iii) there exist r, r_1 such that $0 < r$ and $0 < r_1$ and for every g such that $g \in \mathrm{dom}\, f_2 \cap]x_0, x_0 + r[$ holds $r_1 \leq f_2(g)$.

Then $f_1 f_2$ is right divergent to $+\infty$ in x_0.

(27) (i) If f is left divergent to $+\infty$ in x_0 and $r > 0$, then rf is left divergent to $+\infty$ in x_0,

(ii) if f is left divergent to $+\infty$ in x_0 and $r < 0$, then rf is left divergent to $-\infty$ in x_0,

(iii) if f is left divergent to $-\infty$ in x_0 and $r > 0$, then rf is left divergent to $-\infty$ in x_0,

(iv) if f is left divergent to $-\infty$ in x_0 and $r < 0$, then rf is left divergent to $+\infty$ in x_0.

(28) (i) If f is right divergent to $+\infty$ in x_0 and $r > 0$, then rf is right divergent to $+\infty$ in x_0,

(ii) if f is right divergent to $+\infty$ in x_0 and $r < 0$, then rf is right divergent to $-\infty$ in x_0,

(iii) if f is right divergent to $-\infty$ in x_0 and $r > 0$, then rf is right divergent to $-\infty$ in x_0,

(iv) if f is right divergent to $-\infty$ in x_0 and $r < 0$, then rf is right divergent to $+\infty$ in x_0.

(29) If f is left divergent to $+\infty$ in x_0 or f is left divergent to $-\infty$ in x_0, then $|f|$ is left divergent to $+\infty$ in x_0.

(30) If f is right divergent to $+\infty$ in x_0 or f is right divergent to $-\infty$ in x_0, then $|f|$ is right divergent to $+\infty$ in x_0.

(31) If there exists r such that $0 < r$ and f is non-decreasing on $]x_0 - r, x_0[$ and f is not upper bounded on $]x_0 - r, x_0[$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}\, f$, then f is left divergent to $+\infty$ in x_0.

(32) If there exists r such that $0 < r$ and f is increasing on $]x_0 - r, x_0[$ and f is not upper bounded on $]x_0 - r, x_0[$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}\, f$, then f is left divergent to $+\infty$ in x_0.

(33) If there exists r such that $0 < r$ and f is non-increasing on $]x_0 - r, x_0[$ and f is not lower bounded on $]x_0 - r, x_0[$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}\, f$, then f is left divergent to $-\infty$ in x_0.

(34) If there exists r such that $0 < r$ and f is decreasing on $]x_0 - r, x_0[$ and f is not lower bounded on $]x_0 - r, x_0[$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}\, f$, then f is left

divergent to $-\infty$ in x_0.

(35) If there exists r such that $0 < r$ and f is non-increasing on $]x_0, x_0 + r[$ and f is not upper bounded on $]x_0, x_0+r[$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$, then f is right divergent to $+\infty$ in x_0.

(36) If there exists r such that $0 < r$ and f is decreasing on $]x_0, x_0 + r[$ and f is not upper bounded on $]x_0, x_0 + r[$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$, then f is right divergent to $+\infty$ in x_0.

(37) If there exists r such that $0 < r$ and f is non-decreasing on $]x_0, x_0 + r[$ and f is not lower bounded on $]x_0, x_0+r[$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$, then f is right divergent to $-\infty$ in x_0.

Next we state several propositions:

(38) If there exists r such that $0 < r$ and f is increasing on $]x_0, x_0 + r[$ and f is not lower bounded on $]x_0, x_0 + r[$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$, then f is right divergent to $-\infty$ in x_0.

(39) Suppose that
(i) f_1 is left divergent to $+\infty$ in x_0,
(ii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$,
(iii) there exists r such that $0 < r$ and $\operatorname{dom} f \cap]x_0 - r, x_0[\subseteq \operatorname{dom} f_1 \cap]x_0 - r, x_0[$ and for every g such that $g \in \operatorname{dom} f \cap]x_0 - r, x_0[$ holds $f_1(g) \leq f(g)$.
Then f is left divergent to $+\infty$ in x_0.

(40) Suppose that
(i) f_1 is left divergent to $-\infty$ in x_0,
(ii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$,
(iii) there exists r such that $0 < r$ and $\operatorname{dom} f \cap]x_0 - r, x_0[\subseteq \operatorname{dom} f_1 \cap]x_0 - r, x_0[$ and for every g such that $g \in \operatorname{dom} f \cap]x_0 - r, x_0[$ holds $f(g) \leq f_1(g)$.
Then f is left divergent to $-\infty$ in x_0.

(41) Suppose that
(i) f_1 is right divergent to $+\infty$ in x_0,
(ii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$,
(iii) there exists r such that $0 < r$ and $\operatorname{dom} f \cap]x_0, x_0+r[\subseteq \operatorname{dom} f_1 \cap]x_0, x_0+r[$ and for every g such that $g \in \operatorname{dom} f \cap]x_0, x_0 + r[$ holds $f_1(g) \leq f(g)$.
Then f is right divergent to $+\infty$ in x_0.

(42) Suppose that
(i) f_1 is right divergent to $-\infty$ in x_0,
(ii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$,

(iii) there exists r such that $0 < r$ and $\mathrm{dom}\, f \cap]x_0, x_0+r[\subseteq \mathrm{dom}\, f_1 \cap]x_0, x_0+r[$ and for every g such that $g \in \mathrm{dom}\, f \cap]x_0, x_0+r[$ holds $f(g) \leq f_1(g)$.

Then f is right divergent to $-\infty$ in x_0.

(43) If f_1 is left divergent to $+\infty$ in x_0 and there exists r such that $0 < r$ and $]x_0-r, x_0[\subseteq \mathrm{dom}\, f \cap \mathrm{dom}\, f_1$ and for every g such that $g \in]x_0-r, x_0[$ holds $f_1(g) \leq f(g)$, then f is left divergent to $+\infty$ in x_0.

(44) If f_1 is left divergent to $-\infty$ in x_0 and there exists r such that $0 < r$ and $]x_0-r, x_0[\subseteq \mathrm{dom}\, f \cap \mathrm{dom}\, f_1$ and for every g such that $g \in]x_0-r, x_0[$ holds $f(g) \leq f_1(g)$, then f is left divergent to $-\infty$ in x_0.

(45) If f_1 is right divergent to $+\infty$ in x_0 and there exists r such that $0 < r$ and $]x_0, x_0+r[\subseteq \mathrm{dom}\, f \cap \mathrm{dom}\, f_1$ and for every g such that $g \in]x_0, x_0+r[$ holds $f_1(g) \leq f(g)$, then f is right divergent to $+\infty$ in x_0.

(46) If f_1 is right divergent to $-\infty$ in x_0 and there exists r such that $0 < r$ and $]x_0, x_0+r[\subseteq \mathrm{dom}\, f \cap \mathrm{dom}\, f_1$ and for every g such that $g \in]x_0, x_0+r[$ holds $f(g) \leq f_1(g)$, then f is right divergent to $-\infty$ in x_0.

Let us consider f, x_0. Let us assume that f is left convergent in x_0. The functor $\lim_{x_0^-} f$ yields a real number and is defined by:

(Def.7) for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f \cap]-\infty, x_0[$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = \lim_{x_0^-} f$.

Let us consider f, x_0. Let us assume that f is right convergent in x_0. The functor $\lim_{x_0^+} f$ yields a real number and is defined by:

(Def.8) for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f \cap]x_0, +\infty[$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = \lim_{x_0^+} f$.

One can prove the following propositions:

(47) If f is left convergent in x_0, then $\lim_{x_0^-} f = g$ if and only if for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f \cap]-\infty, x_0[$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

(48) If f is right convergent in x_0, then $\lim_{x_0^+} f = g$ if and only if for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f \cap]x_0, +\infty[$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

(49) If f is left convergent in x_0, then $\lim_{x_0^-} f = g$ if and only if for every g_1 such that $0 < g_1$ there exists r such that $r < x_0$ and for every r_1 such that $r < r_1$ and $r_1 < x_0$ and $r_1 \in \mathrm{dom}\, f$ holds $|f(r_1) - g| < g_1$.

(50) If f is right convergent in x_0, then $\lim_{x_0^+} f = g$ if and only if for every g_1 such that $0 < g_1$ there exists r such that $x_0 < r$ and for every r_1 such that $r_1 < r$ and $x_0 < r_1$ and $r_1 \in \mathrm{dom}\, f$ holds $|f(r_1) - g| < g_1$.

(51) If f is left convergent in x_0, then $r f$ is left convergent in x_0 and $\lim_{x_0^-}(r f) = r \cdot (\lim_{x_0^-} f)$.

(52) If f is left convergent in x_0, then $-f$ is left convergent in x_0 and $\lim_{x_0^-}(-f) = -\lim_{x_0^-} f$.

(53) Suppose f_1 is left convergent in x_0 and f_2 is left convergent in x_0 and for every r such that $r < x_0$ there exists g such that $r < g$ and $g <$

x_0 and $g \in \mathrm{dom}(f_1 + f_2)$. Then $f_1 + f_2$ is left convergent in x_0 and $\lim_{x_0^-}(f_1 + f_2) = \lim_{x_0^-} f_1 + \lim_{x_0^-} f_2$.

(54) Suppose f_1 is left convergent in x_0 and f_2 is left convergent in x_0 and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_1 - f_2)$. Then $f_1 - f_2$ is left convergent in x_0 and $\lim_{x_0^-}(f_1 - f_2) = \lim_{x_0^-} f_1 - \lim_{x_0^-} f_2$.

(55) If f is left convergent in x_0 and $f^{-1}\{0\} = \emptyset$ and $\lim_{x_0^-} f \neq 0$, then $\frac{1}{f}$ is left convergent in x_0 and $\lim_{x_0^-} \frac{1}{f} = (\lim_{x_0^-} f)^{-1}$.

(56) If f is left convergent in x_0, then $|f|$ is left convergent in x_0 and $\lim_{x_0^-} |f| = |\lim_{x_0^-} f|$.

(57) Suppose f is left convergent in x_0 and $\lim_{x_0^-} f \neq 0$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}\, f$ and $f(g) \neq 0$. Then $\frac{1}{f}$ is left convergent in x_0 and $\lim_{x_0^-} \frac{1}{f} = (\lim_{x_0^-} f)^{-1}$.

(58) Suppose f_1 is left convergent in x_0 and f_2 is left convergent in x_0 and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_1 f_2)$. Then $f_1 f_2$ is left convergent in x_0 and $\lim_{x_0^-}(f_1 f_2) = (\lim_{x_0^-} f_1) \cdot (\lim_{x_0^-} f_2)$.

(59) Suppose f_1 is left convergent in x_0 and f_2 is left convergent in x_0 and $\lim_{x_0^-} f_2 \neq 0$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}\, \frac{f_1}{f_2}$. Then $\frac{f_1}{f_2}$ is left convergent in x_0 and $\lim_{x_0^-} \frac{f_1}{f_2} = \frac{\lim_{x_0^-} f_1}{\lim_{x_0^-} f_2}$.

(60) If f is right convergent in x_0, then rf is right convergent in x_0 and $\lim_{x_0^+}(rf) = r \cdot (\lim_{x_0^+} f)$.

(61) If f is right convergent in x_0, then $-f$ is right convergent in x_0 and $\lim_{x_0^+}(-f) = -\lim_{x_0^+} f$.

(62) Suppose f_1 is right convergent in x_0 and f_2 is right convergent in x_0 and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_1 + f_2)$. Then $f_1 + f_2$ is right convergent in x_0 and $\lim_{x_0^+}(f_1 + f_2) = \lim_{x_0^+} f_1 + \lim_{x_0^+} f_2$.

(63) Suppose f_1 is right convergent in x_0 and f_2 is right convergent in x_0 and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_1 - f_2)$. Then $f_1 - f_2$ is right convergent in x_0 and $\lim_{x_0^+}(f_1 - f_2) = \lim_{x_0^+} f_1 - \lim_{x_0^+} f_2$.

(64) If f is right convergent in x_0 and $f^{-1}\{0\} = \emptyset$ and $\lim_{x_0^+} f \neq 0$, then $\frac{1}{f}$ is right convergent in x_0 and $\lim_{x_0^+} \frac{1}{f} = (\lim_{x_0^+} f)^{-1}$.

(65) If f is right convergent in x_0, then $|f|$ is right convergent in x_0 and $\lim_{x_0^+} |f| = |\lim_{x_0^+} f|$.

(66) Suppose f is right convergent in x_0 and $\lim_{x_0^+} f \neq 0$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}\, f$ and $f(g) \neq 0$. Then $\frac{1}{f}$ is right convergent in x_0 and $\lim_{x_0^+} \frac{1}{f} = (\lim_{x_0^+} f)^{-1}$.

(67) Suppose f_1 is right convergent in x_0 and f_2 is right convergent in x_0

and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_1 f_2)$. Then $f_1 f_2$ is right convergent in x_0 and $\lim_{x_0^+}(f_1 f_2) = (\lim_{x_0^+} f_1) \cdot (\lim_{x_0^+} f_2)$.

(68) Suppose f_1 is right convergent in x_0 and f_2 is right convergent in x_0 and $\lim_{x_0^+} f_2 \neq 0$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}\,\frac{f_1}{f_2}$. Then $\frac{f_1}{f_2}$ is right convergent in x_0 and $\lim_{x_0^+} \frac{f_1}{f_2} = \frac{\lim_{x_0^+} f_1}{\lim_{x_0^+} f_2}$.

(69) Suppose f_1 is left convergent in x_0 and $\lim_{x_0^-} f_1 = 0$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_1 f_2)$ and there exists r such that $0 < r$ and f_2 is bounded on $]x_0 - r, x_0[$. Then $f_1 f_2$ is left convergent in x_0 and $\lim_{x_0^-}(f_1 f_2) = 0$.

(70) Suppose f_1 is right convergent in x_0 and $\lim_{x_0^+} f_1 = 0$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_1 f_2)$ and there exists r such that $0 < r$ and f_2 is bounded on $]x_0, x_0 + r[$. Then $f_1 f_2$ is right convergent in x_0 and $\lim_{x_0^+}(f_1 f_2) = 0$.

(71) Suppose that

(i) f_1 is left convergent in x_0,

(ii) f_2 is left convergent in x_0,

(iii) $\lim_{x_0^-} f_1 = \lim_{x_0^-} f_2$,

(iv) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}\, f$,

(v) there exists r such that $0 < r$ and for every g such that $g \in \mathrm{dom}\, f \cap]x_0 - r, x_0[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$ but $\mathrm{dom}\, f_1 \cap]x_0 - r, x_0[\subseteq \mathrm{dom}\, f_2 \cap]x_0 - r, x_0[$ and $\mathrm{dom}\, f \cap]x_0 - r, x_0[\subseteq \mathrm{dom}\, f_1 \cap]x_0 - r, x_0[$ or $\mathrm{dom}\, f_2 \cap]x_0 - r, x_0[\subseteq \mathrm{dom}\, f_1 \cap]x_0 - r, x_0[$ and $\mathrm{dom}\, f \cap]x_0 - r, x_0[\subseteq \mathrm{dom}\, f_2 \cap]x_0 - r, x_0[$.

Then f is left convergent in x_0 and $\lim_{x_0^-} f = \lim_{x_0^-} f_1$.

(72) Suppose that

(i) f_1 is left convergent in x_0,

(ii) f_2 is left convergent in x_0,

(iii) $\lim_{x_0^-} f_1 = \lim_{x_0^-} f_2$,

(iv) there exists r such that $0 < r$ and $]x_0 - r, x_0[\subseteq (\mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2) \cap \mathrm{dom}\, f$ and for every g such that $g \in]x_0 - r, x_0[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$.

Then f is left convergent in x_0 and $\lim_{x_0^-} f = \lim_{x_0^-} f_1$.

(73) Suppose that

(i) f_1 is right convergent in x_0,

(ii) f_2 is right convergent in x_0,

(iii) $\lim_{x_0^+} f_1 = \lim_{x_0^+} f_2$,

(iv) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}\, f$,

(v) there exists r such that $0 < r$ and for every g such that $g \in \mathrm{dom}\, f \cap]x_0, x_0 + r[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$ but $\mathrm{dom}\, f_1 \cap]x_0, x_0 + r[\subseteq$

$\operatorname{dom} f_2 \cap]x_0, x_0 + r[$ and $\operatorname{dom} f \cap]x_0, x_0 + r[\subseteq \operatorname{dom} f_1 \cap]x_0, x_0 + r[$ or $\operatorname{dom} f_2 \cap]x_0, x_0 + r[\subseteq \operatorname{dom} f_1 \cap]x_0, x_0 + r[$ and $\operatorname{dom} f \cap]x_0, x_0 + r[\subseteq \operatorname{dom} f_2 \cap]x_0, x_0 + r[$.

Then f is right convergent in x_0 and $\lim_{x_0^+} f = \lim_{x_0^+} f_1$.

(74) Suppose that
 (i) f_1 is right convergent in x_0,
 (ii) f_2 is right convergent in x_0,
 (iii) $\lim_{x_0^+} f_1 = \lim_{x_0^+} f_2$,
 (iv) there exists r such that $0 < r$ and $]x_0, x_0 + r[\subseteq (\operatorname{dom} f_1 \cap \operatorname{dom} f_2) \cap \operatorname{dom} f$ and for every g such that $g \in]x_0, x_0 + r[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$.

Then f is right convergent in x_0 and $\lim_{x_0^+} f = \lim_{x_0^+} f_1$.

(75) Suppose that
 (i) f_1 is left convergent in x_0,
 (ii) f_2 is left convergent in x_0,
 (iii) there exists r such that $0 < r$ but $\operatorname{dom} f_1 \cap]x_0 - r, x_0[\subseteq \operatorname{dom} f_2 \cap]x_0 - r, x_0[$ and for every g such that $g \in \operatorname{dom} f_1 \cap]x_0 - r, x_0[$ holds $f_1(g) \leq f_2(g)$ or $\operatorname{dom} f_2 \cap]x_0 - r, x_0[\subseteq \operatorname{dom} f_1 \cap]x_0 - r, x_0[$ and for every g such that $g \in \operatorname{dom} f_2 \cap]x_0 - r, x_0[$ holds $f_1(g) \leq f_2(g)$.

Then $\lim_{x_0^-} f_1 \leq \lim_{x_0^-} f_2$.

(76) Suppose that
 (i) f_1 is right convergent in x_0,
 (ii) f_2 is right convergent in x_0,
 (iii) there exists r such that $0 < r$ but $\operatorname{dom} f_1 \cap]x_0, x_0 + r[\subseteq \operatorname{dom} f_2 \cap]x_0, x_0 + r[$ and for every g such that $g \in \operatorname{dom} f_1 \cap]x_0, x_0 + r[$ holds $f_1(g) \leq f_2(g)$ or $\operatorname{dom} f_2 \cap]x_0, x_0 + r[\subseteq \operatorname{dom} f_1 \cap]x_0, x_0 + r[$ and for every g such that $g \in \operatorname{dom} f_2 \cap]x_0, x_0 + r[$ holds $f_1(g) \leq f_2(g)$.

Then $\lim_{x_0^+} f_1 \leq \lim_{x_0^+} f_2$.

(77) If f is left divergent to $+\infty$ in x_0 or f is left divergent to $-\infty$ in x_0 but for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$, then $\frac{1}{f}$ is left convergent in x_0 and $\lim_{x_0^-} \frac{1}{f} = 0$.

One can prove the following propositions:

(78) If f is right divergent to $+\infty$ in x_0 or f is right divergent to $-\infty$ in x_0 but for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$, then $\frac{1}{f}$ is right convergent in x_0 and $\lim_{x_0^+} \frac{1}{f} = 0$.

(79) If f is left convergent in x_0 and $\lim_{x_0^-} f = 0$ and there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap]x_0 - r, x_0[$ holds $0 < f(g)$, then $\frac{1}{f}$ is left divergent to $+\infty$ in x_0.

(80) If f is left convergent in x_0 and $\lim_{x_0^-} f = 0$ and there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap]x_0 - r, x_0[$ holds $f(g) < 0$, then $\frac{1}{f}$ is left divergent to $-\infty$ in x_0.

(81) If f is right convergent in x_0 and $\lim_{x_0^+} f = 0$ and there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap]x_0, x_0+r[$ holds $0 < f(g)$, then $\frac{1}{f}$ is right divergent to $+\infty$ in x_0.

(82) If f is right convergent in x_0 and $\lim_{x_0^+} f = 0$ and there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap]x_0, x_0+r[$ holds $f(g) < 0$, then $\frac{1}{f}$ is right divergent to $-\infty$ in x_0.

(83) Suppose that
(i) f is left convergent in x_0,
(ii) $\lim_{x_0^-} f = 0$,
(iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$,
(iv) there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap]x_0 - r, x_0[$ holds $0 \leq f(g)$.
Then $\frac{1}{f}$ is left divergent to $+\infty$ in x_0.

(84) Suppose that
(i) f is left convergent in x_0,
(ii) $\lim_{x_0^-} f = 0$,
(iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$,
(iv) there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap]x_0 - r, x_0[$ holds $f(g) \leq 0$.
Then $\frac{1}{f}$ is left divergent to $-\infty$ in x_0.

(85) Suppose that
(i) f is right convergent in x_0,
(ii) $\lim_{x_0^+} f = 0$,
(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$,
(iv) there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap]x_0, x_0 + r[$ holds $0 \leq f(g)$.
Then $\frac{1}{f}$ is right divergent to $+\infty$ in x_0.

(86) Suppose that
(i) f is right convergent in x_0,
(ii) $\lim_{x_0^+} f = 0$,
(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$,
(iv) there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap]x_0, x_0 + r[$ holds $f(g) \leq 0$.
Then $\frac{1}{f}$ is right divergent to $-\infty$ in x_0.

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Received August 20, 1990

FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

Lattice of Subgroups of a Group. Frattini Subgroup

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Summary. We define the notion of a subgroup generated by a set of elements of a group and two closely connected notions, namely lattice of subgroups and the Frattini subgroup. The operations on the lattice are the intersection of subgroups (introduced in [18]) and multiplication of subgroups, which result is defined as a subgroup generated by a sum of carriers of the two subgroups. In order to define the Frattini subgroup and to prove theorems concerning it we introduce notion of maximal subgroup and non-generating element of the group (see page 30 in [6]). The Frattini subgroup is defined as in [6] as an intersection of all maximal subgroups. We show that an element of the group belongs to the Frattini subgroup of the group if and only if it is a non-generating element. We also prove theorems that should be proved in [1] but are not.

MML Identifier: GROUP_4.

The notation and terminology used here are introduced in the following articles: [3], [13], [4], [11], [20], [10], [19], [8], [16], [5], [17], [2], [15], [18], [14], [12], [21], [7], [9], and [1]. Let D be a non-empty set, and let F be a finite sequence of elements of D, and let X be a set. Then $F - X$ is a finite sequence of elements of D.

In this article we present several logical schemes. The scheme *SubsetD* deals with a non-empty set $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\{d : \mathcal{P}[d]\}$, where d is an element of $\mathcal{A}$, is a subset of $\mathcal{A}$

for all values of the parameters.

The scheme *MeetSbgEx* deals with a group $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists a subgroup H of $\mathcal{A}$ such that the carrier of $H = \bigcap\{A : \bigvee_K[A =$ the carrier of $K \wedge \mathcal{P}[K]]\}$, where A is a subset of $\mathcal{A}$, and K is a subgroup of $\mathcal{A}$ provided the parameters have the following property:

[1]Supported by RPBP.III-24.C1

ISSN 0777–4028

- there exists a subgroup H of $\mathcal{A}$ such that $\mathcal{P}[H]$.

For simplicity we adopt the following rules: X denotes a set, k, l, m, n denote natural numbers, i, i_1, i_2, i_3, j denote integers, G denotes a group, a, b, c denote elements of G, A, B denote subsets of G, H, H_1, H_2, H_3, K denote subgroups of G, N_1, N_2 denote normal subgroups of G, h denotes an element of H, F, F_1, F_2 denote finite sequences of elements of the carrier of G, and I, I_1, I_2 denote finite sequences of elements of $\mathbb{Z}$. The scheme *SubgrSep* deals with a group $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists X such that $X \subseteq \mathrm{SubGr}\,\mathcal{A}$ and for every subgroup H of $\mathcal{A}$ holds $H \in X$ if and only if $\mathcal{P}[H]$

for all values of the parameters.

Let i be an element of $\mathbb{Z}$. The functor $@i$ yields an integer and is defined by:

(Def.1) $@i = i$.

We now state the proposition

(1) For every element i of $\mathbb{Z}$ holds $@i = i$.

Let us consider i. The functor $@i$ yielding an element of $\mathbb{Z}$ is defined as follows:

(Def.2) $@i = i$.

Next we state several propositions:

(2) $@i = i$.

(3) If $a = h$, then $a^n = h^n$.

(4) If $a = h$, then $a^i = h^i$.

(5) If $a \in H$, then $a^n \in H$.

(6) If $a \in H$, then $a^i \in H$.

Let us consider G, F. The functor $\prod F$ yielding an element of G is defined as follows:

(Def.3) $\prod F =$ the operation of $G \odot F$.

Next we state a number of propositions:

(7) $\prod F =$ the operation of $G \odot F$.

(8) $\prod(F_1 \frown F_2) = \prod F_1 \cdot \prod F_2$.

(9) $\prod(F \frown \langle a \rangle) = \prod F \cdot a$.

(10) $\prod(\langle a \rangle \frown F) = a \cdot \prod F$.

(11) $\prod \varepsilon_{\text{the carrier of } G} = 1_G$.

(12) $\prod \langle a \rangle = a$.

(13) $\prod \langle a, b \rangle = a \cdot b$.

(14) $\prod \langle a, b, c \rangle = (a \cdot b) \cdot c$ and $\prod \langle a, b, c \rangle = a \cdot (b \cdot c)$.

(15) $\prod(n \longmapsto a) = a^n$.

(16) $\prod(F - \{1_G\}) = \prod F$.

(17) If $\mathrm{len}\, F_1 = \mathrm{len}\, F_2$ and for every k such that $k \in \mathrm{Seg}(\mathrm{len}\, F_1)$ holds $F_2((\mathrm{len}\, F_1 - k) + 1) = (\pi_k F_1)^{-1}$, then $\prod F_1 = (\prod F_2)^{-1}$.

(18) If G is an Abelian group, then for every permutation P of $\mathrm{Seg}(\mathrm{len}\, F_1)$ such that $F_2 = F_1 \cdot P$ holds $\prod F_1 = \prod F_2$.

(19) If G is an Abelian group and F_1 is one-to-one and F_2 is one-to-one and $\mathrm{rng}\, F_1 = \mathrm{rng}\, F_2$, then $\prod F_1 = \prod F_2$.

(20) If G is an Abelian group and $\mathrm{len}\, F = \mathrm{len}\, F_1$ and $\mathrm{len}\, F = \mathrm{len}\, F_2$ and for every k such that $k \in \mathrm{Seg}(\mathrm{len}\, F)$ holds $F(k) = \pi_k F_1 \cdot \pi_k F_2$, then $\prod F = \prod F_1 \cdot \prod F_2$.

(21) If $\mathrm{rng}\, F \subseteq \overline{H}$, then $\prod F \in H$.

Let us consider G, I, F. Let us assume that $\mathrm{len}\, F = \mathrm{len}\, I$. The functor F^I yields a finite sequence of elements of the carrier of G and is defined as follows:

(Def.4) $\mathrm{len}(F^I) = \mathrm{len}\, F$ and for every k such that $k \in \mathrm{Seg}(\mathrm{len}\, F)$ holds $(F^I)(k) = \pi_k F^{@(\pi_k I)}$.

One can prove the following propositions:

(22) If $\mathrm{len}\, F = \mathrm{len}\, I$ and $\mathrm{len}\, F_1 = \mathrm{len}\, F$ and for every k such that $k \in \mathrm{Seg}(\mathrm{len}\, F)$ holds $F_1(k) = \pi_k F^{@(\pi_k I)}$, then $F_1 = F^I$.

(23) If $\mathrm{len}\, F = \mathrm{len}\, I$, then for every k such that $k \in \mathrm{Seg}(\mathrm{len}\, F)$ holds $(F^I)(k) = \pi_k F^{@(\pi_k I)}$.

(24) If $\mathrm{len}\, F = \mathrm{len}\, I$, then $\mathrm{len}(F^I) = \mathrm{len}\, F$.

(25) If $\mathrm{len}\, F_1 = \mathrm{len}\, I_1$ and $\mathrm{len}\, F_2 = \mathrm{len}\, I_2$, then $(F_1 \frown F_2)^{I_1 \frown I_2} = F_1^{I_1} \frown F_2^{I_2}$.

(26) If $\mathrm{len}\, F = \mathrm{len}\, I$ and $\mathrm{rng}\, F \subseteq \overline{H}$, then $\prod(F^I) \in H$.

(27) $\varepsilon_{\text{the carrier of } G}^{\varepsilon_{\mathbb{Z}}} = \varepsilon$.

(28) $\langle a \rangle^{\langle @i \rangle} = \langle a^i \rangle$.

(29) $\langle a, b \rangle^{\langle @i, @j \rangle} = \langle a^i, b^j \rangle$.

(30) $\langle a, b, c \rangle^{\langle @i_1, @i_2, @i_3 \rangle} = \langle a^{i_1}, b^{i_2}, c^{i_3} \rangle$.

(31) $F^{\mathrm{len}\, F \longmapsto @(+1)} = F$.

(32) $F^{\mathrm{len}\, F \longmapsto @(+0)} = \mathrm{len}\, F \longmapsto 1_G$.

(33) If $\mathrm{len}\, I = n$, then $(n \longmapsto 1_G)^I = n \longmapsto 1_G$.

Let us consider G, A. The functor $\mathrm{gr}(A)$ yielding a subgroup of G is defined as follows:

(Def.5) $A \subseteq$ the carrier of $\mathrm{gr}(A)$ and for every H such that $A \subseteq$ the carrier of H holds $\mathrm{gr}(A)$ is a subgroup of H.

We now state a number of propositions:

(34) If $A \subseteq$ the carrier of H_1 and for every H_2 such that $A \subseteq$ the carrier of H_2 holds H_1 is a subgroup of H_2, then $H_1 = \mathrm{gr}(A)$.

(35) $A \subseteq$ the carrier of $\mathrm{gr}(A)$.

(36) If $A \subseteq$ the carrier of H, then $\mathrm{gr}(A)$ is a subgroup of H.

(37) $a \in \mathrm{gr}(A)$ if and only if there exist F, I such that $\mathrm{len}\, F = \mathrm{len}\, I$ and $\mathrm{rng}\, F \subseteq A$ and $\prod(F^I) = a$.

(38) If $a \in A$, then $a \in \mathrm{gr}(A)$.

(39) $\mathrm{gr}(\emptyset_{\text{the carrier of } G}) = \{\mathbf{1}\}_G$.

(40) $\mathrm{gr}(\overline{H}) = H$.

(41) If $A \subseteq B$, then $\mathrm{gr}(A)$ is a subgroup of $\mathrm{gr}(B)$.

(42) $\mathrm{gr}(A \cap B)$ is a subgroup of $\mathrm{gr}(A) \cap \mathrm{gr}(B)$.

(43) The carrier of $\mathrm{gr}(A) = \bigcap\{B : \bigvee_H [B =$ the carrier of $H \wedge A \subseteq \overline{H}]\}$.

(44) $\mathrm{gr}(A) = \mathrm{gr}(A \setminus \{1_G\})$.

We now define two new predicates. Let us consider G, a. We say that a is non-generating if and only if:

(Def.6) for every A such that $\mathrm{gr}(A) = G$ holds $\mathrm{gr}(A \setminus \{a\}) = G$.

a is generating stands for a is not non-generating.

We now state the proposition

(46)[2] 1_G is non-generating.

Let us consider G, H. We say that H is maximal if and only if:

(Def.7) $H \neq G$ and for every K such that $H \neq K$ and H is a subgroup of K holds $K = G$.

Next we state the proposition

(48)[3] If H is maximal and $a \notin H$, then $\mathrm{gr}(\overline{H} \cup \{a\}) = G$.

Let us consider G. The functor $\Phi(G)$ yields a subgroup of G and is defined as follows:

(Def.8) the carrier of $\Phi(G) = \bigcap\{A : \bigvee_H [A =$ the carrier of $H \wedge$ H is maximal $]\}$ if there exists H such that H is maximal, $\Phi(G) = G$, otherwise.

We now state several propositions:

(49) If there exists H such that H is maximal and the carrier of $H = \bigcap\{A : \bigvee_K [A =$ the carrier of $K \wedge$ K is maximal $]\}$, then $H = \Phi(G)$.

(50) If for every H holds H is not maximal, then $\Phi(G) = G$.

(51) If there exists H such that H is maximal, then the carrier of $\Phi(G) = \bigcap\{A : \bigvee_K [A =$ the carrier of $K \wedge$ K is maximal $]\}$.

(52) If there exists H such that H is maximal, then $a \in \Phi(G)$ if and only if for every H such that H is maximal holds $a \in H$.

(53) If for every H holds H is not maximal, then $a \in \Phi(G)$.

(54) If H is maximal, then $\Phi(G)$ is a subgroup of H.

(55) The carrier of $\Phi(G) = \{a : a$ is non-generating $\}$.

(56) $a \in \Phi(G)$ if and only if a is non-generating.

Let us consider G, H_1, H_2. The functor $H_1 \cdot H_2$ yielding a subset of G is defined as follows:

(Def.9) $H_1 \cdot H_2 = \overline{H_1} \cdot \overline{H_2}$.

The following propositions are true:

(57) $H_1 \cdot H_2 = \overline{H_1} \cdot \overline{H_2}$ and $H_1 \cdot H_2 = H_1 \cdot \overline{H_2}$ and $H_1 \cdot H_2 = \overline{H_1} \cdot H_2$.

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[2]The proposition (45) was either repeated or obvious.

[3]The proposition (47) was either repeated or obvious.

(58) $H \cdot H = \overline{H}$.

(59) $(H_1 \cdot H_2) \cdot H_3 = H_1 \cdot (H_2 \cdot H_3)$.

(60) $(a \cdot H_1) \cdot H_2 = a \cdot (H_1 \cdot H_2)$.

(61) $(H_1 \cdot H_2) \cdot a = H_1 \cdot (H_2 \cdot a)$.

(62) $(A \cdot H_1) \cdot H_2 = A \cdot (H_1 \cdot H_2)$.

(63) $(H_1 \cdot H_2) \cdot A = H_1 \cdot (H_2 \cdot A)$.

(64) $N_1 \cdot N_2 = N_2 \cdot N_1$.

(65) If G is an Abelian group, then $H_1 \cdot H_2 = H_2 \cdot H_1$.

Let us consider G, H_1, H_2. The functor $H_1 \sqcup H_2$ yielding a subgroup of G is defined as follows:

(Def.10) $H_1 \sqcup H_2 = \mathrm{gr}(\overline{H_1} \cup \overline{H_2})$.

One can prove the following propositions:

(66) $H_1 \sqcup H_2 = \mathrm{gr}(\overline{H_1} \cup \overline{H_2})$.

(67) $a \in H_1 \sqcup H_2$ if and only if there exist F, I such that $\mathrm{len}\, F = \mathrm{len}\, I$ and $\mathrm{rng}\, F \subseteq \overline{H_1} \cup \overline{H_2}$ and $a = \prod(F^I)$.

(68) $H_1 \sqcup H_2 = \mathrm{gr}(H_1 \cdot H_2)$.

(69) If $H_1 \cdot H_2 = H_2 \cdot H_1$, then the carrier of $H_1 \sqcup H_2 = H_1 \cdot H_2$.

(70) If G is an Abelian group, then the carrier of $H_1 \sqcup H_2 = H_1 \cdot H_2$.

(71) The carrier of $N_1 \sqcup N_2 = N_1 \cdot N_2$.

(72) $N_1 \sqcup N_2$ is a normal subgroup of G.

(73) $H \sqcup H = H$.

(74) $H_1 \sqcup H_2 = H_2 \sqcup H_1$.

(75) $(H_1 \sqcup H_2) \sqcup H_3 = H_1 \sqcup (H_2 \sqcup H_3)$.

(76) $\{\mathbf{1}\}_G \sqcup H = H$ and $H \sqcup \{\mathbf{1}\}_G = H$.

(77) $\Omega_G \sqcup H = G$ and $H \sqcup \Omega_G = G$.

(78) H_1 is a subgroup of $H_1 \sqcup H_2$ and H_2 is a subgroup of $H_1 \sqcup H_2$.

(79) H_1 is a subgroup of H_2 if and only if $H_1 \sqcup H_2 = H_2$.

(80) If H_1 is a subgroup of H_2, then H_1 is a subgroup of $H_2 \sqcup H_3$.

(81) If H_1 is a subgroup of H_3 and H_2 is a subgroup of H_3, then $H_1 \sqcup H_2$ is a subgroup of H_3.

(82) If H_1 is a subgroup of H_2, then $H_1 \sqcup H_3$ is a subgroup of $H_2 \sqcup H_3$.

(83) $H_1 \cap H_2$ is a subgroup of $H_1 \sqcup H_2$.

(84) $(H_1 \cap H_2) \sqcup H_2 = H_2$.

(85) $H_1 \cap (H_1 \sqcup H_2) = H_1$.

(86) $H_1 \sqcup H_2 = H_2$ if and only if $H_1 \cap H_2 = H_1$.

In the sequel S_1, S_2 are elements of $\mathrm{SubGr}\, G$ and o is a binary operation on $\mathrm{SubGr}\, G$. Let us consider G. The functor $\mathrm{SubJoin}\, G$ yields a binary operation on $\mathrm{SubGr}\, G$ and is defined by:

(Def.11) for all S_1, S_2, H_1, H_2 such that $S_1 = H_1$ and $S_2 = H_2$ holds $(\mathrm{SubJoin}\, G)(S_1, S_2) = H_1 \sqcup H_2$.

Next we state two propositions:

(87) If for all S_1, S_2, H_1, H_2 such that $S_1 = H_1$ and $S_2 = H_2$ holds $o(S_1, S_2) = H_1 \sqcup H_2$, then $o = \mathrm{SubJoin}\, G$.

(88) If $H_1 = S_1$ and $H_2 = S_2$, then $\mathrm{SubJoin}\, G(S_1, S_2) = H_1 \sqcup H_2$.

Let us consider G. The functor $\mathrm{SubMeet}\, G$ yields a binary operation on $\mathrm{SubGr}\, G$ and is defined as follows:

(Def.12) for all S_1, S_2, H_1, H_2 such that $S_1 = H_1$ and $S_2 = H_2$ holds $(\mathrm{SubMeet}\, G)(S_1, S_2) = H_1 \cap H_2$.

One can prove the following two propositions:

(89) If for all S_1, S_2, H_1, H_2 such that $S_1 = H_1$ and $S_2 = H_2$ holds $o(S_1, S_2) = H_1 \cap H_2$, then $o = \mathrm{SubMeet}\, G$.

(90) If $H_1 = S_1$ and $H_2 = S_2$, then $\mathrm{SubMeet}\, G(S_1, S_2) = H_1 \cap H_2$.

Let us consider G. The functor $\mathbb{L}_G$ yielding a lattice is defined as follows:

(Def.13) $\mathbb{L}_G = \langle \mathrm{SubGr}\, G, \mathrm{SubJoin}\, G, \mathrm{SubMeet}\, G \rangle$.

One can prove the following propositions:

(91) $\mathbb{L}_G = \langle \mathrm{SubGr}\, G, \mathrm{SubJoin}\, G, \mathrm{SubMeet}\, G \rangle$.

(92) The carrier of $\mathbb{L}_G = \mathrm{SubGr}\, G$.

(93) The join operation of $\mathbb{L}_G = \mathrm{SubJoin}\, G$.

(94) The meet operation of $\mathbb{L}_G = \mathrm{SubMeet}\, G$.

(95) $\mathbb{L}_G$ is a lower bound lattice.

(96) $\mathbb{L}_G$ is an upper bound lattice.

(97) $\mathbb{L}_G$ is a bound lattice.

(98) $\perp_{\mathbb{L}_G} = \{\mathbf{1}\}_G$.

(99) $\top_{\mathbb{L}_G} = \Omega_G$.

(100) $n \bmod 2 = 0$ or $n \bmod 2 = 1$.

(101) $k \cdot n \bmod k = 0$ and $k \cdot n \bmod n = 0$.

(102) If $k > 1$, then $1 \bmod k = 1$.

(103) If $k \bmod n = 0$ and $l = k - m \cdot n$, then $l \bmod n = 0$.

(104) If $n \neq 0$ and $k \bmod n = 0$ and $l < n$, then $(k + l) \bmod n = l$.

(105) If $k \bmod n = 0$ and $l \bmod n = 0$, then $(k + l) \bmod n = 0$.

(106) If $n \neq 0$ and $k \bmod n = 0$ and $l \bmod n = 0$, then $(k + l) \div n = (k \div n) + (l \div n)$.

(107) If $k \neq 0$, then $k \cdot n \div k = n$.

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Received August 22, 1990

FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

Equalities and Inequalities in Real Numbers

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Summary. The article is to give a number of useful theorems concerning equalities and inequalities in real numbers. Some of the theorems are extentions of [1] theorems, others were found to be needed in practice.

MML Identifier: `REAL_2`.

The terminology and notation used here are introduced in the following articles: [1], [3], [2], and [4]. In the sequel a, b, d, e will be real numbers. One can prove the following propositions:

(1) If $b+a=b$ or $a+b=b$ or $b-a=b$, then $a=0$.

(2) Suppose that
(i) $a-b=0$ or $a+(-b)=0$ or $(-b)+a=0$ or $-a=-b$ or $a-e=b-e$ or $a-e=b+(-e)$ or $a-e=(-e)+b$ or $e-a=e-b$ or $e-a=e+(-b)$ or $e-a=(-b)+e$.
Then $a=b$.

(3) If $a=-b$, then $a+b=0$ and $b+a=0$ and $-a=b$.

(4) If $a+b=0$ or $b+a=0$, then $a=-b$.

(5) $(-a)-b=(-b)-a$.

(6) $-(a+b)=(-a)+(-b)$ and $-(a+b)=(-b)+(-a)$ and $-(a+b)=(-b)-a$ and $-(a+b)=(-a)-b$.

(7) $a-b=(-b)+a$.

(8) $-(a-b)=(-a)+b$ and $-(a-b)=b-a$ and $-(a-b)=b+(-a)$.

(9) $-((-a)+b)=a-b$ and $-((-a)+b)=a+(-b)$ and $-((-a)+b)=(-b)+a$.

(10) (i) $a+b=-((-a)-b)$,
(ii) $a+b=-((-b)-a)$,
(iii) $a+b=-((-b)+(-a))$,

ISSN 0777–4028

(iv) $a + b = -((-a) + (-b))$,
(v) $a + b = a - (-b)$,
(vi) $a + b = b - (-a)$.

(11) If $a+b = e$ or $b+a = e$, then $a = e-b$ and $a = e+(-b)$ and $a = (-b)+e$.

(12) If $a = e-b$ or $a = e+(-b)$ or $a = (-b)+e$, then $a+b = e$ and $b+a = e$ and $b = e - a$.

(13) If $a + b = e + d$, then $a - e = d - b$ and $a - d = e - b$ and $b - e = d - a$ and $b - d = e - a$.

(14) If $a - e = d - b$, then $a + b = e + d$ and $a + b = d + e$ and $b + a = d + e$ and $b + a = e + d$.

(15) If $a - b = e - d$, then $a - e = b - d$.

(16) If $a + b = e - d$ or $b + a = e - d$, then $a + d = e - b$ and $d + a = e - b$.

(17) (i) $a = a + (b - b)$,
(ii) $a = (a + b) - b$,
(iii) $a = a + (b + (-b))$,
(iv) $a = (a + b) + (-b)$,
(v) $a = a - (b - b)$,
(vi) $a = (a - b) + b$,
(vii) $a = a - (b + (-b))$,
(viii) $a = a + ((-b) + b)$,
(ix) $a = (a + (-b)) + b$,
(x) $a = b + (a - b)$,
(xi) $a = (b + a) - b$,
(xii) $a = b + (a + (-b))$,
(xiii) $a = (b + a) + (-b)$,
(xiv) $a = b - (b - a)$,
(xv) $a = (b - b) + a$,
(xvi) $a = (-b) + (a + b)$,
(xvii) $a = ((-b) + a) + b$,
(xviii) $a = (-b) + (b + a)$,
(xix) $a = ((-b) + b) + a$,
(xx) $a = (-b) - ((-a) - b)$,
(xxi) $a = (-b) - ((-b) - a)$.

(18) $a - (b - e) = a + (e - b)$ and $a + (b - e) = (a + b) - e$.

(20)[1] $a + ((-b) - e) = (a - b) - e$ and $a - ((-b) - e) = (a + b) + e$.

(21) $(a+b)+e = (a+e)+b$ and $(a+b)+e = (b+e)+a$ and $(a+b)+e = (e+a)+b$ and $(a + b) + e = (e + b) + a$.

(22) $(a + b) - e = (a - e) + b$ and $(a + b) - e = (b - e) + a$ and $(a + b) - e = ((-e) + a) + b$ and $(a + b) - e = ((-e) + b) + a$.

(23) $(a-b)+e = (e-b)+a$ and $(a-b)+e = ((-b)+a)+e$ and $(a-b)+e = ((-b) + e) + a$.

[1]The proposition (19) was either repeated or obvious.

(24) (i) $(a-b)-e=(a-e)-b$,
(ii) $(a-b)-e=((-b)+a)-e$,
(iii) $(a-b)-e=((-b)-e)+a$,
(iv) $(a-b)-e=((-e)+a)-b$,
(v) $(a-b)-e=((-e)-b)+a$.

(25) $((-a)+b)-e=((-e)+b)-a$ and $((-a)+b)-e=((-e)-a)+b$.

(26) (i) $((-a)-b)-e=((-a)-e)-b$,
(ii) $((-a)-b)-e=((-b)-e)-a$,
(iii) $((-a)-b)-e=((-e)-a)-b$,
(iv) $((-a)-b)-e=((-e)-b)-a$.

(27) (i) $-((a+b)+e)=((-a)-b)-e$,
(ii) $-((a+b)-e)=((-a)-b)+e$,
(iii) $-((a-b)+e)=((-a)+b)-e$,
(iv) $-(((-a)+b)+e)=(a-b)-e$,
(v) $-((a-b)-e)=((-a)+b)+e$,
(vi) $-(((-a)+b)-e)=(a-b)+e$,
(vii) $-(((-a)-b)+e)=(a+b)-e$,
(viii) $-(((-a)-b)-e)=(a+b)+e$.

(28) (i) $a+e=(a+b)+(e-b)$,
(ii) $a+e=(b+a)+(e-b)$,
(iii) $a+e=(a-b)+(e+b)$,
(iv) $a+e=(a-b)+(b+e)$,
(v) $e+a=(a+b)+(e-b)$,
(vi) $e+a=(b+a)+(e-b)$,
(vii) $e+a=(a-b)+(e+b)$,
(viii) $e+a=(a-b)+(b+e)$,
(ix) $a+e=(a+b)-(b-e)$,
(x) $a+e=(b+a)-(b-e)$,
(xi) $e+a=(b+a)-(b-e)$,
(xii) $e+a=(a+b)-(b-e)$.

(29) (i) $a-e=(a-b)-(e-b)$,
(ii) $a-e=(a-b)+(b-e)$,
(iii) $a-e=(a+b)-(e+b)$,
(iv) $a-e=(b+a)-(e+b)$,
(v) $a-e=(b+a)-(b+e)$.

(30) If $b\neq 0$, then if $\frac{a}{b}=1$ or $a\cdot b^{-1}=1$ or $b^{-1}\cdot a=1$, then $a=b$.

(31) If $e\neq 0$ and $\frac{a}{e}=\frac{b}{e}$, then $a=b$.

Next we state a number of propositions:

(32) If $a\cdot 1=b\cdot 1$ or $a\cdot 1=1\cdot b$ or $1\cdot a=1\cdot b$ or $1\cdot a=b\cdot 1$, then $a=b$.

(33) If $a\neq 0$ and $b\neq 0$, then if $a^{-1}=b^{-1}$ or $\frac{1}{a}=\frac{1}{b}$ or $\frac{1}{a}=b^{-1}$, then $a=b$.

(34) If $b\neq 0$ and $\frac{a}{b}=-1$, then $a=-b$ and $b=-a$.

(35) If $a\cdot b=1$ or $b\cdot a=1$, then $a=\frac{1}{b}$ and $a=b^{-1}$.

(36) If $b \neq 0$, then if $a = \frac{1}{b}$ or $a = b^{-1}$, then $a \cdot b = 1$ and $b \cdot a = 1$ and $a^{-1} = b$ and $b = \frac{1}{a}$.

(37) If $b \neq 0$ but $a \cdot b = b$ or $b \cdot a = b$, then $a = 1$.

(38) If $b \neq 0$ but $a \cdot b = -b$ or $b \cdot a = -b$, then $a = -1$.

(39) If $a \neq 0$ and $b \neq 0$ and $\frac{b}{a} = b$, then $a = 1$.

(40) If $a \neq 0$ and $b \neq 0$ and $\frac{b}{a} = -b$, then $a = -1$.

(41) If $a \neq 0$, then $\frac{1}{a} \neq 0$.

(42) If $a \neq 0$ and $b \neq 0$, then $a \cdot b^{-1} \neq 0$ and $b^{-1} \cdot a \neq 0$ and $\frac{a}{b} \neq 0$ and $a^{-1} \cdot b^{-1} \neq 0$ and $\frac{1}{a \cdot b} \neq 0$.

(43) $\frac{1}{1} = 1$ and $1^{-1} = 1$ and $\frac{1}{-1} = -1$ and $(-1)^{-1} = -1$ and $(-1) \cdot (-1) = 1$.

(44) $\frac{a}{1} = a$ and $a \cdot 1^{-1} = a$ and $1^{-1} \cdot a = a$.

(45) If $a \neq 0$, then $\frac{-a}{a} = -1$ and $\frac{a}{-a} = -1$ and $(-a)^{-1} = -a^{-1}$.

(46) If $a \neq 0$, then if $a = a^{-1}$ or $a = \frac{1}{a}$, then $a = 1$ or $a = -1$.

(47) Suppose $a \neq 0$ and $b \neq 0$. Then
(i) $(a \cdot b^{-1})^{-1} = a^{-1} \cdot b$,
(ii) $(a \cdot b^{-1})^{-1} = b \cdot a^{-1}$,
(iii) $(b^{-1} \cdot a)^{-1} = b \cdot a^{-1}$,
(iv) $(b^{-1} \cdot a)^{-1} = a^{-1} \cdot b$,
(v) $(a^{-1} \cdot b^{-1})^{-1} = a \cdot b$.

(48) If $a \neq 0$ and $b \neq 0$, then $\frac{1}{\frac{a}{b}} = \frac{b}{a}$ and $\frac{a}{b}^{-1} = \frac{b}{a}$.

(49) (i) $(-a) \cdot b = -b \cdot a$,
(ii) $a \cdot (-b) = -b \cdot a$,
(iii) $(-a) \cdot b = (-b) \cdot a$,
(iv) $(-a) \cdot (-b) = a \cdot b$,
(v) $(-a) \cdot (-b) = b \cdot a$,
(vi) $-a \cdot (-b) = a \cdot b$,
(vii) $-a \cdot (-b) = b \cdot a$,
(viii) $-(-a) \cdot b = a \cdot b$,
(ix) $-(-a) \cdot b = b \cdot a$.

(50) If $b \neq 0$, then $\frac{a}{b} = 0$ if and only if $a = 0$.

(51) If $a \neq 0$ and $b \neq 0$, then $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{a \cdot b}$.

(52) If $a \neq 0$, then $\frac{1}{\frac{1}{a}} = a$.

(53) Suppose $e \neq 0$ and $d \neq 0$. Then
(i) $\frac{a}{e} \cdot \frac{b}{d} = \frac{b \cdot a}{e \cdot d}$,
(ii) $\frac{a}{e} \cdot \frac{b}{d} = \frac{b \cdot a}{d \cdot e}$,
(iii) $\frac{a}{e} \cdot \frac{b}{d} = \frac{a \cdot b}{d \cdot e}$,
(iv) $\frac{a}{e} \cdot \frac{b}{d} = \frac{a}{d} \cdot \frac{b}{e}$.

(54) If $a \neq 0$, then $a \cdot \frac{1}{a} = 1$.

(55) Suppose $b \neq 0$ and $e \neq 0$. Then
(i) $\frac{a}{b} = \frac{a \cdot e}{e \cdot b}$,

(ii) $\frac{a}{b} = \frac{e \cdot a}{b \cdot e}$,
(iii) $\frac{a}{b} = \frac{e \cdot a}{e \cdot b}$,
(iv) $\frac{a}{b} = \frac{\frac{a}{e}}{\frac{b}{e}}$,
(v) $\frac{a}{b} = e \cdot \frac{a}{b \cdot e}$,
(vi) $\frac{a}{b} = e \cdot \frac{a}{e \cdot b}$,
(vii) $\frac{a}{b} = \frac{a}{e \cdot b} \cdot e$,
(viii) $\frac{a}{b} = \frac{a}{b \cdot e} \cdot e$,
(ix) $\frac{a}{b} = e \cdot \frac{\frac{a}{e}}{b}$,
(x) $\frac{a}{b} = \frac{\frac{a}{e}}{b} \cdot e$,
(xi) $\frac{a}{b} = \frac{a}{e} \cdot \frac{e}{b}$.

(56) If $b \neq 0$, then $a \cdot \frac{1}{b} = \frac{a}{b}$ and $\frac{1}{b} \cdot a = \frac{a}{b}$.

(57) If $b \neq 0$, then $\frac{a}{\frac{1}{b}} = a \cdot b$ and $\frac{a}{\frac{1}{b}} = b \cdot a$.

(58) If $b \neq 0$, then $-\frac{a}{-b} = \frac{a}{b}$ and $-\frac{-a}{b} = \frac{a}{b}$ and $\frac{-a}{-b} = \frac{a}{b}$ and $\frac{-a}{b} = \frac{a}{-b}$.

(59) If $e \neq 0$, then $\frac{a+b}{e} = \frac{b}{e} + \frac{a}{e}$.

(60) Suppose $e \neq 0$ and $d \neq 0$. Then
(i) $\frac{a}{e} + \frac{b}{d} = \frac{d \cdot a + b \cdot e}{e \cdot d}$,
(ii) $\frac{a}{e} + \frac{b}{d} = \frac{d \cdot a + e \cdot b}{e \cdot d}$,
(iii) $\frac{a}{e} + \frac{b}{d} = \frac{a \cdot d + e \cdot b}{e \cdot d}$,
(iv) $\frac{a}{e} + \frac{b}{d} = \frac{d \cdot a + b \cdot e}{d \cdot e}$,
(v) $\frac{a}{e} + \frac{b}{d} = \frac{d \cdot a + e \cdot b}{d \cdot e}$,
(vi) $\frac{a}{e} + \frac{b}{d} = \frac{a \cdot d + e \cdot b}{d \cdot e}$,
(vii) $\frac{a}{e} - \frac{b}{d} = \frac{d \cdot a - b \cdot e}{e \cdot d}$,
(viii) $\frac{a}{e} - \frac{b}{d} = \frac{d \cdot a - e \cdot b}{e \cdot d}$,
(ix) $\frac{a}{e} - \frac{b}{d} = \frac{a \cdot d - e \cdot b}{e \cdot d}$,
(x) $\frac{a}{e} - \frac{b}{d} = \frac{d \cdot a - b \cdot e}{d \cdot e}$,
(xi) $\frac{a}{e} - \frac{b}{d} = \frac{d \cdot a - e \cdot b}{d \cdot e}$,
(xii) $\frac{a}{e} - \frac{b}{d} = \frac{a \cdot d - e \cdot b}{d \cdot e}$.

(61) Suppose $b \neq 0$ and $e \neq 0$. Then
(i) $\frac{a}{\frac{b}{e}} = \frac{e \cdot a}{b}$,
(ii) $\frac{a}{\frac{b}{e}} = a \cdot \frac{e}{b}$,
(iii) $\frac{a}{\frac{b}{e}} = \frac{e}{b} \cdot a$,
(iv) $\frac{a}{\frac{b}{e}} = e \cdot \frac{a}{b}$,
(v) $\frac{a}{\frac{b}{e}} = \frac{a}{b} \cdot e$.

(62) Suppose $b \neq 0$. Then
(i) $a = a \cdot \frac{b}{b}$,
(ii) $a = \frac{a \cdot b}{b}$,
(iii) $a = a \cdot (b \cdot \frac{1}{b})$,
(iv) $a = (a \cdot b) \cdot \frac{1}{b}$,

(v) $a = \frac{\frac{a}{b}}{b}$,

(vi) $a = \frac{a}{b} \cdot b$,

(vii) $a = \frac{a}{b \cdot \frac{1}{b}}$,

(viii) $a = a \cdot (\frac{1}{b} \cdot b)$,

(ix) $a = (a \cdot \frac{1}{b}) \cdot b$,

(x) $a = b \cdot \frac{a}{b}$,

(xi) $a = \frac{b \cdot a}{b}$,

(xii) $a = b \cdot (a \cdot \frac{1}{b})$,

(xiii) $a = (b \cdot a) \cdot \frac{1}{b}$,

(xiv) $a = \frac{b}{b} \cdot a$,

(xv) $a = (\frac{1}{b} \cdot b) \cdot a$,

(xvi) $a = \frac{1}{b} \cdot (b \cdot a)$,

(xvii) $a = \frac{1}{b} \cdot (a \cdot b)$,

(xviii) $a = (\frac{1}{b} \cdot a) \cdot b$.

The following propositions are true:

(63) For every a, b there exists e such that $a = b - e$.

(64) For all a, b such that $a \neq 0$ and $b \neq 0$ there exists e such that $a = \frac{b}{e}$.

(65) Suppose $b \neq 0$. Then

(i) $\frac{a}{b} + e = \frac{a + e \cdot b}{b}$,

(ii) $\frac{a}{b} + e = \frac{a + b \cdot e}{b}$,

(iii) $\frac{a}{b} + e = \frac{b \cdot e + a}{b}$,

(iv) $\frac{a}{b} + e = \frac{e \cdot b + a}{b}$,

(v) $e + \frac{a}{b} = \frac{e \cdot b + a}{b}$,

(vi) $e + \frac{a}{b} = \frac{a + e \cdot b}{b}$,

(vii) $e + \frac{a}{b} = \frac{a + b \cdot e}{b}$,

(viii) $e + \frac{a}{b} = \frac{b \cdot e + a}{b}$.

(66) Suppose $b \neq 0$. Then

(i) $\frac{a}{b} - e = \frac{a - e \cdot b}{b}$,

(ii) $\frac{a}{b} - e = \frac{a - b \cdot e}{b}$,

(iii) $e - \frac{a}{b} = \frac{e \cdot b - a}{b}$,

(iv) $e - \frac{a}{b} = \frac{b \cdot e - a}{b}$.

(67) Suppose $b \neq 0$ and $e \neq 0$. Then

(i) $\frac{\frac{a}{b}}{e} = \frac{a}{b \cdot e}$,

(ii) $\frac{\frac{a}{b}}{e} = \frac{a}{e \cdot b}$,

(iii) $\frac{\frac{a}{b}}{e} = \frac{\frac{a}{e}}{b}$,

(iv) $\frac{\frac{a}{b}}{e} = \frac{1}{b} \cdot \frac{a}{e}$,

(v) $\frac{\frac{a}{b}}{e} = \frac{a}{e} \cdot \frac{1}{b}$,

(vi) $\frac{\frac{a}{b}}{e} = \frac{1}{e} \cdot \frac{a}{b}$,

(vii) $\frac{\frac{a}{b}}{e} = \frac{a}{b} \cdot \frac{1}{e}$,

(viii) $\frac{1}{e} \cdot \frac{a}{b} = \frac{a}{b \cdot e}$,
(ix) $\frac{1}{e} \cdot \frac{a}{b} = \frac{a}{e \cdot b}$,
(x) $\frac{a}{b} \cdot \frac{1}{e} = \frac{a}{e \cdot b}$,
(xi) $\frac{a}{b} \cdot \frac{1}{e} = \frac{a}{b \cdot e}$.

(68) Suppose $b \neq 0$. Then $e \cdot \frac{a}{b} = \frac{e \cdot a}{b}$ and $e \cdot \frac{a}{b} = \frac{a \cdot e}{b}$ and $\frac{a}{b} \cdot e = \frac{a \cdot e}{b}$ and $\frac{a}{b} \cdot e = \frac{e \cdot a}{b}$.

(69) $(a \cdot b) \cdot e = (a \cdot e) \cdot b$ and $(a \cdot b) \cdot e = (b \cdot e) \cdot a$ and $(a \cdot b) \cdot e = (e \cdot a) \cdot b$ and $(a \cdot b) \cdot e = (e \cdot b) \cdot a$.

(70) Suppose $e \neq 0$ and $d \neq 0$. Then
(i) $\frac{a \cdot b}{e \cdot d} = \frac{\frac{a}{e} \cdot b}{d}$,
(ii) $\frac{a \cdot b}{e \cdot d} = \frac{b \cdot \frac{a}{e}}{d}$,
(iii) $\frac{a \cdot b}{e \cdot d} = \frac{\frac{b}{e} \cdot a}{d}$,
(iv) $\frac{a \cdot b}{e \cdot d} = \frac{a \cdot \frac{b}{e}}{d}$,
(v) $\frac{a \cdot b}{d \cdot e} = \frac{\frac{a}{e} \cdot b}{d}$,
(vi) $\frac{a \cdot b}{d \cdot e} = \frac{b \cdot \frac{a}{e}}{d}$,
(vii) $\frac{a \cdot b}{d \cdot e} = \frac{a \cdot \frac{b}{e}}{d}$,
(viii) $\frac{a \cdot b}{d \cdot e} = \frac{\frac{b}{e} \cdot a}{d}$.

(71) $(-1) \cdot a = -a$ and $a \cdot (-1) = -a$ and $(-a) \cdot (-1) = a$ and $(-1) \cdot (-a) = a$ and $-a = \frac{a}{-1}$ and $a = \frac{-a}{-1}$.

(72) If $e \neq 0$, then if $a \cdot e = b$ or $e \cdot a = b$, then $a = \frac{b}{e}$.

(73) If $e \neq 0$ and $a = \frac{b}{e}$, then $a \cdot e = b$ and $e \cdot a = b$.

(74) If $a \neq 0$ and $e \neq 0$ and $a = \frac{b}{e}$, then $e = \frac{b}{a}$.

(75) If $e \neq 0$ and $d \neq 0$, then if $a \cdot e = b \cdot d$ or $e \cdot a = b \cdot d$ or $e \cdot a = d \cdot b$ or $a \cdot e = d \cdot b$, then $\frac{a}{d} = \frac{b}{e}$.

(76) If $e \neq 0$ and $d \neq 0$ and $\frac{a}{d} = \frac{b}{e}$, then $a \cdot e = b \cdot d$ and $e \cdot a = b \cdot d$ and $e \cdot a = d \cdot b$ and $a \cdot e = d \cdot b$.

(77) If $e \neq 0$ and $d \neq 0$, then if $a \cdot e = \frac{b}{d}$ or $e \cdot a = \frac{b}{d}$, then $a \cdot d = \frac{b}{e}$ and $d \cdot a = \frac{b}{e}$.

(78) Suppose $b \neq 0$. Then
(i) $a \cdot e = (a \cdot b) \cdot \frac{e}{b}$,
(ii) $a \cdot e = (b \cdot a) \cdot \frac{e}{b}$,
(iii) $a \cdot e = \frac{a}{b} \cdot (e \cdot b)$,
(iv) $a \cdot e = \frac{a}{b} \cdot (b \cdot e)$,
(v) $e \cdot a = (a \cdot b) \cdot \frac{e}{b}$,
(vi) $e \cdot a = (b \cdot a) \cdot \frac{e}{b}$,
(vii) $e \cdot a = \frac{a}{b} \cdot (e \cdot b)$,
(viii) $e \cdot a = \frac{a}{b} \cdot (b \cdot e)$.

(79) Suppose $b \neq 0$ and $e \neq 0$. Then $a \cdot e = \frac{a \cdot b}{\frac{b}{e}}$ and $a \cdot e = \frac{b \cdot a}{\frac{b}{e}}$ and $e \cdot a = \frac{b \cdot a}{\frac{b}{e}}$ and $e \cdot a = \frac{a \cdot b}{\frac{b}{e}}$.

(80) If $b \neq 0$, then $\frac{a}{b} \cdot e = \frac{e}{b} \cdot a$ and $\frac{a}{b} \cdot e = (\frac{1}{b} \cdot a) \cdot e$ and $\frac{a}{b} \cdot e = (\frac{1}{b} \cdot e) \cdot a$.

(81) $(-a) \cdot (-b) = a \cdot b$ and $(-a) \cdot (-b) = b \cdot a$.

(82) If $b \neq 0$ and $d \neq 0$ and $b \neq d$ and $\frac{a}{b} = \frac{e}{d}$, then $\frac{a}{b} = \frac{a-e}{b-d}$.

(83) Suppose $b \neq 0$ and $d \neq 0$ and $b \neq -d$ and $\frac{a}{b} = \frac{e}{d}$. Then $\frac{a}{b} = \frac{a+e}{b+d}$ and $\frac{a}{b} = \frac{e+a}{b+d}$ and $\frac{a}{b} = \frac{e+a}{d+b}$ and $\frac{a}{b} = \frac{a+e}{d+b}$.

(84) (i) $e \cdot (a + b) = a \cdot e + e \cdot b$,
(ii) $e \cdot (a + b) = e \cdot a + b \cdot e$,
(iii) $e \cdot (a + b) = a \cdot e + b \cdot e$,
(iv) $(a + b) \cdot e = e \cdot a + b \cdot e$,
(v) $(a + b) \cdot e = a \cdot e + e \cdot b$,
(vi) $(a + b) \cdot e = e \cdot a + e \cdot b$,
(vii) $e \cdot (b + a) = a \cdot e + e \cdot b$,
(viii) $e \cdot (b + a) = e \cdot a + b \cdot e$,
(ix) $e \cdot (b + a) = a \cdot e + b \cdot e$,
(x) $(b + a) \cdot e = e \cdot a + b \cdot e$,
(xi) $(b + a) \cdot e = a \cdot e + e \cdot b$,
(xii) $(b + a) \cdot e = e \cdot a + e \cdot b$,
(xiii) $(a + b) \cdot e = b \cdot e + a \cdot e$,
(xiv) $e \cdot (a + b) = e \cdot b + e \cdot a$.

(85) (i) $e \cdot (a - b) = a \cdot e - e \cdot b$,
(ii) $e \cdot (a - b) = e \cdot a - b \cdot e$,
(iii) $e \cdot (a - b) = a \cdot e - b \cdot e$,
(iv) $(a - b) \cdot e = e \cdot a - b \cdot e$,
(v) $(a - b) \cdot e = a \cdot e - e \cdot b$,
(vi) $(a - b) \cdot e = e \cdot a - e \cdot b$,
(vii) $(a - b) \cdot e = (b - a) \cdot (-e)$,
(viii) $(a - b) \cdot e = -(b - a) \cdot e$,
(ix) $e \cdot (a - b) = (-e) \cdot (b - a)$,
(x) $e \cdot (a - b) = -e \cdot (b - a)$.

(86) If $a \neq 0$, then if $\frac{1}{a} = 1$ or $a^{-1} = 1$, then $a = 1$.

(87) If $a \neq 0$, then if $\frac{1}{a} = -1$ or $a^{-1} = -1$, then $a = -1$.

(88) (i) $2 \cdot a = a + a$,
(ii) $a \cdot 2 = a + a$,
(iii) $3 \cdot a = (a + a) + a$,
(iv) $a \cdot 3 = (a + a) + a$,
(v) $4 \cdot a = ((a + a) + a) + a$,
(vi) $a \cdot 4 = ((a + a) + a) + a$.

(89) $\frac{a+a}{2} = a$ and $\frac{(a+a)+a}{3} = a$ and $\frac{((a+a)+a)+a}{4} = a$ and $\frac{a+a}{4} = \frac{a}{2}$.

(90) (i) $\frac{a}{2}+\frac{a}{2}=a$,
(ii) $(\frac{a}{3}+\frac{a}{3})+\frac{a}{3}=a$,
(iii) $((\frac{a}{4}+\frac{a}{4})+\frac{a}{4})+\frac{a}{4}=a$,
(iv) $\frac{a}{4}+\frac{a}{4}=\frac{a}{2}$.
(91) If $b \neq 0$, then $\frac{a}{2\cdot b}+\frac{a}{2\cdot b}=\frac{a}{b}$ and $(\frac{a}{3\cdot b}+\frac{a}{3\cdot b})+\frac{a}{3\cdot b}=\frac{a}{b}$.
(92) Suppose $e \neq 0$. Then
(i) $a+b=e\cdot(\frac{a}{e}+\frac{b}{e})$,
(ii) $b+a=e\cdot(\frac{a}{e}+\frac{b}{e})$,
(iii) $b+a=(\frac{a}{e}+\frac{b}{e})\cdot e$,
(iv) $a+b=(\frac{a}{e}+\frac{b}{e})\cdot e$.
(93) If $e \neq 0$, then $a-b=e\cdot(\frac{a}{e}-\frac{b}{e})$ and $a-b=(\frac{a}{e}-\frac{b}{e})\cdot e$.

One can prove the following propositions:

(94) Suppose $e \neq 0$. Then
(i) $a+b=\frac{a\cdot e+b\cdot e}{e}$,
(ii) $a+b=\frac{a\cdot e+e\cdot b}{e}$,
(iii) $a+b=\frac{e\cdot a+e\cdot b}{e}$,
(iv) $a+b=\frac{e\cdot a+b\cdot e}{e}$,
(v) $b+a=\frac{e\cdot a+b\cdot e}{e}$,
(vi) $b+a=\frac{e\cdot a+e\cdot b}{e}$,
(vii) $b+a=\frac{a\cdot e+e\cdot b}{e}$,
(viii) $b+a=\frac{a\cdot e+b\cdot e}{e}$.
(95) Suppose $e \neq 0$. Then
(i) $a-b=\frac{a\cdot e-b\cdot e}{e}$,
(ii) $a-b=\frac{a\cdot e-e\cdot b}{e}$,
(iii) $a-b=\frac{e\cdot a-e\cdot b}{e}$,
(iv) $a-b=\frac{e\cdot a-b\cdot e}{e}$.
(96) Suppose $a \neq 0$. Then
(i) $a+b=a\cdot(1+\frac{b}{a})$,
(ii) $a+b=(1+\frac{b}{a})\cdot a$,
(iii) $a+b=(\frac{b}{a}+1)\cdot a$,
(iv) $a+b=a\cdot(\frac{b}{a}+1)$,
(v) $b+a=a\cdot(1+\frac{b}{a})$,
(vi) $b+a=(1+\frac{b}{a})\cdot a$,
(vii) $b+a=(\frac{b}{a}+1)\cdot a$,
(viii) $b+a=a\cdot(\frac{b}{a}+1)$.
(97) If $a \neq 0$, then $a-b=a\cdot(1-\frac{b}{a})$ and $a-b=(1-\frac{b}{a})\cdot a$.
(98) $(a-b)\cdot(e-d)=(b-a)\cdot(d-e)$.
(99) (i) $((a+b)+e)\cdot d=(a\cdot d+b\cdot d)+e\cdot d$,
(ii) $d\cdot((a+b)+e)=(d\cdot a+d\cdot b)+d\cdot e$,
(iii) $((a+b)-e)\cdot d=(a\cdot d+b\cdot d)-e\cdot d$,

(iv) $d \cdot ((a+b)-e) = (d \cdot a + d \cdot b) - d \cdot e$,
(v) $((a-b)+e) \cdot d = (a \cdot d - b \cdot d) + e \cdot d$,
(vi) $d \cdot ((a-b)+e) = (d \cdot a - d \cdot b) + d \cdot e$,
(vii) $((a-b)-e) \cdot d = (a \cdot d - b \cdot d) - e \cdot d$,
(viii) $d \cdot ((a-b)-e) = (d \cdot a - d \cdot b) - d \cdot e$.

(100) Suppose $d \neq 0$. Then
(i) $\frac{(a+b)+e}{d} = (\frac{a}{d} + \frac{b}{d}) + \frac{e}{d}$,
(ii) $\frac{(a+b)-e}{d} = (\frac{a}{d} + \frac{b}{d}) - \frac{e}{d}$,
(iii) $\frac{(a-b)+e}{d} = (\frac{a}{d} - \frac{b}{d}) + \frac{e}{d}$,
(iv) $\frac{(a-b)-e}{d} = (\frac{a}{d} - \frac{b}{d}) - \frac{e}{d}$.

(101) (i) $(a+b) \cdot (e+d) = ((a \cdot e + a \cdot d) + b \cdot e) + b \cdot d$,
(ii) $(a+b) \cdot (e-d) = ((a \cdot e - a \cdot d) + b \cdot e) - b \cdot d$,
(iii) $(a-b) \cdot (e+d) = ((a \cdot e + a \cdot d) - b \cdot e) - b \cdot d$,
(iv) $(a-b) \cdot (e-d) = ((a \cdot e - a \cdot d) - b \cdot e) + b \cdot d$.

(103)[2] If $a \geq b$, then $a + e \geq e + b$ and $e + a \geq e + b$ and $e + a \geq b + e$.

(104) If $a + e \geq b + e$ or $a + e \geq e + b$ or $e + a \geq e + b$ or $e + a \geq b + e$ or $a - e \geq b - e$, then $a \geq b$.

(105) Suppose that
(i) $a - b \leq 0$ or $a + (-b) \leq 0$ or $(-b) + a \leq 0$ or $-a \geq -b$ or $b - a \geq 0$ or $b + (-a) \geq 0$ or $(-a) + b \geq 0$ or $a - e \leq b + (-e)$ or $a - e \leq (-e) + b$ or $a + (-e) \leq b - e$ or $(-e) + a \leq b - e$ or $e - a \geq e - b$.
Then $a \leq b$.

(106) Suppose that
(i) $a - b < 0$ or $a + (-b) < 0$ or $(-b) + a < 0$ or $-a > -b$ or $b - a > 0$ or $b + (-a) > 0$ or $(-a) + b > 0$ or $a - e < b + (-e)$ or $a - e < (-e) + b$ or $a + (-e) < b - e$ or $(-e) + a < b - e$ or $e - a > e - b$.
Then $a < b$.

(107) Suppose $a \leq b$. Then $a - b \leq 0$ and $a + (-b) \leq 0$ and $(-b) + a \leq 0$ and $b - a \geq 0$ and $b + (-a) \geq 0$ and $(-a) + b \geq 0$ and $-a \geq -b$ and $e - a \geq e - b$.

(108) Suppose $a < b$. Then $a - b < 0$ and $a + (-b) < 0$ and $(-b) + a < 0$ and $b - a > 0$ and $b + (-a) > 0$ and $(-a) + b > 0$ and $-a > -b$ and $e - a > e - b$.

(109) If $a \leq -b$, then $a + b \leq 0$ and $b + a \leq 0$ and $-a \geq b$.

(110) If $a < -b$, then $a + b < 0$ and $b + a < 0$ and $-a > b$.

(111) If $-a \leq b$, then $b + a \geq 0$ and $a + b \geq 0$ and $a \geq -b$.

(112) If $-b < a$, then $a + b > 0$ and $b + a > 0$ and $b > -a$.

(113) If $a + b \leq 0$ or $b + a \leq 0$, then $a \leq -b$.

(114) If $a + b < 0$ or $b + a < 0$, then $a < -b$.

(115) If $a + b \geq 0$ or $b + a \geq 0$, then $a \geq -b$.

[2] The proposition (102) was either repeated or obvious.

(116) If $a+b>0$ or $b+a>0$, then $a>-b$.

(117) Suppose $b>0$. Then
(i) if $\frac{a}{b}>1$, then $a>b$,
(ii) if $\frac{a}{b}<1$, then $a<b$,
(iii) if $\frac{a}{b}>-1$, then $a>-b$ and $b>-a$,
(iv) if $\frac{a}{b}<-1$, then $a<-b$ and $b<-a$.

(118) Suppose $b>0$. Then
(i) if $\frac{a}{b}\geq 1$, then $a\geq b$,
(ii) if $\frac{a}{b}\leq 1$, then $a\leq b$,
(iii) if $\frac{a}{b}\geq -1$, then $a\geq -b$ and $b\geq -a$,
(iv) if $\frac{a}{b}\leq -1$, then $a\leq -b$ and $b\leq -a$.

(119) Suppose $b<0$. Then
(i) if $\frac{a}{b}>1$, then $a<b$,
(ii) if $\frac{a}{b}<1$, then $a>b$,
(iii) if $\frac{a}{b}>-1$, then $a<-b$ and $b<-a$,
(iv) if $\frac{a}{b}<-1$, then $a>-b$ and $b>-a$.

(120) Suppose $b<0$. Then
(i) if $\frac{a}{b}\geq 1$, then $a\leq b$,
(ii) if $\frac{a}{b}\leq 1$, then $a\geq b$,
(iii) if $\frac{a}{b}\geq -1$, then $a\leq -b$ and $b\leq -a$,
(iv) if $\frac{a}{b}\leq -1$, then $a\geq -b$ and $b\geq -a$.

(121) If $a\geq 0$ or $a>0$ but $b\geq 0$ or $b>0$ or $a\leq 0$ or $a<0$ but $b\leq 0$ or $b<0$, then $a\cdot b\geq 0$ and $b\cdot a\geq 0$.

(122) If $a<0$ and $b<0$ or $a>0$ and $b>0$, then $a\cdot b>0$.

(123) If $a\geq 0$ or $a>0$ but $b\leq 0$ or $b<0$ or $a\leq 0$ or $a<0$ but $b\geq 0$ or $b>0$, then $a\cdot b\leq 0$ and $b\cdot a\leq 0$.

(124) If $a>0$ and $b<0$, then $a\cdot b<0$ and $b\cdot a<0$.

One can prove the following propositions:

(125) If $a\leq 0$ and $b<0$ or $a\geq 0$ and $b>0$, then $\frac{a}{b}\geq 0$.

(126) If $a\geq 0$ and $b<0$ or $a\leq 0$ and $b>0$, then $\frac{a}{b}\leq 0$.

(127) If $a>0$ and $b>0$ or $a<0$ and $b<0$, then $\frac{a}{b}>0$.

(128) If $a<0$ and $b>0$, then $\frac{a}{b}<0$ and $\frac{b}{a}<0$.

(129) If $a\cdot b\leq 0$, then $a\geq 0$ and $b\leq 0$ or $a\leq 0$ and $b\geq 0$.

(131)[3] If $a\cdot b>0$, then $a>0$ and $b>0$ or $a<0$ and $b<0$.

(132) If $a\cdot b<0$, then $a>0$ and $b<0$ or $a<0$ and $b>0$.

(133) If $b\neq 0$ and $\frac{a}{b}\leq 0$, then $b>0$ and $a\leq 0$ or $b<0$ and $a\geq 0$.

(134) If $b\neq 0$ and $\frac{a}{b}\geq 0$, then $b>0$ and $a\geq 0$ or $b<0$ and $a\leq 0$.

(135) If $b\neq 0$ and $\frac{a}{b}<0$, then $b<0$ and $a>0$ or $b>0$ and $a<0$.

(136) If $b\neq 0$ and $\frac{a}{b}>0$, then $b>0$ and $a>0$ or $b<0$ and $a<0$.

[3]The proposition (130) was either repeated or obvious.

(137) If $a > 1$ but $b > 1$ or $b \geq 1$ or $a < -1$ but $b < -1$ or $b \leq -1$, then $a \cdot b > 1$ and $b \cdot a > 1$.

(138) If $a \geq 1$ and $b \geq 1$ or $a \leq -1$ and $b \leq -1$, then $a \cdot b \geq 1$.

(139) Suppose that

(i) $0 < a$ or $0 \leq a$ but $a < 1$ but $0 < b$ or $0 \leq b$ but $b < 1$ or $b \leq 1$ or $0 > a$ or $0 \geq a$ but $a > -1$ but $0 > b$ or $0 \geq b$ but $b > -1$ or $b \geq -1$. Then $a \cdot b < 1$ and $b \cdot a < 1$.

(140) If $0 \leq a$ and $a \leq 1$ and $0 \leq b$ and $b \leq 1$ or $0 \geq a$ and $a \geq -1$ and $0 \geq b$ and $b \geq -1$, then $a \cdot b \leq 1$.

(141) If $e < 0$ and $a \leq b$ or $e > 0$ and $a \geq b$, then $\frac{a}{e} \geq \frac{b}{e}$.

(142) If $0 < a$ and $a < b$ or $b < a$ and $a < 0$, then $\frac{a}{b} < 1$ and $\frac{b}{a} > 1$.

(143) If $0 < a$ and $a \leq b$ or $b \leq a$ and $a < 0$, then $\frac{a}{b} \leq 1$ and $\frac{b}{a} \geq 1$.

(144) If $a > 0$ and $b > 1$ or $a < 0$ and $b < 1$, then $a \cdot b > a$ and $b \cdot a > a$.

(145) If $a > 0$ and $b < 1$ or $a < 0$ and $b > 1$, then $a \cdot b < a$ and $b \cdot a < a$.

(146) If $a > 0$ or $a \geq 0$ but $b > 1$ or $b \geq 1$ or $a < 0$ or $a \leq 0$ but $b < 1$ or $b \leq 1$, then $a \cdot b \geq a$ and $b \cdot a \geq a$.

(147) If $a > 0$ or $a \geq 0$ but $b < 1$ or $b \leq 1$ or $a < 0$ or $a \leq 0$ but $b > 1$ or $b \geq 1$, then $a \cdot b \leq a$ and $b \cdot a \leq a$.

(148) $a > 0$ if and only if $-a < 0$ but $a \geq 0$ if and only if $-a \leq 0$ but $a \leq 0$ if and only if $-a \geq 0$.

(149) If $a < 0$, then $\frac{1}{a} < 0$ and $a^{-1} < 0$ but if $a > 0$, then $\frac{1}{a} > 0$.

(150) If $a \neq 0$, then if $\frac{1}{a} < 0$, then $a < 0$ but if $\frac{1}{a} > 0$, then $a > 0$.

(151) If $0 < a$ or $b < 0$ but $a < b$, then $\frac{1}{a} > \frac{1}{b}$.

(152) If $0 < a$ or $b < 0$ but $a \leq b$, then $\frac{1}{a} \geq \frac{1}{b}$.

(153) If $a < 0$ and $b > 0$, then $\frac{1}{a} < \frac{1}{b}$.

(154) If $a \neq 0$ and $b \neq 0$ but $\frac{1}{b} > 0$ or $\frac{1}{a} < 0$ and $\frac{1}{a} > \frac{1}{b}$, then $a < b$.

(155) If $a \neq 0$ and $b \neq 0$ but $\frac{1}{b} > 0$ or $\frac{1}{a} < 0$ and $\frac{1}{a} \geq \frac{1}{b}$, then $a \leq b$.

Next we state a number of propositions:

(156) If $a \neq 0$ and $b \neq 0$ and $\frac{1}{a} < 0$ and $\frac{1}{b} > 0$, then $a < b$.

(157) If $a < -1$, then $0 > \frac{1}{a}$ and $\frac{1}{a} > -1$.

(158) If $a \leq -1$, then $0 > \frac{1}{a}$ and $\frac{1}{a} \geq -1$.

(159) If $-1 < a$ and $a < 0$, then $\frac{1}{a} < -1$.

(160) If $-1 \leq a$ and $a < 0$, then $\frac{1}{a} \leq -1$.

(161) If $0 < a$ and $a < 1$, then $\frac{1}{a} > 1$.

(162) If $0 < a$ and $a \leq 1$, then $\frac{1}{a} \geq 1$.

(163) If $1 < a$, then $0 < \frac{1}{a}$ and $\frac{1}{a} < 1$.

(164) If $1 \leq a$, then $0 < \frac{1}{a}$ and $\frac{1}{a} \leq 1$.

(165) If $b \leq e - a$, then $a \leq e - b$ but if $b \geq e - a$, then $a \geq e - b$.

(166) If $b < e - a$, then $a < e - b$ but if $b > e - a$, then $a > e - b$.

(167) If $a+b \leq e+d$, then $a-e \leq d-b$ and $e-a \geq b-d$ and $a-d \leq e-b$ and $d-a \geq b-e$.

(168) If $a+b < e+d$, then $a-e < d-b$ and $e-a > b-d$ and $a-d < e-b$ and $d-a > b-e$.

(169) Suppose $a-b \leq e-d$. Then $a+d \leq e+b$ and $d+a \leq e+b$ and $d+a \leq b+e$ and $a+d \leq b+e$ and $a-e \leq b-d$ and $e-a \geq d-b$ and $b-a \geq d-e$.

(170) Suppose $a-b < e-d$. Then $a+d < e+b$ and $d+a < e+b$ and $d+a < b+e$ and $a+d < b+e$ and $a-e < b-d$ and $e-a > d-b$ and $b-a > d-e$.

(171) (i) If $a+b \leq e-d$ or $b+a \leq e-d$, then $a+d \leq e-b$ and $d+a \leq e-b$,
(ii) if $a+b \geq e-d$ or $b+a \geq e-d$, then $a+d \geq e-b$ and $d+a \geq e-b$.

(172) (i) If $a+b < e-d$ or $b+a < e-d$, then $a+d < e-b$ and $d+a < e-b$,
(ii) if $a+b > e-d$ or $b+a > e-d$, then $a+d > e-b$ and $d+a > e-b$.

(173) If $a < 0$, then $b+a < b$ and $a+b < b$ and $b-a > b$ but if $a+b < b$ or $b+a < b$ or $b-a > b$, then $a < 0$.

(174) If $a \leq 0$, then $b+a \leq b$ and $a+b \leq b$ and $b-a \geq b$ but if $b+a \leq b$ or $a+b \leq b$ or $b-a \geq b$, then $a \leq 0$.

(175) If $a > 0$, then $b+a > b$ and $a+b > b$ and $b-a < b$ but if $b+a > b$ or $a+b > b$ or $b-a < b$, then $a > 0$.

(176) If $a \geq 0$, then $b+a \geq b$ and $a+b \geq b$ and $b-a \leq b$ but if $b+a \geq b$ or $a+b \geq b$ or $b-a \leq b$, then $a \geq 0$.

(177) (i) If $b > 0$ but $a \cdot b \leq e$ or $b \cdot a \leq e$, then $a \leq \frac{e}{b}$,
(ii) if $b < 0$ but $a \cdot b \leq e$ or $b \cdot a \leq e$, then $a \geq \frac{e}{b}$,
(iii) if $b > 0$ but $a \cdot b \geq e$ or $b \cdot a \geq e$, then $a \geq \frac{e}{b}$,
(iv) if $b < 0$ but $a \cdot b \geq e$ or $b \cdot a \geq e$, then $a \leq \frac{e}{b}$.

(178) (i) If $b > 0$ but $a \cdot b < e$ or $b \cdot a < e$, then $a < \frac{e}{b}$,
(ii) if $b < 0$ but $a \cdot b < e$ or $b \cdot a < e$, then $a > \frac{e}{b}$,
(iii) if $b > 0$ but $a \cdot b > e$ or $b \cdot a > e$, then $a > \frac{e}{b}$,
(iv) if $b < 0$ but $a \cdot b > e$ or $b \cdot a > e$, then $a < \frac{e}{b}$.

(179) (i) If $b > 0$ and $a \geq \frac{e}{b}$, then $a \cdot b \geq e$ and $b \cdot a \geq e$,
(ii) if $b > 0$ and $a \leq \frac{e}{b}$, then $a \cdot b \leq e$ and $b \cdot a \leq e$,
(iii) if $b < 0$ and $a \geq \frac{e}{b}$, then $a \cdot b \leq e$ and $b \cdot a \leq e$,
(iv) if $b < 0$ and $a \leq \frac{e}{b}$, then $a \cdot b \geq e$ and $b \cdot a \geq e$.

(180) (i) If $b > 0$ and $a > \frac{e}{b}$, then $a \cdot b > e$ and $b \cdot a > e$,
(ii) if $b > 0$ and $a < \frac{e}{b}$, then $a \cdot b < e$ and $b \cdot a < e$,
(iii) if $b < 0$ and $a > \frac{e}{b}$, then $a \cdot b < e$ and $b \cdot a < e$,
(iv) if $b < 0$ and $a < \frac{e}{b}$, then $a \cdot b > e$ and $b \cdot a > e$.

(181) If for every a such that $a > 0$ holds $b+a \geq e$ or for every a such that $a < 0$ holds $b-a \geq e$, then $b \geq e$.

(182) If for every a such that $a > 0$ holds $b-a \leq e$ or for every a such that $a < 0$ holds $b+a \leq e$, then $b \leq e$.

(183) If for every a such that $a > 1$ holds $b \cdot a \geq e$ or for every a such that $0 < a$ and $a < 1$ holds $\frac{b}{a} \geq e$, then $b \geq e$.

(184) If for every a such that $0 < a$ and $a < 1$ holds $b \cdot a \leq e$ or for every a such that $a > 1$ holds $\frac{b}{a} \leq e$, then $b \leq e$.

(185) Suppose $b > 0$ and $d > 0$ or $b < 0$ and $d < 0$ but $a \cdot d < e \cdot b$ or $d \cdot a < e \cdot b$ or $d \cdot a < b \cdot e$ or $a \cdot d < b \cdot e$. Then $\frac{a}{b} < \frac{e}{d}$.

(186) Suppose $b > 0$ and $d < 0$ or $b < 0$ and $d > 0$ but $a \cdot d < e \cdot b$ or $d \cdot a < e \cdot b$ or $d \cdot a < b \cdot e$ or $a \cdot d < b \cdot e$. Then $\frac{a}{b} > \frac{e}{d}$.

The following propositions are true:

(187) Suppose $b > 0$ and $d > 0$ or $b < 0$ and $d < 0$ but $a \cdot d \leq e \cdot b$ or $d \cdot a \leq e \cdot b$ or $d \cdot a \leq b \cdot e$ or $a \cdot d \leq b \cdot e$. Then $\frac{a}{b} \leq \frac{e}{d}$.

(188) Suppose $b > 0$ and $d < 0$ or $b < 0$ and $d > 0$ but $a \cdot d \leq e \cdot b$ or $d \cdot a \leq e \cdot b$ or $d \cdot a \leq b \cdot e$ or $a \cdot d \leq b \cdot e$. Then $\frac{a}{b} \geq \frac{e}{d}$.

(189) Suppose $b > 0$ and $d > 0$ or $b < 0$ and $d < 0$ but $\frac{a}{b} < \frac{e}{d}$. Then $a \cdot d < e \cdot b$ and $d \cdot a < e \cdot b$ and $d \cdot a < b \cdot e$ and $a \cdot d < b \cdot e$.

(190) Suppose $b < 0$ and $d > 0$ or $b > 0$ and $d < 0$ but $\frac{a}{b} < \frac{e}{d}$. Then $a \cdot d > e \cdot b$ and $d \cdot a > e \cdot b$ and $d \cdot a > b \cdot e$ and $a \cdot d > b \cdot e$.

(191) Suppose $b > 0$ and $d > 0$ or $b < 0$ and $d < 0$ but $\frac{a}{b} \leq \frac{e}{d}$. Then $a \cdot d \leq e \cdot b$ and $d \cdot a \leq e \cdot b$ and $d \cdot a \leq b \cdot e$ and $a \cdot d \leq b \cdot e$.

(192) Suppose $b < 0$ and $d > 0$ or $b > 0$ and $d < 0$ but $\frac{a}{b} \leq \frac{e}{d}$. Then $a \cdot d \geq e \cdot b$ and $d \cdot a \geq e \cdot b$ and $d \cdot a \geq b \cdot e$ and $a \cdot d \geq b \cdot e$.

(193) Suppose $b < 0$ and $d < 0$ or $b > 0$ and $d > 0$. Then
- (i) if $a \cdot b < \frac{e}{d}$ or $b \cdot a < \frac{e}{d}$, then $a \cdot d < \frac{e}{b}$ and $d \cdot a < \frac{e}{b}$,
- (ii) if $a \cdot b > \frac{e}{d}$ or $b \cdot a > \frac{e}{d}$, then $a \cdot d > \frac{e}{b}$ and $d \cdot a > \frac{e}{b}$.

(194) Suppose $b < 0$ and $d > 0$ or $b > 0$ and $d < 0$. Then
- (i) if $a \cdot b < \frac{e}{d}$ or $b \cdot a < \frac{e}{d}$, then $a \cdot d > \frac{e}{b}$ and $d \cdot a > \frac{e}{b}$,
- (ii) if $a \cdot b > \frac{e}{d}$ or $b \cdot a > \frac{e}{d}$, then $a \cdot d < \frac{e}{b}$ and $d \cdot a < \frac{e}{b}$.

(195) Suppose $b < 0$ and $d < 0$ or $b > 0$ and $d > 0$. Then
- (i) if $a \cdot b \leq \frac{e}{d}$ or $b \cdot a \leq \frac{e}{d}$, then $a \cdot d \leq \frac{e}{b}$ and $d \cdot a \leq \frac{e}{b}$,
- (ii) if $a \cdot b \geq \frac{e}{d}$ or $b \cdot a \geq \frac{e}{d}$, then $a \cdot d \geq \frac{e}{b}$ and $d \cdot a \geq \frac{e}{b}$.

(196) Suppose $b < 0$ and $d > 0$ or $b > 0$ and $d < 0$. Then
- (i) if $a \cdot b \leq \frac{e}{d}$ or $b \cdot a \leq \frac{e}{d}$, then $a \cdot d \geq \frac{e}{b}$ and $d \cdot a \geq \frac{e}{b}$,
- (ii) if $a \cdot b \geq \frac{e}{d}$ or $b \cdot a \geq \frac{e}{d}$, then $a \cdot d \leq \frac{e}{b}$ and $d \cdot a \leq \frac{e}{b}$.

(197) Suppose $0 < a$ or $0 \leq a$ but $a < b$ or $a \leq b$ but $0 < e$ or $0 \leq e$ and $e \leq d$. Then $a \cdot e \leq b \cdot d$ and $a \cdot e \leq d \cdot b$ and $e \cdot a \leq d \cdot b$ and $e \cdot a \leq b \cdot d$.

(198) Suppose $0 > a$ or $0 \geq a$ but $a > b$ or $a \geq b$ but $0 > e$ or $0 \geq e$ and $e \geq d$. Then $a \cdot e \leq b \cdot d$ and $a \cdot e \leq d \cdot b$ and $e \cdot a \leq d \cdot b$ and $e \cdot a \leq b \cdot d$.

(199) Suppose $0 < a$ but $a \leq b$ or $a < b$ and $0 < e$ and $e < d$ or $0 > a$ but $a \geq b$ or $a > b$ and $0 > e$ and $e > d$. Then $a \cdot e < b \cdot d$ and $a \cdot e < d \cdot b$ and $e \cdot a < d \cdot b$ and $e \cdot a < b \cdot d$.

(200) If $e > 0$ but $a > 0$ or $b < 0$ and $a < b$, then $\frac{e}{a} > \frac{e}{b}$.

(201) If $e > 0$ or $e \geq 0$ but $a > 0$ or $b < 0$ and $a \leq b$, then $\frac{e}{a} \geq \frac{e}{b}$.

(202) If $e < 0$ but $a > 0$ or $b < 0$ and $a < b$, then $\frac{e}{a} < \frac{e}{b}$.

(203) If $e < 0$ or $e \leq 0$ but $a > 0$ or $b < 0$ and $a \leq b$, then $\frac{e}{a} \leq \frac{e}{b}$.

Next we state the proposition

(204) For all subsets X, Y of $\mathbb{R}$ such that $X \neq \emptyset$ and $Y \neq \emptyset$ and for all a, b such that $a \in X$ and $b \in Y$ holds $a \leq b$ there exists d such that for every a such that $a \in X$ holds $a \leq d$ and for every b such that $b \in Y$ holds $d \leq b$.

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Received September 5, 1990

FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

Countable Sets and Hessenberg's Theorem

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Summary. The concept of countable sets is introduced and there are shown some facts which deal with finite and countable sets. Besides, the article includes theorems and lemmas on the sum and the product of infinite cardinals. The most important of them is Hessenberg's theorem which says that for every infinite cardinal $\mathbf{m}$ the product $\mathbf{m} \cdot \mathbf{m}$ is equal to $\mathbf{m}$.

MML Identifier: `CARD_4`.

The papers [20], [16], [3], [11], [9], [15], [5], [8], [7], [21], [19], [2], [1], [10], [22], [12], [13], [18], [14], [17], [4], and [6] provide the terminology and notation for this paper. For simplicity we follow the rules: X, Y are sets, D is a non-empty set, m, n, n_1, n_2, n_3, m_2, m_1 are natural numbers, A, B are ordinal numbers, L, K, M, N are cardinal numbers, x is arbitrary, and f is a function. Next we state a number of propositions:

(1) X is finite if and only if $\overline{\overline{X}}$ is finite.

(2) X is finite if and only if $\overline{\overline{X}} < \aleph_{\mathbf{0}}$.

(3) If X is finite, then $\overline{\overline{X}} \in \aleph_{\mathbf{0}}$ and $\overline{\overline{X}} \in \omega$.

(4) X is finite if and only if there exists n such that $\overline{\overline{X}} = \overline{\overline{n}}$.

(5) $\operatorname{succ} A \setminus \{A\} = A$.

(6) If $A \approx \operatorname{ord}(n)$, then $A = \operatorname{ord}(n)$.

(7) A is finite if and only if $A \in \omega$.

(8) A is not finite if and only if $\omega \subseteq A$.

(9) M is finite if and only if $M \in \aleph_{\mathbf{0}}$.

(10) M is finite if and only if $M < \aleph_{\mathbf{0}}$.

(11) M is not finite if and only if $\aleph_{\mathbf{0}} \subseteq M$.

(12) M is not finite if and only if $\aleph_{\mathbf{0}} \leq M$.

(13) If N is finite and M is not finite, then $N < M$ and $N \leq M$.

ISSN 0777–4028

(14) X is not finite if and only if there exists Y such that $Y \subseteq X$ and $\overline{\overline{Y}} = \aleph_{\mathbf{0}}$.

(15) ω is not finite and $\mathbb{N}$ is not finite.

(16) $\aleph_{\mathbf{0}}$ is not finite.

(17) $X = \emptyset$ if and only if $\overline{\overline{X}} = \overline{\mathbf{0}}$.

(18) $M \neq \overline{\mathbf{0}}$ if and only if $\overline{\mathbf{0}} < M$.

(19) $\overline{\mathbf{0}} \leq M$.

(20) $\overline{\overline{X}} = \overline{\overline{Y}}$ if and only if $X^{+} = Y^{+}$.

(21) $M = N$ if and only if $N^{+} = M^{+}$.

(22) $N < M$ if and only if $N^{+} \leq M$.

(23) $N < M^{+}$ if and only if $N \leq M$.

(24) $\overline{\mathbf{0}} < M$ if and only if $\overline{\mathbf{1}} \leq M$.

(25) $\overline{\mathbf{1}} < M$ if and only if $\overline{\mathbf{2}} \leq M$.

(26) If M is finite but $N \leq M$ or $N < M$, then N is finite.

(27) A is a limit ordinal number if and only if for all B, n such that $B \in A$ holds $B + \mathrm{ord}(n) \in A$.

(28) $A + \mathrm{succ}\,\mathrm{ord}(n) = \mathrm{succ}\,A + \mathrm{ord}(n)$ and $A + \mathrm{ord}(n+1) = \mathrm{succ}\,A + \mathrm{ord}(n)$.

(29) There exists n such that $A \cdot \mathrm{succ}\,\mathbf{1} = A + \mathrm{ord}(n)$.

(30) If A is a limit ordinal number, then $A \cdot \mathrm{succ}\,\mathbf{1} = A$.

(31) If $\omega \subseteq A$, then $\mathbf{1} + A = A$.

Next we state a number of propositions:

(32) If M is not finite, then $\mathrm{ord}(M)$ is a limit ordinal number.

(33) If M is not finite, then $M + M = M$.

(34) If M is not finite but $N \leq M$ or $N < M$, then $M + N = M$ and $N + M = M$.

(35) If X is not finite but $X \approx Y$ or $Y \approx X$, then $X \cup Y \approx X$ and $\overline{\overline{X \cup Y}} = \overline{\overline{X}}$.

(36) If X is not finite and Y is finite, then $X \cup Y \approx X$ and $\overline{\overline{X \cup Y}} = \overline{\overline{X}}$.

(37) If X is not finite but $\overline{\overline{Y}} < \overline{\overline{X}}$ or $\overline{\overline{Y}} \leq \overline{\overline{X}}$, then $X \cup Y \approx X$ and $\overline{\overline{X \cup Y}} = \overline{\overline{X}}$.

(38) If M is finite and N is finite, then $M + N$ is finite.

(39) If M is not finite, then $M + N$ is not finite and $N + M$ is not finite.

(40) If M is finite and N is finite, then $M \cdot N$ is finite.

(41) If $K < L$ and $M < N$ or $K \leq L$ and $M < N$ or $K < L$ and $M \leq N$ or $K \leq L$ and $M \leq N$, then $K + M \leq L + N$ and $M + K \leq L + N$.

(42) If $M < N$ or $M \leq N$, then $K + M \leq K + N$ and $K + M \leq N + K$ and $M + K \leq K + N$ and $M + K \leq N + K$.

Let us consider X. We say that X is countable if and only if:

(Def.1) $\overline{\overline{X}} \leq \aleph_{\mathbf{0}}$.

One can prove the following propositions:

(43) If X is finite, then X is countable.

(44) ω is countable and $\mathbb{N}$ is countable.

(45) X is countable if and only if there exists f such that $\operatorname{dom} f = \mathbb{N}$ and $X \subseteq \operatorname{rng} f$.

(46) If $Y \subseteq X$ and X is countable, then Y is countable.

(47) If X is countable and Y is countable, then $X \cup Y$ is countable.

(48) If X is countable, then $X \cap Y$ is countable and $Y \cap X$ is countable.

(49) If X is countable, then $X \setminus Y$ is countable.

(50) If X is countable and Y is countable, then $X \dot{-} Y$ is countable.

The scheme *Lambda2N* deals with a binary functor $\mathcal{F}$ yielding a natural number and states that:

there exists a function f from $[\!:\mathbb{N}, \mathbb{N}:\!]$ into $\mathbb{N}$ such that for all n, m holds $f(\langle n, m\rangle) = \mathcal{F}(n, m)$

for all values of the parameter.

In the sequel r will denote a real number. Next we state the proposition

(51) $r \neq 0$ or $n = 0$ if and only if $r^n \neq 0$.

Let m, n be natural numbers. Then m^n is a natural number.

One can prove the following propositions:

(52) If $2^{n_1} \cdot (2 \cdot m_1 + 1) = 2^{n_2} \cdot (2 \cdot m_2 + 1)$, then $n_1 = n_2$ and $m_1 = m_2$.

(53) $[\!:\mathbb{N}, \mathbb{N}:\!] \approx \mathbb{N}$ and $\overline{\overline{\mathbb{N}}} = \overline{\overline{[\!:\mathbb{N}, \mathbb{N}:\!]}}$.

(54) $\aleph_{\mathbf{0}} \cdot \aleph_{\mathbf{0}} = \aleph_{\mathbf{0}}$.

(55) If X is countable and Y is countable, then $[\!: X, Y :\!]$ is countable.

(56) $D^1 \approx D$ and $\overline{\overline{D^1}} = \overline{\overline{D}}$.

We now state a number of propositions:

(57) $[\!: D^n, D^m :\!] \approx D^{n+m}$ and $\overline{\overline{[\!: D^n, D^m :\!]}} = \overline{\overline{D^{n+m}}}$.

(58) If D is countable, then D^n is countable.

(59) If $\overline{\overline{\operatorname{dom} f}} \leq M$ and for every x such that $x \in \operatorname{dom} f$ holds $\overline{\overline{f(x)}} \leq N$, then $\overline{\overline{\bigcup f}} \leq M \cdot N$.

(60) If $\overline{\overline{X}} \leq M$ and for every Y such that $Y \in X$ holds $\overline{\overline{Y}} \leq N$, then $\overline{\overline{\bigcup X}} \leq M \cdot N$.

(61) For every f such that $\operatorname{dom} f$ is countable and for every x such that $x \in \operatorname{dom} f$ holds $f(x)$ is countable holds $\bigcup f$ is countable.

(62) If X is countable and for every Y such that $Y \in X$ holds Y is countable, then $\bigcup X$ is countable.

(63) For every f such that $\operatorname{dom} f$ is finite and for every x such that $x \in \operatorname{dom} f$ holds $f(x)$ is finite holds $\bigcup f$ is finite.

(64) If X is finite and for every Y such that $Y \in X$ holds Y is finite, then $\bigcup X$ is finite.

(65) If D is countable, then D^* is countable.

(66) $\aleph_{\mathbf{0}} \leq \overline{\overline{D^*}}$.

Now we present three schemes. The scheme *FraenCoun1* deals with a unary functor $\mathcal{F}$, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(n) : \mathcal{P}[n]\}$ is countable

for all values of the parameters.

The scheme *FraenCoun2* concerns a binary functor $\mathcal{F}$, and a binary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(n_1, n_2) : \mathcal{P}[n_1, n_2]\}$ is countable

for all values of the parameters.

The scheme *FraenCoun3* concerns a ternary functor $\mathcal{F}$, and a ternary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(n_1, n_2, n_3) : \mathcal{P}[n_1, n_2, n_3]\}$ is countable

for all values of the parameters.

The following propositions are true:

(67) $\aleph_{\mathbf{0}} \cdot \overline{\overline{n}} \leq \aleph_{\mathbf{0}}$ and $\overline{\overline{n}} \cdot \aleph_{\mathbf{0}} \leq \aleph_{\mathbf{0}}$.

(68) If $K < L$ and $M < N$ or $K \leq L$ and $M < N$ or $K < L$ and $M \leq N$ or $K \leq L$ and $M \leq N$, then $K \cdot M \leq L \cdot N$ and $M \cdot K \leq L \cdot N$.

(69) If $M < N$ or $M \leq N$, then $K \cdot M \leq K \cdot N$ and $K \cdot M \leq N \cdot K$ and $M \cdot K \leq K \cdot N$ and $M \cdot K \leq N \cdot K$.

(70) If $K < L$ and $M < N$ or $K \leq L$ and $M < N$ or $K < L$ and $M \leq N$ or $K \leq L$ and $M \leq N$, then $K = \overline{\mathbf{0}}$ or $K^M \leq L^N$.

(71) If $M < N$ or $M \leq N$, then $K = \overline{\mathbf{0}}$ or $K^M \leq K^N$ and $M^K \leq N^K$.

(72) $M \leq M + N$ and $N \leq M + N$.

(73) If $N \neq \overline{\mathbf{0}}$, then $M \leq M \cdot N$ and $M \leq N \cdot M$.

(74) If $K < L$ and $M < N$, then $K + M < L + N$ and $M + K < L + N$.

(75) If $K + M < K + N$, then $M < N$.

(76) If $\overline{\overline{X}} + \overline{\overline{Y}} = \overline{\overline{X}}$ and $\overline{\overline{Y}} < \overline{\overline{X}}$, then $\overline{\overline{X \setminus Y}} = \overline{\overline{X}}$.

One can prove the following propositions:

(77) If M is not finite, then $M \cdot M = M$.

(78) If M is not finite and $\overline{\mathbf{0}} < N$ but $N \leq M$ or $N < M$, then $M \cdot N = M$ and $N \cdot M = M$.

(79) If M is not finite but $N \leq M$ or $N < M$, then $M \cdot N \leq M$ and $N \cdot M \leq M$.

(80) If X is not finite, then $[\!: X, X :\!] \approx X$ and $\overline{\overline{[\!: X, X :\!]}} = \overline{\overline{X}}$.

(81) If X is not finite and Y is finite and $Y \neq \emptyset$, then $[\!: X, Y :\!] \approx X$ and $\overline{\overline{[\!: X, Y :\!]}} = \overline{\overline{X}}$.

(82) If $K < L$ and $M < N$, then $K \cdot M < L \cdot N$ and $M \cdot K < L \cdot N$.

(83) If $K \cdot M < K \cdot N$, then $M < N$.

(84) If X is not finite, then $\overline{\overline{X}} = \aleph_{\mathbf{0}} \cdot \overline{\overline{X}}$.

(85) If $X \neq \emptyset$ and X is finite and Y is not finite, then $\overline{\overline{Y}} \cdot \overline{\overline{X}} = \overline{\overline{Y}}$.

(86) If D is not finite and $n \neq 0$, then $D^n \approx D$ and $\overline{\overline{D^n}} = \overline{\overline{D}}$.

(87) If D is not finite, then $\overline{\overline{D}} = \overline{\overline{D^*}}$.

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Received September 5, 1990

FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

The Limit of a Real Function at a Point

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Summary. We define the proper and the improper limit of a real function at a point. The main properties of the operations on the limit of a function are proved. The connection between the one-side limits and the limit of a function at a point are exposed. Equivalent Cauchy and Heine characterizations of the limit of a real function at a point are proved.

MML Identifier: `LIMFUNC3`.

The papers [17], [5], [1], [2], [3], [15], [13], [6], [8], [14], [18], [16], [4], [10], [11], [12], [7], and [9] provide the notation and terminology for this paper. For simplicity we adopt the following convention: r, r_1, r_2, g, g_1, g_2, x_0 will be real numbers, n, k will be natural numbers, s_1 will be a sequence of real numbers, and f, f_1, f_2 will be partial functions from $\mathbb{R}$ to $\mathbb{R}$. The following propositions are true:

(1) If $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]-\infty, x_0[$ or $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]x_0, +\infty[$, then $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$.

(2) Suppose for every n holds $0 < |x_0 - s_1(n)|$ and $|x_0 - s_1(n)| < \frac{1}{n+1}$ and $s_1(n) \in \operatorname{dom} f$. Then s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$.

(3) Suppose s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$. Then for every r such that $0 < r$ there exists n such that for every k such that $n \leq k$ holds $0 < |x_0 - s_1(k)|$ and $|x_0 - s_1(k)| < r$ and $s_1(k) \in \operatorname{dom} f$.

(4) If $0 < r$, then $]x_0 - r, x_0 + r[\setminus \{x_0\} =]x_0 - r, x_0[\cup]x_0, x_0 + r[$.

(5) Suppose $0 < r_2$ and $]x_0 - r_2, x_0[\cup]x_0, x_0 + r_2[\subseteq \operatorname{dom} f$. Then for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$.

[1]Supported by RPBP.III-24.C8

ISSN 0777–4028

(6) If for every n holds $x_0 - \frac{1}{n+1} < s_1(n)$ and $s_1(n) < x_0$ and $s_1(n) \in \operatorname{dom} f$, then s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$.

(7) If s_1 is convergent and $\lim s_1 = x_0$ and $0 < g$, then there exists k such that for every n such that $k \leq n$ holds $x_0 - g < s_1(n)$ and $s_1(n) < x_0 + g$.

(8) The following conditions are equivalent:
 (i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,
 (ii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$.

We now define three new predicates. Let us consider f, x_0. We say that f is convergent in x_0 if and only if:

(Def.1) (i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,
 (ii) there exists g such that for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

We say that f is divergent to $+\infty$ in x_0 if and only if:

(Def.2) (i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,
 (ii) for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$ holds $f \cdot s_1$ is divergent to $+\infty$.

We say that f is divergent to $-\infty$ in x_0 if and only if:

(Def.3) (i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,
 (ii) for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$ holds $f \cdot s_1$ is divergent to $-\infty$.

The following propositions are true:

(9) f is convergent in x_0 if and only if the following conditions are satisfied:
 (i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,
 (ii) there exists g such that for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

(10) f is divergent to $+\infty$ in x_0 if and only if the following conditions are satisfied:

(i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(ii) for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$ holds $f \cdot s_1$ is divergent to $+\infty$.

(11) f is divergent to $-\infty$ in x_0 if and only if the following conditions are satisfied:

(i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(ii) for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$ holds $f \cdot s_1$ is divergent to $-\infty$.

(12) f is convergent in x_0 if and only if the following conditions are satisfied:

(i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(ii) there exists g such that for every g_1 such that $0 < g_1$ there exists g_2 such that $0 < g_2$ and for every r_1 such that $0 < |x_0 - r_1|$ and $|x_0 - r_1| < g_2$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < g_1$.

(13) f is divergent to $+\infty$ in x_0 if and only if the following conditions are satisfied:

(i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(ii) for every g_1 there exists g_2 such that $0 < g_2$ and for every r_1 such that $0 < |x_0 - r_1|$ and $|x_0 - r_1| < g_2$ and $r_1 \in \operatorname{dom} f$ holds $g_1 < f(r_1)$.

(14) f is divergent to $-\infty$ in x_0 if and only if the following conditions are satisfied:

(i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(ii) for every g_1 there exists g_2 such that $0 < g_2$ and for every r_1 such that $0 < |x_0 - r_1|$ and $|x_0 - r_1| < g_2$ and $r_1 \in \operatorname{dom} f$ holds $f(r_1) < g_1$.

(15) f is divergent to $+\infty$ in x_0 if and only if f is left divergent to $+\infty$ in x_0 and f is right divergent to $+\infty$ in x_0.

(16) f is divergent to $-\infty$ in x_0 if and only if f is left divergent to $-\infty$ in x_0 and f is right divergent to $-\infty$ in x_0.

(17) Suppose that

(i) f_1 is divergent to $+\infty$ in x_0,

(ii) f_2 is divergent to $+\infty$ in x_0,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$.

Then $f_1 + f_2$ is divergent to $+\infty$ in x_0 and $f_1 f_2$ is divergent to $+\infty$ in x_0.

(18) Suppose that
(i) f_1 is divergent to $-\infty$ in x_0,
(ii) f_2 is divergent to $-\infty$ in x_0,
(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$.
Then $f_1 + f_2$ is divergent to $-\infty$ in x_0 and $f_1 f_2$ is divergent to $+\infty$ in x_0.

(19) Suppose that
(i) f_1 is divergent to $+\infty$ in x_0,
(ii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom}(f_1 + f_2)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom}(f_1 + f_2)$,
(iii) there exists r such that $0 < r$ and f_2 is lower bounded on $]x_0 - r, x_0[\cup]x_0, x_0 + r[$.
Then $f_1 + f_2$ is divergent to $+\infty$ in x_0.

(20) Suppose that
(i) f_1 is divergent to $+\infty$ in x_0,
(ii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom}(f_1 f_2)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom}(f_1 f_2)$,
(iii) there exist r, r_1 such that $0 < r$ and $0 < r_1$ and for every g such that $g \in \operatorname{dom} f_2 \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[)$ holds $r_1 \leq f_2(g)$.
Then $f_1 f_2$ is divergent to $+\infty$ in x_0.

(21) (i) If f is divergent to $+\infty$ in x_0 and $r > 0$, then rf is divergent to $+\infty$ in x_0,
(ii) if f is divergent to $+\infty$ in x_0 and $r < 0$, then rf is divergent to $-\infty$ in x_0,
(iii) if f is divergent to $-\infty$ in x_0 and $r > 0$, then rf is divergent to $-\infty$ in x_0,
(iv) if f is divergent to $-\infty$ in x_0 and $r < 0$, then rf is divergent to $+\infty$ in x_0.

(22) If f is divergent to $+\infty$ in x_0 or f is divergent to $-\infty$ in x_0, then $|f|$ is divergent to $+\infty$ in x_0.

(23) Suppose that
(i) there exists r such that $0 < r$ and f is non-decreasing on $]x_0 - r, x_0[$ and f is non-increasing on $]x_0, x_0 + r[$ and f is not upper bounded on $]x_0 - r, x_0[$ and f is not upper bounded on $]x_0, x_0 + r[$,
(ii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$.
Then f is divergent to $+\infty$ in x_0.

(24) Suppose that
(i) there exists r such that $0 < r$ and f is increasing on $]x_0 - r, x_0[$ and f is decreasing on $]x_0, x_0 + r[$ and f is not upper bounded on $]x_0 - r, x_0[$

and f is not upper bounded on $]x_0, x_0 + r[$,

(ii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$.

Then f is divergent to $+\infty$ in x_0.

(25) Suppose that

(i) there exists r such that $0 < r$ and f is non-increasing on $]x_0 - r, x_0[$ and f is non-decreasing on $]x_0, x_0 + r[$ and f is not lower bounded on $]x_0 - r, x_0[$ and f is not lower bounded on $]x_0, x_0 + r[$,

(ii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$.

Then f is divergent to $-\infty$ in x_0.

(26) Suppose that

(i) there exists r such that $0 < r$ and f is decreasing on $]x_0 - r, x_0[$ and f is increasing on $]x_0, x_0 + r[$ and f is not lower bounded on $]x_0 - r, x_0[$ and f is not lower bounded on $]x_0, x_0 + r[$,

(ii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$.

Then f is divergent to $-\infty$ in x_0.

(27) Suppose that

(i) f_1 is divergent to $+\infty$ in x_0,

(ii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(iii) there exists r such that $0 < r$ and $\operatorname{dom} f \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[) \subseteq \operatorname{dom} f_1 \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[)$ and for every g such that $g \in \operatorname{dom} f \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[)$ holds $f_1(g) \leq f(g)$.

Then f is divergent to $+\infty$ in x_0.

(28) Suppose that

(i) f_1 is divergent to $-\infty$ in x_0,

(ii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(iii) there exists r such that $0 < r$ and $\operatorname{dom} f \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[) \subseteq \operatorname{dom} f_1 \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[)$ and for every g such that $g \in \operatorname{dom} f \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[)$ holds $f(g) \leq f_1(g)$.

Then f is divergent to $-\infty$ in x_0.

(29) Suppose that

(i) f_1 is divergent to $+\infty$ in x_0,

(ii) there exists r such that $0 < r$ and $]x_0 - r, x_0[\cup]x_0, x_0 + r[\subseteq \operatorname{dom} f \cap \operatorname{dom} f_1$ and for every g such that $g \in]x_0 - r, x_0[\cup]x_0, x_0 + r[$ holds $f_1(g) \leq f(g)$.

Then f is divergent to $+\infty$ in x_0.

(30) Suppose that

(i) f_1 is divergent to $-\infty$ in x_0,

(ii) there exists r such that $0 < r$ and $]x_0 - r, x_0[\cup]x_0, x_0 + r[\subseteq \mathrm{dom}\, f \cap \mathrm{dom}\, f_1$ and for every g such that $g \in]x_0 - r, x_0[\cup]x_0, x_0 + r[$ holds $f(g) \leq f_1(g)$.

Then f is divergent to $-\infty$ in x_0.

Let us consider f, x_0. Let us assume that f is convergent in x_0. The functor $\lim_{x_0} f$ yields a real number and is defined by:

(Def.4) for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f \setminus \{x_0\}$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = \lim_{x_0} f$.

The following propositions are true:

(31) If f is convergent in x_0, then $\lim_{x_0} f = g$ if and only if for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f \setminus \{x_0\}$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

(32) Suppose f is convergent in x_0. Then $\lim_{x_0} f = g$ if and only if for every g_1 such that $0 < g_1$ there exists g_2 such that $0 < g_2$ and for every r_1 such that $0 < |x_0 - r_1|$ and $|x_0 - r_1| < g_2$ and $r_1 \in \mathrm{dom}\, f$ holds $|f(r_1) - g| < g_1$.

(33) If f is convergent in x_0, then f is left convergent in x_0 and f is right convergent in x_0 and $\lim_{x_0^-} f = \lim_{x_0^+} f$ and $\lim_{x_0} f = \lim_{x_0^-} f$ and $\lim_{x_0} f = \lim_{x_0^+} f$.

(34) If f is left convergent in x_0 and f is right convergent in x_0 and $\lim_{x_0^-} f = \lim_{x_0^+} f$, then f is convergent in x_0 and $\lim_{x_0} f = \lim_{x_0^-} f$ and $\lim_{x_0} f = \lim_{x_0^+} f$.

(35) If f is convergent in x_0, then rf is convergent in x_0 and $\lim_{x_0}(rf) = r \cdot (\lim_{x_0} f)$.

(36) If f is convergent in x_0, then $-f$ is convergent in x_0 and $\lim_{x_0}(-f) = -\lim_{x_0} f$.

(37) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is convergent in x_0,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_1 + f_2)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_1 + f_2)$.

Then $f_1 + f_2$ is convergent in x_0 and $\lim_{x_0}(f_1 + f_2) = \lim_{x_0} f_1 + \lim_{x_0} f_2$.

(38) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is convergent in x_0,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_1 - f_2)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_1 - f_2)$.

Then $f_1 - f_2$ is convergent in x_0 and $\lim_{x_0}(f_1 - f_2) = \lim_{x_0} f_1 - \lim_{x_0} f_2$.

(39) If f is convergent in x_0 and $f^{-1}\{0\} = \emptyset$ and $\lim_{x_0} f \neq 0$, then $\frac{1}{f}$ is convergent in x_0 and $\lim_{x_0} \frac{1}{f} = (\lim_{x_0} f)^{-1}$.

(40) If f is convergent in x_0, then $|f|$ is convergent in x_0 and $\lim_{x_0}|f| = |\lim_{x_0} f|$.

(41) Suppose that

(i) f is convergent in x_0,

(ii) $\lim_{x_0} f \neq 0$,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$ and $f(g_1) \neq 0$ and $f(g_2) \neq 0$.

Then $\frac{1}{f}$ is convergent in x_0 and $\lim_{x_0} \frac{1}{f} = (\lim_{x_0} f)^{-1}$.

(42) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is convergent in x_0,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom}(f_1 f_2)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom}(f_1 f_2)$.

Then $f_1 f_2$ is convergent in x_0 and $\lim_{x_0}(f_1 f_2) = (\lim_{x_0} f_1) \cdot (\lim_{x_0} f_2)$.

(43) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is convergent in x_0,

(iii) $\lim_{x_0} f_2 \neq 0$,

(iv) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} \frac{f_1}{f_2}$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} \frac{f_1}{f_2}$.

Then $\frac{f_1}{f_2}$ is convergent in x_0 and $\lim_{x_0} \frac{f_1}{f_2} = \frac{\lim_{x_0} f_1}{\lim_{x_0} f_2}$.

(44) Suppose that

(i) f_1 is convergent in x_0,

(ii) $\lim_{x_0} f_1 = 0$,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom}(f_1 f_2)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom}(f_1 f_2)$,

(iv) there exists r such that $0 < r$ and f_2 is bounded on $]x_0 - r, x_0[\cup]x_0, x_0 + r[$.

Then $f_1 f_2$ is convergent in x_0 and $\lim_{x_0}(f_1 f_2) = 0$.

(45) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is convergent in x_0,

(iii) $\lim_{x_0} f_1 = \lim_{x_0} f_2$,

(iv) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(v) there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap (]x_0-r, x_0[\cup]x_0, x_0+r[)$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$ but $\operatorname{dom} f_1 \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[) \subseteq \operatorname{dom} f_2 \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[)$ and $\operatorname{dom} f \cap (]x_0-r, x_0[\cup]x_0, x_0+r[) \subseteq \operatorname{dom} f_1 \cap (]x_0-r, x_0[\cup]x_0, x_0+r[)$ or $\operatorname{dom} f_2 \cap (]x_0-r, x_0[\cup]x_0, x_0+r[) \subseteq \operatorname{dom} f_1 \cap (]x_0-r, x_0[\cup]x_0, x_0+r[)$ and $\operatorname{dom} f \cap (]x_0-r, x_0[\cup]x_0, x_0+r[) \subseteq \operatorname{dom} f_2 \cap (]x_0-r, x_0[\cup]x_0, x_0+r[)$.
Then f is convergent in x_0 and $\lim_{x_0} f = \lim_{x_0} f_1$.

(46) Suppose that
(i) f_1 is convergent in x_0,
(ii) f_2 is convergent in x_0,
(iii) $\lim_{x_0} f_1 = \lim_{x_0} f_2$,
(iv) there exists r such that $0 < r$ and $]x_0-r, x_0[\cup]x_0, x_0+r[\subseteq (\operatorname{dom} f_1 \cap \operatorname{dom} f_2) \cap \operatorname{dom} f$ and for every g such that $g \in]x_0-r, x_0[\cup]x_0, x_0+r[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$.
Then f is convergent in x_0 and $\lim_{x_0} f = \lim_{x_0} f_1$.

(47) Suppose that
(i) f_1 is convergent in x_0,
(ii) f_2 is convergent in x_0,
(iii) there exists r such that $0 < r$ but $\operatorname{dom} f_1 \cap (]x_0-r, x_0[\cup]x_0, x_0+r[) \subseteq \operatorname{dom} f_2 \cap (]x_0-r, x_0[\cup]x_0, x_0+r[)$ and for every g such that $g \in \operatorname{dom} f_1 \cap (]x_0-r, x_0[\cup]x_0, x_0+r[)$ holds $f_1(g) \leq f_2(g)$ or $\operatorname{dom} f_2 \cap (]x_0-r, x_0[\cup]x_0, x_0+r[) \subseteq \operatorname{dom} f_1 \cap (]x_0-r, x_0[\cup]x_0, x_0+r[)$ and for every g such that $g \in \operatorname{dom} f_2 \cap (]x_0-r, x_0[\cup]x_0, x_0+r[)$ holds $f_1(g) \leq f_2(g)$.
Then $\lim_{x_0} f_1 \leq \lim_{x_0} f_2$.

(48) Suppose that
(i) f is divergent to $+\infty$ in x_0 or f is divergent to $-\infty$ in x_0,
(ii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$ and $f(g_1) \neq 0$ and $f(g_2) \neq 0$.
Then $\frac{1}{f}$ is convergent in x_0 and $\lim_{x_0} \frac{1}{f} = 0$.

(49) Suppose that
(i) f is convergent in x_0,
(ii) $\lim_{x_0} f = 0$,
(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$ and $f(g_1) \neq 0$ and $f(g_2) \neq 0$,
(iv) there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap (]x_0-r, x_0[\cup]x_0, x_0+r[)$ holds $0 \leq f(g)$.
Then $\frac{1}{f}$ is divergent to $+\infty$ in x_0.

(50) Suppose that
(i) f is convergent in x_0,
(ii) $\lim_{x_0} f = 0$,
(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and

$g_2 \in \operatorname{dom} f$ and $f(g_1) \neq 0$ and $f(g_2) \neq 0$,

(iv) there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[)$ holds $f(g) \leq 0$.

Then $\frac{1}{f}$ is divergent to $-\infty$ in x_0.

(51) If f is convergent in x_0 and $\lim_{x_0} f = 0$ and there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[)$ holds $0 < f(g)$, then $\frac{1}{f}$ is divergent to $+\infty$ in x_0.

(52) If f is convergent in x_0 and $\lim_{x_0} f = 0$ and there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[)$ holds $f(g) < 0$, then $\frac{1}{f}$ is divergent to $-\infty$ in x_0.

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Received September 5, 1990

FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

The Limit of a Composition of Real Functions

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Summary. The theorem on the proper and the improper limit of a composition of real functions at a point, at infinity and one-side limits at a point are presented.

MML Identifier: `LIMFUNC4`.

The terminology and notation used in this paper have been introduced in the following articles: [17], [4], [1], [2], [15], [13], [5], [8], [14], [16], [3], [10], [11], [12], [7], [9], and [6]. We follow a convention: r, r_1, r_2, g, g_1, g_2, x_0 will be real numbers and f_1, f_2 will be partial functions from $\mathbb{R}$ to $\mathbb{R}$. The following propositions are true:

(1) Let s be a sequence of real numbers. Then for every set X such that $\operatorname{rng} s \subseteq \operatorname{dom}(f_2 \cdot f_1) \cap X$ holds $\operatorname{rng} s \subseteq \operatorname{dom}(f_2 \cdot f_1)$ and $\operatorname{rng} s \subseteq X$ and $\operatorname{rng} s \subseteq \operatorname{dom} f_1$ and $\operatorname{rng} s \subseteq \operatorname{dom} f_1 \cap X$ and $\operatorname{rng}(f_1 \cdot s) \subseteq \operatorname{dom} f_2$.

(2) For every sequence of real numbers s and for every set X such that $\operatorname{rng} s \subseteq \operatorname{dom}(f_2 \cdot f_1) \setminus X$ holds $\operatorname{rng} s \subseteq \operatorname{dom}(f_2 \cdot f_1)$ and $\operatorname{rng} s \subseteq \operatorname{dom} f_1$ and $\operatorname{rng} s \subseteq \operatorname{dom} f_1 \setminus X$ and $\operatorname{rng}(f_1 \cdot s) \subseteq \operatorname{dom} f_2$.

(3) If f_1 is divergent in $+\infty$ to $+\infty$ and f_2 is divergent in $+\infty$ to $+\infty$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $+\infty$.

(4) If f_1 is divergent in $+\infty$ to $+\infty$ and f_2 is divergent in $+\infty$ to $-\infty$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $-\infty$.

(5) If f_1 is divergent in $+\infty$ to $-\infty$ and f_2 is divergent in $-\infty$ to $+\infty$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $+\infty$.

[1]Supported by RPBP.III-24.C8

ISSN 0777–4028

(6) If f_1 is divergent in $+\infty$ to $-\infty$ and f_2 is divergent in $-\infty$ to $-\infty$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $-\infty$.

(7) If f_1 is divergent in $-\infty$ to $+\infty$ and f_2 is divergent in $+\infty$ to $+\infty$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $+\infty$.

(8) If f_1 is divergent in $-\infty$ to $+\infty$ and f_2 is divergent in $+\infty$ to $-\infty$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $-\infty$.

(9) If f_1 is divergent in $-\infty$ to $-\infty$ and f_2 is divergent in $-\infty$ to $+\infty$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $+\infty$.

(10) If f_1 is divergent in $-\infty$ to $-\infty$ and f_2 is divergent in $-\infty$ to $-\infty$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $-\infty$.

(11) If f_1 is left divergent to $+\infty$ in x_0 and f_2 is divergent in $+\infty$ to $+\infty$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is left divergent to $+\infty$ in x_0.

(12) If f_1 is left divergent to $+\infty$ in x_0 and f_2 is divergent in $+\infty$ to $-\infty$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is left divergent to $-\infty$ in x_0.

(13) If f_1 is left divergent to $-\infty$ in x_0 and f_2 is divergent in $-\infty$ to $+\infty$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is left divergent to $+\infty$ in x_0.

(14) If f_1 is left divergent to $-\infty$ in x_0 and f_2 is divergent in $-\infty$ to $-\infty$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is left divergent to $-\infty$ in x_0.

(15) If f_1 is right divergent to $+\infty$ in x_0 and f_2 is divergent in $+\infty$ to $+\infty$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is right divergent to $+\infty$ in x_0.

(16) If f_1 is right divergent to $+\infty$ in x_0 and f_2 is divergent in $+\infty$ to $-\infty$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is right divergent to $-\infty$ in x_0.

(17) If f_1 is right divergent to $-\infty$ in x_0 and f_2 is divergent in $-\infty$ to $+\infty$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is right divergent to $+\infty$ in x_0.

(18) If f_1 is right divergent to $-\infty$ in x_0 and f_2 is divergent in $-\infty$ to $-\infty$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is right divergent to $-\infty$ in x_0.

(19) Suppose that

(i) f_1 is left convergent in x_0,

(ii) f_2 is left divergent to $+\infty$ in $\lim_{x_0^-} f_1$,

(iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0 - g, x_0[$ holds $f_1(r) < \lim_{x_0^-} f_1$.
Then $f_2 \cdot f_1$ is left divergent to $+\infty$ in x_0.

(20) Suppose that
(i) f_1 is left convergent in x_0,
(ii) f_2 is left divergent to $-\infty$ in $\lim_{x_0^-} f_1$,
(iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0 - g, x_0[$ holds $f_1(r) < \lim_{x_0^-} f_1$.
Then $f_2 \cdot f_1$ is left divergent to $-\infty$ in x_0.

(21) Suppose that
(i) f_1 is left convergent in x_0,
(ii) f_2 is right divergent to $+\infty$ in $\lim_{x_0^-} f_1$,
(iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0 - g, x_0[$ holds $\lim_{x_0^-} f_1 < f_1(r)$.
Then $f_2 \cdot f_1$ is left divergent to $+\infty$ in x_0.

(22) Suppose that
(i) f_1 is left convergent in x_0,
(ii) f_2 is right divergent to $-\infty$ in $\lim_{x_0^-} f_1$,
(iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0 - g, x_0[$ holds $\lim_{x_0^-} f_1 < f_1(r)$.
Then $f_2 \cdot f_1$ is left divergent to $-\infty$ in x_0.

(23) Suppose that
(i) f_1 is right convergent in x_0,
(ii) f_2 is right divergent to $+\infty$ in $\lim_{x_0^+} f_1$,
(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0, x_0 + g[$ holds $\lim_{x_0^+} f_1 < f_1(r)$.
Then $f_2 \cdot f_1$ is right divergent to $+\infty$ in x_0.

(24) Suppose that
(i) f_1 is right convergent in x_0,
(ii) f_2 is right divergent to $-\infty$ in $\lim_{x_0^+} f_1$,
(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0, x_0 + g[$ holds $\lim_{x_0^+} f_1 < f_1(r)$.

Then $f_2 \cdot f_1$ is right divergent to $-\infty$ in x_0.

(25) Suppose that

(i) f_1 is right convergent in x_0,

(ii) f_2 is left divergent to $+\infty$ in $\lim_{x_0^+} f_1$,

(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0, x_0 + g[$ holds $f_1(r) < \lim_{x_0^+} f_1$.

Then $f_2 \cdot f_1$ is right divergent to $+\infty$ in x_0.

(26) Suppose that

(i) f_1 is right convergent in x_0,

(ii) f_2 is left divergent to $-\infty$ in $\lim_{x_0^+} f_1$,

(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0, x_0 + g[$ holds $f_1(r) < \lim_{x_0^+} f_1$.

Then $f_2 \cdot f_1$ is right divergent to $-\infty$ in x_0.

(27) If f_1 is convergent in $+\infty$ and f_2 is left divergent to $+\infty$ in $\lim_{+\infty} f_1$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]r, +\infty[$ holds $f_1(g) < \lim_{+\infty} f_1$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $+\infty$.

(28) If f_1 is convergent in $+\infty$ and f_2 is left divergent to $-\infty$ in $\lim_{+\infty} f_1$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]r, +\infty[$ holds $f_1(g) < \lim_{+\infty} f_1$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $-\infty$.

(29) If f_1 is convergent in $+\infty$ and f_2 is right divergent to $+\infty$ in $\lim_{+\infty} f_1$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]r, +\infty[$ holds $\lim_{+\infty} f_1 < f_1(g)$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $+\infty$.

(30) If f_1 is convergent in $+\infty$ and f_2 is right divergent to $-\infty$ in $\lim_{+\infty} f_1$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]r, +\infty[$ holds $\lim_{+\infty} f_1 < f_1(g)$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $-\infty$.

(31) If f_1 is convergent in $-\infty$ and f_2 is left divergent to $+\infty$ in $\lim_{-\infty} f_1$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]-\infty, r[$ holds $f_1(g) < \lim_{-\infty} f_1$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $+\infty$.

Next we state a number of propositions:

(32) If f_1 is convergent in $-\infty$ and f_2 is left divergent to $-\infty$ in $\lim_{-\infty} f_1$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]-\infty, r[$ holds $f_1(g) < \lim_{-\infty} f_1$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $-\infty$.

(33) If f_1 is convergent in $-\infty$ and f_2 is right divergent to $+\infty$ in $\lim_{-\infty} f_1$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]-\infty, r[$ holds $\lim_{-\infty} f_1 < f_1(g)$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $+\infty$.

(34) If f_1 is convergent in $-\infty$ and f_2 is right divergent to $-\infty$ in $\lim_{-\infty} f_1$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]-\infty, r[$ holds $\lim_{-\infty} f_1 < f_1(g)$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $-\infty$.

(35) Suppose f_1 is divergent to $+\infty$ in x_0 and f_2 is divergent in $+\infty$ to $+\infty$ and for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$. Then $f_2 \cdot f_1$ is divergent to $+\infty$ in x_0.

(36) Suppose f_1 is divergent to $+\infty$ in x_0 and f_2 is divergent in $+\infty$ to $-\infty$ and for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$. Then $f_2 \cdot f_1$ is divergent to $-\infty$ in x_0.

(37) Suppose f_1 is divergent to $-\infty$ in x_0 and f_2 is divergent in $-\infty$ to $+\infty$ and for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$. Then $f_2 \cdot f_1$ is divergent to $+\infty$ in x_0.

(38) Suppose f_1 is divergent to $-\infty$ in x_0 and f_2 is divergent in $-\infty$ to $-\infty$ and for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$. Then $f_2 \cdot f_1$ is divergent to $-\infty$ in x_0.

(39) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is divergent to $+\infty$ in $\lim_{x_0} f_1$,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap (]x_0 - g, x_0[\cup]x_0, x_0 + g[)$ holds $f_1(r) \neq \lim_{x_0} f_1$.

Then $f_2 \cdot f_1$ is divergent to $+\infty$ in x_0.

(40) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is divergent to $-\infty$ in $\lim_{x_0} f_1$,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap (]x_0 - g, x_0[\cup]x_0, x_0 + g[)$ holds $f_1(r) \neq \lim_{x_0} f_1$.

Then $f_2 \cdot f_1$ is divergent to $-\infty$ in x_0.

(41) Suppose that

(i) f_1 is convergent in x_0,
(ii) f_2 is right divergent to $+\infty$ in $\lim_{x_0} f_1$,
(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap (]x_0 - g, x_0[\cup]x_0, x_0 + g[)$ holds $f_1(r) > \lim_{x_0} f_1$.

Then $f_2 \cdot f_1$ is divergent to $+\infty$ in x_0.

(42) Suppose that
(i) f_1 is convergent in x_0,
(ii) f_2 is right divergent to $-\infty$ in $\lim_{x_0} f_1$,
(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap (]x_0 - g, x_0[\cup]x_0, x_0 + g[)$ holds $f_1(r) > \lim_{x_0} f_1$.

Then $f_2 \cdot f_1$ is divergent to $-\infty$ in x_0.

(43) Suppose that
(i) f_1 is right convergent in x_0,
(ii) f_2 is divergent to $+\infty$ in $\lim_{x_0^+} f_1$,
(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0, x_0 + g[$ holds $f_1(r) \neq \lim_{x_0^+} f_1$.

Then $f_2 \cdot f_1$ is right divergent to $+\infty$ in x_0.

(44) Suppose that
(i) f_1 is right convergent in x_0,
(ii) f_2 is divergent to $-\infty$ in $\lim_{x_0^+} f_1$,
(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0, x_0 + g[$ holds $f_1(r) \neq \lim_{x_0^+} f_1$.

Then $f_2 \cdot f_1$ is right divergent to $-\infty$ in x_0.

(45) If f_1 is convergent in $+\infty$ and f_2 is divergent to $+\infty$ in $\lim_{+\infty} f_1$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]r, +\infty[$ holds $f_1(g) \neq \lim_{+\infty} f_1$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $+\infty$.

(46) If f_1 is convergent in $+\infty$ and f_2 is divergent to $-\infty$ in $\lim_{+\infty} f_1$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]r, +\infty[$ holds $f_1(g) \neq \lim_{+\infty} f_1$, then $f_2 \cdot f_1$ is divergent in $+\infty$ to $-\infty$.

(47) If f_1 is convergent in $-\infty$ and f_2 is divergent to $+\infty$ in $\lim_{-\infty} f_1$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and

there exists r such that for every g such that $g \in \operatorname{dom} f_1 \cap]-\infty, r[$ holds $f_1(g) \neq \lim_{-\infty} f_1$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $+\infty$.

(48) If f_1 is convergent in $-\infty$ and f_2 is divergent to $-\infty$ in $\lim_{-\infty} f_1$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \operatorname{dom} f_1 \cap]-\infty, r[$ holds $f_1(g) \neq \lim_{-\infty} f_1$, then $f_2 \cdot f_1$ is divergent in $-\infty$ to $-\infty$.

(49) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is left divergent to $+\infty$ in $\lim_{x_0} f_1$,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \operatorname{dom} f_1 \cap (]x_0 - g, x_0[\cup]x_0, x_0 + g[)$ holds $f_1(r) < \lim_{x_0} f_1$.

Then $f_2 \cdot f_1$ is divergent to $+\infty$ in x_0.

(50) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is left divergent to $-\infty$ in $\lim_{x_0} f_1$,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \operatorname{dom} f_1 \cap (]x_0 - g, x_0[\cup]x_0, x_0 + g[)$ holds $f_1(r) < \lim_{x_0} f_1$.

Then $f_2 \cdot f_1$ is divergent to $-\infty$ in x_0.

(51) Suppose that

(i) f_1 is left convergent in x_0,

(ii) f_2 is divergent to $+\infty$ in $\lim_{x_0^-} f_1$,

(iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \operatorname{dom} f_1 \cap]x_0 - g, x_0[$ holds $f_1(r) \neq \lim_{x_0^-} f_1$.

Then $f_2 \cdot f_1$ is left divergent to $+\infty$ in x_0.

(52) Suppose that

(i) f_1 is left convergent in x_0,

(ii) f_2 is divergent to $-\infty$ in $\lim_{x_0^-} f_1$,

(iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \operatorname{dom} f_1 \cap]x_0 - g, x_0[$ holds $f_1(r) \neq \lim_{x_0^-} f_1$.

Then $f_2 \cdot f_1$ is left divergent to $-\infty$ in x_0.

(53) If f_1 is divergent in $+\infty$ to $+\infty$ and f_2 is convergent in $+\infty$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is convergent in $+\infty$ and $\lim_{+\infty}(f_2 \cdot f_1) = \lim_{+\infty} f_2$.

(54) If f_1 is divergent in $+\infty$ to $-\infty$ and f_2 is convergent in $-\infty$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is convergent in $+\infty$ and $\lim_{+\infty}(f_2 \cdot f_1) = \lim_{-\infty} f_2$.

(55) If f_1 is divergent in $-\infty$ to $+\infty$ and f_2 is convergent in $+\infty$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is convergent in $-\infty$ and $\lim_{-\infty}(f_2 \cdot f_1) = \lim_{+\infty} f_2$.

(56) If f_1 is divergent in $-\infty$ to $-\infty$ and f_2 is convergent in $-\infty$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is convergent in $-\infty$ and $\lim_{-\infty}(f_2 \cdot f_1) = \lim_{-\infty} f_2$.

(57) If f_1 is left divergent to $+\infty$ in x_0 and f_2 is convergent in $+\infty$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is left convergent in x_0 and $\lim_{x_0^-}(f_2 \cdot f_1) = \lim_{+\infty} f_2$.

(58) If f_1 is left divergent to $-\infty$ in x_0 and f_2 is convergent in $-\infty$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is left convergent in x_0 and $\lim_{x_0^-}(f_2 \cdot f_1) = \lim_{-\infty} f_2$.

(59) If f_1 is right divergent to $+\infty$ in x_0 and f_2 is convergent in $+\infty$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is right convergent in x_0 and $\lim_{x_0^+}(f_2 \cdot f_1) = \lim_{+\infty} f_2$.

(60) If f_1 is right divergent to $-\infty$ in x_0 and f_2 is convergent in $-\infty$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$, then $f_2 \cdot f_1$ is right convergent in x_0 and $\lim_{x_0^+}(f_2 \cdot f_1) = \lim_{-\infty} f_2$.

(61) Suppose that
- (i) f_1 is left convergent in x_0,
- (ii) f_2 is left convergent in $\lim_{x_0^-} f_1$,
- (iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
- (iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0 - g, x_0[$ holds $f_1(r) < \lim_{x_0^-} f_1$.

Then $f_2 \cdot f_1$ is left convergent in x_0 and $\lim_{x_0^-}(f_2 \cdot f_1) = \lim_{\lim_{x_0^-} f_1^-} f_2$.

(62) Suppose that
- (i) f_1 is right convergent in x_0,
- (ii) f_2 is right convergent in $\lim_{x_0^+} f_1$,
- (iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
- (iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0, x_0 + g[$ holds $\lim_{x_0^+} f_1 < f_1(r)$.

Then $f_2 \cdot f_1$ is right convergent in x_0 and $\lim_{x_0^+}(f_2 \cdot f_1) = \lim_{\lim_{x_0^+} f_1^+} f_2$.

One can prove the following propositions:

(63) Suppose that
(i) f_1 is left convergent in x_0,
(ii) f_2 is right convergent in $\lim_{x_0^-} f_1$,
(iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0 - g, x_0[$ holds $\lim_{x_0^-} f_1 < f_1(r)$.
Then $f_2 \cdot f_1$ is left convergent in x_0 and $\lim_{x_0^-}(f_2 \cdot f_1) = \lim_{\lim_{x_0^-} f_1^+} f_2$.

(64) Suppose that
(i) f_1 is right convergent in x_0,
(ii) f_2 is left convergent in $\lim_{x_0^+} f_1$,
(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0, x_0 + g[$ holds $f_1(r) < \lim_{x_0^+} f_1$.
Then $f_2 \cdot f_1$ is right convergent in x_0 and $\lim_{x_0^+}(f_2 \cdot f_1) = \lim_{\lim_{x_0^+} f_1^-} f_2$.

(65) Suppose f_1 is convergent in $+\infty$ and f_2 is left convergent in $\lim_{+\infty} f_1$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]r, +\infty[$ holds $f_1(g) < \lim_{+\infty} f_1$. Then $f_2 \cdot f_1$ is convergent in $+\infty$ and $\lim_{+\infty}(f_2 \cdot f_1) = \lim_{\lim_{+\infty} f_1^-} f_2$.

(66) Suppose f_1 is convergent in $+\infty$ and f_2 is right convergent in $\lim_{+\infty} f_1$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]r, +\infty[$ holds $\lim_{+\infty} f_1 < f_1(g)$. Then $f_2 \cdot f_1$ is convergent in $+\infty$ and $\lim_{+\infty}(f_2 \cdot f_1) = \lim_{\lim_{+\infty} f_1^+} f_2$.

(67) Suppose f_1 is convergent in $-\infty$ and f_2 is left convergent in $\lim_{-\infty} f_1$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]-\infty, r[$ holds $f_1(g) < \lim_{-\infty} f_1$. Then $f_2 \cdot f_1$ is convergent in $-\infty$ and $\lim_{-\infty}(f_2 \cdot f_1) = \lim_{\lim_{-\infty} f_1^-} f_2$.

(68) Suppose f_1 is convergent in $-\infty$ and f_2 is right convergent in $\lim_{-\infty} f_1$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]-\infty, r[$ holds $\lim_{-\infty} f_1 < f_1(g)$. Then $f_2 \cdot f_1$ is convergent in $-\infty$ and $\lim_{-\infty}(f_2 \cdot f_1) = \lim_{\lim_{-\infty} f_1^+} f_2$.

(69) Suppose f_1 is divergent to $+\infty$ in x_0 and f_2 is convergent in $+\infty$ and for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$. Then $f_2 \cdot f_1$ is convergent in x_0 and $\lim_{x_0}(f_2 \cdot f_1) = \lim_{+\infty} f_2$.

(70) Suppose f_1 is divergent to $-\infty$ in x_0 and f_2 is convergent in $-\infty$ and for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that

$r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$. Then $f_2 \cdot f_1$ is convergent in x_0 and $\lim_{x_0}(f_2 \cdot f_1) = \lim_{-\infty} f_2$.

(71) Suppose f_1 is convergent in $+\infty$ and f_2 is convergent in $\lim_{+\infty} f_1$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]r, +\infty[$ holds $f_1(g) \neq \lim_{+\infty} f_1$. Then $f_2 \cdot f_1$ is convergent in $+\infty$ and $\lim_{+\infty}(f_2 \cdot f_1) = \lim_{\lim_{+\infty} f_1} f_2$.

(72) Suppose f_1 is convergent in $-\infty$ and f_2 is convergent in $\lim_{-\infty} f_1$ and for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$ and there exists r such that for every g such that $g \in \mathrm{dom}\, f_1 \cap]-\infty, r[$ holds $f_1(g) \neq \lim_{-\infty} f_1$. Then $f_2 \cdot f_1$ is convergent in $-\infty$ and $\lim_{-\infty}(f_2 \cdot f_1) = \lim_{\lim_{-\infty} f_1} f_2$.

(73) Suppose that
- (i) f_1 is convergent in x_0,
- (ii) f_2 is left convergent in $\lim_{x_0} f_1$,
- (iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$,
- (iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap (]x_0 - g, x_0[\cup]x_0, x_0 + g[)$ holds $f_1(r) < \lim_{x_0} f_1$.

Then $f_2 \cdot f_1$ is convergent in x_0 and $\lim_{x_0}(f_2 \cdot f_1) = \lim_{\lim_{x_0} f_1^-} f_2$.

(74) Suppose that
- (i) f_1 is left convergent in x_0,
- (ii) f_2 is convergent in $\lim_{x_0^-} f_1$,
- (iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,
- (iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0 - g, x_0[$ holds $f_1(r) \neq \lim_{x_0^-} f_1$.

Then $f_2 \cdot f_1$ is left convergent in x_0 and $\lim_{x_0^-}(f_2 \cdot f_1) = \lim_{\lim_{x_0^-} f_1} f_2$.

(75) Suppose that
- (i) f_1 is convergent in x_0,
- (ii) f_2 is right convergent in $\lim_{x_0} f_1$,
- (iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$,
- (iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap (]x_0 - g, x_0[\cup]x_0, x_0 + g[)$ holds $\lim_{x_0} f_1 < f_1(r)$.

Then $f_2 \cdot f_1$ is convergent in x_0 and $\lim_{x_0}(f_2 \cdot f_1) = \lim_{\lim_{x_0} f_1^+} f_2$.

(76) Suppose that
- (i) f_1 is right convergent in x_0,
- (ii) f_2 is convergent in $\lim_{x_0^+} f_1$,

(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap]x_0, x_0 + g[$ holds $f_1(r) \neq \lim_{x_0^+} f_1$.

Then $f_2 \cdot f_1$ is right convergent in x_0 and $\lim_{x_0^+}(f_2 \cdot f_1) = \lim_{\lim_{x_0^+} f_1} f_2$.

(77) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is convergent in $\lim_{x_0} f_1$,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \mathrm{dom}(f_2 \cdot f_1)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \mathrm{dom}(f_2 \cdot f_1)$,

(iv) there exists g such that $0 < g$ and for every r such that $r \in \mathrm{dom}\, f_1 \cap (]x_0 - g, x_0[\cup]x_0, x_0 + g[)$ holds $f_1(r) \neq \lim_{x_0} f_1$.

Then $f_2 \cdot f_1$ is convergent in x_0 and $\lim_{x_0}(f_2 \cdot f_1) = \lim_{\lim_{x_0} f_1} f_2$.

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Received September 5, 1990

FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

Locally Connected Spaces

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Summary. This article is a continuation of [6]. We define a neighbourhood of a point and a neighbourhood of a set and prove some facts about them. Then the definitions of a locally connected space and a locally connected set are introduced. Some theorems on locally connected spaces are given (based on [5]). We also define a quasi-component of a point and prove some of its basic properties.

MML Identifier: CONNSP_2.

The papers [11], [10], [2], [13], [7], [1], [12], [9], [3], [8], [14], [6], and [4] provide the terminology and notation for this paper. Let X be a topological space, and let x be a point of X. A subset of X is called a neighborhood of x if:

(Def.1) $x \in \mathrm{Int}\,\mathrm{it}$.

Let X be a topological space, and let A be a subset of X. A subset of X is called a neighborhood of A if:

(Def.2) $A \subseteq \mathrm{Int}\,\mathrm{it}$.

In the sequel X will denote a topological space, x will denote a point of X, and A, U_1 will denote subsets of X. We now state a number of propositions:

(2)[2] A is a neighborhood of U_1 if and only if $U_1 \subseteq \mathrm{Int}\,A$.

(3) For every x and for all subsets A, B of X such that A is a neighborhood of x and B is a neighborhood of x holds $A \cup B$ is a neighborhood of x.

(4) For every x and for all subsets A, B of X holds A is a neighborhood of x and B is a neighborhood of x if and only if $A \cap B$ is a neighborhood of x.

(5) For every subset U_1 of X and for every point x of X such that U_1 is open and $x \in U_1$ holds U_1 is a neighborhood of x.

(6) For every subset U_1 of X and for every point x of X such that U_1 is a neighborhood of x holds $x \in U_1$.

[1]Supported by RPBP.III-24.C1
[2]The proposition (1) was either repeated or obvious.

ISSN 0777–4028

(7) For all U_1, x such that U_1 is a neighborhood of x there exists a subset V of X such that V is a neighborhood of x and V is open and $V \subseteq U_1$.

(8) For all U_1, x holds U_1 is a neighborhood of x if and only if there exists a subset V of X such that V is open and $V \subseteq U_1$ and $x \in V$.

(9) U_1 is open if and only if for every x such that $x \in U_1$ there exists a subset A of X such that A is a neighborhood of x and $A \subseteq U_1$.

(10) For every subset V of X holds V is a neighborhood of $\{x\}$ if and only if V is a neighborhood of x.

(11) For every subset B of X and for every point x of $X \restriction B$ and for every subset A of $X \restriction B$ and for every subset A_1 of X and for every point x_1 of X such that $B \neq \emptyset_X$ and B is open and A is a neighborhood of x and $A = A_1$ and $x = x_1$ holds A_1 is a neighborhood of x_1.

(12) For every subset B of X and for every point x of $X \restriction B$ and for every subset A of $X \restriction B$ and for every subset A_1 of X and for every point x_1 of X such that A_1 is a neighborhood of x_1 and $A = A_1$ and $x = x_1$ holds A is a neighborhood of x.

(13) For all subsets A, B of X such that A is a component of X and $A \subseteq B$ holds A is a component of B.

(14) For every subspace X_1 of X and for every point x of X and for every point x_1 of X_1 such that $x = x_1$ holds $\mathrm{Component}(x_1) \subseteq \mathrm{Component}(x)$.

Let X be a topological space, and let x be a point of X. We say that X is locally connected in x if and only if:

(Def.3) for every subset U_1 of X such that U_1 is a neighborhood of x there exists a subset V of X such that V is a neighborhood of x and V is connected and $V \subseteq U_1$.

Let X be a topological space. We say that X is locally connected if and only if:

(Def.4) for every point x of X holds X is locally connected in x.

Let X be a topological space, and let A be a subset of X, and let x be a point of X. We say that A is locally connected in x if and only if:

(Def.5) there exists a point x_1 of $X \restriction A$ such that $x_1 = x$ and $X \restriction A$ is locally connected in x_1.

The following proposition is true

(17)[3] A is locally connected in x if and only if there exists a point x_1 of $X \restriction A$ such that $x_1 = x$ and $X \restriction A$ is locally connected in x_1.

Let X be a topological space, and let A be a subset of X. We say that A is locally connected if and only if:

(Def.6) $X \restriction A$ is locally connected.

One can prove the following propositions:

[3]The propositions (15)–(16) were either repeated or obvious.

(19)[4] For every x holds X is locally connected in x if and only if for all subsets V, S of X such that V is a neighborhood of x and S is a component of V and $x \in S$ holds S is a neighborhood of x.

(20) For every x holds X is locally connected in x if and only if for every subset U_1 of X such that U_1 is open and $x \in U_1$ there exists a point x_1 of $X \restriction U_1$ such that $x_1 = x$ and $x \in \mathrm{Int\,Component}(x_1)$.

(21) If X is locally connected, then for every subset S of X such that S is a component of X holds S is open.

(22) If X is locally connected in x, then for every subset A of X such that A is open and $x \in A$ holds A is locally connected in x.

(23) If X is locally connected, then for every subset A of X such that $A \neq \emptyset_X$ and A is open holds A is locally connected.

(24) X is locally connected if and only if for all subsets A, B of X such that $A \neq \emptyset_X$ and A is open and B is a component of A holds B is open.

(25) If X is locally connected, then for every subset E of X and for every subset C of $X \restriction E$ such that $E \neq \emptyset_X$ and $C \neq \emptyset_{X \restriction E}$ and C is connected and C is open there exists a subset H of X such that H is open and H is connected and $C = E \cap H$.

(26) X is a T_4 space if and only if for all subsets A, C of X such that $A \neq \emptyset$ and $C \neq \Omega_X$ and $A \subseteq C$ and A is closed and C is open there exists a subset G of X such that G is open and $A \subseteq G$ and $\overline{G} \subseteq C$.

(27) Suppose X is locally connected and X is a T_4 space. Let A, B be subsets of X. Suppose $A \neq \emptyset$ and $B \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Then if A is connected and B is connected, then there exist subsets R, S of X such that R is connected and S is connected and R is open and S is open and $A \subseteq R$ and $B \subseteq S$ and $R \cap S = \emptyset$.

(28) For every point x of X and for every family F of subsets of X such that for every subset A of X holds $A \in F$ if and only if A is open closed and $x \in A$ holds $F \neq \emptyset$.

Let X be a topological space, and let x be a point of X. The
quasi-component of x
is a subset of X defined by:

(Def.7) there exists a family F of subsets of X such that for every subset A of X holds $A \in F$ if and only if A is open closed and $x \in A$ and $\bigcap F =$ the quasi-component of x.

We now state several propositions:

(29) $A =$ the quasi-component of x if and only if there exists a family F of subsets of X such that for every subset A of X holds $A \in F$ if and only if A is open closed and $x \in A$ and $\bigcap F = A$.

(30) $x \in$ the quasi-component of x.

[4]The proposition (18) was either repeated or obvious.

(31) If A is open closed and $x \in A$, then if $A \subseteq$ the quasi-component of x, then $A =$ the quasi-component of x.

(32) The quasi-component of x is closed.

(33) Component$(x) \subseteq$ the quasi-component of x.

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Received September 5, 1990

FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

Construction of Finite Sequences over Ring and Left-, Right-, and Bi-Modules over a Ring [1]

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Summary. This text includes definitions of finite sequences over rings and left-, right-, and bi-module over a ring, treated as functions defined for *all* natural numbers, but almost everywhere equal to zero. Some elementary theorems are proved, in particular for each category of sequences the schema of existence is proved. In all four cases, *i.e* for rings, left-, right-, and bi-modules are almost exactly the same, hovewer we do not know how to do the job in Mizar in a different way. The paper is mostly based on [2]. In particular the notion of initial segment of natural numbers is introduced which differs from that of [2] by starting with zero. This proved to be more convenient for algebraic purposes.

MML Identifier: ALGSEQ_1.

The notation and terminology used in this paper are introduced in the following papers: [8], [3], [5], [1], [4], [6], and [7]. We adopt the following rules: i, k, l, m, n will be natural numbers and x will be arbitrary. We now state four propositions:

(2)[2] If $m < n + 1$, then $m < n$ or $m = n$.

(4)[3] If $k < 2$, then $k = 0$ or $k = 1$.

(5) For every real number x holds $x < x + 1$.

(7)[4] If $k < l$ and $l \leq k + 1$, then $l = k + 1$.

Let us consider n. The functor $\mathrm{PSeg}\, n$ yields a set and is defined by:

(Def.1) $\mathrm{PSeg}\, n = \{k : k < n\}$.

[1]Supported by RPBP.III-24.C3

[2]The proposition (1) was either repeated or obvious.
[3]The proposition (3) was either repeated or obvious.
[4]The proposition (6) was either repeated or obvious.

ISSN 0777–4028

Let us consider n. Then $\mathrm{PSeg}\, n$ is sets of natural numbers.

We now state a number of propositions:

(8) $\mathrm{PSeg}\, n = \{k : k < n\}$.

(9) If $x \in \mathrm{PSeg}\, n$, then x is a natural number.

(10) $k \in \mathrm{PSeg}\, n$ if and only if $k < n$.

(11) $\mathrm{PSeg}\, 0 = \emptyset$ and $\mathrm{PSeg}\, 1 = \{0\}$ and $\mathrm{PSeg}\, 2 = \{0, 1\}$.

(12) $n \in \mathrm{PSeg}(n + 1)$.

(13) $n \leq m$ if and only if $\mathrm{PSeg}\, n \subseteq \mathrm{PSeg}\, m$.

(14) If $\mathrm{PSeg}\, n = \mathrm{PSeg}\, m$, then $n = m$.

(15) If $k \leq n$, then $\mathrm{PSeg}\, k = \mathrm{PSeg}\, k \cap \mathrm{PSeg}\, n$ and $\mathrm{PSeg}\, k = \mathrm{PSeg}\, n \cap \mathrm{PSeg}\, k$.

(16) If $\mathrm{PSeg}\, k = \mathrm{PSeg}\, k \cap \mathrm{PSeg}\, n$ or $\mathrm{PSeg}\, k = \mathrm{PSeg}\, n \cap \mathrm{PSeg}\, k$, then $k \leq n$.

(17) $\mathrm{PSeg}\, n \cup \{n\} = \mathrm{PSeg}(n + 1)$.

In the sequel R is a field structure and x is a scalar of R. Let us consider R. A function from $\mathbb{N}$ into the carrier of R is said to be an algebraic sequence of R if:

(Def.2) there exists n such that for every i such that $i \geq n$ holds $\mathrm{it}(i) = 0_R$.

In the sequel p, q denote algebraic sequences of R. Next we state the proposition

(19)[5] $\mathrm{dom}\, p = \mathbb{N}$.

Let us consider R, p, k. We say that the length of p is at most k if and only if:

(Def.3) for every i such that $i \geq k$ holds $p(i) = 0_R$.

We now state the proposition

(20) the length of p is at most k if and only if for every i such that $i \geq k$ holds $p(i) = 0_R$.

Let us consider R, p. The functor $\mathrm{len}\, p$ yielding a natural number is defined as follows:

(Def.4) the length of p is at most $\mathrm{len}\, p$ and for every m such that the length of p is at most m holds $\mathrm{len}\, p \leq m$.

We now state several propositions:

(21) $i = \mathrm{len}\, p$ if and only if the length of p is at most i and for every m such that the length of p is at most m holds $i \leq m$.

(22) For every i such that $i \geq \mathrm{len}\, p$ holds $p(i) = 0_R$.

(23) If $p(k) \neq 0_R$, then $\mathrm{len}\, p > k$.

(24) If for every i such that $i < k$ holds $p(i) \neq 0_R$, then $\mathrm{len}\, p \geq k$.

(25) If $\mathrm{len}\, p = k + 1$, then $p(k) \neq 0_R$.

Let us consider R, p. The functor $\mathrm{support}\, p$ yields sets of natural numbers and is defined as follows:

[5]The proposition (18) was either repeated or obvious.

(Def.5) $\operatorname{support} p = \operatorname{PSeg}(\operatorname{len} p)$.

Next we state two propositions:

(26) For every y being sets of natural numbers holds $y = \operatorname{support} p$ if and only if $y = \operatorname{PSeg}(\operatorname{len} p)$.

(27) $k = \operatorname{len} p$ if and only if $\operatorname{PSeg} k = \operatorname{support} p$.

The scheme *AlgSeqLambdaF* concerns field structure $\mathcal{A}$, a natural number $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a scalar of $\mathcal{A}$ and states that:

there exists an algebraic sequence p of $\mathcal{A}$ such that $\operatorname{len} p \leq \mathcal{B}$ and for every k such that $k < \mathcal{B}$ holds $p(k) = \mathcal{F}(k)$
for all values of the parameters.

One can prove the following proposition

(28) If $\operatorname{len} p = \operatorname{len} q$ and for every k such that $k < \operatorname{len} p$ holds $p(k) = q(k)$, then $p = q$.

The following proposition is true

(29) For every R such that the carrier of $R \neq \{0_R\}$ for every k there exists an algebraic sequence p of R such that $\operatorname{len} p = k$.

Let us consider R, x. The functor $\langle x\rangle$ yielding an algebraic sequence of R is defined by:

(Def.6) $\operatorname{len}\langle x\rangle \leq 1$ and $\langle x\rangle(0) = x$.

One can prove the following propositions:

(30) $p = \langle x\rangle$ if and only if $\operatorname{len} p \leq 1$ and $p(0) = x$.

(31) $p = \langle 0_R\rangle$ if and only if $\operatorname{len} p = 0$.

(32) $p = \langle 0_R\rangle$ if and only if $\operatorname{support} p = \emptyset$.

(33) $\langle 0_R\rangle(i) = 0_R$.

(34) $p = \langle 0_R\rangle$ if and only if $\operatorname{rng} p = \{0_R\}$.

In the sequel R will be an associative ring and V will be a left module over R. Let us consider R, V. The functor Θ_V yields a vector of V and is defined by:

(Def.7) $\Theta_V = 0_{\text{the carrier of } V}$.

One can prove the following proposition

(35) $\Theta_V = 0_{\text{the carrier of } V}$.

In the sequel x denotes a vector of V. Let us consider R, V. A function from $\mathbb{N}$ into the carrier of the carrier of V is said to be an algebraic sequence of V if:

(Def.8) there exists n such that for every i such that $i \geq n$ holds $\operatorname{it}(i) = \Theta_V$.

In the sequel p, q will denote algebraic sequences of V. The following proposition is true

(37)[6] $\operatorname{dom} p = \mathbb{N}$.

Let us consider R, V, p, k. We say that the length of p is at most k if and only if:

[6]The proposition (36) was either repeated or obvious.

(Def.9) for every i such that $i \geq k$ holds $p(i) = \Theta_V$.

We now state the proposition

(38) the length of p is at most k if and only if for every i such that $i \geq k$ holds $p(i) = \Theta_V$.

Let us consider R, V, p. The functor $\operatorname{len} p$ yields a natural number and is defined as follows:

(Def.10) the length of p is at most $\operatorname{len} p$ and for every m such that the length of p is at most m holds $\operatorname{len} p \leq m$.

One can prove the following propositions:

(39) $i = \operatorname{len} p$ if and only if the length of p is at most i and for every m such that the length of p is at most m holds $i \leq m$.

(40) For every i such that $i \geq \operatorname{len} p$ holds $p(i) = \Theta_V$.

(41) If $p(k) \neq \Theta_V$, then $\operatorname{len} p > k$.

(42) If for every i such that $i < k$ holds $p(i) \neq \Theta_V$, then $\operatorname{len} p \geq k$.

(43) If $\operatorname{len} p = k + 1$, then $p(k) \neq \Theta_V$.

Let us consider R, V, p. The functor $\operatorname{support} p$ yields sets of natural numbers and is defined by:

(Def.11) $\operatorname{support} p = \operatorname{PSeg}(\operatorname{len} p)$.

We now state two propositions:

(44) For every y being sets of natural numbers holds $y = \operatorname{support} p$ if and only if $y = \operatorname{PSeg}(\operatorname{len} p)$.

(45) $k = \operatorname{len} p$ if and only if $\operatorname{PSeg} k = \operatorname{support} p$.

The scheme *AlgSeqLambdaLM* deals with an associative ring $\mathcal{A}$, a left module $\mathcal{B}$ over $\mathcal{A}$, a natural number $\mathcal{C}$, and a unary functor $\mathcal{F}$ yielding a vector of $\mathcal{B}$ and states that:

there exists an algebraic sequence p of $\mathcal{B}$ such that $\operatorname{len} p \leq \mathcal{C}$ and for every k such that $k < \mathcal{C}$ holds $p(k) = \mathcal{F}(k)$

for all values of the parameters.

The following proposition is true

(46) If $\operatorname{len} p = \operatorname{len} q$ and for every k such that $k < \operatorname{len} p$ holds $p(k) = q(k)$, then $p = q$.

We now state the proposition

(47) For all R, V such that the carrier of the carrier of $V \neq \{\Theta_V\}$ for every k there exists an algebraic sequence p of V such that $\operatorname{len} p = k$.

Let us consider R, V, x. The functor $\langle x \rangle$ yielding an algebraic sequence of V is defined as follows:

(Def.12) $\operatorname{len}\langle x \rangle \leq 1$ and $\langle x \rangle(0) = x$.

One can prove the following propositions:

(48) $p = \langle x \rangle$ if and only if $\operatorname{len} p \leq 1$ and $p(0) = x$.

(49) $p = \langle \Theta_V \rangle$ if and only if $\operatorname{len} p = 0$.

(50) $p = \langle \Theta_V \rangle$ if and only if $\operatorname{support} p = \emptyset$.

(51) $\langle \Theta_V \rangle(i) = \Theta_V$.

(52) $p = \langle \Theta_V \rangle$ if and only if $\operatorname{rng} p = \{\Theta_V\}$.

In the sequel V will denote a right module over R. Let us consider R, V. The functor Θ_V yields a vector of V and is defined as follows:

(Def.13) $\Theta_V = 0_{\text{the carrier of } V}$.

The following proposition is true

(53) $\Theta_V = 0_{\text{the carrier of } V}$.

Let us consider R, V. The functor Θ_V yields a vector of V and is defined as follows:

(Def.14) $\Theta_V = 0_{\text{the carrier of } V}$.

The following proposition is true

(54) $\Theta_V = 0_{\text{the carrier of } V}$.

In the sequel x will denote a vector of V. Let us consider R, V. A function from $\mathbb{N}$ into the carrier of the carrier of V is called an algebraic sequence of V if:

(Def.15) there exists n such that for every i such that $i \geq n$ holds $\operatorname{it}(i) = \Theta_V$.

In the sequel p, q will be algebraic sequences of V. We now state the proposition

(56)[7] $\operatorname{dom} p = \mathbb{N}$.

Let us consider R, V, p, k. We say that the length of p is at most k if and only if:

(Def.16) for every i such that $i \geq k$ holds $p(i) = \Theta_V$.

Next we state the proposition

(57) the length of p is at most k if and only if for every i such that $i \geq k$ holds $p(i) = \Theta_V$.

Let us consider R, V, p. The functor $\operatorname{len} p$ yields a natural number and is defined by:

(Def.17) the length of p is at most $\operatorname{len} p$ and for every m such that the length of p is at most m holds $\operatorname{len} p \leq m$.

Next we state several propositions:

(58) $i = \operatorname{len} p$ if and only if the length of p is at most i and for every m such that the length of p is at most m holds $i \leq m$.

(59) For every i such that $i \geq \operatorname{len} p$ holds $p(i) = \Theta_V$.

(60) If $p(k) \neq \Theta_V$, then $\operatorname{len} p > k$.

(61) If for every i such that $i < k$ holds $p(i) \neq \Theta_V$, then $\operatorname{len} p \geq k$.

(62) If $\operatorname{len} p = k + 1$, then $p(k) \neq \Theta_V$.

[7]The proposition (55) was either repeated or obvious.

Let us consider R, V, p. The functor $\mathrm{support}\, p$ yielding sets of natural numbers is defined by:

(Def.18) $\mathrm{support}\, p = \mathrm{PSeg}(\mathrm{len}\, p)$.

The following propositions are true:

(63) For every y being sets of natural numbers holds $y = \mathrm{support}\, p$ if and only if $y = \mathrm{PSeg}(\mathrm{len}\, p)$.

(64) $k = \mathrm{len}\, p$ if and only if $\mathrm{PSeg}\, k = \mathrm{support}\, p$.

The scheme *AlgSeqLambdaRM* deals with an associative ring $\mathcal{A}$, a right module $\mathcal{B}$ over $\mathcal{A}$, a natural number $\mathcal{C}$, and a unary functor $\mathcal{F}$ yielding a vector of $\mathcal{B}$ and states that:

there exists an algebraic sequence p of $\mathcal{B}$ such that $\mathrm{len}\, p \leq \mathcal{C}$ and for every k such that $k < \mathcal{C}$ holds $p(k) = \mathcal{F}(k)$
for all values of the parameters.

The following proposition is true

(65) If $\mathrm{len}\, p = \mathrm{len}\, q$ and for every k such that $k < \mathrm{len}\, p$ holds $p(k) = q(k)$, then $p = q$.

One can prove the following proposition

(66) For all R, V such that the carrier of the carrier of $V \neq \{\Theta_V\}$ for every k there exists an algebraic sequence p of V such that $\mathrm{len}\, p = k$.

Let us consider R, V, x. The functor $\langle x \rangle$ yielding an algebraic sequence of V is defined by:

(Def.19) $\mathrm{len}\langle x \rangle \leq 1$ and $\langle x \rangle(0) = x$.

We now state several propositions:

(67) $p = \langle x \rangle$ if and only if $\mathrm{len}\, p \leq 1$ and $p(0) = x$.

(68) $p = \langle \Theta_V \rangle$ if and only if $\mathrm{len}\, p = 0$.

(69) $p = \langle \Theta_V \rangle$ if and only if $\mathrm{support}\, p = \emptyset$.

(70) $\langle \Theta_V \rangle(i) = \Theta_V$.

(71) $p = \langle \Theta_V \rangle$ if and only if $\mathrm{rng}\, p = \{\Theta_V\}$.

In the sequel V is a bimodule over R. Let us consider R, V. The functor Θ_V yields a vector of V and is defined as follows:

(Def.20) $\Theta_V = 0_{\text{the carrier of } V}$.

One can prove the following proposition

(72) $\Theta_V = 0_{\text{the carrier of } V}$.

Let us consider R, V. The functor Θ_V yields a vector of V and is defined as follows:

(Def.21) $\Theta_V = 0_{\text{the carrier of } V}$.

We now state the proposition

(73) $\Theta_V = 0_{\text{the carrier of } V}$.

In the sequel x will denote a vector of V. Let us consider R, V. A function from $\mathbb{N}$ into the carrier of the carrier of V is said to be an algebraic sequence of V if:

(Def.22) there exists n such that for every i such that $i \geq n$ holds $\mathrm{it}(i) = \Theta_V$.

In the sequel p, q will be algebraic sequences of V. We now state the proposition

(75)[8] $\mathrm{dom}\, p = \mathbb{N}$.

Let us consider R, V, p, k. We say that the length of p is at most k if and only if:

(Def.23) for every i such that $i \geq k$ holds $p(i) = \Theta_V$.

Next we state the proposition

(76) the length of p is at most k if and only if for every i such that $i \geq k$ holds $p(i) = \Theta_V$.

Let us consider R, V, p. The functor $\mathrm{len}\, p$ yielding a natural number is defined by:

(Def.24) the length of p is at most $\mathrm{len}\, p$ and for every m such that the length of p is at most m holds $\mathrm{len}\, p \leq m$.

One can prove the following propositions:

(77) $i = \mathrm{len}\, p$ if and only if the length of p is at most i and for every m such that the length of p is at most m holds $i \leq m$.

(78) For every i such that $i \geq \mathrm{len}\, p$ holds $p(i) = \Theta_V$.

(79) If $p(k) \neq \Theta_V$, then $\mathrm{len}\, p > k$.

(80) If for every i such that $i < k$ holds $p(i) \neq \Theta_V$, then $\mathrm{len}\, p \geq k$.

(81) If $\mathrm{len}\, p = k + 1$, then $p(k) \neq \Theta_V$.

Let us consider R, V, p. The functor $\mathrm{support}\, p$ yielding sets of natural numbers is defined by:

(Def.25) $\mathrm{support}\, p = \mathrm{PSeg}(\mathrm{len}\, p)$.

We now state two propositions:

(82) For every y being sets of natural numbers holds $y = \mathrm{support}\, p$ if and only if $y = \mathrm{PSeg}(\mathrm{len}\, p)$.

(83) $k = \mathrm{len}\, p$ if and only if $\mathrm{PSeg}\, k = \mathrm{support}\, p$.

The scheme *AlgSeqLambdaBM* concerns an associative ring $\mathcal{A}$, a bimodule $\mathcal{B}$ over $\mathcal{A}$, a natural number $\mathcal{C}$, and a unary functor $\mathcal{F}$ yielding a vector of $\mathcal{B}$ and states that:

there exists an algebraic sequence p of $\mathcal{B}$ such that $\mathrm{len}\, p \leq \mathcal{C}$ and for every k such that $k < \mathcal{C}$ holds $p(k) = \mathcal{F}(k)$
for all values of the parameters.

We now state the proposition

[8] The proposition (74) was either repeated or obvious.

(84) If $\operatorname{len} p = \operatorname{len} q$ and for every k such that $k < \operatorname{len} p$ holds $p(k) = q(k)$, then $p = q$.

The following proposition is true

(85) For all R, V such that the carrier of the carrier of $V \neq \{\Theta_V\}$ for every k there exists an algebraic sequence p of V such that $\operatorname{len} p = k$.

Let us consider R, V, x. The functor $\langle x \rangle$ yields an algebraic sequence of V and is defined by:

(Def.26) $\operatorname{len}\langle x \rangle \leq 1$ and $\langle x \rangle(0) = x$.

Next we state several propositions:

(86) $p = \langle x \rangle$ if and only if $\operatorname{len} p \leq 1$ and $p(0) = x$.

(87) $p = \langle \Theta_V \rangle$ if and only if $\operatorname{len} p = 0$.

(88) $p = \langle \Theta_V \rangle$ if and only if $\operatorname{support} p = \emptyset$.

(89) $\langle \Theta_V \rangle(i) = \Theta_V$.

(90) $p = \langle \Theta_V \rangle$ if and only if $\operatorname{rng} p = \{\Theta_V\}$.

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Received September 13, 1990

FORMALIZED MATHEMATICS
Vol.2,No.1, January–February 1991
Université Catholique de Louvain

Relations of Tolerance [1]

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Summary. Introduces notions of relations of tolerance, tolerance set and neighbourhood of an element. The basic properties of relations of tolerance are proved.

MML Identifier: `TOLER_1`.

The notation and terminology used here have been introduced in the following papers: [2], [3], [4], [5], and [1]. We adopt the following rules: X, Y, Z denote sets, x, y are arbitrary, and R denotes a relation between X and X. The following propositions are true:

(1) field $\varnothing = \emptyset$.

(2) $\varnothing$ is pseudo reflexive.

(3) $\varnothing$ is symmetric.

(4) $\varnothing$ is irreflexive.

(5) $\varnothing$ is antisymmetric.

(6) $\varnothing$ is asymmetric.

(7) $\varnothing$ is connected.

(8) $\varnothing$ is strongly connected.

(9) $\varnothing$ is transitive.

Let us consider X. The functor ∇_X yielding a relation between X and X is defined by:

(Def.1) $\nabla_X = [: X, X :]$.

Let us consider X, R, Y. Then $R\,|^2\, Y$ is a relation between Y and Y.

The following propositions are true:

(10) For every relation R between X and X holds $R = \nabla_X$ if and only if $R = [: X, X :]$.

[1] Supported by Philippe le Hodey Foundation. This work had been done on Mizar Workshop '89 (Fourdrain, France) in Summer '89.

ISSN 0777–4028

(11) $\nabla_X = [: X, X :]$.
(12) $\operatorname{dom} \nabla_X = X$.
(13) $\operatorname{rng} \nabla_X = X$.
(14) $\operatorname{field} \nabla_X = X$.
(15) For all x, y such that $x \in X$ and $y \in X$ holds $\langle x, y\rangle \in \nabla_X$.
(16) For all x, y such that $x \in \operatorname{field} \nabla_X$ and $y \in \operatorname{field} \nabla_X$ holds $\langle x, y\rangle \in \nabla_X$.
(17) ∇_X is pseudo reflexive.
(18) ∇_X is symmetric.
(19) ∇_X is strongly connected.
(20) ∇_X is transitive.
(21) ∇_X is connected.

Let us consider X. A relation between X and X is said to be a tolerance of X if:

(Def.2) it is pseudo reflexive and it is symmetric and $\operatorname{field} \mathrm{it} = X$.

In the sequel T, R denote tolerances of X. The following propositions are true:

(23)[2] For every tolerance R of X holds R is pseudo reflexive and R is symmetric and $\operatorname{field} R = X$.
(24) For every tolerance T of X holds $\operatorname{dom} T = X$.
(25) For every tolerance T of X holds $\operatorname{rng} T = X$.
(26) For every tolerance T of X holds $\operatorname{field} T = X$.
(27) For every tolerance T of X holds $x \in X$ if and only if $\langle x, x\rangle \in T$.
(28) For every tolerance T of X holds T is reflexive in X.
(29) For every tolerance T of X holds T is symmetric in X.
(30) For every tolerance T of X such that $\langle x, y\rangle \in T$ holds $\langle y, x\rangle \in T$.
(31) For every tolerance T of X and for all x, y such that $\langle x, y\rangle \in T$ holds $x \in X$ and $y \in X$.
(32) For every relation R between X and Y such that R is symmetric holds $R\,|^2\, Z$ is symmetric.

Let us consider X, T, and let Y be a subset of X. Then $T\,|^2\, Y$ is a tolerance of Y.

Next we state the proposition

(33) If $Y \subseteq X$, then $T\,|^2\, Y$ is a tolerance of Y.

Let us consider X, and let T be a tolerance of X. A set is called a set of mutually elements w.r.t. T if:

(Def.3) for all x, y such that $x \in \mathrm{it}$ and $y \in \mathrm{it}$ holds $\langle x, y\rangle \in T$.

We now state the proposition

(34) $\emptyset$ is a set of mutually elements w.r.t. T.

[2]The proposition (22) was either repeated or obvious.

Let us consider X, and let T be a tolerance of X. A set of mutually elements w.r.t. T is called a tolerance class of T if:

(Def.4) for every x such that $x \notin$ it and $x \in X$ there exists y such that $y \in$ it and $\langle x, y \rangle \notin T$.

Next we state a number of propositions:

(36)[3] Y is a set of mutually elements w.r.t. T if and only if for all x, y such that $x \in Y$ and $y \in Y$ holds $\langle x, y \rangle \in T$.

(38)[4] For every tolerance T of X such that $\emptyset$ is a tolerance class of T holds $T = \varnothing$.

(39) $\varnothing$ is a tolerance of $\emptyset$.

(40) For all x, y such that $\langle x, y \rangle \in T$ holds $\{x, y\}$ is a set of mutually elements w.r.t. T.

(41) For every x such that $x \in X$ holds $\{x\}$ is a set of mutually elements w.r.t. T.

(42) For all Y, Z such that Y is a set of mutually elements w.r.t. T and Z is a set of mutually elements w.r.t. T holds $Y \cap Z$ is a set of mutually elements w.r.t. T.

(43) If Y is a set of mutually elements w.r.t. T, then $Y \subseteq X$.

(44) If Y is a tolerance class of T, then $Y \subseteq X$.

(45) For every set Y of mutually elements w.r.t. T there exists a tolerance class Z of T such that $Y \subseteq Z$.

(46) For all x, y such that $\langle x, y \rangle \in T$ there exists a tolerance class Z of T such that $x \in Z$ and $y \in Z$.

(47) For every x such that $x \in X$ there exists a tolerance class Z of T such that $x \in Z$.

Let us consider X. Then $\triangle_X$ is a tolerance of X.

We now state three propositions:

(48) ∇_X is a tolerance of X.

(49) $T \subseteq \nabla_X$.

(50) $\triangle_X \subseteq T$.

The scheme *ToleranceEx* concerns a set $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a tolerance T of $\mathcal{A}$ such that for all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{A}$ holds $\langle x, y \rangle \in T$ if and only if $\mathcal{P}[x, y]$

provided the parameters satisfy the following conditions:

- for every x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x, x]$,
- for all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{A}$ and $\mathcal{P}[x, y]$ holds $\mathcal{P}[y, x]$.

One can prove the following propositions:

[3]The proposition (35) was either repeated or obvious.

[4]The proposition (37) was either repeated or obvious.

(51) For every Y there exists a tolerance T of $\bigcup Y$ such that for every Z such that $Z \in Y$ holds Z is a set of mutually elements w.r.t. T.

(52) Let Y be a set. Let T, R be tolerances of $\bigcup Y$. Then if for all x, y holds $\langle x, y \rangle \in T$ if and only if there exists Z such that $Z \in Y$ and $x \in Z$ and $y \in Z$ and for all x, y holds $\langle x, y \rangle \in R$ if and only if there exists Z such that $Z \in Y$ and $x \in Z$ and $y \in Z$, then $T = R$.

(53) For all tolerances T, R of X such that for every Z holds Z is a tolerance class of T if and only if Z is a tolerance class of R holds $T = R$.

Let us consider X, and let T be a tolerance of X, and let us consider x. The functor neighbourhood(x, T) yielding a set is defined by:

(Def.5) for every y holds $y \in \text{neighbourhood}(x, T)$ if and only if $\langle x, y \rangle \in T$.

One can prove the following propositions:

(54) For every tolerance T of X and for every x and for every set Y holds $Y = \text{neighbourhood}(x, T)$ if and only if for every y holds $y \in Y$ if and only if $\langle x, y \rangle \in T$.

(55) For every tolerance T of X holds $y \in \text{neighbourhood}(x, T)$ if and only if $\langle x, y \rangle \in T$.

(56) If $x \in X$, then $x \in \text{neighbourhood}(x, T)$.

(57) $\text{neighbourhood}(x, T) \subseteq X$.

(58) For every Y such that for every set Z holds $Z \in Y$ if and only if $x \in Z$ and Z is a tolerance class of T holds $\text{neighbourhood}(x, T) = \bigcup Y$.

(59) For every Y such that for every Z holds $Z \in Y$ if and only if $x \in Z$ and Z is a set of mutually elements w.r.t. T holds $\text{neighbourhood}(x, T) = \bigcup Y$.

We now define two new functors. Let us consider X, and let T be a tolerance of X. The functor $\text{TolSets}\, T$ yields a set and is defined by:

(Def.6) for every Y holds $Y \in \text{TolSets}\, T$ if and only if Y is a set of mutually elements w.r.t. T.

The functor $\text{TolClasses}\, T$ yields a set and is defined by:

(Def.7) for every Y holds $Y \in \text{TolClasses}\, T$ if and only if Y is a tolerance class of T.

The following propositions are true:

(60) For every set Y and for every tolerance T of X holds $Y = \text{TolSets}\, T$ if and only if for every Z holds $Z \in Y$ if and only if Z is a set of mutually elements w.r.t. T.

(61) For every tolerance T of X and for every Z holds $Z \in \text{TolSets}\, T$ if and only if Z is a set of mutually elements w.r.t. T.

(62) For every set Y and for every tolerance T of X holds $Y = \text{TolClasses}\, T$ if and only if for every Z holds $Z \in Y$ if and only if Z is a tolerance class of T.

(63) For every tolerance T of X holds $Z \in \text{TolClasses}\, T$ if and only if Z is a tolerance class of T.

(64) If TolClasses $R \subseteq$ TolClasses T, then $R \subseteq T$.

(65) For all tolerances T, R of X such that TolClasses T = TolClasses R holds $T = R$.

(66) $\bigcup$(TolClasses T) $= X$.

(67) $\bigcup$(TolSets T) $= X$.

(68) If for every x such that $x \in X$ holds neighbourhood(x, T) is a set of mutually elements w.r.t. T, then T is transitive.

(69) If T is transitive, then for every x such that $x \in X$ holds
neighbourhood(x, T)
is a tolerance class of T.

(70) For every x and for every tolerance class Y of T such that $x \in Y$ holds $Y \subseteq$ neighbourhood(x, T).

(71) TolSets $R \subseteq$ TolSets T if and only if $R \subseteq T$.

(72) TolClasses $T \subseteq$ TolSets T.

(73) If for every x such that $x \in X$ holds
neighbourhood$(x, R) \subseteq$ neighbourhood(x, T),
then $R \subseteq T$.

(74) $T \subseteq T \cdot T$.

(75) If $T = T \cdot T$, then T is transitive.

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Received September 20, 1990

FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

Real Normed Space

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Summary. We construct a real normed space ⟨V, ‖.‖⟩, where V is a real vector space and ‖.‖ is a norm. Auxillary properties of the norm are proved. Next, we introduce the notion of sequence in the real normed space. The basic operations on sequences (addition, substraction, multiplication by real number) are defined. We study some properties of sequences in the real normed space and the operations on them.

MML Identifier: `NORMSP_1`.

The notation and terminology used in this paper have been introduced in the following papers: [5], [13], [16], [3], [4], [1], [2], [17], [11], [12], [9], [7], [8], [10], [15], [14], and [6]. We consider normed structures which are systems

⟨vectors, a norm⟩,

where the vectors constitute a real linear space and the norm is a function from the vectors of the vectors into $\mathbb{R}$.

In the sequel X is a normed structure and a, b are real numbers. Let us consider X. A point of X is an element of the vectors of the vectors of X.

In the sequel x denotes a point of X. Let us consider X, x. The functor $\|x\|$ yields a real number and is defined as follows:

(Def.1) $\|x\| = (\text{the norm of } X)(x)$.

A normed structure is said to be a real normed space if:

(Def.2) for all points x, y of it and for every a holds $\|x\| = 0$ if and only if $x = 0_{\text{the vectors of it}}$ but $\|a \cdot x\| = |a| \cdot \|x\|$ and $\|x + y\| \leq \|x\| + \|y\|$.

We adopt the following rules: R_1 is a real normed space and x, y, z, g are points of R_1. The following propositions are true:

(2)[2] $\|x\| = 0$ if and only if $x = 0_{\text{the vectors of } R_1}$.

[1]Supported by RPBP.III-24.C8
[2]The proposition (1) was either repeated or obvious.

ISSN 0777–4028

(3) $\|a \cdot x\| = |a| \cdot \|x\|$.
(4) $\|x + y\| \leq \|x\| + \|y\|$.
(5) $\|0_{\text{the vectors of } R_1}\| = 0$.
(6) $\|-x\| = \|x\|$.
(7) $\|x - y\| \leq \|x\| + \|y\|$.
(8) $0 \leq \|x\|$.
(9) $\|a \cdot x + b \cdot y\| \leq |a| \cdot \|x\| + |b| \cdot \|y\|$.
(10) $\|x - y\| = 0$ if and only if $x = y$.
(11) $\|x - y\| = \|y - x\|$.
(12) $\|x\| - \|y\| \leq \|x - y\|$.
(13) $|\|x\| - \|y\|| \leq \|x - y\|$.
(14) $\|x - z\| \leq \|x - y\| + \|y - z\|$.
(15) If $x \neq y$, then $\|x - y\| \neq 0$.

Let us consider R_1. A subset of R_1 is a subset of the vectors of the vectors of R_1.

Let us consider R_1. A function is called a sequence of R_1 if:

(Def.3) dom it $= \mathbb{N}$ and rng it $\subseteq$ the vectors of the vectors of R_1.

For simplicity we adopt the following rules: S, S_1, S_2, T are sequences of R_1, k, n, m are natural numbers, r is a real number, f is a function, and d is arbitrary. We now state several propositions:

(17)[3] f is a sequence of R_1 if and only if dom $f = \mathbb{N}$ and for every d such that $d \in \mathbb{N}$ holds $f(d)$ is a point of R_1.
(18) For all S, T such that for every n holds $S(n) = T(n)$ holds $S = T$.
(19) For every x there exists S such that rng $S = \{x\}$.
(20) If there exists x such that for every n holds $S(n) = x$, then there exists x such that rng $S = \{x\}$.
(21) If there exists x such that rng $S = \{x\}$, then for every n holds $S(n) = S(n + 1)$.
(22) If for every n holds $S(n) = S(n + 1)$, then for all n, k holds $S(n) = S(n + k)$.
(23) If for all n, k holds $S(n) = S(n+k)$, then for all n, m holds $S(n) = S(m)$.
(24) If for all n, m holds $S(n) = S(m)$, then there exists x such that for every n holds $S(n) = x$.
(25) There exists S such that rng $S = \{0_{\text{the vectors of } R_1}\}$.

Let us consider R_1, S. We say that S is constant if and only if:

(Def.4) there exists x such that for every n holds $S(n) = x$.

The following propositions are true:

(27)[4] S is constant if and only if there exists x such that rng $S = \{x\}$.

[3]The proposition (16) was either repeated or obvious.
[4]The proposition (26) was either repeated or obvious.

(28) For every n holds $S(n)$ is a point of R_1.

Let us consider R_1, S, n. Then $S(n)$ is a point of R_1.

The scheme *ExRNSSeq* concerns a real normed space $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$ and states that:

there exists a sequence S of $\mathcal{A}$ such that for every n holds $S(n) = \mathcal{F}(n)$ for all values of the parameters.

Let us consider R_1, S_1, S_2. The functor $S_1 + S_2$ yielding a sequence of R_1 is defined as follows:

(Def.5) for every n holds $(S_1 + S_2)(n) = S_1(n) + S_2(n)$.

One can prove the following proposition

(29) $S = S_1 + S_2$ if and only if for every n holds $S(n) = S_1(n) + S_2(n)$.

Let us consider R_1, S_1, S_2. The functor $S_1 - S_2$ yielding a sequence of R_1 is defined as follows:

(Def.6) for every n holds $(S_1 - S_2)(n) = S_1(n) - S_2(n)$.

The following proposition is true

(30) $S = S_1 - S_2$ if and only if for every n holds $S(n) = S_1(n) - S_2(n)$.

Let us consider R_1, S, x. The functor $S - x$ yields a sequence of R_1 and is defined by:

(Def.7) for every n holds $(S - x)(n) = S(n) - x$.

Next we state the proposition

(31) $T = S - x$ if and only if for every n holds $T(n) = S(n) - x$.

Let us consider R_1, S, a. The functor $a \cdot S$ yields a sequence of R_1 and is defined by:

(Def.8) for every n holds $(a \cdot S)(n) = a \cdot S(n)$.

We now state the proposition

(32) $T = a \cdot S$ if and only if for every n holds $T(n) = a \cdot S(n)$.

Let us consider R_1, S. We say that S is convergent if and only if:

(Def.9) there exists g such that for every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\|S(n) - g\| < r$.

One can prove the following propositions:

(34)[5] If S_1 is convergent and S_2 is convergent, then $S_1 + S_2$ is convergent.

(35) If S_1 is convergent and S_2 is convergent, then $S_1 - S_2$ is convergent.

(36) If S is convergent, then $S - x$ is convergent.

(37) If S is convergent, then $a \cdot S$ is convergent.

Let us consider R_1, S. The functor $\|S\|$ yielding a sequence of real numbers is defined by:

(Def.10) for every n holds $\|S\|(n) = \|S(n)\|$.

Next we state two propositions:

[5]The proposition (33) was either repeated or obvious.

(38) $\|S\|$ is a sequence of real numbers if and only if for every n holds $\|S\|(n) = \|S(n)\|$.

(39) If S is convergent, then $\|S\|$ is convergent.

Let us consider R_1, S. Let us assume that S is convergent. The functor $\lim S$ yielding a point of R_1 is defined by:

(Def.11) for every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\|S(n) - (\lim S)\| < r$.

The following propositions are true:

(40) If S is convergent, then $\lim S = g$ if and only if for every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\|S(n) - g\| < r$.

(41) If S is convergent and $\lim S = g$, then $\|S-g\|$ is convergent and $\lim\|S-g\| = 0$.

(42) If S_1 is convergent and S_2 is convergent, then $\lim(S_1 + S_2) = \lim S_1 + \lim S_2$.

(43) If S_1 is convergent and S_2 is convergent, then $\lim(S_1 - S_2) = \lim S_1 - \lim S_2$.

(44) If S is convergent, then $\lim(S - x) = \lim S - x$.

(45) If S is convergent, then $\lim(a \cdot S) = a \cdot (\lim S)$.

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Received September 20, 1990

FORMALIZED MATHEMATICS
Vol.2,No.1, January–February 1991
Université Catholique de Louvain

Schemes of Existence of some Types of Functions

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Summary. We prove some useful shemes of existence of real sequences, partial functions from a domain into a domain, partial functions from a set to a set and functions from a domain into a domain. At the begining we prove some related auxiliary theorems to the article [1].

MML Identifier: `SCHEME1`.

The notation and terminology used here are introduced in the following articles: [9], [5], [1], [2], [3], [8], [6], [4], and [7]. We adopt the following convention: x, y will be arbitrary, n, m will denote natural numbers, and r will denote a real number. Next we state four propositions:

(1) For every n there exists m such that $n = 2 \cdot m$ or $n = 2 \cdot m + 1$.

(2) For every n there exists m such that $n = 3 \cdot m$ or $n = 3 \cdot m + 1$ or $n = 3 \cdot m + 2$.

(3) For every n there exists m such that $n = 4 \cdot m$ or $n = 4 \cdot m + 1$ or $n = 4 \cdot m + 2$ or $n = 4 \cdot m + 3$.

(4) For every n there exists m such that $n = 5 \cdot m$ or $n = 5 \cdot m + 1$ or $n = 5 \cdot m + 2$ or $n = 5 \cdot m + 3$ or $n = 5 \cdot m + 4$.

In this article we present several logical schemes. The scheme *ExRealSubseq* concerns a sequence of real numbers $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists a sequence of real numbers q such that q is a subsequence of $\mathcal{A}$ and for every n holds $\mathcal{P}[q(n)]$ and for every n such that for every r such that $r = \mathcal{A}(n)$ holds $\mathcal{P}[r]$ there exists m such that $\mathcal{A}(n) = q(m)$

provided the following requirement is met:

- for every n there exists m such that $n \leq m$ and $\mathcal{P}[\mathcal{A}(m)]$.

[1]Supported by RPBP.III-24.C8

ISSN 0777–4028

The scheme *ExRealSeq2* deals with a unary functor $\mathcal{F}$ yielding a real number and a unary functor $\mathcal{G}$ yielding a real number and states that:

there exists a sequence of real numbers s such that for every n holds $s(2 \cdot n) = \mathcal{F}(n)$ and $s(2 \cdot n + 1) = \mathcal{G}(n)$

for all values of the parameters.

The scheme *ExRealSeq3* deals with a unary functor $\mathcal{F}$ yielding a real number, a unary functor $\mathcal{G}$ yielding a real number, and a unary functor $\mathcal{H}$ yielding a real number and states that:

there exists a sequence of real numbers s such that for every n holds $s(3 \cdot n) = \mathcal{F}(n)$ and $s(3 \cdot n + 1) = \mathcal{G}(n)$ and $s(3 \cdot n + 2) = \mathcal{H}(n)$

for all values of the parameters.

The scheme *ExRealSeq4* deals with a unary functor $\mathcal{F}$ yielding a real number, a unary functor $\mathcal{G}$ yielding a real number, a unary functor $\mathcal{H}$ yielding a real number, and a unary functor $\mathcal{I}$ yielding a real number and states that:

there exists a sequence of real numbers s such that for every n holds $s(4 \cdot n) = \mathcal{F}(n)$ and $s(4 \cdot n + 1) = \mathcal{G}(n)$ and $s(4 \cdot n + 2) = \mathcal{H}(n)$ and $s(4 \cdot n + 3) = \mathcal{I}(n)$

for all values of the parameters.

The scheme *ExRealSeq5* deals with a unary functor $\mathcal{F}$ yielding a real number, a unary functor $\mathcal{G}$ yielding a real number, a unary functor $\mathcal{H}$ yielding a real number, a unary functor $\mathcal{I}$ yielding a real number, and a unary functor $\mathcal{J}$ yielding a real number and states that:

there exists a sequence of real numbers s such that for every n holds $s(5 \cdot n) = \mathcal{F}(n)$ and $s(5 \cdot n + 1) = \mathcal{G}(n)$ and $s(5 \cdot n + 2) = \mathcal{H}(n)$ and $s(5 \cdot n + 3) = \mathcal{I}(n)$ and $s(5 \cdot n + 4) = \mathcal{J}(n)$

for all values of the parameters.

The scheme *PartFuncExD2* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

there exists a partial function f from $\mathcal{A}$ to $\mathcal{B}$ such that for every element c of $\mathcal{A}$ holds $c \in \operatorname{dom} f$ if and only if $\mathcal{P}[c]$ or $\mathcal{Q}[c]$ and for every element c of $\mathcal{A}$ such that $c \in \operatorname{dom} f$ holds if $\mathcal{P}[c]$, then $f(c) = \mathcal{F}(c)$ but if $\mathcal{Q}[c]$, then $f(c) = \mathcal{G}(c)$

provided the following condition is met:

- for every element c of $\mathcal{A}$ such that $\mathcal{P}[c]$ holds not $\mathcal{Q}[c]$.

The scheme *PartFuncExD2'* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

there exists a partial function f from $\mathcal{A}$ to $\mathcal{B}$ such that for every element c of $\mathcal{A}$ holds $c \in \operatorname{dom} f$ if and only if $\mathcal{P}[c]$ or $\mathcal{Q}[c]$ and for every element c of $\mathcal{A}$ such that $c \in \operatorname{dom} f$ holds if $\mathcal{P}[c]$, then $f(c) = \mathcal{F}(c)$ but if $\mathcal{Q}[c]$, then $f(c) = \mathcal{G}(c)$

provided the following requirement is met:

- for every element c of $\mathcal{A}$ such that $\mathcal{P}[c]$ and $\mathcal{Q}[c]$ holds $\mathcal{F}(c) = \mathcal{G}(c)$.

The scheme *PartFuncExD2"* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

there exists a partial function f from $\mathcal{A}$ to $\mathcal{B}$ such that f is total and for every element c of $\mathcal{A}$ such that $c \in \mathrm{dom}\, f$ holds if $\mathcal{P}[c]$, then $f(c) = \mathcal{F}(c)$ but if not $\mathcal{P}[c]$, then $f(c) = \mathcal{G}(c)$
for all values of the parameters.

The scheme *PartFuncExD3* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{B}$, and three unary predicates $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$, and states that:

there exists a partial function f from $\mathcal{A}$ to $\mathcal{B}$ such that for every element c of $\mathcal{A}$ holds $c \in \mathrm{dom}\, f$ if and only if $\mathcal{P}[c]$ or $\mathcal{Q}[c]$ or $\mathcal{R}[c]$ and for every element c of $\mathcal{A}$ such that $c \in \mathrm{dom}\, f$ holds if $\mathcal{P}[c]$, then $f(c) = \mathcal{F}(c)$ but if $\mathcal{Q}[c]$, then $f(c) = \mathcal{G}(c)$ but if $\mathcal{R}[c]$, then $f(c) = \mathcal{H}(c)$
provided the parameters satisfy the following condition:

- for every element c of $\mathcal{A}$ holds if $\mathcal{P}[c]$, then not $\mathcal{Q}[c]$ but if $\mathcal{P}[c]$, then not $\mathcal{R}[c]$ but if $\mathcal{Q}[c]$, then not $\mathcal{R}[c]$.

The scheme *PartFuncExD3'* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{B}$, and three unary predicates $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$, and states that:

there exists a partial function f from $\mathcal{A}$ to $\mathcal{B}$ such that for every element c of $\mathcal{A}$ holds $c \in \mathrm{dom}\, f$ if and only if $\mathcal{P}[c]$ or $\mathcal{Q}[c]$ or $\mathcal{R}[c]$ and for every element c of $\mathcal{A}$ such that $c \in \mathrm{dom}\, f$ holds if $\mathcal{P}[c]$, then $f(c) = \mathcal{F}(c)$ but if $\mathcal{Q}[c]$, then $f(c) = \mathcal{G}(c)$ but if $\mathcal{R}[c]$, then $f(c) = \mathcal{H}(c)$
provided the following requirement is met:

- for every element c of $\mathcal{A}$ holds if $\mathcal{P}[c]$ and $\mathcal{Q}[c]$, then $\mathcal{F}(c) = \mathcal{G}(c)$ but if $\mathcal{P}[c]$ and $\mathcal{R}[c]$, then $\mathcal{F}(c) = \mathcal{H}(c)$ but if $\mathcal{Q}[c]$ and $\mathcal{R}[c]$, then $\mathcal{G}(c) = \mathcal{H}(c)$.

The scheme *PartFuncExD4* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{B}$, a unary functor $\mathcal{I}$ yielding an element of $\mathcal{B}$, and four unary predicates $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{S}$, and states that:

there exists a partial function f from $\mathcal{A}$ to $\mathcal{B}$ such that for every element c of $\mathcal{A}$ holds $c \in \mathrm{dom}\, f$ if and only if $\mathcal{P}[c]$ or $\mathcal{Q}[c]$ or $\mathcal{R}[c]$ or $\mathcal{S}[c]$ and for every element c of $\mathcal{A}$ such that $c \in \mathrm{dom}\, f$ holds if $\mathcal{P}[c]$, then $f(c) = \mathcal{F}(c)$ but if $\mathcal{Q}[c]$, then $f(c) = \mathcal{G}(c)$ but if $\mathcal{R}[c]$, then $f(c) = \mathcal{H}(c)$ but if $\mathcal{S}[c]$, then $f(c) = \mathcal{I}(c)$
provided the parameters satisfy the following condition:

- for every element c of $\mathcal{A}$ holds if $\mathcal{P}[c]$, then not $\mathcal{Q}[c]$ but if $\mathcal{P}[c]$, then not $\mathcal{R}[c]$ but if $\mathcal{P}[c]$, then not $\mathcal{S}[c]$ but if $\mathcal{Q}[c]$, then not $\mathcal{R}[c]$ but if $\mathcal{Q}[c]$, then not $\mathcal{S}[c]$ but if $\mathcal{R}[c]$, then not $\mathcal{S}[c]$.

The scheme *PartFuncExS2* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, a unary functor $\mathcal{F}$, a unary functor $\mathcal{G}$, and two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

there exists a partial function f from $\mathcal{A}$ to $\mathcal{B}$ such that for every x holds $x \in \mathrm{dom}\, f$ if and only if $x \in \mathcal{A}$ but $\mathcal{P}[x]$ or $\mathcal{Q}[x]$ and for every x such that $x \in \mathrm{dom}\, f$ holds if $\mathcal{P}[x]$, then $f(x) = \mathcal{F}(x)$ but if $\mathcal{Q}[x]$, then $f(x) = \mathcal{G}(x)$

provided the parameters satisfy the following conditions:

- for every x such that $x \in \mathcal{A}$ holds if $\mathcal{P}[x]$, then not $\mathcal{Q}[x]$,
- for every x such that $x \in \mathcal{A}$ and $\mathcal{P}[x]$ holds $\mathcal{F}(x) \in \mathcal{B}$,
- for every x such that $x \in \mathcal{A}$ and $\mathcal{Q}[x]$ holds $\mathcal{G}(x) \in \mathcal{B}$.

The scheme *PartFuncExS3* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, a unary functor $\mathcal{F}$, a unary functor $\mathcal{G}$, a unary functor $\mathcal{H}$, and three unary predicates $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$, and states that:

there exists a partial function f from $\mathcal{A}$ to $\mathcal{B}$ such that for every x holds $x \in \operatorname{dom} f$ if and only if $x \in \mathcal{A}$ but $\mathcal{P}[x]$ or $\mathcal{Q}[x]$ or $\mathcal{R}[x]$ and for every x such that $x \in \operatorname{dom} f$ holds if $\mathcal{P}[x]$, then $f(x) = \mathcal{F}(x)$ but if $\mathcal{Q}[x]$, then $f(x) = \mathcal{G}(x)$ but if $\mathcal{R}[x]$, then $f(x) = \mathcal{H}(x)$

provided the parameters meet the following conditions:

- for every x such that $x \in \mathcal{A}$ holds if $\mathcal{P}[x]$, then not $\mathcal{Q}[x]$ but if $\mathcal{P}[x]$, then not $\mathcal{R}[x]$ but if $\mathcal{Q}[x]$, then not $\mathcal{R}[x]$,
- for every x such that $x \in \mathcal{A}$ and $\mathcal{P}[x]$ holds $\mathcal{F}(x) \in \mathcal{B}$,
- for every x such that $x \in \mathcal{A}$ and $\mathcal{Q}[x]$ holds $\mathcal{G}(x) \in \mathcal{B}$,
- for every x such that $x \in \mathcal{A}$ and $\mathcal{R}[x]$ holds $\mathcal{H}(x) \in \mathcal{B}$.

The scheme *PartFuncExS4* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, a unary functor $\mathcal{F}$, a unary functor $\mathcal{G}$, a unary functor $\mathcal{H}$, a unary functor $\mathcal{I}$, and four unary predicates $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{S}$, and states that:

there exists a partial function f from $\mathcal{A}$ to $\mathcal{B}$ such that for every x holds $x \in \operatorname{dom} f$ if and only if $x \in \mathcal{A}$ but $\mathcal{P}[x]$ or $\mathcal{Q}[x]$ or $\mathcal{R}[x]$ or $\mathcal{S}[x]$ and for every x such that $x \in \operatorname{dom} f$ holds if $\mathcal{P}[x]$, then $f(x) = \mathcal{F}(x)$ but if $\mathcal{Q}[x]$, then $f(x) = \mathcal{G}(x)$ but if $\mathcal{R}[x]$, then $f(x) = \mathcal{H}(x)$ but if $\mathcal{S}[x]$, then $f(x) = \mathcal{I}(x)$

provided the parameters meet the following requirements:

- for every x such that $x \in \mathcal{A}$ holds if $\mathcal{P}[x]$, then not $\mathcal{Q}[x]$ but if $\mathcal{P}[x]$, then not $\mathcal{R}[x]$ but if $\mathcal{P}[x]$, then not $\mathcal{S}[x]$ but if $\mathcal{Q}[x]$, then not $\mathcal{R}[x]$ but if $\mathcal{Q}[x]$, then not $\mathcal{S}[x]$ but if $\mathcal{R}[x]$, then not $\mathcal{S}[x]$,
- for every x such that $x \in \mathcal{A}$ and $\mathcal{P}[x]$ holds $\mathcal{F}(x) \in \mathcal{B}$,
- for every x such that $x \in \mathcal{A}$ and $\mathcal{Q}[x]$ holds $\mathcal{G}(x) \in \mathcal{B}$,
- for every x such that $x \in \mathcal{A}$ and $\mathcal{R}[x]$ holds $\mathcal{H}(x) \in \mathcal{B}$,
- for every x such that $x \in \mathcal{A}$ and $\mathcal{S}[x]$ holds $\mathcal{I}(x) \in \mathcal{B}$.

The scheme *PartFuncExC_D2* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a non-empty set $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, a binary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and two binary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

there exists a partial function f from $[\!: \mathcal{A}, \mathcal{B} :\!]$ to $\mathcal{C}$ such that for every element c of $\mathcal{A}$ and for every element d of $\mathcal{B}$ holds $\langle c, d \rangle \in \operatorname{dom} f$ if and only if $\mathcal{P}[c, d]$ or $\mathcal{Q}[c, d]$ and for every element c of $\mathcal{A}$ and for every element d of $\mathcal{B}$ such that $\langle c, d \rangle \in \operatorname{dom} f$ holds if $\mathcal{P}[c, d]$, then $f(\langle c, d \rangle) = \mathcal{F}(c, d)$ but if $\mathcal{Q}[c, d]$, then $f(\langle c, d \rangle) = \mathcal{G}(c, d)$

provided the parameters meet the following requirement:

- for every element c of $\mathcal{A}$ and for every element d of $\mathcal{B}$ such that $\mathcal{P}[c, d]$ holds not $\mathcal{Q}[c, d]$.

The scheme *PartFuncExC_D3* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a non-empty set $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, a binary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{C}$, and three binary predicates $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$, and states that:

there exists a partial function f from $[:\mathcal{A}, \mathcal{B}:]$ to $\mathcal{C}$ such that for every element c of $\mathcal{A}$ and for every element d of $\mathcal{B}$ holds $\langle c, d\rangle \in \operatorname{dom} f$ if and only if $\mathcal{P}[c, d]$ or $\mathcal{Q}[c, d]$ or $\mathcal{R}[c, d]$ and for every element c of $\mathcal{A}$ and for every element r of $\mathcal{B}$ such that $\langle c, r\rangle \in \operatorname{dom} f$ holds if $\mathcal{P}[c, r]$, then $f(\langle c, r\rangle) = \mathcal{F}(c, r)$ but if $\mathcal{Q}[c, r]$, then $f(\langle c, r\rangle) = \mathcal{G}(c, r)$ but if $\mathcal{R}[c, r]$, then $f(\langle c, r\rangle) = \mathcal{H}(c, r)$

provided the following requirement is met:

- for every element c of $\mathcal{A}$ and for every element s of $\mathcal{B}$ holds if $\mathcal{P}[c, s]$, then not $\mathcal{Q}[c, s]$ but if $\mathcal{P}[c, s]$, then not $\mathcal{R}[c, s]$ but if $\mathcal{Q}[c, s]$, then not $\mathcal{R}[c, s]$.

The scheme *PartFuncExC_S2* concerns a set $\mathcal{A}$, a set $\mathcal{B}$, a set $\mathcal{C}$, a binary functor $\mathcal{F}$, a binary functor $\mathcal{G}$, and two binary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

there exists a partial function f from $[:\mathcal{A}, \mathcal{B}:]$ to $\mathcal{C}$ such that for all x, y holds $\langle x, y\rangle \in \operatorname{dom} f$ if and only if $x \in \mathcal{A}$ and $y \in \mathcal{B}$ but $\mathcal{P}[x, y]$ or $\mathcal{Q}[x, y]$ and for all x, y such that $\langle x, y\rangle \in \operatorname{dom} f$ holds if $\mathcal{P}[x, y]$, then $f(\langle x, y\rangle) = \mathcal{F}(x, y)$ but if $\mathcal{Q}[x, y]$, then $f(\langle x, y\rangle) = \mathcal{G}(x, y)$

provided the following conditions are met:

- for all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ holds if $\mathcal{P}[x, y]$, then not $\mathcal{Q}[x, y]$,
- for all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and $\mathcal{P}[x, y]$ holds $\mathcal{F}(x, y) \in \mathcal{C}$,
- for all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and $\mathcal{Q}[x, y]$ holds $\mathcal{G}(x, y) \in \mathcal{C}$.

The scheme *PartFuncExC_S3* concerns a set $\mathcal{A}$, a set $\mathcal{B}$, a set $\mathcal{C}$, a binary functor $\mathcal{F}$, a binary functor $\mathcal{G}$, a binary functor $\mathcal{H}$, and three binary predicates $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$, and states that:

there exists a partial function f from $[:\mathcal{A}, \mathcal{B}:]$ to $\mathcal{C}$ such that for all x, y holds $\langle x, y\rangle \in \operatorname{dom} f$ if and only if $x \in \mathcal{A}$ and $y \in \mathcal{B}$ but $\mathcal{P}[x, y]$ or $\mathcal{Q}[x, y]$ or $\mathcal{R}[x, y]$ and for all x, y such that $\langle x, y\rangle \in \operatorname{dom} f$ holds if $\mathcal{P}[x, y]$, then $f(\langle x, y\rangle) = \mathcal{F}(x, y)$ but if $\mathcal{Q}[x, y]$, then $f(\langle x, y\rangle) = \mathcal{G}(x, y)$ but if $\mathcal{R}[x, y]$, then $f(\langle x, y\rangle) = \mathcal{H}(x, y)$

provided the following conditions are met:

- for all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ holds if $\mathcal{P}[x, y]$, then not $\mathcal{Q}[x, y]$ but if $\mathcal{P}[x, y]$, then not $\mathcal{R}[x, y]$ but if $\mathcal{Q}[x, y]$, then not $\mathcal{R}[x, y]$,
- for all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ holds if $\mathcal{P}[x, y]$, then $\mathcal{F}(x, y) \in \mathcal{C}$,
- for all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ holds if $\mathcal{Q}[x, y]$, then $\mathcal{G}(x, y) \in \mathcal{C}$,
- for all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ holds if $\mathcal{R}[x, y]$, then $\mathcal{H}(x, y) \in \mathcal{C}$.

The scheme *ExFuncD3* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{B}$, and three unary predicates $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$, and states that:

there exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every element c of $\mathcal{A}$ holds if $\mathcal{P}[c]$, then $f(c) = \mathcal{F}(c)$ but if $\mathcal{Q}[c]$, then $f(c) = \mathcal{G}(c)$ but if $\mathcal{R}[c]$, then $f(c) = \mathcal{H}(c)$

provided the parameters satisfy the following conditions:

- for every element c of $\mathcal{A}$ holds if $\mathcal{P}[c]$, then not $\mathcal{Q}[c]$ but if $\mathcal{P}[c]$, then not $\mathcal{R}[c]$ but if $\mathcal{Q}[c]$, then not $\mathcal{R}[c]$,
- for every element c of $\mathcal{A}$ holds $\mathcal{P}[c]$ or $\mathcal{Q}[c]$ or $\mathcal{R}[c]$.

The scheme *ExFuncD4* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{B}$, a unary functor $\mathcal{I}$ yielding an element of $\mathcal{B}$, and four unary predicates $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{S}$, and states that:

there exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every element c of $\mathcal{A}$ holds if $\mathcal{P}[c]$, then $f(c) = \mathcal{F}(c)$ but if $\mathcal{Q}[c]$, then $f(c) = \mathcal{G}(c)$ but if $\mathcal{R}[c]$, then $f(c) = \mathcal{H}(c)$ but if $\mathcal{S}[c]$, then $f(c) = \mathcal{I}(c)$

provided the following conditions are met:

- for every element c of $\mathcal{A}$ holds if $\mathcal{P}[c]$, then not $\mathcal{Q}[c]$ but if $\mathcal{P}[c]$, then not $\mathcal{R}[c]$ but if $\mathcal{P}[c]$, then not $\mathcal{S}[c]$ but if $\mathcal{Q}[c]$, then not $\mathcal{R}[c]$ but if $\mathcal{Q}[c]$, then not $\mathcal{S}[c]$ but if $\mathcal{R}[c]$, then not $\mathcal{S}[c]$,
- for every element c of $\mathcal{A}$ holds $\mathcal{P}[c]$ or $\mathcal{Q}[c]$ or $\mathcal{R}[c]$ or $\mathcal{S}[c]$.

The scheme *FuncExC_D2* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a non-empty set $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, a binary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a function f from $[\!:\mathcal{A}, \mathcal{B}:\!]$ into $\mathcal{C}$ such that for every element c of $\mathcal{A}$ and for every element d of $\mathcal{B}$ such that $\langle c, d\rangle \in \operatorname{dom} f$ holds if $\mathcal{P}[c, d]$, then $f(\langle c, d\rangle) = \mathcal{F}(c, d)$ but if not $\mathcal{P}[c, d]$, then $f(\langle c, d\rangle) = \mathcal{G}(c, d)$

for all values of the parameters.

The scheme *FuncExC_D3* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a non-empty set $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, a binary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{C}$, and three binary predicates $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$, and states that:

there exists a function f from $[\!:\mathcal{A}, \mathcal{B}:\!]$ into $\mathcal{C}$ such that for every element c of $\mathcal{A}$ and for every element d of $\mathcal{B}$ holds $\langle c, d\rangle \in \operatorname{dom} f$ if and only if $\mathcal{P}[c, d]$ or $\mathcal{Q}[c, d]$ or $\mathcal{R}[c, d]$ and for every element c of $\mathcal{A}$ and for every element d of $\mathcal{B}$ such that $\langle c, d\rangle \in \operatorname{dom} f$ holds if $\mathcal{P}[c, d]$, then $f(\langle c, d\rangle) = \mathcal{F}(c, d)$ but if $\mathcal{Q}[c, d]$, then $f(\langle c, d\rangle) = \mathcal{G}(c, d)$ but if $\mathcal{R}[c, d]$, then $f(\langle c, d\rangle) = \mathcal{H}(c, d)$

provided the parameters have the following properties:

- for every element c of $\mathcal{A}$ and for every element d of $\mathcal{B}$ holds if $\mathcal{P}[c, d]$, then not $\mathcal{Q}[c, d]$ but if $\mathcal{P}[c, d]$, then not $\mathcal{R}[c, d]$ but if $\mathcal{Q}[c, d]$, then not $\mathcal{R}[c, d]$,
- for every element c of $\mathcal{A}$ and for every element d of $\mathcal{B}$ holds $\mathcal{P}[c, d]$ or $\mathcal{Q}[c, d]$ or $\mathcal{R}[c, d]$.

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Received September 21, 1990

FORMALIZED MATHEMATICS
Vol.2,No.1, January–February 1991
Université Catholique de Louvain

Integer and Rational Exponents

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Summary. The article includes definitions and theorems which are needed to define real exponent. The following notions are defined: natural exponent, integer exponent and rational exponent.

MML Identifier: `PREPOWER`.

The terminology and notation used in this paper are introduced in the following papers: [12], [15], [4], [10], [1], [2], [3], [9], [7], [8], [14], [11], [13], [6], and [5]. For simplicity we follow the rules: a, b, c will be real numbers, m, n will be natural numbers, k, l, i will be integers, p, q will be rational numbers, and s_1, s_2 will be sequences of real numbers. The following propositions are true:

(2)[2] If s_1 is convergent and for every n holds $s_1(n) \geq a$, then $\lim s_1 \geq a$.

(3) If s_1 is convergent and for every n holds $s_1(n) \leq a$, then $\lim s_1 \leq a$.

Let us consider a. The functor $(a^\kappa)_{\kappa\in\mathbb{N}}$ yielding a sequence of real numbers is defined as follows:

(Def.1) $((a^\kappa)_{\kappa\in\mathbb{N}})(0) = 1$ and for every m holds $((a^\kappa)_{\kappa\in\mathbb{N}})(m+1) = ((a^\kappa)_{\kappa\in\mathbb{N}})(m) \cdot a$.

Next we state two propositions:

(4) For every sequence of real numbers s and for every a holds $s = (a^\kappa)_{\kappa\in\mathbb{N}}$ if and only if $s(0) = 1$ and for every m holds $s(m+1) = s(m) \cdot a$.

(5) For every a such that $a \neq 0$ for every m holds $(a^\kappa)_{\kappa\in\mathbb{N}}(m) \neq 0$.

Let us consider a, n. The functor $a^n_{\mathbb{N}}$ yields a real number and is defined by:

(Def.2) $a^n_{\mathbb{N}} = (a^\kappa)_{\kappa\in\mathbb{N}}(n)$.

Next we state a number of propositions:

(6) $a^n_{\mathbb{N}} = (a^\kappa)_{\kappa\in\mathbb{N}}(n)$.

[1]Supported by RPBP.III-24.C8

[2]The proposition (1) was either repeated or obvious.

ISSN 0777–4028

(7) $a_{\mathbb{N}}^{n} \cdot a = a_{\mathbb{N}}^{n+1}$.
(8) $1_{\mathbb{N}}^{n} = 1$.
(9) $a_{\mathbb{N}}^{n+m} = a_{\mathbb{N}}^{n} \cdot a_{\mathbb{N}}^{m}$.
(10) $(a \cdot b)_{\mathbb{N}}^{n} = a_{\mathbb{N}}^{n} \cdot b_{\mathbb{N}}^{n}$.
(11) $a_{\mathbb{N}}^{n \cdot m} = (a_{\mathbb{N}}^{n})_{\mathbb{N}}^{m}$.
(12) If $0 \neq a$, then $0 \neq a_{\mathbb{N}}^{n}$.
(13) If $0 < a$, then $0 < a_{\mathbb{N}}^{n}$.
(14) If $a \neq 0$, then $\frac{1}{a}_{\mathbb{N}}^{n} = \frac{1}{a_{\mathbb{N}}^{n}}$.
(15) If $a \neq 0$, then $\frac{b}{a}_{\mathbb{N}}^{n} = \frac{b_{\mathbb{N}}^{n}}{a_{\mathbb{N}}^{n}}$.
(16) If $n \geq 1$, then $0_{\mathbb{N}}^{n} = 0$.
(17) If $0 < a$ and $a \leq b$, then $a_{\mathbb{N}}^{n} \leq b_{\mathbb{N}}^{n}$.
(18) If $0 \leq a$ and $a < b$ and $1 \leq n$, then $a_{\mathbb{N}}^{n} < b_{\mathbb{N}}^{n}$.
(19) If $a \geq 1$, then $a_{\mathbb{N}}^{n} \geq 1$.
(20) If $1 \leq a$ and $1 \leq n$, then $a \leq a_{\mathbb{N}}^{n}$.
(21) If $1 < a$ and $2 \leq n$, then $a < a_{\mathbb{N}}^{n}$.
(22) If $0 < a$ and $a \leq 1$ and $1 \leq n$, then $a_{\mathbb{N}}^{n} \leq a$.
(23) If $0 < a$ and $a < 1$ and $2 \leq n$, then $a_{\mathbb{N}}^{n} < a$.
(24) If $-1 < a$, then $(1+a)_{\mathbb{N}}^{n} \geq 1 + n \cdot a$.
(25) If $0 < a$ and $a < 1$, then $(1+a)_{\mathbb{N}}^{n} \leq 1 + 3_{\mathbb{N}}^{n} \cdot a$.
(26) If s_1 is convergent and for every n holds $s_2(n) = (s_1(n))_{\mathbb{N}}^{m}$, then s_2 is convergent and $\lim s_2 = (\lim s_1)_{\mathbb{N}}^{m}$.

Let us consider n, a. Let us assume that $1 \leq n$. The functor $\mathrm{root}_n(a)$ yields a real number and is defined as follows:

(Def.3) $(\mathrm{root}_n(a))_{\mathbb{N}}^{n} = a$ and $\mathrm{root}_n(a) > 0$ if $a > 0$, $\mathrm{root}_n(a) = 0$ if $a = 0$.

Next we state a number of propositions:

(27) For all a, b, n such that $1 \leq n$ holds if $a > 0$, then $b = \mathrm{root}_n(a)$ if and only if $b_{\mathbb{N}}^{n} = a$ and $b > 0$ but if $a = 0$, then $\mathrm{root}_n(a) = 0$.
(28) If $a \geq 0$ and $n \geq 1$, then $(\mathrm{root}_n(a))_{\mathbb{N}}^{n} = a$ and $\mathrm{root}_n(a_{\mathbb{N}}^{n}) = a$.
(29) If $n \geq 1$, then $\mathrm{root}_n(1) = 1$.
(30) If $a \geq 0$, then $\mathrm{root}_1(a) = a$.
(31) If $a \geq 0$ and $b \geq 0$ and $n \geq 1$, then $\mathrm{root}_n(a \cdot b) = \mathrm{root}_n(a) \cdot \mathrm{root}_n(b)$.
(32) If $a > 0$ and $n \geq 1$, then $\mathrm{root}_n(\frac{1}{a}) = \frac{1}{\mathrm{root}_n(a)}$.
(33) If $a \geq 0$ and $b > 0$ and $n \geq 1$, then $\mathrm{root}_n(\frac{a}{b}) = \frac{\mathrm{root}_n(a)}{\mathrm{root}_n(b)}$.
(34) If $a \geq 0$ and $n \geq 1$ and $m \geq 1$, then $\mathrm{root}_n(\mathrm{root}_m(a)) = \mathrm{root}_{n \cdot m}(a)$.
(35) If $a \geq 0$ and $n \geq 1$ and $m \geq 1$, then $\mathrm{root}_n(a) \cdot \mathrm{root}_m(a) = \mathrm{root}_{n \cdot m}(a_{\mathbb{N}}^{n+m})$.
(36) If $0 \leq a$ and $a \leq b$ and $n \geq 1$, then $\mathrm{root}_n(a) \leq \mathrm{root}_n(b)$.
(37) If $a \geq 0$ and $a < b$ and $n \geq 1$, then $\mathrm{root}_n(a) < \mathrm{root}_n(b)$.
(38) If $a \geq 1$ and $n \geq 1$, then $\mathrm{root}_n(a) \geq 1$ and $a \geq \mathrm{root}_n(a)$.

(39) If $0 \leq a$ and $a < 1$ and $n \geq 1$, then $a \leq \mathrm{root}_n(a)$ and $\mathrm{root}_n(a) < 1$.

(40) If $a > 0$ and $n \geq 1$, then $\mathrm{root}_n(a) - 1 \leq \frac{a-1}{n}$.

(41) If $a \geq 0$, then $\mathrm{root}_2(a) = \sqrt{a}$.

(42) For every sequence of real numbers s and for every a such that $a > 0$ and for every n such that $n \geq 1$ holds $s(n) = \mathrm{root}_n(a)$ holds s is convergent and $\lim s = 1$.

Let us consider a, k. Let us assume that $a \neq 0$. The functor $a^k_{\mathbb{Z}}$ yields a real number and is defined as follows:

(Def.4) $a^k_{\mathbb{Z}} = a^{|k|}_{\mathbb{N}}$ if $k \geq 0$, $a^k_{\mathbb{Z}} = (a^{|k|}_{\mathbb{N}})^{-1}$ if $k < 0$.

We now state a number of propositions:

(43) If $a \neq 0$, then if $k \geq 0$, then $a^k_{\mathbb{Z}} = a^{|k|}_{\mathbb{N}}$ but if $k < 0$, then $a^k_{\mathbb{Z}} = (a^{|k|}_{\mathbb{N}})^{-1}$.

(44) If $a \neq 0$, then for every i such that $i = 0$ holds $a^i_{\mathbb{Z}} = 1$.

(45) If $a \neq 0$, then for every i such that $i = 1$ holds $a^i_{\mathbb{Z}} = a$.

(46) If $a \neq 0$ and $i = n$, then $a^i_{\mathbb{Z}} = a^n_{\mathbb{N}}$.

(47) $1^k_{\mathbb{Z}} = 1$.

(48) If $a \neq 0$, then $a^k_{\mathbb{Z}} \neq 0$.

(49) If $a > 0$, then $a^k_{\mathbb{Z}} > 0$.

(50) If $a \neq 0$ and $b \neq 0$, then $(a \cdot b)^k_{\mathbb{Z}} = a^k_{\mathbb{Z}} \cdot b^k_{\mathbb{Z}}$.

(51) If $a \neq 0$, then $a^{-k}_{\mathbb{Z}} = \frac{1}{a^k_{\mathbb{Z}}}$.

(52) If $a \neq 0$, then $\frac{1}{a}^k_{\mathbb{Z}} = \frac{1}{a^k_{\mathbb{Z}}}$.

(53) If $a \neq 0$, then $a^{m-n}_{\mathbb{Z}} = \frac{a^m_{\mathbb{N}}}{a^n_{\mathbb{N}}}$.

(54) If $a \neq 0$, then $a^{k+l}_{\mathbb{Z}} = a^k_{\mathbb{Z}} \cdot a^l_{\mathbb{Z}}$.

(55) If $a \neq 0$, then $(a^k_{\mathbb{Z}})^l_{\mathbb{Z}} = a^{k \cdot l}_{\mathbb{Z}}$.

(56) If $a > 0$ and $n \geq 1$, then $(\mathrm{root}_n(a))^k_{\mathbb{Z}} = \mathrm{root}_n(a^k_{\mathbb{Z}})$.

Let us consider a, p. Let us assume that $a > 0$. The functor $a^p_{\mathbb{Q}}$ yielding a real number is defined by:

(Def.5) $a^p_{\mathbb{Q}} = \mathrm{root}_{\mathrm{den}\, p}(a^{\mathrm{num}\, p}_{\mathbb{Z}})$.

We now state a number of propositions:

(57) If $a > 0$, then $a^p_{\mathbb{Q}} = \mathrm{root}_{\mathrm{den}\, p}(a^{\mathrm{num}\, p}_{\mathbb{Z}})$.

(58) If $a > 0$ and $p = 0$, then $a^p_{\mathbb{Q}} = 1$.

(59) If $a > 0$ and $p = 1$, then $a^p_{\mathbb{Q}} = a$.

(60) If $a > 0$ and $p = n$, then $a^p_{\mathbb{Q}} = a^n_{\mathbb{N}}$.

(61) If $a > 0$ and $n \geq 1$ and $p = n^{-1}$, then $a^p_{\mathbb{Q}} = \mathrm{root}_n(a)$.

(62) $1^p_{\mathbb{Q}} = 1$.

(63) If $a > 0$, then $a^p_{\mathbb{Q}} > 0$.

(64) If $a > 0$, then $a^p_{\mathbb{Q}} \cdot a^q_{\mathbb{Q}} = a^{p+q}_{\mathbb{Q}}$.

(65) If $a > 0$, then $\frac{1}{a^p_{\mathbb{Q}}} = a^{-p}_{\mathbb{Q}}$.

(66) If $a > 0$, then $\frac{a_{\mathbb{Q}}^{p}}{a_{\mathbb{Q}}^{q}} = a_{\mathbb{Q}}^{p-q}$.

(67) If $a > 0$ and $b > 0$, then $(a \cdot b)_{\mathbb{Q}}^{p} = a_{\mathbb{Q}}^{p} \cdot b_{\mathbb{Q}}^{p}$.

(68) If $a > 0$, then $\frac{1}{a}_{\mathbb{Q}}^{p} = \frac{1}{a_{\mathbb{Q}}^{p}}$.

(69) If $a > 0$ and $b > 0$, then $\frac{a}{b}_{\mathbb{Q}}^{p} = \frac{a_{\mathbb{Q}}^{p}}{b_{\mathbb{Q}}^{p}}$.

(70) If $a > 0$, then $(a_{\mathbb{Q}}^{p})_{\mathbb{Q}}^{q} = a_{\mathbb{Q}}^{p \cdot q}$.

(71) If $a \geq 1$ and $p \geq 0$, then $a_{\mathbb{Q}}^{p} \geq 1$.

(72) If $a \geq 1$ and $p \leq 0$, then $a_{\mathbb{Q}}^{p} \leq 1$.

(73) If $a > 1$ and $p > 0$, then $a_{\mathbb{Q}}^{p} > 1$.

(74) If $a \geq 1$ and $p \geq q$, then $a_{\mathbb{Q}}^{p} \geq a_{\mathbb{Q}}^{q}$.

(75) If $a > 1$ and $p > q$, then $a_{\mathbb{Q}}^{p} > a_{\mathbb{Q}}^{q}$.

(76) If $a > 0$ and $a < 1$ and $p > 0$, then $a_{\mathbb{Q}}^{p} < 1$.

(77) If $a > 0$ and $a \leq 1$ and $p \leq 0$, then $a_{\mathbb{Q}}^{p} \geq 1$.

A sequence of real numbers is called a rational sequence if:

(Def.6) for every n holds it(n) is a rational number.

Let s be a rational sequence, and let us consider n. Then $s(n)$ is a rational number.

Next we state two propositions:

(79)[3] For every a there exists a rational sequence s such that s is convergent and $\lim s = a$ and for every n holds $s(n) \leq a$.

(80) For every a there exists a rational sequence s such that s is convergent and $\lim s = a$ and for every n holds $s(n) \geq a$.

Let us consider a, and let s be a rational sequence. Let us assume that $a > 0$. The functor $a_{\mathbb{Q}}^{s}$ yields a sequence of real numbers and is defined as follows:

(Def.7) for every n holds $(a_{\mathbb{Q}}^{s})(n) = a_{\mathbb{Q}}^{s(n)}$.

The following propositions are true:

(81) For every a and for every rational sequence s and for every s_1 such that $a > 0$ holds $s_1 = a_{\mathbb{Q}}^{s}$ if and only if for every n holds $s_1(n) = a_{\mathbb{Q}}^{s(n)}$.

(82) For every rational sequence s and for every a such that s is convergent and $a > 0$ holds $a_{\mathbb{Q}}^{s}$ is convergent.

(83) For all rational sequences s_1, s_2 and for every a such that s_1 is convergent and s_2 is convergent and $\lim s_1 = \lim s_2$ and $a > 0$ holds $a_{\mathbb{Q}}^{s_1}$ is convergent and $a_{\mathbb{Q}}^{s_2}$ is convergent and $\lim a_{\mathbb{Q}}^{s_1} = \lim a_{\mathbb{Q}}^{s_2}$.

Let us consider a, b. Let us assume that $a > 0$. The functor $a_{\mathbb{R}}^{b}$ yielding a real number is defined by:

(Def.8) there exists a rational sequence s such that s is convergent and $\lim s = b$ and $a_{\mathbb{Q}}^{s}$ is convergent and $\lim a_{\mathbb{Q}}^{s} = a_{\mathbb{R}}^{b}$.

We now state a number of propositions:

[3] The proposition (78) was either repeated or obvious.

(84) For all a, b, c such that $a > 0$ holds $c = a_{\mathbb{R}}^{b}$ if and only if there exists a rational sequence s such that s is convergent and $\lim s = b$ and $a_{\mathbb{Q}}^{s}$ is convergent and $\lim a_{\mathbb{Q}}^{s} = c$.

(85) If $a > 0$, then $a_{\mathbb{R}}^{0} = 1$.

(86) If $a > 0$, then $a_{\mathbb{R}}^{1} = a$.

(87) $1_{\mathbb{R}}^{a} = 1$.

(88) If $a > 0$, then $a_{\mathbb{R}}^{p} = a_{\mathbb{Q}}^{p}$.

(89) If $a > 0$, then $a_{\mathbb{R}}^{b+c} = a_{\mathbb{R}}^{b} \cdot a_{\mathbb{R}}^{c}$.

(90) If $a > 0$, then $a_{\mathbb{R}}^{-c} = \frac{1}{a_{\mathbb{R}}^{c}}$.

(91) If $a > 0$, then $a_{\mathbb{R}}^{b-c} = \frac{a_{\mathbb{R}}^{b}}{a_{\mathbb{R}}^{c}}$.

(92) If $a > 0$ and $b > 0$, then $(a \cdot b)_{\mathbb{R}}^{c} = a_{\mathbb{R}}^{c} \cdot b_{\mathbb{R}}^{c}$.

(93) If $a > 0$, then $\frac{1}{a}_{\mathbb{R}}^{c} = \frac{1}{a_{\mathbb{R}}^{c}}$.

(94) If $a > 0$ and $b > 0$, then $\frac{a}{b}_{\mathbb{R}}^{c} = \frac{a_{\mathbb{R}}^{c}}{b_{\mathbb{R}}^{c}}$.

(95) If $a > 0$, then $a_{\mathbb{R}}^{b} > 0$.

(96) If $a \geq 1$ and $c \geq b$, then $a_{\mathbb{R}}^{c} \geq a_{\mathbb{R}}^{b}$.

(97) If $a > 1$ and $c > b$, then $a_{\mathbb{R}}^{c} > a_{\mathbb{R}}^{b}$.

(98) If $a > 0$ and $a \leq 1$ and $c \geq b$, then $a_{\mathbb{R}}^{c} \leq a_{\mathbb{R}}^{b}$.

(99) If $a \geq 1$ and $b \geq 0$, then $a_{\mathbb{R}}^{b} \geq 1$.

(100) If $a > 1$ and $b > 0$, then $a_{\mathbb{R}}^{b} > 1$.

(101) If $a \geq 1$ and $b \leq 0$, then $a_{\mathbb{R}}^{b} \leq 1$.

(102) If $a > 1$ and $b < 0$, then $a_{\mathbb{R}}^{b} < 1$.

(103) If s_1 is convergent and s_2 is convergent and $\lim s_1 > 0$ and for every n holds $s_1(n) > 0$ and $s_2(n) = (s_1(n))_{\mathbb{Q}}^{p}$, then $\lim s_2 = (\lim s_1)_{\mathbb{Q}}^{p}$.

(104) If $a > 0$ and s_1 is convergent and s_2 is convergent and for every n holds $s_2(n) = a_{\mathbb{R}}^{s_1(n)}$, then $\lim s_2 = a_{\mathbb{R}}^{\lim s_1}$.

(105) If $a > 0$, then $(a_{\mathbb{R}}^{b})_{\mathbb{R}}^{c} = a_{\mathbb{R}}^{b \cdot c}$.

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Received September 21, 1990

FORMALIZED MATHEMATICS
Vol.2,No.1, January–February 1991
Université Catholique de Louvain

Homotheties and Shears in Affine Planes [1]

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Summary. We study connections between Major Desargues Axiom and the transitivity of group of homotheties. A formal proof of the theorem which establishes an equivalence of these two properties of affine planes is given. We also study connections between the trapezium version of Major Desargues Axiom and the existence of the shears in affine planes. The article contains investigations on "Scherungssatz".

MML Identifier: `HOMOTHET`.

The papers [9], [1], [2], [10], [3], [4], [6], [7], [5], and [8] provide the terminology and notation for this paper. For simplicity we adopt the following rules: A_1 will be an affine plane, a, b, o, p, p', q, q', x, y will be elements of the points of A_1, M, K will be subsets of the points of A_1, and f will be a permutation of the points of A_1. We now state four propositions:

(1) Suppose that
(i) not $\mathbf{L}(o,a,p)$,
(ii) $\mathbf{L}(o,a,b)$,
(iii) $\mathbf{L}(o,a,x)$,
(iv) $\mathbf{L}(o,a,y)$,
(v) $\mathbf{L}(o,p,p')$,
(vi) $\mathbf{L}(o,p,q)$,
(vii) $\mathbf{L}(o,p,q')$,
(viii) $p \neq q$,
(ix) $a \neq x$,
(x) $o \neq q$,
(xi) $o \neq x$,
(xii) $a,p \parallel b,p'$,
(xiii) $a,q \parallel b,q'$,
(xiv) $x,p \parallel y,p'$,

[1]Supported by RPBP.III-24.C2

ISSN 0777–4028

(xv) A_1 satisfies **DES**.

Then $x, q \parallel y, q'$.

(2) If for all o, a, b such that $o \neq a$ and $o \neq b$ and $\mathbf{L}(o, a, b)$ there exists f such that f is a dilatation and $f(o) = o$ and $f(a) = b$, then A_1 satisfies **DES**.

(3) If A_1 satisfies **DES**, then for all o, a, b such that $o \neq a$ and $o \neq b$ and $\mathbf{L}(o, a, b)$ there exists f such that f is a dilatation and $f(o) = o$ and $f(a) = b$.

(4) A_1 satisfies **DES** if and only if for all o, a, b such that $o \neq a$ and $o \neq b$ and $\mathbf{L}(o, a, b)$ there exists f such that f is a dilatation and $f(o) = o$ and $f(a) = b$.

Let us consider A_1, f, K. We say that f is Sc K if and only if:

(Def.1) f is a collineation and K is a line and for every x such that $x \in K$ holds $f(x) = x$ and for every x holds $x, f(x) \parallel K$.

One can prove the following propositions:

(5) If f is Sc K and $f(p) = p$ and $p \notin K$, then $f = \mathrm{id}_{\text{the points of } A_1}$.

(6) If for all a, b, K such that $a, b \parallel K$ and $a \notin K$ there exists f such that f is Sc K and $f(a) = b$, then A_1 satisfies **TDES**.

(7) Suppose that

(i) $K \parallel M$,

(ii) $p \in K$,

(iii) $q \in K$,

(iv) $p' \in K$,

(v) $q' \in K$,

(vi) A_1 satisfies **TDES**,

(vii) $a \in M$,

(viii) $b \in M$,

(ix) $x \in M$,

(x) $y \in M$,

(xi) $a \neq b$,

(xii) $q \neq p$,

(xiii) $p, a \parallel p', x$,

(xiv) $p, b \parallel p', y$,

(xv) $q, a \parallel q', x$.

Then $q, b \parallel q', y$.

(8) If $a, b \parallel K$ and $a \notin K$ and A_1 satisfies **TDES**, then there exists f such that f is Sc K and $f(a) = b$.

(9) A_1 satisfies **TDES** if and only if for all a, b, K such that $a, b \parallel K$ and $a \notin K$ there exists f such that f is Sc K and $f(a) = b$.

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Received September 21, 1990

FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

Directed Geometrical Bundles and Their Analytical Representation [1]

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Summary. We introduce the notion of weak directed geometrical bundle. We prove representation theorems for directed and weak directed geometrical bundles which establish a one-to-one correspondence between such structures and appropriate 2-divisible abelian groups. To this aim we construct over an arbitrary weak directed geometrical bundle a group defined entirely in terms of geometrical notions - the group of (abstract) "free vectors".

MML Identifier: `AFVECT0`.

The terminology and notation used here have been introduced in the following articles: [8], [3], [4], [10], [11], [7], [5], [6], [1], [9], and [2]. An affine structure is said to be a weak affine vector space if:

(Def.1) (i) there exist elements a, b of the points of it such that $a \neq b$,

(ii) for all elements a, b, c of the points of it such that $a, b \Rrightarrow c, c$ holds $a = b$,

(iii) for all elements a, b, c, d, p, q of the points of it such that $a, b \Rrightarrow p, q$ and $c, d \Rrightarrow p, q$ holds $a, b \Rrightarrow c, d$,

(iv) for every elements a, b, c of the points of it there exists an element d of the points of it such that $a, b \Rrightarrow c, d$,

(v) for all elements a, b, c, a', b', c' of the points of it such that $a, b \Rrightarrow a', b'$ and $a, c \Rrightarrow a', c'$ holds $b, c \Rrightarrow b', c'$,

(vi) for every elements a, c of the points of it there exists an element b of the points of it such that $a, b \Rrightarrow b, c$,

[1] Supported by RPBP.III-24.C3

ISSN 0777–4028

(vii) for all elements a, b, c, d of the points of it such that $a, b \Rrightarrow c, d$ holds $a, c \Rrightarrow b, d$.

We see that the space of free vectors is a weak affine vector space.

We adopt the following convention: A_1 will be a weak affine vector space and a, b, c, d, f, a', b', c', d', f', p, q, r, o will be elements of the points of A_1. The following propositions are true:

(2)[2] $a, b \Rrightarrow a, b$.

(3) $a, a \Rrightarrow a, a$.

(4) If $a, b \Rrightarrow c, d$, then $c, d \Rrightarrow a, b$.

(5) If $a, b \Rrightarrow a, c$, then $b = c$.

(6) If $a, b \Rrightarrow c, d$ and $a, b \Rrightarrow c, d'$, then $d = d'$.

(7) For all a, b holds $a, a \Rrightarrow b, b$.

(8) If $a, b \Rrightarrow c, d$, then $b, a \Rrightarrow d, c$.

(9) If $a, b \Rrightarrow c, d$ and $a, c \Rrightarrow b', d$, then $b = b'$.

(10) If $b, c \Rrightarrow b', c'$ and $a, d \Rrightarrow b, c$ and $a, d' \Rrightarrow b', c'$, then $d = d'$.

(11) If $a, b \Rrightarrow a', b'$ and $c, d \Rrightarrow b, a$ and $c, d' \Rrightarrow b', a'$, then $d = d'$.

(12) If $a, b \Rrightarrow a', b'$ and $c, d \Rrightarrow c', d'$ and $b, f \Rrightarrow c, d$ and $b', f' \Rrightarrow c', d'$, then $a, f \Rrightarrow a', f'$.

(13) If $a, b \Rrightarrow a', b'$ and $a, c \Rrightarrow c', b'$, then $b, c \Rrightarrow c', a'$.

Let us consider A_1, a, b. We say that a, b are in a maximal distance if and only if:

(Def.2) $a, b \Rrightarrow b, a$ and $a \neq b$.

One can prove the following propositions:

(15)[3] a, a are not in a maximal distance.

(16) There exist a, b such that $a \neq b$ and a, b are not in a maximal distance.

(17) If a, b are in a maximal distance, then b, a are in a maximal distance.

(18) If a, b are in a maximal distance and a, c are in a maximal distance, then $b = c$ or b, c are in a maximal distance.

(19) If a, b are in a maximal distance and $a, b \Rrightarrow c, d$, then c, d are in a maximal distance.

Let us consider A_1, a, b, c. We say that b is a midpoint of a, c if and only if:

(Def.3) $a, b \Rrightarrow b, c$.

We now state a number of propositions:

(21)[4] If b is a midpoint of a, c, then b is a midpoint of c, a.

(22) b is a midpoint of a, b if and only if $a = b$.

(23) b is a midpoint of a, a if and only if $a = b$ or a, b are in a maximal distance.

[2]The proposition (1) was either repeated or obvious.
[3]The proposition (14) was either repeated or obvious.
[4]The proposition (20) was either repeated or obvious.

(24) There exists b such that b is a midpoint of a, c.

(25) If b is a midpoint of a, c and b' is a midpoint of a, c, then $b = b'$ or b, b' are in a maximal distance.

(26) There exists c such that b is a midpoint of a, c.

(27) If b is a midpoint of a, c and b is a midpoint of a, c', then $c = c'$.

(28) If b is a midpoint of a, c and b, b' are in a maximal distance, then b' is a midpoint of a, c.

(29) If b is a midpoint of a, c and b' is a midpoint of a, c' and b, b' are in a maximal distance, then $c = c'$.

(30) If p is a midpoint of a, a' and p is a midpoint of b, b', then $a, b \Rrightarrow b', a'$.

(31) If p is a midpoint of a, a' and q is a midpoint of b, b' and p, q are in a maximal distance, then $a, b \Rrightarrow b', a'$.

Let us consider A_1, a, b. The functor $\mathrm{PSym}(a, b)$ yields an element of the points of A_1 and is defined as follows:

(Def.4) a is a midpoint of b, $\mathrm{PSym}(a, b)$.

One can prove the following propositions:

(32) $\mathrm{PSym}(p, a) = b$ if and only if p is a midpoint of a, b.

(33) $\mathrm{PSym}(p, a) = b$ if and only if $a, p \Rrightarrow p, b$.

(34) p is a midpoint of a, $\mathrm{PSym}(p, a)$.

(35) $\mathrm{PSym}(p, a) = a$ if and only if $a = p$ or a, p are in a maximal distance.

(36) $\mathrm{PSym}(p, \mathrm{PSym}(p, a)) = a$.

(37) If $\mathrm{PSym}(p, a) = \mathrm{PSym}(p, b)$, then $a = b$.

(38) There exists a such that $\mathrm{PSym}(p, a) = b$.

(39) $a, b \Rrightarrow \mathrm{PSym}(p, b), \mathrm{PSym}(p, a)$.

(40) $a, b \Rrightarrow c, d$ if and only if $\mathrm{PSym}(p, a), \mathrm{PSym}(p, b) \Rrightarrow \mathrm{PSym}(p, c), \mathrm{PSym}(p, d)$.

(41) a, b are in a maximal distance if and only if $\mathrm{PSym}(p, a)$, $\mathrm{PSym}(p, b)$ are in a maximal distance.

(42) b is a midpoint of a, c if and only if $\mathrm{PSym}(p, b)$ is a midpoint of $\mathrm{PSym}(p, a)$, $\mathrm{PSym}(p, c)$.

(43) $\mathrm{PSym}(p, a) = \mathrm{PSym}(q, a)$ if and only if $p = q$ or p, q are in a maximal distance.

(44) $\mathrm{PSym}(q, \mathrm{PSym}(p, \mathrm{PSym}(q, a))) = \mathrm{PSym}(\mathrm{PSym}(q, p), a)$.

(45) $\mathrm{PSym}(p, \mathrm{PSym}(q, a)) = \mathrm{PSym}(q, \mathrm{PSym}(p, a))$ if and only if $p = q$ or p, q are in a maximal distance or q, $\mathrm{PSym}(p, q)$ are in a maximal distance.

(46) $\mathrm{PSym}(p, \mathrm{PSym}(q, \mathrm{PSym}(r, a))) = \mathrm{PSym}(r, \mathrm{PSym}(q, \mathrm{PSym}(p, a)))$.

(47) There exists d such that $\mathrm{PSym}(a, \mathrm{PSym}(b, \mathrm{PSym}(c, p))) = \mathrm{PSym}(d, p)$.

(48) There exists c such that $\mathrm{PSym}(a, \mathrm{PSym}(c, p)) = \mathrm{PSym}(c, \mathrm{PSym}(b, p))$.

Let us consider A_1, o, a, b. The functor $\mathrm{Padd}(o, a, b)$ yielding an element of the points of A_1 is defined as follows:

(Def.5) $o, a \Rrightarrow b, \mathrm{Padd}(o, a, b)$.

Next we state the proposition

(49) $\mathrm{Padd}(o, a, b) = c$ if and only if $o, a \Rrightarrow b, c$.

Let us consider A_1, o, a. The functor $\mathrm{Pcom}(o, a)$ yielding an element of the points of A_1 is defined as follows:

(Def.6) o is a midpoint of a, $\mathrm{Pcom}(o, a)$.

One can prove the following propositions:

(50) $\mathrm{Pcom}(o, a) = b$ if and only if o is a midpoint of a, b.

(51) $\mathrm{Pcom}(o, a) = b$ if and only if $a, o \Rrightarrow o, b$.

Let us consider A_1, o. The functor $\mathrm{Padd}\, o$ yielding a binary operation on the points of A_1 is defined as follows:

(Def.7) for all a, b holds $(\mathrm{Padd}\, o)(a,\ b) = \mathrm{Padd}(o, a, b)$.

Let us consider A_1, o. The functor $\mathrm{Pcom}\, o$ yielding a unary operation on the points of A_1 is defined as follows:

(Def.8) for every a holds $(\mathrm{Pcom}\, o)(a) = \mathrm{Pcom}(o, a)$.

The following propositions are true:

(52) For every binary operation O on the points of A_1 holds $O = \mathrm{Padd}\, o$ if and only if for all a, b holds $O(a,\ b) = \mathrm{Padd}(o, a, b)$.

(53) For every unary operation O on the points of A_1 holds $O = \mathrm{Pcom}\, o$ if and only if for every a holds $O(a) = \mathrm{Pcom}(o, a)$.

Let us consider A_1, o. The functor $\mathrm{GroupVect}(A_1, o)$ yields a group structure and is defined by:

(Def.9) $\mathrm{GroupVect}(A_1, o) = \langle$ the points of $A_1, \mathrm{Padd}\, o, \mathrm{Pcom}\, o, o\rangle$.

The following two propositions are true:

(54) For every X being a group structure holds $X = \mathrm{GroupVect}(A_1, o)$ if and only if $X = \langle$ the points of $A_1, \mathrm{Padd}\, o, \mathrm{Pcom}\, o, o\rangle$.

(55) For all A_1, o holds the carrier of $\mathrm{GroupVect}(A_1, o) =$ the points of A_1 and the addition of $\mathrm{GroupVect}(A_1, o) = \mathrm{Padd}\, o$ and the reverse-map of $\mathrm{GroupVect}(A_1, o) = \mathrm{Pcom}\, o$ and the zero of $\mathrm{GroupVect}(A_1, o) = o$.

In the sequel a, b, c will denote elements of $\mathrm{GroupVect}(A_1, o)$. One can prove the following propositions:

(56) For an arbitrary x holds x is an element of the points of A_1 if and only if x is an element of $\mathrm{GroupVect}(A_1, o)$.

(57) For all elements a, b of $\mathrm{GroupVect}(A_1, o)$ and for all elements a', b' of the points of A_1 such that $a = a'$ and $b = b'$ holds $a + b = (\mathrm{Padd}\, o)(a',\ b')$.

(58) For every element a of $\mathrm{GroupVect}(A_1, o)$ and for every element a' of the points of A_1 such that $a = a'$ holds $-a = (\mathrm{Pcom}\, o)(a')$.

(59) $0_{\mathrm{GroupVect}(A_1, o)} = o$.

(60) For every uniquely 2-divisible group A_2 and for all elements a, b of A_2 and for all elements a', b' of the carrier of A_2 such that $a = a'$ and $b = b'$ holds $a + b = a' \# b'$.

(61) $a + b = b + a$.

(62) $(a + b) + c = a + (b + c)$.

(63) $a + 0_{\mathrm{GroupVect}(A_1, o)} = a$.

(64) $a + (-a) = 0_{\mathrm{GroupVect}(A_1, o)}$.

(65) $\mathrm{GroupVect}(A_1, o)$ is an Abelian group.

Let us consider A_1, o. Then $\mathrm{GroupVect}(A_1, o)$ is an Abelian group.

In the sequel a, b will be elements of the carrier of $\mathrm{GroupVect}(A_1, o)$. Next we state the proposition

(66) For every a there exists b such that (the addition of $\mathrm{GroupVect}(A_1, o)$)($b$, b) $= a$.

Let us consider A_1, o. Then $\mathrm{GroupVect}(A_1, o)$ is a 2-divisible group.

In the sequel A_1 will denote a space of free vectors and o will denote an element of the points of A_1. One can prove the following proposition

(67) For every element a of the carrier of $\mathrm{GroupVect}(A_1, o)$ such that (the addition of
$\mathrm{GroupVect}(A_1, o)$)($a$, a) $= 0_{\mathrm{GroupVect}(A_1, o)}$
holds $a = 0_{\mathrm{GroupVect}(A_1, o)}$.

Let us consider A_1, o. Then $\mathrm{GroupVect}(A_1, o)$ is a uniquely 2-divisible group.

A uniquely 2-divisible group is said to be a proper uniquely two divisible group if:

(Def.10) there exist elements a, b of the carrier of it such that $a \neq b$.

The following proposition is true

(69)[5] $\mathrm{GroupVect}(A_1, o)$ is a proper uniquely two divisible group.

Let us consider A_1, o. Then $\mathrm{GroupVect}(A_1, o)$ is a proper uniquely two divisible group.

Next we state the proposition

(70) For every proper uniquely two divisible group A_2 holds $\mathrm{Vectors}(A_2)$ is a space of free vectors.

Let A_2 be a proper uniquely two divisible group. Then $\mathrm{Vectors}(A_2)$ is a space of free vectors.

We now state two propositions:

(71) For every A_1 and for every element o of the points of A_1 holds $A_1 = \mathrm{Vectors}(\mathrm{GroupVect}(A_1, o))$.

(72) For every A_3 being an affine structure holds A_3 is a space of free vectors if and only if there exists a proper uniquely two divisible group A_2 such that $A_3 = \mathrm{Vectors}(A_2)$.

[5]The proposition (68) was either repeated or obvious.

Let X, Y be group structures, and let f be a function from the carrier of X into the carrier of Y. We say that f is an isomorphism of X and Y if and only if:

(Def.11) f is one-to-one and $\operatorname{rng} f =$ the carrier of Y and for all elements a, b of X holds $f(a+b) = f(a) + f(b)$ and $f(0_X) = 0_Y$ and $f(-a) = -f(a)$.

Let X, Y be group structures. We say that X, Y are isomorph if and only if:

(Def.12) there exists a function f from the carrier of X into the carrier of Y such that f is an isomorphism of X and Y.

In the sequel A_2 will be a proper uniquely two divisible group and f will be a function from the carrier of A_2 into the carrier of A_2. The following propositions are true:

(75)[6] Let o' be an element of A_2. Let o be an element of the points of $\mathrm{Vectors}(A_2)$. Suppose for every element x of A_2 holds $f(x) = o' + x$ and $o = o'$. Then for all elements a, b of A_2 holds $f(a+b) = (\mathrm{Padd}\, o)(f(a), f(b))$ and $f(0_{A_2}) = 0_{\mathrm{GroupVect}(\mathrm{Vectors}(A_2),o)}$ and $f(-a) = (\mathrm{Pcom}\, o)(f(a))$.

(76) For every element o' of A_2 such that for every element b of A_2 holds $f(b) = o' + b$ holds f is one-to-one.

(77) For every element o' of A_2 and for every element o of the points of $\mathrm{Vectors}(A_2)$ such that for every element b of A_2 holds $f(b) = o' + b$ and $o = o'$ holds $\operatorname{rng} f =$ the carrier of $\mathrm{GroupVect}(\mathrm{Vectors}(A_2), o)$.

(78) For every proper uniquely two divisible group A_2 and for every element o' of A_2 and for every element o of the points of $\mathrm{Vectors}(A_2)$ such that $o = o'$ holds A_2, $\mathrm{GroupVect}(\mathrm{Vectors}(A_2), o)$ are isomorph.

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Received September 24, 1990

FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

Definable Functions

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Summary. The article is contituation of [6] and [5]. It deals with concepts of variables occuring in a formula and free variables, replacement of variables in a formula and definable functions. The goal is to create a base of facts which are neccesary to show that every model of ZF set theory is a good model, i.e. it is closed under fundamental set-theoretical operations (union, intersection, Cartesian product ect.). The base includes the facts concerning the composition and conditional sum of two definable functions.

MML Identifier: ZFMODEL2.

The notation and terminology used here are introduced in the following articles: [12], [1], [11], [8], [7], [10], [4], [9], [2], [3], [5], and [6]. For simplicity we follow a convention: x, y, z, x_1, x_2, x_3, x_4 will denote variables, M will denote a non-empty set, i, j will denote natural numbers, m, m_1, m_2, m_3, m_4 will denote elements of M, H, H_1, H_2 will denote ZF-formulae, and v, v_1, v_2 will denote functions from VAR into M. One can prove the following propositions:

(1) $\mathrm{Free}(H(\frac{x}{y})) \subseteq (\mathrm{Free}\, H \setminus \{x\}) \cup \{y\}$.

(2) If $y \notin \mathrm{Var}_H$, then if $x \in \mathrm{Free}\, H$, then $\mathrm{Free}(H(\frac{x}{y})) = (\mathrm{Free}\, H \setminus \{x\}) \cup \{y\}$ but if $x \notin \mathrm{Free}\, H$, then $\mathrm{Free}(H(\frac{x}{y})) = \mathrm{Free}\, H$.

(3) Var_H is finite.

(4) There exists i such that for every j such that $x_j \in \mathrm{Var}_H$ holds $j < i$ and there exists x such that $x \notin \mathrm{Var}_H$.

(5) If $x \notin \mathrm{Var}_H$, then $M, v \models H$ if and only if $M, v \models \forall_x H$.

(6) If $x \notin \mathrm{Var}_H$, then $M, v \models H$ if and only if $M, v(\frac{x}{m}) \models H$.

(7) Suppose $x \neq y$ and $y \neq z$ and $z \neq x$. Then $((v(\frac{x}{m_1}))(\frac{y}{m_2}))(\frac{z}{m_3}) = ((v(\frac{z}{m_3}))(\frac{y}{m_2}))(\frac{x}{m_1})$ and $((v(\frac{x}{m_1}))(\frac{y}{m_2}))(\frac{z}{m_3}) = ((v(\frac{y}{m_2}))(\frac{z}{m_3}))(\frac{x}{m_1})$.

(8) Suppose $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_3 \neq x_4$. Then

ISSN 0777–4028

(i) $(((v(\frac{x_1}{m_1}))(\frac{x_2}{m_2}))(\frac{x_3}{m_3}))(\frac{x_4}{m_4}) = (((v(\frac{x_2}{m_2}))(\frac{x_3}{m_3}))(\frac{x_4}{m_4}))(\frac{x_1}{m_1})$,
(ii) $(((v(\frac{x_1}{m_1}))(\frac{x_2}{m_2}))(\frac{x_3}{m_3}))(\frac{x_4}{m_4}) = (((v(\frac{x_3}{m_3}))(\frac{x_4}{m_4}))(\frac{x_1}{m_1}))(\frac{x_2}{m_2})$,
(iii) $(((v(\frac{x_1}{m_1}))(\frac{x_2}{m_2}))(\frac{x_3}{m_3}))(\frac{x_4}{m_4}) = (((v(\frac{x_4}{m_4}))(\frac{x_2}{m_2}))(\frac{x_3}{m_3}))(\frac{x_1}{m_1})$.

(9) (i) $((v(\frac{x_1}{m_1}))(\frac{x_2}{m_2}))(\frac{x_1}{m}) = (v(\frac{x_2}{m_2}))(\frac{x_1}{m})$,
(ii) $(((v(\frac{x_1}{m_1}))(\frac{x_2}{m_2}))(\frac{x_3}{m_3}))(\frac{x_1}{m}) = ((v(\frac{x_2}{m_2}))(\frac{x_3}{m_3}))(\frac{x_1}{m})$,
(iii) $((((v(\frac{x_1}{m_1}))(\frac{x_2}{m_2}))(\frac{x_3}{m_3}))(\frac{x_4}{m_4}))(\frac{x_1}{m}) = (((v(\frac{x_2}{m_2}))(\frac{x_3}{m_3}))(\frac{x_4}{m_4}))(\frac{x_1}{m})$.

(10) If $x \notin \mathrm{Free}\, H$, then $M, v \models H$ if and only if $M, v(\frac{x}{m}) \models H$.

(11) Suppose $x_0 \notin \mathrm{Free}\, H$ and $M, v \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$. Then for all m_1, m_2 holds $\mathrm{f}_H[v](m_1) = m_2$ if and only if $M, (v(\frac{x_3}{m_1}))(\frac{x_4}{m_2}) \models H$.

(12) If $\mathrm{Free}\, H \subseteq \{x_3, x_4\}$ and $M \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$, then $\mathrm{f}_H[v] = \mathrm{f}_H[M]$.

(13) If $x \notin \mathrm{Var}_H$, then $M, v \models H(\frac{y}{x})$ if and only if $M, v(\frac{y}{v(x)}) \models H$.

(14) If $x \notin \mathrm{Var}_H$ and $M, v \models H$, then $M, v(\frac{x}{v(y)}) \models H(\frac{y}{x})$.

(15) Suppose that
(i) $x_0 \notin \mathrm{Free}\, H$,
(ii) $M, v \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$,
(iii) $x \notin \mathrm{Var}_H$,
(iv) $y \neq x_3$,
(v) $y \neq x_4$,
(vi) $y \notin \mathrm{Free}\, H$,
(vii) $x \neq x_0$,
(viii) $x \neq x_3$,
(ix) $x \neq x_4$.
Then
(x) $x_0 \notin \mathrm{Free}(H(\frac{y}{x}))$,
(xi) $M, v(\frac{x}{v(y)}) \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4}(H(\frac{y}{x})) \Leftrightarrow x_4{=}x_0))$,
(xii) $\mathrm{f}_H[v] = \mathrm{f}_{H(\frac{y}{x})}[v(\frac{x}{v(y)})]$.

(16) If $x \notin \mathrm{Var}_H$, then $M \models H(\frac{y}{x})$ if and only if $M \models H$.

(17) Suppose $x_0 \notin \mathrm{Free}\, H_1$ and $M, v_1 \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H_1 \Leftrightarrow x_4{=}x_0))$. Then there exist H_2, v_2 such that for every j such that $j < i$ and $x_j \in \mathrm{Var}_{H_2}$ holds $j = 3$ or $j = 4$ and $x_0 \notin \mathrm{Free}\, H_2$ and $M, v_2 \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H_2 \Leftrightarrow x_4{=}x_0))$ and $\mathrm{f}_{H_1}[v_1] = \mathrm{f}_{H_2}[v_2]$.

(18) Suppose $x_0 \notin \mathrm{Free}\, H_1$ and $M, v_1 \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H_1 \Leftrightarrow x_4{=}x_0))$. Then there exist H_2, v_2 such that $\mathrm{Free}\, H_1 \cap \mathrm{Free}\, H_2 \subseteq \{x_3, x_4\}$ and $x_0 \notin \mathrm{Free}\, H_2$ and $M, v_2 \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H_2 \Leftrightarrow x_4{=}x_0))$ and $\mathrm{f}_{H_1}[v_1] = \mathrm{f}_{H_2}[v_2]$.

In the sequel F, G are functions. One can prove the following propositions:

(19) If F is definable in M and G is definable in M, then $F \cdot G$ is definable in M.

(20) If $x_0 \notin \mathrm{Free}\, H$, then $M, v \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$ if and only if for every m_1 there exists m_2 such that for every m_3 holds $M, (v(\frac{x_3}{m_1}))(\frac{x_4}{m_3}) \models H$ if and only if $m_3 = m_2$.

(21) Suppose F is definable in M and G is definable in M and $\mathrm{Free}\, H \subseteq \{x_3\}$. Let F_1 be a function. Then if $\mathrm{dom}\, F_1 = M$ and for every v holds if $M, v \models H$, then $F_1(v(x_3)) = F(v(x_3))$ but if $M, v \models \neg H$, then $F_1(v(x_3)) = G(v(x_3))$, then F_1 is definable in M.

(22) If F is parametrically definable in M and G is parametrically definable in M, then $G \cdot F$ is parametrically definable in M.

(23) Suppose that
(i) $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\, H_1$,
(ii) $M, v \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H_1 \Leftrightarrow x_4{=}x_0))$,
(iii) $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\, H_2$,
(iv) $M, v \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H_2 \Leftrightarrow x_4{=}x_0))$,
(v) $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\, H$,
(vi) $x_4 \notin \mathrm{Free}\, H$.

Let F_1 be a function. Then if $\mathrm{dom}\, F_1 = M$ and for every m holds if $M, v(\frac{x_3}{m}) \models H$, then $F_1(m) = \mathrm{f}_{H_1}[v](m)$ but if $M, v(\frac{x_3}{m}) \models \neg H$, then $F_1(m) = \mathrm{f}_{H_2}[v](m)$, then F_1 is parametrically definable in M.

(24) id_M is definable in M.

(25) id_M is parametrically definable in M.

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Received September 26, 1990

FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

Propositional Calculus [1]

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Summary. We develop the classical propositional calculus, following [3].

MML Identifier: `LUKASI_1`.

The notation and terminology used here are introduced in the articles [1] and [2]. We follow the rules: p, q, r, s are elements of CQC−WFF and X is a subset of CQC−WFF. We now state a number of propositions:

(1) $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r)) \in \mathrm{Taut}$.

(2) If $p \Rightarrow q \in \mathrm{Taut}$, then $(q \Rightarrow r) \Rightarrow (p \Rightarrow r) \in \mathrm{Taut}$.

(3) If $p \Rightarrow q \in \mathrm{Taut}$ and $q \Rightarrow r \in \mathrm{Taut}$, then $p \Rightarrow r \in \mathrm{Taut}$.

(4) $p \Rightarrow p \in \mathrm{Taut}$.

(5) $q \Rightarrow (p \Rightarrow q) \in \mathrm{Taut}$.

(6) $((p \Rightarrow q) \Rightarrow r) \Rightarrow (q \Rightarrow r) \in \mathrm{Taut}$.

(7) $q \Rightarrow ((q \Rightarrow p) \Rightarrow p) \in \mathrm{Taut}$.

(8) $(s \Rightarrow (q \Rightarrow p)) \Rightarrow (q \Rightarrow (s \Rightarrow p)) \in \mathrm{Taut}$.

(9) $(q \Rightarrow r) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in \mathrm{Taut}$.

(10) $(q \Rightarrow (q \Rightarrow r)) \Rightarrow (q \Rightarrow r) \in \mathrm{Taut}$.

(11) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in \mathrm{Taut}$.

(12) $\neg\,\mathrm{VERUM} \Rightarrow p \in \mathrm{Taut}$.

(13) If $q \in \mathrm{Taut}$, then $p \Rightarrow q \in \mathrm{Taut}$.

(14) If $p \in \mathrm{Taut}$, then $(p \Rightarrow q) \Rightarrow q \in \mathrm{Taut}$.

(15) If $s \Rightarrow (q \Rightarrow p) \in \mathrm{Taut}$, then $q \Rightarrow (s \Rightarrow p) \in \mathrm{Taut}$.

(16) If $s \Rightarrow (q \Rightarrow p) \in \mathrm{Taut}$ and $q \in \mathrm{Taut}$, then $s \Rightarrow p \in \mathrm{Taut}$.

(17) If $s \Rightarrow (q \Rightarrow p) \in \mathrm{Taut}$ and $q \in \mathrm{Taut}$ and $s \in \mathrm{Taut}$, then $p \in \mathrm{Taut}$.

[1] Supported by RPBP.III-24.C1

ISSN 0777–4028

(18) If $q \Rightarrow (q \Rightarrow r) \in \mathrm{Taut}$, then $q \Rightarrow r \in \mathrm{Taut}$.

(19) If $p \Rightarrow (q \Rightarrow r) \in \mathrm{Taut}$, then $(p \Rightarrow q) \Rightarrow (p \Rightarrow r) \in \mathrm{Taut}$.

(20) If $p \Rightarrow (q \Rightarrow r) \in \mathrm{Taut}$ and $p \Rightarrow q \in \mathrm{Taut}$, then $p \Rightarrow r \in \mathrm{Taut}$.

(21) If $p \Rightarrow (q \Rightarrow r) \in \mathrm{Taut}$ and $p \Rightarrow q \in \mathrm{Taut}$ and $p \in \mathrm{Taut}$, then $r \in \mathrm{Taut}$.

(22) If $p \Rightarrow (q \Rightarrow r) \in \mathrm{Taut}$ and $p \Rightarrow (r \Rightarrow s) \in \mathrm{Taut}$, then $p \Rightarrow (q \Rightarrow s) \in \mathrm{Taut}$.

(23) $p \Rightarrow \mathrm{VERUM} \in \mathrm{Taut}$.

(24) $(\neg p \Rightarrow \neg q) \Rightarrow (q \Rightarrow p) \in \mathrm{Taut}$.

(25) $\neg(\neg p) \Rightarrow p \in \mathrm{Taut}$.

(26) $(p \Rightarrow q) \Rightarrow (\neg q \Rightarrow \neg p) \in \mathrm{Taut}$.

(27) $p \Rightarrow \neg(\neg p) \in \mathrm{Taut}$.

(28) $(\neg(\neg p) \Rightarrow q) \Rightarrow (p \Rightarrow q) \in \mathrm{Taut}$ and $(p \Rightarrow q) \Rightarrow (\neg(\neg p) \Rightarrow q) \in \mathrm{Taut}$.

(29) $(p \Rightarrow \neg(\neg q)) \Rightarrow (p \Rightarrow q) \in \mathrm{Taut}$ and $(p \Rightarrow q) \Rightarrow (p \Rightarrow \neg(\neg q)) \in \mathrm{Taut}$.

(30) $(p \Rightarrow \neg q) \Rightarrow (q \Rightarrow \neg p) \in \mathrm{Taut}$.

(31) $(\neg p \Rightarrow q) \Rightarrow (\neg q \Rightarrow p) \in \mathrm{Taut}$.

We now state a number of propositions:

(32) $(p \Rightarrow \neg p) \Rightarrow \neg p \in \mathrm{Taut}$.

(33) $\neg p \Rightarrow (p \Rightarrow q) \in \mathrm{Taut}$.

(34) $p \Rightarrow q \in \mathrm{Taut}$ if and only if $\neg q \Rightarrow \neg p \in \mathrm{Taut}$.

(35) If $\neg p \Rightarrow \neg q \in \mathrm{Taut}$, then $q \Rightarrow p \in \mathrm{Taut}$.

(36) $p \in \mathrm{Taut}$ if and only if $\neg(\neg p) \in \mathrm{Taut}$.

(37) $p \Rightarrow q \in \mathrm{Taut}$ if and only if $p \Rightarrow \neg(\neg q) \in \mathrm{Taut}$.

(38) $p \Rightarrow q \in \mathrm{Taut}$ if and only if $\neg(\neg p) \Rightarrow q \in \mathrm{Taut}$.

(39) If $p \Rightarrow \neg q \in \mathrm{Taut}$, then $q \Rightarrow \neg p \in \mathrm{Taut}$.

(40) If $\neg p \Rightarrow q \in \mathrm{Taut}$, then $\neg q \Rightarrow p \in \mathrm{Taut}$.

(41) $\vdash (p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$.

(42) If $\vdash p \Rightarrow q$, then $\vdash (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$.

(43) If $\vdash p \Rightarrow q$ and $\vdash q \Rightarrow r$, then $\vdash p \Rightarrow r$.

(44) $\vdash p \Rightarrow p$.

(45) $\vdash p \Rightarrow (q \Rightarrow p)$.

(46) If $\vdash p$, then $\vdash q \Rightarrow p$.

(47) $\vdash (s \Rightarrow (q \Rightarrow p)) \Rightarrow (q \Rightarrow (s \Rightarrow p))$.

(48) If $\vdash p \Rightarrow (q \Rightarrow r)$, then $\vdash q \Rightarrow (p \Rightarrow r)$.

(49) If $\vdash p \Rightarrow (q \Rightarrow r)$ and $\vdash q$, then $\vdash p \Rightarrow r$.

(50) $\vdash p \Rightarrow \mathrm{VERUM}$ and $\vdash \neg\,\mathrm{VERUM} \Rightarrow p$.

(51) $\vdash p \Rightarrow ((p \Rightarrow q) \Rightarrow q)$.

(52) $\vdash (q \Rightarrow (q \Rightarrow r)) \Rightarrow (q \Rightarrow r)$.

(53) If $\vdash q \Rightarrow (q \Rightarrow r)$, then $\vdash q \Rightarrow r$.

(54) $\vdash (p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$.

(55) If $\vdash p \Rightarrow (q \Rightarrow r)$, then $\vdash (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$.
(56) If $\vdash p \Rightarrow (q \Rightarrow r)$ and $\vdash p \Rightarrow q$, then $\vdash p \Rightarrow r$.
(57) $\vdash ((p \Rightarrow q) \Rightarrow r) \Rightarrow (q \Rightarrow r)$.
(58) If $\vdash (p \Rightarrow q) \Rightarrow r$, then $\vdash q \Rightarrow r$.
(59) $\vdash (p \Rightarrow q) \Rightarrow ((r \Rightarrow p) \Rightarrow (r \Rightarrow q))$.
(60) If $\vdash p \Rightarrow q$, then $\vdash (r \Rightarrow p) \Rightarrow (r \Rightarrow q)$.
(61) $\vdash (p \Rightarrow q) \Rightarrow (\neg q \Rightarrow \neg p)$.
(62) $\vdash (\neg p \Rightarrow \neg q) \Rightarrow (q \Rightarrow p)$.

The following propositions are true:

(63) $\vdash \neg p \Rightarrow \neg q$ if and only if $\vdash q \Rightarrow p$.
(64) $\vdash p \Rightarrow \neg(\neg p)$.
(65) $\vdash \neg(\neg p) \Rightarrow p$.
(66) $\vdash \neg(\neg p)$ if and only if $\vdash p$.
(67) $\vdash (\neg(\neg p) \Rightarrow q) \Rightarrow (p \Rightarrow q)$.
(68) $\vdash \neg(\neg p) \Rightarrow q$ if and only if $\vdash p \Rightarrow q$.
(69) $\vdash (p \Rightarrow \neg(\neg q)) \Rightarrow (p \Rightarrow q)$.
(70) $\vdash p \Rightarrow \neg(\neg q)$ if and only if $\vdash p \Rightarrow q$.
(71) $\vdash (p \Rightarrow \neg q) \Rightarrow (q \Rightarrow \neg p)$.
(72) If $\vdash p \Rightarrow \neg q$, then $\vdash q \Rightarrow \neg p$.
(73) $\vdash (\neg p \Rightarrow q) \Rightarrow (\neg q \Rightarrow p)$.
(74) If $\vdash \neg p \Rightarrow q$, then $\vdash \neg q \Rightarrow p$.
(75) If $X \vdash p \Rightarrow q$, then $X \vdash (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$.
(76) If $X \vdash p \Rightarrow q$ and $X \vdash q \Rightarrow r$, then $X \vdash p \Rightarrow r$.
(77) $X \vdash p \Rightarrow p$.
(78) If $X \vdash p$, then $X \vdash q \Rightarrow p$.
(79) If $X \vdash p$, then $X \vdash (p \Rightarrow q) \Rightarrow q$.
(80) If $X \vdash p \Rightarrow (q \Rightarrow r)$, then $X \vdash q \Rightarrow (p \Rightarrow r)$.
(81) If $X \vdash p \Rightarrow (q \Rightarrow r)$ and $X \vdash q$, then $X \vdash p \Rightarrow r$.
(82) If $X \vdash p \Rightarrow (p \Rightarrow q)$, then $X \vdash p \Rightarrow q$.
(83) If $X \vdash (p \Rightarrow q) \Rightarrow r$, then $X \vdash q \Rightarrow r$.
(84) If $X \vdash p \Rightarrow (q \Rightarrow r)$, then $X \vdash (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$.
(85) If $X \vdash p \Rightarrow (q \Rightarrow r)$ and $X \vdash p \Rightarrow q$, then $X \vdash p \Rightarrow r$.
(86) $X \vdash \neg p \Rightarrow \neg q$ if and only if $X \vdash q \Rightarrow p$.
(87) $X \vdash \neg(\neg p)$ if and only if $X \vdash p$.
(88) $X \vdash p \Rightarrow \neg(\neg q)$ if and only if $X \vdash p \Rightarrow q$.
(89) $X \vdash \neg(\neg p) \Rightarrow q$ if and only if $X \vdash p \Rightarrow q$.
(90) If $X \vdash p \Rightarrow \neg q$, then $X \vdash q \Rightarrow \neg p$.
(91) If $X \vdash \neg p \Rightarrow q$, then $X \vdash \neg q \Rightarrow p$.
(92) If $X \vdash p \Rightarrow \neg q$ and $X \vdash q$, then $X \vdash \neg p$.

(93) If $X \vdash \neg p \Rightarrow q$ and $X \vdash \neg q$, then $X \vdash p$.

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Received September 26, 1990

FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

Complex Spaces [1]

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Summary. We introduce the concept of n-dimensional complex space. We prove a number of simple but useful theorems concerning addition, multiplication by scalars and similar basic concepts. We introduce metric and topology. We prove that an n-dimensional complex space is a Hausdorf space and that it is regular.

MML Identifier: `COMPLSP1`.

The articles [20], [16], [12], [1], [21], [5], [22], [7], [8], [3], [17], [11], [2], [18], [19], [6], [4], [9], [10], [15], [14], and [13] provide the notation and terminology for this paper. We follow the rules: k, n will be natural numbers, r, r', r_1 will be real numbers, and c, c', c_1, c_2 will be elements of $\mathbb{C}$. In this article we present several logical schemes. The scheme *FuncDefUniq* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$ and states that:

for all functions f_1, f_2 from $\mathcal{A}$ into $\mathcal{B}$ such that for every element x of $\mathcal{A}$ holds $f_1(x) = \mathcal{F}(x)$ and for every element x of $\mathcal{A}$ holds $f_2(x) = \mathcal{F}(x)$ holds $f_1 = f_2$
for all values of the parameters.

The scheme *UnOpDefuniq* deals with a non-empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$ and states that:

for all unary operations u_1, u_2 on $\mathcal{A}$ such that for every element x of $\mathcal{A}$ holds $u_1(x) = \mathcal{F}(x)$ and for every element x of $\mathcal{A}$ holds $u_2(x) = \mathcal{F}(x)$ holds $u_1 = u_2$
for all values of the parameters.

The scheme *BinOpDefuniq* deals with a non-empty set $\mathcal{A}$ and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$ and states that:

for all binary operations o_1, o_2 on $\mathcal{A}$ such that for all elements a, b of $\mathcal{A}$ holds $o_1(a, b) = \mathcal{F}(a, b)$ and for all elements a, b of $\mathcal{A}$ holds $o_2(a, b) = \mathcal{F}(a, b)$ holds $o_1 = o_2$
for all values of the parameters.

[1]Supported by RPBP.III-24.C1

ISSN 0777–4028

The binary operation $+_{\mathbb{C}}$ on $\mathbb{C}$ is defined as follows:

(Def.1) for all c_1, c_2 holds $+_{\mathbb{C}}(c_1, c_2) = c_1 + c_2$.

The following propositions are true:

(1) $+_{\mathbb{C}}$ is commutative.

(2) $+_{\mathbb{C}}$ is associative.

(3) $0_{\mathbb{C}}$ is a unity w.r.t. $+_{\mathbb{C}}$.

(4) $\mathbf{1}_{+_{\mathbb{C}}} = 0_{\mathbb{C}}$.

(5) $+_{\mathbb{C}}$ has a unity.

The unary operation $-_{\mathbb{C}}$ on $\mathbb{C}$ is defined as follows:

(Def.2) for every c holds $-_{\mathbb{C}}(c) = -c$.

Next we state three propositions:

(6) $-_{\mathbb{C}}$ is an inverse operation w.r.t. $+_{\mathbb{C}}$.

(7) $+_{\mathbb{C}}$ has an inverse operation.

(8) The inverse operation w.r.t. $+_{\mathbb{C}} = -_{\mathbb{C}}$.

The binary operation $-_{\mathbb{C}}$ on $\mathbb{C}$ is defined by:

(Def.3) $-_{\mathbb{C}} = +_{\mathbb{C}} \circ (\mathrm{id}_{\mathbb{C}}, -_{\mathbb{C}})$.

The following proposition is true

(9) $-_{\mathbb{C}}(c_1, c_2) = c_1 - c_2$.

The binary operation $\cdot_{\mathbb{C}}$ on $\mathbb{C}$ is defined by:

(Def.4) for all c_1, c_2 holds $\cdot_{\mathbb{C}}(c_1, c_2) = c_1 \cdot c_2$.

The following propositions are true:

(10) $\cdot_{\mathbb{C}}$ is commutative.

(11) $\cdot_{\mathbb{C}}$ is associative.

(12) $1_{\mathbb{C}}$ is a unity w.r.t. $\cdot_{\mathbb{C}}$.

(13) $\mathbf{1}_{\cdot_{\mathbb{C}}} = 1_{\mathbb{C}}$.

(14) $\cdot_{\mathbb{C}}$ has a unity.

(15) $\cdot_{\mathbb{C}}$ is distributive w.r.t. $+_{\mathbb{C}}$.

Let us consider c. The functor $\cdot_{\mathbb{C}}^{c}$ yields a unary operation on $\mathbb{C}$ and is defined by:

(Def.5) $\cdot_{\mathbb{C}}^{c} = \cdot_{\mathbb{C}}^{\circ}(c, \mathrm{id}_{\mathbb{C}})$.

We now state two propositions:

(16) $\cdot_{\mathbb{C}}^{c}(c') = c \cdot c'$.

(17) $\cdot_{\mathbb{C}}^{c}$ is distributive w.r.t. $+_{\mathbb{C}}$.

The function $|\cdot|_{\mathbb{C}}$ from $\mathbb{C}$ into $\mathbb{R}$ is defined by:

(Def.6) for every c holds $|\cdot|_{\mathbb{C}}(c) = |c|$.

In the sequel z, z_1, z_2 will be finite sequences of elements of $\mathbb{C}$. We now define two new functors. Let us consider z_1, z_2. The functor $z_1 + z_2$ yields a finite sequence of elements of $\mathbb{C}$ and is defined by:

(Def.7) $z_1 + z_2 = +^{\circ}_{\mathbb{C}}(z_1, z_2)$.

The functor $z_1 - z_2$ yielding a finite sequence of elements of $\mathbb{C}$ is defined as follows:

(Def.8) $z_1 - z_2 = -^{\circ}_{\mathbb{C}}(z_1, z_2)$.

Let us consider z. The functor $-z$ yielding a finite sequence of elements of $\mathbb{C}$ is defined by:

(Def.9) $-z = -_{\mathbb{C}} \cdot z$.

Let us consider c, z. The functor $c \cdot z$ yielding a finite sequence of elements of $\mathbb{C}$ is defined by:

(Def.10) $c \cdot z = \cdot^{c}_{\mathbb{C}} \cdot z$.

Let us consider z. The functor $|z|$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def.11) $|z| = |\cdot|_{\mathbb{C}} \cdot z$.

Let us consider n. The functor $\mathbb{C}^n$ yielding a non-empty set of finite sequences of $\mathbb{C}$ is defined by:

(Def.12) $\mathbb{C}^n = \mathbb{C}^n$.

We follow a convention: x, z, z_1, z_2, z_3 will denote elements of $\mathbb{C}^n$ and A, B will denote subsets of $\mathbb{C}^n$. One can prove the following propositions:

(18) $\operatorname{len} z = n$.

(19) For every element z of $\mathbb{C}^0$ holds $z = \varepsilon_{\mathbb{C}}$.

(20) $\varepsilon_{\mathbb{C}}$ is an element of $\mathbb{C}^0$.

(21) If $k \in \operatorname{Seg} n$, then $z(k) \in \mathbb{C}$.

(22) If $k \in \operatorname{Seg} n$, then $z(k)$ is an element of $\mathbb{C}$.

(23) If for every k such that $k \in \operatorname{Seg} n$ holds $z_1(k) = z_2(k)$, then $z_1 = z_2$.

Let us consider n, z_1, z_2. Then $z_1 + z_2$ is an element of $\mathbb{C}^n$.

Next we state three propositions:

(24) If $k \in \operatorname{Seg} n$ and $c_1 = z_1(k)$ and $c_2 = z_2(k)$, then $(z_1 + z_2)(k) = c_1 + c_2$.

(25) $z_1 + z_2 = z_2 + z_1$.

(26) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.

Let us consider n. The functor $0^n_{\mathbb{C}}$ yielding a finite sequence of elements of $\mathbb{C}$ is defined by:

(Def.13) $0^n_{\mathbb{C}} = n \longmapsto 0_{\mathbb{C}}$.

Let us consider n. Then $0^n_{\mathbb{C}}$ is an element of $\mathbb{C}^n$.

Next we state two propositions:

(27) If $k \in \operatorname{Seg} n$, then $0^n_{\mathbb{C}}(k) = 0_{\mathbb{C}}$.

(28) $z + 0^n_{\mathbb{C}} = z$ and $z = 0^n_{\mathbb{C}} + z$.

Let us consider n, z. Then $-z$ is an element of $\mathbb{C}^n$.

Next we state several propositions:

(29) If $k \in \operatorname{Seg} n$ and $c = z(k)$, then $(-z)(k) = -c$.

(30) $z + (-z) = 0_{\mathbb{C}}^n$ and $(-z) + z = 0_{\mathbb{C}}^n$.

(31) If $z_1 + z_2 = 0_{\mathbb{C}}^n$, then $z_1 = -z_2$ and $z_2 = -z_1$.

(32) $-(-z) = z$.

(33) If $-z_1 = -z_2$, then $z_1 = z_2$.

(34) If $z_1 + z = z_2 + z$ or $z_1 + z = z + z_2$, then $z_1 = z_2$.

(35) $-(z_1 + z_2) = (-z_1) + (-z_2)$.

Let us consider n, z_1, z_2. Then $z_1 - z_2$ is an element of $\mathbb{C}^n$.

Next we state a number of propositions:

(36) If $k \in \operatorname{Seg} n$ and $c_1 = z_1(k)$ and $c_2 = z_2(k)$, then $(z_1 - z_2)(k) = c_1 - c_2$.

(37) $z_1 - z_2 = z_1 + (-z_2)$.

(38) $z - 0_{\mathbb{C}}^n = z$.

(39) $0_{\mathbb{C}}^n - z = -z$.

(40) $z_1 - (-z_2) = z_1 + z_2$.

(41) $-(z_1 - z_2) = z_2 - z_1$.

(42) $-(z_1 - z_2) = (-z_1) + z_2$.

(43) $z - z = 0_{\mathbb{C}}^n$.

(44) If $z_1 - z_2 = 0_{\mathbb{C}}^n$, then $z_1 = z_2$.

(45) $(z_1 - z_2) - z_3 = z_1 - (z_2 + z_3)$.

(46) $z_1 + (z_2 - z_3) = (z_1 + z_2) - z_3$.

(47) $z_1 - (z_2 - z_3) = (z_1 - z_2) + z_3$.

(48) $(z_1 - z_2) + z_3 = (z_1 + z_3) - z_2$.

(49) $z_1 = (z_1 + z) - z$.

(50) $z_1 + (z_2 - z_1) = z_2$.

(51) $z_1 = (z_1 - z) + z$.

Let us consider n, c, z. Then $c \cdot z$ is an element of $\mathbb{C}^n$.

One can prove the following propositions:

(52) If $k \in \operatorname{Seg} n$ and $c' = z(k)$, then $(c \cdot z)(k) = c \cdot c'$.

(53) $c_1 \cdot (c_2 \cdot z) = (c_1 \cdot c_2) \cdot z$.

(54) $(c_1 + c_2) \cdot z = c_1 \cdot z + c_2 \cdot z$.

(55) $c \cdot (z_1 + z_2) = c \cdot z_1 + c \cdot z_2$.

(56) $1_{\mathbb{C}} \cdot z = z$.

(57) $0_{\mathbb{C}} \cdot z = 0_{\mathbb{C}}^n$.

(58) $(-1_{\mathbb{C}}) \cdot z = -z$.

Let us consider n, z. Then $|z|$ is an element of $\mathbb{R}^n$.

Next we state four propositions:

(59) If $k \in \operatorname{Seg} n$ and $c = z(k)$, then $|z|(k) = |c|$.

(60) $|0_{\mathbb{C}}^n| = n \longmapsto 0$.

(61) $|-z| = |z|$.

(62) $|c \cdot z| = |c| \cdot |z|$.

Let z be a finite sequence of elements of $\mathbb{C}$. The functor $|z|$ yields a real number and is defined by:

(Def.14) $|z| = \sqrt{\sum({}^2|z|)}$.

One can prove the following propositions:

(63) $|0^n_{\mathbb{C}}| = 0$.

(64) If $|z| = 0$, then $z = 0^n_{\mathbb{C}}$.

(65) $0 \leq |z|$.

(66) $|-z| = |z|$.

(67) $|c \cdot z| = |c| \cdot |z|$.

(68) $|z_1 + z_2| \leq |z_1| + |z_2|$.

(69) $|z_1 - z_2| \leq |z_1| + |z_2|$.

(70) $|z_1| - |z_2| \leq |z_1 + z_2|$.

(71) $|z_1| - |z_2| \leq |z_1 - z_2|$.

(72) $|z_1 - z_2| = 0$ if and only if $z_1 = z_2$.

(73) If $z_1 \neq z_2$, then $0 < |z_1 - z_2|$.

(74) $|z_1 - z_2| = |z_2 - z_1|$.

(75) $|z_1 - z_2| \leq |z_1 - z| + |z - z_2|$.

Let us consider n, and let A be an element of $2^{\mathbb{C}^n}$. We say that A is open if and only if:

(Def.15) for every x such that $x \in A$ there exists r such that $0 < r$ and for every z such that $|z| < r$ holds $x + z \in A$.

Let us consider n, and let A be an element of $2^{\mathbb{C}^n}$. We say that A is closed if and only if:

(Def.16) for every x such that for every r such that $r > 0$ there exists z such that $|z| < r$ and $x + z \in A$ holds $x \in A$.

We now state four propositions:

(76) For every element A of $2^{\mathbb{C}^n}$ such that $A = \emptyset$ holds A is open.

(77) For every element A of $2^{\mathbb{C}^n}$ such that $A = \mathbb{C}^n$ holds A is open.

(78) For every family A_1 of subsets of $\mathbb{C}^n$ such that for every element A of $2^{\mathbb{C}^n}$ such that $A \in A_1$ holds A is open for every element A of $2^{\mathbb{C}^n}$ such that $A = \bigcup A_1$ holds A is open.

(79) For all subsets A, B of $\mathbb{C}^n$ such that A is open and B is open for every element C of $2^{\mathbb{C}^n}$ such that $C = A \cap B$ holds C is open.

Let us consider n, x, r. The functor $\mathrm{Ball}(x, r)$ yielding a subset of $\mathbb{C}^n$ is defined by:

(Def.17) $\mathrm{Ball}(x, r) = \{z : |z - x| < r\}$.

The following three propositions are true:

(80) $z \in \mathrm{Ball}(x, r)$ if and only if $|x - z| < r$.

(81) If $0 < r$, then $x \in \mathrm{Ball}(x, r)$.

(82) $\mathrm{Ball}(z_1, r_1)$ is open.

Now we present two schemes. The scheme *SubsetFD* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(x) : \mathcal{P}[x]\}$, where x is an element of $\mathcal{A}$, is a subset of $\mathcal{B}$

for all values of the parameters.

The scheme *SubsetFD2* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a non-empty set $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a binary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(x, y) : \mathcal{P}[x, y]\}$, where x is an element of $\mathcal{A}$, and y is an element of $\mathcal{B}$, is a subset of $\mathcal{C}$

for all values of the parameters.

Let us consider n, x, A. The functor $\rho(x, A)$ yielding a real number is defined by:

(Def.18) for every X being sets of real numbers such that $X = \{|x - z| : z \in A\}$ holds $\rho(x, A) = \inf X$.

Let us consider n, A, r. The functor $\mathrm{Ball}(A, r)$ yields a subset of $\mathbb{C}^n$ and is defined as follows:

(Def.19) $\mathrm{Ball}(A, r) = \{z : \rho(z, A) < r\}$.

Next we state a number of propositions:

(83) If for every r' such that $r' > 0$ holds $r + r' > r_1$, then $r \geq r_1$.

(84) For every X being sets of real numbers and for every r such that $X \neq \emptyset$ and for every r' such that $r' \in X$ holds $r \leq r'$ holds $\inf X \geq r$.

(85) If $A \neq \emptyset$, then $\rho(x, A) \geq 0$.

(86) If $A \neq \emptyset$, then $\rho(x + z, A) \leq \rho(x, A) + |z|$.

(87) If $x \in A$, then $\rho(x, A) = 0$.

(88) If $x \notin A$ and $A \neq \emptyset$ and A is closed, then $\rho(x, A) > 0$.

(89) If $A \neq \emptyset$, then $|z_1 - x| + \rho(x, A) \geq \rho(z_1, A)$.

(90) $z \in \mathrm{Ball}(A, r)$ if and only if $\rho(z, A) < r$.

(91) If $0 < r$ and $x \in A$, then $x \in \mathrm{Ball}(A, r)$.

(92) If $0 < r$, then $A \subseteq \mathrm{Ball}(A, r)$.

(93) If $A \neq \emptyset$, then $\mathrm{Ball}(A, r_1)$ is open.

Let us consider n, A, B. The functor $\rho(A, B)$ yields a real number and is defined as follows:

(Def.20) for every X being sets of real numbers such that $X = \{|x - z| : x \in A \wedge z \in B\}$ holds $\rho(A, B) = \inf X$.

Let X, Y be sets of real numbers. The functor $X + Y$ yields sets of real numbers and is defined as follows:

(Def.21) $X + Y = \{r + r_1 : r \in X \wedge r_1 \in Y\}$.

Next we state several propositions:

(94) For all X, Y being sets of real numbers such that $X \neq \emptyset$ and $Y \neq \emptyset$ holds $X + Y \neq \emptyset$.

(95) For all X, Y being sets of real numbers such that $X \neq \emptyset$ and X is lower bounded and $Y \neq \emptyset$ and Y is lower bounded holds $X + Y$ is lower bounded.

(96) For all X, Y being sets of real numbers such that $X \neq \emptyset$ and X is lower bounded and $Y \neq \emptyset$ and Y is lower bounded holds $\inf(X + Y) = \inf X + \inf Y$.

(97) For all X, Y being sets of real numbers such that Y is lower bounded and $X \neq \emptyset$ and for every r such that $r \in X$ there exists r_1 such that $r_1 \in Y$ and $r_1 \leq r$ holds $\inf X \geq \inf Y$.

(98) If $A \neq \emptyset$ and $B \neq \emptyset$, then $\rho(A, B) \geq 0$.

(99) $\rho(A, B) = \rho(B, A)$.

(100) If $A \neq \emptyset$ and $B \neq \emptyset$, then $\rho(x, A) + \rho(x, B) \geq \rho(A, B)$.

(101) If $A \cap B \neq \emptyset$, then $\rho(A, B) = 0$.

Let us consider n. The open subsets of $\mathbb{C}^n$ constitute a family of subsets of $\mathbb{C}^n$ defined by:

(Def.22) the open subsets of $\mathbb{C}^n = \{A : A$ is open $\}$, where A is an element of $2^{\mathbb{C}^n}$.

The following proposition is true

(102) For every element A of $2^{\mathbb{C}^n}$ holds $A \in$ the open subsets of $\mathbb{C}^n$ if and only if A is open.

Let us consider n. The n-dimensional complex space is a topological space defined by:

(Def.23) the n-dimensional complex space $= \langle \mathbb{C}^n$, the open subsets of $\mathbb{C}^n \rangle$.

We now state two propositions:

(103) The topology of the n-dimensional complex space $=$ the open subsets of $\mathbb{C}^n$.

(104) The carrier of the n-dimensional complex space $= \mathbb{C}^n$.

In the sequel p denotes a point of the n-dimensional complex space and V denotes a subset of the n-dimensional complex space. Next we state several propositions:

(105) p is an element of $\mathbb{C}^n$.

(106) V is a subset of $\mathbb{C}^n$.

(107) For every subset A of $\mathbb{C}^n$ holds A is a subset of the n-dimensional complex space.

(108) For every subset A of $\mathbb{C}^n$ such that $A = V$ holds A is open if and only if V is open.

(109) For every subset A of $\mathbb{C}^n$ holds A is closed if and only if A^{c} is open.

(110) For every subset A of $\mathbb{C}^n$ such that $A = V$ holds A is closed if and only if V is closed.

(111) The n-dimensional complex space is a T_2 space.

(112) The n-dimensional complex space is a T_3 space.

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Received September 27, 1990

FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

Several Properties of Fields. Field Theory

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Summary. The article includes a continuation of the paper [2]. Some simple theorems concerning basic properties of a field are proved.

MML Identifier: `REALSET3`.

The articles [8], [7], [5], [6], [3], [1], [2], and [4] provide the terminology and notation for this paper. The following propositions are true:

(1) For every field F holds $-_F(\mathbf{0}_F) = \mathbf{0}_F$.

(2) For every field F holds $\overset{-1}{_F}(\mathbf{1}_F) = \mathbf{1}_F$.

(3) For every field F and for all elements a, b of the support of F holds $-_F(+_F(\langle a, -_F(b)\rangle)) = +_F(\langle b, -_F(a)\rangle)$.

(4) For every field F and for all elements a, b of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\overset{-1}{_F}(\cdot_F(\langle a, \overset{-1}{_F}(b)\rangle)) = \cdot_F(\langle b, \overset{-1}{_F}(a)\rangle)$.

(5) For every field F and for all elements a, b of the support of F holds $-_F(+_F(\langle a, b\rangle)) = +_F(\langle -_F(a), -_F(b)\rangle)$.

(6) For every field F and for all elements a, b of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\overset{-1}{_F}(\cdot_F(\langle a, b\rangle)) = \cdot_F(\langle \overset{-1}{_F}(a), \overset{-1}{_F}(b)\rangle)$.

(7) For every field F and for all elements a, b, c, d of the support of F holds $+_F(\langle a, -_F(b)\rangle) = +_F(\langle c, -_F(d)\rangle)$ if and only if $+_F(\langle a, d\rangle) = +_F(\langle b, c\rangle)$.

(8) Let F be a field. Then for all elements a, c of the support of F and for all elements b, d of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\cdot_F(\langle a, \overset{-1}{_F}(b)\rangle) = \cdot_F(\langle c, \overset{-1}{_F}(d)\rangle)$ if and only if $\cdot_F(\langle a, d\rangle) = \cdot_F(\langle b, c\rangle)$.

(9) For every field F and for all elements a, b of the support of F holds $\cdot_F(\langle a, b\rangle) = \mathbf{0}_F$ if and only if $a = \mathbf{0}_F$ or $b = \mathbf{0}_F$.

(10) Let F be a field. Let a, b be elements of the support of F. Let c, d be elements of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$. Then $\cdot_F(\langle \cdot_F(\langle a, \overset{-1}{_F}(c)\rangle), \cdot_F(\langle b, \overset{-1}{_F}(d)\rangle)\rangle) = \cdot_F(\langle \cdot_F(\langle a, b\rangle), \overset{-1}{_F}(\cdot_F(\langle c, d\rangle))\rangle)$.

(11) Let F be a field. Let a, b be elements of the support of F. Let c, d be elements of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$.

ISSN 0777–4028

Then $+_F(\langle \cdot_F(\langle a, \overset{-1}{_F}(c)\rangle), \cdot_F(\langle b, \overset{-1}{_F}(d)\rangle)\rangle) =$
$\cdot_F(\langle +_F(\langle \cdot_F(\langle a, d\rangle), \cdot_F(\langle b, c\rangle)\rangle), \overset{-1}{_F}(\cdot_F(\langle c, d\rangle))\rangle)$.

Let F be a field. The functor osf F yielding a binary operation of the support of F is defined as follows:

(Def.1) for all elements x, y of the support of F holds $(\operatorname{osf} F)(\langle x, y\rangle) = +_F(\langle x, -_F(y)\rangle)$.

The following propositions are true:

(12) For every field F and for every binary operation S of the support of F holds $S = \operatorname{osf} F$ if and only if for all elements x, y of the support of F holds $S(\langle x, y\rangle) = +_F(\langle x, -_F(y)\rangle)$.

(13) For every field F and for all elements x, y of the support of F holds $\operatorname{osf} F(\langle x, y\rangle) = +_F(\langle x, -_F(y)\rangle)$.

(14) For every field F and for every element x of the support of F holds $\operatorname{osf} F(\langle x, x\rangle) = \mathbf{0}_F$.

(15) For every field F and for all elements a, b, c of the support of F holds $\cdot_F(\langle a, \operatorname{osf} F(\langle b, c\rangle)\rangle) = \operatorname{osf} F(\langle \cdot_F(\langle a, b\rangle), \cdot_F(\langle a, c\rangle)\rangle)$.

(16) For every field F and for all elements a, b of the support of F holds $\operatorname{osf} F(\langle a, b\rangle)$ is an element of the support of F.

(17) For every field F and for all elements a, b, c of the support of F holds $\cdot_F(\langle \operatorname{osf} F(\langle a, b\rangle), c\rangle) = \operatorname{osf} F(\langle \cdot_F(\langle a, c\rangle), \cdot_F(\langle b, c\rangle)\rangle)$.

(18) For every field F and for all elements a, b of the support of F holds $\operatorname{osf} F(\langle a, b\rangle) = -_F(\operatorname{osf} F(\langle b, a\rangle))$.

(19) For every field F and for all elements a, b of the support of F holds $\operatorname{osf} F(\langle -_F(a), b\rangle) = -_F(+_F(\langle a, b\rangle))$.

(20) For every field F and for all elements a, b, c, d of the support of F holds $\operatorname{osf} F(\langle a, b\rangle) = \operatorname{osf} F(\langle c, d\rangle)$ if and only if $+_F(\langle a, d\rangle) = +_F(\langle b, c\rangle)$.

(21) For every field F and for every element a of the support of F holds $\operatorname{osf} F(\langle \mathbf{0}_F, a\rangle) = -_F(a)$.

(22) For every field F and for every element a of the support of F holds $\operatorname{osf} F(\langle a, \mathbf{0}_F\rangle) = a$.

(23) For every field F and for all elements a, b, c of the support of F holds $+_F(\langle a, b\rangle) = c$ if and only if $\operatorname{osf} F(\langle c, a\rangle) = b$.

(24) For every field F and for all elements a, b, c of the support of F holds $+_F(\langle a, b\rangle) = c$ if and only if $\operatorname{osf} F(\langle c, b\rangle) = a$.

(25) For every field F and for all elements a, b, c of the support of F holds $\operatorname{osf} F(\langle a, \operatorname{osf} F(\langle b, c\rangle)\rangle) = +_F(\langle \operatorname{osf} F(\langle a, b\rangle), c\rangle)$.

(26) For every field F and for all elements a, b, c of the support of F holds $\operatorname{osf} F(\langle a, +_F(\langle b, c\rangle)\rangle) = \operatorname{osf} F(\langle \operatorname{osf} F(\langle a, b\rangle), c\rangle)$.

Let F be a field. The functor ovf F yields a function from
the support of $F\#$(the support of $F \setminus \operatorname{single}(\mathbf{0}_F)$)
into the support of F and is defined as follows:

(Def.2) for every element x of the support of F and for every element y of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $(\mathrm{ovf}\ F)(\langle x, y \rangle) = \cdot_F(\langle x, \overset{-1}{_F}(y) \rangle)$.

Next we state a number of propositions:

(27) Let F be a field. Then for every function D from the support of F#(the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$) into the support of F holds $D = \mathrm{ovf}\ F$ if and only if for every element x of the support of F and for every element y of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $D(\langle x, y \rangle) = \cdot_F(\langle x, \overset{-1}{_F}(y) \rangle)$.

(28) For every field F and for every element x of the support of F and for every element y of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\mathrm{ovf}\ F(\langle x, y \rangle) = \cdot_F(\langle x, \overset{-1}{_F}(y) \rangle)$.

(29) For every field F and for every element x of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\mathrm{ovf}\ F(\langle x, x \rangle) = \mathbf{1}_F$.

(30) For every field F and for every element a of the support of F and for every element b of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\mathrm{ovf}\ F(\langle a, b \rangle)$ is an element of the support of F.

(31) For every field F and for all elements a, b of the support of F and for every element c of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\cdot_F(\langle a, \mathrm{ovf}\ F(\langle b, c \rangle) \rangle) = \mathrm{ovf}\ F(\langle \cdot_F(\langle a, b \rangle), c \rangle)$.

(32) For every field F and for every element a of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\cdot_F(\langle a, \mathrm{ovf}\ F(\langle \mathbf{1}_F, a \rangle) \rangle) = \mathbf{1}_F$ and $\cdot_F(\langle \mathrm{ovf}\ F(\langle \mathbf{1}_F, a \rangle), a \rangle) = \mathbf{1}_F$.

(34)[1] For every field F and for all elements a, b of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\cdot_F(\langle a, \overset{-1}{_F}(b) \rangle) = \overset{-1}{_F}(\cdot_F(\langle b, \overset{-1}{_F}(a) \rangle))$.

(35) For every field F and for all elements a, b of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\mathrm{ovf}\ F(\langle a, b \rangle) = \overset{-1}{_F}(\mathrm{ovf}\ F(\langle b, a \rangle))$.

(36) For every field F and for all elements a, b of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\mathrm{ovf}\ F(\langle \overset{-1}{_F}(a), b \rangle) = \overset{-1}{_F}(\cdot_F(\langle a, b \rangle))$.

(37) For every field F and for all elements a, c of the support of F and for all elements b, d of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\mathrm{ovf}\ F(\langle a, b \rangle) = \mathrm{ovf}\ F(\langle c, d \rangle)$ if and only if $\cdot_F(\langle a, d \rangle) = \cdot_F(\langle b, c \rangle)$.

(38) For every field F and for every element a of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\mathrm{ovf}\ F(\langle \mathbf{1}_F, a \rangle) = \overset{-1}{_F}(a)$.

(39) For every field F and for every element a of the support of F holds $\mathrm{ovf}\ F(\langle a, \mathbf{1}_F \rangle) = a$.

(40) For every field F and for every element a of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ and for all elements b, c of the support of F holds $\cdot_F(\langle a, b \rangle) = c$ if and only if $\mathrm{ovf}\ F(\langle c, a \rangle) = b$.

(41) For every field F and for all elements a, c of the support of F and for every element b of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\cdot_F(\langle a, b \rangle) = c$ if and only if $\mathrm{ovf}\ F(\langle c, b \rangle) = a$.

[1]The proposition (33) was either repeated or obvious.

(42) For every field F and for every element a of the support of F and for all elements b, c of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\mathrm{ovf}\, F(\langle a, \mathrm{ovf}\, F(\langle b, c\rangle)\rangle) = \cdot_F(\langle \mathrm{ovf}\, F(\langle a, b\rangle), c\rangle)$.

(43) For every field F and for every element a of the support of F and for all elements b, c of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\mathrm{ovf}\, F(\langle a, \cdot_F(\langle b, c\rangle)\rangle) = \mathrm{ovf}\, F(\langle \mathrm{ovf}\, F(\langle a, b\rangle), c\rangle)$.

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Received September 27, 1990

FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

Infimum and Supremum of the Set of Real Numbers. Measure Theory

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Summary. We introduce some properties of the least upper bound and the greatest lower bound of the subdomain of $\overline{\mathbb{R}}$ numbers, where $\overline{\mathbb{R}}$ denotes the enlarged set of real numbers, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The paper contains definitions of majorant and minorant elements, bounded from above, bounded from below and bounded sets, sup and inf of set, for nonempty subset of $\overline{\mathbb{R}}$. We prove theorems describing the basic relationships among those definitions. The work is the first part of the series of articles concerning the Lebesgue measure theory.

MML Identifier: `SUPINF_1`.

The terminology and notation used here have been introduced in the following articles: [3], [1], and [2]. The constant $+\infty$ is defined by:

(Def.1) $+\infty = \mathbb{R}$.

The following propositions are true:

(1) $+\infty = \mathbb{R}$.

(2) $+\infty \notin \mathbb{R}$.

A positive infinite number is defined as follows:

(Def.2) it $= +\infty$.

One can prove the following proposition

(4)[1] $+\infty$ is a positive infinite number.

The constant $-\infty$ is defined as follows:

(Def.3) $-\infty = \{\mathbb{R}\}$.

The following propositions are true:

(5) $-\infty = \{\mathbb{R}\}$.

(6) $-\infty \notin \mathbb{R}$.

[1]The proposition (3) was either repeated or obvious.

ISSN 0777–4028

A negative infinite number is defined as follows:

(Def.4) it $= -\infty$.

One can prove the following proposition

(8)[2] $-\infty$ is a negative infinite number.

A *Real number* is defined as follows:

(Def.5) it $\in \mathbb{R} \cup \{-\infty, +\infty\}$.

One can prove the following propositions:

(10)[3] For every real number x holds x is a *Real number*.

(11) For an arbitrary x such that $x = -\infty$ or $x = +\infty$ holds x is a *Real number*.

Let us note that it makes sense to consider the following constant. Then $+\infty$ is a *Real number*.

Let us note that it makes sense to consider the following constant. Then $-\infty$ is a *Real number*.

Next we state the proposition

(14)[4] $-\infty \neq +\infty$.

Let x, y be *Real numbers*. The predicate $x \leq y$ is defined by:

(Def.6) there exist real numbers p, q such that $p = x$ and $q = y$ and $p \leq q$ or there exists a positive infinite number q such that $q = y$ or there exists a negative infinite number p such that $p = x$.

Next we state several propositions:

(16)[5] For all *Real numbers* x, y such that x is a real number and y is a real number holds $x \leq y$ if and only if there exist real numbers p, q such that $p = x$ and $q = y$ and $p \leq q$.

(17) For every *Real number* x such that $x \in \mathbb{R}$ holds $x \not\leq -\infty$.

(18) For every *Real number* x such that $x \in \mathbb{R}$ holds $+\infty \not\leq x$.

(19) $+\infty \not\leq -\infty$.

(20) For every *Real number* x holds $x \leq +\infty$.

(21) For every *Real number* x holds $-\infty \leq x$.

(22) For all *Real numbers* x, y such that $x \leq y$ and $y \leq x$ holds $x = y$.

(23) For every *Real number* x such that $x \leq -\infty$ holds $x = -\infty$.

(24) For every *Real number* x such that $+\infty \leq x$ holds $x = +\infty$.

The scheme *SepR_eal* concerns a unary predicate $\mathcal{P}$, and states that:

there exists a subset X of $\mathbb{R} \cup \{-\infty, +\infty\}$ such that for every *Real number* x holds $x \in X$ if and only if $\mathcal{P}[x]$

for all values of the parameter.

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[2]The proposition (7) was either repeated or obvious.
[3]The proposition (9) was either repeated or obvious.
[4]The propositions (12)–(13) were either repeated or obvious.
[5]The proposition (15) was either repeated or obvious.

The set $\overline{\mathbb{R}}$ is defined as follows:

(Def.7) $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

We now state several propositions:

(25) $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

(26) $\overline{\mathbb{R}}$ is a non-empty set.

(27) For an arbitrary x holds x is a *Real number* if and only if $x \in \overline{\mathbb{R}}$.

(28) For every *Real number* x holds $x \leq x$.

(29) For all *Real numbers* x, y, z such that $x \leq y$ and $y \leq z$ holds $x \leq z$.

Let us note that it makes sense to consider the following constant. Then $\overline{\mathbb{R}}$ is a non-empty set.

Let x, y be *Real numbers*. The predicate $x < y$ is defined by:

(Def.8) $x \leq y$ and $x \neq y$.

The following proposition is true

(31)[6] For every *Real number* x such that $x \in \mathbb{R}$ holds $-\infty < x$ and $x < +\infty$.

Let X be a non-empty subset of $\overline{\mathbb{R}}$. A *Real number* is said to be a majorant of X if:

(Def.9) for every *Real number* x such that $x \in X$ holds $x \leq$ it.

We now state two propositions:

(33)[7] For every non-empty subset X of $\overline{\mathbb{R}}$ holds $+\infty$ is a majorant of X.

(34) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ such that $X \subseteq Y$ for every *Real number* U_1 such that U_1 is a majorant of Y holds U_1 is a majorant of X.

Let X be a non-empty subset of $\overline{\mathbb{R}}$. A *Real number* is said to be a minorant of X if:

(Def.10) for every *Real number* x such that $x \in X$ holds it $\leq x$.

We now state four propositions:

(36)[8] For every non-empty subset X of $\overline{\mathbb{R}}$ holds $-\infty$ is a minorant of X.

(37) For every non-empty subset X of $\overline{\mathbb{R}}$ such that $X = \overline{\mathbb{R}}$ holds $+\infty$ is a majorant of X.

(38) For every non-empty subset X of $\overline{\mathbb{R}}$ such that $X = \overline{\mathbb{R}}$ holds $-\infty$ is a minorant of X.

(39) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ such that $X \subseteq Y$ for every *Real number* L_1 such that L_1 is a minorant of Y holds L_1 is a minorant of X.

Let us note that it makes sense to consider the following constant. Then $\mathbb{R}$ is a non-empty subset of $\overline{\mathbb{R}}$.

One can prove the following propositions:

(41)[9] $+\infty$ is a majorant of $\mathbb{R}$.

[6] The proposition (30) was either repeated or obvious.
[7] The proposition (32) was either repeated or obvious.
[8] The proposition (35) was either repeated or obvious.
[9] The proposition (40) was either repeated or obvious.

(42) $-\infty$ is a minorant of $\mathbb{R}$.

Let X be a non-empty subset of $\overline{\mathbb{R}}$. We say that X is upper bounded if and only if:

(Def.11) there exists a majorant U_1 of X such that $U_1 \in \mathbb{R}$.

The following two propositions are true:

(44)[10] For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ such that $X \subseteq Y$ holds if Y is upper bounded, then X is upper bounded.

(45) $\mathbb{R}$ is not upper bounded.

Let X be a non-empty subset of $\overline{\mathbb{R}}$. We say that X is lower bounded if and only if:

(Def.12) there exists a minorant L_1 of X such that $L_1 \in \mathbb{R}$.

The following two propositions are true:

(47)[11] For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ such that $X \subseteq Y$ holds if Y is lower bounded, then X is lower bounded.

(48) $\mathbb{R}$ is not lower bounded.

Let X be a non-empty subset of $\overline{\mathbb{R}}$. We say that X is bounded if and only if:

(Def.13) X is upper bounded and X is lower bounded.

The following two propositions are true:

(50)[12] For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ such that $X \subseteq Y$ holds if Y is bounded, then X is bounded.

(51) For every non-empty subset X of $\overline{\mathbb{R}}$ there exists a non-empty subset Y of $\overline{\mathbb{R}}$ such that for every *Real number* x holds $x \in Y$ if and only if x is a majorant of X.

Let X be a non-empty subset of $\overline{\mathbb{R}}$. The functor $\overline{X}$ yields a non-empty subset of $\overline{\mathbb{R}}$ and is defined as follows:

(Def.14) for every *Real number* x holds $x \in \overline{X}$ if and only if x is a majorant of X.

One can prove the following four propositions:

(52) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every non-empty subset Y of $\overline{\mathbb{R}}$ holds $Y = \overline{X}$ if and only if for every *Real number* x holds $x \in Y$ if and only if x is a majorant of X.

(53) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every *Real number* x holds $x \in \overline{X}$ if and only if x is a majorant of X.

(54) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ such that $X \subseteq Y$ for every *Real number* x such that $x \in \overline{Y}$ holds $x \in \overline{X}$.

(55) For every non-empty subset X of $\overline{\mathbb{R}}$ there exists a non-empty subset Y of $\overline{\mathbb{R}}$ such that for every *Real number* x holds $x \in Y$ if and only if x is a minorant of X.

[10]The proposition (43) was either repeated or obvious.
[11]The proposition (46) was either repeated or obvious.
[12]The proposition (49) was either repeated or obvious.

Let X be a non-empty subset of $\overline{\mathbb{R}}$. The functor $\underline{X}$ yields a non-empty subset of $\overline{\mathbb{R}}$ and is defined by:

(Def.15) for every *Real number* x holds $x \in \underline{X}$ if and only if x is a minorant of X.

We now state a number of propositions:

(56) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every non-empty subset Y of $\overline{\mathbb{R}}$ holds $Y = \underline{X}$ if and only if for every *Real number* x holds $x \in Y$ if and only if x is a minorant of X.

(57) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every *Real number* x holds $x \in \underline{X}$ if and only if x is a minorant of X.

(58) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ such that $X \subseteq Y$ for every *Real number* x such that $x \in \underline{Y}$ holds $x \in \underline{X}$.

(59) For every non-empty subset X of $\overline{\mathbb{R}}$ such that X is upper bounded and $X \neq \{-\infty\}$ there exists a real number x such that $x \in X$ and $x \neq -\infty$.

(60) For every non-empty subset X of $\overline{\mathbb{R}}$ such that X is lower bounded and $X \neq \{+\infty\}$ there exists a real number x such that $x \in X$ and $x \neq +\infty$.

(62)[13] For every non-empty subset X of $\overline{\mathbb{R}}$ such that X is upper bounded and $X \neq \{-\infty\}$ there exists a *Real number* U_1 such that U_1 is a majorant of X and for every *Real number* y such that y is a majorant of X holds $U_1 \leq y$.

(63) For every non-empty subset X of $\overline{\mathbb{R}}$ such that X is lower bounded and $X \neq \{+\infty\}$ there exists a *Real number* L_1 such that L_1 is a minorant of X and for every *Real number* y such that y is a minorant of X holds $y \leq L_1$.

(64) For every non-empty subset X of $\overline{\mathbb{R}}$ such that $X = \{-\infty\}$ holds X is upper bounded.

(65) For every non-empty subset X of $\overline{\mathbb{R}}$ such that $X = \{+\infty\}$ holds X is lower bounded.

(66) For every non-empty subset X of $\overline{\mathbb{R}}$ such that $X = \{-\infty\}$ there exists a *Real number* U_1 such that U_1 is a majorant of X and for every *Real number* y such that y is a majorant of X holds $U_1 \leq y$.

(67) For every non-empty subset X of $\overline{\mathbb{R}}$ such that $X = \{+\infty\}$ there exists a *Real number* L_1 such that L_1 is a minorant of X and for every *Real number* y such that y is a minorant of X holds $y \leq L_1$.

(68) For every non-empty subset X of $\overline{\mathbb{R}}$ such that X is upper bounded there exists a *Real number* U_1 such that U_1 is a majorant of X and for every *Real number* y such that y is a majorant of X holds $U_1 \leq y$.

(69) For every non-empty subset X of $\overline{\mathbb{R}}$ such that X is lower bounded there exists a *Real number* L_1 such that L_1 is a minorant of X and for every *Real number* y such that y is a minorant of X holds $y \leq L_1$.

[13]The proposition (61) was either repeated or obvious.

(70) For every non-empty subset X of $\overline{\mathbb{R}}$ such that X is not upper bounded for every *Real number* y such that y is a majorant of X holds $y = +\infty$.

(71) For every non-empty subset X of $\overline{\mathbb{R}}$ such that X is not lower bounded for every *Real number* y such that y is a minorant of X holds $y = -\infty$.

(72) For every non-empty subset X of $\overline{\mathbb{R}}$ there exists a *Real number* U_1 such that U_1 is a majorant of X and for every *Real number* y such that y is a majorant of X holds $U_1 \leq y$.

(73) For every non-empty subset X of $\overline{\mathbb{R}}$ there exists a *Real number* L_1 such that L_1 is a minorant of X and for every *Real number* y such that y is a minorant of X holds $y \leq L_1$.

Let X be a non-empty subset of $\overline{\mathbb{R}}$. The functor $\sup X$ yields a *Real number* and is defined as follows:

(Def.16) $\sup X$ is a majorant of X and for every *Real number* y such that y is a majorant of X holds $\sup X \leq y$.

The following propositions are true:

(74) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every *Real number* S holds $S = \sup X$ if and only if S is a majorant of X and for every *Real number* y such that y is a majorant of X holds $S \leq y$.

(75) For every non-empty subset X of $\overline{\mathbb{R}}$ holds $\sup X$ is a majorant of X and for every *Real number* y such that y is a majorant of X holds $\sup X \leq y$.

(76) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every *Real number* x such that $x \in X$ holds $x \leq \sup X$.

Let X be a non-empty subset of $\overline{\mathbb{R}}$. The functor $\inf X$ yields a *Real number* and is defined by:

(Def.17) $\inf X$ is a minorant of X and for every *Real number* y such that y is a minorant of X holds $y \leq \inf X$.

The following propositions are true:

(77) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every *Real number* S holds $S = \inf X$ if and only if S is a minorant of X and for every *Real number* y such that y is a minorant of X holds $y \leq S$.

(78) For every non-empty subset X of $\overline{\mathbb{R}}$ holds $\inf X$ is a minorant of X and for every *Real number* y such that y is a minorant of X holds $y \leq \inf X$.

(79) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every *Real number* x such that $x \in X$ holds $\inf X \leq x$.

(80) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every majorant x of X such that $x \in X$ holds $x = \sup X$.

(81) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every minorant x of X such that $x \in X$ holds $x = \inf X$.

(82) For every non-empty subset X of $\overline{\mathbb{R}}$ holds $\sup X = \inf \overline{X}$ and $\inf X = \sup \underline{X}$.

(83) For every non-empty subset X of $\overline{\mathbb{R}}$ such that X is upper bounded and $X \neq \{-\infty\}$ holds $\sup X \in \mathbb{R}$.

(84) For every non-empty subset X of $\overline{\mathbb{R}}$ such that X is lower bounded and $X \neq \{+\infty\}$ holds $\inf X \in \mathbb{R}$.

Let x be a *Real number*. Then $\{x\}$ is a non-empty subset of $\overline{\mathbb{R}}$.

Let x, y be *Real numbers*. Then $\{x, y\}$ is a non-empty subset of $\overline{\mathbb{R}}$.

We now state a number of propositions:

(85) For every *Real number* x holds $\sup\{x\} = x$.

(86) For every *Real number* x holds $\inf\{x\} = x$.

(87) $\sup\{-\infty\} = -\infty$.

(88) $\sup\{+\infty\} = +\infty$.

(89) $\inf\{-\infty\} = -\infty$.

(90) $\inf\{+\infty\} = +\infty$.

(91) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ such that $X \subseteq Y$ holds $\sup X \leq \sup Y$.

(92) For all *Real numbers* x, y and for every *Real number* a such that $x \leq a$ and $y \leq a$ holds $\sup\{x, y\} \leq a$.

(93) For all *Real numbers* x, y holds if $x \leq y$, then $\sup\{x, y\} = y$ but if $y \leq x$, then $\sup\{x, y\} = x$.

(94) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ such that $X \subseteq Y$ holds $\inf Y \leq \inf X$.

(95) For all *Real numbers* x, y and for every *Real number* a such that $a \leq x$ and $a \leq y$ holds $a \leq \inf\{x, y\}$.

(96) For all *Real numbers* x, y holds if $x \leq y$, then $\inf\{x, y\} = x$ but if $y \leq x$, then $\inf\{x, y\} = y$.

(97) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every *Real number* x such that there exists a *Real number* y such that $y \in X$ and $x \leq y$ holds $x \leq \sup X$.

(98) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every *Real number* x such that there exists a *Real number* y such that $y \in X$ and $y \leq x$ holds $\inf X \leq x$.

(99) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ such that for every *Real number* x such that $x \in X$ there exists a *Real number* y such that $y \in Y$ and $x \leq y$ holds $\sup X \leq \sup Y$.

(100) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ such that for every *Real number* y such that $y \in Y$ there exists a *Real number* x such that $x \in X$ and $x \leq y$ holds $\inf X \leq \inf Y$.

Let X, Y be non-empty subsets of $\overline{\mathbb{R}}$. Then $X \cup Y$ is a non-empty subset of $\overline{\mathbb{R}}$.

One can prove the following propositions:

(101) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ and for every majorant U_2 of X and for every majorant U_3 of Y holds $\sup\{U_2, U_3\}$ is a majorant of $X \cup Y$.

(102) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ and for every minorant L_2 of X and for every minorant L_3 of Y holds $\inf\{L_2, L_3\}$ is a minorant of $X \cup Y$.

(103) For all non-empty subsets X, Y, S of $\overline{\mathbb{R}}$ and for every majorant U_2 of X and for every majorant U_3 of Y such that $S = X \cap Y$ holds $\inf\{U_2, U_3\}$ is a majorant of S.

(104) For all non-empty subsets X, Y, S of $\overline{\mathbb{R}}$ and for every minorant L_2 of X and for every minorant L_3 of Y such that $S = X \cap Y$ holds $\sup\{L_2, L_3\}$ is a minorant of S.

(105) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ holds $\sup(X \cup Y) = \sup\{\sup X, \sup Y\}$.

(106) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ holds $\inf(X \cup Y) = \inf\{\inf X, \inf Y\}$.

(107) For all non-empty subsets X, Y, S of $\overline{\mathbb{R}}$ such that $S = X \cap Y$ holds $\sup S \leq \inf\{\sup X, \sup Y\}$.

(108) For all non-empty subsets X, Y, S of $\overline{\mathbb{R}}$ such that $S = X \cap Y$ holds $\sup\{\inf X, \inf Y\} \leq \inf S$.

Let X be a non-empty set. A set is called a non-empty set of non-empty subsets of X if:

(Def.18) it is a non-empty subset of 2^X and for every set A such that $A \in$ it holds A is a non-empty set.

Let F be a non-empty set of non-empty subsets of $\overline{\mathbb{R}}$. The functor $\sup_{\overline{\mathbb{R}}} F$ yielding a non-empty subset of $\overline{\mathbb{R}}$ is defined as follows:

(Def.19) for every *Real number* a holds $a \in \sup_{\overline{\mathbb{R}}} F$ if and only if there exists a non-empty subset A of $\overline{\mathbb{R}}$ such that $A \in F$ and $a = \sup A$.

We now state several propositions:

(110)[14] For every non-empty set F of non-empty subsets of $\overline{\mathbb{R}}$ and for every non-empty subset S of $\overline{\mathbb{R}}$ holds $S = \sup_{\overline{\mathbb{R}}} F$ if and only if for every *Real number* a holds $a \in S$ if and only if there exists a non-empty subset A of $\overline{\mathbb{R}}$ such that $A \in F$ and $a = \sup A$.

(111) For every non-empty set F of non-empty subsets of $\overline{\mathbb{R}}$ and for every *Real number* a holds $a \in \sup_{\overline{\mathbb{R}}} F$ if and only if there exists a non-empty subset A of $\overline{\mathbb{R}}$ such that $A \in F$ and $a = \sup A$.

(112) For every non-empty set F of non-empty subsets of $\overline{\mathbb{R}}$ and for every non-empty subset S of $\overline{\mathbb{R}}$ such that $S = \bigcup F$ holds $\sup S$ is a majorant of $\sup_{\overline{\mathbb{R}}} F$.

(113) For every non-empty set F of non-empty subsets of $\overline{\mathbb{R}}$ and for every non-empty subset S of $\overline{\mathbb{R}}$ such that $S = \bigcup F$ holds $\sup(\sup_{\overline{\mathbb{R}}} F)$ is a majorant of S.

[14]The proposition (109) was either repeated or obvious.

(114) For every non-empty set F of non-empty subsets of $\overline{\mathbb{R}}$ and for every non-empty subset S of $\overline{\mathbb{R}}$ such that $S = \bigcup F$ holds $\sup S = \sup(\sup_{\overline{\mathbb{R}}} F)$.

Let F be a non-empty set of non-empty subsets of $\overline{\mathbb{R}}$. The functor $\inf_{\overline{\mathbb{R}}} F$ yields a non-empty subset of $\overline{\mathbb{R}}$ and is defined as follows:

(Def.20) for every *Real number* a holds $a \in \inf_{\overline{\mathbb{R}}} F$ if and only if there exists a non-empty subset A of $\overline{\mathbb{R}}$ such that $A \in F$ and $a = \inf A$.

We now state several propositions:

(115) For every non-empty set F of non-empty subsets of $\overline{\mathbb{R}}$ and for every non-empty subset S of $\overline{\mathbb{R}}$ holds $S = \inf_{\overline{\mathbb{R}}} F$ if and only if for every *Real number* a holds $a \in S$ if and only if there exists a non-empty subset A of $\overline{\mathbb{R}}$ such that $A \in F$ and $a = \inf A$.

(116) For every non-empty set F of non-empty subsets of $\overline{\mathbb{R}}$ and for every *Real number* a holds $a \in \inf_{\overline{\mathbb{R}}} F$ if and only if there exists a non-empty subset A of $\overline{\mathbb{R}}$ such that $A \in F$ and $a = \inf A$.

(117) For every non-empty set F of non-empty subsets of $\overline{\mathbb{R}}$ and for every non-empty subset S of $\overline{\mathbb{R}}$ such that $S = \bigcup F$ holds $\inf S$ is a minorant of $\inf_{\overline{\mathbb{R}}} F$.

(118) For every non-empty set F of non-empty subsets of $\overline{\mathbb{R}}$ and for every non-empty subset S of $\overline{\mathbb{R}}$ such that $S = \bigcup F$ holds $\inf(\inf_{\overline{\mathbb{R}}} F)$ is a minorant of S.

(119) For every non-empty set F of non-empty subsets of $\overline{\mathbb{R}}$ and for every non-empty subset S of $\overline{\mathbb{R}}$ such that $S = \bigcup F$ holds $\inf S = \inf(\inf_{\overline{\mathbb{R}}} F)$.

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Received September 27, 1990

FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

Series of Positive Real Numbers. Measure Theory

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Summary. We introduce properties of a series of nonnegative $\overline{\mathbb{R}}$ numbers, where $\overline{\mathbb{R}}$ denotes the enlarged set of real numbers, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The paper contains definitions of sup F and inf F, for F being function, and a definition of a sumable subset of $\overline{\mathbb{R}}$. We prove the basic theorems regarding the definitions mentioned above. The work is the second part of a series of articles concerning the Lebesgue measure theory.

MML Identifier: `SUPINF_2`.

The notation and terminology used here are introduced in the following articles: [6], [5], [2], [3], [4], and [1]. Let x, y be *Real numbers*. Let us assume that neither $x = +\infty$ and $y = -\infty$ nor $x = -\infty$ and $y = +\infty$. The functor $x + y$ yielding a *Real number* is defined by:

(Def.1) there exist real numbers a, b such that $x = a$ and $y = b$ and $x+y = a+b$ or $x = +\infty$ and $x + y = +\infty$ or $y = +\infty$ and $x + y = +\infty$ or $x = -\infty$ and $x + y = -\infty$ or $y = -\infty$ and $x + y = -\infty$.

Next we state four propositions:

(1) Let x, y be *Real numbers*. Suppose neither $x = +\infty$ and $y = -\infty$ nor $x = -\infty$ and $y = +\infty$. Then
(i) there exist real numbers a, b such that $x = a$ and $y = b$ and $x+y = a+b$, or
(ii) $x = +\infty$ and $x + y = +\infty$, or
(iii) $y = +\infty$ and $x + y = +\infty$, or
(iv) $x = -\infty$ and $x + y = -\infty$, or
(v) $y = -\infty$ and $x + y = -\infty$.

(2) For all *Real numbers* x, y and for all real numbers a, b such that $x = a$ and $y = b$ holds $x + y = a + b$.

ISSN 0777–4028

(3) For every *Real number* x such that $x \neq -\infty$ holds $+\infty + x = +\infty$ and $x + +\infty = +\infty$.

(4) For every *Real number* x such that $x \neq +\infty$ holds $-\infty + x = -\infty$ and $x + -\infty = -\infty$.

Let x, y be *Real numbers*. Let us assume that neither $x = +\infty$ and $y = +\infty$ nor $x = -\infty$ and $y = -\infty$. The functor $x - y$ yielding a *Real number* is defined by:

(Def.2) there exist real numbers a, b such that $x = a$ and $y = b$ and $x - y = a - b$ or $x = +\infty$ and $x - y = +\infty$ or $y = +\infty$ and $x - y = -\infty$ or $x = -\infty$ and $x - y = -\infty$ or $y = -\infty$ and $x - y = +\infty$.

We now state a number of propositions:

(5) Let x, y be *Real numbers*. Suppose neither $x = +\infty$ and $y = +\infty$ nor $x = -\infty$ and $y = -\infty$. Then

(i) there exist real numbers a, b such that $x = a$ and $y = b$ and $x - y = a - b$, or

(ii) $x = +\infty$ and $x - y = +\infty$, or

(iii) $y = +\infty$ and $x - y = -\infty$, or

(iv) $x = -\infty$ and $x - y = -\infty$, or

(v) $y = -\infty$ and $x - y = +\infty$.

(6) For all *Real numbers* x, y and for all real numbers a, b such that $x = a$ and $y = b$ holds $x - y = a - b$.

(7) For every *Real number* x such that $x \neq +\infty$ holds $+\infty - x = +\infty$ and $x - +\infty = -\infty$.

(8) For every *Real number* x such that $x \neq -\infty$ holds $-\infty - x = -\infty$ and $x - -\infty = +\infty$.

(9) For all *Real numbers* x, s such that $x + s = +\infty$ holds $x = +\infty$ or $s = +\infty$.

(10) For all *Real numbers* x, s such that $x + s = -\infty$ holds $x = -\infty$ or $s = -\infty$.

(11) For all *Real numbers* x, s such that $x - s = +\infty$ holds $x = +\infty$ or $s = -\infty$.

(12) For all *Real numbers* x, s such that $x - s = -\infty$ holds $x = -\infty$ or $s = +\infty$.

(13) For all *Real numbers* x, s such that neither $x = +\infty$ and $s = -\infty$ nor $x = -\infty$ and $s = +\infty$ and $x + s \in \mathbb{R}$ holds $x \in \mathbb{R}$ and $s \in \mathbb{R}$.

(14) For all *Real numbers* x, s such that neither $x = +\infty$ and $s = +\infty$ nor $x = -\infty$ and $s = -\infty$ and $x - s \in \mathbb{R}$ holds $x \in \mathbb{R}$ and $s \in \mathbb{R}$.

(15) Let x, y, s, t be *Real numbers*. Then if neither $x = +\infty$ and $s = -\infty$ nor $x = -\infty$ and $s = +\infty$ and neither $y = +\infty$ and $t = -\infty$ nor $y = -\infty$ and $t = +\infty$ and $x \leq y$ and $s \leq t$, then $x + s \leq y + t$.

(16) Let x, y, s, t be *Real numbers*. Then if neither $x = +\infty$ and $t = +\infty$ nor $x = -\infty$ and $t = -\infty$ and neither $y = +\infty$ and $s = +\infty$ nor $y = -\infty$

and $s = -\infty$ and $x \leq y$ and $s \leq t$, then $x - t \leq y - s$.

Let x be a *Real number*. The functor $-x$ yields a *Real number* and is defined by:

(Def.3) there exists a real number a such that $x = a$ and $-x = -a$ or $x = +\infty$ and $-x = -\infty$ or $x = -\infty$ and $-x = +\infty$.

We now state several propositions:

(17) For every *Real number* x and for every *Real number* z holds $z = -x$ if and only if there exists a real number a such that $x = a$ and $z = -a$ or $x = +\infty$ and $z = -\infty$ or $x = -\infty$ and $z = +\infty$.

(18) For every *Real number* x holds there exists a real number a such that $x = a$ and $-x = -a$ or $x = +\infty$ and $-x = -\infty$ or $x = -\infty$ and $-x = +\infty$.

(19) For every *Real number* x and for every real number a such that $x = a$ holds $-x = -a$.

(20) For every *Real number* x holds if $x = +\infty$, then $-x = -\infty$ but if $x = -\infty$, then $-x = +\infty$.

(21) For every *Real number* x holds $-(-x) = x$.

(22) For all *Real numbers* x, y holds $x \leq y$ if and only if $-y \leq -x$.

(23) For all *Real numbers* x, y holds $x < y$ if and only if $-y < -x$.

(24) For all *Real numbers* x, y such that $x = y$ holds $x \leq y$.

The *Real number* $0_{\overline{\mathbb{R}}}$ is defined by:

(Def.4) $0_{\overline{\mathbb{R}}} = 0$.

We now state several propositions:

(25) $0_{\overline{\mathbb{R}}} = 0$.

(26) For every *Real number* x holds $x + 0_{\overline{\mathbb{R}}} = x$ and $0_{\overline{\mathbb{R}}} + x = x$.

(27) $-\infty < 0_{\overline{\mathbb{R}}}$ and $0_{\overline{\mathbb{R}}} < +\infty$.

(28) For all *Real numbers* x, y, z such that $0_{\overline{\mathbb{R}}} \leq z$ and $0_{\overline{\mathbb{R}}} \leq x$ and $y = x + z$ holds $x \leq y$.

(29) For every real number x such that $x \in \mathbb{N}$ holds $0 \leq x$.

(30) For every *Real number* x such that $x \in \mathbb{N}$ holds $0_{\overline{\mathbb{R}}} \leq x$.

Let X, Y be non-empty subsets of $\overline{\mathbb{R}}$. Let us assume that neither $-\infty \in X$ and $+\infty \in Y$ nor $+\infty \in X$ and $-\infty \in Y$. The functor $X + Y$ yielding a non-empty subset of $\overline{\mathbb{R}}$ is defined as follows:

(Def.5) for every *Real number* z holds $z \in X + Y$ if and only if there exist *Real numbers* x, y such that $x \in X$ and $y \in Y$ and $z = x + y$.

We now state two propositions:

(31) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ such that neither $-\infty \in X$ and $+\infty \in Y$ nor $+\infty \in X$ and $-\infty \in Y$ for every *Real number* z holds $z \in X + Y$ if and only if there exist *Real numbers* x, y such that $x \in X$ and $y \in Y$ and $z = x + y$.

(32) Let X, Y, Z be non-empty subsets of $\overline{\mathbb{R}}$. Then if neither $-\infty \in X$ and $+\infty \in Y$ nor $+\infty \in X$ and $-\infty \in Y$, then $Z = X + Y$ if and only if for every *Real number* z holds $z \in Z$ if and only if there exist *Real numbers* x, y such that $x \in X$ and $y \in Y$ and $z = x + y$.

Let X be a non-empty subset of $\overline{\mathbb{R}}$. The functor $-X$ yielding a non-empty subset of $\overline{\mathbb{R}}$ is defined as follows:

(Def.6) for every *Real number* z holds $z \in -X$ if and only if there exists a *Real number* x such that $x \in X$ and $z = -x$.

Next we state a number of propositions:

(33) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every *Real number* z holds $z \in -X$ if and only if there exists a *Real number* x such that $x \in X$ and $z = -x$.

(34) For all non-empty subsets X, Z of $\overline{\mathbb{R}}$ holds $Z = -X$ if and only if for every *Real number* z holds $z \in Z$ if and only if there exists a *Real number* x such that $x \in X$ and $z = -x$.

(35) For every non-empty subset X of $\overline{\mathbb{R}}$ holds $-(-X) = X$.

(36) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every *Real number* z holds $z \in X$ if and only if $-z \in -X$.

(37) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ holds $X \subseteq Y$ if and only if $-X \subseteq -Y$.

(38) For every *Real number* z holds $z \in \mathbb{R}$ if and only if $-z \in \mathbb{R}$.

(39) Let X, Y be non-empty subsets of $\overline{\mathbb{R}}$. Then if neither $-\infty \in X$ and $+\infty \in Y$ nor $+\infty \in X$ and $-\infty \in Y$ and neither $\sup X = +\infty$ and $\sup Y = -\infty$ nor $\sup X = -\infty$ and $\sup Y = +\infty$, then $\sup(X + Y) \leq \sup X + \sup Y$.

(40) Let X, Y be non-empty subsets of $\overline{\mathbb{R}}$. Then if neither $-\infty \in X$ and $+\infty \in Y$ nor $+\infty \in X$ and $-\infty \in Y$ and neither $\inf X = +\infty$ and $\inf Y = -\infty$ nor $\inf X = -\infty$ and $\inf Y = +\infty$, then $\inf X + \inf Y \leq \inf(X + Y)$.

(41) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ such that X is upper bounded and Y is upper bounded holds $\sup(X + Y) \leq \sup X + \sup Y$.

(42) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ such that X is lower bounded and Y is lower bounded holds $\inf X + \inf Y \leq \inf(X + Y)$.

(43) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every *Real number* a holds a is a majorant of X if and only if $-a$ is a minorant of $-X$.

(44) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every *Real number* a holds a is a minorant of X if and only if $-a$ is a majorant of $-X$.

(45) For every non-empty subset X of $\overline{\mathbb{R}}$ holds $\inf(-X) = -\sup X$.

(46) For every non-empty subset X of $\overline{\mathbb{R}}$ holds $\sup(-X) = -\inf X$.

Let X be a non-empty set, and let Y be a non-empty subset of $\overline{\mathbb{R}}$, and let F be a function from X into Y. Then $\operatorname{rng} F$ is a non-empty subset of $\overline{\mathbb{R}}$.

Let X be a non-empty set, and let Y be a non-empty subset of $\overline{\mathbb{R}}$, and let F be a function from X into Y. The functor $\sup F$ yielding a *Real number* is defined by:

(Def.7) $\sup F = \sup(\operatorname{rng} F)$.

The following proposition is true

(47) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y holds $\sup F = \sup(\operatorname{rng} F)$.

Let X be a non-empty set, and let Y be a non-empty subset of $\overline{\mathbb{R}}$, and let F be a function from X into Y. The functor $\inf F$ yields a *Real number* and is defined by:

(Def.8) $\inf F = \inf(\operatorname{rng} F)$.

Next we state the proposition

(48) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y holds $\inf F = \inf(\operatorname{rng} F)$.

Let X be a non-empty set, and let Y be a non-empty subset of $\overline{\mathbb{R}}$, and let F be a function from X into Y, and let x be an element of X. Then $F(x)$ is a *Real number*.

The scheme *FunctR_ealEx* concerns a non-empty set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ and states that:

there exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every element x of $\mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

provided the parameters have the following property:

- for every element x of $\mathcal{A}$ holds $\mathcal{F}(x) \in \mathcal{B}$.

Let X be a non-empty set, and let Y, Z be non-empty subsets of $\overline{\mathbb{R}}$, and let F be a function from X into Y, and let G be a function from X into Z. Let us assume that neither $-\infty \in Y$ and $+\infty \in Z$ nor $+\infty \in Y$ and $-\infty \in Z$. The functor $F + G$ yields a function from X into $Y + Z$ and is defined by:

(Def.9) for every element x of X holds $(F + G)(x) = F(x) + G(x)$.

Next we state several propositions:

(49) Let X be a non-empty set. Let Y, Z be non-empty subsets of $\overline{\mathbb{R}}$. Suppose neither $-\infty \in Y$ and $+\infty \in Z$ nor $+\infty \in Y$ and $-\infty \in Z$. Then for every function F from X into Y and for every function G from X into Z and for every function H from X into $Y + Z$ holds $H = F + G$ if and only if for every element x of X holds $H(x) = F(x) + G(x)$.

(50) Let X be a non-empty set. Then for all non-empty subsets Y, Z of $\overline{\mathbb{R}}$ such that neither $-\infty \in Y$ and $+\infty \in Z$ nor $+\infty \in Y$ and $-\infty \in Z$ for every function F from X into Y and for every function G from X into Z and for every element x of X holds $(F + G)(x) = F(x) + G(x)$.

(51) For every non-empty set X and for all non-empty subsets Y, Z of $\overline{\mathbb{R}}$ such that neither $-\infty \in Y$ and $+\infty \in Z$ nor $+\infty \in Y$ and $-\infty \in Z$ for every function F from X into Y and for every function G from X into Z holds $\operatorname{rng}(F + G) \subseteq \operatorname{rng} F + \operatorname{rng} G$.

(52) Let X be a non-empty set. Let Y, Z be non-empty subsets of $\overline{\mathbb{R}}$. Suppose neither $-\infty \in Y$ and $+\infty \in Z$ nor $+\infty \in Y$ and $-\infty \in Z$. Then for

every function F from X into Y and for every function G from X into Z such that neither $\sup F = +\infty$ and $\sup G = -\infty$ nor $\sup F = -\infty$ and $\sup G = +\infty$ holds $\sup(F + G) \leq \sup F + \sup G$.

(53) Let X be a non-empty set. Let Y, Z be non-empty subsets of $\overline{\mathbb{R}}$. Suppose neither $-\infty \in Y$ and $+\infty \in Z$ nor $+\infty \in Y$ and $-\infty \in Z$. Then for every function F from X into Y and for every function G from X into Z such that neither $\inf F = +\infty$ and $\inf G = -\infty$ nor $\inf F = -\infty$ and $\inf G = +\infty$ holds $\inf F + \inf G \leq \inf(F + G)$.

Let X be a non-empty set, and let Y be a non-empty subset of $\overline{\mathbb{R}}$, and let F be a function from X into Y. The functor $-F$ yielding a function from X into $-Y$ is defined by:

(Def.10) for every element x of X holds $(-F)(x) = -F(x)$.

One can prove the following three propositions:

(54) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y and for every function G from X into $-Y$ holds $G = -F$ if and only if for every element x of X holds $G(x) = -F(x)$.

(55) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y holds $\operatorname{rng}(-F) = -\operatorname{rng} F$.

(56) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y holds $\inf(-F) = -\sup F$ and $\sup(-F) = -\inf F$.

Let X be a non-empty set, and let Y be a non-empty subset of $\overline{\mathbb{R}}$, and let F be a function from X into Y. We say that F is upper bounded if and only if:

(Def.11) $\sup F < +\infty$.

Let X be a non-empty set, and let Y be a non-empty subset of $\overline{\mathbb{R}}$, and let F be a function from X into Y. We say that F is lower bounded if and only if:

(Def.12) $-\infty < \inf F$.

Let X be a non-empty set, and let Y be a non-empty subset of $\overline{\mathbb{R}}$, and let F be a function from X into Y. We say that F is bounded if and only if:

(Def.13) F is upper bounded and F is lower bounded.

We now state a number of propositions:

(60)[1] For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y holds F is bounded if and only if $\sup F < +\infty$ and $-\infty < \inf F$.

(61) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y holds F is upper bounded if and only if $-F$ is lower bounded.

[1]The propositions (57)–(59) were either repeated or obvious.

(62) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y holds F is lower bounded if and only if $-F$ is upper bounded.

(63) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y holds F is bounded if and only if $-F$ is bounded.

(64) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y and for every element x of X holds $-\infty \leq F(x)$ and $F(x) \leq +\infty$.

(65) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y and for every element x of X such that $Y \subseteq \mathbb{R}$ holds $-\infty < F(x)$ and $F(x) < +\infty$.

(66) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y and for every element x of X holds $\inf F \leq F(x)$ and $F(x) \leq \sup F$.

(67) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y such that $Y \subseteq \mathbb{R}$ holds F is upper bounded if and only if $\sup F \in \mathbb{R}$.

(68) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y such that $Y \subseteq \mathbb{R}$ holds F is lower bounded if and only if $\inf F \in \mathbb{R}$.

(69) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y such that $Y \subseteq \mathbb{R}$ holds F is bounded if and only if $\inf F \in \mathbb{R}$ and $\sup F \in \mathbb{R}$.

(70) For every non-empty set X and for all non-empty subsets Y, Z of $\overline{\mathbb{R}}$ such that $Y \subseteq \mathbb{R}$ and $Z \subseteq \mathbb{R}$ for every function F_1 from X into Y and for every function F_2 from X into Z such that F_1 is upper bounded and F_2 is upper bounded holds $F_1 + F_2$ is upper bounded.

(71) For every non-empty set X and for all non-empty subsets Y, Z of $\overline{\mathbb{R}}$ such that $Y \subseteq \mathbb{R}$ and $Z \subseteq \mathbb{R}$ for every function F_1 from X into Y and for every function F_2 from X into Z such that F_1 is lower bounded and F_2 is lower bounded holds $F_1 + F_2$ is lower bounded.

(72) For every non-empty set X and for all non-empty subsets Y, Z of $\overline{\mathbb{R}}$ such that $Y \subseteq \mathbb{R}$ and $Z \subseteq \mathbb{R}$ for every function F_1 from X into Y and for every function F_2 from X into Z such that F_1 is bounded and F_2 is bounded holds $F_1 + F_2$ is bounded.

(73) There exists a function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is one-to-one and $\mathbb{N} = \operatorname{rng} F$ and $\operatorname{rng} F$ is a non-empty subset of $\overline{\mathbb{R}}$.

A non-empty subset of $\overline{\mathbb{R}}$ is called a denumerable set of larged real if:

(Def.14) there exists a function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that it $= \operatorname{rng} F$.

Next we state the proposition

(75)[2] $\mathbb{N}$ is a denumerable set of larged real.

A denumerable set of larged real is said to be a denumerable set of positive larged real if:

(Def.15) for every *Real number* x such that $x \in$ it holds $0_{\overline{\mathbb{R}}} \leq x$.

Let D be a denumerable set of larged real. A function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ is said to be a numeration of D if:

(Def.16) $D = \operatorname{rng}$ it.

One can prove the following proposition

(78)[3] For every denumerable set D of positive larged real and for every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ holds F is a numeration of D if and only if $D = \operatorname{rng} F$.

Let N be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$, and let n be a natural number. Then $N(n)$ is a *Real number*.

We see that the *Real number* is an element of $\overline{\mathbb{R}}$.

The scheme *RecFuncExR_eal* concerns a *Real number* $\mathcal{A}$ and a binary functor $\mathcal{F}$ yielding a *Real number* and states that:

there exists a function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that $F(0) = \mathcal{A}$ and for every natural number n and for every *Real number* x such that $x = F(n)$ holds $F(n+1) = \mathcal{F}(n, x)$

for all values of the parameters.

We now state the proposition

(79) For every denumerable set D of larged real and for every numeration N of D there exists a function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that $F(0) = N(0)$ and for every natural number n and for every *Real number* y such that $y = F(n)$ holds $F(n+1) = y + N(n+1)$.

Let D be a denumerable set of larged real, and let N be a numeration of D. The functor $\operatorname{Ser}(D, N)$ yields a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and is defined by:

(Def.17) $\operatorname{Ser}(D, N)(0) = N(0)$ and for every natural number n and for every *Real number* y such that $y = \operatorname{Ser}(D, N)(n)$ holds $\operatorname{Ser}(D, N)(n+1) = y + N(n+1)$.

The following propositions are true:

(80) Let D be a denumerable set of larged real. Then for every numeration N of D and for every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ holds $F = \operatorname{Ser}(D, N)$ if and only if $F(0) = N(0)$ and for every natural number n and for every *Real number* y such that $y = F(n)$ holds $F(n+1) = y + N(n+1)$.

(81) For every denumerable set D of larged real and for every numeration N of D holds $\operatorname{Ser}(D, N)(0) = N(0)$ and for every natural number n and for every *Real number* y such that $y = \operatorname{Ser}(D, N)(n)$ holds $\operatorname{Ser}(D, N)(n+1) = y + N(n+1)$.

(82) For every denumerable set D of positive larged real and for every numeration N of D and for every natural number n holds $0_{\overline{\mathbb{R}}} \leq N(n)$.

[2]The proposition (74) was either repeated or obvious.

[3]The propositions (76)–(77) were either repeated or obvious.

(83) For every denumerable set D of positive larged real and for every numeration N of D and for every natural number n holds $\mathrm{Ser}(D,N)(n) \leq \mathrm{Ser}(D,N)(n+1)$ and $0_{\overline{\mathbb{R}}} \leq \mathrm{Ser}(D,N)(n)$.

(84) For every denumerable set D of positive larged real and for every numeration N of D and for all natural numbers n, m holds $\mathrm{Ser}(D,N)(n) \leq \mathrm{Ser}(D,N)(n+m)$.

Let D be a denumerable set of larged real. A non-empty subset of $\overline{\mathbb{R}}$ is called a set of series of D if:

(Def.18) there exists a numeration N of D such that it $= \mathrm{rng}\,\mathrm{Ser}(D,N)$.

Let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Then $\mathrm{rng}\,F$ is a non-empty subset of $\overline{\mathbb{R}}$.

Let D be a denumerable set of positive larged real, and let N be a numeration of D. The functor $\sum_D N$ yields a *Real number* and is defined as follows:

(Def.19) $\sum_D N = \sup(\mathrm{rng}\,\mathrm{Ser}(D,N))$.

One can prove the following propositions:

(86)[4] For every denumerable set D of positive larged real and for every numeration N of D and for every *Real number* s holds $s = \sum_D N$ if and only if $s = \sup(\mathrm{rng}\,\mathrm{Ser}(D,N))$.

(87) For every denumerable set D of positive larged real and for every numeration N of D holds $\sum_D N = \sup(\mathrm{rng}\,\mathrm{Ser}(D,N))$.

Let D be a denumerable set of positive larged real, and let N be a numeration of D. We say that D is N sumable if and only if:

(Def.20) $\sum_D N \in \mathbb{R}$.

One can prove the following proposition

(89)[5] For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ holds $\mathrm{rng}\,F$ is a denumerable set of larged real.

Let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Then $\mathrm{rng}\,F$ is a denumerable set of larged real.

Next we state the proposition

(90) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ holds F is a numeration of $\mathrm{rng}\,F$.

Let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. The functor $\mathrm{Ser}\,F$ yields a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and is defined by:

(Def.21) for every numeration N of $\mathrm{rng}\,F$ such that $N = F$ holds $\mathrm{Ser}\,F = \mathrm{Ser}(\mathrm{rng}\,F, N)$.

We now state the proposition

(91) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and for every numeration N of $\mathrm{rng}\,F$ such that $N = F$ holds $\mathrm{Ser}\,F = \mathrm{Ser}(\mathrm{rng}\,F, N)$.

Let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. We say that F is non-negative if and only if:

[4] The proposition (85) was either repeated or obvious.
[5] The proposition (88) was either repeated or obvious.

(Def.22) $\operatorname{rng} F$ is a denumerable set of positive larged real.

Let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Let us assume that F is non-negative. The functor $\sum F$ yields a *Real number* and is defined by:

(Def.23) $\sum F = \sup(\operatorname{rng} \operatorname{Ser} F)$.

The following propositions are true:

(93)[6] For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is non-negative holds $\sum F = \sup(\operatorname{rng} \operatorname{Ser} F)$.

(94) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ holds F is non-negative if and only if for every natural number n holds $0_{\overline{\mathbb{R}}} \leq F(n)$.

(95) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and for every natural number n such that F is non-negative holds $\operatorname{Ser} F(n) \leq \operatorname{Ser} F(n+1)$ and $0_{\overline{\mathbb{R}}} \leq \operatorname{Ser} F(n)$.

(96) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is non-negative for all natural numbers n, m holds $\operatorname{Ser} F(n) \leq \operatorname{Ser} F(n+m)$.

(97) For all functions F_1, F_2 from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F_1 is non-negative holds if for every natural number n holds $F_1(n) \leq F_2(n)$, then for every natural number n holds $\operatorname{Ser} F_1(n) \leq \operatorname{Ser} F_2(n)$.

(98) For all functions F_1, F_2 from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F_1 is non-negative holds if for every natural number n holds $F_1(n) \leq F_2(n)$, then $\sum F_1 \leq \sum F_2$.

(99) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ holds $\operatorname{Ser} F(0) = F(0)$ and for every natural number n and for every *Real number* y such that $y = \operatorname{Ser} F(n)$ holds $\operatorname{Ser} F(n+1) = y + F(n+1)$.

(100) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is non-negative holds if there exists a natural number n such that $F(n) = +\infty$, then $\sum F = +\infty$.

Let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Let us assume that F is non-negative. We say that F is sumable if and only if:

(Def.24) $\sum F \in \mathbb{R}$.

One can prove the following propositions:

(102)[7] For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is non-negative holds if there exists a natural number n such that $F(n) = +\infty$, then F is not sumable.

(103) For all functions F_1, F_2 from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F_1 is non-negative holds if for every natural number n holds $F_1(n) \leq F_2(n)$, then if F_2 is sumable, then F_1 is sumable.

(104) For all functions F_1, F_2 from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F_1 is non-negative holds if for every natural number n holds $F_1(n) \leq F_2(n)$, then if F_1 is not sumable, then F_2 is not sumable.

(105) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is non-negative for every natural number n such that for every natural number r such that $n \leq r$ holds $F(r) = 0_{\overline{\mathbb{R}}}$ holds $\sum F = \operatorname{Ser} F(n)$.

[6]The proposition (92) was either repeated or obvious.
[7]The proposition (101) was either repeated or obvious.

(106) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that for every natural number n holds $F(n) \in \mathbb{R}$ for every natural number n holds $\mathrm{Ser}\, F(n) \in \mathbb{R}$.

(107) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is non-negative holds if there exists a natural number n such that for every natural number k such that $n \leq k$ holds $F(k) = 0_{\overline{\mathbb{R}}}$ and for every natural number k such that $k \leq n$ holds $F(k) \neq +\infty$, then F is sumable.

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Received September 27, 1990

FORMALIZED MATHEMATICS
Vol.2, No.1, January–February 1991
Université Catholique de Louvain

From Double Loops to Fields

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Summary. This paper contains the second part of the set of articles concerning the theory of algebraic structures, based on the [9], pp. 9-12 (pages 4-6 of the English edition).

First the basic structure $\langle F, +, \cdot, 1, 0\rangle$ is defined. Following it the consecutive structures are contemplated in detail, including double loop, left quasi-field, right quasi-field, double sided quasi-field, skew field and field. These structures are created by gradually augmenting the basic structure with new axioms of commutativity, associativity, distributivity etc. Each part of the article begins with the set of auxiliary theorems related to the structure under consideration that are necessary for the existence proof of each defined mode. Next the mode and proof of its existence is included. If the current set of axioms may be replaced with a different and equivalent one, the appropriate proof is performed following the definition of that mode. With the introduction of double loop the "$-$" function is defined. Some interesting features of this function are also included.

MML Identifier: `ALGSTR_2`.

The terminology and notation used here have been introduced in the following articles: [11], [10], [3], [4], [1], [2], [6], [5], [7], and [8]. We consider double loop structures which are systems

$\langle$a carrier, an addition, a multiplication, a unity, a zero$\rangle$,

where the carrier is a non-empty set, the addition is a binary operation on the carrier, the multiplication is a binary operation on the carrier, the unity is an element of the carrier, and the zero is an element of the carrier.

In the sequel G_1 will be a double loop structure and L will be a double loop structure. Let us consider G_1. An element of G_1 is an element of the carrier of G_1.

In the sequel a, b will denote elements of G_1. Let us consider G_1, a, b. The functor $a + b$ yields an element of G_1 and is defined by:

[1]Supported by RPBP.III-24.B5

ISSN 0777–4028

(Def.1) $a + b =$ (the addition of G_1)(a, b).

Let us consider G_1, a, b. The functor $a \cdot b$ yields an element of G_1 and is defined by:

(Def.2) $a \cdot b =$ (the multiplication of G_1)(a, b).

One can prove the following propositions:

(1) $a + b =$ (the addition of G_1)(a, b).

(2) $a \cdot b =$ (the multiplication of G_1)(a, b).

Let us consider G_1. The functor 0_{G_1} yielding an element of G_1 is defined as follows:

(Def.3) $0_{G_1} =$ the zero of G_1.

Let us consider G_1. The functor 1_{G_1} yields an element of G_1 and is defined as follows:

(Def.4) $1_{G_1} =$ the unity of G_1.

The following two propositions are true:

(3) $0_{G_1} =$ the zero of G_1.

(4) $1_{G_1} =$ the unity of G_1.

The double loop structure $\mathrm{loop}_{\mathbb{R}}$ is defined by:

(Def.5) $\mathrm{loop}_{\mathbb{R}} = \langle \mathbb{R}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, 1, 0 \rangle$.

One can prove the following three propositions:

(5) $\mathrm{loop}_{\mathbb{R}} = \langle \mathbb{R}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, 1, 0 \rangle$.

(6) For every real numbers q, p there exists a real number y such that $p = q + y$.

(7) For every real numbers q, p there exists a real number y such that $p = y + q$.

A double loop structure is said to be a double loop if:

(Def.6) (i) for every element a of it holds $a + 0_{\mathrm{it}} = a$,

(ii) for every element a of it holds $0_{\mathrm{it}} + a = a$,

(iii) for every elements a, b of it there exists an element x of it such that $a + x = b$,

(iv) for every elements a, b of it there exists an element x of it such that $x + a = b$,

(v) for all elements a, x, y of it such that $a + x = a + y$ holds $x = y$,

(vi) for all elements a, x, y of it such that $x + a = y + a$ holds $x = y$,

(vii) $0_{\mathrm{it}} \neq 1_{\mathrm{it}}$,

(viii) for every element a of it holds $a \cdot (1_{\mathrm{it}}) = a$,

(ix) for every element a of it holds $(1_{\mathrm{it}}) \cdot a = a$,

(x) for all elements a, b of it such that $a \neq 0_{\mathrm{it}}$ there exists an element x of it such that $a \cdot x = b$,

(xi) for all elements a, b of it such that $a \neq 0_{\mathrm{it}}$ there exists an element x of it such that $x \cdot a = b$,

(xii) for all elements a, x, y of it such that $a \neq 0_{\rm it}$ holds if $a \cdot x = a \cdot y$, then $x = y$,

(xiii) for all elements a, x, y of it such that $a \neq 0_{\rm it}$ holds if $x \cdot a = y \cdot a$, then $x = y$,

(xiv) for every element a of it holds $a \cdot 0_{\rm it} = 0_{\rm it}$,

(xv) for every element a of it holds $0_{\rm it} \cdot a = 0_{\rm it}$.

Let us note that it makes sense to consider the following constant. Then $\mathrm{loop}_{\mathbb{R}}$ is a double loop.

Let L be a double loop, and let a be an element of L. The functor $-a$ yielding an element of L is defined as follows:

(Def.7) $a + (-a) = 0_L$.

Next we state the proposition

(9)[2] For every double loop L and for every element a of L holds $a + (-a) = 0_L$.

Let L be a double loop, and let a, b be elements of L. The functor $a - b$ yielding an element of L is defined by:

(Def.8) $a - b = a + (-b)$.

We now state the proposition

(10) For every double loop L and for all elements a, b of L holds $a - b = a + (-b)$.

A double loop is said to be a left quasi-field if:

(Def.9) (i) for all elements a, b, c of it holds $(a + b) + c = a + (b + c)$,

(ii) for all elements a, b of it holds $a + b = b + a$,

(iii) for all elements a, b, c of it holds $a \cdot (b + c) = a \cdot b + a \cdot c$.

In the sequel a, b, c, x, y are elements of L. The following proposition is true

(12)[3] L is a left quasi-field if and only if the following conditions are satisfied:

(i) for every a holds $a + 0_L = a$,

(ii) for every a there exists x such that $a + x = 0_L$,

(iii) for all a, b, c holds $(a + b) + c = a + (b + c)$,

(iv) for all a, b holds $a + b = b + a$,

(v) $0_L \neq 1_L$,

(vi) for every a holds $a \cdot (1_L) = a$,

(vii) for every a holds $(1_L) \cdot a = a$,

(viii) for all a, b such that $a \neq 0_L$ there exists x such that $a \cdot x = b$,

(ix) for all a, b such that $a \neq 0_L$ there exists x such that $x \cdot a = b$,

(x) for all a, x, y such that $a \neq 0_L$ holds if $a \cdot x = a \cdot y$, then $x = y$,

(xi) for all a, x, y such that $a \neq 0_L$ holds if $x \cdot a = y \cdot a$, then $x = y$,

(xii) for every a holds $a \cdot 0_L = 0_L$,

(xiii) for every a holds $0_L \cdot a = 0_L$,

(xiv) for all a, b, c holds $a \cdot (b + c) = a \cdot b + a \cdot c$.

[2]The proposition (8) was either repeated or obvious.

[3]The proposition (11) was either repeated or obvious.

We follow the rules: G will be a left quasi-field and a, b, x, y will be elements of G. We now state several propositions:

(13) $a + (-a) = 0_G$ and $(-a) + a = 0_G$.

(14) $a \cdot (-b) = -a \cdot b$.

(15) $-(-a) = a$.

(16) $(-1_G) \cdot (-1_G) = 1_G$.

(17) $a \cdot (x - y) = a \cdot x - a \cdot y$.

A double loop is called a right quasi-field if:

(Def.10) (i) for all elements a, b, c of it holds $(a + b) + c = a + (b + c)$,
(ii) for all elements a, b of it holds $a + b = b + a$,
(iii) for all elements a, b, c of it holds $(b + c) \cdot a = b \cdot a + c \cdot a$.

In the sequel a, b, c, x, y are elements of L. One can prove the following proposition

(19)[4] L is a right quasi-field if and only if the following conditions are satisfied:
(i) for every a holds $a + 0_L = a$,
(ii) for every a there exists x such that $a + x = 0_L$,
(iii) for all a, b, c holds $(a + b) + c = a + (b + c)$,
(iv) for all a, b holds $a + b = b + a$,
(v) $0_L \neq 1_L$,
(vi) for every a holds $a \cdot (1_L) = a$,
(vii) for every a holds $(1_L) \cdot a = a$,
(viii) for all a, b such that $a \neq 0_L$ there exists x such that $a \cdot x = b$,
(ix) for all a, b such that $a \neq 0_L$ there exists x such that $x \cdot a = b$,
(x) for all a, x, y such that $a \neq 0_L$ holds if $a \cdot x = a \cdot y$, then $x = y$,
(xi) for all a, x, y such that $a \neq 0_L$ holds if $x \cdot a = y \cdot a$, then $x = y$,
(xii) for every a holds $a \cdot 0_L = 0_L$,
(xiii) for every a holds $0_L \cdot a = 0_L$,
(xiv) for all a, b, c holds $(b + c) \cdot a = b \cdot a + c \cdot a$.

We adopt the following rules: G will be a right quasi-field and a, b, x, y will be elements of G. We now state several propositions:

(20) $a + (-a) = 0_G$ and $(-a) + a = 0_G$.

(21) $(-b) \cdot a = -b \cdot a$.

(22) $-(-a) = a$.

(23) $(-1_G) \cdot (-1_G) = 1_G$.

(24) $(x - y) \cdot a = x \cdot a - y \cdot a$.

In the sequel a, b, c, x, y will denote elements of L. A double loop is called a double sided quasi-field if:

(Def.11) (i) for all elements a, b, c of it holds $(a + b) + c = a + (b + c)$,
(ii) for all elements a, b of it holds $a + b = b + a$,
(iii) for all elements a, b, c of it holds $a \cdot (b + c) = a \cdot b + a \cdot c$,
(iv) for all elements a, b, c of it holds $(b + c) \cdot a = b \cdot a + c \cdot a$.

[4]The proposition (18) was either repeated or obvious.

Let us note that it makes sense to consider the following constant. Then $\mathrm{loop}_{\mathbb{R}}$ is a double sided quasi-field.

The following propositions are true:

(26)[5] L is a double sided quasi-field if and only if the following conditions are satisfied:

(i) for every a holds $a + 0_L = a$,
(ii) for every a there exists x such that $a + x = 0_L$,
(iii) for all a, b, c holds $(a + b) + c = a + (b + c)$,
(iv) for all a, b holds $a + b = b + a$,
(v) $0_L \neq 1_L$,
(vi) for every a holds $a \cdot (1_L) = a$,
(vii) for every a holds $(1_L) \cdot a = a$,
(viii) for all a, b such that $a \neq 0_L$ there exists x such that $a \cdot x = b$,
(ix) for all a, b such that $a \neq 0_L$ there exists x such that $x \cdot a = b$,
(x) for all a, x, y such that $a \neq 0_L$ holds if $a \cdot x = a \cdot y$, then $x = y$,
(xi) for all a, x, y such that $a \neq 0_L$ holds if $x \cdot a = y \cdot a$, then $x = y$,
(xii) for every a holds $a \cdot 0_L = 0_L$,
(xiii) for every a holds $0_L \cdot a = 0_L$,
(xiv) for all a, b, c holds $a \cdot (b + c) = a \cdot b + a \cdot c$,
(xv) for all a, b, c holds $(b + c) \cdot a = b \cdot a + c \cdot a$.

(27) For every double sided quasi-field L holds L is a left quasi-field.

(28) For every double sided quasi-field L holds L is a right quasi-field.

We adopt the following rules: G will be a double sided quasi-field and a, b, x, y will be elements of G. Next we state two propositions:

(29) $a \cdot (-b) = -a \cdot b$ and $(-b) \cdot a = -b \cdot a$.

(30) $a \cdot (x - y) = a \cdot x - a \cdot y$ and $(x - y) \cdot a = x \cdot a - y \cdot a$.

We see that the double sided quasi-field is a left quasi-field.

In the sequel a, b, c, x will be elements of L. A double sided quasi-field is called a skew field if:

(Def.12) for all elements a, b, c of it holds $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Let us note that it makes sense to consider the following constant. Then $\mathrm{loop}_{\mathbb{R}}$ is a skew field.

The following proposition is true

(32)[6] L is a skew field if and only if the following conditions are satisfied:

(i) for every a holds $a + 0_L = a$,
(ii) for every a there exists x such that $a + x = 0_L$,
(iii) for all a, b, c holds $(a + b) + c = a + (b + c)$,
(iv) for all a, b holds $a + b = b + a$,
(v) $0_L \neq 1_L$,
(vi) for every a holds $a \cdot (1_L) = a$,

[5] The proposition (25) was either repeated or obvious.
[6] The proposition (31) was either repeated or obvious.

(vii) for every a such that $a \neq 0_L$ there exists x such that $a \cdot x = 1_L$,
(viii) for every a holds $a \cdot 0_L = 0_L$,
(ix) for every a holds $0_L \cdot a = 0_L$,
(x) for all a, b, c holds $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
(xi) for all a, b, c holds $a \cdot (b + c) = a \cdot b + a \cdot c$,
(xii) for all a, b, c holds $(b + c) \cdot a = b \cdot a + c \cdot a$.

A skew field is said to be a field if:

(Def.13) for all elements a, b of it holds $a \cdot b = b \cdot a$.

Let us note that it makes sense to consider the following constant. Then $\mathrm{loop}_{\mathbb{R}}$ is a field.

The following proposition is true

(34)[7] L is a field if and only if the following conditions are satisfied:
(i) for every a holds $a + 0_L = a$,
(ii) for every a there exists x such that $a + x = 0_L$,
(iii) for all a, b, c holds $(a + b) + c = a + (b + c)$,
(iv) for all a, b holds $a + b = b + a$,
(v) $0_L \neq 1_L$,
(vi) for every a holds $a \cdot (1_L) = a$,
(vii) for every a such that $a \neq 0_L$ there exists x such that $a \cdot x = 1_L$,
(viii) for every a holds $a \cdot 0_L = 0_L$,
(ix) for all a, b, c holds $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
(x) for all a, b, c holds $a \cdot (b + c) = a \cdot b + a \cdot c$,
(xi) for all a, b holds $a \cdot b = b \cdot a$.

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Received September 27, 1990

Index of MML Identifiers

Contents

Form. Math. 2 (**1**)

Continued on inside back cover