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Families of Subsets, Subspaces and Mappings in Topological Spaces

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Summary. This article is a continuation of [8]. Some basic theorems about families of sets in a topological space have been proved. Following redefinitions have been made: singleton of a set as a family in the topological space and results of boolean operations on families as a family of the topological space. Notion of a family of complements of sets and a closed (open) family have been also introduced. Next some theorems refer to subspaces in a topological space: some facts about types in a subspace, theorems about open and closed sets and families in a subspace. A notion of restriction of a family has been also introduced and basic properties of this notion have been proved. The last part of the article is about mappings. There are proved necessary and sufficient conditions for a mapping to be continuous. A notion of homeomorphism has been defined next. Theorems about homeomorphisms of topological spaces have been also proved.

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The articles [7], [2], [3], [1], [5], [4], and [6] provide the notation and terminology for this paper. For simplicity we follow the rules: x will be arbitrary, T, S, V will denote topological spaces, P, Q, R will denote subsets of T, F, G will denote families of subsets of T, P_1 will denote a subset of S, and H will denote a family of subsets of S. We now state several propositions:

(1) $F \subseteq 2^{\Omega_T}$.

(2) If $x \in F$, then x is a subset of T.

(3) For every set X such that $X \subseteq F$ holds X is a family of subsets of T.

(4) $F = G$ if and only if for every P holds $P \in F$ if and only if $P \in G$.

(5) If F is a cover of T, then $F \neq \emptyset$.

Let us consider T, P. Then $\{P\}$ is a family of subsets of T.

Let us consider T, F, G. Then $F \cup G$ is a family of subsets of T. Then $F \cap G$ is a family of subsets of T. Then $F \setminus G$ is a family of subsets of T.

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The following propositions are true:

(6) $\bigcup F \setminus \bigcup G \subseteq \bigcup (F \setminus G)$.

(7) $F^{\mathrm{c}} = (F$ **qua** a family of subsets of the carrier of $T)^{\mathrm{c}}$.

(8) $P \in F^{\mathrm{c}}$ if and only if $P^{\mathrm{c}} \in F$.

(9) $(F^{\mathrm{c}})^{\mathrm{c}} = F$.

(10) $F \neq \emptyset$ if and only if $F^{\mathrm{c}} \neq \emptyset$.

(11) If $F \neq \emptyset$, then $\bigcap F^{\mathrm{c}} = (\bigcup F)^{\mathrm{c}}$.

(12) If $F \neq \emptyset$, then $\bigcup F^{\mathrm{c}} = (\bigcap F)^{\mathrm{c}}$.

(13) F^{c} is finite if and only if F is finite.

We now define two new predicates. Let us consider T, F. The predicate F is open is defined by:

if $P \in F$, then P is open.

The predicate F is closed is defined by:

if $P \in F$, then P is closed.

One can prove the following propositions:

(14) F is open if and only if for every P such that $P \in F$ holds P is open.

(15) F is closed if and only if for every P such that $P \in F$ holds P is closed.

(16) F is closed if and only if F^{c} is open.

(17) F is open if and only if F^{c} is closed.

(18) If $F \subseteq G$ and G is open, then F is open.

(19) If $F \subseteq G$ and G is closed, then F is closed.

(20) If F is open and G is open, then $F \cup G$ is open.

(21) If F is open, then $F \cap G$ is open.

(22) If F is open, then $F \setminus G$ is open.

(23) If F is closed and G is closed, then $F \cup G$ is closed.

(24) If F is closed, then $F \cap G$ is closed.

(25) If F is closed, then $F \setminus G$ is closed.

(26) If F is open, then $\bigcup F$ is open.

(27) If F is open and F is finite, then $\bigcap F$ is open.

(28) If F is closed and F is finite, then $\bigcup F$ is closed.

(29) If F is closed, then $\bigcap F$ is closed.

In the sequel A will be a subspace of T. The following propositions are true:

(30) For every subset B of A holds B is a subset of T.

(31) For every family F of subsets of A holds F is a family of subsets of T.

(32) For every subset B of A holds B is open if and only if there exists C being a subset of T such that C is open and $C \cap \Omega_A = B$.

(33) For every subset Q of T such that Q is open for every subset P of A such that $P = Q$ holds P is open.

(34) For every subset Q of T such that Q is closed for every subset P of A such that $P = Q$ holds P is closed.

(35) If F is open, then for every family G of subsets of A such that $G = F$ holds G is open.

(36) If F is closed, then for every family G of subsets of A such that $G = F$ holds G is closed.

(37) If $P = \Omega_A$, then $T \upharpoonright P = A$.

(38) If $P \neq \emptyset$, then $Q \cap P$ is a subset of $T \upharpoonright P$.

Let us consider T, P, F. The functor $F \upharpoonright P$ yields a family of subsets of $T \upharpoonright P$ and is defined by:

for every subset Q of $T \upharpoonright P$ holds $Q \in F \upharpoonright P$ if and only if there exists R such that $R \in F$ and $R \cap P = Q$.

We now state a number of propositions:

(39) For every subset Q of $T \upharpoonright P$ holds $Q \in F \upharpoonright P$ if and only if there exists R being a subset of T such that $R \in F$ and $R \cap P = Q$.

(40) If $F \subseteq G$, then $F \upharpoonright P \subseteq G \upharpoonright P$.

(41) If $P \neq \emptyset$ and $Q \in F$, then $Q \cap P \in F \upharpoonright P$.

(42) If $Q \subseteq \bigcup F$, then $Q \cap P \subseteq \bigcup (F \upharpoonright P)$.

(43) If $P \subseteq \bigcup F$, then $P = \bigcup (F \upharpoonright P)$.

(44) $\bigcup (F \upharpoonright P) \subseteq \bigcup F$.

(45) If $P \subseteq \bigcup (F \upharpoonright P)$, then $P \subseteq \bigcup F$.

(46) If $P \neq \emptyset$ and F is finite, then $F \upharpoonright P$ is finite.

(47) If $P \neq \emptyset$ and F is open, then $F \upharpoonright P$ is open.

(48) If $P \neq \emptyset$ and F is closed, then $F \upharpoonright P$ is closed.

(49) For every family F of subsets of A such that F is open there exists G being a family of subsets of T such that G is open and for every subset AA of T such that $AA = \Omega_A$ holds $F = G \upharpoonright AA$.

(50) If $P \neq \emptyset$, then there exists f being a function such that $\operatorname{dom} f = F$ and $\operatorname{rng} f = F \upharpoonright P$ and for every x such that $x \in F$ for every Q such that $Q = x$ holds $f(x) = Q \cap P$.

In the sequel f will denote a map from T into S. We now state several propositions:

(51) $\operatorname{dom} f = \Omega_T$ and $\operatorname{rng} f \subseteq \Omega_S$.

(52) $f^{-1}(\Omega_S) = \Omega_T$.

(53) $(^\circ f)^\circ F$ is a family of subsets of S.

(54) $^{-1} f^\circ H$ is a family of subsets of T.

(55) f is continuous if and only if for every P_1 such that P_1 is open holds $f^{-1} P_1$ is open.

(56) f is continuous if and only if for every P_1 holds $\overline{f^{-1} P_1} \subseteq f^{-1} \overline{P_1}$.

(57) f is continuous if and only if for every P holds $f^\circ \overline{P} \subseteq \overline{f^\circ P}$.

The arguments of the notions defined below are the following: T, S, V which are objects of the type reserved above; f which is a map from T into S; g which is a map from S into V. Then $g \cdot f$ is a map from T into V.

One can prove the following propositions:

(58) For every map f from T into S for every map g from S into V such that f is continuous and g is continuous holds $g \cdot f$ is continuous.

(59) If f is continuous and H is open, then for every F such that $F = {}^{-1} f^{\circ} H$ holds F is open.

(60) If f is continuous and H is closed, then for every F such that $F = {}^{-1} f^{\circ} H$ holds F is closed.

Let us consider T, S, f. Let us assume that $\operatorname{rng} f = \Omega_S$ and f is one-to-one. The functor f^{-1} yielding a map from S into T, is defined by:

$f^{-1} = f^{-1}$.

One can prove the following propositions:

(61) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then $f^{-1} = (f\,\mathbf{qua}\,\text{a function})^{-1}$.

(62) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then $\operatorname{dom}(f^{-1}) = \Omega_S$ and $\operatorname{rng}(f^{-1}) = \Omega_T$.

(63) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then f^{-1} is one-to-one.

(64) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then $(f^{-1})^{-1} = f$.

(65) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then $f^{-1} \cdot f = \mathrm{id}_{\operatorname{dom} f}$ and $f \cdot f^{-1} = \mathrm{id}_{\operatorname{rng} f}$.

(66) For every map f from T into S for every map g from S into V such that $\operatorname{rng} f = \Omega_S$ and f is one-to-one and $\operatorname{rng} g = \Omega_V$ and g is one-to-one holds $(g \cdot f)^{-1} = f^{-1} \cdot g^{-1}$.

(67) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then $f^{\circ} P = (f^{-1})^{-1} P$.

(68) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then $f^{-1} P_1 = f^{-1\,\circ} P_1$.

Let us consider T, S, f. The predicate f is a homeomorphism is defined by: $\operatorname{rng} f = \Omega_S$ and f is one-to-one and f is continuous and f^{-1} is continuous.

One can prove the following propositions:

(69) f is a homeomorphism if and only if $\operatorname{rng} f = \Omega_S$ and f is one-to-one and f is continuous and f^{-1} is continuous.

(70) If f is a homeomorphism, then f^{-1} is a homeomorphism.

(71) For every map f from T into S for every map g from S into V such that f is a homeomorphism and g is a homeomorphism holds $g \cdot f$ is a homeomorphism.

(72) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then f is a homeomorphism if and only if for every P holds P is closed if and only if $f^{\circ} P$ is closed.

(73) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then f is a homeomorphism if and only if for every P_1 holds $f^{-1}\, \overline{P_1} = \overline{f^{-1}\, P_1}$.

(74) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then f is a homeomorphism if and only if for every P holds $f^{\circ}\, \overline{P} = \overline{f^{\circ} P}$.

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