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Connectives and Subformulae of the First Order Language

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Summary. In the article the development of the first order language defined in [5] is continued. The following connectives are introduced: implication (⇒), disjunction (∨), and equivalence (⇔). We introduce also the existential quantifier (∃) and FALSUM. Some theorems on disjunctive, conditional, biconditional and existential formulae are proved and their selector functors are introduced. The second part of the article deals with notions of subformula, proper subformula and immediate constituent of a QC-formula.

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The papers [7], [6], [3], [4], [1], [2], and [5] provide the terminology and notation for this paper. We adopt the following convention: x, y, z will be bound variables and p, q, p_1, p_2, q_1 will be elements of WFF. One can prove the following propositions:

(1) If $\neg p = \neg q$, then $p = q$.

(2) $\mathrm{Arg}(\neg p) = p$.

(3) If $p \wedge q = p_1 \wedge q_1$, then $p = p_1$ and $q = q_1$.

(4) If p is conjunctive, then $p = \mathrm{LeftArg}(p) \wedge \mathrm{RightArg}(p)$.

(5) $\mathrm{LeftArg}(p \wedge q) = p$ and $\mathrm{RightArg}(p \wedge q) = q$.

(6) If $\forall_x p = \forall_y q$, then $x = y$ and $p = q$.

(7) If p is universal, then $p = \forall_{\mathrm{Bound}(p)} \mathrm{Scope}(p)$.

(8) $\mathrm{Bound}(\forall_x p) = x$ and $\mathrm{Scope}(\forall_x p) = p$.

We now define three new functors. The formula FALSUM is defined as follows:

FALSUM $= \neg$ VERUM.

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Let p, q be elements of WFF. The functor $p \Rightarrow q$ yields a formula and is defined by:

$p \Rightarrow q = \neg(p \wedge \neg q)$.

The functor $p \vee q$ yields a formula and is defined as follows:

$p \vee q = \neg(\neg p \wedge \neg q)$.

Let p, q be elements of WFF. The functor $p \Leftrightarrow q$ yielding a formula, is defined as follows:

$p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$.

Let x be a bound variable, and let p be an element of WFF. The functor $\exists_x p$ yielding a formula, is defined as follows:

$\exists_x p = \neg(\forall_x \neg p)$.

The following propositions are true:

(9) FALSUM $= \neg$ VERUM.

(10) $p \Rightarrow q = \neg(p \wedge \neg q)$.

(11) $p \vee q = \neg(\neg p \wedge \neg q)$.

(12) $p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$.

(13) FALSUM is negative and Arg(FALSUM) = VERUM.

(14) $p \vee q = \neg p \Rightarrow q$.

(15) $\exists_x p = \neg(\forall_x \neg p)$.

(16) If $p \vee q = p_1 \vee q_1$, then $p = p_1$ and $q = q_1$.

(17) If $p \Rightarrow q = p_1 \Rightarrow q_1$, then $p = p_1$ and $q = q_1$.

(18) If $p \Leftrightarrow q = p_1 \Leftrightarrow q_1$, then $p = p_1$ and $q = q_1$.

(19) If $\exists_x p = \exists_y q$, then $x = y$ and $p = q$.

We now define two new functors. Let x, y be bound variables, and let p be an element of WFF. The functor $\forall_{x,y} p$ yielding a formula, is defined by:

$\forall_{x,y} p = \forall_x (\forall_y p)$.

The functor $\exists_{x,y} p$ yields a formula and is defined by:

$\exists_{x,y} p = \exists_x (\exists_y p)$.

Next we state several propositions:

(20) $\forall_{x,y} p = \forall_x (\forall_y p)$ and $\exists_{x,y} p = \exists_x (\exists_y p)$.

(21) For all bound variables x_1, x_2, y_1, y_2 such that $\forall_{x_1,y_1} p_1 = \forall_{x_2,y_2} p_2$ holds $x_1 = x_2$ and $y_1 = y_2$ and $p_1 = p_2$.

(22) If $\forall_{x,y} p = \forall_z q$, then $x = z$ and $\forall_y p = q$.

(23) For all bound variables x_1, x_2, y_1, y_2 such that $\exists_{x_1,y_1} p_1 = \exists_{x_2,y_2} p_2$ holds $x_1 = x_2$ and $y_1 = y_2$ and $p_1 = p_2$.

(24) If $\exists_{x,y} p = \exists_z q$, then $x = z$ and $\exists_y p = q$.

(25) $\forall_{x,y} p$ is universal and Bound$(\forall_{x,y} p) = x$ and Scope$(\forall_{x,y} p) = \forall_y p$.

We now define two new functors. Let x, y, z be bound variables, and let p be an element of WFF. The functor $\forall_{x,y,z} p$ yields a formula and is defined by:

$\forall_{x,y,z} p = \forall_x (\forall_{y,z} p)$.

The functor $\exists_{x,y,z} p$ yields a formula and is defined by:

$\exists_{x,y,z} p = \exists_x(\exists_{y,z} p)$.

The following propositions are true:

(26) $\forall_{x,y,z} p = \forall_x(\forall_{y,z} p)$ and $\exists_{x,y,z} p = \exists_x(\exists_{y,z} p)$.

(27) For all bound variables x_1, x_2, y_1, y_2, z_1, z_2 such that $\forall_{x_1,y_1,z_1} p_1 = \forall_{x_2,y_2,z_2} p_2$ holds $x_1 = x_2$ and $y_1 = y_2$ and $z_1 = z_2$ and $p_1 = p_2$.

In the sequel s, t will be bound variables. We now state several propositions:

(28) If $\forall_{x,y,z} p = \forall_t q$, then $x = t$ and $\forall_{y,z} p = q$.

(29) If $\forall_{x,y,z} p = \forall_{t,s} q$, then $x = t$ and $y = s$ and $\forall_z p = q$.

(30) For all bound variables x_1, x_2, y_1, y_2, z_1, z_2 such that $\exists_{x_1,y_1,z_1} p_1 = \exists_{x_2,y_2,z_2} p_2$ holds $x_1 = x_2$ and $y_1 = y_2$ and $z_1 = z_2$ and $p_1 = p_2$.

(31) If $\exists_{x,y,z} p = \exists_t q$, then $x = t$ and $\exists_{y,z} p = q$.

(32) If $\exists_{x,y,z} p = \exists_{t,s} q$, then $x = t$ and $y = s$ and $\exists_z p = q$.

(33) $\forall_{x,y,z} p$ is universal and $\mathrm{Bound}(\forall_{x,y,z} p) = x$ and $\mathrm{Scope}(\forall_{x,y,z} p) = \forall_{y,z} p$.

We now define four new predicates. Let H be an element of WFF. We say that H is disjunctive if and only if:

there exist elements p, q of WFF such that $H = p \vee q$.

We say that H is conditional if and only if:

there exist elements p, q of WFF such that $H = p \Rightarrow q$.

We say that H is biconditional if and only if:

there exist elements p, q of WFF such that $H = p \Leftrightarrow q$.

We say that H is existential if and only if:

there exists a bound variable x and there exists an element p of WFF such that $H = \exists_x p$.

We now state several propositions:

(34) For every element H of WFF holds H is disjunctive if and only if there exist elements p, q of WFF such that $H = p \vee q$.

(35) For every element H of WFF holds H is conditional if and only if there exist elements p, q of WFF such that $H = p \Rightarrow q$.

(36) For every element H of WFF holds H is biconditional if and only if there exist elements p, q of WFF such that $H = p \Leftrightarrow q$.

(37) For every element H of WFF holds H is existential if and only if there exists a bound variable x and there exists an element p of WFF such that $H = \exists_x p$.

(38) $\exists_{x,y} p$ is existential and $\exists_{x,y,z} p$ is existential.

We now define four new functors. Let H be an element of WFF. The functor $\mathrm{LeftDisj}(H)$ yields a formula and is defined by:

$\mathrm{LeftDisj}(H) = \mathrm{Arg}(\mathrm{LeftArg}(\mathrm{Arg}(H)))$.

The functor $\mathrm{RightDisj}(H)$ yielding a formula, is defined as follows:

$\mathrm{RightDisj}(H) = \mathrm{Arg}(\mathrm{RightArg}(\mathrm{Arg}(H)))$.

The functor $\mathrm{Antecedent}(H)$ yields a formula and is defined by:

$\mathrm{Antecedent}(H) = \mathrm{LeftArg}(\mathrm{Arg}(H))$.

The functor $\mathrm{Consequent}(H)$ yields a formula and is defined by:

Consequent(H) = Arg(RightArg(Arg(H))).

We now define two new functors. Let H be an element of WFF. The functor LeftSide(H) yields a formula and is defined by:

LeftSide(H) = Antecedent(LeftArg(H)).

The functor RightSide(H) yielding a formula, is defined as follows:

RightSide(H) = Consequent(LeftArg(H)).

The following propositions are true:

(39) For every element H of WFF holds LeftDisj(H) = Arg(LeftArg(Arg(H))) .

(40) For every element H of WFF holds RightDisj(H) = Arg(RightArg(Arg(H))) .

(41) For every element H of WFF holds Antecedent(H) = LeftArg(Arg(H)).

(42) For every element H of WFF holds Consequent(H) = Arg(RightArg(Arg(H))) .

(43) For every element H of WFF holds LeftSide(H) = Antecedent(LeftArg(H)) .

(44) For every element H of WFF holds RightSide(H) = Consequent(LeftArg(H)) .

In the sequel F, G, H will be elements of WFF. We now state a number of propositions:

(45) LeftDisj$(F \vee G) = F$ and RightDisj$(F \vee G) = G$ and Arg$(F \vee G) = \neg F \wedge \neg G$.

(46) Antecedent$(F \Rightarrow G) = F$ and Consequent$(F \Rightarrow G) = G$ and Arg$(F \Rightarrow G) = F \wedge \neg G$.

(47) LeftSide$(F \Leftrightarrow G) = F$ and RightSide$(F \Leftrightarrow G) = G$ and LeftArg$(F \Leftrightarrow G) = F \Rightarrow G$ and RightArg$(F \Leftrightarrow G) = G \Rightarrow F$.

(48) Arg$(\exists_x H) = \forall_x \neg H$.

(49) If H is disjunctive, then H is conditional and H is negative and Arg(H) is conjunctive and LeftArg(Arg(H)) is negative and RightArg(Arg(H)) is negative.

(50) If H is conditional, then H is negative and Arg(H) is conjunctive and RightArg(Arg(H)) is negative.

(51) If H is biconditional, then H is conjunctive and LeftArg(H) is conditional and RightArg(H) is conditional.

(52) If H is existential, then H is negative and Arg(H) is universal and Scope(Arg(H)) is negative.

(53) If H is disjunctive, then H = LeftDisj$(H) \vee$ RightDisj(H).

(54) If H is conditional, then H = Antecedent$(H) \Rightarrow$ Consequent(H).

(55) If H is biconditional, then H = LeftSide$(H) \Leftrightarrow$ RightSide(H).

(56) If H is existential, then $H = \exists_{\mathrm{Bound}(\mathrm{Arg}(H))}$ Arg(Scope(Arg(H))).

Let G, H be elements of WFF. We say that G is an immediate constituent of H if and only if:

$H = \neg G$ or there exists an element F of WFF such that $H = G \wedge F$ or $H = F \wedge G$ or there exists a bound variable x such that $H = \forall_x G$.

For simplicity we adopt the following convention: x is a bound variable, k, n are natural numbers, P is a k-ary predicate symbol, and V is a list of variables of the length k. One can prove the following propositions:

(57) G is an immediate constituent of H if and only if $H = \neg G$ or there exists F such that $H = G \wedge F$ or $H = F \wedge G$ or there exists x such that $H = \forall_x G$.

(58) H is not an immediate constituent of VERUM.

(59) H is not an immediate constituent of $P[V]$.

(60) F is an immediate constituent of $\neg H$ if and only if $F = H$.

(61) H is an immediate constituent of FALSUM if and only if $H =$ VERUM.

(62) F is an immediate constituent of $G \wedge H$ if and only if $F = G$ or $F = H$.

(63) F is an immediate constituent of $\forall_x H$ if and only if $F = H$.

(64) If H is atomic, then F is not an immediate constituent of H.

(65) If H is negative, then F is an immediate constituent of H if and only if $F = \mathrm{Arg}(H)$.

(66) If H is conjunctive, then F is an immediate constituent of H if and only if $F = \mathrm{LeftArg}(H)$ or $F = \mathrm{RightArg}(H)$.

(67) If H is universal, then F is an immediate constituent of H if and only if $F = \mathrm{Scope}(H)$.

In the sequel L denotes a finite sequence. Let us consider G, H. We say that G is a subformula of H if and only if:

there exist n, L such that $1 \leq n$ and $\mathrm{len}\, L = n$ and $L(1) = G$ and $L(n) = H$ and for every k such that $1 \leq k$ and $k < n$ there exist elements G_1, H_1 of WFF such that $L(k) = G_1$ and $L(k+1) = H_1$ and G_1 is an immediate constituent of H_1.

We now state two propositions:

(68) G is a subformula of H if and only if there exist n, L such that $1 \leq n$ and $\mathrm{len}\, L = n$ and $L(1) = G$ and $L(n) = H$ and for every k such that $1 \leq k$ and $k < n$ there exist elements G_1, H_1 of WFF such that $L(k) = G_1$ and $L(k+1) = H_1$ and G_1 is an immediate constituent of H_1.

(69) H is a subformula of H.

Let us consider H, F. We say that H is a proper subformula of F if and only if:

H is a subformula of F and $H \neq F$.

One can prove the following propositions:

(70) H is a proper subformula of F if and only if H is a subformula of F and $H \neq F$.

(71) If H is an immediate constituent of F, then $\mathrm{len}(@H) < \mathrm{len}(@F)$.

(72) If H is an immediate constituent of F, then H is a subformula of F.

(73) If H is an immediate constituent of F, then H is a proper subformula of F.

(74) If H is a proper subformula of F, then $\mathrm{len}(@H) < \mathrm{len}(@F)$.

(75) If H is a proper subformula of F, then there exists G such that G is an immediate constituent of F.

(76) If F is a proper subformula of G and G is a proper subformula of H, then F is a proper subformula of H.

(77) If F is a subformula of G and G is a subformula of H, then F is a subformula of H.

(78) If G is a subformula of H and H is a subformula of G, then $G = H$.

(79) It is not true that: G is a proper subformula of H and H is a subformula of G.

(80) It is not true that: G is a proper subformula of H and H is a proper subformula of G.

(81) It is not true that: G is a subformula of H and H is an immediate constituent of G.

(82) It is not true that: G is a proper subformula of H and H is an immediate constituent of G.

(83) Suppose F is a proper subformula of G and G is a subformula of H or F is a subformula of G and G is a proper subformula of H or F is a subformula of G and G is an immediate constituent of H or F is an immediate constituent of G and G is a subformula of H or F is a proper subformula of G and G is an immediate constituent of H or F is an immediate constituent of G and G is a proper subformula of H. Then F is a proper subformula of H.

(84) F is not a proper subformula of VERUM.

(85) F is not a proper subformula of $P[V]$.

(86) F is a subformula of H if and only if F is a proper subformula of $\neg H$.

(87) If $\neg F$ is a subformula of H, then F is a proper subformula of H.

(88) F is a proper subformula of FALSUM if and only if F is a subformula of VERUM.

(89) F is a subformula of G or F is a subformula of H if and only if F is a proper subformula of $G \wedge H$.

(90) If $F \wedge G$ is a subformula of H, then F is a proper subformula of H and G is a proper subformula of H.

(91) F is a subformula of H if and only if F is a proper subformula of $\forall_x H$.

(92) If $\forall_x H$ is a subformula of F, then H is a proper subformula of F.

(93) $F \wedge \neg G$ is a proper subformula of $F \Rightarrow G$ and F is a proper subformula of $F \Rightarrow G$ and $\neg G$ is a proper subformula of $F \Rightarrow G$ and G is a proper subformula of $F \Rightarrow G$.

(94) $\neg F \wedge \neg G$ is a proper subformula of $F \vee G$ and $\neg F$ is a proper subformula of $F \vee G$ and $\neg G$ is a proper subformula of $F \vee G$ and F is a proper subformula of $F \vee G$ and G is a proper subformula of $F \vee G$.

(95) If H is atomic, then F is not a proper subformula of H.

(96) If H is negative, then $\mathrm{Arg}(H)$ is a proper subformula of H.

(97) If H is conjunctive, then $\mathrm{LeftArg}(H)$ is a proper subformula of H and $\mathrm{RightArg}(H)$ is a proper subformula of H.

(98) If H is universal, then $\mathrm{Scope}(H)$ is a proper subformula of H.

(99) H is a subformula of VERUM if and only if $H = \mathrm{VERUM}$.

(100) H is a subformula of $P[V]$ if and only if $H = P[V]$.

(101) H is a subformula of FALSUM if and only if $H = \mathrm{FALSUM}$ or $H = \mathrm{VERUM}$.

Let us consider H. The functor $\mathrm{Subformulae}\, H$ yields a set and is defined by:

for arbitrary a holds $a \in \mathrm{Subformulae}\, H$ if and only if there exists F such that $F = a$ and F is a subformula of H.

Next we state a number of propositions:

(102) For arbitrary a holds $a \in \mathrm{Subformulae}\, H$ if and only if there exists F such that $F = a$ and F is a subformula of H.

(103) If $G \in \mathrm{Subformulae}\, H$, then G is a subformula of H.

(104) If F is a subformula of H, then $\mathrm{Subformulae}\, F \subseteq \mathrm{Subformulae}\, H$.

(105) If $G \in \mathrm{Subformulae}\, H$, then $\mathrm{Subformulae}\, G \subseteq \mathrm{Subformulae}\, H$.

(106) $H \in \mathrm{Subformulae}\, H$.

(107) $\mathrm{Subformulae}\, \mathrm{VERUM} = \{\mathrm{VERUM}\}$.

(108) $\mathrm{Subformulae}(P[V]) = \{P[V]\}$.

(109) $\mathrm{Subformulae}\, \mathrm{FALSUM} = \{\mathrm{VERUM}, \mathrm{FALSUM}\}$.

(110) $\mathrm{Subformulae}\, \neg H = \mathrm{Subformulae}\, H \cup \{\neg H\}$.

(111) $\mathrm{Subformulae}\, H \wedge F = (\mathrm{Subformulae}\, H \cup \mathrm{Subformulae}\, F) \cup \{H \wedge F\}$.

(112) $\mathrm{Subformulae}\, \forall_x H = \mathrm{Subformulae}\, H \cup \{\forall_x H\}$.

(113) $\mathrm{Subformulae}\, F \Rightarrow G = (\mathrm{Subformulae}\, F \cup \mathrm{Subformulae}\, G) \cup \{\neg G, F \wedge \neg G, F \Rightarrow G\}$.

(114) $\mathrm{Subformulae}\, F \vee G = (\mathrm{Subformulae}\, F \cup \mathrm{Subformulae}\, G) \cup \{\neg G, \neg F, \neg F \wedge \neg G, F \vee G\}$.

(115) $\mathrm{Subformulae}\, F \Leftrightarrow G = (\mathrm{Subformulae}\, F \cup \mathrm{Subformulae}\, G) \cup \{\neg G, F \wedge \neg G, F \Rightarrow G, \neg F, G \wedge \neg F, G \Rightarrow F, F \Leftrightarrow G\}$.

(116) $H = \mathrm{VERUM}$ or H is atomic if and only if $\mathrm{Subformulae}\, H = \{H\}$.

(117) If H is negative, then $\mathrm{Subformulae}\, H = \mathrm{Subformulae}\, \mathrm{Arg}(H) \cup \{H\}$.

(118) If H is conjunctive, then $\mathrm{Subformulae}\, H = (\mathrm{Subformulae}\, \mathrm{LeftArg}(H) \cup \mathrm{Subformulae}\, \mathrm{RightArg}(H)) \cup \{H\}$.

(119) If H is universal, then $\mathrm{Subformulae}\, H = \mathrm{Subformulae}\, \mathrm{Scope}(H) \cup \{H\}$.

(120) If H is an immediate constituent of G or H is a proper subformula of G or H is a subformula of G but $G \in \mathrm{Subformulae}\, F$, then $H \in \mathrm{Subformulae}\, F$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[5] Piotr Rudnicki and Andrzej Trybulec. A first order language. *Formalized Mathematics*, 1(**2**):303–311, 1990.

[6] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.

[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

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