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Replacing of Variables in Formulas of ZF Theory

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Summary. Part one is a supplement to papers [1], [2], and [3]. It deals with concepts of selector functions, atomic, negative, conjunctive formulas and etc., subformulas, free variables, satisfiability and models (it is shown that axioms of the predicate and the quantifier calculus are satisfied in an arbitrary set). In part two there are introduced notions of variables occurring in a formula and replacing of variables in a formula.

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The terminology and notation used in this paper have been introduced in the following articles: [9], [8], [5], [6], [4], [7], [1], and [2]. For simplicity we adopt the following rules: p, p_1, p_2, q, r, F, G, G_1, G_2, H, H_1, H_2 will be ZF-formulae, x, x_1, x_2, y, y_1, y_2, z, z_1, z_2, s, t will be variables, a will be arbitrary, and X will be a set. Next we state a number of propositions:

(1) $\mathrm{Var}_1(x{=}y) = x$ and $\mathrm{Var}_2(x{=}y) = y$.

(2) $\mathrm{Var}_1(x\epsilon y) = x$ and $\mathrm{Var}_2(x\epsilon y) = y$.

(3) $\mathrm{Arg}(\neg p) = p$.

(4) $\mathrm{LeftArg}(p \wedge q) = p$ and $\mathrm{RightArg}(p \wedge q) = q$.

(5) $\mathrm{LeftArg}(p \vee q) = p$ and $\mathrm{RightArg}(p \vee q) = q$.

(6) $\mathrm{Antecedent}(p \Rightarrow q) = p$ and $\mathrm{Consequent}(p \Rightarrow q) = q$.

(7) $\mathrm{LeftSide}(p \Leftrightarrow q) = p$ and $\mathrm{RightSide}(p \Leftrightarrow q) = q$.

(8) $\mathrm{Bound}(\forall_x p) = x$ and $\mathrm{Scope}(\forall_x p) = p$.

(9) $\mathrm{Bound}(\exists_x p) = x$ and $\mathrm{Scope}(\exists_x p) = p$.

(10) $p \vee q = \neg p \Rightarrow q$.

(11) If $\forall_{x,y} p = \forall_z q$, then $x = z$ and $\forall_y p = q$.

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(12) If $\exists_{x,y}p = \exists_z q$, then $x = z$ and $\exists_y p = q$.

(13) $\forall_{x,y}p$ is universal and $\mathrm{Bound}(\forall_{x,y}p) = x$ and $\mathrm{Scope}(\forall_{x,y}p) = \forall_y p$.

(14) $\exists_{x,y}p$ is existential and $\mathrm{Bound}(\exists_{x,y}p) = x$ and $\mathrm{Scope}(\exists_{x,y}p) = \exists_y p$.

(15) $\forall_{x,y,z}p = \forall_x(\forall_y(\forall_z p))$ and $\forall_{x,y,z}p = \forall_{x,y}(\forall_z p)$.

(16) If $\forall_{x_1,y_1}p_1 = \forall_{x_2,y_2}p_2$, then $x_1 = x_2$ and $y_1 = y_2$ and $p_1 = p_2$.

(17) If $\forall_{x_1,y_1,z_1}p_1 = \forall_{x_2,y_2,z_2}p_2$, then $x_1 = x_2$ and $y_1 = y_2$ and $z_1 = z_2$ and $p_1 = p_2$.

(18) If $\forall_{x,y,z}p = \forall_t q$, then $x = t$ and $\forall_{y,z}p = q$.

(19) If $\forall_{x,y,z}p = \forall_{t,s}q$, then $x = t$ and $y = s$ and $\forall_z p = q$.

(20) If $\exists_{x_1,y_1}p_1 = \exists_{x_2,y_2}p_2$, then $x_1 = x_2$ and $y_1 = y_2$ and $p_1 = p_2$.

(21) $\exists_{x,y,z}p = \exists_x(\exists_y(\exists_z p))$ and $\exists_{x,y,z}p = \exists_{x,y}(\exists_z p)$.

(22) If $\exists_{x_1,y_1,z_1}p_1 = \exists_{x_2,y_2,z_2}p_2$, then $x_1 = x_2$ and $y_1 = y_2$ and $z_1 = z_2$ and $p_1 = p_2$.

(23) If $\exists_{x,y,z}p = \exists_t q$, then $x = t$ and $\exists_{y,z}p = q$.

(24) If $\exists_{x,y,z}p = \exists_{t,s}q$, then $x = t$ and $y = s$ and $\exists_z p = q$.

(25) $\forall_{x,y,z}p$ is universal and $\mathrm{Bound}(\forall_{x,y,z}p) = x$ and $\mathrm{Scope}(\forall_{x,y,z}p) = \forall_{y,z}p$.

(26) $\exists_{x,y,z}p$ is existential and $\mathrm{Bound}(\exists_{x,y,z}p) = x$ and $\mathrm{Scope}(\exists_{x,y,z}p) = \exists_{y,z}p$.

(27) If H is disjunctive, then $\mathrm{LeftArg}(H) = \mathrm{Arg}(\mathrm{LeftArg}(\mathrm{Arg}(H)))$.

(28) If H is disjunctive, then $\mathrm{RightArg}(H) = \mathrm{Arg}(\mathrm{RightArg}(\mathrm{Arg}(H)))$.

(29) If H is conditional, then $\mathrm{Antecedent}(H) = \mathrm{LeftArg}(\mathrm{Arg}(H))$.

(30) If H is conditional, then $\mathrm{Consequent}(H) = \mathrm{Arg}(\mathrm{RightArg}(\mathrm{Arg}(H)))$.

(31) If H is biconditional, then $\mathrm{LeftSide}(H) = \mathrm{Antecedent}(\mathrm{LeftArg}(H))$ and $\mathrm{LeftSide}(H) = \mathrm{Consequent}(\mathrm{RightArg}(H))$.

(32) If H is biconditional, then $\mathrm{RightSide}(H) = \mathrm{Consequent}(\mathrm{LeftArg}(H))$ and $\mathrm{RightSide}(H) = \mathrm{Antecedent}(\mathrm{RightArg}(H))$.

(33) If H is existential, then $\mathrm{Bound}(H) = \mathrm{Bound}(\mathrm{Arg}(H))$ and $\mathrm{Scope}(H) = \mathrm{Arg}(\mathrm{Scope}(\mathrm{Arg}(H)))$.

(34) $\mathrm{Arg}(F \vee G) = \neg F \wedge \neg G$ and $\mathrm{Antecedent}(F \vee G) = \neg F$ and $\mathrm{Consequent}(F \vee G) = G$.

(35) $\mathrm{Arg}(F \Rightarrow G) = F \wedge \neg G$.

(36) $\mathrm{LeftArg}(F \Leftrightarrow G) = F \Rightarrow G$ and $\mathrm{RightArg}(F \Leftrightarrow G) = G \Rightarrow F$.

(37) $\mathrm{Arg}(\exists_x H) = \forall_x \neg H$.

(38) If H is disjunctive, then H is conditional and H is negative and $\mathrm{Arg}(H)$ is conjunctive and $\mathrm{LeftArg}(\mathrm{Arg}(H))$ is negative and $\mathrm{RightArg}(\mathrm{Arg}(H))$ is negative.

(39) If H is conditional, then H is negative and $\mathrm{Arg}(H)$ is conjunctive and $\mathrm{RightArg}(\mathrm{Arg}(H))$ is negative.

(40) If H is biconditional, then H is conjunctive and $\mathrm{LeftArg}(H)$ is conditional and $\mathrm{RightArg}(H)$ is conditional.

(41) If H is existential, then H is negative and $\operatorname{Arg}(H)$ is universal and $\operatorname{Scope}(\operatorname{Arg}(H))$ is negative.

(42) It is not true that: H is an equality and H is a membership or H is negative or H is conjunctive or H is universal and it is not true that: H is a membership and H is negative or H is conjunctive or H is universal and it is not true that: H is negative and H is conjunctive or H is universal and it is not true that: H is conjunctive and H is universal.

(43) If F is a subformula of G, then $\operatorname{len} F \leq \operatorname{len} G$.

(44) Suppose F is a proper subformula of G and G is a subformula of H or F is a subformula of G and G is a proper subformula of H or F is a subformula of G and G is an immediate constituent of H or F is an immediate constituent of G and G is a subformula of H or F is a proper subformula of G and G is an immediate constituent of H or F is an immediate constituent of G and G is a proper subformula of H. Then F is a proper subformula of H.

(45) H is not a proper subformula of H.

(46) H is not an immediate constituent of H.

(47) It is not true that: G is a proper subformula of H and H is a subformula of G.

(48) It is not true that: G is a proper subformula of H and H is a proper subformula of G.

(49) It is not true that: G is a subformula of H and H is an immediate constituent of G.

(50) It is not true that: G is a proper subformula of H and H is an immediate constituent of G.

(51) If $\neg F$ is a subformula of H, then F is a proper subformula of H.

(52) If $F \wedge G$ is a subformula of H, then F is a proper subformula of H and G is a proper subformula of H.

(53) If $\forall_x H$ is a subformula of F, then H is a proper subformula of F.

(54) $F \wedge \neg G$ is a proper subformula of $F \Rightarrow G$ and F is a proper subformula of $F \Rightarrow G$ and $\neg G$ is a proper subformula of $F \Rightarrow G$ and G is a proper subformula of $F \Rightarrow G$.

(55) $\neg F \wedge \neg G$ is a proper subformula of $F \vee G$ and $\neg F$ is a proper subformula of $F \vee G$ and $\neg G$ is a proper subformula of $F \vee G$ and F is a proper subformula of $F \vee G$ and G is a proper subformula of $F \vee G$.

(56) $\forall_x \neg H$ is a proper subformula of $\exists_x H$ and $\neg H$ is a proper subformula of $\exists_x H$.

(57) G is a subformula of H if and only if $G \in \operatorname{Subformulae} H$.

(58) If $G \in \operatorname{Subformulae} H$, then $\operatorname{Subformulae} G \subseteq \operatorname{Subformulae} H$.

(59) $H \in \operatorname{Subformulae} H$.

(60) $\operatorname{Subformulae} F \Rightarrow G = (\operatorname{Subformulae} F \cup \operatorname{Subformulae} G) \cup \{\neg G, F \wedge \neg G, F \Rightarrow G\}$.

(61) Subformulae $F \vee G = ($Subformulae $F \cup$ Subformulae $G) \cup \{\neg G, \neg F, \neg F \wedge \neg G, F \vee G\}$.

(62) Subformulae $F \Leftrightarrow G = ($Subformulae $F \cup$ Subformulae $G) \cup \{\neg G, F \wedge \neg G, F \Rightarrow G, \neg F, G \wedge \neg F, G \Rightarrow F, F \Leftrightarrow G\}$.

(63) $\mathrm{Free}(x{=}y) = \{x, y\}$.

(64) $\mathrm{Free}(x \epsilon y) = \{x, y\}$.

(65) $\mathrm{Free}(\neg p) = \mathrm{Free}\, p$.

(66) $\mathrm{Free}(p \wedge q) = \mathrm{Free}\, p \cup \mathrm{Free}\, q$.

(67) $\mathrm{Free}(\forall_x p) = \mathrm{Free}\, p \setminus \{x\}$.

(68) $\mathrm{Free}(p \vee q) = \mathrm{Free}\, p \cup \mathrm{Free}\, q$.

(69) $\mathrm{Free}(p \Rightarrow q) = \mathrm{Free}\, p \cup \mathrm{Free}\, q$.

(70) $\mathrm{Free}(p \Leftrightarrow q) = \mathrm{Free}\, p \cup \mathrm{Free}\, q$.

(71) $\mathrm{Free}(\exists_x p) = \mathrm{Free}\, p \setminus \{x\}$.

(72) $\mathrm{Free}(\forall_{x,y} p) = \mathrm{Free}\, p \setminus \{x, y\}$.

(73) $\mathrm{Free}(\forall_{x,y,z} p) = \mathrm{Free}\, p \setminus \{x, y, z\}$.

(74) $\mathrm{Free}(\exists_{x,y} p) = \mathrm{Free}\, p \setminus \{x, y\}$.

(75) $\mathrm{Free}(\exists_{x,y,z} p) = \mathrm{Free}\, p \setminus \{x, y, z\}$.

The scheme *ZF_Induction* deals with a unary predicate $\mathcal{P}$, and states that:

for every H holds $\mathcal{P}[H]$

provided the parameter satisfies the following conditions:

- for all x_1, x_2 holds $\mathcal{P}[x_1{=}x_2]$ and $\mathcal{P}[x_1 \epsilon x_2]$,
- for every H such that $\mathcal{P}[H]$ holds $\mathcal{P}[\neg H]$,
- for all H_1, H_2 such that $\mathcal{P}[H_1]$ and $\mathcal{P}[H_2]$ holds $\mathcal{P}[H_1 \wedge H_2]$,
- for all H, x such that $\mathcal{P}[H]$ holds $\mathcal{P}[\forall_x H]$.

For simplicity we adopt the following rules: M, E will denote non-empty families of sets, e will denote an element of E, m, m' will denote elements of M, f, g will denote functions from VAR into E, and v, v' will denote functions from VAR into M. Let us consider E, f, x, e. The functor $f(\frac{x}{e})$ yields a function from VAR into E and is defined by:

(Def.1) $(f(\frac{x}{e}))(x) = e$ and for every y such that $(f(\frac{x}{e}))(y) \neq f(y)$ holds $x = y$.

The following proposition is true

(76) $g = f(\frac{x}{e})$ if and only if $g(x) = e$ and for every y such that $g(y) \neq f(y)$ holds $x = y$.

Let D, D_1, D_2 be non-empty sets, and let f be a function from D into D_1. Let us assume that $D_1 \subseteq D_2$. The functor $D_2[f]$ yields a function from D into D_2 and is defined as follows:

(Def.2) $D_2[f] = f$.

Next we state several propositions:

(77) For all non-empty sets D, D_1, D_2 and for every function f from D into D_1 such that $D_1 \subseteq D_2$ holds $D_2[f] = f$.

(78) $(v(\frac{x}{m'}))(\frac{x}{m}) = v(\frac{x}{m})$ and $v(\frac{x}{v(x)}) = v$.

(79) If $x \neq y$, then $(v(\frac{x}{m}))(\frac{y}{m'}) = (v(\frac{y}{m'}))(\frac{x}{m})$.

(80) $M, v \models \forall_x H$ if and only if for every m holds $M, v(\frac{x}{m}) \models H$.

(81) $M, v \models \forall_x H$ if and only if $M, v(\frac{x}{m}) \models \forall_x H$.

(82) $M, v \models \exists_x H$ if and only if there exists m such that $M, v(\frac{x}{m}) \models H$.

(83) $M, v \models \exists_x H$ if and only if $M, v(\frac{x}{m}) \models \exists_x H$.

(84) For all v, v' such that for every x such that $x \in \mathrm{Free}\, H$ holds $v'(x) = v(x)$ holds if $M, v \models H$, then $M, v' \models H$.

(85) $\mathrm{Free}\, H$ is finite.

In the sequel i, j will denote natural numbers. The following propositions are true:

(86) If $x_i = x_j$, then $i = j$.

(87) There exists i such that $x = x_i$.

(88) x is a natural number and $x \in \mathbb{N}$.

(89) $M, v \models x{=}x$.

(90) $M \models x{=}x$.

(91) $M, v \not\models x \epsilon x$.

(92) $M \not\models x \epsilon x$ and $M \models \neg x \epsilon x$.

(93) $M \models x{=}y$ if and only if $x = y$ or there exists a such that $\{a\} = M$.

(94) $M \models \neg x \epsilon y$ if and only if $x = y$ or for every X such that $X \in M$ holds $X \cap M = \emptyset$.

(95) If H is an equality, then $M, v \models H$ if and only if $v(\mathrm{Var}_1(H)) = v(\mathrm{Var}_2(H))$.

(96) If H is a membership, then $M, v \models H$ if and only if $v(\mathrm{Var}_1(H)) \in v(\mathrm{Var}_2(H))$.

(97) If H is negative, then $M, v \models H$ if and only if $M, v \not\models \mathrm{Arg}(H)$.

(98) If H is conjunctive, then $M, v \models H$ if and only if $M, v \models \mathrm{LeftArg}(H)$ and $M, v \models \mathrm{RightArg}(H)$.

(99) If H is universal, then $M, v \models H$ if and only if for every m holds $M, v(\frac{\mathrm{Bound}(H)}{m}) \models \mathrm{Scope}(H)$.

(100) If H is disjunctive, then $M, v \models H$ if and only if $M, v \models \mathrm{LeftArg}(H)$ or $M, v \models \mathrm{RightArg}(H)$.

(101) If H is conditional, then $M, v \models H$ if and only if if
$M, v \models \mathrm{Antecedent}(H)$,
then $M, v \models \mathrm{Consequent}(H)$.

(102) If H is biconditional, then $M, v \models H$ if and only if $M, v \models \mathrm{LeftSide}(H)$ if and only if $M, v \models \mathrm{RightSide}(H)$.

(103) If H is existential, then $M, v \models H$ if and only if there exists m such that $M, v(\frac{\mathrm{Bound}(H)}{m}) \models \mathrm{Scope}(H)$.

(104) $M \models \exists_x H$ if and only if for every v there exists m such that $M, v(\frac{x}{m}) \models H$.

(105) If $M \models H$, then $M \models \exists_x H$.

(106) $M \models H$ if and only if $M \models \forall_{x,y} H$.

(107) If $M \models H$, then $M \models \exists_{x,y} H$.

(108) $M \models H$ if and only if $M \models \forall_{x,y,z} H$.

(109) If $M \models H$, then $M \models \exists_{x,y,z} H$.

(110) $M, v \models (p \Leftrightarrow q) \Rightarrow (p \Rightarrow q)$ and $M \models (p \Leftrightarrow q) \Rightarrow (p \Rightarrow q)$.

(111) $M, v \models (p \Leftrightarrow q) \Rightarrow (q \Rightarrow p)$ and $M \models (p \Leftrightarrow q) \Rightarrow (q \Rightarrow p)$.

(112) $M \models (p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$.

(113) If $M, v \models p \Rightarrow q$ and $M, v \models q \Rightarrow r$, then $M, v \models p \Rightarrow r$.

(114) If $M \models p \Rightarrow q$ and $M \models q \Rightarrow r$, then $M \models p \Rightarrow r$.

(115) $M, v \models (p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$ and $M \models (p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$.

(116) $M, v \models p \Rightarrow (q \Rightarrow p)$ and $M \models p \Rightarrow (q \Rightarrow p)$.

(117) $M, v \models (p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ and $M \models (p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$.

(118) $M, v \models p \wedge q \Rightarrow p$ and $M \models p \wedge q \Rightarrow p$.

(119) $M, v \models p \wedge q \Rightarrow q$ and $M \models p \wedge q \Rightarrow q$.

(120) $M, v \models p \wedge q \Rightarrow q \wedge p$ and $M \models p \wedge q \Rightarrow q \wedge p$.

(121) $M, v \models p \Rightarrow p \wedge p$ and $M \models p \Rightarrow p \wedge p$.

(122) $M, v \models (p \Rightarrow q) \Rightarrow ((p \Rightarrow r) \Rightarrow (p \Rightarrow q \wedge r))$ and $M \models (p \Rightarrow q) \Rightarrow ((p \Rightarrow r) \Rightarrow (p \Rightarrow q \wedge r))$.

(123) $M, v \models p \Rightarrow p \vee q$ and $M \models p \Rightarrow p \vee q$.

(124) $M, v \models q \Rightarrow p \vee q$ and $M \models q \Rightarrow p \vee q$.

(125) $M, v \models p \vee q \Rightarrow q \vee p$ and $M \models p \vee q \Rightarrow q \vee p$.

(126) $M, v \models p \Rightarrow p \vee p$ and $M \models p \Rightarrow p \vee p$.

(127) $M, v \models (p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r))$ and $M \models (p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r))$.

(128) $M, v \models (p \Rightarrow r) \wedge (q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)$ and $M \models (p \Rightarrow r) \wedge (q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)$.

(129) $M, v \models (p \Rightarrow \neg q) \Rightarrow (q \Rightarrow \neg p)$ and $M \models (p \Rightarrow \neg q) \Rightarrow (q \Rightarrow \neg p)$.

(130) $M, v \models \neg p \Rightarrow (p \Rightarrow q)$ and $M \models \neg p \Rightarrow (p \Rightarrow q)$.

(131) $M, v \models (p \Rightarrow q) \wedge (p \Rightarrow \neg q) \Rightarrow \neg p$ and $M \models (p \Rightarrow q) \wedge (p \Rightarrow \neg q) \Rightarrow \neg p$.

(132) If $M, v \models p \Rightarrow q$ and $M, v \models p$, then $M, v \models q$.

(133) If $M \models p \Rightarrow q$ and $M \models p$, then $M \models q$.

(134) $M, v \models \neg(p \wedge q) \Rightarrow \neg p \vee \neg q$ and $M \models \neg(p \wedge q) \Rightarrow \neg p \vee \neg q$.

(135) $M, v \models \neg p \vee \neg q \Rightarrow \neg(p \wedge q)$ and $M \models \neg p \vee \neg q \Rightarrow \neg(p \wedge q)$.

(136) $M, v \models \neg(p \vee q) \Rightarrow \neg p \wedge \neg q$ and $M \models \neg(p \vee q) \Rightarrow \neg p \wedge \neg q$.

(137) $M, v \models \neg p \wedge \neg q \Rightarrow \neg(p \vee q)$ and $M \models \neg p \wedge \neg q \Rightarrow \neg(p \vee q)$.

(138) $M \models (\forall_x H) \Rightarrow H$.

(139) $M \models H \Rightarrow (\exists_x H)$.

(140) If $x \notin \operatorname{Free} H_1$, then $M \models (\forall_x H_1 \Rightarrow H_2) \Rightarrow (H_1 \Rightarrow (\forall_x H_2))$.

(141) If $x \notin \operatorname{Free} H_1$ and $M \models H_1 \Rightarrow H_2$, then $M \models H_1 \Rightarrow (\forall_x H_2)$.

(142) If $x \notin \operatorname{Free} H_2$, then $M \models (\forall_x H_1 \Rightarrow H_2) \Rightarrow ((\exists_x H_1) \Rightarrow H_2)$.

(143) If $x \notin \operatorname{Free} H_2$ and $M \models H_1 \Rightarrow H_2$, then $M \models (\exists_x H_1) \Rightarrow H_2$.

(144) If $M \models H_1 \Rightarrow (\forall_x H_2)$, then $M \models H_1 \Rightarrow H_2$.

(145) If $M \models (\exists_x H_1) \Rightarrow H_2$, then $M \models H_1 \Rightarrow H_2$.

(146) $\mathrm{WFF} \subseteq 2^{[\mathbb{N}, \mathbb{N}]}$.

Let us consider H. The functor Var_H yields a set and is defined by:

(Def.3) $\mathrm{Var}_H = \operatorname{rng} H \setminus \{0, 1, 2, 3, 4\}$.

We now state a number of propositions:

(147) $\mathrm{Var}_H = \operatorname{rng} H \setminus \{0, 1, 2, 3, 4\}$.

(148) $x \neq 0$ and $x \neq 1$ and $x \neq 2$ and $x \neq 3$ and $x \neq 4$.

(149) $x \notin \{0, 1, 2, 3, 4\}$.

(150) If $a \in \mathrm{Var}_H$, then $a \neq 0$ and $a \neq 1$ and $a \neq 2$ and $a \neq 3$ and $a \neq 4$.

(151) $\mathrm{Var}_{x=y} = \{x, y\}$.

(152) $\mathrm{Var}_{x \epsilon y} = \{x, y\}$.

(153) $\mathrm{Var}_{\neg H} = \mathrm{Var}_H$.

(154) $\mathrm{Var}_{H_1 \wedge H_2} = \mathrm{Var}_{H_1} \cup \mathrm{Var}_{H_2}$.

(155) $\mathrm{Var}_{\forall_x H} = \mathrm{Var}_H \cup \{x\}$.

(156) $\mathrm{Var}_{H_1 \vee H_2} = \mathrm{Var}_{H_1} \cup \mathrm{Var}_{H_2}$.

(157) $\mathrm{Var}_{H_1 \Rightarrow H_2} = \mathrm{Var}_{H_1} \cup \mathrm{Var}_{H_2}$.

(158) $\mathrm{Var}_{H_1 \Leftrightarrow H_2} = \mathrm{Var}_{H_1} \cup \mathrm{Var}_{H_2}$.

(159) $\mathrm{Var}_{\exists_x H} = \mathrm{Var}_H \cup \{x\}$.

(160) $\mathrm{Var}_{\forall_{x,y} H} = \mathrm{Var}_H \cup \{x, y\}$.

(161) $\mathrm{Var}_{\exists_{x,y} H} = \mathrm{Var}_H \cup \{x, y\}$.

(162) $\mathrm{Var}_{\forall_{x,y,z} H} = \mathrm{Var}_H \cup \{x, y, z\}$.

(163) $\mathrm{Var}_{\exists_{x,y,z} H} = \mathrm{Var}_H \cup \{x, y, z\}$.

(164) $\operatorname{Free} H \subseteq \mathrm{Var}_H$.

Let us consider H. Then Var_H is a non-empty subset of VAR.

Let us consider H, x, y. The functor $H(\frac{x}{y})$ yields a function and is defined by:

(Def.4) $\operatorname{dom}(H(\frac{x}{y})) = \operatorname{dom} H$ and for every a such that $a \in \operatorname{dom} H$ holds if $H(a) = x$, then $(H(\frac{x}{y}))(a) = y$ but if $H(a) \neq x$, then $(H(\frac{x}{y}))(a) = H(a)$.

One can prove the following propositions:

(165) For every function f holds $f = H(\frac{x}{y})$ if and only if $\operatorname{dom} f = \operatorname{dom} H$ and for every a such that $a \in \operatorname{dom} H$ holds if $H(a) = x$, then $f(a) = y$ but if $H(a) \neq x$, then $f(a) = H(a)$.

(166) $x_1{=}x_2(\frac{y_1}{y_2}) = z_1{=}z_2$ if and only if $x_1 \neq y_1$ and $x_2 \neq y_1$ and $z_1 = x_1$ and $z_2 = x_2$ or $x_1 = y_1$ and $x_2 \neq y_1$ and $z_1 = y_2$ and $z_2 = x_2$ or $x_1 \neq y_1$ and $x_2 = y_1$ and $z_1 = x_1$ and $z_2 = y_2$ or $x_1 = y_1$ and $x_2 = y_1$ and $z_1 = y_2$ and $z_2 = y_2$.

(167) There exist z_1, z_2 such that $x_1{=}x_2(\frac{y_1}{y_2}) = z_1{=}z_2$.

(168) $x_1 \epsilon x_2(\frac{y_1}{y_2}) = z_1 \epsilon z_2$ if and only if $x_1 \neq y_1$ and $x_2 \neq y_1$ and $z_1 = x_1$ and $z_2 = x_2$ or $x_1 = y_1$ and $x_2 \neq y_1$ and $z_1 = y_2$ and $z_2 = x_2$ or $x_1 \neq y_1$ and $x_2 = y_1$ and $z_1 = x_1$ and $z_2 = y_2$ or $x_1 = y_1$ and $x_2 = y_1$ and $z_1 = y_2$ and $z_2 = y_2$.

(169) There exist z_1, z_2 such that $x_1 \epsilon x_2(\frac{y_1}{y_2}) = z_1 \epsilon z_2$.

(170) $\neg F = (\neg H)(\frac{x}{y})$ if and only if $F = H(\frac{x}{y})$.

(171) $H(\frac{x}{y}) \in \mathrm{WFF}$.

Let us consider H, x, y. Then $H(\frac{x}{y})$ is a ZF-formula.

The following propositions are true:

(172) $G_1 \wedge G_2 = (H_1 \wedge H_2)(\frac{x}{y})$ if and only if $G_1 = H_1(\frac{x}{y})$ and $G_2 = H_2(\frac{x}{y})$.

(173) If $z \neq x$, then $\forall_z G = (\forall_z H)(\frac{x}{y})$ if and only if $G = H(\frac{x}{y})$.

(174) $\forall_y G = (\forall_x H)(\frac{x}{y})$ if and only if $G = H(\frac{x}{y})$.

(175) $G_1 \vee G_2 = (H_1 \vee H_2)(\frac{x}{y})$ if and only if $G_1 = H_1(\frac{x}{y})$ and $G_2 = H_2(\frac{x}{y})$.

(176) $G_1 \Rightarrow G_2 = (H_1 \Rightarrow H_2)(\frac{x}{y})$ if and only if $G_1 = H_1(\frac{x}{y})$ and $G_2 = H_2(\frac{x}{y})$.

(177) $G_1 \Leftrightarrow G_2 = (H_1 \Leftrightarrow H_2)(\frac{x}{y})$ if and only if $G_1 = H_1(\frac{x}{y})$ and $G_2 = H_2(\frac{x}{y})$.

(178) If $z \neq x$, then $\exists_z G = (\exists_z H)(\frac{x}{y})$ if and only if $G = H(\frac{x}{y})$.

(179) $\exists_y G = (\exists_x H)(\frac{x}{y})$ if and only if $G = H(\frac{x}{y})$.

(180) H is an equality if and only if $H(\frac{x}{y})$ is an equality.

(181) H is a membership if and only if $H(\frac{x}{y})$ is a membership.

(182) H is negative if and only if $H(\frac{x}{y})$ is negative.

(183) H is conjunctive if and only if $H(\frac{x}{y})$ is conjunctive.

(184) H is universal if and only if $H(\frac{x}{y})$ is universal.

(185) If H is negative, then $\mathrm{Arg}(H(\frac{x}{y})) = \mathrm{Arg}(H)(\frac{x}{y})$.

(186) If H is conjunctive, then $\mathrm{LeftArg}(H(\frac{x}{y})) = \mathrm{LeftArg}(H)(\frac{x}{y})$ and $\mathrm{RightArg}(H(\frac{x}{y})) = \mathrm{RightArg}(H)(\frac{x}{y})$.

(187) If H is universal, then $\mathrm{Scope}(H(\frac{x}{y})) = \mathrm{Scope}(H)(\frac{x}{y})$ but if $\mathrm{Bound}(H) = x$, then $\mathrm{Bound}(H(\frac{x}{y})) = y$ but if $\mathrm{Bound}(H) \neq x$, then $\mathrm{Bound}(H(\frac{x}{y})) = \mathrm{Bound}(H)$.

(188) H is disjunctive if and only if $H(\frac{x}{y})$ is disjunctive.

(189) H is conditional if and only if $H(\frac{x}{y})$ is conditional.

(190) If H is biconditional, then $H(\frac{x}{y})$ is biconditional.

(191) H is existential if and only if $H(\frac{x}{y})$ is existential.

(192) If H is disjunctive, then $\mathrm{LeftArg}(H(\frac{x}{y})) = \mathrm{LeftArg}(H)(\frac{x}{y})$ and $\mathrm{RightArg}(H(\frac{x}{y})) = \mathrm{RightArg}(H)(\frac{x}{y})$.

(193) If H is conditional, then $\mathrm{Antecedent}(H(\frac{x}{y})) = \mathrm{Antecedent}(H)(\frac{x}{y})$ and $\mathrm{Consequent}(H(\frac{x}{y})) = \mathrm{Consequent}(H)(\frac{x}{y})$.

(194) If H is biconditional, then $\mathrm{LeftSide}(H(\frac{x}{y})) = \mathrm{LeftSide}(H)(\frac{x}{y})$ and $\mathrm{RightSide}(H(\frac{x}{y})) = \mathrm{RightSide}(H)(\frac{x}{y})$.

(195) If H is existential, then $\mathrm{Scope}(H(\frac{x}{y})) = \mathrm{Scope}(H)(\frac{x}{y})$ but if $\mathrm{Bound}(H) = x$, then $\mathrm{Bound}(H(\frac{x}{y})) = y$ but if $\mathrm{Bound}(H) \neq x$, then $\mathrm{Bound}(H(\frac{x}{y})) = \mathrm{Bound}(H)$.

(196) If $x \notin \mathrm{Var}_H$, then $H(\frac{x}{y}) = H$.

(197) $H(\frac{x}{x}) = H$.

(198) If $x \neq y$, then $x \notin \mathrm{Var}_{H(\frac{x}{y})}$.

(199) If $x \in \mathrm{Var}_H$, then $y \in \mathrm{Var}_{H(\frac{x}{y})}$.

(200) If $x \neq y$, then $(H(\frac{x}{y}))(\frac{x}{z}) = H(\frac{x}{y})$.

(201) $\mathrm{Var}_{H(\frac{x}{y})} \subseteq (\mathrm{Var}_H \setminus \{x\}) \cup \{y\}$.

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