Boolean Domains

Andrzej Trybulec\textsuperscript{1} \quad Agata Darmochwa/suppress l\textsuperscript{2}
Warsaw University \quad Warsaw University
Bia/suppress lystok \quad Bia/suppress lystok

Summary. BOOLE DOMAIN is a SET DOMAIN that is closed under union and difference. This condition is equivalent to being closed under symmetric difference and one of the following operations: union, intersection or difference. We introduce the set of all finite subsets of a set $A$, denoted by $\text{Fin} \ A$. The mode Finite Subset of a set $A$ is introduced with the mother type: Element of $\text{Fin} \ A$. In consequence, “Finite Subset of . . .” is an elementary type, therefore one may use such types as “set of Finite Subset of $A$”, “[Finite Subset of $A$], Finite Subset of $A$]”, and so on. The article begins with some auxiliary theorems that belong really to [5] or [1] but are missing there. Moreover, bool $A$ is redefined as a SET DOMAIN, for an arbitrary set $A$.

The articles [4], [5], [3], and [2] provide the notation and terminology for this paper. In the sequel $X, Y$ will denote objects of the type set. The following propositions are true:

\begin{align*}
(1) \quad X \text{ misses } Y & \text{ implies } X \setminus Y = X \land Y \setminus X = Y, \\
(2) \quad X \text{ misses } Y & \text{ implies } (X \cup Y) \setminus Y = X \land (X \cup Y) \setminus X = Y, \\
(3) \quad X \cup Y & = X \uplus (Y \setminus X), \\
(4) \quad X \cup Y & = X \uplus Y \uplus X \cap Y, \\
(5) \quad X \setminus Y & = X \uplus (X \cap Y), \\
(6) \quad X \cap Y & = X \uplus Y \uplus (X \cup Y), \\
(7) \quad (\text{for } x \text{ being set st } x \in X \text{ holds } x \in Y) & \text{ implies } X \subseteq Y.
\end{align*}

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Let us consider $X$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\text{bool } X \quad \text{is} \quad \text{SET\_DOMAIN}.$$  

The following proposition is true

$$(8)\quad \text{for } Y \text{ being Element of } \text{bool } X \text{ holds } Y \subseteq X.$$  

The mode

$$\text{BOOLE\_DOMAIN},$$  

which widens to the type \text{SET\_DOMAIN}, is defined by

$$\text{for } X,Y \text{ being Element of it holds } X \cup Y \in \text{it} \land X \setminus Y \in \text{it}.$$  

The following proposition is true

$$(9)\quad \text{for } A \text{ being SET\_DOMAIN holds } A \text{ is BOOLE\_DOMAIN}$$  

iff for $X,Y$ being Element of $A$ holds $X \cup Y \in A \land X \setminus Y \in A$.

In the sequel $A$ will denote an object of the type \text{BOOLE\_DOMAIN}. One can prove the following propositions:

$$(10)\quad X \in A \land Y \in A \text{ implies } X \cup Y \in A \land X \setminus Y \in A,$$

$$(11)\quad X \text{ is Element of } A \land Y \text{ is Element of } A \text{ implies } X \cup Y \text{ is Element of } A,$$

$$(12)\quad X \text{ is Element of } A \land Y \text{ is Element of } A \text{ implies } X \setminus Y \text{ is Element of } A.$$  

The arguments of the notions defined below are the following: $A$ which is an object of the type reserved above; $X, Y$ which are objects of the type Element of $A$. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$X \cup Y \quad \text{is} \quad \text{Element of } A,$$

$$X \setminus Y \quad \text{is} \quad \text{Element of } A.$$  

The following propositions are true:

$$(13)\quad X \text{ is Element of } A \land Y \text{ is Element of } A \text{ implies } X \cap Y \text{ is Element of } A,$$

$$(14)\quad X \text{ is Element of } A \land Y \text{ is Element of } A \text{ implies } X \div Y \text{ is Element of } A,$$

$$(15)\quad \text{for } A \text{ being SET\_DOMAIN st}$$  

for $X,Y$ being Element of $A$ holds $X \div Y \in A \land X \setminus Y \in A$

holds $A$ is BOOLE\_DOMAIN,
for $A$ being SET\_DOMAIN st
for $X,Y$ being Element of $A$ holds $X \setminus Y \in A \& X \cap Y \in A$
holds $A$ is BOOLE\_DOMAIN,

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$X \cap Y$ is Element of $A$,

$X \setminus Y$ is Element of $A$.

We now state four propositions:

(18) $\emptyset \in A$,

(19) $\emptyset$ is Element of $A$,

(20) bool $A$ is BOOLE\_DOMAIN,

(21) for $A,B$ being BOOLE\_DOMAIN holds $A \cap B$ is BOOLE\_DOMAIN.

In the sequel $A, B$ will denote objects of the type set. Let us consider $A$. The functor

$$\text{Fin} A,$$

with values of the type BOOLE\_DOMAIN, is defined by

for $X$ being set holds $X \in \text{it}$ iff $X \subseteq A \& X$ is finite.

The following propositions are true:

(22) $B \in \text{Fin} A$ iff $B \subseteq A \& B$ is finite,

(23) $A \subseteq B$ implies $\text{Fin} A \subseteq \text{Fin} B$,

(24) $\text{Fin} (A \cap B) = \text{Fin} A \cap \text{Fin} B$,

(25) $\text{Fin} A \cup \text{Fin} B \subseteq \text{Fin} (A \cup B)$,

(26) $\text{Fin} A \subseteq \text{bool} A$,

(27) $A$ is finite implies $\text{Fin} A = \text{bool} A$,

(28) $\text{Fin} \emptyset = \{ \emptyset \}$. 