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Ordered Affine Spaces Defined in Terms of Directed Parallelity - part I [1]

Henryk Oryszczyszyn[2]
Warsaw University
Białystok

Krzysztof Prażmowski[3]
Warsaw University
Białystok

Summary. In the article we consider several geometrical relations in given arbitrary ordered affine space defined in terms of directed parallelity. In particular we introduce the notions of the nondirected parallelity of segments, of collinearity, and the betweenness relation determined by the given relation of directed parallelity. The obtained structures satisfy commonly accepted axioms for affine spaces. At the end of the article we introduce a formal definition of affine space and affine plane (defined in terms of parallelity of segments).

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The notation and terminology used in this paper are introduced in the articles [2] and [1]. In the sequel X is a non-empty set. Let us consider X, and let R be a relation on $[\!: X, X :\!]$. The functor $\lambda(R)$ yielding a relation on $[\!: X, X :\!]$, is defined as follows:

for all elements a, b, c, d of X holds $\langle\langle a,b\rangle,\langle c,d\rangle\rangle \in \lambda(R)$ if and only if $\langle\langle a,b\rangle,\langle c,d\rangle\rangle \in R$ or $\langle\langle a,b\rangle,\langle d,c\rangle\rangle \in R$.

One can prove the following two propositions:

(1) For all relations R, R' on $[\!: X, X :\!]$ holds $R' = \lambda(R)$ if and only if for all elements a, b, c, d of X holds $\langle\langle a,b\rangle,\langle c,d\rangle\rangle \in R'$ if and only if $\langle\langle a,b\rangle,\langle c,d\rangle\rangle \in R$ or $\langle\langle a,b\rangle,\langle d,c\rangle\rangle \in R$.

(2) For every relation R on $[\!: X, X :\!]$ and for all elements a, b, c, d of X holds $\langle\langle a,b\rangle,\langle c,d\rangle\rangle \in \lambda(R)$ if and only if $\langle\langle a,b\rangle,\langle c,d\rangle\rangle \in R$ or $\langle\langle a,b\rangle,\langle d,c\rangle\rangle \in R$.

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Let S be an affine structure. The functor $\Lambda(S)$ yielding an affine structure, is defined as follows:

$\Lambda(S) = \langle$ the points of $S, \lambda($ the congruence of $S)\rangle$.

One can prove the following proposition

(3) For all S, S' being affine structures holds $\Lambda(S) = S'$ if and only if $S' = \langle$ the points of $S, \lambda($ the congruence of $S)\rangle$.

We adopt the following convention: S will be an ordered affine space and a, b, c, d, x, y, z, t, u, w will be elements of the points of S. The following propositions are true:

(4) $x, y \upharpoonleft\upharpoonright x, y$.

(5) If $x, y \upharpoonleft\upharpoonright z, t$, then $y, x \upharpoonleft\upharpoonright t, z$ and $z, t \upharpoonleft\upharpoonright x, y$ and $t, z \upharpoonleft\upharpoonright y, x$.

(6) If $z \neq t$ and $x, y \upharpoonleft\upharpoonright z, t$ and $z, t \upharpoonleft\upharpoonright u, w$, then $x, y \upharpoonleft\upharpoonright u, w$.

(7) $x, x \upharpoonleft\upharpoonright y, z$ and $y, z \upharpoonleft\upharpoonright x, x$.

(8) If $x, y \upharpoonleft\upharpoonright z, t$ and $x, y \upharpoonleft\upharpoonright t, z$, then $x = y$ or $z = t$.

(9) $x, y \upharpoonleft\upharpoonright x, z$ if and only if $x, y \upharpoonleft\upharpoonright y, z$ or $x, z \upharpoonleft\upharpoonright z, y$.

Let us consider S, a, b, c. The predicate $\mathbf{B}(a, b, c)$ is defined as follows:

$a, b \upharpoonleft\upharpoonright b, c$.

The following propositions are true:

(10) $\mathbf{B}(a, b, c)$ if and only if $a, b \upharpoonleft\upharpoonright b, c$.

(11) $x, y \upharpoonleft\upharpoonright x, z$ if and only if $\mathbf{B}(x, y, z)$ or $\mathbf{B}(x, z, y)$.

(12) If $\mathbf{B}(a, b, a)$, then $a = b$.

(13) If $\mathbf{B}(a, b, c)$, then $\mathbf{B}(c, b, a)$.

(14) $\mathbf{B}(x, x, y)$ and $\mathbf{B}(x, y, y)$.

(15) If $\mathbf{B}(a, b, c)$ and $\mathbf{B}(a, c, d)$, then $\mathbf{B}(b, c, d)$.

(16) If $b \neq c$ and $\mathbf{B}(a, b, c)$ and $\mathbf{B}(b, c, d)$, then $\mathbf{B}(a, c, d)$.

(17) There exists z such that $\mathbf{B}(x, y, z)$ and $y \neq z$.

(18) If $\mathbf{B}(x, y, z)$ and $\mathbf{B}(y, x, z)$, then $x = y$.

(19) If $x \neq y$ and $\mathbf{B}(x, y, z)$ and $\mathbf{B}(x, y, t)$, then $\mathbf{B}(y, z, t)$ or $\mathbf{B}(y, t, z)$.

(20) If $x \neq y$ and $\mathbf{B}(x, y, z)$ and $\mathbf{B}(x, y, t)$, then $\mathbf{B}(x, z, t)$ or $\mathbf{B}(x, t, z)$.

(21) If $\mathbf{B}(x, y, t)$ and $\mathbf{B}(x, z, t)$, then $\mathbf{B}(x, y, z)$ or $\mathbf{B}(x, z, y)$.

Let us consider S, a, b, c, d. The predicate $a, b \parallel c, d$ is defined as follows:

$a, b \upharpoonleft\upharpoonright c, d$ or $a, b \upharpoonleft\upharpoonright d, c$.

One can prove the following propositions:

(22) $a, b \parallel c, d$ if and only if $a, b \upharpoonleft\upharpoonright c, d$ or $a, b \upharpoonleft\upharpoonright d, c$.

(23) $a, b \parallel c, d$ if and only if $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in \lambda($ the congruence of $S)$.

(24) $x, y \parallel y, x$ and $x, y \parallel x, y$.

(25) $x, y \parallel z, z$ and $z, z \parallel x, y$.

(26) If $x, y \parallel x, z$, then $y, x \parallel y, z$.

(27) If $x, y \parallel z, t$, then $x, y \parallel t, z$ and $y, x \parallel z, t$ and $y, x \parallel t, z$ and $z, t \parallel x, y$ and $z, t \parallel y, x$ and $t, z \parallel x, y$ and $t, z \parallel y, x$.

(28) Suppose that

(i) $a \neq b$,

(ii) $a, b \parallel x, y$ and $a, b \parallel z, t$ or $a, b \parallel x, y$ and $z, t \parallel a, b$ or $x, y \parallel a, b$ and $z, t \parallel a, b$ or $x, y \parallel a, b$ and $a, b \parallel z, t$.

Then $x, y \parallel z, t$.

(29) There exist x, y, z such that $x, y \not\parallel x, z$.

(30) There exists t such that $x, z \parallel y, t$ and $y \neq t$.

(31) There exists t such that $x, y \parallel z, t$ and $x, z \parallel y, t$.

(32) If $z, x \parallel x, t$ and $x \neq z$, then there exists u such that $y, x \parallel x, u$ and $y, z \parallel t, u$.

Let us consider S, a, b, c. The predicate $\mathbf{L}(a, b, c)$ is defined as follows: $a, b \parallel a, c$.

One can prove the following propositions:

(33) $\mathbf{L}(a, b, c)$ if and only if $a, b \parallel a, c$.

(34) If $\mathbf{B}(a, b, c)$, then $\mathbf{L}(a, b, c)$.

(35) If $\mathbf{L}(a, b, c)$, then $\mathbf{B}(a, b, c)$ or $\mathbf{B}(b, a, c)$ or $\mathbf{B}(a, c, b)$.

(36) If $\mathbf{L}(x, y, z)$, then $\mathbf{L}(x, z, y)$ and $\mathbf{L}(y, x, z)$ and $\mathbf{L}(y, z, x)$ and $\mathbf{L}(z, x, y)$ and $\mathbf{L}(z, y, x)$.

(37) $\mathbf{L}(x, x, y)$ and $\mathbf{L}(x, y, y)$ and $\mathbf{L}(x, y, x)$.

(38) If $x \neq y$ and $\mathbf{L}(x, y, z)$ and $\mathbf{L}(x, y, t)$ and $\mathbf{L}(x, y, u)$, then $\mathbf{L}(z, t, u)$.

(39) If $x \neq y$ and $\mathbf{L}(x, y, z)$ and $x, y \parallel z, t$, then $\mathbf{L}(x, y, t)$.

(40) If $\mathbf{L}(x, y, z)$ and $\mathbf{L}(x, y, t)$, then $x, y \parallel z, t$.

(41) If $u \neq z$ and $\mathbf{L}(x, y, u)$ and $\mathbf{L}(x, y, z)$ and $\mathbf{L}(u, z, w)$, then $\mathbf{L}(x, y, w)$.

(42) There exist x, y, z such that not $\mathbf{L}(x, y, z)$.

(43) If $x \neq y$, then there exists z such that not $\mathbf{L}(x, y, z)$.

In the sequel AS will denote an affine structure. Let us consider AS, and let a, b, c, d be elements of the points of AS. The predicate $a, b \parallel c, d$ is defined as follows:

$\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the congruence of AS.

The following propositions are true:

(44) For all elements a, b, c, d of the points of AS holds $a, b \parallel c, d$ if and only if $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the congruence of AS.

(45) If $AS = \Lambda(S)$, then for all elements a, b, c, d of the points of S and for all elements a', b', c', d' of the points of AS such that $a = a'$ and $b = b'$ and $c = c'$ and $d = d'$ holds $a', b' \parallel c', d'$ if and only if $a, b \parallel c, d$.

(46) Suppose $AS = \Lambda(S)$. Then

(i) there exist elements x, y of the points of AS such that $x \neq y$,

(ii) for all elements x, y, z, t, u, w of the points of AS holds $x, y \parallel y, x$ and $x, y \parallel z, z$ but if $x \neq y$ and $x, y \parallel z, t$ and $x, y \parallel u, w$, then $z, t \parallel u, w$ but if $x, y \parallel x, z$, then $y, x \parallel y, z$,

(iii) there exist elements x, y, z of the points of AS such that $x, y \nparallel x, z$,

(iv) for every elements x, y, z of the points of AS there exists an element t of the points of AS such that $x, z \parallel y, t$ and $y \neq t$,

(v) for every elements x, y, z of the points of AS there exists an element t of the points of AS such that $x, y \parallel z, t$ and $x, z \parallel y, t$,

(vi) for all elements x, y, z, t of the points of AS such that $z, x \parallel x, t$ and $x \neq z$ there exists an element u of the points of AS such that $y, x \parallel x, u$ and $y, z \parallel t, u$.

An affine structure is said to be an affine space if:

(i) there exist elements x, y of the points of it such that $x \neq y$,

(ii) for all elements x, y, z, t, u, w of the points of it holds $x, y \parallel y, x$ and $x, y \parallel z, z$ but if $x \neq y$ and $x, y \parallel z, t$ and $x, y \parallel u, w$, then $z, t \parallel u, w$ but if $x, y \parallel x, z$, then $y, x \parallel y, z$,

(iii) there exist elements x, y, z of the points of it such that $x, y \nparallel x, z$,

(iv) for every elements x, y, z of the points of it there exists an element t of the points of it such that $x, z \parallel y, t$ and $y \neq t$,

(v) for every elements x, y, z of the points of it there exists an element t of the points of it such that $x, y \parallel z, t$ and $x, z \parallel y, t$,

(vi) for all elements x, y, z, t of the points of it such that $z, x \parallel x, t$ and $x \neq z$ there exists an element u of the points of it such that $y, x \parallel x, u$ and $y, z \parallel t, u$.

The following three propositions are true:

(47) Let AS be an affine space. Then

(i) there exist elements x, y of the points of AS such that $x \neq y$,

(ii) for all elements x, y, z, t, u, w of the points of AS holds $x, y \parallel y, x$ and $x, y \parallel z, z$ but if $x \neq y$ and $x, y \parallel z, t$ and $x, y \parallel u, w$, then $z, t \parallel u, w$ but if $x, y \parallel x, z$, then $y, x \parallel y, z$,

(iii) there exist elements x, y, z of the points of AS such that $x, y \nparallel x, z$,

(iv) for every elements x, y, z of the points of AS there exists an element t of the points of AS such that $x, z \parallel y, t$ and $y \neq t$,

(v) for every elements x, y, z of the points of AS there exists an element t of the points of AS such that $x, y \parallel z, t$ and $x, z \parallel y, t$,

(vi) for all elements x, y, z, t of the points of AS such that $z, x \parallel x, t$ and $x \neq z$ there exists an element u of the points of AS such that $y, x \parallel x, u$ and $y, z \parallel t, u$.

(48) $\Lambda(S)$ is an affine space.

(49) The following conditions are equivalent:

(i) there exist elements x, y of the points of AS such that $x \neq y$ and for all elements x, y, z, t, u, w of the points of AS holds $x, y \parallel y, x$ and $x, y \parallel z, z$ but if $x \neq y$ and $x, y \parallel z, t$ and $x, y \parallel u, w$, then $z, t \parallel u, w$ but if $x, y \parallel x, z$, then $y, x \parallel y, z$ and there exist elements x, y, z of the points of AS such that $x, y \nparallel x, z$ and for every elements x, y, z of the points of AS there exists an element t of the points of AS such that $x, z \parallel y, t$ and $y \neq t$ and for every elements x, y, z of the points of AS there exists an element t of the points of AS such that $x, y \parallel z, t$ and $x, z \parallel y, t$ and for all elements x, y, z, t of the points of AS such that $z, x \parallel x, t$ and $x \neq z$

there exists an element u of the points of AS such that $y, x \parallel x, u$ and $y, z \parallel t, u$,

(ii) AS is an affine space.

We follow the rules: S will be an ordered affine plane and x, y, z, t, u will be elements of the points of S. We now state two propositions:

(50) If $x, y \nparallel z, t$, then there exists u such that $x, y \parallel x, u$ and $z, t \parallel z, u$.

(51) If $AS = \Lambda(S)$, then for all elements x, y, z, t of the points of AS such that $x, y \nparallel z, t$ there exists an element u of the points of AS such that $x, y \parallel x, u$ and $z, t \parallel z, u$.

An affine space is said to be an affine plane if:
for all elements x, y, z, t of the points of it such that $x, y \nparallel z, t$ there exists an element u of the points of it such that $x, y \parallel x, u$ and $z, t \parallel z, u$.

In the sequel ASP will denote an affine space. Next we state three propositions:

(52) ASP is an affine plane if and only if for all elements x, y, z, t of the points of ASP such that $x, y \nparallel z, t$ there exists an element u of the points of ASP such that $x, y \parallel x, u$ and $z, t \parallel z, u$.

(53) $\Lambda(S)$ is an affine plane.

(54) AS is an affine plane if and only if the following conditions are satisfied:

(i) there exist elements x, y of the points of AS such that $x \neq y$,

(ii) for all elements x, y, z, t, u, w of the points of AS holds $x, y \parallel y, x$ and $x, y \parallel z, z$ but if $x \neq y$ and $x, y \parallel z, t$ and $x, y \parallel u, w$, then $z, t \parallel u, w$ but if $x, y \parallel x, z$, then $y, x \parallel y, z$,

(iii) there exist elements x, y, z of the points of AS such that $x, y \nparallel x, z$,

(iv) for every elements x, y, z of the points of AS there exists an element t of the points of AS such that $x, z \parallel y, t$ and $y \neq t$,

(v) for every elements x, y, z of the points of AS there exists an element t of the points of AS such that $x, y \parallel z, t$ and $x, z \parallel y, t$,

(vi) for all elements x, y, z, t of the points of AS such that $z, x \parallel x, t$ and $x \neq z$ there exists an element u of the points of AS such that $y, x \parallel x, u$ and $y, z \parallel t, u$,

(vii) for all elements x, y, z, t of the points of AS such that $x, y \nparallel z, t$ there exists an element u of the points of AS such that $x, y \parallel x, u$ and $z, t \parallel z, u$.

References

[1] Henryk Oryszczyszyn and Krzysztof Prażmowski. Analytical ordered affine spaces. *Formalized Mathematics*, 1(**3**):601–605, 1990.

[2] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

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