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A Construction of an Abstract Space of Congruence of Vectors [1]

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Summary. In the class of abelian groups a subclass of two-divisible-groups is singled out, and in the latter, a subclass of uniquely-two-divisible-groups. With every such a group a special geometrical structure, more precisely the structure of "congruence of vectors" is correlated. The notion of "affine vector space" (denoted by AffVect) is introduced. This term is defined by means of suitable axiom system. It is proved that every structure of the congruence of vectors determined by a non trivial uniquely two divisible group is a affine vector space.

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The articles [5], [1], [4], [2], and [3] provide the notation and terminology for this paper. In the sequel AG denotes an Abelian group and G denotes a group structure. One can prove the following propositions:

(1) $\mathbb{R}_{\rm G}$ is an Abelian group.

(2) If $G = \mathbb{R}_{\rm G}$, then for every element a of the carrier of G there exists an element b of the carrier of G such that (the addition of G)$(b, b) = a$.

(3) If $G = \mathbb{R}_{\rm G}$, then for every element a of the carrier of G such that (the addition of G)$(a, a) = 0_G$ holds $a = 0_G$.

An Abelian group is called a 2-divisible group if:
for every element a of the carrier of it there exists an element b of the carrier of it such that (the addition of it)$(b, b) = a$.

The following two propositions are true:

(4) For every AG holds AG is a 2-divisible group if and only if for every element a of the carrier of AG there exists an element b of the carrier of AG such that (the addition of AG)$(b, b) = a$.

(5) $\mathbb{R}_{\rm G}$ is a 2-divisible group.

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A 2-divisible group is said to be a uniquely 2-divisible group if:

for every element a of the carrier of it such that (the addition of it)$(a, a) = 0_{\rm it}$ holds $a = 0_{\rm it}$.

One can prove the following three propositions:

(6) For every 2-divisible group AG holds AG is a uniquely 2-divisible group if and only if for every element a of the carrier of AG such that (the addition of $AG)(a, a) = 0_{AG}$ holds $a = 0_{AG}$.

(7) For every AG holds AG is a uniquely 2-divisible group if and only if for every element a of the carrier of AG there exists an element b of the carrier of AG such that (the addition of $AG)(b, b) = a$ and for every element a of the carrier of AG such that (the addition of $AG)(a, a) = 0_{AG}$ holds $a = 0_{AG}$.

(8) $\mathbb{R}_{\rm G}$ is a uniquely 2-divisible group.

We adopt the following rules: ADG is a uniquely 2-divisible group and a, b, c, d, a', b', c', p, q are elements of the carrier of ADG. Let us consider ADG, a, b. The functor $a\#b$ yielding an element of the carrier of ADG is defined as follows:

$a\#b =$ (the addition of $ADG)(a, b)$.

Let us consider ADG. The functor Congr_{ADG} yields a binary relation on [: the carrier of ADG, the carrier of ADG :] and is defined as follows:

for all a, b, c, d holds $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in \mathrm{Congr}_{ADG}$ if and only if $a\#d = b\#c$.

Let us consider ADG. The functor $\mathrm{Vectors}(ADG)$ yielding an affine structure is defined by:

$\mathrm{Vectors}(ADG) = \langle$ the carrier of $ADG, \mathrm{Congr}_{ADG}\rangle$.

Next we state the proposition

(9) The points of $\mathrm{Vectors}(ADG) =$ the carrier of ADG and the congruence of $\mathrm{Vectors}(ADG) = \mathrm{Congr}_{ADG}$.

Let us consider ADG, a, b, c, d. The predicate $a, b \Rrightarrow c, d$ is defined by:

$\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the congruence of $\mathrm{Vectors}(ADG)$.

Next we state a number of propositions:

(10) $a, b \Rrightarrow c, d$ if and only if $a\#d = b\#c$.

(11) If $G = \mathbb{R}_{\rm G}$, then there exist elements a, b of the carrier of G such that $a \neq b$.

(12) There exists ADG and there exist a, b such that $a \neq b$.

(13) If $a, b \Rrightarrow c, c$, then $a = b$.

(14) If $a, b \Rrightarrow p, q$ and $c, d \Rrightarrow p, q$, then $a, b \Rrightarrow c, d$.

(15) There exists d such that $a, b \Rrightarrow c, d$.

(16) If $a, b \Rrightarrow a', b'$ and $a, c \Rrightarrow a', c'$, then $b, c \Rrightarrow b', c'$.

(17) There exists b such that $a, b \Rrightarrow b, c$.

(18) If $a, b \Rrightarrow b, c$ and $a, b' \Rrightarrow b', c$, then $b = b'$.

(19) If $a, b \Rrightarrow c, d$, then $a, c \Rrightarrow b, d$.

In the sequel AS denotes an affine structure. Let us consider AS, and let a, b, c, d be elements of the points of AS. The predicate $a,b \Rrightarrow c,d$ is defined by: $\langle\langle a,b\rangle,\langle c,d\rangle\rangle \in$ the congruence of AS.

One can prove the following proposition

(20) Suppose there exist elements a, b of the carrier of ADG such that $a \neq b$. Then
  (i) there exist elements a, b of the points of Vectors(ADG) such that $a \neq b$,
  (ii) for all elements a, b, c of the points of Vectors(ADG) such that $a,b \Rrightarrow c,c$ holds $a = b$,
  (iii) for all elements a, b, c, d, p, q of the points of Vectors(ADG) such that $a,b \Rrightarrow p,q$ and $c,d \Rrightarrow p,q$ holds $a,b \Rrightarrow c,d$,
  (iv) for every elements a, b, c of the points of Vectors(ADG) there exists an element d of the points of Vectors(ADG) such that $a,b \Rrightarrow c,d$,
  (v) for all elements a, b, c, a', b', c' of the points of Vectors(ADG) such that $a,b \Rrightarrow a',b'$ and $a,c \Rrightarrow a',c'$ holds $b,c \Rrightarrow b',c'$,
  (vi) for every elements a, c of the points of Vectors(ADG) there exists an element b of the points of Vectors(ADG) such that $a,b \Rrightarrow b,c$,
  (vii) for all elements a, b, c, b' of the points of Vectors(ADG) such that $a,b \Rrightarrow b,c$ and $a,b' \Rrightarrow b',c$ holds $b = b'$,
  (viii) for all elements a, b, c, d of the points of Vectors(ADG) such that $a,b \Rrightarrow c,d$ holds $a,c \Rrightarrow b,d$.

An affine structure is said to be a space of free vectors if:

(i) there exist elements a, b of the points of it such that $a \neq b$,
(ii) for all elements a, b, c of the points of it such that $a,b \Rrightarrow c,c$ holds $a = b$,
(iii) for all elements a, b, c, d, p, q of the points of it such that $a,b \Rrightarrow p,q$ and $c,d \Rrightarrow p,q$ holds $a,b \Rrightarrow c,d$,
(iv) for every elements a, b, c of the points of it there exists an element d of the points of it such that $a,b \Rrightarrow c,d$,
(v) for all elements a, b, c, a', b', c' of the points of it such that $a,b \Rrightarrow a',b'$ and $a,c \Rrightarrow a',c'$ holds $b,c \Rrightarrow b',c'$,
(vi) for every elements a, c of the points of it there exists an element b of the points of it such that $a,b \Rrightarrow b,c$,
(vii) for all elements a, b, c, b' of the points of it such that $a,b \Rrightarrow b,c$ and $a,b' \Rrightarrow b',c$ holds $b = b'$,
(viii) for all elements a, b, c, d of the points of it such that $a,b \Rrightarrow c,d$ holds $a,c \Rrightarrow b,d$.

We now state several propositions:

(21) Given AS. Then the following conditions are equivalent:
  (i) there exist elements a, b of the points of AS such that $a \neq b$ and for all elements a, b, c of the points of AS such that $a,b \Rrightarrow c,c$ holds $a = b$ and for all elements a, b, c, d, p, q of the points of AS such that $a,b \Rrightarrow p,q$ and $c,d \Rrightarrow p,q$ holds $a,b \Rrightarrow c,d$ and for every elements a, b, c of the points of AS there exists an element d of the points of AS such that $a,b \Rrightarrow c,d$ and for all elements a, b, c, a', b', c' of the points of AS such that $a,b \Rrightarrow a',b'$

and $a, c \Rrightarrow a', c'$ holds $b, c \Rrightarrow b', c'$ and for every elements a, c of the points of AS there exists an element b of the points of AS such that $a, b \Rrightarrow b, c$ and for all elements a, b, c, b' of the points of AS such that $a, b \Rrightarrow b, c$ and $a, b' \Rrightarrow b', c$ holds $b = b'$ and for all elements a, b, c, d of the points of AS such that $a, b \Rrightarrow c, d$ holds $a, c \Rrightarrow b, d$,

(ii) AS is a space of free vectors.

(22) If there exist elements a, b of the carrier of ADG such that $a \neq b$, then Vectors(ADG) is a space of free vectors.

(23) For every ADG and for all elements a, b of the carrier of ADG holds $a\#b =$ (the addition of ADG)(a, b).

(24) For every ADG and for every binary relation R on $[\!:$ the carrier of ADG, the carrier of $ADG :\!]$ holds $R = \mathrm{Congr}_{ADG}$ if and only if for all elements a, b, c, d of the carrier of ADG holds $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in R$ if and only if $a\#d = b\#c$.

(25) For every ADG and for every AS being an affine structure holds $AS =$ Vectors(ADG) if and only if $AS = \langle$ the carrier of $ADG, \mathrm{Congr}_{ADG}\rangle$.

(26) For every ADG and for all elements a, b, c, d of the carrier of ADG holds $a, b \Rrightarrow c, d$ if and only if $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the congruence of Vectors(ADG).

(27) For every AS being an affine structure and for all elements a, b, c, d of the points of AS holds $a, b \Rrightarrow c, d$ if and only if $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the congruence of AS.

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