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Pigeon Hole Principle

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Summary. We introduce the notion of a predicate that states that a function is one-to-one at a given element of it's domain (i.e. counter image of image of the element is equal to its singleton). We also introduce some rather technical functors concerning finite sequences: the lowest index of the given element of the range of the finite sequence, the substring preceding (and succeeding) the first occurrence of given element of the range. At the end of the article we prove the pigeon hole principle.

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The notation and terminology used here are introduced in the following papers: [8], [4], [3], [6], [7], [1], [5], [9], [2], and [10]. For simplicity we adopt the following convention: f is a function, p, q are finite sequences, x, y, z are arbitrary, i, k, n are natural numbers, and A, B are sets. Let us consider f, x. We say that f is one-to-one at x if and only if:

$f^{-1}(f^{\circ}\{x\}) = \{x\}$.

We now state several propositions:

(1) f is one-to-one at x if and only if $f^{-1}(f^{\circ}\{x\}) = \{x\}$.

(2) If f is one-to-one at x, then $x \in \operatorname{dom} f$.

(3) f is one-to-one at x if and only if $x \in \operatorname{dom} f$ and $f^{-1}\{f(x)\} = \{x\}$.

(4) f is one-to-one at x if and only if $x \in \operatorname{dom} f$ and for every z such that $z \in \operatorname{dom} f$ and $x \neq z$ holds $f(x) \neq f(z)$.

(5) For every x such that $x \in \operatorname{dom} f$ holds f is one-to-one at x if and only if f is one-to-one.

Let us consider f, y. We say that f yields y just once if and only if:

$f^{-1}\{y\}$ is finite and $\operatorname{card}(f^{-1}\{y\}) = 1$.

Next we state several propositions:

(6) f yields y just once if and only if $f^{-1}\{y\}$ is finite and $\operatorname{card}(f^{-1}\{y\}) = 1$.

(7) If f yields y just once, then $y \in \operatorname{rng} f$.

(8) f yields y just once if and only if there exists x such that $\{x\} = f^{-1}\{y\}$.

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(9) f yields y just once if and only if there exists x such that $x \in \operatorname{dom} f$ and $y = f(x)$ and for every z such that $z \in \operatorname{dom} f$ and $z \neq x$ holds $f(z) \neq y$.

(10) f is one-to-one if and only if for every y such that $y \in \operatorname{rng} f$ holds f yields y just once.

(11) f is one-to-one at x if and only if $x \in \operatorname{dom} f$ and f yields $f(x)$ just once.

Let us consider f, y. Let us assume that f yields y just once. The functor $f^{-1}(y)$ is defined as follows:

$f^{-1}(y) \in \operatorname{dom} f$ and $f(f^{-1}(y)) = y$.

One can prove the following propositions:

(12) If f yields y just once and $x \in \operatorname{dom} f$ and $f(x) = y$, then $x = f^{-1}(y)$.

(13) If f yields y just once, then $f^{-1}(y) \in \operatorname{dom} f$.

(14) If f yields y just once, then $f(f^{-1}(y)) = y$.

(15) If f yields y just once, then for every x such that $x \in \operatorname{dom} f$ and $x \neq f^{-1}(y)$ holds $f(x) \neq y$.

(16) If f yields y just once, then $f \circ \{f^{-1}(y)\} = \{y\}$.

(17) If f yields y just once, then $f^{-1} \{y\} = \{f^{-1}(y)\}$.

(18) If f is one-to-one and $y \in \operatorname{rng} f$, then $f^{-1}(y) = f^{-1}(y)$.

(19) If $x \in \operatorname{dom} f$ and f yields $f(x)$ just once, then $f^{-1}(f(x)) = x$.

(20) If f is one-to-one at x, then $f^{-1}(f(x)) = x$.

(21) If f yields y just once, then f is one-to-one at $f^{-1}(y)$.

We adopt the following convention: D will be a non-empty set, d, d_1, d_2, d_3 will be elements of D, and P will be a finite sequence of elements of D. Let us consider D, d_1, d_2. Then $\langle d_1, d_2 \rangle$ is a finite sequence of elements of D.

Let us consider D, d_1, d_2, d_3. Then $\langle d_1, d_2, d_3 \rangle$ is a finite sequence of elements of D.

Let us consider D, P, i. Let us assume that $i \in \operatorname{dom} P$. The functor $\pi_i P$ yielding an element of D, is defined as follows:

$\pi_i P = P(i)$.

Next we state several propositions:

(22) If $i \in \operatorname{dom} P$, then $\pi_i P = P(i)$.

(23) If $i \in \operatorname{Seg}(\operatorname{len} P)$, then $\pi_i P = P(i)$.

(24) If $1 \leq i$ and $i \leq \operatorname{len} P$, then $\pi_i P = P(i)$.

(25) $\pi_1 \langle d \rangle = d$.

(26) $\pi_1 \langle d_1, d_2 \rangle = d_1$ and $\pi_2 \langle d_1, d_2 \rangle = d_2$.

(27) $\pi_1 \langle d_1, d_2, d_3 \rangle = d_1$ and $\pi_2 \langle d_1, d_2, d_3 \rangle = d_2$ and $\pi_3 \langle d_1, d_2, d_3 \rangle = d_3$.

Let us consider p, x. Let us assume that $x \in \operatorname{rng} p$. The functor $x \looparrowleft p$ yields a natural number and is defined by:

$x \looparrowleft p = \operatorname{Sgm}(p^{-1} \{x\})(1)$.

Next we state a number of propositions:

(28) If $x \in \operatorname{rng} p$, then $x \looparrowleft p = \operatorname{Sgm}(p^{-1} \{x\})(1)$.

(29) If $x \in \operatorname{rng} p$, then $p(x \hookleftarrow p) = x$.

(30) If $x \in \operatorname{rng} p$, then $x \hookleftarrow p \in \operatorname{dom} p$.

(31) If $x \in \operatorname{rng} p$, then $1 \leq x \hookleftarrow p$ and $x \hookleftarrow p \leq \operatorname{len} p$.

(32) If $x \in \operatorname{rng} p$, then $x \hookleftarrow p - 1$ is a natural number and $\operatorname{len} p - x \hookleftarrow p$ is a natural number.

(33) If $x \in \operatorname{rng} p$, then $x \hookleftarrow p \in p^{-1} \{x\}$.

(34) If $x \in \operatorname{rng} p$, then for every k such that $k \in \operatorname{dom} p$ and $k < x \hookleftarrow p$ holds $p(k) \neq x$.

(35) If p yields x just once, then $p^{-1}(x) = x \hookleftarrow p$.

(36) If p yields x just once, then for every k such that $k \in \operatorname{dom} p$ and $k \neq x \hookleftarrow p$ holds $p(k) \neq x$.

(37) If $x \in \operatorname{rng} p$ and for every k such that $k \in \operatorname{dom} p$ and $k \neq x \hookleftarrow p$ holds $p(k) \neq x$, then p yields x just once.

(38) p yields x just once if and only if $x \in \operatorname{rng} p$ and $\{x \hookleftarrow p\} = p^{-1} \{x\}$.

(39) If p is one-to-one and $x \in \operatorname{rng} p$, then $\{x \hookleftarrow p\} = p^{-1} \{x\}$.

(40) p yields x just once if and only if $\operatorname{len}(p - \{x\}) = \operatorname{len} p - 1$.

(41) If p yields x just once, then for every k such that $k \in \operatorname{dom}(p - \{x\})$ holds if $k < x \hookleftarrow p$, then $(p - \{x\})(k) = p(k)$ but if $x \hookleftarrow p \leq k$, then $(p - \{x\})(k) = p(k + 1)$.

(42) Suppose p is one-to-one and $x \in \operatorname{rng} p$. Then for every k such that $k \in \operatorname{dom}(p - \{x\})$ holds $(p - \{x\})(k) = p(k)$ if and only if $k < x \hookleftarrow p$ but $(p - \{x\})(k) = p(k + 1)$ if and only if $x \hookleftarrow p \leq k$.

Let us consider p, x. Let us assume that $x \in \operatorname{rng} p$. The functor $p \leftarrow x$ yields a finite sequence and is defined as follows:

there exists n such that $n = x \hookleftarrow p - 1$ and $p \leftarrow x = p \restriction \operatorname{Seg} n$.

One can prove the following propositions:

(43) If $x \in \operatorname{rng} p$, then there exists n such that $n = x \hookleftarrow p - 1$ and $p \restriction \operatorname{Seg} n = p \leftarrow x$.

(44) If $x \in \operatorname{rng} p$ and there exists n such that $n = x \hookleftarrow p - 1$ and $p \restriction \operatorname{Seg} n = q$, then $q = p \leftarrow x$.

(45) If $x \in \operatorname{rng} p$ and $n = x \hookleftarrow p - 1$, then $p \restriction \operatorname{Seg} n = p \leftarrow x$.

(46) If $x \in \operatorname{rng} p$, then $\operatorname{len}(p \leftarrow x) = x \hookleftarrow p - 1$.

(47) If $x \in \operatorname{rng} p$ and $n = x \hookleftarrow p - 1$, then $\operatorname{dom}(p \leftarrow x) = \operatorname{Seg} n$.

(48) If $x \in \operatorname{rng} p$ and $k \in \operatorname{dom}(p \leftarrow x)$, then $p(k) = (p \leftarrow x)(k)$.

(49) If $x \in \operatorname{rng} p$, then $x \notin \operatorname{rng}(p \leftarrow x)$.

(50) If $x \in \operatorname{rng} p$, then $\operatorname{rng}(p \leftarrow x)$ misses $\{x\}$.

(51) If $x \in \operatorname{rng} p$, then $\operatorname{rng}(p \leftarrow x) \subseteq \operatorname{rng} p$.

(52) If $x \in \operatorname{rng} p$, then $x \hookleftarrow p = 1$ if and only if $p \leftarrow x = \varepsilon$.

(53) If $x \in \operatorname{rng} p$ and p is a finite sequence of elements of D, then $p \leftarrow x$ is a finite sequence of elements of D.

Let us consider p, x. Let us assume that $x \in \operatorname{rng} p$. The functor $p \rightarrow x$ yields a finite sequence and is defined as follows:

$\operatorname{len}(p \rightarrow x) = \operatorname{len} p - x \hookleftarrow p$ and for every k such that $k \in \operatorname{dom}(p \rightarrow x)$ holds $(p \rightarrow x)(k) = p(k + x \hookleftarrow p)$.

One can prove the following propositions:

(54) If $x \in \operatorname{rng} p$ and $\operatorname{len} q = \operatorname{len} p - x \hookleftarrow p$ and for every k such that $k \in \operatorname{dom} q$ holds $q(k) = p(k + x \hookleftarrow p)$, then $q = p \rightarrow x$.

(55) If $x \in \operatorname{rng} p$, then $\operatorname{len}(p \rightarrow x) = \operatorname{len} p - x \hookleftarrow p$.

(56) If $x \in \operatorname{rng} p$, then for every k such that $k \in \operatorname{dom}(p \rightarrow x)$ holds $(p \rightarrow x)(k) = p(k + x \hookleftarrow p)$.

(57) If $x \in \operatorname{rng} p$ and $n = \operatorname{len} p - x \hookleftarrow p$, then $\operatorname{dom}(p \rightarrow x) = \operatorname{Seg} n$.

(58) If $x \in \operatorname{rng} p$ and $n \in \operatorname{dom}(p \rightarrow x)$, then $n + x \hookleftarrow p \in \operatorname{dom} p$.

(59) If $x \in \operatorname{rng} p$, then $\operatorname{rng}(p \rightarrow x) \subseteq \operatorname{rng} p$.

(60) p yields x just once if and only if $x \in \operatorname{rng} p$ and $x \notin \operatorname{rng}(p \rightarrow x)$.

(61) If $x \in \operatorname{rng} p$ and p is one-to-one, then $x \notin \operatorname{rng}(p \rightarrow x)$.

(62) p yields x just once if and only if $x \in \operatorname{rng} p$ and $\operatorname{rng}(p \rightarrow x)$ misses $\{x\}$.

(63) If $x \in \operatorname{rng} p$ and p is one-to-one, then $\operatorname{rng}(p \rightarrow x)$ misses $\{x\}$.

(64) If $x \in \operatorname{rng} p$, then $x \hookleftarrow p = \operatorname{len} p$ if and only if $p \rightarrow x = \varepsilon$.

(65) If $x \in \operatorname{rng} p$ and p is a finite sequence of elements of D, then $p \rightarrow x$ is a finite sequence of elements of D.

(66) If $x \in \operatorname{rng} p$, then $p = ((p \leftarrow x) ^\frown \langle x \rangle) ^\frown (p \rightarrow x)$.

(67) If $x \in \operatorname{rng} p$ and p is one-to-one, then $p \leftarrow x$ is one-to-one.

(68) If $x \in \operatorname{rng} p$ and p is one-to-one, then $p \rightarrow x$ is one-to-one.

(69) p yields x just once if and only if $x \in \operatorname{rng} p$ and $p - \{x\} = (p \leftarrow x) ^\frown (p \rightarrow x)$.

(70) If $x \in \operatorname{rng} p$ and p is one-to-one, then $p - \{x\} = (p \leftarrow x) ^\frown (p \rightarrow x)$.

(71) If $x \in \operatorname{rng} p$ and $p - \{x\}$ is one-to-one and $p - \{x\} = (p \leftarrow x) ^\frown (p \rightarrow x)$, then p is one-to-one.

(72) If $x \in \operatorname{rng} p$ and p is one-to-one, then $\operatorname{rng}(p \leftarrow x)$ misses $\operatorname{rng}(p \rightarrow x)$.

(73) If A is finite, then there exists p such that $\operatorname{rng} p = A$ and p is one-to-one.

(74) If $\operatorname{rng} p \subseteq \operatorname{dom} p$ and p is one-to-one, then $\operatorname{rng} p = \operatorname{dom} p$.

(75) If $\operatorname{rng} p = \operatorname{dom} p$, then p is one-to-one.

(76) If $\operatorname{rng} p = \operatorname{rng} q$ and $\operatorname{len} p = \operatorname{len} q$ and q is one-to-one, then p is one-to-one.

(77) p is one-to-one if and only if $\operatorname{card}(\operatorname{rng} p) = \operatorname{len} p$.

In the sequel f denotes a function from A into B. The following propositions are true:

(78) If $\operatorname{card} A = \operatorname{card} B$ and A is finite and B is finite and f is one-to-one, then $\operatorname{rng} f = B$.

(79) If $\mathrm{card}\, A = \mathrm{card}\, B$ and A is finite and B is finite and $\mathrm{rng}\, f = B$, then f is one-to-one.

(80) If $\overline{\overline{B}} < \overline{\overline{A}}$ and $B \neq \emptyset$, then there exist x, y such that $x \in A$ and $y \in A$ and $x \neq y$ and $f(x) = f(y)$.

(81) If $\overline{\overline{A}} < \overline{\overline{B}}$, then there exists x such that $x \in B$ and for every y such that $y \in A$ holds $f(y) \neq x$.

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