Preface

As was stated in [5], Mizar articles are being published in this periodical using special technology. They are written in High Level Formalized Language for Mathematics called *Mizar.* Their logical and mathematical correctness is verified by the PC Mizar system (distributed by Mizar Users Group owing to the grant received from the Philippe le Hodey Foundation). Mizar articles are submitted to the Mizar Users Association. For the both addresses see the second page of the cover. Articles form the Main Mizar Library (MML), which is continually enlarged and updated. The power of the PC Mizar system lies in the automatic processing of cross-references between articles. Main Mizar Library together with PC Mizar system constitute a system for collecting, formalizing and verifying mathematical knowledge. MML forms a basis of a Knowledge Management System for Mathematics supplied with Mizar articles.

Programs have being written which process Mizar articles and translate them automatically into English, and generate texts in TEX. Next LATEX uses a special format called MizTEX to produce them. While we do try to obtain such English which would not resemble a machine language, for instance by making use of the generator of random numbers to improve the style, obtaining good English is not the main objective of our work. What we do aim at is to obtain a readable text owing to which one can watch the development of MML, and also publish new mathematical results (for the time being in a small number). Another interesting problem is how to develop mathematics formal enough to make it verifiable by the computer. This accounts for certain peculiarities of the articles. For instance, the conceptual apparatus is strongly developed, which can best be illustrated by so-called casting functions, which without changing their respective arguments change their types. Other peculiarities include the occurrence of minor propositions resulting from detailed formalization, and repetitions of definitions in the form of so-called definitional theorems (see proposition 1 in [1], page 265).

As was explained in [5], what is translated is not Mizar articles themselves but so-called Mizar abstracts. An article includes certain elements which are transferred to the data base, such as theorems and definitions. But it also contains fragments which are not transferred there, such as proofs of theorems and lemmas. Mizar abstracts contain only those parts of Mizar articles which are transferred to the data base. This has been due to the fact that the material published at first was intended to facilitate to the Mizar users the use of the data base. In the future, as non-trivial proofs are offered, we plan to publish the translations of full Mizar articles.

It must be explained at this point that both PC Mizar system and Main Mizar Library are systematically developed. In the case of PC Mizar it is mainly the *Mizar* language which is enriched, which makes it more convenient to write articles; the same may by said of the proof-checker, which enables one to write shorter proofs. In the case of MML its development is linked above all to the ever new enhancers of texts, put in operation, which automatically improve the quality of the articles, usually by shortening them. As a simple example one can mention the program `Inacc` (inaccessible fragments), which removes from the text of the article those fragments which are not transferred to the data base and are not used in the article (for instance, when the author writes a lemma which she/he supposes to use later one but then changes the conception of the proofs and makes no reference to that lemma). Another improvement is that of `Chkrprem` (relevant premisses): it removes from the text the unnecessary references in justifications of statements. Since both processes interact that causes considerable changes in the articles. For instance, the strengthening of the proof-checker may result in that certain items of information function are understood by default and reference to a sentence A in the justification of a statement B ceases to be necessary. Hence in the successive verifications of the quality of the paper that reference is removed. We have to do with a special case of such interaction when a theorem has become obvious for the proof-checker, which is to say that when checking whether a given sentence is true the proof-checker needs not refer to other theorems and lemmas. The improver `Trivtheo` (trivial theorems) pin-points such theorems and eliminates them since as a rule they are not useful. But the numbering of the propositions in a given article is retained. Hence a gap in the numbering and the related information in the footnote: "The propositon (8) became obvious" (see [2], page 670).

Note also that in the present issue the symbol of multiplication of real functions has been changed from $\cdot$ into $\diamond$ (see [4] and [3]).

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Classical Configurations in Affine Planes [1]

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Summary. The classical sequence of implications which hold between Desargues and Pappus Axioms id proved. Formally Minor and Major Desargues Axiom (as suitable properties - predicates - of an affine plane) together with all its indirect forms are introduced; the same procedure is applied to Pappus Axioms. The so called Trapezium Desargues Axiom is also considered.

MML Identifier: `AFF_2`.

The articles [1], and [2] provide the notation and terminology for this paper. We follow the rules: AP will denote an affine plane, a, a', b, b', c, c', o will denote elements of the points of AP, and A, C, K, M, N, P will denote subsets of the points of AP. Let us consider AP. We say that AP satisfies **PPAP** if and only if:

Given M, N, a, b, c, a', b', c'. Then if M is a line and N is a line and $a \in M$ and $b \in M$ and $c \in M$ and $a' \in N$ and $b' \in N$ and $c' \in N$ and $a, b' \parallel b, a'$ and $b, c' \parallel c, b'$, then $a, c' \parallel c, a'$.

We now state the proposition

(1) Given AP. Then AP satisfies **PPAP** if and only if for all M, N, a, b, c, a', b', c' such that M is a line and N is a line and $a \in M$ and $b \in M$ and $c \in M$ and $a' \in N$ and $b' \in N$ and $c' \in N$ and $a, b' \parallel b, a'$ and $b, c' \parallel c, b'$ holds $a, c' \parallel c, a'$.

Let us consider AP. We say that AP satisfies **PAP** if and only if:

Given M, N, o, a, b, c, a', b', c'. Suppose that

(i) M is a line,
(ii) N is a line,
(iii) $M \neq N$,
(iv) $o \in M$,

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(v) $o \in N$,
(vi) $o \neq a$,
(vii) $o \neq a'$,
(viii) $o \neq b$,
(ix) $o \neq b'$,
(x) $o \neq c$,
(xi) $o \neq c'$,
(xii) $a \in M$,
(xiii) $b \in M$,
(xiv) $c \in M$,
(xv) $a' \in N$,
(xvi) $b' \in N$,
(xvii) $c' \in N$,
(xviii) $a, b' \parallel b, a'$,
(xix) $b, c' \parallel c, b'$.

Then $a, c' \parallel c, a'$.

The following proposition is true

(2) Given AP. Then AP satisfies **PAP** if and only if for all M, N, o, a, b, c, a', b', c' such that M is a line and N is a line and $M \neq N$ and $o \in M$ and $o \in N$ and $o \neq a$ and $o \neq a'$ and $o \neq b$ and $o \neq b'$ and $o \neq c$ and $o \neq c'$ and $a \in M$ and $b \in M$ and $c \in M$ and $a' \in N$ and $b' \in N$ and $c' \in N$ and $a, b' \parallel b, a'$ and $b, c' \parallel c, b'$ holds $a, c' \parallel c, a'$.

Let us consider AP. We say that AP satisfies $\mathbf{PAP_1}$ if and only if:

Given M, N, o, a, b, c, a', b', c'. Suppose that

(i) M is a line,
(ii) N is a line,
(iii) $M \neq N$,
(iv) $o \in M$,
(v) $o \in N$,
(vi) $o \neq a$,
(vii) $o \neq a'$,
(viii) $o \neq b$,
(ix) $o \neq b'$,
(x) $o \neq c$,
(xi) $o \neq c'$,
(xii) $a \in M$,
(xiii) $b \in M$,
(xiv) $c \in M$,
(xv) $b' \in N$,
(xvi) $c' \in N$,
(xvii) $a, b' \parallel b, a'$,
(xviii) $b, c' \parallel c, b'$,
(xix) $a, c' \parallel c, a'$,
(xx) $b \neq c$.

Then $a' \in N$.

One can prove the following proposition

(3) Given AP. Then AP satisfies $\mathbf{PAP_1}$ if and only if for all M, N, o, a, b, c, a', b', c' such that M is a line and N is a line and $M \neq N$ and $o \in M$ and $o \in N$ and $o \neq a$ and $o \neq a'$ and $o \neq b$ and $o \neq b'$ and $o \neq c$ and $o \neq c'$ and $a \in M$ and $b \in M$ and $c \in M$ and $b' \in N$ and $c' \in N$ and $a, b' \parallel b, a'$ and $b, c' \parallel c, b'$ and $a, c' \parallel c, a'$ and $b \neq c$ holds $a' \in N$.

Let us consider AP. We say that AP satisfies **DES** if and only if:

Given A, P, C, o, a, b, c, a', b', c'. Suppose that

(i) $o \in A$,
(ii) $o \in P$,
(iii) $o \in C$,
(iv) $o \neq a$,
(v) $o \neq b$,
(vi) $o \neq c$,
(vii) $a \in A$,
(viii) $a' \in A$,
(ix) $b \in P$,
(x) $b' \in P$,
(xi) $c \in C$,
(xii) $c' \in C$,
(xiii) A is a line,
(xiv) P is a line,
(xv) C is a line,
(xvi) $A \neq P$,
(xvii) $A \neq C$,
(xviii) $a, b \parallel a', b'$,
(xix) $a, c \parallel a', c'$.

Then $b, c \parallel b', c'$.

We now state the proposition

(4) Given AP. Then AP satisfies **DES** if and only if for all A, P, C, o, a, b, c, a', b', c' such that $o \in A$ and $o \in P$ and $o \in C$ and $o \neq a$ and $o \neq b$ and $o \neq c$ and $a \in A$ and $a' \in A$ and $b \in P$ and $b' \in P$ and $c \in C$ and $c' \in C$ and A is a line and P is a line and C is a line and $A \neq P$ and $A \neq C$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ holds $b, c \parallel b', c'$.

Let us consider AP. We say that AP satisfies $\mathbf{DES_1}$ if and only if:

Given A, P, C, o, a, b, c, a', b', c'. Suppose that

(i) $o \in A$,
(ii) $o \in P$,
(iii) $o \neq a$,
(iv) $o \neq b$,
(v) $o \neq c$,
(vi) $a \in A$,
(vii) $a' \in A$,

(viii) $b \in P$,
(ix) $b' \in P$,
(x) $c \in C$,
(xi) $c' \in C$,
(xii) A is a line,
(xiii) P is a line,
(xiv) C is a line,
(xv) $A \neq P$,
(xvi) $A \neq C$,
(xvii) $a, b \parallel a', b'$,
(xviii) $a, c \parallel a', c'$,
(xix) $b, c \parallel b', c'$,
(xx) not $\mathbf{L}(a, b, c)$,
(xxi) $c \neq c'$.

Then $o \in C$.

One can prove the following proposition

(5) Given AP. Then AP satisfies $\mathbf{DES_1}$ if and only if for all A, P, C, o, a, b, c, a', b', c' such that $o \in A$ and $o \in P$ and $o \neq a$ and $o \neq b$ and $o \neq c$ and $a \in A$ and $a' \in A$ and $b \in P$ and $b' \in P$ and $c \in C$ and $c' \in C$ and A is a line and P is a line and C is a line and $A \neq P$ and $A \neq C$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ and $b, c \parallel b', c'$ and not $\mathbf{L}(a, b, c)$ and $c \neq c'$ holds $o \in C$.

Let us consider AP. We say that AP satisfies $\mathbf{DES_2}$ if and only if:

Given A, P, C, o, a, b, c, a', b', c'. Suppose that

(i) $o \in A$,
(ii) $o \in P$,
(iii) $o \in C$,
(iv) $o \neq a$,
(v) $o \neq b$,
(vi) $o \neq c$,
(vii) $a \in A$,
(viii) $a' \in A$,
(ix) $b \in P$,
(x) $b' \in P$,
(xi) $c \in C$,
(xii) A is a line,
(xiii) P is a line,
(xiv) C is a line,
(xv) $A \neq P$,
(xvi) $A \neq C$,
(xvii) $a, b \parallel a', b'$,
(xviii) $a, c \parallel a', c'$,
(xix) $b, c \parallel b', c'$.

Then $c' \in C$.

One can prove the following proposition

(6) Given AP. Then AP satisfies $\mathbf{DES_2}$ if and only if for all A, P, C, o, a, b, c, a', b', c' such that $o \in A$ and $o \in P$ and $o \in C$ and $o \neq a$ and $o \neq b$ and $o \neq c$ and $a \in A$ and $a' \in A$ and $b \in P$ and $b' \in P$ and $c \in C$ and A is a line and P is a line and C is a line and $A \neq P$ and $A \neq C$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ and $b, c \parallel b', c'$ holds $c' \in C$.

Let us consider AP. We say that AP satisfies **TDES** if and only if:

Given K, o, a, b, c, a', b', c'. Suppose that

(i) K is a line,
(ii) $o \in K$,
(iii) $c \in K$,
(iv) $c' \in K$,
(v) $a \notin K$,
(vi) $o \neq c$,
(vii) $a \neq b$,
(viii) $\mathbf{L}(o, a, a')$,
(ix) $\mathbf{L}(o, b, b')$,
(x) $a, b \parallel a', b'$,
(xi) $a, c \parallel a', c'$,
(xii) $a, b \parallel K$.

Then $b, c \parallel b', c'$.

We now state the proposition

(7) Given AP. Then AP satisfies **TDES** if and only if for all K, o, a, b, c, a', b', c' such that K is a line and $o \in K$ and $c \in K$ and $c' \in K$ and $a \notin K$ and $o \neq c$ and $a \neq b$ and $\mathbf{L}(o, a, a')$ and $\mathbf{L}(o, b, b')$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ and $a, b \parallel K$ holds $b, c \parallel b', c'$.

Let us consider AP. We say that AP satisfies $\mathbf{TDES_1}$ if and only if:

Given K, o, a, b, c, a', b', c'. Suppose that

(i) K is a line,
(ii) $o \in K$,
(iii) $c \in K$,
(iv) $c' \in K$,
(v) $a \notin K$,
(vi) $o \neq c$,
(vii) $a \neq b$,
(viii) $\mathbf{L}(o, a, a')$,
(ix) $a, b \parallel a', b'$,
(x) $b, c \parallel b', c'$,
(xi) $a, c \parallel a', c'$,
(xii) $a, b \parallel K$.

Then $\mathbf{L}(o, b, b')$.

One can prove the following proposition

(8) Given AP. Then AP satisfies $\mathbf{TDES_1}$ if and only if for all K, o, a, b, c, a', b', c' such that K is a line and $o \in K$ and $c \in K$ and $c' \in K$ and $a \notin K$ and $o \neq c$ and $a \neq b$ and $\mathbf{L}(o, a, a')$ and $a, b \parallel a', b'$ and $b, c \parallel b', c'$ and $a, c \parallel a', c'$ and $a, b \parallel K$ holds $\mathbf{L}(o, b, b')$.

Let us consider AP. We say that AP satisfies $\mathbf{TDES_2}$ if and only if:

Given K, o, a, b, c, a', b', c'. Suppose that

(i) K is a line,
(ii) $o \in K$,
(iii) $c \in K$,
(iv) $c' \in K$,
(v) $a \notin K$,
(vi) $o \neq c$,
(vii) $a \neq b$,
(viii) $\mathbf{L}(o, a, a')$,
(ix) $\mathbf{L}(o, b, b')$,
(x) $b, c \parallel b', c'$,
(xi) $a, c \parallel a', c'$,
(xii) $a, b \parallel K$.

Then $a, b \parallel a', b'$.

The following proposition is true

(9) Given AP. Then AP satisfies $\mathbf{TDES_2}$ if and only if for all K, o, a, b, c, a', b', c' such that K is a line and $o \in K$ and $c \in K$ and $c' \in K$ and $a \notin K$ and $o \neq c$ and $a \neq b$ and $\mathbf{L}(o, a, a')$ and $\mathbf{L}(o, b, b')$ and $b, c \parallel b', c'$ and $a, c \parallel a', c'$ and $a, b \parallel K$ holds $a, b \parallel a', b'$.

Let us consider AP. We say that AP satisfies $\mathbf{TDES_3}$ if and only if:

Given K, o, a, b, c, a', b', c'. Suppose that

(i) K is a line,
(ii) $o \in K$,
(iii) $c \in K$,
(iv) $a \notin K$,
(v) $o \neq c$,
(vi) $a \neq b$,
(vii) $\mathbf{L}(o, a, a')$,
(viii) $\mathbf{L}(o, b, b')$,
(ix) $a, b \parallel a', b'$,
(x) $a, c \parallel a', c'$,
(xi) $b, c \parallel b', c'$,
(xii) $a, b \parallel K$.

Then $c' \in K$.

We now state the proposition

(10) Given AP. Then AP satisfies $\mathbf{TDES_3}$ if and only if for all K, o, a, b, c, a', b', c' such that K is a line and $o \in K$ and $c \in K$ and $a \notin K$ and $o \neq c$ and $a \neq b$ and $\mathbf{L}(o, a, a')$ and $\mathbf{L}(o, b, b')$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ and $b, c \parallel b', c'$ and $a, b \parallel K$ holds $c' \in K$.

Let us consider AP. We say that AP satisfies **des** if and only if:

Given A, P, C, a, b, c, a', b', c'. Suppose that

(i) $A \parallel P$,
(ii) $A \parallel C$,
(iii) $a \in A$,
(iv) $a' \in A$,
(v) $b \in P$,
(vi) $b' \in P$,
(vii) $c \in C$,
(viii) $c' \in C$,
(ix) A is a line,
(x) P is a line,
(xi) C is a line,
(xii) $A \neq P$,
(xiii) $A \neq C$,
(xiv) $a, b \parallel a', b'$,
(xv) $a, c \parallel a', c'$.

Then $b, c \parallel b', c'$.

The following proposition is true

(11) Given AP. Then AP satisfies **des** if and only if for all A, P, C, a, b, c, a', b', c' such that $A \parallel P$ and $A \parallel C$ and $a \in A$ and $a' \in A$ and $b \in P$ and $b' \in P$ and $c \in C$ and $c' \in C$ and A is a line and P is a line and C is a line and $A \neq P$ and $A \neq C$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ holds $b, c \parallel b', c'$.

Let us consider AP. We say that AP satisfies $\mathbf{des_1}$ if and only if:

Given A, P, C, a, b, c, a', b', c'. Suppose that

(i) $A \parallel P$,
(ii) $a \in A$,
(iii) $a' \in A$,
(iv) $b \in P$,
(v) $b' \in P$,
(vi) $c \in C$,
(vii) $c' \in C$,
(viii) A is a line,
(ix) P is a line,
(x) C is a line,
(xi) $A \neq P$,
(xii) $A \neq C$,
(xiii) $a, b \parallel a', b'$,
(xiv) $a, c \parallel a', c'$,
(xv) $b, c \parallel b', c'$,
(xvi) not $\mathbf{L}(a, b, c)$,
(xvii) $c \neq c'$.

Then $A \parallel C$.

The following proposition is true

(12) Given AP. Then AP satisfies $\mathbf{des_1}$ if and only if for all A, P, C, a, b, c, a', b', c' such that $A \parallel P$ and $a \in A$ and $a' \in A$ and $b \in P$ and $b' \in P$ and $c \in C$ and $c' \in C$ and A is a line and P is a line and C is a line and $A \neq P$ and $A \neq C$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ and $b, c \parallel b', c'$ and not $\mathbf{L}(a, b, c)$ and $c \neq c'$ holds $A \parallel C$.

Let us consider AP. We say that AP satisfies **pap** if and only if:

Given M, N, a, b, c, a', b', c'. Suppose M is a line and N is a line and $a \in M$ and $b \in M$ and $c \in M$ and $M \parallel N$ and $M \neq N$ and $a' \in N$ and $b' \in N$ and $c' \in N$ and $a, b' \parallel b, a'$ and $b, c' \parallel c, b'$. Then $a, c' \parallel c, a'$.

The following proposition is true

(13) Given AP. Then AP satisfies **pap** if and only if for all M, N, a, b, c, a', b', c' such that M is a line and N is a line and $a \in M$ and $b \in M$ and $c \in M$ and $M \parallel N$ and $M \neq N$ and $a' \in N$ and $b' \in N$ and $c' \in N$ and $a, b' \parallel b, a'$ and $b, c' \parallel c, b'$ holds $a, c' \parallel c, a'$.

Let us consider AP. We say that AP satisfies $\mathbf{pap_1}$ if and only if:

Given M, N, a, b, c, a', b', c'. Suppose that

(i) M is a line,
(ii) N is a line,
(iii) $a \in M$,
(iv) $b \in M$,
(v) $c \in M$,
(vi) $M \parallel N$,
(vii) $M \neq N$,
(viii) $a' \in N$,
(ix) $b' \in N$,
(x) $a, b' \parallel b, a'$,
(xi) $b, c' \parallel c, b'$,
(xii) $a, c' \parallel c, a'$,
(xiii) $a' \neq b'$.

Then $c' \in N$.

We now state a number of propositions:

(14) Given AP. Then AP satisfies $\mathbf{pap_1}$ if and only if for all M, N, a, b, c, a', b', c' such that M is a line and N is a line and $a \in M$ and $b \in M$ and $c \in M$ and $M \parallel N$ and $M \neq N$ and $a' \in N$ and $b' \in N$ and $a, b' \parallel b, a'$ and $b, c' \parallel c, b'$ and $a, c' \parallel c, a'$ and $a' \neq b'$ holds $c' \in N$.

(15) AP satisfies **PAP** if and only if AP satisfies $\mathbf{PAP_1}$.

(16) AP satisfies **DES** if and only if AP satisfies $\mathbf{DES_1}$.

(17) If AP satisfies **TDES**, then AP satisfies $\mathbf{TDES_1}$.

(18) If AP satisfies $\mathbf{TDES_1}$, then AP satisfies $\mathbf{TDES_2}$.

(19) If AP satisfies $\mathbf{TDES_2}$, then AP satisfies $\mathbf{TDES_3}$.

(20) If AP satisfies $\mathbf{TDES_3}$, then AP satisfies **TDES**.

(21) AP satisfies **des** if and only if AP satisfies $\mathbf{des_1}$.

(22) AP satisfies **pap** if and only if AP satisfies $\mathbf{pap_1}$.

(23) If AP satisfies **PAP**, then AP satisfies **pap**.

(24) AP satisfies **PPAP** if and only if AP satisfies **PAP** and AP satisfies **pap**.

(25) If AP satisfies **PAP**, then AP satisfies **DES**.

(26) If AP satisfies **DES**, then AP satisfies **TDES**.

(27) If AP satisfies $\mathbf{TDES_1}$, then AP satisfies $\mathbf{des_1}$.

(28) If AP satisfies **TDES**, then AP satisfies **des**.

(29) If AP satisfies **des**, then AP satisfies **pap**.

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Affine Localizations of Desargues Axiom [1]

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Summary. Several affine localizations of Major Desargues Axiom together with its indirect forms are introduced. Logical relationships between these formulas and between them and the classical Desargues Axiom are demonstrated.

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The articles [1], [3], and [2] provide the notation and terminology for this paper. We follow a convention: AP denotes an affine plane, a, a', b, b', c, c', o, p, q denote elements of the points of AP, and A, C, P denote subsets of the points of AP. Let us consider AP. We say that AP satisfies **DES1** if and only if:

Given A, P, C, o, a, a', b, b', c, c', p, q. Suppose that

(i) A is a line,
(ii) P is a line,
(iii) C is a line,
(iv) $P \neq A$,
(v) $P \neq C$,
(vi) $A \neq C$,
(vii) $o \in A$,
(viii) $a \in A$,
(ix) $a' \in A$,
(x) $o \in P$,
(xi) $b \in P$,
(xii) $b' \in P$,

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(xiii) $o \in C$,
(xiv) $c \in C$,
(xv) $c' \in C$,
(xvi) $o \neq a$,
(xvii) $o \neq b$,
(xviii) $o \neq c$,
(xix) $p \neq q$,
(xx) not $\mathbf{L}(b, a, c)$,
(xxi) not $\mathbf{L}(b', a', c')$,
(xxii) $a \neq a'$,
(xxiii) $\mathbf{L}(b, a, p)$,
(xxiv) $\mathbf{L}(b', a', p)$,
(xxv) $\mathbf{L}(b, c, q)$,
(xxvi) $\mathbf{L}(b', c', q)$,
(xxvii) $a, c \parallel a', c'$.

Then $a, c \parallel p, q$.

We now state the proposition

(1) Given AP. Then AP satisfies **DES1** if and only if for all A, P, C, o, a, a', b, b', c, c', p, q such that A is a line and P is a line and C is a line and $P \neq A$ and $P \neq C$ and $A \neq C$ and $o \in A$ and $a \in A$ and $a' \in A$ and $o \in P$ and $b \in P$ and $b' \in P$ and $o \in C$ and $c \in C$ and $c' \in C$ and $o \neq a$ and $o \neq b$ and $o \neq c$ and $p \neq q$ and not $\mathbf{L}(b, a, c)$ and not $\mathbf{L}(b', a', c')$ and $a \neq a'$ and $\mathbf{L}(b, a, p)$ and $\mathbf{L}(b', a', p)$ and $\mathbf{L}(b, c, q)$ and $\mathbf{L}(b', c', q)$ and $a, c \parallel a', c'$ holds $a, c \parallel p, q$.

Let us consider AP. We say that AP satisfies $\mathbf{DES1_1}$ if and only if:

Given A, P, C, o, a, a', b, b', c, c', p, q. Suppose that

(i) A is a line,
(ii) P is a line,
(iii) C is a line,
(iv) $P \neq A$,
(v) $P \neq C$,
(vi) $A \neq C$,
(vii) $o \in A$,
(viii) $a \in A$,
(ix) $a' \in A$,
(x) $o \in P$,
(xi) $b \in P$,
(xii) $b' \in P$,
(xiii) $o \in C$,
(xiv) $c \in C$,
(xv) $c' \in C$,
(xvi) $o \neq a$,
(xvii) $o \neq b$,
(xviii) $o \neq c$,

(xix) $p \neq q$,
(xx) $c \neq q$,
(xxi) not $\mathbf{L}(b, a, c)$,
(xxii) not $\mathbf{L}(b', a', c')$,
(xxiii) $\mathbf{L}(b, a, p)$,
(xxiv) $\mathbf{L}(b', a', p)$,
(xxv) $\mathbf{L}(b, c, q)$,
(xxvi) $\mathbf{L}(b', c', q)$,
(xxvii) $a, c \parallel p, q$.

Then $a, c \parallel a', c'$.

The following proposition is true

(2) Given AP. Then AP satisfies $\mathbf{DES1_1}$ if and only if for all A, P, C, o, a, a', b, b', c, c', p, q such that A is a line and P is a line and C is a line and $P \neq A$ and $P \neq C$ and $A \neq C$ and $o \in A$ and $a \in A$ and $a' \in A$ and $o \in P$ and $b \in P$ and $b' \in P$ and $o \in C$ and $c \in C$ and $c' \in C$ and $o \neq a$ and $o \neq b$ and $o \neq c$ and $p \neq q$ and $c \neq q$ and not $\mathbf{L}(b, a, c)$ and not $\mathbf{L}(b', a', c')$ and $\mathbf{L}(b, a, p)$ and $\mathbf{L}(b', a', p)$ and $\mathbf{L}(b, c, q)$ and $\mathbf{L}(b', c', q)$ and $a, c \parallel p, q$ holds $a, c \parallel a', c'$.

Let us consider AP. We say that AP satisfies $\mathbf{DES1_2}$ if and only if:

Given A, P, C, o, a, a', b, b', c, c', p, q. Suppose that

(i) A is a line,
(ii) P is a line,
(iii) C is a line,
(iv) $P \neq A$,
(v) $P \neq C$,
(vi) $A \neq C$,
(vii) $o \in A$,
(viii) $a \in A$,
(ix) $a' \in A$,
(x) $o \in P$,
(xi) $b \in P$,
(xii) $b' \in P$,
(xiii) $c \in C$,
(xiv) $c' \in C$,
(xv) $o \neq a$,
(xvi) $o \neq b$,
(xvii) $o \neq c$,
(xviii) $p \neq q$,
(xix) not $\mathbf{L}(b, a, c)$,
(xx) not $\mathbf{L}(b', a', c')$,
(xxi) $c \neq c'$,
(xxii) $\mathbf{L}(b, a, p)$,
(xxiii) $\mathbf{L}(b', a', p)$,
(xxiv) $\mathbf{L}(b, c, q)$,

(xxv) $\mathbf{L}(b', c', q)$,
(xxvi) $a, c \parallel a', c'$,
(xxvii) $a, c \parallel p, q$.
Then $o \in C$.

Next we state the proposition

(3) Given AP. Then AP satisfies $\mathbf{DES1_2}$ if and only if for all A, P, C, o, a, a', b, b', c, c', p, q such that A is a line and P is a line and C is a line and $P \neq A$ and $P \neq C$ and $A \neq C$ and $o \in A$ and $a \in A$ and $a' \in A$ and $o \in P$ and $b \in P$ and $b' \in P$ and $c \in C$ and $c' \in C$ and $o \neq a$ and $o \neq b$ and $o \neq c$ and $p \neq q$ and not $\mathbf{L}(b, a, c)$ and not $\mathbf{L}(b', a', c')$ and $c \neq c'$ and $\mathbf{L}(b, a, p)$ and $\mathbf{L}(b', a', p)$ and $\mathbf{L}(b, c, q)$ and $\mathbf{L}(b', c', q)$ and $a, c \parallel a', c'$ and $a, c \parallel p, q$ holds $o \in C$.

Let us consider AP. We say that AP satisfies $\mathbf{DES1_3}$ if and only if:
Given A, P, C, o, a, a', b, b', c, c', p, q. Suppose that
(i) A is a line,
(ii) P is a line,
(iii) C is a line,
(iv) $P \neq A$,
(v) $P \neq C$,
(vi) $A \neq C$,
(vii) $o \in A$,
(viii) $a \in A$,
(ix) $a' \in A$,
(x) $b \in P$,
(xi) $b' \in P$,
(xii) $o \in C$,
(xiii) $c \in C$,
(xiv) $c' \in C$,
(xv) $o \neq a$,
(xvi) $o \neq b$,
(xvii) $o \neq c$,
(xviii) $p \neq q$,
(xix) not $\mathbf{L}(b, a, c)$,
(xx) not $\mathbf{L}(b', a', c')$,
(xxi) $b \neq b'$,
(xxii) $a \neq a'$,
(xxiii) $\mathbf{L}(b, a, p)$,
(xxiv) $\mathbf{L}(b', a', p)$,
(xxv) $\mathbf{L}(b, c, q)$,
(xxvi) $\mathbf{L}(b', c', q)$,
(xxvii) $a, c \parallel a', c'$,
(xxviii) $a, c \parallel p, q$.
Then $o \in P$.

Next we state the proposition

(4) Given AP. Then AP satisfies $\mathbf{DES1_3}$ if and only if for all A, P, C, o, a, a', b, b', c, c', p, q such that A is a line and P is a line and C is a line and $P \neq A$ and $P \neq C$ and $A \neq C$ and $o \in A$ and $a \in A$ and $a' \in A$ and $b \in P$ and $b' \in P$ and $o \in C$ and $c \in C$ and $c' \in C$ and $o \neq a$ and $o \neq b$ and $o \neq c$ and $p \neq q$ and not $\mathbf{L}(b, a, c)$ and not $\mathbf{L}(b', a', c')$ and $b \neq b'$ and $a \neq a'$ and $\mathbf{L}(b, a, p)$ and $\mathbf{L}(b', a', p)$ and $\mathbf{L}(b, c, q)$ and $\mathbf{L}(b', c', q)$ and $a, c \parallel a', c'$ and $a, c \parallel p, q$ holds $o \in P$.

Let us consider AP. We say that AP satisfies **DES2** if and only if:

Given A, P, C, a, a', b, b', c, c', p, q. Suppose that

(i) A is a line,
(ii) P is a line,
(iii) C is a line,
(iv) $A \neq P$,
(v) $A \neq C$,
(vi) $P \neq C$,
(vii) $a \in A$,
(viii) $a' \in A$,
(ix) $b \in P$,
(x) $b' \in P$,
(xi) $c \in C$,
(xii) $c' \in C$,
(xiii) $A \parallel P$,
(xiv) $A \parallel C$,
(xv) not $\mathbf{L}(b, a, c)$,
(xvi) not $\mathbf{L}(b', a', c')$,
(xvii) $p \neq q$,
(xviii) $a \neq a'$,
(xix) $\mathbf{L}(b, a, p)$,
(xx) $\mathbf{L}(b', a', p)$,
(xxi) $\mathbf{L}(b, c, q)$,
(xxii) $\mathbf{L}(b', c', q)$,
(xxiii) $a, c \parallel a', c'$.

Then $a, c \parallel p, q$.

We now state the proposition

(5) Given AP. Then AP satisfies **DES2** if and only if for all A, P, C, a, a', b, b', c, c', p, q such that A is a line and P is a line and C is a line and $A \neq P$ and $A \neq C$ and $P \neq C$ and $a \in A$ and $a' \in A$ and $b \in P$ and $b' \in P$ and $c \in C$ and $c' \in C$ and $A \parallel P$ and $A \parallel C$ and not $\mathbf{L}(b, a, c)$ and not $\mathbf{L}(b', a', c')$ and $p \neq q$ and $a \neq a'$ and $\mathbf{L}(b, a, p)$ and $\mathbf{L}(b', a', p)$ and $\mathbf{L}(b, c, q)$ and $\mathbf{L}(b', c', q)$ and $a, c \parallel a', c'$ holds $a, c \parallel p, q$.

Let us consider AP. We say that AP satisfies $\mathbf{DES2_1}$ if and only if:

Given A, P, C, a, a', b, b', c, c', p, q. Suppose that

(i) A is a line,
(ii) P is a line,

(iii) C is a line,
(iv) $A \neq P$,
(v) $A \neq C$,
(vi) $P \neq C$,
(vii) $a \in A$,
(viii) $a' \in A$,
(ix) $b \in P$,
(x) $b' \in P$,
(xi) $c \in C$,
(xii) $c' \in C$,
(xiii) $A \parallel P$,
(xiv) $A \parallel C$,
(xv) not $\mathbf{L}(b, a, c)$,
(xvi) not $\mathbf{L}(b', a', c')$,
(xvii) $p \neq q$,
(xviii) $\mathbf{L}(b, a, p)$,
(xix) $\mathbf{L}(b', a', p)$,
(xx) $\mathbf{L}(b, c, q)$,
(xxi) $\mathbf{L}(b', c', q)$,
(xxii) $a, c \parallel p, q$.

Then $a, c \parallel a', c'$.

We now state the proposition

(6) Given AP. Then AP satisfies $\mathbf{DES2_1}$ if and only if for all A, P, C, a, a', b, b', c, c', p, q such that A is a line and P is a line and C is a line and $A \neq P$ and $A \neq C$ and $P \neq C$ and $a \in A$ and $a' \in A$ and $b \in P$ and $b' \in P$ and $c \in C$ and $c' \in C$ and $A \parallel P$ and $A \parallel C$ and not $\mathbf{L}(b, a, c)$ and not $\mathbf{L}(b', a', c')$ and $p \neq q$ and $\mathbf{L}(b, a, p)$ and $\mathbf{L}(b', a', p)$ and $\mathbf{L}(b, c, q)$ and $\mathbf{L}(b', c', q)$ and $a, c \parallel p, q$ holds $a, c \parallel a', c'$.

Let us consider AP. We say that AP satisfies $\mathbf{DES2_2}$ if and only if:

Given A, P, C, a, a', b, b', c, c', p, q. Suppose that

(i) A is a line,
(ii) P is a line,
(iii) C is a line,
(iv) $A \neq P$,
(v) $A \neq C$,
(vi) $P \neq C$,
(vii) $a \in A$,
(viii) $a' \in A$,
(ix) $b \in P$,
(x) $b' \in P$,
(xi) $c \in C$,
(xii) $c' \in C$,
(xiii) $A \parallel C$,
(xiv) not $\mathbf{L}(b, a, c)$,

(xv) not $\mathbf{L}(b', a', c')$,
(xvi) $p \neq q$,
(xvii) $a \neq a'$,
(xviii) $\mathbf{L}(b, a, p)$,
(xix) $\mathbf{L}(b', a', p)$,
(xx) $\mathbf{L}(b, c, q)$,
(xxi) $\mathbf{L}(b', c', q)$,
(xxii) $a, c \parallel a', c'$,
(xxiii) $a, c \parallel p, q$.

Then $A \parallel P$.

Next we state the proposition

(7) Given AP. Then AP satisfies $\mathbf{DES2_2}$ if and only if for all A, P, C, a, a', b, b', c, c', p, q such that A is a line and P is a line and C is a line and $A \neq P$ and $A \neq C$ and $P \neq C$ and $a \in A$ and $a' \in A$ and $b \in P$ and $b' \in P$ and $c \in C$ and $c' \in C$ and $A \parallel C$ and not $\mathbf{L}(b, a, c)$ and not $\mathbf{L}(b', a', c')$ and $p \neq q$ and $a \neq a'$ and $\mathbf{L}(b, a, p)$ and $\mathbf{L}(b', a', p)$ and $\mathbf{L}(b, c, q)$ and $\mathbf{L}(b', c', q)$ and $a, c \parallel a', c'$ and $a, c \parallel p, q$ holds $A \parallel P$.

Let us consider AP. We say that AP satisfies $\mathbf{DES2_3}$ if and only if:

Given A, P, C, a, a', b, b', c, c', p, q. Suppose that

(i) A is a line,
(ii) P is a line,
(iii) C is a line,
(iv) $A \neq P$,
(v) $A \neq C$,
(vi) $P \neq C$,
(vii) $a \in A$,
(viii) $a' \in A$,
(ix) $b \in P$,
(x) $b' \in P$,
(xi) $c \in C$,
(xii) $c' \in C$,
(xiii) $A \parallel P$,
(xiv) not $\mathbf{L}(b, a, c)$,
(xv) not $\mathbf{L}(b', a', c')$,
(xvi) $p \neq q$,
(xvii) $c \neq c'$,
(xviii) $\mathbf{L}(b, a, p)$,
(xix) $\mathbf{L}(b', a', p)$,
(xx) $\mathbf{L}(b, c, q)$,
(xxi) $\mathbf{L}(b', c', q)$,
(xxii) $a, c \parallel a', c'$,
(xxiii) $a, c \parallel p, q$.

Then $A \parallel C$.

We now state a number of propositions:

(8) Given AP. Then AP satisfies $\mathbf{DES2_3}$ if and only if for all A, P, C, a, a', b, b', c, c', p, q such that A is a line and P is a line and C is a line and $A \neq P$ and $A \neq C$ and $P \neq C$ and $a \in A$ and $a' \in A$ and $b \in P$ and $b' \in P$ and $c \in C$ and $c' \in C$ and $A \parallel P$ and not $\mathbf{L}(b, a, c)$ and not $\mathbf{L}(b', a', c')$ and $p \neq q$ and $c \neq c'$ and $\mathbf{L}(b, a, p)$ and $\mathbf{L}(b', a', p)$ and $\mathbf{L}(b, c, q)$ and $\mathbf{L}(b', c', q)$ and $a, c \parallel a', c'$ and $a, c \parallel p, q$ holds $A \parallel C$.

(9) If AP satisfies **DES1**, then AP satisfies $\mathbf{DES1_1}$.

(10) If AP satisfies $\mathbf{DES1_1}$, then AP satisfies **DES1**.

(11) If AP satisfies **DES**, then AP satisfies **DES1**.

(12) If AP satisfies **DES**, then AP satisfies $\mathbf{DES1_2}$.

(13) If AP satisfies $\mathbf{DES1_2}$, then AP satisfies $\mathbf{DES1_3}$.

(14) If AP satisfies $\mathbf{DES1_2}$, then AP satisfies **DES**.

(15) If AP satisfies $\mathbf{DES2_1}$, then AP satisfies **DES2**.

(16) AP satisfies $\mathbf{DES2_1}$ if and only if AP satisfies $\mathbf{DES2_3}$.

(17) AP satisfies **DES2** if and only if AP satisfies $\mathbf{DES2_2}$.

(18) If AP satisfies $\mathbf{DES1_3}$, then AP satisfies $\mathbf{DES2_1}$.

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Binary Operations Applied to Finite Sequences

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Summary. The article contains some propositions and theorems related to [7] and [4]. The notions introduced in [7] are extended to finite sequences. A number additional propositions related to this notions are proved. There are also proved some properties of distributive operations and unary operations. The notation and propositions for inverses are introduced.

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The notation and terminology used in this paper are introduced in the following articles: [9], [1], [5], [3], [2], [6], [7], [4], and [8]. For simplicity we adopt the following convention: x, y will be arbitrary, C, C', D, D', E will be non-empty sets, c will be an element of C, c' will be an element of C', d, d_1, d_2, d_3, d_4, e will be elements of D, and d' will be an element of D'. Next we state several propositions:

(1) For every function f holds $\langle \square, f \rangle = \square$ and $\langle f, \square \rangle = \square$.

(2) For every function f holds $[\!:\square, f:\!] = \square$ and $[\!:f, \square:\!] = \square$.

(3) $(C \longmapsto d)(c) = d$.

(4) For all functions F, f holds $F^\circ(\square, f) = \square$ and $F^\circ(f, \square) = \square$.

(5) For every function F holds $F^\circ(\square, x) = \square$.

(6) For every function F holds $F^\circ(x, \square) = \square$.

(7) For every set X and for arbitrary x_1, x_2 holds $\langle X \longmapsto x_1, X \longmapsto x_2 \rangle = X \longmapsto \langle x_1, x_2 \rangle$.

(8) For every function F and for every set X and for arbitrary x_1, x_2 such that $\langle x_1, x_2 \rangle \in \operatorname{dom} F$ holds $F^\circ(X \longmapsto x_1, X \longmapsto x_2) = X \longmapsto F(\langle x_1, x_2 \rangle)$.

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For simplicity we adopt the following rules: i, j will denote natural numbers, F will denote a function from $[: D, D' :]$ into E, p, q will denote finite sequences of elements of D, and p', q' will denote finite sequences of elements of D'. Let us consider D, D', E, F, p, p'. Then $F^\circ(p, p')$ is a finite sequence of elements of E.

Let us consider D, D', E, F, p, d'. Then $F^\circ(p, d')$ is a finite sequence of elements of E.

Let us consider D, D', E, F, d, p'. Then $F^\circ(d, p')$ is a finite sequence of elements of E.

Let us consider D, i, d. Then $i \longmapsto d$ is an element of D^i.

In the sequel f, f' are functions from C into D and h is a function from D into E. Let us consider D, E, p, h. Then $h \cdot p$ is a finite sequence of elements of E.

Next we state two propositions:

(9) $h \cdot (p \frown \langle d \rangle) = (h \cdot p) \frown \langle h(d) \rangle$.

(10) $h \cdot (p \frown q) = (h \cdot p) \frown (h \cdot q)$.

For simplicity we follow a convention: T, T_1, T_2, T_3 denote elements of D^i, T' denotes an element of D'^i, S denotes an element of D^j, and S' denotes an element of D'^j. Next we state a number of propositions:

(11) $F^\circ(T \frown \langle d \rangle, T' \frown \langle d' \rangle) = F^\circ(T, T') \frown \langle F(d, d') \rangle$.

(12) $F^\circ(T \frown S, T' \frown S') = F^\circ(T, T') \frown F^\circ(S, S')$.

(13) $F^\circ(d, p' \frown \langle d' \rangle) = F^\circ(d, p') \frown \langle F(d, d') \rangle$.

(14) $F^\circ(d, p' \frown q') = F^\circ(d, p') \frown F^\circ(d, q')$.

(15) $F^\circ(p \frown \langle d \rangle, d') = F^\circ(p, d') \frown \langle F(d, d') \rangle$.

(16) $F^\circ(p \frown q, d') = F^\circ(p, d') \frown F^\circ(q, d')$.

(17) For every function h from D into E holds $h \cdot (i \longmapsto d) = i \longmapsto h(d)$.

(18) $F^\circ(i \longmapsto d, i \longmapsto d') = i \longmapsto F(d, d')$.

(19) $F^\circ(d, i \longmapsto d') = i \longmapsto F(d, d')$.

(20) $F^\circ(i \longmapsto d, d') = i \longmapsto F(d, d')$.

(21) $F^\circ(i \longmapsto d, T') = F^\circ(d, T')$.

(22) $F^\circ(T, i \longmapsto d) = F^\circ(T, d)$.

(23) $F^\circ(d, T') = F^\circ(d, \mathrm{id}_{D'}) \cdot T'$.

(24) $F^\circ(T, d) = F^\circ(\mathrm{id}_D, d) \cdot T$.

In the sequel F, G are binary operations on D, u is a unary operation on D, and H is a binary operation on E. One can prove the following propositions:

(25) If F is associative, then $F^\circ(d, \mathrm{id}_D) \cdot F^\circ(f, f') = F^\circ(F^\circ(d, \mathrm{id}_D) \cdot f, f')$.

(26) If F is associative, then $F^\circ(\mathrm{id}_D, d) \cdot F^\circ(f, f') = F^\circ(f, F^\circ(\mathrm{id}_D, d) \cdot f')$.

(27) If F is associative, then $F^\circ(d, \mathrm{id}_D) \cdot F^\circ(T_1, T_2) = F^\circ(F^\circ(d, \mathrm{id}_D) \cdot T_1, T_2)$.

(28) If F is associative, then $F^\circ(\mathrm{id}_D, d) \cdot F^\circ(T_1, T_2) = F^\circ(T_1, F^\circ(\mathrm{id}_D, d) \cdot T_2)$.

(29) If F is associative, then $F^\circ(F^\circ(T_1, T_2), T_3) = F^\circ(T_1, F^\circ(T_2, T_3))$.

(30) If F is associative, then $F^\circ(F^\circ(d_1, T), d_2) = F^\circ(d_1, F^\circ(T, d_2))$.

(31) If F is associative, then $F^\circ(F^\circ(T_1, d), T_2) = F^\circ(T_1, F^\circ(d, T_2))$.

(32) If F is associative, then $F^\circ(F(d_1, d_2), T) = F^\circ(d_1, F^\circ(d_2, T))$.

(33) If F is associative, then $F^\circ(T, F(d_1, d_2)) = F^\circ(F^\circ(T, d_1), d_2)$.

(34) If F is commutative, then $F^\circ(T_1, T_2) = F^\circ(T_2, T_1)$.

(35) If F is commutative, then $F^\circ(d, T) = F^\circ(T, d)$.

(36) If F is distributive w.r.t. G, then $F^\circ(G(d_1, d_2), f) = G^\circ(F^\circ(d_1, f), F^\circ(d_2, f))$.

(37) If F is distributive w.r.t. G, then $F^\circ(f, G(d_1, d_2)) = G^\circ(F^\circ(f, d_1), F^\circ(f, d_2))$.

(38) If for all d_1, d_2 holds $h(F(d_1, d_2)) = H(h(d_1), h(d_2))$, then $h \cdot F^\circ(f, f') = H^\circ(h \cdot f, h \cdot f')$.

(39) If for all d_1, d_2 holds $h(F(d_1, d_2)) = H(h(d_1), h(d_2))$, then $h \cdot F^\circ(d, f) = H^\circ(h(d), h \cdot f)$.

(40) If for all d_1, d_2 holds $h(F(d_1, d_2)) = H(h(d_1), h(d_2))$, then $h \cdot F^\circ(f, d) = H^\circ(h \cdot f, h(d))$.

(41) If u is distributive w.r.t. F, then $u \cdot F^\circ(f, f') = F^\circ(u \cdot f, u \cdot f')$.

(42) If u is distributive w.r.t. F, then $u \cdot F^\circ(d, f) = F^\circ(u(d), u \cdot f)$.

(43) If u is distributive w.r.t. F, then $u \cdot F^\circ(f, d) = F^\circ(u \cdot f, u(d))$.

(44) If F has a unity, then $F^\circ(C \longmapsto \mathbf{1}_F, f) = f$ and $F^\circ(f, C \longmapsto \mathbf{1}_F) = f$.

(45) If F has a unity, then $F^\circ(\mathbf{1}_F, f) = f$.

(46) If F has a unity, then $F^\circ(f, \mathbf{1}_F) = f$.

(47) If F is distributive w.r.t. G, then $F^\circ(G(d_1, d_2), T) = G^\circ(F^\circ(d_1, T), F^\circ(d_2, T))$.

(48) If F is distributive w.r.t. G, then $F^\circ(T, G(d_1, d_2)) = G^\circ(F^\circ(T, d_1), F^\circ(T, d_2))$.

(49) If for all d_1, d_2 holds $h(F(d_1, d_2)) = H(h(d_1), h(d_2))$, then $h \cdot F^\circ(T_1, T_2) = H^\circ(h \cdot T_1, h \cdot T_2)$.

(50) If for all d_1, d_2 holds $h(F(d_1, d_2)) = H(h(d_1), h(d_2))$, then $h \cdot F^\circ(d, T) = H^\circ(h(d), h \cdot T)$.

(51) If for all d_1, d_2 holds $h(F(d_1, d_2)) = H(h(d_1), h(d_2))$, then $h \cdot F^\circ(T, d) = H^\circ(h \cdot T, h(d))$.

(52) If u is distributive w.r.t. F, then $u \cdot F^\circ(T_1, T_2) = F^\circ(u \cdot T_1, u \cdot T_2)$.

(53) If u is distributive w.r.t. F, then $u \cdot F^\circ(d, T) = F^\circ(u(d), u \cdot T)$.

(54) If u is distributive w.r.t. F, then $u \cdot F^\circ(T, d) = F^\circ(u \cdot T, u(d))$.

(55) If G is distributive w.r.t. F and $u = G^\circ(d, \mathrm{id}_D)$, then u is distributive w.r.t. F.

(56) If G is distributive w.r.t. F and $u = G^\circ(\mathrm{id}_D, d)$, then u is distributive w.r.t. F.

(57) If F has a unity, then $F^{\circ}(i \longmapsto \mathbf{1}_F,\, T) = T$ and $F^{\circ}(T,\, i \longmapsto \mathbf{1}_F) = T$.

(58) If F has a unity, then $F^{\circ}(\mathbf{1}_F, T) = T$.

(59) If F has a unity, then $F^{\circ}(T, \mathbf{1}_F) = T$.

Let us consider D, u, F. We say that u is an inverse operation w.r.t. F if and only if:

for every d holds $F(d,\, u(d)) = \mathbf{1}_F$ and $F(u(d),\, d) = \mathbf{1}_F$.

One can prove the following proposition

(60) u is an inverse operation w.r.t. F if and only if for every d holds $F(d,\, u(d)) = \mathbf{1}_F$ and $F(u(d),\, d) = \mathbf{1}_F$.

Let us consider D, F. We say that F has an inverse operation if and only if: there exists u such that u is an inverse operation w.r.t. F.

Next we state the proposition

(61) F has an inverse operation if and only if there exists u such that u is an inverse operation w.r.t. F.

Let us consider D, F. Let us assume that F has a unity and F is associative and F has an inverse operation. The inverse operation w.r.t.F yields a unary operation on D and is defined as follows:

the inverse operation w.r.t.F is an inverse operation w.r.t. F.

We now state a number of propositions:

(62) If F has a unity and F is associative and F has an inverse operation, then for every u holds $u =$ the inverse operation w.r.t.F if and only if u is an inverse operation w.r.t. F.

(63) If F has a unity and F is associative and F has an inverse operation, then $F(($the inverse operation w.r.t.F$)(\mathrm{d}),\, \mathrm{d}) = \mathbf{1}_{\mathrm{F}}$ and $F(d,\, ($the inverse operation w.r.t.F$)(\mathrm{d})) = \mathbf{1}_{\mathrm{F}}$.

(64) If F has a unity and F is associative and F has an inverse operation and $F(d_1,\, d_2) = \mathbf{1}_F$, then $d_1 = ($the inverse operation w.r.t.F$)(\mathrm{d}_2)$ and $($the inverse operation w.r.t.F$)(\mathrm{d}_1) = \mathrm{d}_2$.

(65) If F has a unity and F is associative and F has an inverse operation, then $($the inverse operation w.r.t.F$)(\mathbf{1}_{\mathrm{F}}) = \mathbf{1}_{\mathrm{F}}$.

(66) If F has a unity and F is associative and F has an inverse operation, then $($the inverse operation w.r.t.F$)(($the inverse operation w.r.t.F$)(\mathrm{d})) = \mathrm{d}$.

(67) If F has a unity and F is associative and F is commutative and F has an inverse operation, then the inverse operation w.r.t.F is distributive w.r.t. F.

(68) If F has a unity and F is associative and F has an inverse operation but $F(d,\, d_1) = F(d,\, d_2)$ or $F(d_1,\, d) = F(d_2,\, d)$, then $d_1 = d_2$.

(69) If F has a unity and F is associative and F has an inverse operation but $F(d_1,\, d_2) = d_2$ or $F(d_2,\, d_1) = d_2$, then $d_1 = \mathbf{1}_F$.

(70) If F is associative and F has a unity and F has an inverse operation and G is distributive w.r.t. F and $e = \mathbf{1}_F$, then for every d holds $G(e,\, d) = e$ and $G(d,\, e) = e$.

(71) If F has a unity and F is associative and F has an inverse operation and $u =$ the inverse operation w.r.t.F and G is distributive w.r.t. F, then $u(G(d_1, d_2)) = G(u(d_1), d_2)$ and $u(G(d_1, d_2)) = G(d_1, u(d_2))$.

(72) If F has a unity and F is associative and F has an inverse operation and $u =$ the inverse operation w.r.t.F and G is distributive w.r.t. F and G has a unity, then $G^{\circ}(u(\mathbf{1}_G), \mathrm{id}_D) = u$.

(73) If F is associative and F has a unity and F has an inverse operation and G is distributive w.r.t. F, then $(G^{\circ}(d, \mathrm{id}_D))(\mathbf{1}_F) = \mathbf{1}_F$.

(74) If F is associative and F has a unity and F has an inverse operation and G is distributive w.r.t. F, then $(G^{\circ}(\mathrm{id}_D, d))(\mathbf{1}_F) = \mathbf{1}_F$.

(75) If F has a unity and F is associative and F has an inverse operation, then $F^{\circ}(f,$ (the inverse operation w.r.t.F) $\cdot$ f) $=$ C $\longmapsto \mathbf{1}_{\mathrm{F}}$ and F°((the inverse operation w.r.t.F) $\cdot$ f, f) $=$ C $\longmapsto \mathbf{1}_{\mathrm{F}}$.

(76) If F is associative and F has an inverse operation and F has a unity and $F^{\circ}(f, f') = C \longmapsto \mathbf{1}_F$, then $f =$ (the inverse operation w.r.t.F) $\cdot$ f′ and (the inverse operation w.r.t.F) $\cdot$ f $=$ f′.

(77) If F has a unity and F is associative and F has an inverse operation, then $F^{\circ}(T,$ (the inverse operation w.r.t.F) $\cdot$ T) $=$ i $\longmapsto \mathbf{1}_{\mathrm{F}}$ and F°((the inverse operation w.r.t.F) $\cdot$ T, T) $=$ i $\longmapsto \mathbf{1}_{\mathrm{F}}$.

(78) If F is associative and F has an inverse operation and F has a unity and $F^{\circ}(T_1, T_2) = i \longmapsto \mathbf{1}_F$, then $T_1 =$ (the inverse operation w.r.t.F) $\cdot$ T_2 and (the inverse operation w.r.t.F) $\cdot$ $\mathrm{T}_1 = \mathrm{T}_2$.

(79) If F is associative and F has a unity and $e = \mathbf{1}_F$ and F has an inverse operation and G is distributive w.r.t. F, then $G^{\circ}(e, f) = C \longmapsto e$.

(80) If F is associative and F has a unity and $e = \mathbf{1}_F$ and F has an inverse operation and G is distributive w.r.t. F, then $G^{\circ}(e, T) = i \longmapsto e$.

Let F, f, g be functions. The functor $F \circ (f, g)$ yielding a function is defined by:

$F \circ (f, g) = F \cdot [\!: f, g :\!]$.

Next we state several propositions:

(81) For all functions F, f, g holds $F \circ (f, g) = F \cdot [\!: f, g :\!]$.

(82) For all functions F, f, g such that $\langle x, y \rangle \in \mathrm{dom}(F \circ (f, g))$ holds $(F \circ (f, g))(\langle x, y \rangle) = F(\langle f(x), g(y) \rangle)$.

(83) For all functions F, f, g such that $\langle x, y \rangle \in \mathrm{dom}(F \circ (f, g))$ holds $(F \circ (f, g))(x, y) = F(f(x), g(y))$.

(84) For every function F from $[\!: D, D' :\!]$ into E and for every function f from C into D and for every function g from C' into D' holds $F \circ (f, g)$ is a function from $[\!: C, C' :\!]$ into E.

(85) For all functions u, u' from D into D holds $F \circ (u, u')$ is a binary operation on D.

Let us consider D, F, and let f, f' be functions from D into D. Then $F \circ (f, f')$ is a binary operation on D.

The following propositions are true:

(86) For every function F from $[: D, D' :]$ into E and for every function f from C into D and for every function g from C' into D' holds $(F \circ (f, g))(c, c') = F(f(c), g(c'))$.

(87) For every function u from D into D holds $(F \circ (\mathrm{id}_D, u))(d_1, d_2) = F(d_1, u(d_2))$ and $(F \circ (u, \mathrm{id}_D))(d_1, d_2) = F(u(d_1), d_2)$.

(88) $(F \circ (\mathrm{id}_D, u))^\circ(f, f') = F^\circ(f, u \cdot f')$.

(89) $(F \circ (\mathrm{id}_D, u))^\circ(T_1, T_2) = F^\circ(T_1, u \cdot T_2)$.

(90) Suppose F is associative and F has a unity and F is commutative and F has an inverse operation and $u = \mathrm{the\,inverse\,operation\,w.r.t.F}$. Then $u((F \circ (\mathrm{id}_D, u))(d_1, d_2)) = (F \circ (u, \mathrm{id}_D))(d_1, d_2)$ and $(F \circ (\mathrm{id}_D, u))(d_1, d_2) = u((F \circ (u, \mathrm{id}_D))(d_1, d_2))$.

(91) If F is associative and F has a unity and F has an inverse operation, then $(F \circ (\mathrm{id}_D, \mathrm{the\,inverse\,operation\,w.r.t.F}))(\mathrm{d}, \mathrm{d}) = \mathbf{1}_{\mathrm{F}}$.

(92) If F is associative and F has a unity and F has an inverse operation, then $(F \circ (\mathrm{id}_D, \mathrm{the\,inverse\,operation\,w.r.t.F}))(\mathrm{d}, \mathbf{1}_{\mathrm{F}}) = \mathrm{d}$.

(93) If F is associative and F has a unity and F has an inverse operation and $u = \mathrm{the\,inverse\,operation\,w.r.t.F}$, then $(F \circ (\mathrm{id}_D, u))(\mathbf{1}_F, d) = u(d)$.

(94) If F is commutative and F is associative and F has a unity and F has an inverse operation and $G = F \circ (\mathrm{id}_D, \mathrm{the\,inverse\,operation\,w.r.t.F})$, then for all d_1, d_2, d_3, d_4 holds $F(G(d_1, d_2), G(d_3, d_4)) = G(F(d_1, d_3), F(d_2, d_4))$.

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Semigroup operations on finite subsets

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Summary. A continuation of [10]. The propositions and theorems proved in [10] are extended to finite sequences. Several additional theorems related to semigroup operations of functions not included in [10] are proved. The special notation for operations on finite sequences is introduced.

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The articles [11], [1], [9], [6], [2], [12], [7], [3], [13], [8], [10], [5], and [4] provide the terminology and notation for this paper. For simplicity we adopt the following rules: x will be arbitrary, C, C', D, E will denote non-empty sets, c, c_1, c_2, c_3 will denote elements of C, B, B_1, B_2 will denote elements of $\mathrm{Fin}\, C$, A will denote an element of $\mathrm{Fin}\, C'$, d, d_1, d_2, d_3, d_4, e will denote elements of D, F, G will denote binary operations on D, u will denote a unary operation on D, f, f' will denote functions from C into D, g will denote a function from C' into D, H will denote a binary operation on E, h will denote a function from D into E, i, j will denote natural numbers, s will denote a function, p, p_1, p_2, q will denote finite sequences of elements of D, and T_1, T_2 will denote elements of D^i. We now state a number of propositions:

(1) $\mathrm{Seg}\, i$ is an element of $\mathrm{Fin}\, \mathbb{N}$.

(2) $i + j \longmapsto x = (i \longmapsto x) \frown (j \longmapsto x)$.

(3) If F is commutative and F is associative and $c_1 \neq c_2$, then $F\text{-}\sum_{\{c_1,c_2\}} f = F(f(c_1), f(c_2))$.

(4) If F is commutative and F is associative but $B \neq \emptyset$ or F has a unity and $c \notin B$, then $F\text{-}\sum_{B\cup\{c\}} f = F(F\text{-}\sum_B f, f(c))$.

(5) If F is commutative and F is associative and $c_1 \neq c_2$ and $c_1 \neq c_3$ and $c_2 \neq c_3$, then $F\text{-}\sum_{\{c_1,c_2,c_3\}} f = F(F(f(c_1), f(c_2)), f(c_3))$.

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(6) If F is commutative and F is associative but $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$ or F has a unity and $B_1 \cap B_2 = \emptyset$, then $F\text{-}\sum_{B_1 \cup B_2} f = F(F\text{-}\sum_{B_1} f,\ F\text{-}\sum_{B_2} f)$.

(7) If F is commutative and F is associative but $A \neq \emptyset$ or F has a unity and there exists s such that $\operatorname{dom} s = A$ and $\operatorname{rng} s = B$ and s is one-to-one and $g \restriction A = f \cdot s$, then $F\text{-}\sum_A g = F\text{-}\sum_B f$.

(8) If H is commutative and H is associative but $B \neq \emptyset$ or H has a unity and f is one-to-one, then $H\text{-}\sum_{f^\circ B} h = H\text{-}\sum_B (h \cdot f)$.

(9) If F is commutative and F is associative but $B \neq \emptyset$ or F has a unity and $f \restriction B = f' \restriction B$, then $F\text{-}\sum_B f = F\text{-}\sum_B f'$.

(10) If F is commutative and F is associative and F has a unity and $e = \mathbf{1}_F$ and $f\ ^\circ\ B = \{e\}$, then $F\text{-}\sum_B f = e$.

(11) Suppose F is commutative and F is associative and F has a unity and $e = \mathbf{1}_F$ and $G(e, e) = e$ and for all d_1, d_2, d_3, d_4 holds $F(G(d_1, d_2), G(d_3, d_4)) = G(F(d_1, d_3), F(d_2, d_4))$. Then $G(F\text{-}\sum_B f,\ F\text{-}\sum_B f') = F\text{-}\sum_B G^\circ(f, f')$.

(12) If F is commutative and F is associative and F has a unity, then $F(F\text{-}\sum_B f,\ F\text{-}\sum_B f') = F\text{-}\sum_B F^\circ(f, f')$.

(13) If F is commutative and F is associative and F has a unity and F has an inverse operation and $G = F \circ (\mathrm{id}_D, \text{the inverse operation w.r.t.F})$, then $G(F\text{-}\sum_B f,\ F\text{-}\sum_B f') = F\text{-}\sum_B G^\circ(f, f')$.

(14) If F is commutative and F is associative and F has a unity and $e = \mathbf{1}_F$ and G is distributive w.r.t. F and $G(d, e) = e$, then $G(d,\ F\text{-}\sum_B f) = F\text{-}\sum_B (G^\circ(d, f))$.

(15) If F is commutative and F is associative and F has a unity and $e = \mathbf{1}_F$ and G is distributive w.r.t. F and $G(e, d) = e$, then $G(F\text{-}\sum_B f,\ d) = F\text{-}\sum_B (G^\circ(f, d))$.

(16) If F is commutative and F is associative and F has a unity and F has an inverse operation and G is distributive w.r.t. F, then $G(d,\ F\text{-}\sum_B f) = F\text{-}\sum_B (G^\circ(d, f))$.

(17) If F is commutative and F is associative and F has a unity and F has an inverse operation and G is distributive w.r.t. F, then $G(F\text{-}\sum_B f,\ d) = F\text{-}\sum_B (G^\circ(f, d))$.

(18) Suppose F is commutative and F is associative and F has a unity and H is commutative and H is associative and H has a unity and $h(\mathbf{1}_F) = \mathbf{1}_H$ and for all d_1, d_2 holds $h(F(d_1, d_2)) = H(h(d_1), h(d_2))$. Then $h(F\text{-}\sum_B f) = H\text{-}\sum_B (h \cdot f)$.

(19) If F is commutative and F is associative and F has a unity and $u(\mathbf{1}_F) = \mathbf{1}_F$ and u is distributive w.r.t. F, then $u(F\text{-}\sum_B f) = F\text{-}\sum_B (u \cdot f)$.

(20) If F is commutative and F is associative and F has a unity and F has an inverse operation and G is distributive w.r.t. F, then $(G^\circ(d, \mathrm{id}_D))(F\text{-}\sum_B f) = F\text{-}\sum_B (G^\circ(d, \mathrm{id}_D) \cdot f)$.

(21) If F is commutative and F is associative and F has a unity and F

has an inverse operation, then (the inverse operation w.r.t.F)(F-$\sum_{\rm B}$ f) = F-$\sum_{\rm B}$((the inverse operation w.r.t.F) $\cdot$ f).

Let us consider D, p, d. The functor $\Omega_d(p)$ yields a function from $\mathbb{N}$ into D and is defined by:

if $i \in \mathrm{Seg}(\mathrm{len}\,p)$, then $(\Omega_d(p))(i) = p(i)$ but if $i \notin \mathrm{Seg}(\mathrm{len}\,p)$, then $(\Omega_d(p))(i) = d$.

Next we state several propositions:

(22) For every function h from $\mathbb{N}$ into D holds $h = \Omega_d(p)$ if and only if for every i holds if $i \in \mathrm{Seg}(\mathrm{len}\,p)$, then $h(i) = p(i)$ but if $i \notin \mathrm{Seg}(\mathrm{len}\,p)$, then $h(i) = d$.

(23) $\Omega_d(p) \restriction \mathrm{Seg}(\mathrm{len}\,p) = p$.

(24) $\Omega_d((p \frown q)) \restriction \mathrm{Seg}(\mathrm{len}\,p) = p$.

(25) $\mathrm{rng}(\Omega_d(p)) = \mathrm{rng}\,p \cup \{d\}$.

(26) $h \cdot \Omega_d(p) = \Omega_{h(d)}((h \cdot p))$.

Let us consider i. Then $\mathrm{Seg}\,i$ is an element of $\mathrm{Fin}\,\mathbb{N}$.

Let X be a non-empty subset of $\mathbb{R}$, and let x be an element of X. Then $\{x\}$ is an element of $\mathrm{Fin}\,X$. Let y be an element of X. Then $\{x, y\}$ is an element of $\mathrm{Fin}\,X$. Let z be an element of X. Then $\{x, y, z\}$ is an element of $\mathrm{Fin}\,X$.

Let us consider D, F, p. The functor $F \circledast p$ yielding an element of D is defined by:

$F \circledast p = F\text{-}\sum_{\mathrm{Seg}(\mathrm{len}\,p)} \Omega_{\mathbf{1}_F}(p)$.

Next we state several propositions:

(27) $F \circledast p = F\text{-}\sum_{\mathrm{Seg}(\mathrm{len}\,p)} \Omega_{\mathbf{1}_F}(p)$.

(28) If F is commutative and F is associative and F has a unity, then $F \circledast \varepsilon_D = \mathbf{1}_F$.

(29) If F is commutative and F is associative, then $F \circledast \langle d \rangle = d$.

(30) If F is commutative and F is associative but $\mathrm{len}\,p \neq 0$ or F has a unity, then $F \circledast (p \frown \langle d \rangle) = F(F \circledast p,\, d)$.

(31) If F is commutative and F is associative but $\mathrm{len}\,p_1 \neq 0$ and $\mathrm{len}\,p_2 \neq 0$ or F has a unity, then $F \circledast (p_1 \frown p_2) = F(F \circledast p_1,\, F \circledast p_2)$.

(32) If F is commutative and F is associative but $\mathrm{len}\,p \neq 0$ or F has a unity, then $F \circledast (\langle d \rangle \frown p) = F(d,\, F \circledast p)$.

Let us consider D, d_1, d_2. Then $\langle d_1, d_2 \rangle$ is a finite sequence of elements of D.

One can prove the following proposition

(33) If F is commutative and F is associative, then $F \circledast \langle d_1, d_2 \rangle = F(d_1,\, d_2)$.

Let us consider D, d_1, d_2, d_3. Then $\langle d_1, d_2, d_3 \rangle$ is a finite sequence of elements of D.

We now state a number of propositions:

(34) If F is commutative and F is associative, then $F \circledast \langle d_1, d_2, d_3 \rangle = F(F(d_1,\, d_2),\, d_3)$.

(35) If F is commutative and F is associative and F has a unity and $e = \mathbf{1}_F$, then $F \circledast (i \longmapsto e) = e$.

(36) If F is commutative and F is associative, then $F \circledast (1 \longmapsto d) = d$.

(37) If F is commutative and F is associative but $i \neq 0$ and $j \neq 0$ or F has a unity, then $F \circledast (i + j \longmapsto d) = F(F \circledast (i \longmapsto d), F \circledast (j \longmapsto d))$.

(38) If F is commutative and F is associative but $i \neq 0$ and $j \neq 0$ or F has a unity, then $F \circledast (i \cdot j \longmapsto d) = F \circledast (j \longmapsto F \circledast (i \longmapsto d))$.

(39) Suppose F is commutative and F is associative and F has a unity and H is commutative and H is associative and H has a unity and $h(\mathbf{1}_F) = \mathbf{1}_H$ and for all d_1, d_2 holds $h(F(d_1, d_2)) = H(h(d_1), h(d_2))$. Then $h(F \circledast p) = H \circledast (h \cdot p)$.

(40) If F is commutative and F is associative and F has a unity and $u(\mathbf{1}_F) = \mathbf{1}_F$ and u is distributive w.r.t. F, then $u(F \circledast p) = F \circledast (u \cdot p)$.

(41) If F is commutative and F is associative and F has a unity and F has an inverse operation and G is distributive w.r.t. F, then $(G^{\circ}(d, \mathrm{id}_D))(F \circledast p) = F \circledast (G^{\circ}(d, \mathrm{id}_D) \cdot p)$.

(42) If F is commutative and F is associative and F has a unity and F has an inverse operation, then (the inverse operation w.r.t.F)(F⊛p) = F⊛((the inverse operation w.r.t.F) · p).

(43) Suppose that
- (i) F is commutative,
- (ii) F is associative,
- (iii) F has a unity,
- (iv) $e = \mathbf{1}_F$,
- (v) $G(e, e) = e$,
- (vi) for all d_1, d_2, d_3, d_4 holds $F(G(d_1, d_2), G(d_3, d_4)) = G(F(d_1, d_3), F(d_2, d_4))$,
- (vii) $\operatorname{len} p = \operatorname{len} q$.

Then $G(F \circledast p, F \circledast q) = F \circledast G^{\circ}(p, q)$.

(44) Suppose F is commutative and F is associative and F has a unity and $e = \mathbf{1}_F$ and $G(e, e) = e$ and for all d_1, d_2, d_3, d_4 holds $F(G(d_1, d_2), G(d_3, d_4)) = G(F(d_1, d_3), F(d_2, d_4))$. Then $G(F \circledast T_1, F \circledast T_2) = F \circledast G^{\circ}(T_1, T_2)$.

(45) If F is commutative and F is associative and F has a unity and $\operatorname{len} p = \operatorname{len} q$, then $F(F \circledast p, F \circledast q) = F \circledast F^{\circ}(p, q)$.

(46) If F is commutative and F is associative and F has a unity, then $F(F \circledast T_1, F \circledast T_2) = F \circledast F^{\circ}(T_1, T_2)$.

(47) If F is commutative and F is associative and F has a unity, then $F \circledast (i \longmapsto F(d_1, d_2)) = F(F \circledast (i \longmapsto d_1), F \circledast (i \longmapsto d_2))$.

(48) If F is commutative and F is associative and F has a unity and F has an inverse operation and $G = F \circ (\mathrm{id}_D, \text{the inverse operation w.r.t.F})$, then $G(F \circledast T_1, F \circledast T_2) = F \circledast G^{\circ}(T_1, T_2)$.

(49) If F is commutative and F is associative and F has a unity and $e = \mathbf{1}_F$ and G is distributive w.r.t. F and $G(d, e) = e$, then $G(d, F \circledast p) = F \circledast (G^{\circ}(d, p))$.

(50) If F is commutative and F is associative and F has a unity and $e = \mathbf{1}_F$ and G is distributive w.r.t. F and $G(e, d) = e$, then $G(F \circledast p, d) = F \circledast (G^{\circ}(p, d))$.

(51) If F is commutative and F is associative and F has a unity and F has an inverse operation and G is distributive w.r.t. F, then $G(d, F \circledast p) = F \circledast (G^{\circ}(d, p))$.

(52) If F is commutative and F is associative and F has a unity and F has an inverse operation and G is distributive w.r.t. F, then $G(F \circledast p, d) = F \circledast (G^{\circ}(p, d))$.

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The Collinearity Structure

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Summary. The text includes basic axioms and theorems concerning the collinearity structure based on Wanda Szmielew [1], pp. 18-20. Collinearity is defined as a relation on Cartesian product $[: S, S, S :]$ of set S. The basic text is preceeded with a few auxiliary theorems (e.g: ternary relation). Then come the two basic axioms of the collinearity structure: A1.1.1 and A1.1.2 and a few theorems. Another axiom: Aks dim, which states that there exist at least 3 non-collinear points, excludes the trivial structures (i.e. pairs $\langle S, [: S, S, S :]\rangle$). Following it the notion of a line is included and several additional theorems are appended.

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The articles [3], and [2] provide the notation and terminology for this paper. In the sequel R, X will denote sets. Let us consider X. A set is said to be a ternary relation on X if:

it $\subseteq [: X, X, X :]$.

Next we state two propositions:

(1) R is a ternary relation on X if and only if $R \subseteq [: X, X, X :]$.

(2) $X = \emptyset$ or there exists arbitrary a such that $\{a\} = X$ or there exist arbitrary a, b such that $a \neq b$ and $a \in X$ and $b \in X$.

We consider collinearity structures which are systems

$\langle$points, a collinearity relation$\rangle$,

where the points constitute a non-empty set and the collinearity relation is a ternary relation on the points. In the sequel CS is a collinearity structure. Let us consider CS. A point of CS is an element of the points of CS.

In the sequel a, b, c denote points of CS. Let us consider CS, a, b, c. We say that a, b and c are collinear if and only if:

$\langle a, b, c\rangle \in$ the collinearity relation of CS.

The following proposition is true

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(5)[2] a, b and c are collinear if and only if $\langle a, b, c\rangle \in$ the collinearity relation of CS.

A collinearity structure is said to be a collinearity space if:

Let a, b, c, p, q, r be points of it . Then

(i) if $a = b$ or $a = c$ or $b = c$, then $\langle a, b, c\rangle \in$ the collinearity relation of it,

(ii) if $a \neq b$ and $\langle a, b, p\rangle \in$ the collinearity relation of it and $\langle a, b, q\rangle \in$ the collinearity relation of it and $\langle a, b, r\rangle \in$ the collinearity relation of it, then $\langle p, q, r\rangle \in$ the collinearity relation of it.

Next we state the proposition

(6) CS is a collinearity space if and only if for all points a, b, c, p, q, r of CS holds if $a = b$ or $a = c$ or $b = c$, then $\langle a, b, c\rangle \in$ the collinearity relation of CS but if $a \neq b$ and $\langle a, b, p\rangle \in$ the collinearity relation of CS and $\langle a, b, q\rangle \in$ the collinearity relation of CS and $\langle a, b, r\rangle \in$ the collinearity relation of CS, then $\langle p, q, r\rangle \in$ the collinearity relation of CS.

We adopt the following rules: $CLSP$ is a collinearity space and a, b, c, d, p, q, r are points of $CLSP$. We now state several propositions:

(7) If $a = b$ or $a = c$ or $b = c$, then a, b and c are collinear.

(8) If $a \neq b$ and a, b and p are collinear and a, b and q are collinear and a, b and r are collinear, then p, q and r are collinear.

(9) If a, b and c are collinear, then b, a and c are collinear and a, c and b are collinear.

(10) a, b and a are collinear.

(11) If $a \neq b$ and a, b and c are collinear and a, b and d are collinear, then a, c and d are collinear.

(12) If a, b and c are collinear, then b, a and c are collinear.

(13) If a, b and c are collinear, then b, c and a are collinear.

(14) If $p \neq q$ and a, b and p are collinear and a, b and q are collinear and p, q and r are collinear, then a, b and r are collinear.

Let us consider $CLSP$, a, b. The functor $\mathrm{Line}(a, b)$ yields a set and is defined as follows:

$\mathrm{Line}(a, b) = \{p : a, b$ and p are collinear $\}$.

One can prove the following propositions:

(15) $\mathrm{Line}(a, b) = \{p : a, b$ and p are collinear $\}$.

(16) $a \in \mathrm{Line}(a, b)$ and $b \in \mathrm{Line}(a, b)$.

(17) a, b and r are collinear if and only if $r \in \mathrm{Line}(a, b)$.

A collinearity space is said to be a proper collinearity space if:

there exist points a, b, c of it such that a, b and c are not collinear.

The following proposition is true

(18) $CLSP$ is a proper collinearity space if and only if there exist a, b, c such that a, b and c are not collinear.

[2]The propositions (3)–(4) became obvious.

We follow a convention: $CLSP$ will be a proper collinearity space and a, b, p, q, r will be points of $CLSP$. We now state the proposition

(19) For all p, q such that $p \neq q$ there exists r such that p, q and r are not collinear.

Let us consider $CLSP$. A set is called a line of $CLSP$ if:
there exist a, b such that $a \neq b$ and it $= \mathrm{Line}(a, b)$.

The following propositions are true:

(20) For every set X holds X is a line of $CLSP$ if and only if there exist a, b such that $a \neq b$ and $X = \mathrm{Line}(a, b)$.

(21) If $a \neq b$, then $\mathrm{Line}(a, b)$ is a line of $CLSP$.

In the sequel P, Q are lines of $CLSP$. The following propositions are true:

(22) If $a = b$, then $\mathrm{Line}(a, b) =$ the points of $CLSP$.

(23) For every P there exist a, b such that $a \neq b$ and $a \in P$ and $b \in P$.

(24) If $a \neq b$, then there exists P such that $a \in P$ and $b \in P$.

(25) If $p \in P$ and $q \in P$ and $r \in P$, then p, q and r are collinear.

(26) If $P \subseteq Q$, then $P = Q$.

(27) If $p \neq q$ and $p \in P$ and $q \in P$, then $\mathrm{Line}(p, q) \subseteq P$.

(28) If $p \neq q$ and $p \in P$ and $q \in P$, then $\mathrm{Line}(p, q) = P$.

(29) If $p \neq q$ and $p \in P$ and $q \in P$ and $p \in Q$ and $q \in Q$, then $P = Q$.

(30) $P = Q$ or $P \cap Q = \emptyset$ or there exists p such that $P \cap Q = \{p\}$.

(31) If $a \neq b$, then $\mathrm{Line}(a, b) \neq$ the points of $CLSP$.

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The Sum and Product of Finite Sequences of Real Numbers

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Summary. Some operations on the set of n-tuples of real numbers are introduced. Addition, difference of such n-tuples, complement of a n-tuple and multiplication of these by real numbers are defined. In these definitions more general properties of binary operations applied to finite sequences from [3] are used. Then the fact that certain properties are satisfied by those operations is demonstrated directly from [3]. Moreover some properties can be recognized as being those of real vector space. Multiplication of n-tuples of real numbers and square power of n-tuple of real numbers using for notation of some properties of finite sums and products of real numbers are defined, followed by definitions of the finite sum and product of n-tuples of real numbers using notions and properties introduced in [7]. A number of propositions and theorems on sum and product of finite sequences of real numbers are proved. As a additional properties there are proved some properties of real numbers and set representations of binary operations on real numbers.

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The papers [8], [12], [5], [6], [1], [2], [13], [10], [9], [11], [4], [3], and [7] provide the terminology and notation for this paper. For simplicity we follow the rules: i, j, k are natural numbers, r, r', r_1, r_2, r_3 are real numbers, x is an element of $\mathbb{R}$, F, F_1, F_2 are finite sequences of elements of $\mathbb{R}$, and R, R_1, R_2, R_3 are elements of $\mathbb{R}^i$. Next we state the proposition

(1) $-(r_1 + r_2) = (-r_1) + (-r_2)$.

Let us consider x. The functor $@x$ yields a real number and is defined by: $@x = x$.

The following propositions are true:

(2) $@x = x$.

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(3) 0 is a unity w.r.t. $+_{\mathbb{R}}$.

(4) $\mathbf{1}_{+_{\mathbb{R}}} = 0$.

(5) $+_{\mathbb{R}}$ has a unity.

(6) $+_{\mathbb{R}}$ is commutative.

(7) $+_{\mathbb{R}}$ is associative.

The binary operation $-_{\mathbb{R}}$ on $\mathbb{R}$ is defined as follows:
$-_{\mathbb{R}} = +_{\mathbb{R}} \circ (\mathrm{id}_{\mathbb{R}}, -_{\mathbb{R}})$.

We now state two propositions:

(8) $-_{\mathbb{R}} = +_{\mathbb{R}} \circ (\mathrm{id}_{\mathbb{R}}, -_{\mathbb{R}})$.

(9) $-_{\mathbb{R}}(r_1, r_2) = r_1 - r_2$.

The unary operation $\mathrm{sqr}_{\mathbb{R}}$ on $\mathbb{R}$ is defined as follows:
for every r holds $\mathrm{sqr}_{\mathbb{R}}(r) = r^{\mathbf{2}}$.

The following propositions are true:

(10) For every unary operation u on $\mathbb{R}$ holds $u = \mathrm{sqr}_{\mathbb{R}}$ if and only if for every r holds $u(r) = r^{\mathbf{2}}$.

(11) $\cdot_{\mathbb{R}}$ is commutative.

(12) $\cdot_{\mathbb{R}}$ is associative.

(13) 1 is a unity w.r.t. $\cdot_{\mathbb{R}}$.

(14) $\mathbf{1}_{\cdot_{\mathbb{R}}} = 1$.

(15) $\cdot_{\mathbb{R}}$ has a unity.

(16) $\cdot_{\mathbb{R}}$ is distributive w.r.t. $+_{\mathbb{R}}$.

(17) $\mathrm{sqr}_{\mathbb{R}}$ is distributive w.r.t. $\cdot_{\mathbb{R}}$.

Let us consider x. The functor $\cdot_{\mathbb{R}}^{x}$ yielding a unary operation on $\mathbb{R}$ is defined by:
$\cdot_{\mathbb{R}}^{x} = \cdot_{\mathbb{R}}{}^{\circ}(x, \mathrm{id}_{\mathbb{R}})$.

Next we state several propositions:

(18) $\cdot_{\mathbb{R}}^{x} = \cdot_{\mathbb{R}}{}^{\circ}(x, \mathrm{id}_{\mathbb{R}})$.

(19) $\cdot_{\mathbb{R}}^{r}(x) = r \cdot x$.

(20) $\cdot_{\mathbb{R}}^{r}$ is distributive w.r.t. $+_{\mathbb{R}}$.

(21) $-_{\mathbb{R}}$ is an inverse operation w.r.t. $+_{\mathbb{R}}$.

(22) $+_{\mathbb{R}}$ has an inverse operation.

(23) The inverse operation w.r.t.$+_{\mathbb{R}} = -_{\mathbb{R}}$.

(24) $-_{\mathbb{R}}$ is distributive w.r.t. $+_{\mathbb{R}}$.

Let us consider F_1, F_2. The functor $F_1 + F_2$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:
$F_1 + F_2 = +_{\mathbb{R}}{}^{\circ}(F_1, F_2)$.

We now state two propositions:

(25) $F_1 + F_2 = +_{\mathbb{R}}{}^{\circ}(F_1, F_2)$.

(26) If $i \in \mathrm{Seg}(\mathrm{len}(F_1 + F_2))$ and $r_1 = F_1(i)$ and $r_2 = F_2(i)$, then $(F_1 + F_2)(i) = r_1 + r_2$.

Let us consider i, R_1, R_2. Then $R_1 + R_2$ is an element of $\mathbb{R}^i$.

We now state several propositions:

(27) If $j \in \mathrm{Seg}\, i$ and $r_1 = R_1(j)$ and $r_2 = R_2(j)$, then $(R_1+R_2)(j) = r_1+r_2$.

(28) $\varepsilon_{\mathbb{R}} + F = \varepsilon_{\mathbb{R}}$ and $F + \varepsilon_{\mathbb{R}} = \varepsilon_{\mathbb{R}}$.

(29) $\langle r_1 \rangle + \langle r_2 \rangle = \langle r_1 + r_2 \rangle$.

(30) $(i \longmapsto r_1) + (i \longmapsto r_2) = i \longmapsto r_1 + r_2$.

(31) $R_1 + R_2 = R_2 + R_1$.

(32) $R_1 + (R_2 + R_3) = (R_1 + R_2) + R_3$.

(33) $R + (i \longmapsto (0\,\textbf{qua}\,\text{a real number})) = R$ and $R = (i \longmapsto (0\,\textbf{qua}\,\text{a real number})) + R$.

Let us consider F. The functor $-F$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

$-F = -_{\mathbb{R}} \cdot F$.

We now state two propositions:

(34) $-F = -_{\mathbb{R}} \cdot F$.

(35) If $i \in \mathrm{Seg}(\mathrm{len}(-F))$ and $r = F(i)$, then $(-F)(i) = -r$.

Let us consider i, R. Then $-R$ is an element of $\mathbb{R}^i$.

The following propositions are true:

(36) If $j \in \mathrm{Seg}\, i$ and $r = R(j)$, then $(-R)(j) = -r$.

(37) $-\varepsilon_{\mathbb{R}} = \varepsilon_{\mathbb{R}}$.

(38) $-\langle r \rangle = \langle -r \rangle$.

(39) $-(i \longmapsto r) = i \longmapsto -r$.

(40) $R + (-R) = i \longmapsto 0$ and $(-R) + R = i \longmapsto 0$.

(41) If $R_1 + R_2 = i \longmapsto 0$, then $R_1 = -R_2$ and $R_2 = -R_1$.

(42) $-(-R) = R$.

(43) If $-R_1 = -R_2$, then $R_1 = R_2$.

(44) If $R_1 + R = R_2 + R$ or $R_1 + R = R + R_2$, then $R_1 = R_2$.

(45) $-(R_1 + R_2) = (-R_1) + (-R_2)$.

Let us consider F_1, F_2. The functor $F_1 - F_2$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

$F_1 - F_2 = -_{\mathbb{R}}{}^\circ(F_1, F_2)$.

The following two propositions are true:

(46) $F_1 - F_2 = -_{\mathbb{R}}{}^\circ(F_1, F_2)$.

(47) If $i \in \mathrm{Seg}(\mathrm{len}(F_1 - F_2))$ and $r_1 = F_1(i)$ and $r_2 = F_2(i)$, then $(F_1 - F_2)(i) = r_1 - r_2$.

Let us consider i, R_1, R_2. Then $R_1 - R_2$ is an element of $\mathbb{R}^i$.

One can prove the following propositions:

(48) If $j \in \mathrm{Seg}\, i$ and $r_1 = R_1(j)$ and $r_2 = R_2(j)$, then $(R_1-R_2)(j) = r_1-r_2$.

(49) $\varepsilon_{\mathbb{R}} - F = \varepsilon_{\mathbb{R}}$ and $F - \varepsilon_{\mathbb{R}} = \varepsilon_{\mathbb{R}}$.

(50) $\langle r_1 \rangle - \langle r_2 \rangle = \langle r_1 - r_2 \rangle$.

(51) $(i \longmapsto r_1) - (i \longmapsto r_2) = i \longmapsto r_1 - r_2$.

(52) $R_1 - R_2 = R_1 + (-R_2)$.

(53) $R - (i \longmapsto (0 \textbf{ qua } \text{a real number})) = R$.

(54) $(i \longmapsto (0 \textbf{ qua } \text{a real number})) - R = -R$.

(55) $R_1 - (-R_2) = R_1 + R_2$.

(56) $-(R_1 - R_2) = R_2 - R_1$.

(57) $-(R_1 - R_2) = (-R_1) + R_2$.

(58) $R - R = i \longmapsto 0$.

(59) If $R_1 - R_2 = i \longmapsto 0$, then $R_1 = R_2$.

(60) $(R_1 - R_2) - R_3 = R_1 - (R_2 + R_3)$.

(61) $R_1 + (R_2 - R_3) = (R_1 + R_2) - R_3$.

(62) $R_1 - (R_2 - R_3) = (R_1 - R_2) + R_3$.

(63) $R_1 = (R_1 + R) - R$.

(64) $R_1 = (R_1 - R) + R$.

Let us consider r, F. The functor $r \cdot F$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

$r \cdot F = \cdot_{\mathbb{R}}^{r} \cdot F$.

We now state two propositions:

(65) $r \cdot F = \cdot_{\mathbb{R}}^{r} \cdot F$.

(66) If $i \in \mathrm{Seg}(\mathrm{len}(r \cdot F))$ and $r' = F(i)$, then $(r \cdot F)(i) = r \cdot r'$.

Let us consider i, r, R. Then $r \cdot R$ is an element of $\mathbb{R}^i$.

Next we state a number of propositions:

(67) If $j \in \mathrm{Seg}\, i$ and $r' = R(j)$, then $(r \cdot R)(j) = r \cdot r'$.

(68) $r \cdot \varepsilon_{\mathbb{R}} = \varepsilon_{\mathbb{R}}$.

(69) $r \cdot \langle r_1 \rangle = \langle r \cdot r_1 \rangle$.

(70) $r_1 \cdot (i \longmapsto r_2) = i \longmapsto r_1 \cdot r_2$.

(71) $(r_1 \cdot r_2) \cdot R = r_1 \cdot (r_2 \cdot R)$.

(72) $(r_1 + r_2) \cdot R = r_1 \cdot R + r_2 \cdot R$.

(73) $r \cdot (R_1 + R_2) = r \cdot R_1 + r \cdot R_2$.

(74) $1 \cdot R = R$.

(75) $0 \cdot R = i \longmapsto 0$.

(76) $(-1) \cdot R = -R$.

Let us consider F. The functor ${}^2F$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

${}^2F = \mathrm{sqr}_{\mathbb{R}} \cdot F$.

Next we state two propositions:

(77) ${}^2F = \mathrm{sqr}_{\mathbb{R}} \cdot F$.

(78) If $i \in \mathrm{Seg}(\mathrm{len}({}^2F))$ and $r = F(i)$, then ${}^2F(i) = r^{\mathbf{2}}$.

Let us consider i, R. Then ${}^2R$ is an element of $\mathbb{R}^i$.

Next we state several propositions:

(79) If $j \in \mathrm{Seg}\, i$ and $r = R(j)$, then ${}^2R(j) = r^{\mathbf{2}}$.

(80) ${}^2\varepsilon_{\mathbb{R}} = \varepsilon_{\mathbb{R}}$.

(81) ${}^2\langle r \rangle = \langle r^{\mathbf{2}} \rangle$.

(82) ${}^2(i \longmapsto r) = i \longmapsto r^{\mathbf{2}}$.

(83) ${}^2(-R) = {}^2R$.

(84) ${}^2(r \cdot R) = r^{\mathbf{2}} \cdot {}^2R$.

Let us consider F_1, F_2. The functor $F_1 \bullet F_2$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

$F_1 \bullet F_2 = \cdot_{\mathbb{R}}{}^{\circ}(F_1, F_2)$.

One can prove the following two propositions:

(85) $F_1 \bullet F_2 = \cdot_{\mathbb{R}}{}^{\circ}(F_1, F_2)$.

(86) If $i \in \mathrm{Seg}(\mathrm{len}(F_1 \bullet F_2))$ and $r_1 = F_1(i)$ and $r_2 = F_2(i)$, then $F_1 \bullet F_2(i) = r_1 \cdot r_2$.

Let us consider i, R_1, R_2. Then $R_1 \bullet R_2$ is an element of $\mathbb{R}^i$.

The following propositions are true:

(87) If $j \in \mathrm{Seg}\, i$ and $r_1 = R_1(j)$ and $r_2 = R_2(j)$, then $R_1 \bullet R_2(j) = r_1 \cdot r_2$.

(88) $\varepsilon_{\mathbb{R}} \bullet F = \varepsilon_{\mathbb{R}}$ and $F \bullet \varepsilon_{\mathbb{R}} = \varepsilon_{\mathbb{R}}$.

(89) $\langle r_1 \rangle \bullet \langle r_2 \rangle = \langle r_1 \cdot r_2 \rangle$.

(90) $R_1 \bullet R_2 = R_2 \bullet R_1$.

(91) $R_1 \bullet (R_2 \bullet R_3) = (R_1 \bullet R_2) \bullet R_3$.

(92) $(i \longmapsto r) \bullet R = r \cdot R$ and $R \bullet (i \longmapsto r) = r \cdot R$.

(93) $(i \longmapsto r_1) \bullet (i \longmapsto r_2) = i \longmapsto r_1 \cdot r_2$.

(94) $r \cdot R_1 \bullet R_2 = (r \cdot R_1) \bullet R_2$.

(95) $r \cdot R_1 \bullet R_2 = (r \cdot R_1) \bullet R_2$ and $r \cdot R_1 \bullet R_2 = R_1 \bullet (r \cdot R_2)$.

(96) $r \cdot R = (i \longmapsto r) \bullet R$.

(97) ${}^2R = R \bullet R$.

(98) ${}^2(R_1 + R_2) = ({}^2R_1 + 2 \cdot R_1 \bullet R_2) + {}^2R_2$.

(99) ${}^2(R_1 - R_2) = ({}^2R_1 - 2 \cdot R_1 \bullet R_2) + {}^2R_2$.

(100) ${}^2(R_1 \bullet R_2) = ({}^2R_1) \bullet ({}^2R_2)$.

Let F be a finite sequence of elements of $\mathbb{R}$. The functor $\sum F$ yields a real number and is defined by:

$\sum F = +_{\mathbb{R}} \circledast F$.

One can prove the following propositions:

(101) $\sum F = +_{\mathbb{R}} \circledast F$.

(102) $\sum \varepsilon_{\mathbb{R}} = 0$.

(103) $\sum \langle r \rangle = r$.

(104) $\sum (F \frown \langle r \rangle) = \sum F + r$.

(105) $\sum(F_1 \frown F_2) = \sum F_1 + \sum F_2$.

(106) $\sum(\langle r \rangle \frown F) = r + \sum F$.

(107) $\sum\langle r_1, r_2 \rangle = r_1 + r_2$.

(108) $\sum\langle r_1, r_2, r_3 \rangle = (r_1 + r_2) + r_3$.

(109) For every element R of $\mathbb{R}^0$ holds $\sum R = 0$.

(110) $\sum(i \longmapsto r) = i \cdot r$.

(111) $\sum(i \longmapsto (0\,\mathbf{qua}\,\text{a real number})) = 0$.

(112) If for all j, r_1, r_2 such that $j \in \operatorname{Seg} i$ and $r_1 = R_1(j)$ and $r_2 = R_2(j)$ holds $r_1 \leq r_2$, then $\sum R_1 \leq \sum R_2$.

(113) Suppose for all j, r_1, r_2 such that $j \in \operatorname{Seg} i$ and $r_1 = R_1(j)$ and $r_2 = R_2(j)$ holds $r_1 \leq r_2$ and there exist j, r_1, r_2 such that $j \in \operatorname{Seg} i$ and $r_1 = R_1(j)$ and $r_2 = R_2(j)$ and $r_1 < r_2$. Then $\sum R_1 < \sum R_2$.

(114) If for all i, r such that $i \in \operatorname{Seg}(\operatorname{len} F)$ and $r = F(i)$ holds $0 \leq r$, then $0 \leq \sum F$.

(115) If for all i, r such that $i \in \operatorname{Seg}(\operatorname{len} F)$ and $r = F(i)$ holds $0 \leq r$ and there exist i, r such that $i \in \operatorname{Seg}(\operatorname{len} F)$ and $r = F(i)$ and $0 < r$, then $0 < \sum F$.

(116) $0 \leq \sum({}^2F)$.

(117) $\sum(r \cdot F) = r \cdot \sum F$.

(118) $\sum(-F) = -\sum F$.

(119) $\sum(R_1 + R_2) = \sum R_1 + \sum R_2$.

(120) $\sum(R_1 - R_2) = \sum R_1 - \sum R_2$.

(121) If $\sum({}^2R) = 0$, then $R = i \longmapsto 0$.

(122) $(\sum(R_1 \bullet R_2))^{\mathbf{2}} \leq \sum({}^2R_1) \cdot \sum({}^2R_2)$.

Let F be a finite sequence of elements of $\mathbb{R}$. The functor $\prod F$ yields a real number and is defined as follows:

$\prod F = \cdot_{\mathbb{R}} \circledast F$.

Next we state a number of propositions:

(123) $\prod F = \cdot_{\mathbb{R}} \circledast F$.

(124) $\prod \varepsilon_{\mathbb{R}} = 1$.

(125) $\prod\langle r \rangle = r$.

(126) $\prod(F \frown \langle r \rangle) = \prod F \cdot r$.

(127) $\prod(F_1 \frown F_2) = \prod F_1 \cdot \prod F_2$.

(128) $\prod(\langle r \rangle \frown F) = r \cdot \prod F$.

(129) $\prod\langle r_1, r_2 \rangle = r_1 \cdot r_2$.

(130) $\prod\langle r_1, r_2, r_3 \rangle = (r_1 \cdot r_2) \cdot r_3$.

(131) For every element R of $\mathbb{R}^0$ holds $\prod R = 1$.

(132) $\prod(i \longmapsto (1\,\mathbf{qua}\,\text{a real number})) = 1$.

(133) There exists k such that $k \in \operatorname{Seg}(\operatorname{len} F)$ and $F(k) = 0$ if and only if $\prod F = 0$.

(134) $\prod(i+j \longmapsto r) = \prod(i \longmapsto r) \cdot \prod(j \longmapsto r)$.

(135) $\prod(i \cdot j \longmapsto r) = \prod(j \longmapsto \prod(i \longmapsto r))$.

(136) $\prod(i \longmapsto r_1 \cdot r_2) = \prod(i \longmapsto r_1) \cdot \prod(i \longmapsto r_2)$.

(137) $\prod(R_1 \bullet R_2) = \prod R_1 \cdot \prod R_2$.

(138) $\prod(r \cdot R) = \prod(i \longmapsto r) \cdot \prod R$.

(139) $\prod({}^2R) = (\prod R)^{\mathbf{2}}$.

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A Classical First Order Language

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Summary. The aim is to construct a language for the classical predicate calculus. The language is defined as a subset of the language constructed in [8]. Well-formed formulas of this language are defined and some usual connectives and quantifiers of [8,1] are accordingly. We prove inductive and definitional schemes for formulas of our language. Substitution for individual variables in formulas of the introduced language is defined. This definition is borrowed from [7]. For such purpose some auxiliary notation and propositions are introduced.

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The articles [10], [3], [4], [5], [9], [2], [8], [1], and [6] provide the notation and terminology for this paper. In the sequel i, j, k will denote natural numbers. One can prove the following proposition

(1) For every non-empty set D and for every finite sequence l of elements of D such that $k \in \mathrm{Seg}(\mathrm{len}\, l)$ holds $l(k) \in D$.

Let x, y, a, b be arbitrary. The functor $(x = y \to a, b)$ is defined as follows: $(x = y \to a, b) = a$ if $x = y$, $(x = y \to a, b) = b$, otherwise.

One can prove the following propositions:

(2) For arbitrary x, y, a, b such that $x = y$ holds $(x = y \to a, b) = a$.

(3) For arbitrary x, y, a, b such that $x \neq y$ holds $(x = y \to a, b) = b$.

Let x, y be arbitrary. The functor $x \dot{\longmapsto} y$ yields a function and is defined as follows:

$x \dot{\longmapsto} y = \{x\} \longmapsto y$.

One can prove the following three propositions:

(4) For arbitrary x, y holds $x \dot{\longmapsto} y = \{x\} \longmapsto y$.

(5) For arbitrary x, y holds $\mathrm{dom}(x \dot{\longmapsto} y) = \{x\}$ and $\mathrm{rng}(x \dot{\longmapsto} y) = \{y\}$.

(6) For arbitrary x, y holds $(x \dot{\longmapsto} y)(x) = y$.

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For simplicity we follow the rules: x, y are bound variables, a is a free variable, p, q are elements of WFF, l, ll are finite sequences of elements of Var, and P is a predicate symbol. Let F be a function from WFF into WFF, and let us consider p. Then $F(p)$ is an element of WFF.

One can prove the following proposition

(7) For an arbitrary x holds $x \in \mathrm{Var}$ if and only if $x \in \mathrm{FixedVar}$ or $x \in \mathrm{FreeVar}$ or $x \in \mathrm{BoundVar}$.

A substitution is a partial function from FreeVar to Var.

In the sequel f will be a substitution. Let us consider l, f. The functor $l[f]$ yielding a finite sequence of elements of Var is defined as follows:

$\mathrm{len}(l[f]) = \mathrm{len}\, l$ and for every k such that $1 \leq k$ and $k \leq \mathrm{len}\, l$ holds if $l(k) \in \mathrm{dom}\, f$, then $(l[f])(k) = f(l(k))$ but if $l(k) \notin \mathrm{dom}\, f$, then $(l[f])(k) = l(k)$.

The following proposition is true

(9)[2] $ll = l[f]$ if and only if the following conditions are satisfied:

(i) $\mathrm{len}\, ll = \mathrm{len}\, l$,

(ii) for every k such that $1 \leq k$ and $k \leq \mathrm{len}\, l$ holds if $l(k) \in \mathrm{dom}\, f$, then $ll(k) = f(l(k))$ but if $l(k) \notin \mathrm{dom}\, f$, then $ll(k) = l(k)$.

Let us consider k, and let l be a list of variables of the length k, and let us consider f. Then $l[f]$ is a list of variables of the length k.

One can prove the following proposition

(10) $a \dot{\longmapsto} x$ is a substitution.

Let us consider a, x. Then $a \dot{\longmapsto} x$ is a substitution.

We now state the proposition

(11) If $f = a \dot{\longmapsto} x$ and $ll = l[f]$ and $1 \leq k$ and $k \leq \mathrm{len}\, l$, then if $l(k) = a$, then $ll(k) = x$ but if $l(k) \neq a$, then $ll(k) = l(k)$.

Let A be a non-empty subset of WFF. We see that it makes sense to consider the following mode for restricted scopes of arguments. Then all the objests of the mode element of A are a formula.

The non-empty subset $\mathrm{WFF}_{\mathrm{CQC}}$ of WFF is defined as follows:

$\mathrm{WFF}_{\mathrm{CQC}} = \{s : \mathrm{Fixed}\, s = \emptyset \wedge \mathrm{Free}\, s = \emptyset\}$.

The following propositions are true:

(12) $\mathrm{WFF}_{\mathrm{CQC}} = \{s : \mathrm{Fixed}\, s = \emptyset \wedge \mathrm{Free}\, s = \emptyset\}$.

(13) p is an element of $\mathrm{WFF}_{\mathrm{CQC}}$ if and only if $\mathrm{Fixed}\, p = \emptyset$ and $\mathrm{Free}\, p = \emptyset$.

Let us consider k. A list of variables of the length k is said to be a variables list of k if:

$\{\mathrm{it}(i) : 1 \leq i \wedge i \leq \mathrm{len}\,\mathrm{it}\} \subseteq \mathrm{BoundVar}$.

One can prove the following propositions:

(14) For every list of variables l of the length k holds l is a variables list of k if and only if $\{l(i) : 1 \leq i \wedge i \leq \mathrm{len}\, l\} \subseteq \mathrm{BoundVar}$.

[2] The proposition (8) became obvious.

(15) Let l be a list of variables of the length k. Then l is a variables list of k if and only if $\{l(i) : 1 \leq i \wedge i \leq \operatorname{len} l \wedge l(i) \in \mathrm{FreeVar}\} = \emptyset$ and $\{l(j) : 1 \leq j \wedge j \leq \operatorname{len} l \wedge l(j) \in \mathrm{FixedVar}\} = \emptyset$.

In the sequel r, s denote elements of $\mathrm{WFF}_{\mathrm{CQC}}$. Next we state two propositions:

(16) VERUM is an element of $\mathrm{WFF}_{\mathrm{CQC}}$.

(17) Let P be a k-ary predicate symbol. Let l be a list of variables of the length k. Then $P[l]$ is an element of $\mathrm{WFF}_{\mathrm{CQC}}$ if and only if $\{l(i) : 1 \leq i \wedge i \leq \operatorname{len} l \wedge l(i) \in \mathrm{FreeVar}\} = \emptyset$ and $\{l(j) : 1 \leq j \wedge j \leq \operatorname{len} l \wedge l(j) \in \mathrm{FixedVar}\} = \emptyset$.

Let us consider k, and let P be a k-ary predicate symbol, and let l be a variables list of k. Then $P[l]$ is an element of $\mathrm{WFF}_{\mathrm{CQC}}$.

We now state two propositions:

(18) $\neg p$ is an element of $\mathrm{WFF}_{\mathrm{CQC}}$ if and only if p is an element of $\mathrm{WFF}_{\mathrm{CQC}}$.

(19) $p \wedge q$ is an element of $\mathrm{WFF}_{\mathrm{CQC}}$ if and only if p is an element of $\mathrm{WFF}_{\mathrm{CQC}}$ and q is an element of $\mathrm{WFF}_{\mathrm{CQC}}$.

Let us note that it makes sense to consider the following constant. Then VERUM is an element of $\mathrm{WFF}_{\mathrm{CQC}}$. Let us consider r. Then $\neg r$ is an element of $\mathrm{WFF}_{\mathrm{CQC}}$. Let us consider s. Then $r \wedge s$ is an element of $\mathrm{WFF}_{\mathrm{CQC}}$.

One can prove the following three propositions:

(20) $r \Rightarrow s$ is an element of $\mathrm{WFF}_{\mathrm{CQC}}$.

(21) $r \vee s$ is an element of $\mathrm{WFF}_{\mathrm{CQC}}$.

(22) $r \Leftrightarrow s$ is an element of $\mathrm{WFF}_{\mathrm{CQC}}$.

Let us consider r, s. Then $r \Rightarrow s$ is an element of $\mathrm{WFF}_{\mathrm{CQC}}$. Then $r \vee s$ is an element of $\mathrm{WFF}_{\mathrm{CQC}}$. Then $r \Leftrightarrow s$ is an element of $\mathrm{WFF}_{\mathrm{CQC}}$.

We now state the proposition

(23) $\forall_x p$ is an element of $\mathrm{WFF}_{\mathrm{CQC}}$ if and only if p is an element of $\mathrm{WFF}_{\mathrm{CQC}}$.

Let us consider x, r. Then $\forall_x r$ is an element of $\mathrm{WFF}_{\mathrm{CQC}}$.

We now state the proposition

(24) $\exists_x r$ is an element of $\mathrm{WFF}_{\mathrm{CQC}}$.

Let us consider x, r. Then $\exists_x r$ is an element of $\mathrm{WFF}_{\mathrm{CQC}}$.

Let D be a non-empty set, and let F be a function from $\mathrm{WFF}_{\mathrm{CQC}}$ into D, and let us consider r. Then $F(r)$ is an element of D.

In this article we present several logical schemes. The scheme *CQC_Ind* concerns a unary predicate $\mathcal{P}$, and states that:

for every r holds $\mathcal{P}[r]$

provided the parameter satisfies the following condition:

- for all r, s, x, k and for every variables list l of k and for every k-ary predicate symbol P holds $\mathcal{P}[\mathrm{VERUM}]$ and $\mathcal{P}[P[l]]$ but if $\mathcal{P}[r]$, then $\mathcal{P}[\neg r]$ but if $\mathcal{P}[r]$ and $\mathcal{P}[s]$, then $\mathcal{P}[r \wedge s]$ but if $\mathcal{P}[r]$, then $\mathcal{P}[\forall_x r]$.

The scheme *CQC_Func_Ex* concerns a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a ternary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$ and states that:

there exists a function F from $\mathrm{WFF}_{\mathrm{CQC}}$ into $\mathcal{A}$ such that for all r, s, x, k and for every variables list l of k and for every k-ary predicate symbol P and for all elements r', s' of $\mathcal{A}$ such that $r' = F(r)$ and $s' = F(s)$ holds $F(\mathrm{VERUM}) = \mathcal{B}$ and $F(P[l]) = \mathcal{F}(k, P, l)$ and $F(\neg r) = \mathcal{G}(r')$ and $F(r \wedge s) = \mathcal{H}(r', s')$ and $F(\forall_x r) = \mathcal{I}(x, r')$
for all values of the parameters.

The scheme *CQC_Func_Uniq* concerns a non-empty set $\mathcal{A}$, a function $\mathcal{B}$ from $\mathrm{WFF}_{\mathrm{CQC}}$ into $\mathcal{A}$, a function $\mathcal{C}$ from $\mathrm{WFF}_{\mathrm{CQC}}$ into $\mathcal{A}$, an element $\mathcal{D}$ of $\mathcal{A}$, a ternary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{B} = \mathcal{C}$
provided the parameters satisfy the following conditions:

- Given r, s, x, k. Let l be a variables list of k. Let P be a k-ary predicate symbol. Let r', s' be elements of $\mathcal{A}$. Suppose $r' = \mathcal{B}(r)$ and $s' = \mathcal{B}(s)$. Then $\mathcal{B}(\mathrm{VERUM}) = \mathcal{D}$ and $\mathcal{B}(P[l]) = \mathcal{F}(k, P, l)$ and $\mathcal{B}(\neg r) = \mathcal{G}(r')$ and $\mathcal{B}(r \wedge s) = \mathcal{H}(r', s')$ and $\mathcal{B}(\forall_x r) = \mathcal{I}(x, r')$,
- Given r, s, x, k. Let l be a variables list of k. Let P be a k-ary predicate symbol. Let r', s' be elements of $\mathcal{A}$. Suppose $r' = \mathcal{C}(r)$ and $s' = \mathcal{C}(s)$. Then $\mathcal{C}(\mathrm{VERUM}) = \mathcal{D}$ and $\mathcal{C}(P[l]) = \mathcal{F}(k, P, l)$ and $\mathcal{C}(\neg r) = \mathcal{G}(r')$ and $\mathcal{C}(r \wedge s) = \mathcal{H}(r', s')$ and $\mathcal{C}(\forall_x r) = \mathcal{I}(x, r')$.

The scheme *CQC_Def_correctn* concerns a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathrm{WFF}_{\mathrm{CQC}}$, an element $\mathcal{C}$ of $\mathcal{A}$, a ternary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$ and states that:

(i) there exists an element d of $\mathcal{A}$ and there exists a function F from $\mathrm{WFF}_{\mathrm{CQC}}$ into $\mathcal{A}$ such that $d = F(\mathcal{B})$ and for all r, s, x, k and for every variables list l of k and for every k-ary predicate symbol P and for all elements r', s' of $\mathcal{A}$ such that $r' = F(r)$ and $s' = F(s)$ holds $F(\mathrm{VERUM}) = \mathcal{C}$ and $F(P[l]) = \mathcal{F}(k, P, l)$ and $F(\neg r) = \mathcal{G}(r')$ and $F(r \wedge s) = \mathcal{H}(r', s')$ and $F(\forall_x r) = \mathcal{I}(x, r')$,

(ii) for all elements d_1, d_2 of $\mathcal{A}$ such that there exists a function F from $\mathrm{WFF}_{\mathrm{CQC}}$ into $\mathcal{A}$ such that $d_1 = F(\mathcal{B})$ and for all r, s, x, k and for every variables list l of k and for every k-ary predicate symbol P and for all elements r', s' of $\mathcal{A}$ such that $r' = F(r)$ and $s' = F(s)$ holds $F(\mathrm{VERUM}) = \mathcal{C}$ and $F(P[l]) = \mathcal{F}(k, P, l)$ and $F(\neg r) = \mathcal{G}(r')$ and $F(r \wedge s) = \mathcal{H}(r', s')$ and $F(\forall_x r) = \mathcal{I}(x, r')$ and there exists a function F from $\mathrm{WFF}_{\mathrm{CQC}}$ into $\mathcal{A}$ such that $d_2 = F(\mathcal{B})$ and for all r, s, x, k and for every variables list l of k and for every k-ary predicate symbol P and for all elements r', s' of $\mathcal{A}$ such that $r' = F(r)$ and $s' = F(s)$ holds $F(\mathrm{VERUM}) = \mathcal{C}$ and $F(P[l]) = \mathcal{F}(k, P, l)$ and $F(\neg r) = \mathcal{G}(r')$ and $F(r \wedge s) = \mathcal{H}(r', s')$ and $F(\forall_x r) = \mathcal{I}(x, r')$ holds $d_1 = d_2$
for all values of the parameters.

The scheme *CQC_Def_VERUM* concerns a non-empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{F}(\mathrm{VERUM}) = \mathcal{B}$

provided the parameters satisfy the following condition:

- Let p be an element of $\mathrm{WFF}_{\mathrm{CQC}}$. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{F}(p)$ if and only if there exists a function F from $\mathrm{WFF}_{\mathrm{CQC}}$ into $\mathcal{A}$ such that $d = F(p)$ and for all r, s, x, k and for every variables list l of k and for every k-ary predicate symbol P and for all elements r', s' of $\mathcal{A}$ such that $r' = F(r)$ and $s' = F(s)$ holds $F(\mathrm{VERUM}) = \mathcal{B}$ and $F(P[l]) = \mathcal{G}(k, P, l)$ and $F(\neg r) = \mathcal{H}(r')$ and $F(r \wedge s) = \mathcal{I}(r', s')$ and $F(\forall_x r) = \mathcal{J}(x, r')$.

The scheme *CQC_Def_atomic* concerns a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a natural number $\mathcal{C}$, a $\mathcal{C}$-ary predicate symbol $\mathcal{D}$, a variables list $\mathcal{E}$ of $\mathcal{C}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{F}(\mathcal{D}[\mathcal{E}]) = \mathcal{G}(\mathcal{C}, \mathcal{D}, \mathcal{E})$

provided the following requirement is met:

- Let p be an element of $\mathrm{WFF}_{\mathrm{CQC}}$. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{F}(p)$ if and only if there exists a function F from $\mathrm{WFF}_{\mathrm{CQC}}$ into $\mathcal{A}$ such that $d = F(p)$ and for all r, s, x, k and for every variables list l of k and for every k-ary predicate symbol P and for all elements r', s' of $\mathcal{A}$ such that $r' = F(r)$ and $s' = F(s)$ holds $F(\mathrm{VERUM}) = \mathcal{B}$ and $F(P[l]) = \mathcal{G}(k, P, l)$ and $F(\neg r) = \mathcal{H}(r')$ and $F(r \wedge s) = \mathcal{I}(r', s')$ and $F(\forall_x r) = \mathcal{J}(x, r')$.

The scheme *CQC_Def_negative* deals with a non-empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, an element $\mathcal{C}$ of $\mathrm{WFF}_{\mathrm{CQC}}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{F}(\neg\mathcal{C}) = \mathcal{H}(\mathcal{F}(\mathcal{C}))$

provided the parameters satisfy the following condition:

- Let p be an element of $\mathrm{WFF}_{\mathrm{CQC}}$. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{F}(p)$ if and only if there exists a function F from $\mathrm{WFF}_{\mathrm{CQC}}$ into $\mathcal{A}$ such that $d = F(p)$ and for all r, s, x, k and for every variables list l of k and for every k-ary predicate symbol P and for all elements r', s' of $\mathcal{A}$ such that $r' = F(r)$ and $s' = F(s)$ holds $F(\mathrm{VERUM}) = \mathcal{B}$ and $F(P[l]) = \mathcal{G}(k, P, l)$ and $F(\neg r) = \mathcal{H}(r')$ and $F(r \wedge s) = \mathcal{I}(r', s')$ and $F(\forall_x r) = \mathcal{J}(x, r')$.

The scheme *QC_Def_conjuncti* concerns a non-empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, an element $\mathcal{C}$ of $\mathrm{WFF}_{\mathrm{CQC}}$, an element $\mathcal{D}$ of $\mathrm{WFF}_{\mathrm{CQC}}$, and a binary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{F}(\mathcal{C} \wedge \mathcal{D}) = \mathcal{I}(\mathcal{F}(\mathcal{C}), \mathcal{F}(\mathcal{D}))$

provided the following condition is satisfied:

- Let p be an element of $\mathrm{WFF}_{\mathrm{CQC}}$. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{F}(p)$ if and only if there exists a function F from $\mathrm{WFF}_{\mathrm{CQC}}$ into $\mathcal{A}$ such that $d = F(p)$ and for all r, s, x, k and for every variables list l of k and for every k-ary predicate symbol P and for all elements r', s' of $\mathcal{A}$ such that $r' = F(r)$ and $s' = F(s)$ holds $F(\mathrm{VERUM}) = \mathcal{B}$ and $F(P[l]) = \mathcal{G}(k, P, l)$ and $F(\neg r) = \mathcal{H}(r')$ and $F(r \wedge s) = \mathcal{I}(r', s')$ and $F(\forall_x r) = \mathcal{J}(x, r')$.

The scheme *QC_Def_universal* concerns a non-empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$, a bound variable $\mathcal{C}$, and an element $\mathcal{D}$ of $\mathrm{WFF}_{\mathrm{CQC}}$ and states that:

$\mathcal{F}(\forall_{\mathcal{C}} \mathcal{D}) = \mathcal{J}(\mathcal{C}, \mathcal{F}(\mathcal{D}))$

provided the following condition is satisfied:

- Let p be an element of $\mathrm{WFF}_{\mathrm{CQC}}$. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{F}(p)$ if and only if there exists a function F from $\mathrm{WFF}_{\mathrm{CQC}}$ into $\mathcal{A}$ such that $d = F(p)$ and for all r, s, x, k and for every variables list l of k and for every k-ary predicate symbol P and for all elements r', s' of $\mathcal{A}$ such that $r' = F(r)$ and $s' = F(s)$ holds $F(\mathrm{VERUM}) = \mathcal{B}$ and $F(P[l]) = \mathcal{G}(k, P, l)$ and $F(\neg r) = \mathcal{H}(r')$ and $F(r \wedge s) = \mathcal{I}(r', s')$ and $F(\forall_x r) = \mathcal{J}(x, r')$.

We now state the proposition

(25) If $\mathrm{Arity}(P) = \mathrm{len}\, l$, then $P[l] = \langle P \rangle \frown l$.

Let us consider x, y, p, q. Then $(x = y \to p, q)$ is an element of WFF.

Let us consider p, x. The functor $p(x)$ yields an element of WFF and is defined as follows:

there exists a function F from WFF into WFF such that $p(x) = F(p)$ and for every q holds $F(\mathrm{VERUM}) = \mathrm{VERUM}$ but if q is atomic, then $F(q) = \mathrm{PredSym}(q)[\mathrm{Args}(q)[\mathbf{a}_0 \longmapsto x]]$ but if q is negative, then $F(q) = \neg(F(\mathrm{Arg}(q)))$ but if q is conjunctive, then $F(q) = (F(\mathrm{LeftArg}(q))) \wedge (F(\mathrm{RightArg}(q)))$ but if q is universal, then $F(q) = (\mathrm{Bound}(q) = x \to q, \forall_{\mathrm{Bound}(q)}(F(\mathrm{Scope}(q))))$.

We now state a number of propositions:

(27)[3] Let r be an element of WFF. Then $r = p(x)$ if and only if there exists a function F from WFF into WFF such that $r = F(p)$ and for every q holds $F(\mathrm{VERUM}) = \mathrm{VERUM}$ but if q is atomic, then $F(q) = \mathrm{PredSym}(q)[\mathrm{Args}(q)[\mathbf{a}_0 \longmapsto x]]$ but if q is negative, then $F(q) = \neg(F(\mathrm{Arg}(q)))$ but if q is conjunctive, then $F(q) = (F(\mathrm{LeftArg}(q))) \wedge (F(\mathrm{RightArg}(q)))$ but if q is universal, then

[3]The proposition (26) became obvious.

$F(q) = (\mathrm{Bound}(q) = x \rightarrow q, \forall_{\mathrm{Bound}(q)}(F(\mathrm{Scope}(q))))$.

(28) $\mathrm{VERUM}(x) = \mathrm{VERUM}$.

(29) If p is atomic, then $p(x) = \mathrm{PredSym}(p)[\mathrm{Args}(p)[\mathbf{a}_0 \stackrel{\cdot}{\longmapsto} x]]$.

(30) For every k-ary predicate symbol P and for every list of variables l of the length k holds $(P[l])(x) = P[l[\mathbf{a}_0 \stackrel{\cdot}{\longmapsto} x]]$.

(31) If p is negative, then $p(x) = \neg(\mathrm{Arg}(p)(x))$.

(32) $\neg p(x) = \neg(p(x))$.

(33) If p is conjunctive, then $p(x) = (\mathrm{LeftArg}(p)(x)) \wedge (\mathrm{RightArg}(p)(x))$.

(34) $p \wedge q(x) = (p(x)) \wedge (q(x))$.

(35) If p is universal and $\mathrm{Bound}(p) = x$, then $p(x) = p$.

(36) If p is universal and $\mathrm{Bound}(p) \neq x$, then $p(x) = \forall_{\mathrm{Bound}(p)}(\mathrm{Scope}(p)(x))$.

(37) $\forall_x p(x) = \forall_x p$.

(38) If $x \neq y$, then $\forall_x p(y) = \forall_x (p(y))$.

(39) If $\mathrm{Free}\, p = \emptyset$, then $p(x) = p$.

(40) $r(x) = r$.

(41) $\mathrm{Fixed}(p(x)) = \mathrm{Fixed}\, p$.

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Classical and Non–classical Pasch Configurations in Ordered Affine Planes [1]

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Summary. Several configuration axioms, which are commonly called in the literature "Pasch Axioms" are introduced; three of them were investigated by Szmielew and concern invariantability of the betweenness relation under parallel projections, and two other were introduced by Tarski. It is demonstrated that they all are consequences of the trapezium axiom, adopted to characterize ordered affine spaces.

MML Identifier: PASCH.

The papers [1] and [2] provide the notation and terminology for this paper. We adopt the following rules: OAS will be an ordered affine space and a, a', b, b', c, c', d, d_1, d_2, p, p', x, y, z, t, u will be elements of the points of OAS. Let us consider OAS. We say that OAS satisfies inner invariancy of betweenness relation under parallel projections if and only if:

for all a, b, c, d, p such that not $\mathbf{L}(p,b,c)$ and $\mathbf{B}(b,p,a)$ and $\mathbf{L}(p,c,d)$ and $b,c \parallel d,a$ holds $\mathbf{B}(c,p,d)$.

We now state the proposition

(1) For every OAS holds OAS satisfies inner invariancy of betweenness relation under parallel projections if and only if for all a, b, c, d, p such that not $\mathbf{L}(p,b,c)$ and $\mathbf{B}(b,p,a)$ and $\mathbf{L}(p,c,d)$ and $b,c \parallel d,a$ holds $\mathbf{B}(c,p,d)$.

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Let us consider OAS. We say that OAS satisfies outer invariancy of betweenness relation under parallel projections if and only if:

for all a, b, c, d, p such that $\mathbf{B}(p,b,c)$ and $\mathbf{L}(p,a,d)$ and $a,b \parallel c,d$ and not $\mathbf{L}(p,a,b)$ holds $\mathbf{B}(p,a,d)$.

We now state the proposition

(2) For every OAS holds OAS satisfies outer invariancy of betweenness relation under parallel projections if and only if for all a, b, c, d, p such that $\mathbf{B}(p,b,c)$ and $\mathbf{L}(p,a,d)$ and $a,b \parallel c,d$ and not $\mathbf{L}(p,a,b)$ holds $\mathbf{B}(p,a,d)$.

Let us consider OAS. We say that OAS satisfies general invariancy of betweenness relation under parallel projections if and only if:

for all a, b, c, a', b', c' such that not $\mathbf{L}(a,b,a')$ and $a,a' \parallel b,b'$ and $a,a' \parallel c,c'$ and $\mathbf{B}(a,b,c)$ and $\mathbf{L}(a',b',c')$ holds $\mathbf{B}(a',b',c')$.

We now state the proposition

(3) For every OAS holds OAS satisfies general invariancy of betweenness relation under parallel projections if and only if for all a, b, c, a', b', c' such that not $\mathbf{L}(a,b,a')$ and $a,a' \parallel b,b'$ and $a,a' \parallel c,c'$ and $\mathbf{B}(a,b,c)$ and $\mathbf{L}(a',b',c')$ holds $\mathbf{B}(a',b',c')$.

Let us consider OAS. We say that OAS satisfies outer form of Pasch' Axiom if and only if:

for all a, b, c, d, x, y such that $\mathbf{B}(a,b,d)$ and $\mathbf{B}(b,x,c)$ and not $\mathbf{L}(a,b,c)$ there exists y such that $\mathbf{B}(a,y,c)$ and $\mathbf{B}(y,x,d)$.

The following proposition is true

(4) For every OAS holds OAS satisfies outer form of Pasch' Axiom if and only if for all a, b, c, d, x, y such that $\mathbf{B}(a,b,d)$ and $\mathbf{B}(b,x,c)$ and not $\mathbf{L}(a,b,c)$ there exists y such that $\mathbf{B}(a,y,c)$ and $\mathbf{B}(y,x,d)$.

Let us consider OAS. We say that OAS satisfies inner form of Pasch' Axiom if and only if:

for all a, b, c, d, x, y such that $\mathbf{B}(a,b,d)$ and $\mathbf{B}(a,x,c)$ and not $\mathbf{L}(a,b,c)$ there exists y such that $\mathbf{B}(b,y,c)$ and $\mathbf{B}(x,y,d)$.

The following proposition is true

(5) For every OAS holds OAS satisfies inner form of Pasch' Axiom if and only if for all a, b, c, d, x, y such that $\mathbf{B}(a,b,d)$ and $\mathbf{B}(a,x,c)$ and not $\mathbf{L}(a,b,c)$ there exists y such that $\mathbf{B}(b,y,c)$ and $\mathbf{B}(x,y,d)$.

Let us consider OAS. We say that OAS satisfies Fano Axiom if and only if:

for all a, b, c, d such that $a,b \upharpoonleft\!\upharpoonright c,d$ and $a,c \upharpoonleft\!\upharpoonright b,d$ and not $\mathbf{L}(a,b,c)$ there exists x such that $\mathbf{B}(a,x,d)$ and $\mathbf{B}(b,x,c)$.

We now state a number of propositions:

(6) For every OAS holds OAS satisfies Fano Axiom if and only if for all a, b, c, d such that $a,b \upharpoonleft\!\upharpoonright c,d$ and $a,c \upharpoonleft\!\upharpoonright b,d$ and not $\mathbf{L}(a,b,c)$ there exists x such that $\mathbf{B}(a,x,d)$ and $\mathbf{B}(b,x,c)$.

(7) If $b,p \upharpoonleft\!\upharpoonright p,c$ and $p \neq c$ and $b \neq p$, then there exists d such that $a,p \upharpoonleft\!\upharpoonright p,d$ and $a,b \parallel c,d$ and $c \neq d$ and $p \neq d$.

(8) If $p, b \upuparrows p, c$ and $p \neq c$ and $b \neq p$, then there exists d such that $p, a \upuparrows p, d$ and $a, b \parallel c, d$ and $c \neq d$.

(9) If $p, b \parallel p, c$ and $p \neq b$, then there exists d such that $p, a \parallel p, d$ and $a, b \parallel c, d$.

(10) If $z, x \parallel x, t$ and $x \neq z$, then there exists u such that $y, x \parallel x, u$ and $y, z \parallel t, u$.

(11) If not $\mathbf{L}(p, a, b)$ and $\mathbf{L}(p, b, c)$ and $\mathbf{L}(p, a, d_1)$ and $\mathbf{L}(p, a, d_2)$ and $a, b \parallel c, d_1$ and $a, b \parallel c, d_2$, then $d_1 = d_2$.

(12) If not $\mathbf{L}(a, b, c)$ and $a, b \parallel c, d_1$ and $a, b \parallel c, d_2$ and $a, c \parallel b, d_1$ and $a, c \parallel b, d_2$, then $d_1 = d_2$.

(13) If not $\mathbf{L}(p, b, c)$ and $\mathbf{B}(b, p, a)$ and $\mathbf{L}(p, c, d)$ and $b, c \parallel d, a$, then $\mathbf{B}(c, p, d)$.

(14) *OAS* satisfies inner invariancy of betweenness relation under parallel projections.

(15) If $\mathbf{B}(p, b, c)$ and $\mathbf{L}(p, a, d)$ and $a, b \parallel c, d$ and not $\mathbf{L}(p, a, b)$, then $\mathbf{B}(p, a, d)$.

(16) *OAS* satisfies outer invariancy of betweenness relation under parallel projections.

(17) If not $\mathbf{L}(a, b, a')$ and $a, a' \parallel b, b'$ and $a, a' \parallel c, c'$ and $\mathbf{B}(a, b, c)$ and $\mathbf{L}(a', b', c')$, then $\mathbf{B}(a', b', c')$.

(18) *OAS* satisfies general invariancy of betweenness relation under parallel projections.

(19) If not $\mathbf{L}(p, a, b)$ and $a, p \upuparrows p, a'$ and $b, p \upuparrows p, b'$ and $a, b \parallel a', b'$, then $a, b \upuparrows b', a'$.

(20) If not $\mathbf{L}(p, a, a')$ and $p, a \upuparrows p, b$ and $p, a' \upuparrows p, b'$ and $a, a' \parallel b, b'$, then $a, a' \upuparrows b, b'$.

(21) If not $\mathbf{L}(p, a, b)$ and $p, a \parallel b, c$ and $p, b \parallel a, c$, then $p, a \upuparrows b, c$ and $p, b \upuparrows a, c$.

(22) If $\mathbf{B}(p, c, b)$ and $c, d \upuparrows b, a$ and $p, d \upuparrows p, a$ and not $\mathbf{L}(p, a, b)$ and $p \neq c$, then $\mathbf{B}(p, d, a)$.

(23) If $\mathbf{B}(p, d, a)$ and $c, d \upuparrows b, a$ and $p, c \upuparrows p, b$ and not $\mathbf{L}(p, a, b)$ and $p \neq c$, then $\mathbf{B}(p, c, b)$.

(24) If not $\mathbf{L}(p, a, b)$ and $p, b \upuparrows p, c$ and $b, a \upuparrows c, d$ and $\mathbf{L}(a, p, d)$ and $p \neq d$, then not $\mathbf{B}(a, p, d)$.

(25) If $p, b \upuparrows p, c$ and $b \neq p$, then there exists x such that $p, a \upuparrows p, x$ and $b, a \upuparrows c, x$.

(26) If $\mathbf{B}(p, c, b)$, then there exists x such that $\mathbf{B}(p, x, a)$ and $b, a \upuparrows c, x$.

(27) If $p \neq b$ and $\mathbf{B}(p, b, c)$, then there exists x such that $\mathbf{B}(p, a, x)$ and $b, a \upuparrows c, x$.

(28) If not $\mathbf{L}(p, a, b)$ and $\mathbf{B}(p, c, b)$, then there exists x such that $\mathbf{B}(p, x, a)$ and $a, b \upuparrows x, c$.

(29) There exists x such that $a, x \upuparrows b, c$ and $a, b \upuparrows x, c$.

(30) If $a, b \upuparrows c, d$ and not $\mathbf{L}(a, b, c)$, then there exists x such that $\mathbf{B}(a, x, d)$ and $\mathbf{B}(b, x, c)$.

(31) If $a, b \upuparrows c, d$ and $a, c \upuparrows b, d$ and not $\mathbf{L}(a, b, c)$, then there exists x such that $\mathbf{B}(a, x, d)$ and $\mathbf{B}(b, x, c)$.

(32) *OAS* satisfies Fano Axiom.

(33) If $a, b \parallel c, d$ and $a, c \parallel b, d$ and not $\mathbf{L}(a, b, c)$, then there exists x such that $\mathbf{L}(x, a, d)$ and $\mathbf{L}(x, b, c)$.

(34) If $a, b \parallel c, d$ and $a, c \parallel b, d$ and not $\mathbf{L}(a, b, c)$ and $\mathbf{L}(p, a, d)$ and $\mathbf{L}(p, b, c)$, then not $\mathbf{L}(p, a, b)$.

(35) If $\mathbf{B}(a, b, d)$ and $\mathbf{B}(b, x, c)$ and not $\mathbf{L}(a, b, c)$, then there exists y such that $\mathbf{B}(a, y, c)$ and $\mathbf{B}(y, x, d)$.

(36) *OAS* satisfies outer form of Pasch' Axiom.

(37) If $\mathbf{B}(a, b, d)$ and $\mathbf{B}(a, x, c)$ and not $\mathbf{L}(a, b, c)$, then there exists y such that $\mathbf{B}(b, y, c)$ and $\mathbf{B}(x, y, d)$.

(38) *OAS* satisfies inner form of Pasch' Axiom.

(39) If $\mathbf{B}(p, a, b)$ and $p, a \upuparrows p', a'$ and not $\mathbf{L}(p, a, p')$ and $\mathbf{L}(p', a', b')$ and $p, p' \upuparrows a, a'$ and $p, p' \upuparrows b, b'$, then $\mathbf{B}(p', a', b')$.

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The Lattice of Real Numbers. The Lattice of Real Functions

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Summary. A proof of the fact , that $\langle\mathbb{R}, \max, \min\rangle$ is a lattice (real lattice). Some basic properties (real lattice is distributive and modular) of it are proved. The same is done for the set $\mathbb{R}^A$ with operations: max(f(A)) and min(f(A)), where $\mathbb{R}^A$ means the set of all functions from A (being non-empty set) to $\mathbb{R}$, f is just such a function.

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The articles [4], [1], [3], and [2] provide the terminology and notation for this paper. In the sequel x, y will denote real numbers. Let x be an element of $\mathbb{R}$. The functor $@x$ yielding a real number is defined by:

$@x = x$.

We now state the proposition

(1) For every element x of $\mathbb{R}$ holds $@x = x$.

We now define two new functors. The binary operation $\min_{\mathbb{R}}$ on $\mathbb{R}$ is defined by:

$\min_{\mathbb{R}}(x, y) = \min(x, y)$.

The binary operation $\max_{\mathbb{R}}$ on $\mathbb{R}$ is defined by:

$\max_{\mathbb{R}}(x, y) = \max(x, y)$.

The following propositions are true:

(2) $\min_{\mathbb{R}}(x, y) = \min(x, y)$.

(3) $\max_{\mathbb{R}}(x, y) = \max(x, y)$.

In the sequel p, q will denote elements of the carrier of $\langle\mathbb{R}, \max_{\mathbb{R}}, \min_{\mathbb{R}}\rangle$. Let x be an element of the carrier of $\langle\mathbb{R}, \max_{\mathbb{R}}, \min_{\mathbb{R}}\rangle$. The functor $@x$ yields a real number and is defined by:

$@x = x$.

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Next we state three propositions:

(4) For every element x of the carrier of $\langle \mathbb{R}, \max_{\mathbb{R}}, \min_{\mathbb{R}} \rangle$ holds $@x = x$.

(5) $p \sqcup q = \max_{\mathbb{R}}(p, q)$.

(6) $p \sqcap q = \min_{\mathbb{R}}(p, q)$.

The lattice $\mathbb{R}_{\mathrm{L}}$ is defined as follows:

$\mathbb{R}_{\mathrm{L}} = \langle \mathbb{R}, \max_{\mathbb{R}}, \min_{\mathbb{R}} \rangle$.

One can prove the following proposition

(7) $\mathbb{R}_{\mathrm{L}} = \langle \mathbb{R}, \max_{\mathbb{R}}, \min_{\mathbb{R}} \rangle$.

In the sequel p, q, r denote elements of the carrier of $\mathbb{R}_{\mathrm{L}}$. One can prove the following propositions:

(8) $\max_{\mathbb{R}}(p, q) = \max_{\mathbb{R}}(q, p)$.

(9) $\min_{\mathbb{R}}(p, q) = \min_{\mathbb{R}}(q, p)$.

(10) (i) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(q, r), p)$,

(ii) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(p, q), r)$,

(iii) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(q, p), r)$,

(iv) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(r, p), q)$,

(v) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(r, q), p)$,

(vi) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(p, r), q)$.

(11) (i) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(q, r), p)$,

(ii) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(p, q), r)$,

(iii) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(q, p), r)$,

(iv) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(r, p), q)$,

(v) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(r, q), p)$,

(vi) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(p, r), q)$.

(12) $\max_{\mathbb{R}}(\min_{\mathbb{R}}(p, q), q) = q$ and $\max_{\mathbb{R}}(q, \min_{\mathbb{R}}(p, q)) = q$ and $\max_{\mathbb{R}}(q, \min_{\mathbb{R}}(q, p)) = q$ and $\max_{\mathbb{R}}(\min_{\mathbb{R}}(q, p), q) = q$.

(13) $\min_{\mathbb{R}}(q, \max_{\mathbb{R}}(q, p)) = q$ and $\min_{\mathbb{R}}(\max_{\mathbb{R}}(p, q), q) = q$ and $\min_{\mathbb{R}}(q, \max_{\mathbb{R}}(p, q)) = q$ and $\min_{\mathbb{R}}(\max_{\mathbb{R}}(q, p), q) = q$.

(14) $\min_{\mathbb{R}}(q, \max_{\mathbb{R}}(p, r)) = \max_{\mathbb{R}}(\min_{\mathbb{R}}(q, p), \min_{\mathbb{R}}(q, r))$.

(15) $\mathbb{R}_{\mathrm{L}}$ is a distributive lattice.

In the sequel L will be a distributive lattice. We now state the proposition

(16) L is a modular lattice.

In the sequel A will denote a non-empty set and f, g, h will denote elements of $\mathbb{R}^A$. Let A be a non-empty set, and let x be an element of $\mathbb{R}^A$. The functor $@x$ yielding an element of $\mathbb{R}^A$ **qua** a non-empty set is defined as follows:

$@x = x$.

We now state the proposition

(17) For every element f of $\mathbb{R}^A$ holds $@f = f$.

We now define two new functors. Let us consider A. The functor $\max_{\mathbb{R}^A}$ yielding a binary operation on $\mathbb{R}^A$ is defined by:

$\max_{\mathbb{R}^A}(f, g) = \max_{\mathbb{R}} {}^\circ(f, g)$.

The functor $\min_{\mathbb{R}^A}$ yields a binary operation on $\mathbb{R}^A$ and is defined as follows:
$\min_{\mathbb{R}^A}(f, g) = \min_{\mathbb{R}}{}^\circ(f, g)$.

Next we state a number of propositions:

(18) $\max_{\mathbb{R}^A}(f, g) = \max_{\mathbb{R}}{}^\circ(f, g)$.

(19) $\min_{\mathbb{R}^A}(f, g) = \min_{\mathbb{R}}{}^\circ(f, g)$.

(20) $\max_{\mathbb{R}^A}(f, g) = \max_{\mathbb{R}^A}(g, f)$.

(21) $\min_{\mathbb{R}^A}(f, g) = \min_{\mathbb{R}^A}(g, f)$.

(22) $\max_{\mathbb{R}^A}(\max_{\mathbb{R}^A}(f, g), h) = \max_{\mathbb{R}^A}(f, \max_{\mathbb{R}^A}(g, h))$.

(23) $\min_{\mathbb{R}^A}(\min_{\mathbb{R}^A}(f, g), h) = \min_{\mathbb{R}^A}(f, \min_{\mathbb{R}^A}(g, h))$.

(24) $\max_{\mathbb{R}^A}(f, \min_{\mathbb{R}^A}(f, g)) = f$.

(25) $\max_{\mathbb{R}^A}(\min_{\mathbb{R}^A}(f, g), f) = f$.

(26) $\max_{\mathbb{R}^A}(\min_{\mathbb{R}^A}(g, f), f) = f$.

(27) $\max_{\mathbb{R}^A}(f, \min_{\mathbb{R}^A}(g, f)) = f$.

(28) $\min_{\mathbb{R}^A}(f, \max_{\mathbb{R}^A}(f, g)) = f$.

(29) $\min_{\mathbb{R}^A}(f, \max_{\mathbb{R}^A}(g, f)) = f$.

(30) $\min_{\mathbb{R}^A}(\max_{\mathbb{R}^A}(g, f), f) = f$.

(31) $\min_{\mathbb{R}^A}(\max_{\mathbb{R}^A}(f, g), f) = f$.

(32) $\min_{\mathbb{R}^A}(f, \max_{\mathbb{R}^A}(g, h)) = \max_{\mathbb{R}^A}(\min_{\mathbb{R}^A}(f, g), \min_{\mathbb{R}^A}(f, h))$.

We now define two new functors. Let us consider A. The functor $\mathbf{max}_{\mathbb{R}^A}$ yields a binary operation on $\mathbb{R}^A$ and is defined by:
$\mathbf{max}_{\mathbb{R}^A}(f, g) = \max_{\mathbb{R}^A}(f, g)$.
The functor $\mathbf{min}_{\mathbb{R}^A}$ yields a binary operation on $\mathbb{R}^A$ and is defined as follows:
$\mathbf{min}_{\mathbb{R}^A}(f, g) = \min_{\mathbb{R}^A}(f, g)$.

The following two propositions are true:

(33) $\mathbf{max}_{\mathbb{R}^A}(f, g) = \max_{\mathbb{R}^A}(f, g)$.

(34) $\mathbf{min}_{\mathbb{R}^A}(f, g) = \min_{\mathbb{R}^A}(f, g)$.

In the sequel p, q are elements of the carrier of $\langle \mathbb{R}^A, \mathbf{max}_{\mathbb{R}^A}, \mathbf{min}_{\mathbb{R}^A} \rangle$. Let us consider A, and let x be an element of the carrier of $\langle \mathbb{R}^A, \mathbf{max}_{\mathbb{R}^A}, \mathbf{min}_{\mathbb{R}^A} \rangle$. The functor $@x$ yields an element of $\mathbb{R}^A$ and is defined as follows:
$@x = x$.

The following propositions are true:

(35) $p \sqcup q = \max_{\mathbb{R}^A}(p, q)$.

(36) $p \sqcup q = \mathbf{max}_{\mathbb{R}^A}(p, q)$.

(37) $p \sqcap q = \min_{\mathbb{R}^A}(p, q)$.

(38) $p \sqcap q = \mathbf{min}_{\mathbb{R}^A}(p, q)$.

Let us consider A. The functor $\mathbb{R}^A_{\mathrm{L}}$ yields a lattice and is defined by:
$\mathbb{R}^A_{\mathrm{L}} = \langle \mathbb{R}^A, \mathbf{max}_{\mathbb{R}^A}, \mathbf{min}_{\mathbb{R}^A} \rangle$.

One can prove the following proposition

(39) $\mathbb{R}^A_{\mathrm{L}} = \langle \mathbb{R}^A, \mathbf{max}_{\mathbb{R}^A}, \mathbf{min}_{\mathbb{R}^A} \rangle$.

In the sequel p, q, r will denote elements of the carrier of $\mathbb{R}^A_{\mathrm{L}}$. We now state several propositions:

(40) $\mathbf{max}_{\mathbb{R}^A}(p, q) = \mathbf{max}_{\mathbb{R}^A}(q, p)$.

(41) $\mathbf{min}_{\mathbb{R}^A}(p, q) = \mathbf{min}_{\mathbb{R}^A}(q, p)$.

(42) (i) $\mathbf{max}_{\mathbb{R}^A}(p, \mathbf{max}_{\mathbb{R}^A}(q, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(q, r), p)$,

(ii) $\mathbf{max}_{\mathbb{R}^A}(p, \mathbf{max}_{\mathbb{R}^A}(q, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(p, q), r)$,

(iii) $\mathbf{max}_{\mathbb{R}^A}(p, \mathbf{max}_{\mathbb{R}^A}(q, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(q, p), r)$,

(iv) $\mathbf{max}_{\mathbb{R}^A}(p, \mathbf{max}_{\mathbb{R}^A}(q, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(r, p), q)$,

(v) $\mathbf{max}_{\mathbb{R}^A}(p, \mathbf{max}_{\mathbb{R}^A}(q, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(r, q), p)$,

(vi) $\mathbf{max}_{\mathbb{R}^A}(p, \mathbf{max}_{\mathbb{R}^A}(q, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(p, r), q)$.

(43) (i) $\mathbf{min}_{\mathbb{R}^A}(p, \mathbf{min}_{\mathbb{R}^A}(q, r)) = \mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(q, r), p)$,

(ii) $\mathbf{min}_{\mathbb{R}^A}(p, \mathbf{min}_{\mathbb{R}^A}(q, r)) = \mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(p, q), r)$,

(iii) $\mathbf{min}_{\mathbb{R}^A}(p, \mathbf{min}_{\mathbb{R}^A}(q, r)) = \mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(q, p), r)$,

(iv) $\mathbf{min}_{\mathbb{R}^A}(p, \mathbf{min}_{\mathbb{R}^A}(q, r)) = \mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(r, p), q)$,

(v) $\mathbf{min}_{\mathbb{R}^A}(p, \mathbf{min}_{\mathbb{R}^A}(q, r)) = \mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(r, q), p)$,

(vi) $\mathbf{min}_{\mathbb{R}^A}(p, \mathbf{min}_{\mathbb{R}^A}(q, r)) = \mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(p, r), q)$.

(44) $\mathbf{max}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(p, q), q) = q$ and $\mathbf{max}_{\mathbb{R}^A}(q, \mathbf{min}_{\mathbb{R}^A}(p, q)) = q$ and $\mathbf{max}_{\mathbb{R}^A}(q, \mathbf{min}_{\mathbb{R}^A}(q, p)) = q$ and $\mathbf{max}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(q, p), q) = q$.

(45) $\mathbf{min}_{\mathbb{R}^A}(q, \mathbf{max}_{\mathbb{R}^A}(q, p)) = q$ and $\mathbf{min}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(p, q), q) = q$ and $\mathbf{min}_{\mathbb{R}^A}(q, \mathbf{max}_{\mathbb{R}^A}(p, q)) = q$ and $\mathbf{min}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(q, p), q) = q$.

(46) $\mathbf{min}_{\mathbb{R}^A}(q, \mathbf{max}_{\mathbb{R}^A}(p, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(q, p), \mathbf{min}_{\mathbb{R}^A}(q, r))$.

(47) $\mathbb{R}^A_{\mathrm{L}}$ is a distributive lattice.

In the sequel F will denote a distributive lattice. We now state the proposition

(48) F is a modular lattice.

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A Construction of an Abstract Space of Congruence of Vectors [1]

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Summary. In the class of abelian groups a subclass of two-divisible-groups is singled out, and in the latter, a subclass of uniquely-two-divisible-groups. With every such a group a special geometrical structure, more precisely the structure of "congruence of vectors" is correlated. The notion of "affine vector space" (denoted by AffVect) is introduced. This term is defined by means of suitable axiom system. It is proved that every structure of the congruence of vectors determined by a non trivial uniquely two divisible group is a affine vector space.

MML Identifier: TDGROUP.

The articles [5], [1], [4], [2], and [3] provide the notation and terminology for this paper. In the sequel AG denotes an Abelian group and G denotes a group structure. One can prove the following propositions:

(1) $\mathbb{R}_{\rm G}$ is an Abelian group.

(2) If $G = \mathbb{R}_{\rm G}$, then for every element a of the carrier of G there exists an element b of the carrier of G such that (the addition of G)(b, b) $= a$.

(3) If $G = \mathbb{R}_{\rm G}$, then for every element a of the carrier of G such that (the addition of G)(a, a) $= 0_G$ holds $a = 0_G$.

An Abelian group is called a 2-divisible group if:

for every element a of the carrier of it there exists an element b of the carrier of it such that (the addition of it)(b, b) $= a$.

The following two propositions are true:

(4) For every AG holds AG is a 2-divisible group if and only if for every element a of the carrier of AG there exists an element b of the carrier of AG such that (the addition of AG)(b, b) $= a$.

(5) $\mathbb{R}_{\rm G}$ is a 2-divisible group.

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A 2-divisible group is said to be a uniquely 2-divisible group if:

for every element a of the carrier of it such that (the addition of it)$(a, a) = 0_{\rm it}$ holds $a = 0_{\rm it}$.

One can prove the following three propositions:

(6) For every 2-divisible group AG holds AG is a uniquely 2-divisible group if and only if for every element a of the carrier of AG such that (the addition of AG)$(a, a) = 0_{AG}$ holds $a = 0_{AG}$.

(7) For every AG holds AG is a uniquely 2-divisible group if and only if for every element a of the carrier of AG there exists an element b of the carrier of AG such that (the addition of AG)$(b, b) = a$ and for every element a of the carrier of AG such that (the addition of AG)$(a, a) = 0_{AG}$ holds $a = 0_{AG}$.

(8) $\mathbb{R}_{\rm G}$ is a uniquely 2-divisible group.

We adopt the following rules: ADG is a uniquely 2-divisible group and a, b, c, d, a', b', c', p, q are elements of the carrier of ADG. Let us consider ADG, a, b. The functor $a \# b$ yielding an element of the carrier of ADG is defined as follows:

$a \# b =$ (the addition of ADG)(a, b).

Let us consider ADG. The functor ${\rm Congr}_{ADG}$ yields a binary relation on $[\!:$ the carrier of ADG, the carrier of $ADG\,:\!]$ and is defined as follows:

for all a, b, c, d holds $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in {\rm Congr}_{ADG}$ if and only if $a \# d = b \# c$.

Let us consider ADG. The functor Vectors(ADG) yielding an affine structure is defined by:

Vectors$(ADG) = \langle$ the carrier of $ADG, {\rm Congr}_{ADG}\rangle$.

Next we state the proposition

(9) The points of Vectors(ADG) = the carrier of ADG and the congruence of Vectors$(ADG) = {\rm Congr}_{ADG}$.

Let us consider ADG, a, b, c, d. The predicate $a, b \Rrightarrow c, d$ is defined by:

$\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the congruence of Vectors(ADG).

Next we state a number of propositions:

(10) $a, b \Rrightarrow c, d$ if and only if $a \# d = b \# c$.

(11) If $G = \mathbb{R}_{\rm G}$, then there exist elements a, b of the carrier of G such that $a \neq b$.

(12) There exists ADG and there exist a, b such that $a \neq b$.

(13) If $a, b \Rrightarrow c, c$, then $a = b$.

(14) If $a, b \Rrightarrow p, q$ and $c, d \Rrightarrow p, q$, then $a, b \Rrightarrow c, d$.

(15) There exists d such that $a, b \Rrightarrow c, d$.

(16) If $a, b \Rrightarrow a', b'$ and $a, c \Rrightarrow a', c'$, then $b, c \Rrightarrow b', c'$.

(17) There exists b such that $a, b \Rrightarrow b, c$.

(18) If $a, b \Rrightarrow b, c$ and $a, b' \Rrightarrow b', c$, then $b = b'$.

(19) If $a, b \Rrightarrow c, d$, then $a, c \Rrightarrow b, d$.

In the sequel AS denotes an affine structure. Let us consider AS, and let a, b, c, d be elements of the points of AS. The predicate $a, b \Rrightarrow c, d$ is defined by:

$\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the congruence of AS.

One can prove the following proposition

(20) Suppose there exist elements a, b of the carrier of ADG such that $a \neq b$. Then

(i) there exist elements a, b of the points of Vectors(ADG) such that $a \neq b$,

(ii) for all elements a, b, c of the points of Vectors(ADG) such that $a, b \Rrightarrow c, c$ holds $a = b$,

(iii) for all elements a, b, c, d, p, q of the points of Vectors(ADG) such that $a, b \Rrightarrow p, q$ and $c, d \Rrightarrow p, q$ holds $a, b \Rrightarrow c, d$,

(iv) for every elements a, b, c of the points of Vectors(ADG) there exists an element d of the points of Vectors(ADG) such that $a, b \Rrightarrow c, d$,

(v) for all elements a, b, c, a', b', c' of the points of Vectors(ADG) such that $a, b \Rrightarrow a', b'$ and $a, c \Rrightarrow a', c'$ holds $b, c \Rrightarrow b', c'$,

(vi) for every elements a, c of the points of Vectors(ADG) there exists an element b of the points of Vectors(ADG) such that $a, b \Rrightarrow b, c$,

(vii) for all elements a, b, c, b' of the points of Vectors(ADG) such that $a, b \Rrightarrow b, c$ and $a, b' \Rrightarrow b', c$ holds $b = b'$,

(viii) for all elements a, b, c, d of the points of Vectors(ADG) such that $a, b \Rrightarrow c, d$ holds $a, c \Rrightarrow b, d$.

An affine structure is said to be a space of free vectors if:

(i) there exist elements a, b of the points of it such that $a \neq b$,

(ii) for all elements a, b, c of the points of it such that $a, b \Rrightarrow c, c$ holds $a = b$,

(iii) for all elements a, b, c, d, p, q of the points of it such that $a, b \Rrightarrow p, q$ and $c, d \Rrightarrow p, q$ holds $a, b \Rrightarrow c, d$,

(iv) for every elements a, b, c of the points of it there exists an element d of the points of it such that $a, b \Rrightarrow c, d$,

(v) for all elements a, b, c, a', b', c' of the points of it such that $a, b \Rrightarrow a', b'$ and $a, c \Rrightarrow a', c'$ holds $b, c \Rrightarrow b', c'$,

(vi) for every elements a, c of the points of it there exists an element b of the points of it such that $a, b \Rrightarrow b, c$,

(vii) for all elements a, b, c, b' of the points of it such that $a, b \Rrightarrow b, c$ and $a, b' \Rrightarrow b', c$ holds $b = b'$,

(viii) for all elements a, b, c, d of the points of it such that $a, b \Rrightarrow c, d$ holds $a, c \Rrightarrow b, d$.

We now state several propositions:

(21) Given AS. Then the following conditions are equivalent:

(i) there exist elements a, b of the points of AS such that $a \neq b$ and for all elements a, b, c of the points of AS such that $a, b \Rrightarrow c, c$ holds $a = b$ and for all elements a, b, c, d, p, q of the points of AS such that $a, b \Rrightarrow p, q$ and $c, d \Rrightarrow p, q$ holds $a, b \Rrightarrow c, d$ and for every elements a, b, c of the points of AS there exists an element d of the points of AS such that $a, b \Rrightarrow c, d$ and for all elements a, b, c, a', b', c' of the points of AS such that $a, b \Rrightarrow a', b'$

and $a, c \Rrightarrow a', c'$ holds $b, c \Rrightarrow b', c'$ and for every elements a, c of the points of AS there exists an element b of the points of AS such that $a, b \Rrightarrow b, c$ and for all elements a, b, c, b' of the points of AS such that $a, b \Rrightarrow b, c$ and $a, b' \Rrightarrow b', c$ holds $b = b'$ and for all elements a, b, c, d of the points of AS such that $a, b \Rrightarrow c, d$ holds $a, c \Rrightarrow b, d$,

(ii) AS is a space of free vectors.

(22) If there exist elements a, b of the carrier of ADG such that $a \neq b$, then Vectors(ADG) is a space of free vectors.

(23) For every ADG and for all elements a, b of the carrier of ADG holds $a\#b =$ (the addition of ADG)(a, b).

(24) For every ADG and for every binary relation R on $[\!:$ the carrier of ADG, the carrier of $ADG\,:\!]$ holds $R = \mathrm{Congr}_{ADG}$ if and only if for all elements a, b, c, d of the carrier of ADG holds $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in R$ if and only if $a\#d = b\#c$.

(25) For every ADG and for every AS being an affine structure holds $AS =$ Vectors(ADG) if and only if $AS = \langle$ the carrier of $ADG, \mathrm{Congr}_{ADG}\rangle$.

(26) For every ADG and for all elements a, b, c, d of the carrier of ADG holds $a, b \Rrightarrow c, d$ if and only if $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the congruence of Vectors(ADG).

(27) For every AS being an affine structure and for all elements a, b, c, d of the points of AS holds $a, b \Rrightarrow c, d$ if and only if $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the congruence of AS.

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A First-Order Predicate Calculus

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Summary. A continuation of [3], with an axiom system of first-order predicate theory. The consequence Cn of a set of formulas X is defined as the intersection of all theories containing X and some basic properties of it has been proved (monotonicity, idempotency, completness etc.). The notion of a proof of given formula is also introduced and it is shown that $\mathrm{Cn}X = \{ p : p$ has a proof w.r.t. $X\}$. First 14 theorems are rather simply facts. I just wanted them to be included in the data base.

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The papers [11], [10], [9], [8], [4], [6], [1], [5], [2], [7], and [3] provide the terminology and notation for this paper. In the sequel i, j, n, k, l will be natural numbers. One can prove the following propositions:

(1) If $n \leq 0$, then $n = 0$.

(2) If $n \leq 1$, then $n = 0$ or $n = 1$.

(3) If $n \leq 2$, then $n = 0$ or $n = 1$ or $n = 2$.

(4) If $n \leq 3$, then $n = 0$ or $n = 1$ or $n = 2$ or $n = 3$.

(5) If $n \leq 4$, then $n = 0$ or $n = 1$ or $n = 2$ or $n = 3$ or $n = 4$.

(6) If $n \leq 5$, then $n = 0$ or $n = 1$ or $n = 2$ or $n = 3$ or $n = 4$ or $n = 5$.

(7) If $n \leq 6$, then $n = 0$ or $n = 1$ or $n = 2$ or $n = 3$ or $n = 4$ or $n = 5$ or $n = 6$.

(8) If $n \leq 7$, then $n = 0$ or $n = 1$ or $n = 2$ or $n = 3$ or $n = 4$ or $n = 5$ or $n = 6$ or $n = 7$.

(9) If $n \leq 8$, then $n = 0$ or $n = 1$ or $n = 2$ or $n = 3$ or $n = 4$ or $n = 5$ or $n = 6$ or $n = 7$ or $n = 8$.

(10) If $n \leq 9$, then $n = 0$ or $n = 1$ or $n = 2$ or $n = 3$ or $n = 4$ or $n = 5$ or $n = 6$ or $n = 7$ or $n = 8$ or $n = 9$.

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Next we state two propositions:

(11) $\{k : k \leq n+1\} = \{i : i \leq n\} \cup \{n+1\}$.

(12) For every n holds $\{k : k \leq n\}$ is finite.

In the sequel X, Y, Z denote sets. One can prove the following two propositions:

(13) If X is finite and $X \subseteq [\!:Y, Z:\!]$, then there exist sets A, B such that A is finite and $A \subseteq Y$ and B is finite and $B \subseteq Z$ and $X \subseteq [\!:A, B:\!]$.

(14) If X is finite and Z is finite and $X \subseteq [\!:Y, Z:\!]$, then there exists a set A such that A is finite and $A \subseteq Y$ and $X \subseteq [\!:A, Z:\!]$.

For simplicity we adopt the following convention: T, S, X, Y will be subsets of $\mathrm{WFF}_{\mathrm{CQC}}$, p, q, r, t, F will be elements of $\mathrm{WFF}_{\mathrm{CQC}}$, s will be a formula, and x, y will be bound variables. Let us consider T. We say that T is a theory if and only if:

(i) $\mathrm{VERUM} \in T$,

(ii) for all p, q, r, s, x, y holds $(\neg p \Rightarrow p) \Rightarrow p \in T$ and $p \Rightarrow (\neg p \Rightarrow q) \in T$ and $(p \Rightarrow q) \Rightarrow (\neg(q \wedge r) \Rightarrow \neg(p \wedge r)) \in T$ and $p \wedge q \Rightarrow q \wedge p \in T$ but if $p \in T$ and $p \Rightarrow q \in T$, then $q \in T$ and $\forall_x p \Rightarrow p \in T$ but if $p \Rightarrow q \in T$ and $x \notin \mathrm{snb}(p)$, then $p \Rightarrow \forall_x q \in T$ but if $s(x) \in \mathrm{WFF}_{\mathrm{CQC}}$ and $s(y) \in \mathrm{WFF}_{\mathrm{CQC}}$ and $x \notin \mathrm{snb}(s)$ and $s(x) \in T$, then $s(y) \in T$.

Next we state a number of propositions:

(15) Suppose that

(i) $\mathrm{VERUM} \in T$,

(ii) for all p, q, r, s, x, y holds $(\neg p \Rightarrow p) \Rightarrow p \in T$ and $p \Rightarrow (\neg p \Rightarrow q) \in T$ and $(p \Rightarrow q) \Rightarrow (\neg(q \wedge r) \Rightarrow \neg(p \wedge r)) \in T$ and $p \wedge q \Rightarrow q \wedge p \in T$ but if $p \in T$ and $p \Rightarrow q \in T$, then $q \in T$ and $\forall_x p \Rightarrow p \in T$ but if $p \Rightarrow q \in T$ and $x \notin \mathrm{snb}(p)$, then $p \Rightarrow \forall_x q \in T$ but if $s(x) \in \mathrm{WFF}_{\mathrm{CQC}}$ and $s(y) \in \mathrm{WFF}_{\mathrm{CQC}}$ and $x \notin \mathrm{snb}(s)$ and $s(x) \in T$, then $s(y) \in T$.

Then T is a theory.

(16) If T is a theory, then $\mathrm{VERUM} \in T$.

(17) If T is a theory, then $(\neg p \Rightarrow p) \Rightarrow p \in T$.

(18) If T is a theory, then $p \Rightarrow (\neg p \Rightarrow q) \in T$.

(19) If T is a theory, then $(p \Rightarrow q) \Rightarrow (\neg(q \wedge r) \Rightarrow \neg(p \wedge r)) \in T$.

(20) If T is a theory, then $p \wedge q \Rightarrow q \wedge p \in T$.

(21) If T is a theory and $p \in T$ and $p \Rightarrow q \in T$, then $q \in T$.

(22) If T is a theory, then $\forall_x p \Rightarrow p \in T$.

(23) If T is a theory and $p \Rightarrow q \in T$ and $x \notin \mathrm{snb}(p)$, then $p \Rightarrow \forall_x q \in T$.

(24) If T is a theory and $s(x) \in \mathrm{WFF}_{\mathrm{CQC}}$ and $s(y) \in \mathrm{WFF}_{\mathrm{CQC}}$ and $x \notin \mathrm{snb}(s)$ and $s(x) \in T$, then $s(y) \in T$.

Let us consider T, S. Then $T \cup S$ is a subset of $\mathrm{WFF}_{\mathrm{CQC}}$. Then $T \cap S$ is a subset of $\mathrm{WFF}_{\mathrm{CQC}}$. Then $T \setminus S$ is a subset of $\mathrm{WFF}_{\mathrm{CQC}}$.

Let us consider p. Then $\{p\}$ is a subset of $\mathrm{WFF}_{\mathrm{CQC}}$.

Next we state the proposition

(25) If T is a theory and S is a theory, then $T \cap S$ is a theory.

Let us consider X. The functor $\mathrm{Cn}\, X$ yielding a subset of $\mathrm{WFF}_{\mathrm{CQC}}$ is defined as follows:

$t \in \mathrm{Cn}\, X$ if and only if for every T such that T is a theory and $X \subseteq T$ holds $t \in T$.

We now state a number of propositions:

(26) $Y = \mathrm{Cn}\, X$ if and only if for every t holds $t \in Y$ if and only if for every T such that T is a theory and $X \subseteq T$ holds $t \in T$.

(27) $\mathrm{VERUM} \in \mathrm{Cn}\, X$.

(28) $(\neg p \Rightarrow p) \Rightarrow p \in \mathrm{Cn}\, X$.

(29) $p \Rightarrow (\neg p \Rightarrow q) \in \mathrm{Cn}\, X$.

(30) $(p \Rightarrow q) \Rightarrow (\neg(q \wedge r) \Rightarrow \neg(p \wedge r)) \in \mathrm{Cn}\, X$.

(31) $p \wedge q \Rightarrow q \wedge p \in \mathrm{Cn}\, X$.

(32) If $p \in \mathrm{Cn}\, X$ and $p \Rightarrow q \in \mathrm{Cn}\, X$, then $q \in \mathrm{Cn}\, X$.

(33) $\forall_x p \Rightarrow p \in \mathrm{Cn}\, X$.

(34) If $p \Rightarrow q \in \mathrm{Cn}\, X$ and $x \notin \mathrm{snb}(p)$, then $p \Rightarrow \forall_x q \in \mathrm{Cn}\, X$.

(35) If $s(x) \in \mathrm{WFF}_{\mathrm{CQC}}$ and $s(y) \in \mathrm{WFF}_{\mathrm{CQC}}$ and $x \notin \mathrm{snb}(s)$ and $s(x) \in \mathrm{Cn}\, X$, then $s(y) \in \mathrm{Cn}\, X$.

(36) $\mathrm{Cn}\, X$ is a theory.

(37) If T is a theory and $X \subseteq T$, then $\mathrm{Cn}\, X \subseteq T$.

(38) $X \subseteq \mathrm{Cn}\, X$.

(39) If $X \subseteq Y$, then $\mathrm{Cn}\, X \subseteq \mathrm{Cn}\, Y$.

(40) $\mathrm{Cn}(\mathrm{Cn}\, X) = \mathrm{Cn}\, X$.

(41) T is a theory if and only if $\mathrm{Cn}\, T = T$.

The non-empty set $\mathbb{K}$ is defined by:

$\mathbb{K} = \{k : k \leq 9\}$.

Next we state three propositions:

(42) $\mathbb{K} = \{k : k \leq 9\}$.

(43) $0 \in \mathbb{K}$ and $1 \in \mathbb{K}$ and $2 \in \mathbb{K}$ and $3 \in \mathbb{K}$ and $4 \in \mathbb{K}$ and $5 \in \mathbb{K}$ and $6 \in \mathbb{K}$ and $7 \in \mathbb{K}$ and $8 \in \mathbb{K}$ and $9 \in \mathbb{K}$.

(44) $\mathbb{K}$ is finite.

In the sequel f, g are finite sequences of elements of $[\!:\mathrm{WFF}_{\mathrm{CQC}}, \mathbb{K}:\!]$. The following proposition is true

(45) Suppose $1 \leq n$ and $n \leq \mathrm{len}\, f$. Then

(i) $(f(n))_{\mathbf{2}} = 0$, or
(ii) $(f(n))_{\mathbf{2}} = 1$, or
(iii) $(f(n))_{\mathbf{2}} = 2$, or
(iv) $(f(n))_{\mathbf{2}} = 3$, or
(v) $(f(n))_{\mathbf{2}} = 4$, or
(vi) $(f(n))_{\mathbf{2}} = 5$, or
(vii) $(f(n))_{\mathbf{2}} = 6$, or

(viii) $(f(n))_{\mathbf{2}} = 7$, or
(ix) $(f(n))_{\mathbf{2}} = 8$, or
(x) $(f(n))_{\mathbf{2}} = 9$.

Let PR be a finite sequence of elements of $[:\mathrm{WFF}_{\mathrm{CQC}}, \mathbb{K}:]$, and let us consider n, X. Let us assume that $1 \leq n$ and $n \leq \operatorname{len} PR$. We say that $PR(n)$ is a correct proof step w.r.t. X if and only if:

$(PR(n))_{\mathbf{1}} \in X$ if $(PR(n))_{\mathbf{2}} = 0$, $(PR(n))_{\mathbf{1}} = \mathrm{VERUM}$ if $(PR(n))_{\mathbf{2}} = 1$, there exists p such that $(PR(n))_{\mathbf{1}} = (\neg p \Rightarrow p) \Rightarrow p$ if $(PR(n))_{\mathbf{2}} = 2$, there exist p, q such that $(PR(n))_{\mathbf{1}} = p \Rightarrow (\neg p \Rightarrow q)$ if $(PR(n))_{\mathbf{2}} = 3$, there exist p, q, r such that $(PR(n))_{\mathbf{1}} = (p \Rightarrow q) \Rightarrow (\neg(q \wedge r) \Rightarrow \neg(p \wedge r))$ if $(PR(n))_{\mathbf{2}} = 4$, there exist p, q such that $(PR(n))_{\mathbf{1}} = p \wedge q \Rightarrow q \wedge p$ if $(PR(n))_{\mathbf{2}} = 5$, there exist p, x such that $(PR(n))_{\mathbf{1}} = \forall_x p \Rightarrow p$ if $(PR(n))_{\mathbf{2}} = 6$, there exist i, j, p, q such that $1 \leq i$ and $i < n$ and $1 \leq j$ and $j < i$ and $p = (PR(j))_{\mathbf{1}}$ and $q = (PR(n))_{\mathbf{1}}$ and $(PR(i))_{\mathbf{1}} = p \Rightarrow q$ if $(PR(n))_{\mathbf{2}} = 7$, there exist i, p, q, x such that $1 \leq i$ and $i < n$ and $(PR(i))_{\mathbf{1}} = p \Rightarrow q$ and $x \notin \mathrm{snb}(p)$ and $(PR(n))_{\mathbf{1}} = p \Rightarrow \forall_x q$ if $(PR(n))_{\mathbf{2}} = 8$, there exist i, x, y, s such that $1 \leq i$ and $i < n$ and $s(x) \in \mathrm{WFF}_{\mathrm{CQC}}$ and $s(y) \in \mathrm{WFF}_{\mathrm{CQC}}$ and $x \notin \mathrm{snb}(s)$ and $s(x) = (PR(i))_{\mathbf{1}}$ and $s(y) = (PR(n))_{\mathbf{1}}$ if $(PR(n))_{\mathbf{2}} = 9$.

The following propositions are true:

(46) If $1 \leq n$ and $n \leq \operatorname{len} f$ and $(f(n))_{\mathbf{2}} = 0$, then $f(n)$ is a correct proof step w.r.t. X if and only if $(f(n))_{\mathbf{1}} \in X$.

(47) If $1 \leq n$ and $n \leq \operatorname{len} f$ and $(f(n))_{\mathbf{2}} = 1$, then $f(n)$ is a correct proof step w.r.t. X if and only if $(f(n))_{\mathbf{1}} = \mathrm{VERUM}$.

(48) If $1 \leq n$ and $n \leq \operatorname{len} f$ and $(f(n))_{\mathbf{2}} = 2$, then $f(n)$ is a correct proof step w.r.t. X if and only if there exists p such that $(f(n))_{\mathbf{1}} = (\neg p \Rightarrow p) \Rightarrow p$.

(49) If $1 \leq n$ and $n \leq \operatorname{len} f$ and $(f(n))_{\mathbf{2}} = 3$, then $f(n)$ is a correct proof step w.r.t. X if and only if there exist p, q such that $(f(n))_{\mathbf{1}} = p \Rightarrow (\neg p \Rightarrow q)$.

(50) If $1 \leq n$ and $n \leq \operatorname{len} f$ and $(f(n))_{\mathbf{2}} = 4$, then $f(n)$ is a correct proof step w.r.t. X if and only if there exist p, q, r such that $(f(n))_{\mathbf{1}} = (p \Rightarrow q) \Rightarrow (\neg(q \wedge r) \Rightarrow \neg(p \wedge r))$.

(51) If $1 \leq n$ and $n \leq \operatorname{len} f$ and $(f(n))_{\mathbf{2}} = 5$, then $f(n)$ is a correct proof step w.r.t. X if and only if there exist p, q such that $(f(n))_{\mathbf{1}} = p \wedge q \Rightarrow q \wedge p$.

(52) If $1 \leq n$ and $n \leq \operatorname{len} f$ and $(f(n))_{\mathbf{2}} = 6$, then $f(n)$ is a correct proof step w.r.t. X if and only if there exist p, x such that $(f(n))_{\mathbf{1}} = \forall_x p \Rightarrow p$.

(53) Suppose $1 \leq n$ and $n \leq \operatorname{len} f$ and $(f(n))_{\mathbf{2}} = 7$. Then $f(n)$ is a correct proof step w.r.t. X if and only if there exist i, j, p, q such that $1 \leq i$ and $i < n$ and $1 \leq j$ and $j < i$ and $p = (f(j))_{\mathbf{1}}$ and $q = (f(n))_{\mathbf{1}}$ and $(f(i))_{\mathbf{1}} = p \Rightarrow q$.

(54) Suppose $1 \leq n$ and $n \leq \operatorname{len} f$ and $(f(n))_{\mathbf{2}} = 8$. Then $f(n)$ is a correct proof step w.r.t. X if and only if there exist i, p, q, x such that $1 \leq i$ and $i < n$ and $(f(i))_{\mathbf{1}} = p \Rightarrow q$ and $x \notin \mathrm{snb}(p)$ and $(f(n))_{\mathbf{1}} = p \Rightarrow \forall_x q$.

(55) Suppose $1 \leq n$ and $n \leq \operatorname{len} f$ and $(f(n))_{\mathbf{2}} = 9$. Then $f(n)$ is a correct proof step w.r.t. X if and only if there exist i, x, y, s such that $1 \leq i$

and $i < n$ and $s(x) \in \mathrm{WFF}_{\mathrm{CQC}}$ and $s(y) \in \mathrm{WFF}_{\mathrm{CQC}}$ and $x \notin \mathrm{snb}(s)$ and $s(x) = (f(i))_{\mathbf{1}}$ and $(f(n))_{\mathbf{1}} = s(y)$.

Let us consider X, f. We say that f is a proof w.r.t. X if and only if:

$f \neq \varepsilon$ and for every n such that $1 \leq n$ and $n \leq \operatorname{len} f$ holds $f(n)$ is a correct proof step w.r.t. X.

The following propositions are true:

(56) f is a proof w.r.t. X if and only if $f \neq \varepsilon$ and for every n such that $1 \leq n$ and $n \leq \operatorname{len} f$ holds $f(n)$ is a correct proof step w.r.t. X.

(57) If f is a proof w.r.t. X, then $\operatorname{rng} f \neq \emptyset$.

(58) If f is a proof w.r.t. X, then $1 \leq \operatorname{len} f$.

(59) Suppose f is a proof w.r.t. X. Then $(f(1))_{\mathbf{2}} = 0$ or $(f(1))_{\mathbf{2}} = 1$ or $(f(1))_{\mathbf{2}} = 2$ or $(f(1))_{\mathbf{2}} = 3$ or $(f(1))_{\mathbf{2}} = 4$ or $(f(1))_{\mathbf{2}} = 5$ or $(f(1))_{\mathbf{2}} = 6$.

(60) If $1 \leq n$ and $n \leq \operatorname{len} f$, then $f(n)$ is a correct proof step w.r.t. X if and only if $f \frown g(n)$ is a correct proof step w.r.t. X.

(61) If $1 \leq n$ and $n \leq \operatorname{len} g$ and $g(n)$ is a correct proof step w.r.t. X, then $f \frown g(n + \operatorname{len} f)$ is a correct proof step w.r.t. X.

(62) If f is a proof w.r.t. X and g is a proof w.r.t. X, then $f \frown g$ is a proof w.r.t. X.

(63) If f is a proof w.r.t. X and $X \subseteq Y$, then f is a proof w.r.t. Y.

(64) If f is a proof w.r.t. X and $1 \leq l$ and $l \leq \operatorname{len} f$, then $(f(l))_{\mathbf{1}} \in \operatorname{Cn} X$.

Let us consider f. Let us assume that $f \neq \varepsilon$. The functor $\operatorname{Effect} f$ yields an element of $\mathrm{WFF}_{\mathrm{CQC}}$ and is defined as follows:

$\operatorname{Effect} f = (f(\operatorname{len} f))_{\mathbf{1}}$.

The following propositions are true:

(65) If $f \neq \varepsilon$, then $\operatorname{Effect} f = (f(\operatorname{len} f))_{\mathbf{1}}$.

(66) If f is a proof w.r.t. X, then $\operatorname{Effect} f \in \operatorname{Cn} X$.

(67) $X \subseteq \{F : \bigvee_f [\, f$ is a proof w.r.t. $X \wedge \operatorname{Effect} f = F]\}$.

(68) For every X such that $Y = \{p : \bigvee_f [\, f$ is a proof w.r.t. $X \wedge \operatorname{Effect} f = p]\}$ holds Y is a theory.

(69) For every X holds $\{p : \bigvee_f [\, f$ is a proof w.r.t. $X \wedge \operatorname{Effect} f = p]\} = \operatorname{Cn} X$.

(70) $p \in \operatorname{Cn} X$ if and only if there exists f such that f is a proof w.r.t. X and $\operatorname{Effect} f = p$.

(71) If $p \in \operatorname{Cn} X$, then there exists Y such that $Y \subseteq X$ and Y is finite and $p \in \operatorname{Cn} Y$.

The subset $\emptyset_{\mathrm{CQC}}$ of $\mathrm{WFF}_{\mathrm{CQC}}$ is defined by:

$\emptyset_{\mathrm{CQC}} = \emptyset_{\mathrm{WFF}_{\mathrm{CQC}}}$.

We now state the proposition

(72) $\emptyset_{\mathrm{CQC}} = \emptyset_{\mathrm{WFF}_{\mathrm{CQC}}}$.

The subset Taut of $\mathrm{WFF}_{\mathrm{CQC}}$ is defined as follows:

$\mathrm{Taut} = \operatorname{Cn} \emptyset_{\mathrm{CQC}}$.

One can prove the following propositions:

(73) Taut = $\mathrm{Cn}\,\emptyset_{\mathrm{CQC}}$.

(74) If T is a theory, then Taut $\subseteq T$.

(75) Taut $\subseteq \mathrm{Cn}\,X$.

(76) Taut is a theory.

(77) VERUM $\in$ Taut.

(78) $(\neg p \Rightarrow p) \Rightarrow p \in$ Taut.

(79) $p \Rightarrow (\neg p \Rightarrow q) \in$ Taut.

(80) $(p \Rightarrow q) \Rightarrow (\neg(q \wedge r) \Rightarrow \neg(p \wedge r)) \in$ Taut.

(81) $p \wedge q \Rightarrow q \wedge p \in$ Taut.

(82) If $p \in$ Taut and $p \Rightarrow q \in$ Taut, then $q \in$ Taut.

(83) $\forall_x p \Rightarrow p \in$ Taut.

(84) If $p \Rightarrow q \in$ Taut and $x \notin \mathrm{snb}(p)$, then $p \Rightarrow \forall_x q \in$ Taut.

(85) If $s(x) \in \mathrm{WFF}_{\mathrm{CQC}}$ and $s(y) \in \mathrm{WFF}_{\mathrm{CQC}}$ and $x \notin \mathrm{snb}(s)$ and $s(x) \in$ Taut, then $s(y) \in$ Taut.

Let us consider X, s. The predicate $X \vdash s$ is defined as follows:
$s \in \mathrm{Cn}\,X$.

Next we state a number of propositions:

(86) $X \vdash s$ if and only if $s \in \mathrm{Cn}\,X$.

(87) $X \vdash$ VERUM.

(88) $X \vdash (\neg p \Rightarrow p) \Rightarrow p$.

(89) $X \vdash p \Rightarrow (\neg p \Rightarrow q)$.

(90) $X \vdash (p \Rightarrow q) \Rightarrow (\neg(q \wedge r) \Rightarrow \neg(p \wedge r))$.

(91) $X \vdash p \wedge q \Rightarrow q \wedge p$.

(92) If $X \vdash p$ and $X \vdash p \Rightarrow q$, then $X \vdash q$.

(93) $X \vdash \forall_x p \Rightarrow p$.

(94) If $X \vdash p \Rightarrow q$ and $x \notin \mathrm{snb}(p)$, then $X \vdash p \Rightarrow \forall_x q$.

(95) If $s(x) \in \mathrm{WFF}_{\mathrm{CQC}}$ and $s(y) \in \mathrm{WFF}_{\mathrm{CQC}}$ and $x \notin \mathrm{snb}(s)$ and $X \vdash s(x)$, then $X \vdash s(y)$.

Let us consider s. The predicate $\vdash s$ is defined as follows:
$\emptyset_{\mathrm{CQC}} \vdash s$.

Next we state two propositions:

(96) $\vdash s$ if and only if $\emptyset_{\mathrm{CQC}} \vdash s$.

(97) $\vdash s$ if and only if $s \in$ Taut.

Let us consider s. Let us note that one can characterize the predicate $\vdash s$ by the following (equivalent) condition: $s \in$ Taut.

We now state a number of propositions:

(98) If $\vdash p$, then $X \vdash p$.

(99) $\vdash$ VERUM.

(100) $\vdash (\neg p \Rightarrow p) \Rightarrow p$.

(101) $\vdash p \Rightarrow (\neg p \Rightarrow q)$.

(102) $\vdash (p \Rightarrow q) \Rightarrow (\neg(q \wedge r) \Rightarrow \neg(p \wedge r))$.

(103) $\vdash p \wedge q \Rightarrow q \wedge p$.

(104) If $\vdash p$ and $\vdash p \Rightarrow q$, then $\vdash q$.

(105) $\vdash \forall_x p \Rightarrow p$.

(106) If $\vdash p \Rightarrow q$ and $x \notin \mathrm{snb}(p)$, then $\vdash p \Rightarrow \forall_x q$.

(107) If $s(x) \in \mathrm{WFF}_{\mathrm{CQC}}$ and $s(y) \in \mathrm{WFF}_{\mathrm{CQC}}$ and $x \notin \mathrm{snb}(s)$ and $\vdash s(x)$, then $\vdash s(y)$.

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Partial Functions from a Domain to a Domain

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Summary. The value of a partial function from a domain to a domain and a inverse partial function are introduced. The value and inverse function were defined in the article [1], but new definitions are introduced. The basic properties of the value, the inverse partial function, the identity partial function, the composition of partial funtion, the 1−1 partial function, the restriction of a partial function, the image, the inverse image and the graph are proved. Constant partial function are introduced, too.

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The terminology and notation used here are introduced in the following papers: [5], [1], [2], [6], [4], and [3]. For simplicity we follow the rules: x, y are arbitrary, X, Y denote sets, C, D, E denote non-empty sets, SC denotes a subset of C, SD denotes a subset of D, SE denotes a subset of E, c, c_1, c_2 denote elements of C, d denotes an element of D, e denotes an element of E, f, f_1, g denote partial functions from C to D, t denotes a partial function from D to C, s denotes a partial function from D to E, h denotes a partial function from C to E, and F denotes a partial function from D to D. The following proposition is true

(1) x is an element of E if and only if $x \in E$.

Let us consider C, D, f, c. Let us assume that $c \in \mathrm{dom}\, f$. The functor $f(c)$ yielding an element of D is defined by:

$f(c) = (f$ **qua** a function$)(c)$.

Next we state four propositions:

(2) If $c \in \mathrm{dom}\, f$, then $f(c) = (f$ **qua** a function$)(c)$.

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(3) If $\operatorname{dom} f = \operatorname{dom} g$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = g(c)$, then $f = g$.

(4) $y \in \operatorname{rng} f$ if and only if there exists c such that $c \in \operatorname{dom} f$ and $y = f(c)$.

(5) If $c \in \operatorname{dom} f$, then $f(c) \in \operatorname{rng} f$.

Let us consider D, C, f. Then $\operatorname{dom} f$ is a subset of C. Then $\operatorname{rng} f$ is a subset of D.

The following propositions are true:

(6) $h = s \cdot f$ if and only if for every c holds $c \in \operatorname{dom} h$ if and only if $c \in \operatorname{dom} f$ and $f(c) \in \operatorname{dom} s$ and for every c such that $c \in \operatorname{dom} h$ holds $h(c) = s(f(c))$.

(7) $c \in \operatorname{dom}(s \cdot f)$ if and only if $c \in \operatorname{dom} f$ and $f(c) \in \operatorname{dom} s$.

(8) If $c \in \operatorname{dom}(s \cdot f)$, then $(s \cdot f)(c) = s(f(c))$.

(9) If $c \in \operatorname{dom} f$ and $f(c) \in \operatorname{dom} s$, then $(s \cdot f)(c) = s(f(c))$.

(10) If $\operatorname{rng} f \subseteq \operatorname{dom} s$ and $c \in \operatorname{dom} f$, then $(s \cdot f)(c) = s(f(c))$.

(11) If $\operatorname{rng} f = \operatorname{dom} s$ and $c \in \operatorname{dom} f$, then $(s \cdot f)(c) = s(f(c))$.

Let us consider D, SD. Then id_{SD} is a partial function from D to D.

Next we state several propositions:

(12) $F = \mathrm{id}_{SD}$ if and only if $\operatorname{dom} F = SD$ and for every d such that $d \in SD$ holds $F(d) = d$.

(13) If $d \in SD$, then $\mathrm{id}_{SD}(d) = d$.

(14) If $d \in \operatorname{dom} F \cap SD$, then $F(d) = (F \cdot \mathrm{id}_{SD})(d)$.

(15) $d \in \operatorname{dom}(\mathrm{id}_{SD} \cdot F)$ if and only if $d \in \operatorname{dom} F$ and $F(d) \in SD$.

(16) f is one-to-one if and only if for all c_1, c_2 such that $c_1 \in \operatorname{dom} f$ and $c_2 \in \operatorname{dom} f$ and $f(c_1) = f(c_2)$ holds $c_1 = c_2$.

Let us consider C, D, and let f be a partial function from C to D. Let us assume that f is one-to-one. The functor f^{-1} yields a partial function from D to C and is defined as follows:

$f^{-1} = (f \textbf{ qua } \text{a function})^{-1}$.

One can prove the following propositions:

(17) If f is one-to-one, then for every partial function g from D to C holds $g = f^{-1}$ if and only if $g = (f \textbf{ qua } \text{a function})^{-1}$.

(18) If f is one-to-one, then for every partial function g from D to C holds $g = f^{-1}$ if and only if $\operatorname{dom} g = \operatorname{rng} f$ and for all d, c holds $d \in \operatorname{rng} f$ and $c = g(d)$ if and only if $c \in \operatorname{dom} f$ and $d = f(c)$.

(19) If f is one-to-one, then $\operatorname{rng} f = \operatorname{dom}(f^{-1})$ and $\operatorname{dom} f = \operatorname{rng}(f^{-1})$.

(20) If f is one-to-one, then $\operatorname{dom}(f^{-1} \cdot f) = \operatorname{dom} f$ and $\operatorname{rng}(f^{-1} \cdot f) = \operatorname{dom} f$.

(21) If f is one-to-one, then $\operatorname{dom}(f \cdot f^{-1}) = \operatorname{rng} f$ and $\operatorname{rng}(f \cdot f^{-1}) = \operatorname{rng} f$.

(22) If f is one-to-one and $c \in \operatorname{dom} f$, then $c = f^{-1}(f(c))$ and $c = (f^{-1} \cdot f)(c)$.

(23) If f is one-to-one and $d \in \operatorname{rng} f$, then $d = f(f^{-1}(d))$ and $d = (f \cdot f^{-1})(d)$.

(24) If f is one-to-one and $\operatorname{dom} f = \operatorname{rng} t$ and $\operatorname{rng} f = \operatorname{dom} t$ and for all c, d such that $c \in \operatorname{dom} f$ and $d \in \operatorname{dom} t$ holds $f(c) = d$ if and only if $t(d) = c$, then $t = f^{-1}$.

(25) If f is one-to-one, then $f^{-1} \cdot f = \mathrm{id}_{\operatorname{dom} f}$ and $f \cdot f^{-1} = \mathrm{id}_{\operatorname{rng} f}$.

(26) If f is one-to-one, then f^{-1} is one-to-one.

(27) If f is one-to-one and $\operatorname{rng} f = \operatorname{dom} s$ and $s \cdot f = \mathrm{id}_{\operatorname{dom} f}$, then $s = f^{-1}$.

(28) If f is one-to-one and $\operatorname{rng} s = \operatorname{dom} f$ and $f \cdot s = \mathrm{id}_{\operatorname{rng} f}$, then $s = f^{-1}$.

(29) If f is one-to-one, then $(f^{-1})^{-1} = f$.

(30) If f is one-to-one and s is one-to-one, then $(s \cdot f)^{-1} = f^{-1} \cdot s^{-1}$.

(31) $(\mathrm{id}_{SC})^{-1} = \mathrm{id}_{SC}$.

Let us consider C, D, f, X. Then $f \restriction X$ is a partial function from C to D.

We now state several propositions:

(32) $g = f \restriction X$ if and only if $\operatorname{dom} g = \operatorname{dom} f \cap X$ and for every c such that $c \in \operatorname{dom} g$ holds $g(c) = f(c)$.

(33) If $c \in \operatorname{dom}(f \restriction X)$, then $(f \restriction X)(c) = f(c)$.

(34) If $c \in \operatorname{dom} f \cap X$, then $(f \restriction X)(c) = f(c)$.

(35) If $c \in \operatorname{dom} f$ and $c \in X$, then $(f \restriction X)(c) = f(c)$.

(36) If $c \in \operatorname{dom} f$ and $c \in X$, then $f(c) \in \operatorname{rng}(f \restriction X)$.

Let us consider C, D, X, f. Then $X \restriction f$ is a partial function from C to D.

The following three propositions are true:

(37) $g = X \restriction f$ if and only if for every c holds $c \in \operatorname{dom} g$ if and only if $c \in \operatorname{dom} f$ and $f(c) \in X$ and for every c such that $c \in \operatorname{dom} g$ holds $g(c) = f(c)$.

(38) $c \in \operatorname{dom}(X \restriction f)$ if and only if $c \in \operatorname{dom} f$ and $f(c) \in X$.

(39) If $c \in \operatorname{dom}(X \restriction f)$, then $(X \restriction f)(c) = f(c)$.

Let us consider C, D, f, X. Then $f \circ X$ is a subset of D.

The following propositions are true:

(40) $SD = f \circ X$ if and only if for every d holds $d \in SD$ if and only if there exists c such that $c \in \operatorname{dom} f$ and $c \in X$ and $d = f(c)$.

(41) $d \in f \circ X$ if and only if there exists c such that $c \in \operatorname{dom} f$ and $c \in X$ and $d = f(c)$.

(42) If $c \in \operatorname{dom} f$, then $f \circ \{c\} = \{f(c)\}$.

(43) If $c_1 \in \operatorname{dom} f$ and $c_2 \in \operatorname{dom} f$, then $f \circ \{c_1, c_2\} = \{f(c_1), f(c_2)\}$.

Let us consider C, D, f, X. Then $f^{-1} X$ is a subset of C.

The following propositions are true:

(44) $SC = f^{-1} X$ if and only if for every c holds $c \in SC$ if and only if $c \in \operatorname{dom} f$ and $f(c) \in X$.

(45) $c \in f^{-1} X$ if and only if $c \in \operatorname{dom} f$ and $f(c) \in X$.

(46) For every f there exists a function g from C into D such that for every c such that $c \in \operatorname{dom} f$ holds $g(c) = f(c)$.

(47) $f \approx g$ if and only if for every c such that $c \in \operatorname{dom} f \cap \operatorname{dom} g$ holds $f(c) = g(c)$.

In this article we present several logical schemes. The scheme *PartFuncExD* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a partial function f from $\mathcal{A}$ to $\mathcal{B}$ such that for every element d of $\mathcal{A}$ holds $d \in \operatorname{dom} f$ if and only if there exists an element c of $\mathcal{B}$ such that $\mathcal{P}[d, c]$ and for every element d of $\mathcal{A}$ such that $d \in \operatorname{dom} f$ holds $\mathcal{P}[d, f(d)]$

provided the following condition is satisfied:

- for every element d of $\mathcal{A}$ and for all elements c_1, c_2 of $\mathcal{B}$ such that $\mathcal{P}[d, c_1]$ and $\mathcal{P}[d, c_2]$ holds $c_1 = c_2$.

The scheme *LambdaPFD* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

there exists a partial function f from $\mathcal{A}$ to $\mathcal{B}$ such that for every element d of $\mathcal{A}$ holds $d \in \operatorname{dom} f$ if and only if $\mathcal{P}[d]$ and for every element d of $\mathcal{A}$ such that $d \in \operatorname{dom} f$ holds $f(d) = \mathcal{F}(d)$

for all values of the parameters.

The scheme *UnPartFuncD* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a set $\mathcal{C}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$ and states that:

Let f, g be partial functions from $\mathcal{A}$ to $\mathcal{B}$. Then if $\operatorname{dom} f = \mathcal{C}$ and for every element c of $\mathcal{A}$ such that $c \in \operatorname{dom} f$ holds $f(c) = \mathcal{F}(c)$ and $\operatorname{dom} g = \mathcal{C}$ and for every element c of $\mathcal{A}$ such that $c \in \operatorname{dom} g$ holds $g(c) = \mathcal{F}(c)$, then $f = g$

for all values of the parameters.

Let us consider C, D, SC, d. Then $SC \longmapsto d$ is a partial function from C to D.

The following propositions are true:

(48) If $c \in SC$, then $(SC \longmapsto d)(c) = d$.

(49) If for every c such that $c \in \operatorname{dom} f$ holds $f(c) = d$, then $f = \operatorname{dom} f \longmapsto d$.

(50) If $c \in \operatorname{dom} f$, then $f \cdot (SE \longmapsto c) = SE \longmapsto f(c)$.

(51) id_{SC} is total if and only if $SC = C$.

(52) If $SC \longmapsto d$ is total, then $SC \neq \emptyset$.

(53) $SC \longmapsto d$ is total if and only if $SC = C$.

Let us consider C, D, f, X. We say that f is a constant on X if and only if: there exists d such that for every c such that $c \in X \cap \operatorname{dom} f$ holds $f(c) = d$.

Next we state a number of propositions:

(54) f is a constant on X if and only if there exists d such that for every c such that $c \in X \cap \operatorname{dom} f$ holds $f(c) = d$.

(55) f is a constant on X if and only if for all c_1, c_2 such that $c_1 \in X \cap \operatorname{dom} f$ and $c_2 \in X \cap \operatorname{dom} f$ holds $f(c_1) = f(c_2)$.

(56) If $X \cap \operatorname{dom} f \neq \emptyset$, then f is a constant on X if and only if there exists d such that $\operatorname{rng}(f \upharpoonright X) = \{d\}$.

(57) If f is a constant on X and $Y \subseteq X$, then f is a constant on Y.

(58) If $X \cap \operatorname{dom} f = \emptyset$, then f is a constant on X.

(59) If $f \restriction SC = \operatorname{dom}(f \restriction SC) \longmapsto d$, then f is a constant on SC.

(60) f is a constant on $\{x\}$.

(61) If f is a constant on X and f is a constant on Y and $(X \cap Y) \cap \operatorname{dom} f \neq \emptyset$, then f is a constant on $X \cup Y$.

(62) If f is a constant on Y, then $f \restriction X$ is a constant on Y.

(63) $SC \longmapsto d$ is a constant on SC.

(64) $\operatorname{graph} f \subseteq \operatorname{graph} g$ if and only if $\operatorname{dom} f \subseteq \operatorname{dom} g$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = g(c)$.

(65) $c \in \operatorname{dom} f$ and $d = f(c)$ if and only if $\langle c, d \rangle \in \operatorname{graph} f$.

(66) If $\langle c, e \rangle \in \operatorname{graph}(s \cdot f)$, then $\langle c, f(c) \rangle \in \operatorname{graph} f$ and $\langle f(c), e \rangle \in \operatorname{graph} s$.

(67) If $\operatorname{graph} f = \{\langle c, d \rangle\}$, then $f(c) = d$.

(68) If $\operatorname{dom} f = \{c\}$, then $\operatorname{graph} f = \{\langle c, f(c) \rangle\}$.

(69) If $\operatorname{graph} f_1 = \operatorname{graph} f \cap \operatorname{graph} g$ and $c \in \operatorname{dom} f_1$, then $f_1(c) = f(c)$ and $f_1(c) = g(c)$.

(70) If $c \in \operatorname{dom} f$ and $\operatorname{graph} f_1 = \operatorname{graph} f \cup \operatorname{graph} g$, then $f_1(c) = f(c)$.

(71) If $c \in \operatorname{dom} g$ and $\operatorname{graph} f_1 = \operatorname{graph} f \cup \operatorname{graph} g$, then $f_1(c) = g(c)$.

(72) If $c \in \operatorname{dom} f_1$ and $\operatorname{graph} f_1 = \operatorname{graph} f \cup \operatorname{graph} g$, then $f_1(c) = f(c)$ or $f_1(c) = g(c)$.

(73) $c \in \operatorname{dom} f$ and $c \in SC$ if and only if $\langle c, f(c) \rangle \in \operatorname{graph}(f \restriction SC)$.

(74) $c \in \operatorname{dom} f$ and $f(c) \in SD$ if and only if $\langle c, f(c) \rangle \in \operatorname{graph}(SD \restriction f)$.

(75) $c \in f^{-1} SD$ if and only if $\langle c, f(c) \rangle \in \operatorname{graph} f$ and $f(c) \in SD$.

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Partial Functions from a Domain to the Set of Real Numbers

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Summary. Basic operations in the set of partial functions which map a domain to the set of all real numbers are introduced. They include adition, substraction, multiplication, division, multipication by a real number and also module. Main properties of these operations are proved. A definition of the partial function bounded on a set (bounded below and bounded above) is presented. There are theorems showing the laws of conservation of totality and boundeness for operations of partial functions. The characteristic function of a subset of a domain as a partial function is redefined and a few properties are proved.

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The papers [6], [3], [1], [7], [5], [2], and [4] provide the terminology and notation for this paper. For simplicity we follow the rules: X, Y will be sets, C will be a non-empty set, c will be an element of C, f, f_1, f_2, f_3, g, g_1 will be partial functions from C to $\mathbb{R}$, and r, r_1, p, p_1 will be real numbers. We now state two propositions:

(1) $(-1)^{-1} = -1$.

(2) If $0 \leq p$ and $0 \leq r$ and $p \leq p_1$ and $r \leq r_1$, then $p \cdot r \leq p_1 \cdot r_1$.

We now define four new functors. Let us consider C, f_1, f_2. The functor $f_1 + f_2$ yields a partial function from C to $\mathbb{R}$ and is defined as follows:

$\mathrm{dom}(f_1 + f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}(f_1 + f_2)$ holds $(f_1 + f_2)(c) = f_1(c) + f_2(c)$.

The functor $f_1 - f_2$ yielding a partial function from C to $\mathbb{R}$ is defined as follows:

$\mathrm{dom}(f_1 - f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}(f_1 - f_2)$ holds $(f_1 - f_2)(c) = f_1(c) - f_2(c)$.

The functor $f_1 \diamond f_2$ yielding a partial function from C to $\mathbb{R}$ is defined by:

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$\mathrm{dom}(f_1 \diamond f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}(f_1 \diamond f_2)$ holds $(f_1 \diamond f_2)(c) = f_1(c) \cdot f_2(c)$.

The functor $\frac{f_1}{f_2}$ yielding a partial function from C to $\mathbb{R}$ is defined by:

$\mathrm{dom}\, \frac{f_1}{f_2} = \mathrm{dom}\, f_1 \cap (\mathrm{dom}\, f_2 \setminus f_2{}^{-1}\, \{0\})$ and for every c such that $c \in \mathrm{dom}\, \frac{f_1}{f_2}$ holds $\frac{f_1}{f_2}(c) = f_1(c) \cdot (f_2(c))^{-1}$.

Let us consider C, f, r. The functor $r \diamond f$ yields a partial function from C to $\mathbb{R}$ and is defined by:

$\mathrm{dom}(r \diamond f) = \mathrm{dom}\, f$ and for every c such that $c \in \mathrm{dom}(r \diamond f)$ holds $(r \diamond f)(c) = r \cdot f(c)$.

We now define three new functors. Let us consider C, f. The functor $|f|$ yields a partial function from C to $\mathbb{R}$ and is defined by:

$\mathrm{dom}\, |f| = \mathrm{dom}\, f$ and for every c such that $c \in \mathrm{dom}\, |f|$ holds $|f|(c) = |f(c)|$.

The functor $-f$ yields a partial function from C to $\mathbb{R}$ and is defined by:

$\mathrm{dom}(-f) = \mathrm{dom}\, f$ and for every c such that $c \in \mathrm{dom}(-f)$ holds $(-f)(c) = -f(c)$.

The functor $\frac{1}{f}$ yielding a partial function from C to $\mathbb{R}$ is defined by:

$\mathrm{dom}\, \frac{1}{f} = \mathrm{dom}\, f \setminus f^{-1}\, \{0\}$ and for every c such that $c \in \mathrm{dom}\, \frac{1}{f}$ holds $\frac{1}{f}(c) = (f(c))^{-1}$.

One can prove the following propositions:

(3) $f = f_1 + f_2$ if and only if $\mathrm{dom}\, f = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}\, f$ holds $f(c) = f_1(c) + f_2(c)$.

(4) $f = f_1 - f_2$ if and only if $\mathrm{dom}\, f = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}\, f$ holds $f(c) = f_1(c) - f_2(c)$.

(5) $f = f_1 \diamond f_2$ if and only if $\mathrm{dom}\, f = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}\, f$ holds $f(c) = f_1(c) \cdot f_2(c)$.

(6) $f = \frac{f_1}{f_2}$ if and only if $\mathrm{dom}\, f = \mathrm{dom}\, f_1 \cap (\mathrm{dom}\, f_2 \setminus f_2{}^{-1}\, \{0\})$ and for every c such that $c \in \mathrm{dom}\, f$ holds $f(c) = f_1(c) \cdot (f_2(c))^{-1}$.

(7) $f = r \diamond f_1$ if and only if $\mathrm{dom}\, f = \mathrm{dom}\, f_1$ and for every c such that $c \in \mathrm{dom}\, f$ holds $f(c) = r \cdot f_1(c)$.

(8) $f = |f_1|$ if and only if $\mathrm{dom}\, f = \mathrm{dom}\, f_1$ and for every c such that $c \in \mathrm{dom}\, f$ holds $f(c) = |f_1(c)|$.

(9) $f = -f_1$ if and only if $\mathrm{dom}\, f = \mathrm{dom}\, f_1$ and for every c such that $c \in \mathrm{dom}\, f$ holds $f(c) = -f_1(c)$.

(10) $f_1 = \frac{1}{f}$ if and only if $\mathrm{dom}\, f_1 = \mathrm{dom}\, f \setminus f^{-1}\, \{0\}$ and for every c such that $c \in \mathrm{dom}\, f_1$ holds $f_1(c) = (f(c))^{-1}$.

(11) $\mathrm{dom}\, \frac{1}{g} \subseteq \mathrm{dom}\, g$ and $\mathrm{dom}\, g \cap (\mathrm{dom}\, g \setminus g^{-1}\, \{0\}) = \mathrm{dom}\, g \setminus g^{-1}\, \{0\}$.

(12) $\mathrm{dom}(f_1 \diamond f_2) \setminus (f_1 \diamond f_2)^{-1}\, \{0\} = (\mathrm{dom}\, f_1 \setminus f_1{}^{-1}\, \{0\}) \cap (\mathrm{dom}\, f_2 \setminus f_2{}^{-1}\, \{0\})$.

(13) If $c \in \mathrm{dom}\, \frac{1}{f}$, then $f(c) \neq 0$.

(14) $\frac{1}{f}{}^{-1}\, \{0\} = \emptyset$.

(15) $|f|^{-1}\, \{0\} = f^{-1}\, \{0\}$ and $(-f)^{-1}\, \{0\} = f^{-1}\, \{0\}$.

(16) $\operatorname{dom} \frac{1}{\frac{1}{f}} = \operatorname{dom}(f \restriction \operatorname{dom} \frac{1}{f})$.

(17) If $r \neq 0$, then $(r \diamond f)^{-1} \{0\} = f^{-1} \{0\}$.

(18) $f_1 + f_2 = f_2 + f_1$.

(19) $(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$.

(20) $f_1 \diamond f_2 = f_2 \diamond f_1$.

(21) $(f_1 \diamond f_2) \diamond f_3 = f_1 \diamond (f_2 \diamond f_3)$.

(22) $(f_1 + f_2) \diamond f_3 = f_1 \diamond f_3 + f_2 \diamond f_3$.

(23) $f_3 \diamond (f_1 + f_2) = f_3 \diamond f_1 + f_3 \diamond f_2$.

(24) $r \diamond (f_1 \diamond f_2) = (r \diamond f_1) \diamond f_2$.

(25) $r \diamond (f_1 \diamond f_2) = f_1 \diamond (r \diamond f_2)$.

(26) $(f_1 - f_2) \diamond f_3 = f_1 \diamond f_3 - f_2 \diamond f_3$.

(27) $f_3 \diamond f_1 - f_3 \diamond f_2 = f_3 \diamond (f_1 - f_2)$.

(28) $r \diamond (f_1 + f_2) = r \diamond f_1 + r \diamond f_2$.

(29) $(r \cdot p) \diamond f = r \diamond (p \diamond f)$.

(30) $r \diamond (f_1 - f_2) = r \diamond f_1 - r \diamond f_2$.

(31) $f_1 - f_2 = (-1) \diamond (f_2 - f_1)$.

(32) $f_1 - (f_2 + f_3) = (f_1 - f_2) - f_3$.

(33) $1 \diamond f = f$.

(34) $f_1 - (f_2 - f_3) = (f_1 - f_2) + f_3$.

(35) $f_1 + (f_2 - f_3) = (f_1 + f_2) - f_3$.

(36) $|f_1 \diamond f_2| = |f_1| \diamond |f_2|$.

(37) $|r \diamond f| = |r| \diamond |f|$.

(38) $-f = (-1) \diamond f$.

(39) $-(-f) = f$.

(40) $f_1 - f_2 = f_1 + (-f_2)$.

(41) $f_1 - (-f_2) = f_1 + f_2$.

(42) $\frac{1}{\frac{1}{f}} = f \restriction \operatorname{dom} \frac{1}{f}$.

(43) $\frac{1}{f_1 \diamond f_2} = \frac{1}{f_1} \diamond \frac{1}{f_2}$.

(44) If $r \neq 0$, then $\frac{1}{r \diamond f} = r^{-1} \diamond \frac{1}{f}$.

(45) $\frac{1}{-f} = (-1) \diamond \frac{1}{f}$.

(46) $\frac{1}{|f|} = |\frac{1}{f}|$.

(47) $\frac{f}{g} = f \diamond \frac{1}{g}$.

(48) $r \diamond \frac{g}{f} = \frac{r \diamond g}{f}$.

(49) $\frac{f}{g} \diamond g = f \restriction \operatorname{dom} \frac{1}{g}$.

(50) $\frac{f}{g} \diamond \frac{f_1}{g_1} = \frac{f \diamond f_1}{g \diamond g_1}$.

(51) $\frac{1}{\frac{f_1}{f_2}} = \frac{f_2 \restriction \mathrm{dom} \frac{1}{f_2}}{f_1}$.

(52) $g \diamond \frac{f_1}{f_2} = \frac{g \diamond f_1}{f_2}$.

(53) $\frac{g}{\frac{f_1}{f_2}} = \frac{g \diamond f_2 \restriction \mathrm{dom} \frac{1}{f_2}}{f_1}$.

(54) $-\frac{f}{g} = \frac{-f}{g}$ and $\frac{f}{-g} = -\frac{f}{g}$.

(55) $\frac{f_1}{f} + \frac{f_2}{f} = \frac{f_1+f_2}{f}$ and $\frac{f_1}{f} - \frac{f_2}{f} = \frac{f_1-f_2}{f}$.

(56) $\frac{f_1}{f} + \frac{g_1}{g} = \frac{f_1 \diamond g + g_1 \diamond f}{f \diamond g}$.

(57) $\frac{\frac{f}{g}}{\frac{f_1}{g_1}} = \frac{f \diamond g_1 \restriction \mathrm{dom} \frac{1}{g_1}}{g \diamond f_1}$.

(58) $\frac{f_1}{f} - \frac{g_1}{g} = \frac{f_1 \diamond g - g_1 \diamond f}{f \diamond g}$.

(59) $|\frac{f_1}{f_2}| = \frac{|f_1|}{|f_2|}$.

(60) $(f_1 + f_2) \restriction X = f_1 \restriction X + f_2 \restriction X$ and $(f_1 + f_2) \restriction X = f_1 \restriction X + f_2$ and $(f_1 + f_2) \restriction X = f_1 + f_2 \restriction X$.

(61) $(f_1 \diamond f_2) \restriction X = f_1 \restriction X \diamond f_2 \restriction X$ and $(f_1 \diamond f_2) \restriction X = f_1 \restriction X \diamond f_2$ and $(f_1 \diamond f_2) \restriction X = f_1 \diamond f_2 \restriction X$.

(62) $(-f) \restriction X = -f \restriction X$ and $\frac{1}{f} \restriction X = \frac{1}{f \restriction X}$ and $|f| \restriction X = |f \restriction X|$.

(63) $(f_1 - f_2) \restriction X = f_1 \restriction X - f_2 \restriction X$ and $(f_1 - f_2) \restriction X = f_1 \restriction X - f_2$ and $(f_1 - f_2) \restriction X = f_1 - f_2 \restriction X$.

(64) $\frac{f_1}{f_2} \restriction X = \frac{f_1 \restriction X}{f_2 \restriction X}$ and $\frac{f_1}{f_2} \restriction X = \frac{f_1 \restriction X}{f_2}$ and $\frac{f_1}{f_2} \restriction X = \frac{f_1}{f_2 \restriction X}$.

(65) $(r \diamond f) \restriction X = r \diamond f \restriction X$.

(66) f_1 is total and f_2 is total if and only if $f_1 + f_2$ is total but f_1 is total and f_2 is total if and only if $f_1 - f_2$ is total but f_1 is total and f_2 is total if and only if $f_1 \diamond f_2$ is total.

(67) f is total if and only if $r \diamond f$ is total.

(68) f is total if and only if $-f$ is total.

(69) f is total if and only if $|f|$ is total.

(70) $\frac{1}{f}$ is total if and only if $f^{-1}\{0\} = \emptyset$ and f is total.

(71) f_1 is total and $f_2{}^{-1}\{0\} = \emptyset$ and f_2 is total if and only if $\frac{f_1}{f_2}$ is total.

(72) If f_1 is total and f_2 is total, then $(f_1 + f_2)(c) = f_1(c) + f_2(c)$ and $(f_1 - f_2)(c) = f_1(c) - f_2(c)$ and $(f_1 \diamond f_2)(c) = f_1(c) \cdot f_2(c)$.

(73) If f is total, then $(r \diamond f)(c) = r \cdot f(c)$.

(74) If f is total, then $(-f)(c) = -f(c)$ and $|f|(c) = |f(c)|$.

(75) If $\frac{1}{f}$ is total, then $\frac{1}{f}(c) = (f(c))^{-1}$.

(76) If f_1 is total and $\frac{1}{f_2}$ is total, then $\frac{f_1}{f_2}(c) = f_1(c) \cdot (f_2(c))^{-1}$.

Let us consider X, C. Then $\chi_{X,C}$ is a partial function from C to $\mathbb{R}$.

Next we state a number of propositions:

(77) $f = \chi_{X,C}$ if and only if $\operatorname{dom} f = C$ and for every c holds if $c \in X$, then $f(c) = 1$ but if $c \notin X$, then $f(c) = 0$.

(78) $\chi_{X,C}$ is total.

(79) $c \in X$ if and only if $\chi_{X,C}(c) = 1$.

(80) $c \notin X$ if and only if $\chi_{X,C}(c) = 0$.

(81) $c \in C \setminus X$ if and only if $\chi_{X,C}(c) = 0$.

(82) $\chi_{\emptyset,C}(c) = 0$.

(83) $\chi_{C,C}(c) = 1$.

(84) $\chi_{X,C}(c) \neq 1$ if and only if $\chi_{X,C}(c) = 0$.

(85) If $X \cap Y = \emptyset$, then $\chi_{X,C} + \chi_{Y,C} = \chi_{X \cup Y,C}$.

(86) $\chi_{X,C} \diamond \chi_{Y,C} = \chi_{X \cap Y,C}$.

We now define two new predicates. Let us consider C, f, Y. We say that f is upper bounded on Y if and only if:

there exists r such that for every c such that $c \in Y \cap \operatorname{dom} f$ holds $f(c) \leq r$.

We say that f is lower bounded on Y if and only if:

there exists r such that for every c such that $c \in Y \cap \operatorname{dom} f$ holds $r \leq f(c)$.

Let us consider C, f, Y. We say that f is bounded on Y if and only if:

f is upper bounded on Y and f is lower bounded on Y.

The following propositions are true:

(87) f is upper bounded on Y if and only if there exists r such that for every c such that $c \in Y \cap \operatorname{dom} f$ holds $f(c) \leq r$.

(88) f is lower bounded on Y if and only if there exists r such that for every c such that $c \in Y \cap \operatorname{dom} f$ holds $r \leq f(c)$.

(89) f is bounded on Y if and only if f is upper bounded on Y and f is lower bounded on Y.

(90) f is bounded on Y if and only if there exists r such that for every c such that $c \in Y \cap \operatorname{dom} f$ holds $|f(c)| \leq r$.

(91) If $Y \subseteq X$ and f is upper bounded on X, then f is upper bounded on Y but if $Y \subseteq X$ and f is lower bounded on X, then f is lower bounded on Y but if $Y \subseteq X$ and f is bounded on X, then f is bounded on Y.

(92) If f is upper bounded on X and f is lower bounded on Y, then f is bounded on $X \cap Y$.

(93) If $X \cap \operatorname{dom} f = \emptyset$, then f is bounded on X.

(94) If $0 = r$, then $r \diamond f$ is bounded on Y.

(95) If f is upper bounded on Y and $0 \leq r$, then $r \diamond f$ is upper bounded on Y but if f is upper bounded on Y and $r \leq 0$, then $r \diamond f$ is lower bounded on Y.

(96) If f is lower bounded on Y and $0 \leq r$, then $r \diamond f$ is lower bounded on Y but if f is lower bounded on Y and $r \leq 0$, then $r \diamond f$ is upper bounded on Y.

(97) If f is bounded on Y, then $r \diamond f$ is bounded on Y.

(98) $|f|$ is lower bounded on X.

(99) If f is bounded on Y, then $|f|$ is bounded on Y and $-f$ is bounded on Y.

(100) If f_1 is upper bounded on X and f_2 is upper bounded on Y, then f_1+f_2 is upper bounded on $X \cap Y$ but if f_1 is lower bounded on X and f_2 is lower bounded on Y, then $f_1 + f_2$ is lower bounded on $X \cap Y$ but if f_1 is bounded on X and f_2 is bounded on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(101) If f_1 is bounded on X and f_2 is bounded on Y, then $f_1 \diamond f_2$ is bounded on $X \cap Y$ and $f_1 - f_2$ is bounded on $X \cap Y$.

(102) If f is upper bounded on X and f is upper bounded on Y, then f is upper bounded on $X \cup Y$.

(103) If f is lower bounded on X and f is lower bounded on Y, then f is lower bounded on $X \cup Y$.

(104) If f is bounded on X and f is bounded on Y, then f is bounded on $X \cup Y$.

(105) If f_1 is a constant on X and f_2 is a constant on Y, then $f_1 + f_2$ is a constant on $X \cap Y$ and $f_1 - f_2$ is a constant on $X \cap Y$ and $f_1 \diamond f_2$ is a constant on $X \cap Y$.

(106) If f is a constant on Y, then $p \diamond f$ is a constant on Y.

(107) If f is a constant on Y, then $|f|$ is a constant on Y and $-f$ is a constant on Y.

(108) If f is a constant on Y, then f is bounded on Y.

(109) If f is a constant on Y, then for every r holds $r \diamond f$ is bounded on Y and $-f$ is bounded on Y and $|f|$ is bounded on Y.

(110) If f_1 is upper bounded on X and f_2 is a constant on Y, then $f_1 + f_2$ is upper bounded on $X \cap Y$ but if f_1 is lower bounded on X and f_2 is a constant on Y, then $f_1 + f_2$ is lower bounded on $X \cap Y$ but if f_1 is bounded on X and f_2 is a constant on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(111) (i) If f_1 is upper bounded on X and f_2 is a constant on Y, then $f_1 - f_2$ is upper bounded on $X \cap Y$,

(ii) if f_1 is lower bounded on X and f_2 is a constant on Y, then $f_1 - f_2$ is lower bounded on $X \cap Y$,

(iii) if f_1 is bounded on X and f_2 is a constant on Y, then $f_1 - f_2$ is bounded on $X \cap Y$ and $f_2 - f_1$ is bounded on $X \cap Y$ and $f_1 \diamond f_2$ is bounded on $X \cap Y$.

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Increasing and Continuous Ordinal Sequences

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Summary. Concatenation of two ordinal sequences, the mode of all ordinals belonging to a universe and the mode of sequences of them with length equal to the rank of the universe are introduced. Besides, the increasing and continuous transfinite sequences, the limes of ordinal sequences and the power of ordinals, and the fact that every increasing and continuous transfinite sequence has critical numbers (fixed points) are discussed.

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The terminology and notation used here have been introduced in the following papers: [6], [4], [2], [3], [1], and [5]. We adopt the following convention: phi, fi, psi are sequences of ordinal numbers and A, B, C are ordinal numbers. The following proposition is true

(1) If $\mathrm{dom}\, fi = \mathrm{succ}\, A$, then $(\mathrm{last}\, fi)$ (as an ordinal) is the limit of fi and $\lim fi = \mathrm{last}\, fi$.

Let us consider fi, psi. The functor $fi \frown psi$ yields a sequence of ordinal numbers and is defined as follows:

$\mathrm{dom}(fi \frown psi) = \mathrm{dom}\, fi + \mathrm{dom}\, psi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $(fi \frown psi)(A) = fi(A)$ and for every A such that $A \in \mathrm{dom}\, psi$ holds $(fi \frown psi)(\mathrm{dom}\, fi + A) = psi(A)$.

The following propositions are true:

(2) Let chi be a sequence of ordinal numbers. Then $chi = fi \frown psi$ if and only if $\mathrm{dom}\, chi = \mathrm{dom}\, fi + \mathrm{dom}\, psi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $chi(A) = fi(A)$ and for every A such that $A \in \mathrm{dom}\, psi$ holds $chi(\mathrm{dom}\, fi + A) = psi(A)$.

(3) If A is the limit of psi, then A is the limit of $fi \frown psi$.

(4) If A is the limit of fi, then $B + A$ is the limit of $B + fi$.

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(5) If A is the limit of fi, then $A \cdot B$ is the limit of $fi \cdot B$.

(6) If $\operatorname{dom} fi = \operatorname{dom} psi$ and B is the limit of fi and C is the limit of psi but for every A such that $A \in \operatorname{dom} fi$ holds $fi(A) \subseteq psi(A)$ or for every A such that $A \in \operatorname{dom} fi$ holds $fi(A) \in psi(A)$, then $B \subseteq C$.

In the sequel f_1, f_2 denote sequences of ordinal numbers. One can prove the following propositions:

(7) If $\operatorname{dom} f_1 = \operatorname{dom} fi$ and $\operatorname{dom} fi = \operatorname{dom} f_2$ and A is the limit of f_1 and A is the limit of f_2 and for every A such that $A \in \operatorname{dom} fi$ holds $f_1(A) \subseteq fi(A)$ and $fi(A) \subseteq f_2(A)$, then A is the limit of fi.

(8) If $\operatorname{dom} fi \neq \mathbf{0}$ and $\operatorname{dom} fi$ is a limit ordinal number and fi is increasing, then $\sup fi$ is the limit of fi and $\lim fi = \sup fi$.

(9) If fi is increasing and $A \subseteq B$ and $B \in \operatorname{dom} fi$, then $fi(A) \subseteq fi(B)$.

(10) If fi is increasing and $A \in \operatorname{dom} fi$, then $A \subseteq fi(A)$.

(11) If phi is increasing, then $phi^{-1} A$ is an ordinal number.

(12) If f_1 is increasing, then $f_2 \cdot f_1$ is a sequence of ordinal numbers.

(13) If f_1 is increasing and f_2 is increasing, then there exists phi such that $phi = f_1 \cdot f_2$ and phi is increasing.

(14) If f_1 is increasing and A is the limit of f_2 and $\sup(\operatorname{rng} f_1) = \operatorname{dom} f_2$ and $fi = f_2 \cdot f_1$, then A is the limit of fi.

(15) If phi is increasing, then $phi \restriction A$ is increasing.

(16) If phi is increasing and $\operatorname{dom} phi$ is a limit ordinal number, then $\sup phi$ is a limit ordinal number.

(17) If fi is increasing and fi is continuous and psi is continuous and $phi = psi \cdot fi$, then phi is continuous.

(18) If for every A such that $A \in \operatorname{dom} fi$ holds $fi(A) = C + A$, then fi is increasing.

(19) If $C \neq \mathbf{0}$ and for every A such that $A \in \operatorname{dom} fi$ holds $fi(A) = A \cdot C$, then fi is increasing.

(20) If $A \neq \mathbf{0}$, then $\mathbf{0}^A = \mathbf{0}$.

(21) If $A \neq \mathbf{0}$ and A is a limit ordinal number, then for every fi such that $\operatorname{dom} fi = A$ and for every B such that $B \in A$ holds $fi(B) = C^B$ holds C^A is the limit of fi.

(22) If $C \neq \mathbf{0}$, then $C^A \neq \mathbf{0}$.

(23) If $\mathbf{1} \in C$, then $C^A \in C^{\operatorname{succ} A}$.

(24) If $\mathbf{1} \in C$ and $A \in B$, then $C^A \in C^B$.

(25) If $\mathbf{1} \in C$ and for every A such that $A \in \operatorname{dom} fi$ holds $fi(A) = C^A$, then fi is increasing.

(26) If $\mathbf{1} \in C$ and $A \neq \mathbf{0}$ and A is a limit ordinal number, then for every fi such that $\operatorname{dom} fi = A$ and for every B such that $B \in A$ holds $fi(B) = C^B$ holds $C^A = \sup fi$.

(27) If $C \neq \mathbf{0}$ and $A \subseteq B$, then $C^A \subseteq C^B$.

(28) If $A \subseteq B$, then $A^C \subseteq B^C$.

(29) If $\mathbf{1} \in C$ and $A \neq \mathbf{0}$, then $\mathbf{1} \in C^A$.

(30) $C^{A+B} = (C^B) \cdot (C^A)$.

(31) $(C^A)^B = C^{B \cdot A}$.

(32) If $\mathbf{1} \in C$, then $A \subseteq C^A$.

The scheme *CriticalNumber* concerns a unary functor $\mathcal{F}$ yielding an ordinal number and states that:

there exists A such that $\mathcal{F}(A) = A$

provided the parameter meets the following conditions:

- for all A, B such that $A \in B$ holds $\mathcal{F}(A) \in \mathcal{F}(B)$,
- for every A such that $A \neq \mathbf{0}$ and A is a limit ordinal number for every *phi* such that $\mathrm{dom}\, fi = A$ and for every B such that $B \in A$ holds $phi(B) = \mathcal{F}(B)$ holds $\mathcal{F}(A)$ is the limit of *phi*.

In the sequel W will be a universal class. We now define two new modes. Let us consider W. An ordinal number is said to be an ordinal of W if:

it $\in W$.

A sequence of ordinal numbers is called a transfinite sequence of ordinals of W if:

dom it $= \mathrm{On}\, W$ and rng it $\subseteq \mathrm{On}\, W$.

We now state two propositions:

(33) A is an ordinal of W if and only if $A \in W$.

(34) *phi* is a transfinite sequence of ordinals of W if and only if $\mathrm{dom}\, phi = \mathrm{On}\, W$ and $\mathrm{rng}\, phi \subseteq \mathrm{On}\, W$.

In the sequel A_1, B_1 will be ordinals of W and *phi* will be a transfinite sequence of ordinals of W. The scheme *UOS_Lambda* concerns a universal class $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an ordinal of $\mathcal{A}$ and states that:

there exists a transfinite sequence *phi* of ordinals of $\mathcal{A}$ such that for every ordinal a of $\mathcal{A}$ holds $phi(a) = \mathcal{F}(a)$

for all values of the parameters.

We now define two new functors. Let us consider W. The functor $\mathbf{0}_W$ yielding an ordinal of W is defined as follows:

$\mathbf{0}_W = \mathbf{0}$.

The functor $\mathbf{1}_W$ yields an ordinal of W and is defined by:

$\mathbf{1}_W = \mathbf{1}$.

Let us consider *phi*, A_1. Then $phi(A_1)$ is an ordinal of W.

Let us consider W, and let p_2, p_1 be transfinite sequences of ordinals of W. Then $p_1 \cdot p_2$ is a transfinite sequence of ordinals of W.

We now state the proposition

(35) $\mathbf{0}_W = \mathbf{0}$ and $\mathbf{1}_W = \mathbf{1}$.

Let us consider W, A_1. Then $\mathrm{succ}\, A_1$ is an ordinal of W. Let us consider B_1. Then $A_1 + B_1$ is an ordinal of W.

Let us consider W, A_1, B_1. Then $A_1 \cdot B_1$ is an ordinal of W.

The following propositions are true:

(36) $A_1 \in \operatorname{dom} phi$.

(37) If $\operatorname{dom} fi \in W$ and $\operatorname{rng} fi \subseteq W$, then $\sup fi \in W$.

We now state the proposition

(38) If phi is increasing and phi is continuous and $\omega \in W$, then there exists A such that $A \in \operatorname{dom} phi$ and $phi(A) = A$.

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Transformations in Affine Spaces [1]

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Summary. Two classes of bijections of its point universe are correlated with every affine structure. The first class consists of the transformations, called formal isometries, which map every segment onto congruent segment, the second class consists of the automorphisms of such a structure. Each of these two classes of bijections forms a group for a given affine structure, if it satisfies a very weak axiom system (models of these axioms are called congruence spaces); formal isometries form a normal subgroup in the group of automorphism. In particular ordered affine spaces and affine spaces are congruence spaces; therefore formal isometries of these structures can be considered. They are called positive dilatations and dilatations, resp. For convenience the class of negative dilatations, transformations which map every "vector" onto parallel "vector", but with opposite sense, is singled out. The class of translations is distinguished as well. Basic facts concerning all these types of transformations are established, like rigidity, decomposition principle, introductory group-theoretical properties. At the end collineations of affine spaces and their properties are investigated; for affine planes it is proved that the class of collineationms coincides with the class of bijections preserving lines.

MML Identifier: TRANSGEO.

The papers [7], [1], [8], [2], [3], [4], [5], and [6] provide the notation and terminology for this paper. We adopt the following convention: A denotes a non-empty set, a, b, x, y, z, t denote elements of A, and f, f_1, f_2, g, h denote permutations of A. Let us consider A, and let us consider f, and let x be an element of A. Then $f(x)$ is an element of A.

Let us consider A, and let us consider f, and let X be a subset of A. Then $f \circ X$ is a subset of A.

Let us consider A, f, g. Then $g \cdot f$ is a permutation of A.

One can prove the following propositions:

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(1) For all f_1, f_2 such that for every x holds $f_1(x) = f_2(x)$ holds $f_1 = f_2$.
(2) There exists x such that $f(x) = y$.
(3) If $f(x) = f(y)$, then $x = y$.
(4) $f(x) = y$ if and only if $f^{-1}(y) = x$.
(5) $(f \cdot g)(x) = f(g(x))$.

Let us consider A, f, g. The functor $f \setminus g$ yields a permutation of A and is defined by:

$f \setminus g = (g \cdot f) \cdot g^{-1}$.

One can prove the following proposition

(6) $f \setminus g = (g \cdot f) \cdot g^{-1}$.

The scheme *EXPermutation* deals with a non-empty set $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a permutation f of $\mathcal{A}$ such that for all elements x, y of $\mathcal{A}$ holds $f(x) = y$ if and only if $\mathcal{P}[x, y]$

provided the following requirements are met:

- for every element x of $\mathcal{A}$ there exists an element y of $\mathcal{A}$ such that $\mathcal{P}[x, y]$,
- for every element y of $\mathcal{A}$ there exists an element x of $\mathcal{A}$ such that $\mathcal{P}[x, y]$,
- for all elements x, y, x' of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{P}[x', y]$ holds $x = x'$,
- for all elements x, y, y' of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{P}[x, y']$ holds $y = y'$.

Next we state a number of propositions:

(7) $(\mathrm{id}_A)^{-1} = \mathrm{id}_A$.
(8) $f \cdot f^{-1} = \mathrm{id}_A$ and $f^{-1} \cdot f = \mathrm{id}_A$.
(9) $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.
(10) $\mathrm{id}_A \cdot f = f$ and $f \cdot \mathrm{id}_A = f$.
(11) $f \cdot \mathrm{id}_A = \mathrm{id}_A \cdot f$.
(12) $f \cdot (g \cdot h) = (f \cdot g) \cdot h$.
(13) If $g \cdot f = h \cdot f$ or $f \cdot g = f \cdot h$, then $g = h$.
(14) $(f \cdot g)^{-1} = g^{-1} \cdot f^{-1}$.
(15) $(f^{-1})^{-1} = f$.
(16) $f \cdot g \setminus h = (f \setminus h) \cdot (g \setminus h)$.
(17) $f^{-1} \setminus g = (f \setminus g)^{-1}$.
(18) $f \setminus g \cdot h = (f \setminus h) \setminus g$.
(19) $\mathrm{id}_A \setminus f = \mathrm{id}_A$.
(20) $f \setminus \mathrm{id}_A = f$.
(21) If $f(a) = a$, then $(f \setminus g)(g(a)) = g(a)$.

In the sequel R will denote a binary relation on $[\!: A, A :\!]$. Let us consider A, f, R. We say that f is a formal isometry of R if and only if:

for all x, y holds $\langle\langle x, y\rangle, \langle f(x), f(y)\rangle\rangle \in R$.

The following propositions are true:

(22) f is a formal isometry of R if and only if for all x, y holds $\langle\langle x, y\rangle, \langle f(x), f(y)\rangle\rangle \in R$.

(23) If R is reflexive in $[: A, A :]$, then id_A is a formal isometry of R.

(24) If R is symmetric in $[: A, A :]$ and f is a formal isometry of R, then f^{-1} is a formal isometry of R.

(25) If R is transitive in $[: A, A :]$ and f is a formal isometry of R and g is a formal isometry of R, then $f \cdot g$ is a formal isometry of R.

(26) Suppose that

(i) for all a, b, x, y, z, t such that $\langle\langle x, y\rangle, \langle a, b\rangle\rangle \in R$ and $\langle\langle a, b\rangle, \langle z, t\rangle\rangle \in R$ and $a \neq b$ holds $\langle\langle x, y\rangle, \langle z, t\rangle\rangle \in R$,

(ii) for all x, y, z holds $\langle\langle x, x\rangle, \langle y, z\rangle\rangle \in R$,

(iii) f is a formal isometry of R,

(iv) g is a formal isometry of R.

Then $f \cdot g$ is a formal isometry of R.

Let us consider A, f, R. We say that f is an automorphism of R if and only if:

for all x, y, z, t holds $\langle\langle x, y\rangle, \langle z, t\rangle\rangle \in R$ if and only if $\langle\langle f(x), f(y)\rangle, \langle f(z), f(t)\rangle\rangle \in R$.

The following propositions are true:

(27) For all a, f, R holds f is an automorphism of R if and only if for all x, y, z, t holds $\langle\langle x, y\rangle, \langle z, t\rangle\rangle \in R$ if and only if $\langle\langle f(x), f(y)\rangle, \langle f(z), f(t)\rangle\rangle \in R$.

(28) id_A is an automorphism of R.

(29) If f is an automorphism of R, then f^{-1} is an automorphism of R.

(30) If f is an automorphism of R and g is an automorphism of R, then $g \cdot f$ is an automorphism of R.

(31) If R is symmetric in $[: A, A :]$ and R is transitive in $[: A, A :]$ and f is a formal isometry of R, then f is an automorphism of R.

(32) Suppose that

(i) for all a, b, x, y, z, t such that $\langle\langle x, y\rangle, \langle a, b\rangle\rangle \in R$ and $\langle\langle a, b\rangle, \langle z, t\rangle\rangle \in R$ and $a \neq b$ holds $\langle\langle x, y\rangle, \langle z, t\rangle\rangle \in R$,

(ii) for all x, y, z holds $\langle\langle x, x\rangle, \langle y, z\rangle\rangle \in R$,

(iii) R is symmetric in $[: A, A :]$,

(iv) f is a formal isometry of R.

Then f is an automorphism of R.

(33) If f is a formal isometry of R and g is an automorphism of R, then $f \setminus g$ is a formal isometry of R.

In the sequel AS will be an affine structure. Let us consider AS, and let f be a permutation of the points of AS. We say that f is a dilatation of AS if and only if:

f is a formal isometry of the congruence of AS.

The following proposition is true

(34) For every permutation f of the points of AS holds f is a dilatation of AS if and only if f is a formal isometry of the congruence of AS.

In the sequel a, b denote elements of the points of AS. Next we state the proposition

(35) For every permutation f of the points of AS holds f is a dilatation of AS if and only if for all a, b holds $a, b \upuparrows f(a), f(b)$.

An affine structure is said to be a congruence space if:

(i) for all elements x, y, z, t, a, b of the points of it such that $x, y \upuparrows a, b$ and $a, b \upuparrows z, t$ and $a \neq b$ holds $x, y \upuparrows z, t$,

(ii) for all elements x, y, z of the points of it holds $x, x \upuparrows y, z$,

(iii) for all elements x, y, z, t of the points of it such that $x, y \upuparrows z, t$ holds $z, t \upuparrows x, y$,

(iv) for all elements x, y of the points of it holds $x, y \upuparrows x, y$.

One can prove the following proposition

(36) Let AS be an affine structure. Then AS is a congruence space if and only if the following conditions are satisfied:

  (i) for all elements x, y, z, t, a, b of the points of AS such that $x, y \upuparrows a, b$ and $a, b \upuparrows z, t$ and $a \neq b$ holds $x, y \upuparrows z, t$,

  (ii) for all elements x, y, z of the points of AS holds $x, x \upuparrows y, z$,

  (iii) for all elements x, y, z, t of the points of AS such that $x, y \upuparrows z, t$ holds $z, t \upuparrows x, y$,

  (iv) for all elements x, y of the points of AS holds $x, y \upuparrows x, y$.

In the sequel CS denotes a congruence space. One can prove the following three propositions:

(37) $\mathrm{id}_{\text{the points of } CS}$ is a dilatation of CS.

(38) For every permutation f of the points of CS such that f is a dilatation of CS holds f^{-1} is a dilatation of CS.

(39) For all permutations f, g of the points of CS such that f is a dilatation of CS and g is a dilatation of CS holds $f \cdot g$ is a dilatation of CS.

We follow the rules: OAS denotes an ordered affine space and a, b, c, d, p, q, x, y, z denote elements of the points of OAS. Next we state the proposition

(40) OAS is a congruence space.

In the sequel f, g are permutations of the points of OAS. Let us consider OAS, and let f be a permutation of the points of OAS. We say that f is a positive dilatation if and only if:

f is a dilatation of OAS.

We now state two propositions:

(41) For every permutation f of the points of OAS holds f is a positive dilatation if and only if f is a dilatation of OAS.

(42) For every permutation f of the points of OAS holds f is a positive dilatation if and only if for all a, b holds $a, b \upuparrows f(a), f(b)$.

Let us consider OAS, and let f be a permutation of the points of OAS. We say that f is a negative dilatation if and only if:

for all a, b holds $a, b \upuparrows f(b), f(a)$.

The following propositions are true:

(43) For every permutation f of the points of OAS holds f is a negative dilatation if and only if for all a, b holds $a, b \upuparrows f(b), f(a)$.

(44) $\mathrm{id}_{\text{the points of } OAS}$ is a positive dilatation.

(45) For every permutation f of the points of OAS such that f is a positive dilatation holds f^{-1} is a positive dilatation.

(46) For all permutations f, g of the points of OAS such that f is a positive dilatation and g is a positive dilatation holds $f \cdot g$ is a positive dilatation.

(47) For no f holds f is a negative dilatation and f is a positive dilatation.

(48) If f is a negative dilatation, then f^{-1} is a negative dilatation.

(49) If f is a positive dilatation and g is a negative dilatation, then $f \cdot g$ is a negative dilatation and $g \cdot f$ is a negative dilatation.

Let us consider OAS, and let f be a permutation of the points of OAS. We say that f is a dilatation if and only if:

f is a formal isometry of $\lambda($ the congruence of $OAS)$.

Next we state a number of propositions:

(50) For every permutation f of the points of OAS holds f is a dilatation if and only if f is a formal isometry of $\lambda($ the congruence of $OAS)$.

(51) For every permutation f of the points of OAS holds f is a dilatation if and only if for all a, b holds $a, b \parallel f(a), f(b)$.

(52) If f is a positive dilatation or f is a negative dilatation, then f is a dilatation.

(53) For every permutation f of the points of OAS such that f is a dilatation there exists a permutation f' of the points of $\Lambda(OAS)$ such that $f = f'$ and f' is a dilatation of $\Lambda(OAS)$.

(54) For every permutation f of the points of $\Lambda(OAS)$ such that f is a dilatation of $\Lambda(OAS)$ there exists a permutation f' of the points of OAS such that $f = f'$ and f' is a dilatation.

(55) $\mathrm{id}_{\text{the points of } OAS}$ is a dilatation.

(56) If f is a dilatation, then f^{-1} is a dilatation.

(57) If f is a dilatation and g is a dilatation, then $f \cdot g$ is a dilatation.

(58) If f is a dilatation, then for all a, b, c, d holds $a, b \parallel c, d$ if and only if $f(a), f(b) \parallel f(c), f(d)$.

(59) If f is a dilatation, then for all a, b, c holds $\mathbf{L}(a, b, c)$ if and only if $\mathbf{L}(f(a), f(b), f(c))$.

(60) If f is a dilatation and $\mathbf{L}(x, f(x), y)$, then $\mathbf{L}(x, f(x), f(y))$.

(61) If $a, b \parallel c, d$, then $a, c \parallel b, d$ or there exists x such that $\mathbf{L}(a, c, x)$ and $\mathbf{L}(b, d, x)$.

(62) If f is a dilatation, then $f = \mathrm{id}_{\text{the points of } OAS}$ or for every x holds $f(x) \neq x$ if and only if for all x, y holds $x, f(x) \parallel y, f(y)$.

(63) If f is a dilatation and $a \neq b$ and $f(a) = a$ and $f(b) = b$ and not $\mathbf{L}(a, b, x)$, then $f(x) = x$.

(64) If f is a dilatation and $f(a) = a$ and $f(b) = b$ and $a \neq b$, then $f = \mathrm{id}_{\text{the points of } OAS}$.

(65) If f is a dilatation and g is a dilatation and $f(a) = g(a)$ and $f(b) = g(b)$, then $a = b$ or $f = g$.

Let us consider OAS, and let f be a permutation of the points of OAS. We say that f is a translation if and only if:

f is a dilatation but $f = \mathrm{id}_{\text{the points of } OAS}$ or for every a holds $a \neq f(a)$.

One can prove the following propositions:

(66) For every permutation f of the points of OAS holds f is a translation if and only if f is a dilatation but $f = \mathrm{id}_{\text{the points of } OAS}$ or for every a holds $a \neq f(a)$.

(67) If f is a dilatation, then f is a translation if and only if for all x, y holds $x, f(x) \parallel y, f(y)$.

(68) If f is a translation and $f(a) = a$, then $f = \mathrm{id}_{\text{the points of } OAS}$.

(69) If f is a translation and g is a translation and $f(a) = g(a)$ and $f(a) \neq a$ and not $\mathbf{L}(a, f(a), x)$, then $f(x) = g(x)$.

(70) If f is a translation and g is a translation and $f(a) = g(a)$, then $f = g$.

(71) If f is a translation, then f^{-1} is a translation.

(72) If f is a translation and g is a translation, then $f \cdot g$ is a translation.

(73) If f is a translation, then f is a positive dilatation.

(74) If f is a dilatation and $f(p) = p$ and $\mathbf{B}(q, p, f(q))$ and not $\mathbf{L}(p, q, x)$, then $\mathbf{B}(x, p, f(x))$.

(75) If f is a dilatation and $f(p) = p$ and $\mathbf{B}(q, p, f(q))$ and $q \neq p$, then $\mathbf{B}(x, p, f(x))$.

(76) If f is a dilatation and $f(p) = p$ and $q \neq p$ and $\mathbf{B}(q, p, f(q))$ and not $\mathbf{L}(p, x, y)$, then $x, y \upharpoonleft\!\upharpoonright f(y), f(x)$.

(77) If f is a dilatation and $f(p) = p$ and $q \neq p$ and $\mathbf{B}(q, p, f(q))$ and $\mathbf{L}(p, x, y)$, then $x, y \upharpoonleft\!\upharpoonright f(y), f(x)$.

(78) If f is a dilatation and $f(p) = p$ and $q \neq p$ and $\mathbf{B}(q, p, f(q))$, then f is a negative dilatation.

(79) If f is a dilatation and $f(p) = p$ and for every x holds $p, x \upharpoonleft\!\upharpoonright p, f(x)$, then for all y, z holds $y, z \upharpoonleft\!\upharpoonright f(y), f(z)$.

(80) If f is a dilatation, then f is a positive dilatation or f is a negative dilatation.

We follow the rules: AFS will be an affine space and a, b, c, d, d_1, d_2, x, y, z, t will be elements of the points of AFS. The following propositions are true:

(81) For all a, b, c, d holds $a, b \upharpoonleft\!\upharpoonright c, d$ if and only if $a, b \parallel c, d$.

(82) AFS is a congruence space.

(83) $\Lambda(OAS)$ is a congruence space.

In the sequel f, g denote permutations of the points of AFS. Let us consider AFS, f. We say that f is a dilatation if and only if:

f is a dilatation of AFS.

Next we state a number of propositions:

(84) For every f holds f is a dilatation if and only if f is a dilatation of AFS.

(85) f is a dilatation if and only if for all a, b holds $a, b \parallel f(a), f(b)$.

(86) $\mathrm{id}_{\text{the points of } AFS}$ is a dilatation.

(87) If f is a dilatation, then f^{-1} is a dilatation.

(88) If f is a dilatation and g is a dilatation, then $f \cdot g$ is a dilatation.

(89) If f is a dilatation, then for all a, b, c, d holds $a, b \parallel c, d$ if and only if $f(a), f(b) \parallel f(c), f(d)$.

(90) If f is a dilatation, then for all a, b, c holds $\mathbf{L}(a, b, c)$ if and only if $\mathbf{L}(f(a), f(b), f(c))$.

(91) If f is a dilatation and $\mathbf{L}(x, f(x), y)$, then $\mathbf{L}(x, f(x), f(y))$.

(92) If $a, b \parallel c, d$, then $a, c \parallel b, d$ or there exists x such that $\mathbf{L}(a, c, x)$ and $\mathbf{L}(b, d, x)$.

(93) If f is a dilatation, then $f = \mathrm{id}_{\text{the points of } AFS}$ or for every x holds $f(x) \neq x$ if and only if for all x, y holds $x, f(x) \parallel y, f(y)$.

(94) If f is a dilatation and $a \neq b$ and $f(a) = a$ and $f(b) = b$ and not $\mathbf{L}(a, b, x)$, then $f(x) = x$.

(95) If f is a dilatation and $f(a) = a$ and $f(b) = b$ and $a \neq b$, then $f = \mathrm{id}_{\text{the points of } AFS}$.

(96) If f is a dilatation and g is a dilatation and $f(a) = g(a)$ and $f(b) = g(b)$, then $a = b$ or $f = g$.

(97) If not $\mathbf{L}(a, b, c)$ and $a, b \parallel c, d_1$ and $a, b \parallel c, d_2$ and $a, c \parallel b, d_1$ and $a, c \parallel b, d_2$, then $d_1 = d_2$.

Let us consider AFS, f. We say that f is a translation if and only if:

f is a dilatation but $f = \mathrm{id}_{\text{the points of } AFS}$ or for every a holds $a \neq f(a)$.

One can prove the following propositions:

(98) For every f holds f is a translation if and only if f is a dilatation but $f = \mathrm{id}_{\text{the points of } AFS}$ or for every a holds $a \neq f(a)$.

(99) $\mathrm{id}_{\text{the points of } AFS}$ is a translation.

(100) If f is a dilatation, then f is a translation if and only if for all x, y holds $x, f(x) \parallel y, f(y)$.

(101) If f is a translation and $f(a) = a$, then $f = \mathrm{id}_{\text{the points of } AFS}$.

(102) If f is a translation and g is a translation and $f(a) = g(a)$ and $f(a) \neq a$ and not $\mathbf{L}(a, f(a), x)$, then $f(x) = g(x)$.

(103) If f is a translation and g is a translation and $f(a) = g(a)$, then $f = g$.

(104) If f is a translation, then f^{-1} is a translation.

(105) If f is a translation and g is a translation, then $f \cdot g$ is a translation.

Let us consider AFS, f. We say that f is a collineation if and only if: f is an automorphism of the congruence of AFS.

Next we state four propositions:

(106) f is a collineation if and only if f is an automorphism of the congruence of AFS.

(107) f is a collineation if and only if for all x, y, z, t holds $x, y \parallel z, t$ if and only if $f(x), f(y) \parallel f(z), f(t)$.

(108) If f is a collineation, then $\mathbf{L}(x, y, z)$ if and only if $\mathbf{L}(f(x), f(y), f(z))$.

(109) If f is a collineation and g is a collineation, then f^{-1} is a collineation and $f \cdot g$ is a collineation and $\mathrm{id}_{\text{the points of } AFS}$ is a collineation.

In the sequel A, C, K will denote subsets of the points of AFS. Next we state several propositions:

(110) If $a \in A$, then $f(a) \in f^\circ A$.

(111) $x \in f^\circ A$ if and only if there exists y such that $y \in A$ and $f(y) = x$.

(112) If $f^\circ A = f^\circ C$, then $A = C$.

(113) If f is a collineation, then $f^\circ \operatorname{Line}(a, b) = \operatorname{Line}(f(a), f(b))$.

(114) If f is a collineation and K is a line, then $f^\circ K$ is a line.

(115) If f is a collineation and $A \parallel C$, then $f^\circ A \parallel f^\circ C$.

For simplicity we follow the rules: AFP is an affine plane, A, K are subsets of the points of AFP, p, x are elements of the points of AFP, and f is a permutation of the points of AFP. We now state two propositions:

(116) If for every A such that A is a line holds $f^\circ A$ is a line, then f is a collineation.

(117) If f is a collineation and K is a line and for every x such that $x \in K$ holds $f(x) = x$ and $p \notin K$ and $f(p) = p$, then $f = \mathrm{id}_{\text{the points of } AFP}$.

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Subcategories and Products of Categories

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Summary. The *subcategory* of a category and product of categories is defined. The *inclusion functor* is the injection (inclusion) map $\overset{E}{\hookrightarrow}$ which sends each object and each arrow of a Subcategory E of a category C to itself (in C). The inclusion functor is faithful. *Full subcategories* of C, that is, those subcategories E of C such that $\mathrm{Hom}_E(a,b) = \mathrm{Hom}_C(b,b)$ for any objects a, b of E, are defined. A subcategory E of C is full when the inclusion functor $\overset{E}{\hookrightarrow}$ is full. The proposition that a full subcategory is determined by giving the set of objects of a category is proved. The product of two categories B and C is constructed in the usual way. Moreover, some simple facts on *bifunctors* (functors from a product category) are proved. The final notions in this article are that of projection functors and product of two functors (*complex* functors and *product* functors).

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The terminology and notation used in this paper have been introduced in the following articles: [10], [8], [3], [4], [7], [2], [6], [1], [11], [9], and [5]. For simplicity we follow the rules: X denotes a set, C, D, E denote non-empty sets, c denotes an element of C, and d denotes an element of D. Let us consider D, X, E, and let F be a non-empty set of functions from X to E, and let f be a function from D into F, and let d be an element of D. Then $f(d)$ is an element of F.

In the sequel f denotes a function from $[\!:C,\ D:\!]$ into E. The following propositions are true:

(1) curry f is a function from C into E^D.

(2) curry′ f is a function from D into E^C.

Let us consider C, D, E, f. Then curry f is a function from C into E^D. Then curry′ f is a function from D into E^C.

The following two propositions are true:

(3) $f(\langle c, d\rangle) = (\text{curry}\, f(c))(d)$.

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(4) $f(\langle c,d\rangle) = (\text{curry}' f(d))(c)$.

In the sequel B, C, D, C', D' denote categories. Let us consider B, C, and let c be an object of C. The functor $B \longmapsto c$ yielding a functor from B to C is defined as follows:

$B \longmapsto c =$ (the morphisms of B) $\longmapsto \mathrm{id}_c$.

One can prove the following propositions:

(5) For every object c of C holds $B \longmapsto c =$ (the morphisms of B) $\longmapsto \mathrm{id}_c$.

(6) For every object c of C and for every morphism f of B holds $(B \longmapsto c)(f) = \mathrm{id}_c$.

(7) For every object c of C and for every object b of B holds $(\mathrm{Obj}(B \longmapsto c))(b) = c$.

Let us consider C, D. The functor $\mathrm{Funct}(C,D)$ yields a non-empty set and is defined by:

for an arbitrary x holds $x \in \mathrm{Funct}(C,D)$ if and only if x is a functor from C to D.

Next we state two propositions:

(8) For every non-empty set F holds $F = \mathrm{Funct}(C,D)$ if and only if for an arbitrary x holds $x \in F$ if and only if x is a functor from C to D.

(9) For every element T of $\mathrm{Funct}(C,D)$ holds T is a functor from C to D.

Let us consider C, D. A non-empty set is called a non-empty set of functors from C into D if:

for every element x of it holds x is a functor from C to D.

The following proposition is true

(10) For every non-empty set F holds F is a non-empty set of functors from C into D if and only if for every element x of F holds x is a functor from C to D.

Let us consider C, D, and let F be a non-empty set of functors from C into D. We see that it makes sense to consider the following mode for restricted scopes of arguments. Then all the objests of the mode element of F are a functor from C to D.

Let A be a non-empty set, and let us consider C, D, and let F be a non-empty set of functors from C into D, and let T be a function from A into F, and let x be an element of A. Then $T(x)$ is an element of F.

Let us consider C, D. Then $\mathrm{Funct}(C,D)$ is a non-empty set of functors from C into D.

Let us consider C. A category is said to be a subcategory of C if:

(i) the objects of it $\subseteq$ the objects of C,

(ii) for all objects a, b of it and for all objects a', b' of C such that $a = a'$ and $b = b'$ holds $\hom(a,b) \subseteq \hom(a',b')$,

(iii) the composition of it $\leq$ the composition of C,

(iv) for every object a of it and for every object a' of C such that $a = a'$ holds $\mathrm{id}_a = \mathrm{id}_{a'}$.

Next we state the proposition

(11) Given C, D. Then D is a subcategory of C if and only if the following conditions are satisfied:

(i) the objects of $D \subseteq$ the objects of C,

(ii) for all objects a, b of D and for all objects a', b' of C such that $a = a'$ and $b = b'$ holds $\hom(a, b) \subseteq \hom(a', b')$,

(iii) the composition of $D \leq$ the composition of C,

(iv) for every object a of D and for every object a' of C such that $a = a'$ holds $\mathrm{id}_a = \mathrm{id}_{a'}$.

In the sequel E will be a subcategory of C. We now state several propositions:

(12) For every object e of E holds e is an object of C.

(13) The morphisms of $E \subseteq$ the morphisms of C.

(14) For every morphism f of E holds f is a morphism of C.

(15) For every morphism f of E and for every morphism f' of C such that $f = f'$ holds $\mathrm{dom}\, f = \mathrm{dom}\, f'$ and $\mathrm{cod}\, f = \mathrm{cod}\, f'$.

(16) For all objects a, b of E and for all objects a', b' of C and for every morphism f from a to b such that $a = a'$ and $b = b'$ and $\hom(a, b) \neq \emptyset$ holds f is a morphism from a' to b'.

(17) For all morphisms f, g of E and for all morphisms f', g' of C such that $f = f'$ and $g = g'$ and $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $g \cdot f = g' \cdot f'$.

(18) C is a subcategory of C.

(19) id_E is a functor from E to C.

Let us consider C, E. The functor $\overset{E}{\hookrightarrow}$ yielding a functor from E to C is defined as follows:

$\overset{E}{\hookrightarrow} = \mathrm{id}_E$.

The following propositions are true:

(20) $\overset{E}{\hookrightarrow} = \mathrm{id}_E$.

(21) For every morphism f of E holds $\overset{E}{\hookrightarrow}(f) = f$.

(22) For every object a of E holds $(\mathrm{Obj} \overset{E}{\hookrightarrow})(a) = a$.

(23) For every object a of E holds $\overset{E}{\hookrightarrow}(a) = a$.

(24) $\overset{E}{\hookrightarrow}$ is faithful.

(25) $\overset{E}{\hookrightarrow}$ is full if and only if for all objects a, b of E and for all objects a', b' of C such that $a = a'$ and $b = b'$ holds $\hom(a, b) = \hom(a', b')$.

Let C be a category structure, and let us consider D. We say that C is full subcategory of D if and only if:

C is a subcategory of D and for all objects a, b of C and for all objects a', b' of D such that $a = a'$ and $b = b'$ holds $\hom(a, b) = \hom(a', b')$.

The following propositions are true:

(26) For every C being a category structure and for every D holds C is full subcategory of D if and only if C is a subcategory of D and for all objects

a, b of C and for all objects a', b' of D such that $a = a'$ and $b = b'$ holds $\hom(a, b) = \hom(a', b')$.

(27) E is full subcategory of C if and only if $\overset{E}{\hookrightarrow}$ is full.

(28) For every non-empty subset O of the objects of C holds $\bigcup\{\hom(a, b) : a \in O \wedge b \in O\}$ is a non-empty subset of the morphisms of C.

(29) Let O be a non-empty subset of the objects of C. Let M be a non-empty set. Suppose $M = \bigcup\{\hom(a, b) : a \in O \wedge b \in O\}$. Then (the dom-map of C) $\restriction M$ is a function from M into O and (the cod-map of C) $\restriction M$ is a function from M into O and (the composition of C) $\restriction [\!: M, M :\!]$ is a partial function from $[\!: M, M :\!]$ to M and (the id-map of C) $\restriction O$ is a function from O into M.

(30) Let O be a non-empty subset of the objects of C. Let M be a non-empty set. Let d, c be functions from M into O. Let p be a partial function from $[\!: M, M :\!]$ to M. Let i be a function from O into M. Suppose $M = \bigcup\{\hom(a, b) : a \in O \wedge b \in O\}$ and $d =$ (the dom-map of C) $\restriction M$ and $c =$ (the cod-map of C) $\restriction M$ and $p =$ (the composition of C) $\restriction [\!: M, M :\!]$ and $i =$ (the id-map of C) $\restriction O$. Then $\langle O, M, d, c, p, i\rangle$ is full subcategory of C.

(31) Let O be a non-empty subset of the objects of C. Let M be a non-empty set. Let d, c be functions from M into O. Let p be a partial function from $[\!: M, M :\!]$ to M. Let i be a function from O into M. Suppose $\langle O, M, d, c, p, i\rangle$ is full subcategory of C. Then $M = \bigcup\{\hom(a, b) : a \in O \wedge b \in O\}$ and $d =$ (the dom-map of C) $\restriction M$ and $c =$ (the cod-map of C) $\restriction M$ and $p =$ (the composition of C) $\restriction [\!: M, M :\!]$ and $i =$ (the id-map of C) $\restriction O$.

Let X_1, X_2, Y_1, Y_2 be non-empty sets, and let f_1 be a function from X_1 into Y_1, and let f_2 be a function from X_2 into Y_2. Then $[\!: f_1, f_2 :\!]$ is a function from $[\!: X_1, X_2 :\!]$ into $[\!: Y_1, Y_2 :\!]$.

Let A, B be non-empty sets, and let f be a partial function from $[\!: A, A :\!]$ to A, and let g be a partial function from $[\!: B, B :\!]$ to B. Then $|\!: f, g :\!|$ is a partial function from $[\!: [\!: A, B :\!], [\!: A, B :\!] :\!]$ to $[\!: A, B :\!]$.

Let us consider C, D. The functor $[\!: C, D :\!]$ yielding a category is defined as follows:

$[\!: C, D :\!] = \langle [\!:$ the objects of C, the objects of $D :\!], [\!:$ the morphisms of C, the morphisms of $D :\!], [\!:$ the dom-map of C, the dom-map of $D :\!], [\!:$ the cod-map of C, the cod-map of $D :\!], |\!:$ the composition of C, the composition of $D :\!|, [\!:$ the id-map of C, the id-map of $D :\!]\rangle$.

Next we state three propositions:

(32) $[\!: C, D :\!] = \langle [\!:$ the objects of C, the objects of $D :\!], [\!:$ the morphisms of C, the morphisms of $D :\!], [\!:$ the dom-map of C, the dom-map of $D :\!], [\!:$ the cod-map of C, the cod-map of $D :\!], |\!:$ the composition of C, the composition of $D :\!|, [\!:$ the id-map of C, the id-map of $D :\!]\rangle$.

(33) (i) The objects of $[\!: C, D :\!] = [\!:$ the objects of C, the objects of $D :\!]$,

(ii) the morphisms of $[\!: C, D :\!] = [\!:$ the morphisms of C, the morphisms of $D :\!]$,

(iii) the dom-map of $[\!: C, D :\!] = [\!:$ the dom-map of C, the dom-map of $D :\!]$,

(iv) the cod-map of $[\!: C, D :\!] = [\!:$ the cod-map of C, the cod-map of $D :\!]$,

(v) the composition of $[\!: C, D :\!] = |\!:$ the composition of C, the composition of $D\!:\!|$,

(vi) the id-map of $[\!: C, D :\!] = [\!:$ the id-map of C, the id-map of $D :\!]$.

(34) For every object c of C and for every object d of D holds $\langle c, d \rangle$ is an object of $[\!: C, D :\!]$.

Let us consider C, D, and let c be an object of C, and let d be an object of D. Then $\langle c, d \rangle$ is an object of $[\!: C, D :\!]$.

One can prove the following propositions:

(35) For every object cd of $[\!: C, D :\!]$ there exists an object c of C and there exists an object d of D such that $cd = \langle c, d \rangle$.

(36) For every morphism f of C and for every morphism g of D holds $\langle f, g \rangle$ is a morphism of $[\!: C, D :\!]$.

Let us consider C, D, and let f be a morphism of C, and let g be a morphism of D. Then $\langle f, g \rangle$ is a morphism of $[\!: C, D :\!]$.

The following propositions are true:

(37) For every morphism fg of $[\!: C, D :\!]$ there exists a morphism f of C and there exists a morphism g of D such that $fg = \langle f, g \rangle$.

(38) For every morphism f of C and for every morphism g of D holds $\operatorname{dom} \langle f, g \rangle = \langle \operatorname{dom} f, \operatorname{dom} g \rangle$ and $\operatorname{cod} \langle f, g \rangle = \langle \operatorname{cod} f, \operatorname{cod} g \rangle$.

(39) For all morphisms f, f' of C and for all morphisms g, g' of D such that $\operatorname{dom} f' = \operatorname{cod} f$ and $\operatorname{dom} g' = \operatorname{cod} g$ holds $\langle f', g' \rangle \cdot \langle f, g \rangle = \langle f' \cdot f, g' \cdot g \rangle$.

(40) For all morphisms f, f' of C and for all morphisms g, g' of D such that $\operatorname{dom} \langle f', g' \rangle = \operatorname{cod} \langle f, g \rangle$ holds $\langle f', g' \rangle \cdot \langle f, g \rangle = \langle f' \cdot f, g' \cdot g \rangle$.

(41) For every object c of C and for every object d of D holds $\mathrm{id}_{\langle c, d \rangle} = \langle \mathrm{id}_c, \mathrm{id}_d \rangle$.

(42) For all objects c, c' of C and for all objects d, d' of D holds $\hom(\langle c, d \rangle, \langle c', d' \rangle) = [\!: \hom(c, c'), \hom(d, d') :\!]$.

(43) For all objects c, c' of C and for every morphism f from c to c' and for all objects d, d' of D and for every morphism g from d to d' such that $\hom(c, c') \neq \emptyset$ and $\hom(d, d') \neq \emptyset$ holds $\langle f, g \rangle$ is a morphism from $\langle c, d \rangle$ to $\langle c', d' \rangle$.

(44) For every functor S from $[\!: C, C' :\!]$ to D and for every object c of C holds $\operatorname{curry} S(\mathrm{id}_c)$ is a functor from C' to D.

(45) For every functor S from $[\!: C, C' :\!]$ to D and for every object c' of C' holds $\operatorname{curry}' S(\mathrm{id}_{c'})$ is a functor from C to D.

Let us consider C, C', D, and let S be a functor from $[\!: C, C' :\!]$ to D, and let c be an object of C. The functor $S(c, -)$ yields a functor from C' to D and is defined as follows:

$S(c, -) = \operatorname{curry} S(\mathrm{id}_c)$.

The following three propositions are true:

(46) For every functor S from $[\!: C, C' :\!]$ to D and for every object c of C holds $S(c, -) = \operatorname{curry} S(\mathrm{id}_c)$.

(47) For every functor S from $[\!: C, C' :\!]$ to D and for every object c of C and for every morphism f of C' holds $S(c, -)(f) = S(\langle \mathrm{id}_c, f \rangle)$.

(48) For every functor S from $[\!: C, C' :\!]$ to D and for every object c of C and for every object c' of C' holds $(\mathrm{Obj}\, S(c, -))(c') = (\mathrm{Obj}\, S)(\langle c, c' \rangle)$.

Let us consider C, C', D, and let S be a functor from $[\!: C, C' :\!]$ to D, and let c' be an object of C'. The functor $S(-, c')$ yielding a functor from C to D is defined by:

$S(-, c') = \operatorname{curry}' S(\mathrm{id}_{c'})$.

We now state several propositions:

(49) For every functor S from $[\!: C, C' :\!]$ to D and for every object c' of C' holds $S(-, c') = \operatorname{curry}' S(\mathrm{id}_{c'})$.

(50) For every functor S from $[\!: C, C' :\!]$ to D and for every object c' of C' and for every morphism f of C holds $S(-, c')(f) = S(\langle f, \mathrm{id}_{c'} \rangle)$.

(51) For every functor S from $[\!: C, C' :\!]$ to D and for every object c of C and for every object c' of C' holds $(\mathrm{Obj}\, S(-, c'))(c) = (\mathrm{Obj}\, S)(\langle c, c' \rangle)$.

(52) Let L be a function from the objects of C into $\mathrm{Funct}(B, D)$. Let M be a function from the objects of B into $\mathrm{Funct}(C, D)$. Suppose that

(i) for every object c of C and for every object b of B holds $(M(b))(\mathrm{id}_c) = (L(c))(\mathrm{id}_b)$,

(ii) for every morphism f of B and for every morphism g of C holds $(M(\operatorname{cod} f))(g) \cdot (L(\operatorname{dom} g))(f) = (L(\operatorname{cod} g))(f) \cdot (M(\operatorname{dom} f))(g)$.

Then there exists a functor S from $[\!: B, C :\!]$ to D such that for every morphism f of B and for every morphism g of C holds $S(\langle f, g \rangle) = (L(\operatorname{cod} g))(f) \cdot (M(\operatorname{dom} f))(g)$.

(53) Let L be a function from the objects of C into $\mathrm{Funct}(B, D)$. Let M be a function from the objects of B into $\mathrm{Funct}(C, D)$. Suppose there exists a functor S from $[\!: B, C :\!]$ to D such that for every object c of C and for every object b of B holds $S(-, c) = L(c)$ and $S(b, -) = M(b)$. Then for every morphism f of B and for every morphism g of C holds $(M(\operatorname{cod} f))(g) \cdot (L(\operatorname{dom} g))(f) = (L(\operatorname{cod} g))(f) \cdot (M(\operatorname{dom} f))(g)$.

(54) $\pi_1($ (the morphisms of $C)\times$ (the morphisms of $D))$ is a functor from $[\!: C, D :\!]$ to C.

(55) $\pi_2($ (the morphisms of $C)\times$ (the morphisms of $D))$ is a functor from $[\!: C, D :\!]$ to D.

We now define two new functors. Let us consider C, D. The functor $\pi_1(C \times D)$ yields a functor from $[\!: C, D :\!]$ to C and is defined as follows:

$\pi_1(C \times D) = \pi_1($ (the morphisms of $C)\times$ (the morphisms of $D))$.

The functor $\pi_2(C \times D)$ yielding a functor from $[\!: C, D :\!]$ to D is defined as follows:

$\pi_2(C \times D) = \pi_2($ (the morphisms of $C)\times$ (the morphisms of $D))$.

One can prove the following propositions:

(56) $\pi_1(C \times D) = \pi_1($ (the morphisms of $C) \times$ (the morphisms of D)).

(57) $\pi_2(C \times D) = \pi_2($ (the morphisms of $C) \times$ (the morphisms of D)).

(58) For every morphism f of C and for every morphism g of D holds $\pi_1(C \times D)(\langle f, g \rangle) = f$.

(59) For every object c of C and for every object d of D holds $(\mathrm{Obj}\,\pi_1(C \times D))(\langle c, d \rangle) = c$.

(60) For every morphism f of C and for every morphism g of D holds $\pi_2(C \times D)(\langle f, g \rangle) = g$.

(61) For every object c of C and for every object d of D holds $(\mathrm{Obj}\,\pi_2(C \times D))(\langle c, d \rangle) = d$.

(62) For every functor T from C to D and for every functor T' from C to D' holds $\langle T, T' \rangle$ is a functor from C to $[\!: D, D' :\!]$.

Let us consider C, D, D', and let T be a functor from C to D, and let T' be a functor from C to D'. Then $\langle T, T' \rangle$ is a functor from C to $[\!: D, D' :\!]$.

One can prove the following propositions:

(63) For every functor T from C to D and for every functor T' from C to D' and for every object c of C holds $(\mathrm{Obj}\langle T, T' \rangle)(c) = \langle (\mathrm{Obj}\,T)(c), (\mathrm{Obj}\,T')(c) \rangle$.

(64) For every functor T from C to D and for every functor T' from C' to D' holds $[\!: T, T' :\!] = \langle T \cdot \pi_1(C \times C'), T' \cdot \pi_2(C \times C') \rangle$.

(65) For every functor T from C to D and for every functor T' from C' to D' holds $[\!: T, T' :\!]$ is a functor from $[\!: C, C' :\!]$ to $[\!: D, D' :\!]$.

Let us consider C, C', D, D', and let T be a functor from C to D, and let T' be a functor from C' to D'. Then $[\!: T, T' :\!]$ is a functor from $[\!: C, C' :\!]$ to $[\!: D, D' :\!]$.

One can prove the following proposition

(66) For every functor T from C to D and for every functor T' from C' to D' and for every object c of C and for every object c' of C' holds $(\mathrm{Obj}[\!: T, T' :\!])(\langle c, c' \rangle) = \langle (\mathrm{Obj}\,T)(c), (\mathrm{Obj}\,T')(c') \rangle$.

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Many-Argument Relations

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Summary. Definitions of relations based on finite sequences. The arity of relation, the set of logical values *Boolean* consisting of *false* and *true* and the operations of negation and conjunction on them are defined.

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The notation and terminology used in this paper have been introduced in the following papers: [5], [2], [1], [3], and [4]. In the sequel x, y will be arbitrary, k will denote a natural number, and D will denote a non-empty set. Let B, A be non-empty sets, and let b be an element of B. Then $A \longmapsto b$ is an element of B^A.

A set is said to be a relation if:
for an arbitrary x such that $x \in$ it holds x is a finite sequence and for all finite sequences a, b such that $a \in$ it and $b \in$ it holds $\operatorname{len} a = \operatorname{len} b$.

We follow a convention: X denotes a set, p, r denote relations, and a, b denote finite sequences. We now state several propositions:

(4)[2] For every X such that for every x such that $x \in X$ holds x is a finite sequence and for all a, b such that $a \in X$ and $b \in X$ holds $\operatorname{len} a = \operatorname{len} b$ holds X is a relation.

(5) If $x \in p$, then x is a finite sequence.

(6) If $a \in p$ and $b \in p$, then $\operatorname{len} a = \operatorname{len} b$.

(7) If $X \subseteq p$, then X is a relation.

(8) $\{a\}$ is a relation.

(9) $\{\langle x, y\rangle\}$ is a relation.

The scheme *rel_exist* concerns a set $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

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[2]The propositions (1)–(3) became obvious.

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there exists r such that for every a holds $a \in r$ if and only if $a \in \mathcal{A}$ and $\mathcal{P}[a]$ provided the parameters satisfy the following condition:

- for all a, b such that $\mathcal{P}[a]$ and $\mathcal{P}[b]$ holds $\operatorname{len} a = \operatorname{len} b$.

Let us consider p, r. Let us note that one can characterize the predicate $p = r$ by the following (equivalent) condition: for every a holds $a \in p$ if and only if $a \in r$.

We now state the proposition

(10) $p = r$ if and only if for every a holds $a \in p$ if and only if $a \in r$.

The relation $\varnothing$ is defined by:

$a \notin \varnothing$.

One can prove the following propositions:

(11) $a \notin \varnothing$.

(12) $p = \varnothing$ if and only if for no a holds $a \in p$.

(13) $\varnothing = \emptyset$.

Let us consider p. Let us assume that $p \neq \varnothing$. The functor $\mathrm{Arity}(p)$ yielding a natural number is defined by:

for every a such that $a \in p$ holds $\mathrm{Arity}(p) = \operatorname{len} a$.

We now state two propositions:

(14) If $p \neq \varnothing$, then for every k holds $k = \mathrm{Arity}(p)$ if and only if for every a such that $a \in p$ holds $k = \operatorname{len} a$.

(15) If $a \in p$ and $p \neq \varnothing$, then $\mathrm{Arity}(p) = \operatorname{len} a$.

Let us consider k. A relation is called a k-ary relation if:

for every a such that $a \in$ it holds $\operatorname{len} a = k$.

One can prove the following two propositions:

(16) For all k, r such that for every a such that $a \in r$ holds $\operatorname{len} a = k$ holds r is a k-ary relation.

(17) For every k-ary relation r such that $a \in r$ holds $\operatorname{len} a = k$.

Let X be a set. A relation is called a relation on X if:

for every a such that $a \in$ it holds $\operatorname{rng} a \subseteq X$.

In the sequel X denotes a set. Next we state four propositions:

(18) For all X, r such that for every a such that $a \in r$ holds $\operatorname{rng} a \subseteq X$ holds r is a relation on X.

(19) For every relation r on X such that $a \in r$ holds $\operatorname{rng} a \subseteq X$.

(20) $\varnothing$ is a relation on X.

(21) $\varnothing$ is a k-ary relation.

Let us consider X, k. A relation is called a k-ary relation of X if:

it is a relation on X and it is a k-ary relation.

Next we state two propositions:

(22) For every relation r holds r is a k-ary relation of X if and only if r is a relation on X and r is a k-ary relation.

(23) For every k-ary relation R of X holds R is a relation on X and R is a k-ary relation.

Let us consider D. The functor $\mathrm{Rel}(D)$ yielding a non-empty family of sets is defined as follows:

for every X holds $X \in \mathrm{Rel}(D)$ if and only if $X \subseteq D^*$ and for all finite sequences a, b of elements of D such that $a \in X$ and $b \in X$ holds $\mathrm{len}\, a = \mathrm{len}\, b$.

The following propositions are true:

(24) For every non-empty set D and for every non-empty family S of sets holds $S = \mathrm{Rel}(D)$ if and only if for every X holds $X \in S$ if and only if $X \subseteq D^*$ and for all finite sequences a, b of elements of D such that $a \in X$ and $b \in X$ holds $\mathrm{len}\, a = \mathrm{len}\, b$.

(25) $X \in \mathrm{Rel}(D)$ if and only if $X \subseteq D^*$ and for all finite sequences a, b of elements of D such that $a \in X$ and $b \in X$ holds $\mathrm{len}\, a = \mathrm{len}\, b$.

Let D be a non-empty set. A relation on D is an element of $\mathrm{Rel}(D)$.

In the sequel a will denote a finite sequence of elements of D and p, r will denote elements of $\mathrm{Rel}(D)$. Next we state three propositions:

(26) If $X \subseteq r$, then X is an element of $\mathrm{Rel}(D)$.

(27) $\{a\}$ is an element of $\mathrm{Rel}(D)$.

(28) For all elements x, y of D holds $\{\langle x, y\rangle\}$ is an element of $\mathrm{Rel}(D)$.

Let us consider D, p, r. Let us note that one can characterize the predicate $p = r$ by the following (equivalent) condition: for every a holds $a \in p$ if and only if $a \in r$.

One can prove the following proposition

(29) $p = r$ if and only if for every a holds $a \in p$ if and only if $a \in r$.

The scheme *rel_D_exist* deals with a non-empty set $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists an element r of $\mathrm{Rel}(\mathcal{A})$ such that for every finite sequence a of elements of $\mathcal{A}$ holds $a \in r$ if and only if $\mathcal{P}[a]$

provided the parameters satisfy the following condition:

- for all finite sequences a, b of elements of $\mathcal{A}$ such that $\mathcal{P}[a]$ and $\mathcal{P}[b]$ holds $\mathrm{len}\, a = \mathrm{len}\, b$.

Let us consider D. The functor $\varnothing_D$ yielding an element of $\mathrm{Rel}(D)$ is defined as follows:

$a \notin \varnothing_D$.

The following three propositions are true:

(30) $r = \varnothing_D$ if and only if for no a holds $a \in r$.

(31) $a \notin \varnothing_D$.

(32) $\varnothing_D = \emptyset$.

Let us consider D, p. Let us assume that $p \neq \varnothing_D$. The functor $\mathrm{Arity}(p)$ yielding a natural number is defined by:

if $a \in p$, then $\mathrm{Arity}(p) = \mathrm{len}\, a$.

Next we state two propositions:

(33) If $p \neq \varnothing_D$, then for every k holds $k = \mathrm{Arity}(p)$ if and only if for every a such that $a \in p$ holds $k = \mathrm{len}\, a$.

(34) If $a \in p$ and $p \neq \varnothing_D$, then $\mathrm{Arity}(p) = \mathrm{len}\, a$.

The scheme *rel_D_exist2* concerns a non-empty set $\mathcal{A}$, a natural number $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

there exists an element r of $\mathrm{Rel}(\mathcal{A})$ such that for every finite sequence a of elements of $\mathcal{A}$ such that $\mathrm{len}\, a = \mathcal{B}$ holds $a \in r$ if and only if $\mathcal{P}[a]$
for all values of the parameters.

The non-empty set *Boolean* is defined by:

Boolean $= \{0, 1\}$.

We now define two new functors. The element *false* of *Boolean* is defined by:

false $= 0$.

The element *true* of *Boolean* is defined as follows:

true $= 1$.

The following four propositions are true:

(35) *Boolean* $= \{0, 1\}$.

(36) *false* $= 0$ and *true* $= 1$.

(37) *Boolean* $= \{$*false*, *true*$\}$.

(38) *false* $\neq$ *true*.

In the sequel u, v, w will denote elements of *Boolean*. Next we state the proposition

(39) $v =$ *false* or $v =$ *true*.

We now define two new functors. Let us consider v. The functor $\neg v$ yielding an element of *Boolean* is defined by:

$\neg v =$ *true* if $v =$ *false*, $\neg v =$ *false* if $v =$ *true*.

Let us consider w. The functor $v \wedge w$ yielding an element of *Boolean* is defined by:

$v \wedge w =$ *true* if $v =$ *true* and $w =$ *true*, $v \wedge w =$ *false*, otherwise.

The following propositions are true:

(40) $\neg(\neg v) = v$.

(41) $v =$ *false* if and only if $\neg v =$ *true* but $v =$ *true* if and only if $\neg v =$ *false*.

(42) If $v \neq$ *false*, then $v =$ *true* but if $v \neq$ *true*, then $v =$ *false*.

(43) $v \neq$ *true* if and only if $v =$ *false*.

(44) It is not true that: $v =$ *true* and $w =$ *true* if and only if $v =$ *false* or $w =$ *false*.

(45) $v \wedge w =$ *true* if and only if $v =$ *true* and $w =$ *true* but $v \wedge w =$ *false* if and only if $v =$ *false* or $w =$ *false*.

(46) $v \wedge \neg v =$ *false*.

(47) $\neg(v \wedge \neg v) =$ *true*.

(48) $v \wedge w = w \wedge v$.

(49) *false* $\wedge\, v =$ *false*.

(50) *true* $\wedge v = v$.

(51) If $v \wedge v =$ *false*, then $v =$ *false*.

(52) $v \wedge (w \wedge u) = (v \wedge w) \wedge u$.

Let us consider X. The functor *Boolean*(*false* $\notin X$) yields an element of *Boolean* and is defined as follows:

Boolean(*false* $\notin X$) $=$ *true* if *false* $\notin X$, *Boolean*(*false* $\notin X$) $=$ *false*, otherwise.

One can prove the following proposition

(53) *false* $\notin X$ if and only if *Boolean*(*false* $\notin X$) $=$ *true* but *false* $\in X$ if and only if *Boolean*(*false* $\notin X$) $=$ *false*.

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Interpretation and Satisfiability in the First Order Logic

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Summary. The main notion discussed is satisfiability. Interpretation and some auxiliary concepts are also introduced.

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The articles [6], [3], [1], [5], [4], [2], and [7] provide the notation and terminology for this paper. In the sequel i, k are natural numbers and A, D are non-empty sets. Let us consider A. The functor $\boldsymbol{V}(A)$ yields a non-empty set of functions and is defined by:

$\boldsymbol{V}(A) = A^{\mathrm{BoundVar}}$.

The following propositions are true:

(1) $\boldsymbol{V}(A) = A^{\mathrm{BoundVar}}$.

(2) For an arbitrary x such that x is an element of $\boldsymbol{V}(A)$ holds x is a function from BoundVar into A.

Let us consider A. Then $\boldsymbol{V}(A)$ is a non-empty set of functions from BoundVar to A.

In the sequel x, y will be bound variables and v, v_1 will be elements of $\boldsymbol{V}(A)$. Let us consider A, v, x. Then $v(x)$ is an element of A.

We now define two new functors. Let us consider A, and let p be an element of $Boolean^A$. The functor $\neg p$ yields an element of $Boolean^A$ and is defined by:

for every element x of A holds $(\neg p)(x) = \neg(p(x))$.

Let q be an element of $Boolean^A$. The functor $p \wedge q$ yielding an element of $Boolean^A$ is defined as follows:

for every element x of A holds $(p \wedge q)(x) = (p(x)) \wedge (q(x))$.

We now state two propositions:

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(4)[2] For every element p of $Boolean^A$ and for every element x of A holds $\neg p(x) = \neg(p(x))$.

(5) For all elements p, q of $Boolean^A$ and for every element x of A holds $p \wedge q(x) = (p(x)) \wedge (q(x))$.

Let us consider A, and let f be an element of $Boolean^{\boldsymbol{V}(A)}$, and let us consider v. Then $f(v)$ is an element of $Boolean$.

Let us consider A, x, and let p be an element of $Boolean^{\boldsymbol{V}(A)}$. The functor $\bigwedge_x p$ yields an element of $Boolean^{\boldsymbol{V}(A)}$ and is defined as follows:

for every v holds $(\bigwedge_x p)(v) = Boolean(false \notin \{p(v') : \bigwedge_y [x \neq y \Rightarrow v'(y) = v(y)]\})$.

Next we state three propositions:

(6) For all x, v and for every element p of $Boolean^{\boldsymbol{V}(A)}$ holds $(\bigwedge_x p)(v) = Boolean(false \notin \{p(v') : \bigwedge[x \neq y \Rightarrow v'(y) = v(y)]\})$.

(7) For every element p of $Boolean^{\boldsymbol{V}(A)}$ holds $(\bigwedge_x p)(v) = false$ if and only if there exists v_1 such that $p(v_1) = false$ and for every y such that $x \neq y$ holds $v_1(y) = v(y)$.

(8) For every element p of $Boolean^{\boldsymbol{V}(A)}$ holds $(\bigwedge_x p)(v) = true$ if and only if for every v_1 such that for every y such that $x \neq y$ holds $v_1(y) = v(y)$ holds $p(v_1) = true$.

In the sequel ll is a variables list of k. Let us consider A, v, k, ll. The functor $ll[v]$ yielding a finite sequence of elements of A is defined as follows:

$\mathrm{len}(ll[v]) = k$ and for every i such that $1 \leq i$ and $i \leq k$ holds $(ll[v])(i) = v(ll(i))$.

We now state the proposition

(9) For all v, k, ll holds $\mathrm{len}(ll[v]) = k$ and for every natural number i such that $1 \leq i$ and $i \leq k$ holds $ll[v](i) = v(ll(i))$.

Let us consider A, k, ll, and let r be an element of $\mathrm{Rel}(A)$. The functor $ll \epsilon r$ yields an element of $Boolean^{\boldsymbol{V}(A)}$ and is defined by:

for every element v of $\boldsymbol{V}(A)$ holds if $ll[v] \in r$, then $(ll \epsilon r)(v) = true$ but if $ll[v] \notin r$, then $(ll \epsilon r)(v) = false$.

Next we state the proposition

(10) For all k, ll, v and for every element r of $\mathrm{Rel}(A)$ holds if $ll[v] \in r$, then $ll \epsilon r(v) = true$ but if $ll[v] \notin r$, then $ll \epsilon r(v) = false$.

Let us consider A, and let F be a function from $\mathrm{WFF}_{\mathrm{CQC}}$ into $Boolean^{\boldsymbol{V}(A)}$, and let p be an element of $\mathrm{WFF}_{\mathrm{CQC}}$. Then $F(p)$ is an element of $Boolean^{\boldsymbol{V}(A)}$.

Let us consider D. A function from PredSym into $\mathrm{Rel}(D)$ is called an interpretation of D if:

for every element P of PredSym and for every element r of $\mathrm{Rel}(D)$ such that $\mathrm{it}(P) = r$ holds $r = \emptyset_D$ or $\mathrm{Arity}(P) = \mathrm{Arity}(r)$.

[2]The proposition (3) became obvious.

Next we state two propositions:

(11) For every non-empty set D and for every function F from PredSym into Rel(D) such that for every element P of PredSym and for every element r of Rel(D) such that $F(P) = r$ holds $r = \varnothing_D$ or Arity(P) = Arity(r) holds F is an interpretation of D.

(12) For every D and for every interpretation J of D and for every element P of PredSym and for every element r of Rel(D) such that $J(P) = r$ holds $r = \varnothing_D$ or Arity(P) = Arity(r).

Let us consider A, and let J be an interpretation of A, and let p be an element of PredSym. Then $J(p)$ is a set.

For simplicity we adopt the following rules: p, q, t will be elements of $\mathrm{WFF}_{\mathrm{CQC}}$, J will be an interpretation of A, P will be a k-ary predicate symbol, and r will be an element of Rel(A). Let us consider A, k, J, P. Then $J(P)$ is an element of Rel(A).

Let us consider A, J, p. The functor Valid(p, J) yielding an element of $\mathit{Boolean}^{\boldsymbol{V}(A)}$ is defined by:

there exists a function F from $\mathrm{WFF}_{\mathrm{CQC}}$ into $\mathit{Boolean}^{\boldsymbol{V}(A)}$ such that Valid$(p, J) = F(p)$ and for all elements p, q of $\mathrm{WFF}_{\mathrm{CQC}}$ and for every bound variable x and for every natural number k and for every variables list ll of k and for every k-ary predicate symbol P and for all elements p', q' of $\mathit{Boolean}^{\boldsymbol{V}(A)}$ such that $p' = F(p)$ and $q' = F(q)$ holds

$F(\mathrm{VERUM}) = \boldsymbol{V}(A) \longmapsto \mathit{true}$

and $F(P[ll]) = ll\epsilon(J(P))$ and $F(\neg p) = \neg p'$ and $F(p \wedge q) = p' \wedge q'$ and $F(\forall_x p) = \bigwedge_x p'$.

We now state a number of propositions:

(13) Valid(VERUM, J) = $\boldsymbol{V}(A) \longmapsto \mathit{true}$.

(14) Valid(VERUM, $J)(v) = \mathit{true}$.

(15) Valid$(P[ll], J) = ll\epsilon(J(P))$.

(16) If $p = P[ll]$ and $r = J(P)$, then $ll[v] \in r$ if and only if Valid$(p, J)(v) = \mathit{true}$.

(17) If $p = P[ll]$ and $r = J(P)$, then $ll[v] \notin r$ if and only if Valid$(p, J)(v) = \mathit{false}$.

(18) If $p = P[ll]$ and $r = J(P)$, then $ll[v] \notin r$ if and only if Valid$(p, J)(v) = \mathit{false}$.

(19) Valid$(\neg p, J) = \neg$ Valid(p, J).

(20) Valid$(\neg p, J)(v) = \neg($Valid$(p, J)(v))$.

(21) Valid$(p \wedge q, J)$ = Valid$(p, J) \wedge$ Valid(q, J).

(22) Valid$(p \wedge q, J)(v) = ($Valid$(p, J)(v)) \wedge ($Valid$(q, J)(v))$.

(23) Valid$(\forall_x p, J) = \bigwedge_x$ Valid(p, J).

(24) Valid$(p \wedge \neg p, J)(v) = \mathit{false}$.

(25) Valid$(\neg(p \wedge \neg p), J)(v) = \mathit{true}$.

Let us consider A, p, J, v. The predicate $J, v \models p$ is defined by:
$\mathrm{Valid}(p, J)(v) = \mathit{true}$.

The following propositions are true:

(26) $J, v \models p$ if and only if $\mathrm{Valid}(p, J)(v) = \mathit{true}$.

(27) $J, v \models P[ll]$ if and only if $ll\epsilon(J(P))(v) = \mathit{true}$.

(28) $J, v \models \neg p$ if and only if $J, v \not\models p$.

(29) $J, v \models p \wedge q$ if and only if $J, v \models p$ and $J, v \models q$.

(30) $J, v \models \forall_x p$ if and only if $(\bigwedge_x \mathrm{Valid}(p, J))(v) = \mathit{true}$.

(31) $J, v \models \forall_x p$ if and only if for every v_1 such that for every y such that $x \neq y$ holds $v_1(y) = v(y)$ holds $\mathrm{Valid}(p, J)(v_1) = \mathit{true}$.

(32) $\mathrm{Valid}(\neg(\neg p), J) = \mathrm{Valid}(p, J)$.

(33) $\mathrm{Valid}(p \wedge p, J) = \mathrm{Valid}(p, J)$.

(34) $\mathrm{Valid}(p \wedge p, J)(v) = \mathrm{Valid}(p, J)(v)$.

(35) $J, v \models p \Rightarrow q$ if and only if $\mathrm{Valid}(p, J)(v) = \mathit{false}$ or $\mathrm{Valid}(q, J)(v) = \mathit{true}$.

(36) $J, v \models p \Rightarrow q$ if and only if if $J, v \models p$, then $J, v \models q$.

(37) For every element p of $\mathit{Boolean}^{\boldsymbol{V}(A)}$ such that $(\bigwedge_x p)(v) = \mathit{true}$ holds $p(v) = \mathit{true}$.

Let us consider A, J, p. The predicate $J \models p$ is defined by:
for every v holds $J, v \models p$.

One can prove the following proposition

(38) $J \models p$ if and only if for every v holds $J, v \models p$.

In the sequel w denotes an element of $\boldsymbol{V}(A)$. The scheme *Lambda_Val* deals with a non-empty set $\mathcal{A}$, a bound variable $\mathcal{B}$, a bound variable $\mathcal{C}$, an element $\mathcal{D}$ of $\boldsymbol{V}(\mathcal{A})$, and an element $\mathcal{E}$ of $\boldsymbol{V}(\mathcal{A})$ and states that:

there exists an element v of $\boldsymbol{V}(\mathcal{A})$ such that for every bound variable x such that $x \neq \mathcal{B}$ holds $v(x) = \mathcal{D}(x)$ and $v(\mathcal{B}) = \mathcal{E}(\mathcal{C})$
for all values of the parameters.

One can prove the following three propositions:

(39) If $x \notin \mathrm{snb}(p)$, then for all v, w such that for every y such that $x \neq y$ holds $w(y) = v(y)$ holds $\mathrm{Valid}(p, J)(v) = \mathrm{Valid}(p, J)(w)$.

(40) If $J, v \models p$ and $x \notin \mathrm{snb}(p)$, then for every w such that for every y such that $x \neq y$ holds $w(y) = v(y)$ holds $J, w \models p$.

(41) $J, v \models \forall_x p$ if and only if for every w such that for every y such that $x \neq y$ holds $w(y) = v(y)$ holds $J, w \models p$.

In the sequel s' will be a formula. We now state a number of propositions:

(42) If $x \neq y$ and $p = s'(x)$ and $q = s'(y)$, then for every v such that $v(x) = v(y)$ holds $\mathrm{Valid}(p, J)(v) = \mathrm{Valid}(q, J)(v)$.

(43) If $x \neq y$ and $x \notin \mathrm{snb}(s')$, then $x \notin \mathrm{snb}(s'(y))$.

(44) $J, v \models \mathrm{VERUM}$.

(45) $J, v \models p \wedge q \Rightarrow q \wedge p$.

(46) $J, v \models (\neg p \Rightarrow p) \Rightarrow p$.
(47) $J, v \models p \Rightarrow (\neg p \Rightarrow q)$.
(48) $J, v \models (p \Rightarrow q) \Rightarrow (\neg(q \wedge t) \Rightarrow \neg(p \wedge t))$.
(49) If $J, v \models p$ and $J, v \models p \Rightarrow q$, then $J, v \models q$.
(50) $J, v \models (\forall_x p) \Rightarrow p$.
(51) $J \models \mathrm{VERUM}$.
(52) $J \models p \wedge q \Rightarrow q \wedge p$.
(53) $J \models (\neg p \Rightarrow p) \Rightarrow p$.
(54) $J \models p \Rightarrow (\neg p \Rightarrow q)$.
(55) $J \models (p \Rightarrow q) \Rightarrow (\neg(q \wedge t) \Rightarrow \neg(p \wedge t))$.
(56) If $J \models p$ and $J \models p \Rightarrow q$, then $J \models q$.
(57) $J \models (\forall_x p) \Rightarrow p$.
(58) If $J \models p \Rightarrow q$ and $x \notin \mathrm{snb}(p)$, then $J \models p \Rightarrow (\forall_x q)$.
(59) For every formula s such that $p = s(x)$ and $q = s(y)$ and $x \notin \mathrm{snb}(s)$ and $J \models p$ holds $J \models q$.

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Probability

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Summary. Some further theorems concerning probability, among them the equivalent definition of probability are discussed, followed by notions of independence of events and conditional probability and basic theorems on them.

MML Identifier: PROB_2.

The notation and terminology used in this paper have been introduced in the following papers: [8], [2], [4], [3], [6], [5], [9], [7], and [1]. For simplicity we adopt the following convention: *Omega* denotes a non-empty set, f denotes a function, m, n denote natural numbers, r, r_1, r_2, r_3 denote real numbers, *seq*, seq_1 denote sequences of real numbers, *Sigma* denotes a σ-field of subsets of *Omega*, *ASeq*, *BSeq* denote sequences of subsets of *Sigma*, P, P_1 denote probabilities on *Sigma*, and A, B, C, A_1, A_2, A_3 denote events of *Sigma*. One can prove the following propositions:

(1) $(r - r_1) + r_2 = (r + r_2) - r_1$.

(2) $r \leq r_1$ if and only if $r < r_1$ or $r = r_1$.

(3) For all r, r_1, r_2 such that $0 < r$ and $r_1 \leq r_2$ holds $\frac{r_1}{r} \leq \frac{r_2}{r}$.

(4) For all r, r_1, r_2, r_3 such that $r \neq 0$ and $r_1 \neq 0$ holds $\frac{r_3}{r_1} = \frac{r_2}{r}$ if and only if $r_3 \cdot r = r_2 \cdot r_1$.

(5) If *seq* is convergent and for every n holds $seq_1(n) = r - seq(n)$, then seq_1 is convergent and $\lim seq_1 = r - \lim seq$.

(6) $A \cap Omega = A$ and $Omega \cap A = A$ and $A \cap \Omega_{Sigma} = A$ and $\Omega_{Sigma} \cap A = A$.

(7) If B misses C, then $A \cap B$ misses $A \cap C$ and $B \cap A$ misses $C \cap A$.

The scheme *SeqEx* concerns a unary functor $\mathcal{F}$ and states that:
there exists f such that $\operatorname{dom} f = \mathbb{N}$ and for every n holds $f(n) = \mathcal{F}(n)$

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for all values of the parameter.

Let us consider *Omega*, *Sigma*, *ASeq*, n. Then $ASeq(n)$ is an event of *Sigma*.

Let us consider *Omega*, *Sigma*, *ASeq*. The functor $\bigcap ASeq$ yielding an event of *Sigma* is defined by:

$\bigcap ASeq = \text{Intersection}\, ASeq$.

One can prove the following propositions:

(8) $\bigcap ASeq = \text{Intersection}\, ASeq$.

(9) For every B, *ASeq* there exists *BSeq* such that for every n holds $BSeq(n) = ASeq(n) \cap B$.

(10) For all B, *ASeq*, *BSeq* such that *ASeq* is nonincreasing and for every n holds $BSeq(n) = ASeq(n) \cap B$ holds *BSeq* is nonincreasing.

(11) For every function f from *Sigma* into $\mathbb{R}$ and for all *ASeq*, n holds $(f \cdot ASeq)(n) = f(ASeq(n))$.

(12) For all *ASeq*, *BSeq*, B such that for every n holds $BSeq(n) = ASeq(n) \cap B$ holds $(\text{Intersection}\, ASeq) \cap B = \text{Intersection}\, BSeq$.

(13) For all P, P_1 such that for every A holds $P(A) = P_1(A)$ holds $P = P_1$.

(14) For every *Omega* and for every sequence *ASeq* of subsets of *Omega* holds *ASeq* is nonincreasing if and only if for every n holds $ASeq(n+1) \subseteq ASeq(n)$.

(15) For every sequence *ASeq* of subsets of *Omega* holds *ASeq* is nondecreasing if and only if for every n holds $ASeq(n) \subseteq ASeq(n+1)$.

(16) For all sequences *ASeq*, *BSeq* of subsets of *Omega* such that for every n holds $ASeq(n) = BSeq(n)$ holds $ASeq = BSeq$.

(17) For every sequence *ASeq* of subsets of *Omega* holds *ASeq* is nonincreasing if and only if Complement *ASeq* is nondecreasing.

Let us consider *Omega*, *Sigma*, *ASeq*. The functor $ASeq^{\mathbf{c}}$ yields a sequence of subsets of *Sigma* and is defined by:

$ASeq^{\mathbf{c}} = \text{Complement}\, ASeq$.

The following proposition is true

(18) $ASeq^{\mathbf{c}} = \text{Complement}\, ASeq$.

Let us consider *Omega*, *Sigma*, *ASeq*. We say that *ASeq* is pairwise disjoint if and only if:

for all m, n such that $m \neq n$ holds $ASeq(m)$ misses $ASeq(n)$.

We now state a number of propositions:

(19) *ASeq* is pairwise disjoint if and only if for all m, n such that $m \neq n$ holds $ASeq(m)$ misses $ASeq(n)$.

(20) Let P be a function from *Sigma* into $\mathbb{R}$. Then P is a probability on *Sigma* if and only if the following conditions are satisfied:

(i) for every A holds $0 \leq P(A)$,

(ii) $P(Omega) = 1$,

(iii) for all A, B such that A misses B holds $P(A \cup B) = P(A) + P(B)$,

(iv) for every $ASeq$ such that $ASeq$ is nondecreasing holds $P \cdot ASeq$ is convergent and $\lim(P \cdot ASeq) = P(\mathrm{Union}\,ASeq)$.

(21) $P((A \cup B) \cup C) = (((P(A) + P(B)) + P(C)) - ((P(A \cap B) + P(B \cap C)) + P(A \cap C))) + P((A \cap B) \cap C)$.

(22) $P(A \setminus A \cap B) = P(A) - P(A \cap B)$.

(23) For all P, A, B holds $P(A \cap B) \leq P(B)$ and $P(A \cap B) \leq P(A)$.

(24) For all P, A, B, C such that $C = B^{\mathrm{c}}$ holds $P(A) = P(A \cap B) + P(A \cap C)$.

(25) For all P, A, B holds $(P(A) + P(B)) - 1 \leq P(A \cap B)$.

(26) For all P, A holds $P(A) = 1 - P(\Omega_{Sigma} \setminus A)$.

(27) For all P, A holds $P(A) < 1$ if and only if $0 < P(\Omega_{Sigma} \setminus A)$.

(28) For all P, A holds $P(\Omega_{Sigma} \setminus A) < 1$ if and only if $0 < P(A)$.

We now define two new predicates. Let us consider $Omega$, $Sigma$, P, A, B. We say that A and B are independent w.r.t P if and only if:

$P(A \cap B) = P(A) \cdot P(B)$.

Let us consider C. We say that A, B and C are independent w.r.t P if and only if:

(i) $P((A \cap B) \cap C) = (P(A) \cdot P(B)) \cdot P(C)$,
(ii) $P(A \cap B) = P(A) \cdot P(B)$,
(iii) $P(A \cap C) = P(A) \cdot P(C)$,
(iv) $P(B \cap C) = P(B) \cdot P(C)$.

We now state a number of propositions:

(29) A and B are independent w.r.t P if and only if $P(A \cap B) = P(A) \cdot P(B)$.

(30) A, B and C are independent w.r.t P if and only if the following conditions are satisfied:
(i) $P((A \cap B) \cap C) = (P(A) \cdot P(B)) \cdot P(C)$,
(ii) $P(A \cap B) = P(A) \cdot P(B)$,
(iii) $P(A \cap C) = P(A) \cdot P(C)$,
(iv) $P(B \cap C) = P(B) \cdot P(C)$.

(31) For all A, B, P holds A and B are independent w.r.t P if and only if B and A are independent w.r.t P.

(32) For all A, B, C, P holds A, B and C are independent w.r.t P if and only if $P((A \cap B) \cap C) = (P(A) \cdot P(B)) \cdot P(C)$ and A and B are independent w.r.t P and B and C are independent w.r.t P and A and C are independent w.r.t P.

(33) For all A, B, C, P such that A, B and C are independent w.r.t P holds B, A and C are independent w.r.t P.

(34) For all A, B, C, P such that A, B and C are independent w.r.t P holds A, C and B are independent w.r.t P.

(35) A and $\emptyset_{Sigma}$ are independent w.r.t P.

(36) A and Ω_{Sigma} are independent w.r.t P.

(37) For all A, B, P such that A and B are independent w.r.t P holds A and $\Omega_{Sigma} \setminus B$ are independent w.r.t P.

(38) For all A, B, P such that A and B are independent w.r.t P holds $\Omega_{Sigma} \setminus A$ and $\Omega_{Sigma} \setminus B$ are independent w.r.t P.

(39) For all A, B, C, P such that A and B are independent w.r.t P and A and C are independent w.r.t P and B misses C holds A and $B \cup C$ are independent w.r.t P.

(40) For all P, A, B such that A and B are independent w.r.t P and $P(A) < 1$ and $P(B) < 1$ holds $P(A \cup B) < 1$.

Let us consider *Omega*, *Sigma*, P, B. Let us assume that $0 < P(B)$. The functor $\mathrm{P}(P/B)$ yielding a probability on *Sigma* is defined by:

for every A holds $(\mathrm{P}(P/B))(A) = \frac{P(A \cap B)}{P(B)}$.

Next we state a number of propositions:

(41) For all P, B such that $0 < P(B)$ for every A holds $\mathrm{P}(P/B)(A) = \frac{P(A \cap B)}{P(B)}$.

(42) For all P, B, A such that $0 < P(B)$ holds $P(A \cap B) = \mathrm{P}(P/B)(A) \cdot P(B)$.

(43) For all P, A, B, C such that $0 < P(A \cap B)$ holds $P((A \cap B) \cap C) = (P(A) \cdot \mathrm{P}(P/A)(B)) \cdot \mathrm{P}(P/(A \cap B))(C)$.

(44) For all P, A, B, C such that $C = B^{\mathrm{c}}$ and $0 < P(B)$ and $0 < P(C)$ holds $P(A) = \mathrm{P}(P/B)(A) \cdot P(B) + \mathrm{P}(P/C)(A) \cdot P(C)$.

(45) Given P, A, A_1, A_2, A_3. Suppose A_1 misses A_2 and $A_3 = (A_1 \cup A_2)^{\mathrm{c}}$ and $0 < P(A_1)$ and $0 < P(A_2)$ and $0 < P(A_3)$. Then $P(A) = (\mathrm{P}(P/A_1)(A) \cdot P(A_1) + \mathrm{P}(P/A_2)(A) \cdot P(A_2)) + \mathrm{P}(P/A_3)(A) \cdot P(A_3)$.

(46) For all P, A, B such that $0 < P(B)$ holds $\mathrm{P}(P/B)(A) = P(A)$ if and only if A and B are independent w.r.t P.

(47) For all P, A, B such that $0 < P(B)$ and $P(B) < 1$ and $\mathrm{P}(P/B)(A) = \mathrm{P}(P/(\Omega_{Sigma} \setminus B))(A)$ holds A and B are independent w.r.t P.

(48) For all P, A, B such that $0 < P(B)$ holds $\frac{(P(A)+P(B))-1}{P(B)} \leq \mathrm{P}(P/B)(A)$.

(49) For all A, B, P such that $0 < P(A)$ and $0 < P(B)$ holds $\mathrm{P}(P/B)(A) = \frac{\mathrm{P}(P/A)(B) \cdot P(A)}{P(B)}$.

(50) Given B, A_1, A_2, P. Suppose $0 < P(B)$ and $A_2 = {A_1}^{\mathrm{c}}$ and $0 < P(A_1)$ and $0 < P(A_2)$. Then

  (i) $\mathrm{P}(P/B)(A_1) = \frac{\mathrm{P}(P/A_1)(B) \cdot P(A_1)}{\mathrm{P}(P/A_1)(B) \cdot P(A_1) + \mathrm{P}(P/A_2)(B) \cdot P(A_2)}$,

  (ii) $\mathrm{P}(P/B)(A_2) = \frac{\mathrm{P}(P/A_2)(B) \cdot P(A_2)}{\mathrm{P}(P/A_1)(B) \cdot P(A_1) + \mathrm{P}(P/A_2)(B) \cdot P(A_2)}$.

(51) Given B, A_1, A_2, A_3, P. Suppose $0 < P(B)$ and $0 < P(A_1)$ and $0 < P(A_2)$ and $0 < P(A_3)$ and A_1 misses A_2 and $A_3 = (A_1 \cup A_2)^{\mathrm{c}}$. Then

  (i) $\mathrm{P}(P/B)(A_1) = \frac{\mathrm{P}(P/A_1)(B) \cdot P(A_1)}{(\mathrm{P}(P/A_1)(B) \cdot P(A_1) + \mathrm{P}(P/A_2)(B) \cdot P(A_2)) + \mathrm{P}(P/A_3)(B) \cdot P(A_3)}$,

  (ii) $\mathrm{P}(P/B)(A_2) = \frac{\mathrm{P}(P/A_2)(B) \cdot P(A_2)}{(\mathrm{P}(P/A_1)(B) \cdot P(A_1) + \mathrm{P}(P/A_2)(B) \cdot P(A_2)) + \mathrm{P}(P/A_3)(B) \cdot P(A_3)}$,

  (iii) $\mathrm{P}(P/B)(A_3) = \frac{\mathrm{P}(P/A_3)(B) \cdot P(A_3)}{(\mathrm{P}(P/A_1)(B) \cdot P(A_1) + \mathrm{P}(P/A_2)(B) \cdot P(A_2)) + \mathrm{P}(P/A_3)(B) \cdot P(A_3)}$.

(52) For all A, B, P such that $0 < P(B)$ holds $1 - \frac{P(\Omega_{Sigma} \setminus A)}{P(B)} \leq \mathrm{P}(P/B)(A)$.

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Translations in Affine Planes [1]

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Summary. Connections between Minor Desargues Axiom and the transitivity of translation groups are investigated. A formal proof of the theorem which establishes the equivalence of these two properties of affine planes is given. We also prove that, under additional requirement, the plane in question satisfies Fano Axiom; its translation group is uniquely two-divisible.

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The terminology and notation used in this paper are introduced in the following papers: [1], [3], [4], [2], and [5]. We adopt the following rules: AS is an affine space and a, b, c, d, p, q, r, x are elements of the points of AS. Let us consider AS. We say that AS satisfies Fano Axiom if and only if:

for all a, b, c, d such that $a,b \parallel c,d$ and $a,c \parallel b,d$ and $a,d \parallel b,c$ holds $\mathbf{L}(a,b,c)$.

The following propositions are true:

(1) AS satisfies Fano Axiom if and only if for all a, b, c, d such that $a,b \parallel c,d$ and $a,c \parallel b,d$ and $a,d \parallel b,c$ holds $\mathbf{L}(a,b,c)$.

(2) If there exist a, b, c such that $\mathbf{L}(a,b,c)$ and $a \neq b$ and $a \neq c$ and $b \neq c$, then for all p, q such that $p \neq q$ there exists r such that $\mathbf{L}(p,q,r)$ and $p \neq r$ and $q \neq r$.

(3) If there exist a, b such that $a \neq b$ and for every x such that $\mathbf{L}(a,b,x)$ holds $x = a$ or $x = b$, then for all p, q, r such that $p \neq q$ and $\mathbf{L}(p,q,r)$ holds $r = p$ or $r = q$.

We follow a convention: AFP is an affine plane, a, a', b, b', c, c', d, p, q, r, x, y are elements of the points of AFP, and f, g, f_1, f_2 are permutations of the points of AFP. We now state a number of propositions:

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(4) If AFP satisfies Fano Axiom and $a, b \parallel c, d$ and $a, c \parallel b, d$ and not $\mathbf{L}(a, b, c)$,
then there exists p such that $\mathbf{L}(b, c, p)$ and $\mathbf{L}(a, d, p)$.

(5) If f is a translation and not $\mathbf{L}(a, f(a), x)$ and $a, f(a) \parallel x, y$ and $a, x \parallel f(a), y$, then $y = f(x)$.

(6) AFP satisfies **des** if and only if for all a, a', b, c, b', c' such that not $\mathbf{L}(a, a', b)$ and not $\mathbf{L}(a, a', c)$ and $a, a' \parallel b, b'$ and $a, a' \parallel c, c'$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ holds $b, c \parallel b', c'$.

(7) There exists f such that f is a translation and $f(a) = a$.

(8) If for all p, q, r such that $p \neq q$ and $\mathbf{L}(p, q, r)$ holds $r = p$ or $r = q$ and $a, b \parallel p, q$ and $a, p \parallel b, q$ and not $\mathbf{L}(a, b, p)$, then $a, q \parallel b, p$.

(9) If AFP satisfies **des**, then there exists f such that f is a translation and $f(a) = b$.

(10) If for every a, b there exists f such that f is a translation and $f(a) = b$, then AFP satisfies **des**.

(11) If f is a translation and g is a translation and not $\mathbf{L}(a, f(a), g(a))$, then $f \cdot g = g \cdot f$.

(12) If AFP satisfies **des** and f is a translation and g is a translation, then $f \cdot g = g \cdot f$.

(13) If f is a translation and g is a translation and $p, f(p) \parallel p, g(p)$, then $p, f(p) \parallel p, (f \cdot g)(p)$.

(14) If AFP satisfies Fano Axiom and AFP satisfies **des** and f is a translation, then there exists g such that g is a translation and $g \cdot g = f$.

(15) If AFP satisfies Fano Axiom and f is a translation and $f \cdot f = \mathrm{id}_{\text{the points of } AFP}$,
then $f = \mathrm{id}_{\text{the points of } AFP}$.

(16) If AFP satisfies **des** and AFP satisfies Fano Axiom and g is a translation and f_1 is a translation and f_2 is a translation and $g = f_1 \cdot f_1$ and $g = f_2 \cdot f_2$, then $f_1 = f_2$.

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Introduction to Probability

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Summary. Definitions of Elementary Event and Event in any sample space E are given. Next, the probability of an Event when E is finite is introduced and some properties of this function are investigated. Last part of the paper is devoted to the conditional probability and essential properties of this function (Bayes Theorem).

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The articles [7], [8], [3], [6], [5], [2], [4], and [1] provide the terminology and notation for this paper. For simplicity we follow the rules: E will denote a non-empty set, a will denote an element of E, A, B, B_1, B_2, B_3, C will denote subsets of E, X, Y will denote sets, and p will denote a finite sequence. Let us consider E. A subset of E is called an elementary event of E if:

it $\subseteq E$ and it $\neq \emptyset$ but $Y \subseteq$ it if and only if $Y = \emptyset$ or $Y =$ it.

In the sequel e, e_1, e_2 will denote elementary events of E. One can prove the following propositions:

(1) If e is an elementary event of E, then $e \subseteq E$.

(2) If e is an elementary event of E, then $e \neq \emptyset$.

(3) For every e such that e is an elementary event of E holds $Y \subseteq e$ if and only if $Y = \emptyset$ or $Y = e$.

(4) e is an elementary event of E if and only if $e \subseteq E$ and $e \neq \emptyset$ but $Y \subseteq e$ if and only if $Y = \emptyset$ or $Y = e$.

(5) If e is an elementary event of E and $e = A \cup B$ and $A \neq B$, then $A = \emptyset$ and $B = e$ or $A = e$ and $B = \emptyset$.

(6) If e is an elementary event of E and $e = A \cup B$, then $A = e$ and $B = e$ or $A = e$ and $B = \emptyset$ or $A = \emptyset$ and $B = e$.

(7) If $a \in E$, then $\{a\}$ is an elementary event of E.

(8) If $\{a\}$ is an elementary event of E, then $a \in E$.

(9) $a \in E$ if and only if $\{a\}$ is an elementary event of E.

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(10) If e_1 is an elementary event of E and e_2 is an elementary event of E and $e_1 \subseteq e_2$, then $e_1 = e_2$.

(11) If e is an elementary event of E, then there exists a such that $a \in E$ and $e = \{a\}$.

(12) For every E there exists e such that e is an elementary event of E.

(13) For every E such that e is an elementary event of E holds e is finite.

(14) If e is an elementary event of E, then there exists p such that p is a finite sequence of elements of E and $\operatorname{rng} p = e$ and $\operatorname{len} p = 1$.

Let us consider E. An event of E is a subset of E.

The following propositions are true:

(15) For every subset X of E holds X is an event of E.

(16) $\emptyset$ is an event of E.

(17) E is an event of E.

(18) If A is an event of E and B is an event of E, then $A \cap B$ is an event of E.

(19) If A is an event of E and B is an event of E, then $A \cup B$ is an event of E.

(20) If $A \subseteq B$ and B is an event of E, then A is an event of E.

(21) If A is an event of E, then A^{c} is an event of E.

(22) If e is an elementary event of E and A is an event of E, then $e \cap A = \emptyset$ or $e \cap A = e$.

(23) If A is an event of E and B is an event of E, then $A \setminus B$ is an event of E.

(24) If e is an elementary event of E, then e is an event of E.

(25) If A is an event of E and $A \neq \emptyset$, then there exists e such that e is an elementary event of E and $e \subseteq A$.

(26) If e is an elementary event of E and A is an event of E and $e \subseteq A \cup A^{\mathrm{c}}$, then $e \subseteq A$ or $e \subseteq A^{\mathrm{c}}$.

(27) If e_1 is an elementary event of E and e_2 is an elementary event of E, then $e_1 = e_2$ or $e_1 \cap e_2 = \emptyset$.

Let us consider X, Y. We say that X exclude Y if and only if: $X \cap Y = \emptyset$.

Next we state several propositions:

(28) X exclude Y if and only if $X \cap Y = \emptyset$.

(29) If X exclude Y, then Y exclude X.

(30) A exclude A^{c}.

(31) For every A holds A exclude $\emptyset$.

(32) A exclude B if and only if $A \setminus B = A$.

(33) $A \cap B$ exclude $A \setminus B$.

(34) $A \cap B$ exclude $A \cap B^{\mathrm{c}}$.

(35) If A exclude B, then A exclude $B \cap C$.

(36) If A exclude B, then $A \cap C$ exclude $B \cap C$.

Let us consider E. Let us assume that E is finite. Let us consider A. The functor $\mathrm{P}(A)$ yields a real number and is defined as follows:

$\mathrm{P}(A) = \frac{\operatorname{card} A}{\operatorname{card} E}$.

Let us consider E. Then Ω_E is an event of E. Then $\emptyset_E$ is an event of E.

The following propositions are true:

(37) If E is finite and A is an event of E, then $\mathrm{P}(A) = \frac{\operatorname{card} A}{\operatorname{card} E}$.

(38) If E is finite and e is an elementary event of E, then $\mathrm{P}(e) = \frac{1}{\operatorname{card} E}$.

(39) If E is finite, then $\mathrm{P}(\Omega_E) = 1$.

(40) If E is finite, then $\mathrm{P}(\emptyset_E) = 0$.

(41) If E is finite and A is an event of E and B is an event of E and A exclude B, then $\mathrm{P}(A \cap B) = 0$.

(42) If E is finite and A is an event of E, then $\mathrm{P}(A) \leq 1$.

(43) If E is finite and A is an event of E, then $0 \leq \mathrm{P}(A)$.

(44) If E is finite and A is an event of E and B is an event of E and $A \subseteq B$, then $\mathrm{P}(A) \leq \mathrm{P}(B)$.

(46)[1] If E is finite and A is an event of E and B is an event of E, then $\mathrm{P}(A \cup B) = (\mathrm{P}(A) + \mathrm{P}(B)) - \mathrm{P}(A \cap B)$.

(47) If E is finite and A is an event of E and B is an event of E and A exclude B, then $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$.

(48) If E is finite and A is an event of E, then $\mathrm{P}(A) = 1 - \mathrm{P}(A^{\mathrm{c}})$ and $\mathrm{P}(A^{\mathrm{c}}) = 1 - \mathrm{P}(A)$.

(49) If E is finite and A is an event of E and B is an event of E, then $\mathrm{P}(A \setminus B) = \mathrm{P}(A) - \mathrm{P}(A \cap B)$.

(50) If E is finite and A is an event of E and B is an event of E and $B \subseteq A$, then $\mathrm{P}(A \setminus B) = \mathrm{P}(A) - \mathrm{P}(B)$.

(51) If E is finite and A is an event of E and B is an event of E, then $\mathrm{P}(A \cup B) \leq \mathrm{P}(A) + \mathrm{P}(B)$.

(52) If E is finite and A is an event of E and B is an event of E, then $\mathrm{P}(A \setminus B) = \mathrm{P}(A \cap B^{\mathrm{c}})$.

(53) If E is finite and A is an event of E and B is an event of E, then $\mathrm{P}(A) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B^{\mathrm{c}})$.

(54) If E is finite and A is an event of E and B is an event of E, then $\mathrm{P}(A) = \mathrm{P}(A \cup B) - \mathrm{P}(B \setminus A)$.

(55) If E is finite and A is an event of E and B is an event of E, then $\mathrm{P}(A) + \mathrm{P}(A^{\mathrm{c}} \cap B) = \mathrm{P}(B) + \mathrm{P}(B^{\mathrm{c}} \cap A)$.

(56) Suppose E is finite and A is an event of E and B is an event of E and C is an event of E. Then $\mathrm{P}((A \cup B) \cup C) = (((\mathrm{P}(A) + \mathrm{P}(B)) + \mathrm{P}(C)) - ((\mathrm{P}(A \cap B) + \mathrm{P}(A \cap C)) + \mathrm{P}(B \cap C))) + \mathrm{P}((A \cap B) \cap C)$.

[1]The proposition (45) became obvious.

(57) If E is finite and A is an event of E and B is an event of E and C is an event of E and A exclude B and A exclude C and B exclude C, then $\mathrm{P}((A \cup B) \cup C) = (\mathrm{P}(A) + \mathrm{P}(B)) + \mathrm{P}(C)$.

(58) If E is finite and A is an event of E and B is an event of E, then $\mathrm{P}(A) - \mathrm{P}(B) \leq \mathrm{P}(A \setminus B)$.

Let us consider E. Let us assume that E is finite. Let us consider B. Let us assume that $0 < \mathrm{P}(B)$. Let us consider A. The functor $\mathrm{P}(A/B)$ yielding a real number is defined by:

$\mathrm{P}(A/B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$.

One can prove the following propositions:

(59) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$, then $\mathrm{P}(A/B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$.

(60) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$, then $\mathrm{P}(A \cap B) = \mathrm{P}(A/B) \cdot \mathrm{P}(B)$.

(61) If E is finite and A is an event of E, then $\mathrm{P}(A/\Omega_E) = \mathrm{P}(A)$.

(62) If E is finite, then $\mathrm{P}(\Omega_E/\Omega_E) = 1$.

(63) If E is finite, then $\mathrm{P}(\emptyset_E/\Omega_E) = 0$.

(64) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$, then $\mathrm{P}(A/B) \leq 1$.

(65) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$, then $0 \leq \mathrm{P}(A/B)$.

(66) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$, then $\mathrm{P}(A/B) = 1 - \frac{\mathrm{P}(B \setminus A)}{\mathrm{P}(B)}$.

(67) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$ and $A \subseteq B$, then $\mathrm{P}(A/B) = \frac{\mathrm{P}(A)}{\mathrm{P}(B)}$.

(68) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$ and A exclude B, then $\mathrm{P}(A/B) = 0$.

(69) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(A)$ and $0 < \mathrm{P}(B)$, then $\mathrm{P}(A) \cdot \mathrm{P}(B/A) = \mathrm{P}(B) \cdot \mathrm{P}(A/B)$.

(70) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$, then $\mathrm{P}(A/B) = 1 - \mathrm{P}(A^{\mathrm{c}}/B)$ and $\mathrm{P}(A^{\mathrm{c}}/B) = 1 - \mathrm{P}(A/B)$.

(71) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$ and $B \subseteq A$, then $\mathrm{P}(A/B) = 1$.

(72) If E is finite and B is an event of E and $0 < \mathrm{P}(B)$, then $\mathrm{P}(\Omega_E/B) = 1$.

(73) If E is finite and A is an event of E and $0 < \mathrm{P}(A)$, then $\mathrm{P}(A^{\mathrm{c}}/A) = 0$.

(74) If E is finite and A is an event of E and $\mathrm{P}(A) < 1$, then $\mathrm{P}(A/A^{\mathrm{c}}) = 0$.

(75) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$ and A exclude B, then $\mathrm{P}(A^{\mathrm{c}}/B) = 1$.

(76) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(A)$ and $\mathrm{P}(B) < 1$ and A exclude B, then $\mathrm{P}(A/B^{\mathrm{c}}) = \frac{\mathrm{P}(A)}{1 - \mathrm{P}(B)}$.

(77) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(A)$ and $\mathrm{P}(B) < 1$ and A exclude B, then $\mathrm{P}(A^{\mathrm{c}}/B^{\mathrm{c}}) = 1 - \frac{\mathrm{P}(A)}{1-\mathrm{P}(B)}$.

(78) If E is finite and A is an event of E and B is an event of E and C is an event of E and $0 < \mathrm{P}(B \cap C)$ and $0 < \mathrm{P}(C)$, then $\mathrm{P}((A \cap B) \cap C) = (\mathrm{P}(A/(B \cap C)) \cdot \mathrm{P}(B/C)) \cdot \mathrm{P}(C)$.

(79) If E is finite and A is an event of E and B is an event of E and $0 < \mathrm{P}(B)$ and $\mathrm{P}(B) < 1$, then $\mathrm{P}(A) = \mathrm{P}(A/B) \cdot \mathrm{P}(B) + \mathrm{P}(A/B^{\mathrm{c}}) \cdot \mathrm{P}(B^{\mathrm{c}})$.

(80) Suppose E is finite and A is an event of E and B_1 is an event of E and B_2 is an event of E and $0 < \mathrm{P}(B_1)$ and $0 < \mathrm{P}(B_2)$ and $B_1 \cup B_2 = E$ and $B_1 \cap B_2 = \emptyset$. Then $\mathrm{P}(A) = \mathrm{P}(A/B_1) \cdot \mathrm{P}(B_1) + \mathrm{P}(A/B_2) \cdot \mathrm{P}(B_2)$.

(81) Suppose that
- (i) E is finite,
- (ii) A is an event of E,
- (iii) B_1 is an event of E,
- (iv) B_2 is an event of E,
- (v) B_3 is an event of E,
- (vi) $0 < \mathrm{P}(B_1)$,
- (vii) $0 < \mathrm{P}(B_2)$,
- (viii) $0 < \mathrm{P}(B_3)$,
- (ix) $(B_1 \cup B_2) \cup B_3 = E$,
- (x) $B_1 \cap B_2 = \emptyset$,
- (xi) $B_1 \cap B_3 = \emptyset$,
- (xii) $B_2 \cap B_3 = \emptyset$.

Then $\mathrm{P}(A) = (\mathrm{P}(A/B_1) \cdot \mathrm{P}(B_1) + \mathrm{P}(A/B_2) \cdot \mathrm{P}(B_2)) + \mathrm{P}(A/B_3) \cdot \mathrm{P}(B_3)$.

(82) Suppose E is finite and A is an event of E and B_1 is an event of E and B_2 is an event of E and $0 < \mathrm{P}(A)$ and $0 < \mathrm{P}(B_1)$ and $0 < \mathrm{P}(B_2)$ and $B_1 \cup B_2 = E$ and $B_1 \cap B_2 = \emptyset$. Then $\mathrm{P}(B_1/A) = \frac{\mathrm{P}(A/B_1) \cdot \mathrm{P}(B_1)}{\mathrm{P}(A/B_1) \cdot \mathrm{P}(B_1) + \mathrm{P}(A/B_2) \cdot \mathrm{P}(B_2)}$.

(83) Suppose that
- (i) E is finite,
- (ii) A is an event of E,
- (iii) B_1 is an event of E,
- (iv) B_2 is an event of E,
- (v) B_3 is an event of E,
- (vi) $0 < \mathrm{P}(A)$,
- (vii) $0 < \mathrm{P}(B_1)$,
- (viii) $0 < \mathrm{P}(B_2)$,
- (ix) $0 < \mathrm{P}(B_3)$,
- (x) $(B_1 \cup B_2) \cup B_3 = E$,
- (xi) $B_1 \cap B_2 = \emptyset$,
- (xii) $B_1 \cap B_3 = \emptyset$,
- (xiii) $B_2 \cap B_3 = \emptyset$.

Then $\mathrm{P}(B_1/A) = \frac{\mathrm{P}(A/B_1) \cdot \mathrm{P}(B_1)}{(\mathrm{P}(A/B_1) \cdot \mathrm{P}(B_1) + \mathrm{P}(A/B_2) \cdot \mathrm{P}(B_2)) + \mathrm{P}(A/B_3) \cdot \mathrm{P}(B_3)}$.

Let us consider E, A, B. We say that A and B are independent if and only if:

$P(A \cap B) = P(A) \cdot P(B)$.

The following propositions are true:

(84) A and B are independent if and only if $P(A \cap B) = P(A) \cdot P(B)$.

(85) If A and B are independent, then B and A are independent.

(86) If E is finite and A is an event of E and B is an event of E and $0 < P(B)$ and A and B are independent, then $P(A/B) = P(A)$.

(87) If E is finite and A is an event of E and B is an event of E and $P(B) = 0$, then A and B are independent.

(88) If E is finite and A is an event of E and B is an event of E and A and B are independent, then A^c and B are independent.

(89) If E is finite and A is an event of E and B is an event of E and A exclude B and A and B are independent, then $P(A) = 0$ or $P(B) = 0$.

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A Construction of Analytical Projective Space

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Summary. The collinearity structure denoted by ProjectiveSpace(V) is correlated with a given vector space V (over the field of Reals). It is a formalization of the standard construction of a projective space, where points are interpreted as equivalence classes of the relation of proportionality considered in the set of all non-zero vectors. Then the relation of collinearity corresponds to the relation of linear dependence of vectors. Several facts concerning vectors are proved, which correspond in this language to some classical axioms of projective geometry.

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The notation and terminology used here are introduced in the following articles: [7], [8], [6], [2], [3], [4], [5], [1], and [9]. We adopt the following rules: V is a real linear space, p, q, r, u, v, w, y, u_1, v_1, w_1 are vectors of V, and a, b, c, a_1, b_1, c_1, a_2, b_2, c_2 are real numbers. Let us consider V, p. We say that p is a proper vector if and only if:

$p \neq 0_V$.

The following proposition is true

(1) p is a proper vector if and only if $p \neq 0_V$.

Let us consider V, p, q. We say that p and q are proportional if and only if: there exist a, b such that $a \cdot p = b \cdot q$ and $a \neq 0$ and $b \neq 0$.

One can prove the following propositions:

(2) p and q are proportional if and only if there exist a, b such that $a \cdot p = b \cdot q$ and $a \neq 0$ and $b \neq 0$.

(3) p and p are proportional.

(4) If p and q are proportional, then q and p are proportional.

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(5) p and q are proportional if and only if there exists a such that $a \neq 0$ and $p = a \cdot q$.

(6) If p and u are proportional and u and q are proportional, then p and q are proportional.

(7) p and 0_V are proportional if and only if $p = 0_V$.

Let us consider V, u, v, w. We say that u, v and w are lineary dependent if and only if:

there exist a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ but $a \neq 0$ or $b \neq 0$ or $c \neq 0$.

We now state a number of propositions:

(8) u, v and w are lineary dependent if and only if there exist a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ but $a \neq 0$ or $b \neq 0$ or $c \neq 0$.

(9) If u and u_1 are proportional and v and v_1 are proportional and w and w_1 are proportional and u, v and w are lineary dependent, then u_1, v_1 and w_1 are lineary dependent.

(10) If u, v and w are lineary dependent, then u, w and v are lineary dependent and v, u and w are lineary dependent and w, v and u are lineary dependent and w, u and v are lineary dependent and v, w and u are lineary dependent.

(11) If v is a proper vector and w is a proper vector and v and w are not proportional, then v, w and u are lineary dependent if and only if there exist a, b such that $u = a \cdot v + b \cdot w$.

(12) If p and q are not proportional and $a_1 \cdot p + b_1 \cdot q = a_2 \cdot p + b_2 \cdot q$ and p is a proper vector and q is a proper vector, then $a_1 = a_2$ and $b_1 = b_2$.

(13) If u, v and w are not lineary dependent and $(a_1 \cdot u + b_1 \cdot v) + c_1 \cdot w = (a_2 \cdot u + b_2 \cdot v) + c_2 \cdot w$, then $a_1 = a_2$ and $b_1 = b_2$ and $c_1 = c_2$.

(14) Suppose p and q are not proportional and $u = a_1 \cdot p + b_1 \cdot q$ and $v = a_2 \cdot p + b_2 \cdot q$ and $a_1 \cdot b_2 - a_2 \cdot b_1 = 0$ and p is a proper vector and q is a proper vector. Then u and v are proportional or $u = 0_V$ or $v = 0_V$.

(15) If $u = 0_V$ or $v = 0_V$ or $w = 0_V$, then u, v and w are lineary dependent.

(16) If u and v are proportional or w and u are proportional or v and w are proportional, then w, u and v are lineary dependent.

(17) If u, v and w are not lineary dependent, then u is a proper vector and v is a proper vector and w is a proper vector and u and v are not proportional and v and w are not proportional and w and u are not proportional.

(18) If $p + q = 0_V$, then p and q are proportional.

(19) If p and q are not proportional and p, q and u are lineary dependent and p, q and v are lineary dependent and p, q and w are lineary dependent and p is a proper vector and q is a proper vector, then u, v and w are lineary dependent.

(20) If u, v and w are not lineary dependent and u, v and p are lineary dependent and v, w and q are lineary dependent, then there exists y such

that u, w and y are lineary dependent and p, q and y are lineary dependent and y is a proper vector.

(21) If p and q are not proportional and p is a proper vector and q is a proper vector, then for every u, v there exists y such that y is a proper vector and u, v and y are lineary dependent and u and y are not proportional and v and y are not proportional.

(22) If p, q and r are not lineary dependent, then for all u, v such that u is a proper vector and v is a proper vector and u and v are not proportional there exists y such that y is a proper vector and u, v and y are not lineary dependent.

(23) Suppose u, v and q are lineary dependent and w, y and q are lineary dependent and u, w and p are lineary dependent and v, y and p are lineary dependent and u, y and r are lineary dependent and v, w and r are lineary dependent and p, q and r are lineary dependent and p is a proper vector and q is a proper vector and r is a proper vector. Then u, v and y are lineary dependent or u, v and w are lineary dependent or u, w and y are lineary dependent or v, w and y are lineary dependent.

In the sequel x, y, z are arbitrary and X denotes a set. Let us consider V. The proper vectors ofV yields a set and is defined as follows:

for an arbitrary u holds $u \in$ the proper vectors ofV if and only if $u \neq 0_V$ and u is a vector of V.

Next we state three propositions:

(24) For every X holds $X =$ the proper vectors ofV if and only if for an arbitrary u holds $u \in X$ if and only if $u \neq 0_V$ and u is a vector of V.

(25) For an arbitrary u such that $u \in$ the proper vectors ofV holds u is a vector of V.

(26) For every u holds $u \in$ the proper vectors ofV if and only if u is a proper vector.

Let us consider V. The proportionality in V yields an equivalence relation of the proper vectors ofV and is defined as follows:

for all x, y holds $\langle x, y \rangle \in$ the proportionality in V if and only if $x \in$ the proper vectors ofV and $y \in$ the proper vectors ofV and there exist vectors u, v of V such that $x = u$ and $y = v$ and u and v are proportional.

We now state three propositions:

(27) For every equivalence relation R of the proper vectors ofV holds $R =$ the proportionality in V if and only if for all x, y holds $\langle x, y \rangle \in R$ if and only if $x \in$ the proper vectors ofV and $y \in$ the proper vectors ofV and there exist vectors u, v of V such that $x = u$ and $y = v$ and u and v are proportional.

(28) If $\langle x, y \rangle \in$ the proportionality in V, then x is a vector of V and y is a vector of V.

(29) $\langle u, v \rangle \in$ the proportionality in V if and only if u is a proper vector and v is a proper vector and u and v are proportional.

Let us consider V, v. Let us assume that v is a proper vector. The direction of v yields a subset of the proper vectors ofV and is defined by:

the direction of $v = [v]_{\text{the proportionality in } V}$.

We now state the proposition

(30) If v is a proper vector, then the direction of $v = [v]_{\text{the proportionality in } V}$.

Let us consider V. The projective points overV yields a set and is defined as follows:

there exists a family Y of subsets of the proper vectors ofV such that $Y =$ Classes(the proportionality in V) and the projective points overV = Y.

The following proposition is true

(31) For every X holds $X =$ the projective points overV if and only if there exists a family Y of subsets of the proper vectors ofV such that $Y =$ Classes(the proportionality in V) and $X = Y$.

A real linear space is said to be a non-trivial real linear space if:

there exists a vector u of it such that $u \neq 0_{\text{it}}$.

The following two propositions are true:

(32) For every real linear space V holds V is a non-trivial real linear space if and only if there exists a vector u of V such that $u \neq 0_V$.

(33) For every real linear space V holds V is a non-trivial real linear space if and only if there exists u such that $u \in$ the proper vectors ofV.

We follow the rules: V will denote a non-trivial real linear space, p, q, r, u, v, w will denote vectors of V, and y will be arbitrary. Let us consider V. Then the proper vectors ofV is a non-empty set.

Let us consider V. Then the projective points overV is a non-empty set.

Next we state two propositions:

(34) If p is a proper vector, then the direction of p is an element of the projective points overV.

(35) If p is a proper vector and q is a proper vector, then the direction of $p =$ the direction of q if and only if p and q are proportional.

Let us consider V. The projective collinearity overV yielding a ternary relation on the projective points overV is defined by:

for arbitrary x, y, z holds $\langle x, y, z\rangle \in$ the projective collinearity overV if and only if there exist p, q, r such that $x =$ the direction of p and $y =$ the direction of q and $z =$ the direction of r and p is a proper vector and q is a proper vector and r is a proper vector and p, q and r are lineary dependent.

We now state the proposition

(36) Let R be a ternary relation on the projective points overV. Then $R =$ the projective collinearity overV if and only if for arbitrary x, y, z holds $\langle x, y, z\rangle \in R$ if and only if there exist p, q, r such that $x =$ the direction of p and $y =$ the direction of q and $z =$ the direction of r and p is a proper vector and q is a proper vector and r is a proper vector and p, q and r are lineary dependent.

Let us consider V. The projective space over V yields a collinearity structure and is defined by:

the projective space over $V = \langle$the projective points overV, the projective collinearity overV$\rangle$.

In the sequel CS will be a collinearity structure. One can prove the following propositions:

(37) For every CS holds $CS =$ the projective space over V if and only if $CS = \langle$the projective points overV, the projective collinearity overV$\rangle$.

(38) The projective space over $V = \langle$the projective points overV, the projective collinearity overV$\rangle$.

(39) For every V holds the points of the projective space over $V =$ the projective points overV and the collinearity relation of the projective space over $V =$ the projective collinearity overV.

(40) If $\langle x, y, z\rangle \in$ the collinearity relation of the projective space over V, then there exist p, q, r such that $x =$ the direction of p and $y =$ the direction of q and $z =$ the direction of r and p is a proper vector and q is a proper vector and r is a proper vector and p, q and r are lineary dependent.

(41) If u is a proper vector and v is a proper vector and w is a proper vector, then $\langle$ the direction of u, the direction of v, the direction of $w\rangle \in$ the collinearity relation of the projective space over V if and only if u, v and w are lineary dependent.

(42) x is an element of the points of the projective space over V if and only if there exists u such that u is a proper vector and $x =$ the direction of u.

(43) For every real linear space V and for every vector v of V such that v is a proper vector for every subset X of the proper vectors ofV holds $X =$ the direction of v if and only if $X = [v]_{\text{the proportionality in } V}$.

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Projective Spaces - Part I

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Summary. In the class of all collinearity structures a subclass of (dimension free) projective spaces, defined by means of a suitable axiom system, is singled out. Whenever a real vector space V is at least 3-dimensional, the structure ProjectiveSpace(V) is a projective space in the above meaning. Some narrower classes of projective spaces are defined: Fano projective spaces, projective planes, and Fano projective planes. For any of the above classes an explicit axiom system is given, as well as an analytical example. There is also a construction a of 3-dimensional and a 4-dimensional real vector space; these are needed to show appropriate examples of projective spaces.

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The notation and terminology used here are introduced in the following papers: [1], [5], [7], [6], [3], [4], and [2]. For simplicity we adopt the following rules: V will denote a real linear space, p, q, r, u, v, w, y, u_1, v_1 will denote vectors of V, a, b, c, d, a_1, b_1 will denote real numbers, and z will be arbitrary. We now state three propositions:

(1) Suppose for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$. Then u is a proper vector and v is a proper vector and w is a proper vector and u, v and w are not lineary dependent and u and v are not proportional.

(2) Given u, v, u_1, v_1. Suppose for all a, b, a_1, b_1 such that $((a \cdot u + b \cdot v) + a_1 \cdot u_1) + b_1 \cdot v_1 = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$. Then u is a proper vector and v is a proper vector and u and v are not proportional and u_1 is a proper vector and v_1 is a proper vector and u_1 and v_1 are not proportional and u, v and u_1 are not lineary dependent and u_1, v_1 and u are not lineary dependent.

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(3) Suppose for every w there exist a, b, c such that $w = (a \cdot p + b \cdot q) + c \cdot r$ and for all a, b, c such that $(a \cdot p + b \cdot q) + c \cdot r = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$. Then for every u, u_1 there exists y such that p, q and y are lineary dependent and u, u_1 and y are lineary dependent and y is a proper vector.

We follow a convention: A is a non-empty set, f, g, h, f_1 are elements of $\mathbb{R}^A$, and x_1, x_2, x_3, x_4 are elements of A. We now state a number of propositions:

(4) Suppose $x_1 \in A$ and $x_2 \in A$ and $x_3 \in A$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_2 \neq x_3$. Then there exist f, g, h such that for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ but if $z \neq x_1$, then $f(z) = 0$ and for every z such that $z \in A$ holds if $z = x_2$, then $g(z) = 1$ but if $z \neq x_2$, then $g(z) = 0$ and for every z such that $z \in A$ holds if $z = x_3$, then $h(z) = 1$ but if $z \neq x_3$, then $h(z) = 0$.

(5) Suppose that

(i) $x_1 \in A$,
(ii) $x_2 \in A$,
(iii) $x_3 \in A$,
(iv) $x_1 \neq x_2$,
(v) $x_1 \neq x_3$,
(vi) $x_2 \neq x_3$,
(vii) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ but if $z \neq x_1$, then $f(z) = 0$,
(viii) for every z such that $z \in A$ holds if $z = x_2$, then $g(z) = 1$ but if $z \neq x_2$, then $g(z) = 0$,
(ix) for every z such that $z \in A$ holds if $z = x_3$, then $h(z) = 1$ but if $z \neq x_3$, then $h(z) = 0$.

Then for all a, b, c such that
$+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g \rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h \rangle)) = \mathbf{0}_{\mathbb{R}^A}$
holds $a = 0$ and $b = 0$ and $c = 0$.

(6) Suppose $x_1 \in A$ and $x_2 \in A$ and $x_3 \in A$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_2 \neq x_3$. Then there exist f, g, h such that for all a, b, c such that $+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g \rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h \rangle)) = \mathbf{0}_{\mathbb{R}^A}$ holds $a = 0$ and $b = 0$ and $c = 0$.

(7) Suppose that

(i) $A = \{x_1, x_2, x_3\}$,
(ii) $x_1 \neq x_2$,
(iii) $x_1 \neq x_3$,
(iv) $x_2 \neq x_3$,
(v) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ but if $z \neq x_1$, then $f(z) = 0$,
(vi) for every z such that $z \in A$ holds if $z = x_2$, then $g(z) = 1$ but if $z \neq x_2$, then $g(z) = 0$,
(vii) for every z such that $z \in A$ holds if $z = x_3$, then $h(z) = 1$ but if $z \neq x_3$, then $h(z) = 0$.

Then for every element h' of $\mathbb{R}^A$ there exist a, b, c such that $h' = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle))$.

(8) Suppose $A = \{x_1, x_2, x_3\}$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_2 \neq x_3$. Then there exist f, g, h such that for every element h' of $\mathbb{R}^A$ there exist a, b, c such that $h' = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle))$.

(9) Suppose $A = \{x_1, x_2, x_3\}$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_2 \neq x_3$. Then there exist f, g, h such that for all a, b, c such that $+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)) = \mathbf{0}_{\mathbb{R}^A}$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every element h' of $\mathbb{R}^A$ there exist a, b, c such that $h' = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle))$.

(10) There exists a non-trivial real linear space V and there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

(11) Suppose $x_1 \in A$ and $x_2 \in A$ and $x_3 \in A$ and $x_4 \in A$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_3 \neq x_4$. Then there exist f, g, h, f_1 such that for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ but if $z \neq x_1$, then $f(z) = 0$ and for every z such that $z \in A$ holds if $z = x_2$, then $g(z) = 1$ but if $z \neq x_2$, then $g(z) = 0$ and for every z such that $z \in A$ holds if $z = x_3$, then $h(z) = 1$ but if $z \neq x_3$, then $h(z) = 0$ and for every z such that $z \in A$ holds if $z = x_4$, then $f_1(z) = 1$ but if $z \neq x_4$, then $f_1(z) = 0$.

(12) Suppose that
- (i) $x_1 \in A$,
- (ii) $x_2 \in A$,
- (iii) $x_3 \in A$,
- (iv) $x_4 \in A$,
- (v) $x_1 \neq x_2$,
- (vi) $x_1 \neq x_3$,
- (vii) $x_1 \neq x_4$,
- (viii) $x_2 \neq x_3$,
- (ix) $x_2 \neq x_4$,
- (x) $x_3 \neq x_4$,
- (xi) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ but if $z \neq x_1$, then $f(z) = 0$,
- (xii) for every z such that $z \in A$ holds if $z = x_2$, then $g(z) = 1$ but if $z \neq x_2$, then $g(z) = 0$,
- (xiii) for every z such that $z \in A$ holds if $z = x_3$, then $h(z) = 1$ but if $z \neq x_3$, then $h(z) = 0$,
- (xiv) for every z such that $z \in A$ holds if $z = x_4$, then $f_1(z) = 1$ but if $z \neq x_4$, then $f_1(z) = 0$.

Given a, b, c, d. Suppose
$+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle d, f_1\rangle)) = \mathbf{0}_{\mathbb{R}^A}$.
Then $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$.

(13) Suppose $x_1 \in A$ and $x_2 \in A$ and $x_3 \in A$ and $x_4 \in A$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_3 \neq x_4$. Then there exist f, g, h, f_1 such that for all a, b, c, d such that $+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle d, f_1\rangle)) = \mathbf{0}_{\mathbb{R}^A}$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$.

(14) Suppose that

(i) $A = \{x_1, x_2, x_3, x_4\}$,

(ii) $x_1 \neq x_2$,

(iii) $x_1 \neq x_3$,

(iv) $x_1 \neq x_4$,

(v) $x_2 \neq x_3$,

(vi) $x_2 \neq x_4$,

(vii) $x_3 \neq x_4$,

(viii) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ but if $z \neq x_1$, then $f(z) = 0$,

(ix) for every z such that $z \in A$ holds if $z = x_2$, then $g(z) = 1$ but if $z \neq x_2$, then $g(z) = 0$,

(x) for every z such that $z \in A$ holds if $z = x_3$, then $h(z) = 1$ but if $z \neq x_3$, then $h(z) = 0$,

(xi) for every z such that $z \in A$ holds if $z = x_4$, then $f_1(z) = 1$ but if $z \neq x_4$, then $f_1(z) = 0$.

Then for every element h' of $\mathbb{R}^A$ there exist a, b, c, d such that $h' = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle d, f_1\rangle))$.

(15) Suppose $A = \{x_1, x_2, x_3, x_4\}$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_3 \neq x_4$. Then there exist f, g, h, f_1 such that for every element h' of $\mathbb{R}^A$ there exist a, b, c, d such that $h' = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle d, f_1\rangle))$.

(16) Suppose $A = \{x_1, x_2, x_3, x_4\}$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_3 \neq x_4$. Then there exist f, g, h, f_1 such that for all a, b, c, d such that $+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle d, f_1\rangle)) = \mathbf{0}_{\mathbb{R}^A}$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and for every element h' of $\mathbb{R}^A$ there exist a, b, c, d such that $h' = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle d, f_1\rangle))$.

(17) There exists a non-trivial real linear space V and there exist u, v, w, u_1 such that for all a, b, c, d such that $((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1 = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and for every y there exist a, b, c, d such that $y = ((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1$.

We follow the rules: V is a non-trivial real linear space, u, v, w, y, w_1 are vectors of V, and p, p_1, p_2, q, q_1, r, r_1, r_2 are elements of the points of the projective space over V. The following propositions are true:

(18) p, q and r are collinear if and only if there exist u, v, w such that $p =$ the direction of u and $q =$ the direction of v and $r =$ the direction of w and u is a proper vector and v is a proper vector and w is a proper vector and u, v and w are lineary dependent.

(19) p, q and p are collinear and p, p and q are collinear and p, q and q are collinear.

(20) If $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear, then r, r_1 and r_2 are collinear.

(21) If p, q and r are collinear, then p, r and q are collinear and q, p and r are collinear and r, q and p are collinear and r, p and q are collinear and q, r and p are collinear.

(22) If p, p_1 and p_2 are collinear and p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear, then there exists r_2 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear.

(23) If p, p_1 and p_2 are not collinear and p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear, then there exists r_2 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear.

(24) If p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear, then there exists r_2 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear.

(25) Suppose p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear. Then p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(26) If there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$, then there exist p, q, r such that p, q and r are not collinear.

(27) Suppose there exist u, v, w_1 such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w_1 = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$. Then for every p, q there exists r such that $p \neq r$ and $q \neq r$ and p, q and r are collinear.

(28) Suppose that

(i) there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

Then there exist elements x_1, x_2 of the points of the projective space over V such that $x_1 \neq x_2$ and for every r_1, r_2 there exists q such that x_1, x_2 and q are collinear and r_1, r_2 and q are collinear.

(29) If there exist elements x_1, x_2 of the points of the projective space over V such that $x_1 \neq x_2$ and for every r_1, r_2 there exists q such that x_1, x_2 and q are collinear and r_1, r_2 and q are collinear, then for every p, p_1, q, q_1 there exists r such that p, p_1 and r are collinear and q, q_1 and r are collinear.

(30) Suppose that

(i) there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

Then for every p, p_1, q, q_1 there exists r such that p, p_1 and r are collinear and q, q_1 and r are collinear.

(31) Suppose that

(i) there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

Then there exist p, q, r such that p, q and r are not collinear and for every p, p_1, q, q_1 there exists r such that p, p_1 and r are collinear and q, q_1 and r are collinear.

A collinearity structure is said to be a projective space defined in terms of collinearity if:

(i) for all elements p, q, r, r_1, r_2 of the points of it such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of it holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for all elements p, p_1, p_2, r, r_1 of the points of it such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of it such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(iv) for every elements p, q of the points of it there exists an element r of the points of it such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(v) there exist elements p, q, r of the points of it such that p, q and r are not collinear.

Next we state three propositions:

(32) Let CS be a collinearity structure. Then CS is a projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of CS such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of CS holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for all elements p, p_1, p_2, r, r_1 of the points of CS such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of CS such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(iv) for every elements p, q of the points of CS there exists an element r of the points of CS such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(v) there exist elements p, q, r of the points of CS such that p, q and r are not collinear.

(33) For every projective space CS defined in terms of collinearity holds CS is a proper collinearity space.

(34) If there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$, then the projective space over V is a

projective space defined in terms of collinearity.

A projective space defined in terms of collinearity is called a Fanoian projective space defined in terms of collinearity if:

Let p_1, r_2, q, r_1, q_1, p, r be elements of the points of it . Suppose p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear. Then p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

The following propositions are true:

(35) Let CS be a projective space defined in terms of collinearity. Then CS is a Fanoian projective space defined in terms of collinearity if and only if for all elements p_1, r_2, q, r_1, q_1, p, r of the points of CS such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(36) Let CS be a collinearity structure. Then CS is a Fanoian projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of CS such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of CS holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for all elements p, p_1, p_2, r, r_1 of the points of CS such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of CS such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(iv) for every elements p, q of the points of CS there exists an element r of the points of CS such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(v) there exist elements p, q, r of the points of CS such that p, q and r are not collinear,

(vi) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of CS such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(37) If there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$, then the projective space over V is a Fanoian projective space defined in terms of collinearity.

A projective space defined in terms of collinearity is called a projective plane defined in terms of collinearity if:

for every elements p, p_1, q, q_1 of the points of it there exists an element r of the points of it such that p, p_1 and r are collinear and q, q_1 and r are collinear.

We now state three propositions:

(38) For every projective space CPS defined in terms of collinearity holds CPS is a projective plane defined in terms of collinearity if and only if for every elements p, p_1, q, q_1 of the points of CPS there exists an element r of the points of CPS such that p, p_1 and r are collinear and q, q_1 and r are collinear.

(39) Let CS be a collinearity structure. Then CS is a projective plane defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of CS such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of CS holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for every elements p, q of the points of CS there exists an element r of the points of CS such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) there exist elements p, q, r of the points of CS such that p, q and r are not collinear,

(v) for every elements p, p_1, q, q_1 of the points of CS there exists an element r of the points of CS such that p, p_1 and r are collinear and q, q_1 and r are collinear.

(40) Suppose that

(i) there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

Then the projective space over V is a projective plane defined in terms of collinearity.

A projective plane defined in terms of collinearity is said to be a Fanoian projective plane defined in terms of collinearity if:

Let p_1, r_2, q, r_1, q_1, p, r be elements of the points of it . Suppose p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear. Then p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

Next we state four propositions:

(41) Let CS be a projective plane defined in terms of collinearity. Then CS is a Fanoian projective plane defined in terms of collinearity if and only if for all elements p_1, r_2, q, r_1, q_1, p, r of the points of CS such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1

are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(42) Let CS be a collinearity structure. Then CS is a Fanoian projective plane defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of CS such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of CS holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for every elements p, q of the points of CS there exists an element r of the points of CS such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) there exist elements p, q, r of the points of CS such that p, q and r are not collinear,

(v) for every elements p, p_1, q, q_1 of the points of CS there exists an element r of the points of CS such that p, p_1 and r are collinear and q, q_1 and r are collinear,

(vi) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of CS such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(43) Suppose that

(i) there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

Then the projective space over V is a Fanoian projective plane defined in terms of collinearity.

(44) For every CS being a collinearity structure holds CS is a Fanoian projective plane defined in terms of collinearity if and only if CS is a Fanoian projective space defined in terms of collinearity and for every elements p, p_1, q, q_1 of the points of CS there exists an element r of the points of CS such that p, p_1 and r are collinear and q, q_1 and r are collinear.

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Topological Properties of Subsets in Real Numbers [1]

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Summary. The following notions for real subsets are defined: open set, closed set, compact set, intervals and neighbourhoods. In the sequel some theorems involving above mentioned notions are proved.

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The notation and terminology used in this paper have been introduced in the following articles: [9], [3], [10], [1], [2], [7], [5], [6], [4], and [8]. For simplicity we adopt the following convention: n, m are natural numbers, x is arbitrary, s, g, g_1, g_2, r, p, q are real numbers, s_1, s_2 are sequences of real numbers, and X, Y, Y_1 are subsets of $\mathbb{R}$. In this article we present several logical schemes. The scheme *SeqChoice* concerns a non-empty set $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a function f from $\mathbb{N}$ into $\mathcal{A}$ such that for every element t of $\mathbb{N}$ holds $\mathcal{P}[t, f(t)]$

provided the following requirement is met:

- for every element t of $\mathbb{N}$ there exists an element ff of $\mathcal{A}$ such that $\mathcal{P}[t, ff]$.

The scheme *RealSeqChoice* concerns a binary predicate $\mathcal{P}$, and states that:

there exists s_1 such that for every n holds $\mathcal{P}[n, s_1(n)]$

provided the parameter meets the following requirement:

- for every n there exists r such that $\mathcal{P}[n, r]$.

We now state several propositions:

(1) $X \subseteq Y$ if and only if for every r such that $r \in X$ holds $r \in Y$.

(2) $r \in X$ if and only if $r \notin X^{\mathrm{c}}$.

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(3) If there exists x such that $x \in Y_1$ and $Y_1 \subseteq Y$ and Y is lower bounded, then Y_1 is lower bounded.

(4) If there exists x such that $x \in Y_1$ and $Y_1 \subseteq Y$ and Y is upper bounded, then Y_1 is upper bounded.

(5) If there exists x such that $x \in Y_1$ and $Y_1 \subseteq Y$ and Y is bounded, then Y_1 is bounded.

Let us consider g, s. The functor $[g, s]$ yields a subset of $\mathbb{R}$ and is defined by:
$[g, s] = \{r : g \leq r \wedge r \leq s\}$.

Next we state the proposition

(6) $[g, s] = \{r : g \leq r \wedge r \leq s\}$.

Let us consider g, s. The functor $]g, s[$ yields a subset of $\mathbb{R}$ and is defined as follows:
$]g, s[= \{r : g < r \wedge r < s\}$.

Next we state a number of propositions:

(7) $]g, s[= \{r : g < r \wedge r < s\}$.

(8) $r \in]p - g, p + g[$ if and only if $|r - p| < g$.

(9) $r \in [p, g]$ if and only if $|(p + g) - 2 \cdot r| \leq g - p$.

(10) $r \in]p, g[$ if and only if $|(p + g) - 2 \cdot r| < g - p$.

(11) For all g, s such that $g \leq s$ holds $[g, s] =]g, s[\cup \{g, s\}$.

(12) If $p \leq g$, then $]g, p[= \emptyset$.

(13) If $p < g$, then $[g, p] = \emptyset$.

(14) If $p = g$, then $[p, g] = \{p\}$ and $[g, p] = \{p\}$ and $]p, g[= \emptyset$.

(15) If $p < g$, then $]p, g[\neq \emptyset$ but if $p \leq g$, then $p \in [p, g]$ and $g \in [p, g]$ and $[p, g] \neq \emptyset$ and $]p, g[\subseteq [p, g]$.

(16) If $r \in [p, g]$ and $s \in [p, g]$, then $[r, s] \subseteq [p, g]$.

(17) If $r \in]p, g[$ and $s \in]p, g[$, then $[r, s] \subseteq]p, g[$.

(18) If $p \leq g$, then $[p, g] = [p, g] \cup [g, p]$.

Let us consider X. We say that X is compact if and only if:
for every s_1 such that $\operatorname{rng} s_1 \subseteq X$ there exists s_2 such that s_2 is a subsequence of s_1 and s_2 is convergent and $\lim s_2 \in X$.

Next we state the proposition

(19) X is compact if and only if for every s_1 such that $\operatorname{rng} s_1 \subseteq X$ there exists s_2 such that s_2 is a subsequence of s_1 and s_2 is convergent and $\lim s_2 \in X$.

Let us consider X. We say that X is closed if and only if:
for every s_1 such that $\operatorname{rng} s_1 \subseteq X$ and s_1 is convergent holds $\lim s_1 \in X$.

The following proposition is true

(20) X is closed if and only if for every s_1 such that $\operatorname{rng} s_1 \subseteq X$ and s_1 is convergent holds $\lim s_1 \in X$.

Let A be a non-empty set, and let X be a subset of A. Then X^{c} is a subset of A.

Let us consider X. We say that X is open if and only if:
X^{c} is closed.

One can prove the following propositions:

(21) X is open if and only if X^{c} is closed.

(22) For all s, g such that $s \leq g$ for every s_1 such that $\operatorname{rng} s_1 \subseteq [s, g]$ holds s_1 is bounded.

(23) For all s, g such that $s \leq g$ holds $[s, g]$ is closed.

(24) For all s, g such that $s \leq g$ holds $[s, g]$ is compact.

(25) For all p, q such that $p < q$ holds $]p, q[$ is open.

(26) If X is compact, then X is closed.

(27) Given X, s_1. Suppose $X \neq \emptyset$ and $\operatorname{rng} s_1 \subseteq X$ and for every p such that $p \in X$ there exist r, n such that $0 < r$ and for every m such that $n < m$ holds $r < |s_1(m) - p|$. Then for every s_2 such that s_2 is a subsequence of s_1 holds it is not true that: s_2 is convergent and $\lim s_2 \in X$.

(28) If there exists r such that $r \in X$ and X is compact, then X is bounded.

(29) If there exists r such that $r \in X$, then X is compact if and only if X is bounded and X is closed.

(30) For every X such that $X \neq \emptyset$ and X is closed and X is upper bounded holds $\sup X \in X$.

(31) For every X such that $X \neq \emptyset$ and X is closed and X is lower bounded holds $\inf X \in X$.

(32) For every X such that $X \neq \emptyset$ and X is compact holds $\sup X \in X$ and $\inf X \in X$.

(33) If X is compact and for all g_1, g_2 such that $g_1 \in X$ and $g_2 \in X$ holds $[g_1, g_2] \subseteq X$, then there exist p, g such that $X = [p, g]$.

A subset of $\mathbb{R}$ is called a real open subset if:
it is open.

We now state the proposition

(34) For every subset X of $\mathbb{R}$ holds X is a real open subset if and only if X is open.

Let us consider r. A real open subset is said to be a neighbourhood of r if:
there exists g such that $0 < g$ and it $=]r - g, r + g[$.

One can prove the following propositions:

(35) For every r and for every real open subset X holds X is a neighbourhood of r if and only if there exists g such that $0 < g$ and $X =]r - g, r + g[$.

(36) For all r, X holds X is a neighbourhood of r if and only if there exists g such that $0 < g$ and $X =]r - g, r + g[$.

(37) For every r and for every neighbourhood N of r holds $r \in N$.

(38) For every r and for every neighbourhoods N_1, N_2 of r there exists a neighbourhood N of r such that $N \subseteq N_1$ and $N \subseteq N_2$.

(39) For every real open subset X and for every r such that $r \in X$ there exists a neighbourhood N of r such that $N \subseteq X$.

(40) For every real open subset X and for every r such that $r \in X$ there exists g such that $0 < g$ and $]r - g, r + g[\subseteq X$.

(41) For every X such that for every r such that $r \in X$ there exists a neighbourhood N of r such that $N \subseteq X$ holds X is open.

(42) For every X holds for every r such that $r \in X$ there exists a neighbourhood N of r such that $N \subseteq X$ if and only if X is open.

(43) If $X \neq \emptyset$ and X is open and X is upper bounded, then $\sup X \notin X$.

(44) If $X \neq \emptyset$ and X is open and X is lower bounded, then $\inf X \notin X$.

(45) If X is open and X is bounded and for all g_1, g_2 such that $g_1 \in X$ and $g_2 \in X$ holds $[g_1, g_2] \subseteq X$, then there exist p, g such that $X =]p, g[$.

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Properties of Real Functions

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Summary. The list of theorems concerning properties of real sequences and functions is enlarged. (See e.g. [9], [4], [8]). The monotone real functions are introduced and their properties are discussed.

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The papers [11], [3], [1], [9], [5], [6], [4], [2], [7], [10], and [8] provide the terminology and notation for this paper. For simplicity we follow a convention: x is arbitrary, X, X_1, Y denote sets, g, r, r_1, r_2, p denote real numbers, R denotes a subset of $\mathbb{R}$, seq, seq_1, seq_2, seq_3 denote sequences of real numbers, Ns denotes an increasing sequence of naturals, n denotes a natural number, and h, h_1, h_2 denote partial functions from $\mathbb{R}$ to $\mathbb{R}$. The following propositions are true:

(1) For all functions F, G and for every X such that $X \subseteq \operatorname{dom} F$ and $F \circ X \subseteq \operatorname{dom} G$ holds $X \subseteq \operatorname{dom}(G \cdot F)$.

(2) For all functions F, G and for every X holds $G \restriction (F \circ X) \cdot F \restriction X = (G \cdot F) \restriction X$.

(3) For all functions F, G and for all X, X_1 holds $G \restriction X_1 \cdot F \restriction X = (G \cdot F) \restriction (X \cap F^{-1} X_1)$.

(4) For all functions F, G and for every X holds $X \subseteq \operatorname{dom}(G \cdot F)$ if and only if $X \subseteq \operatorname{dom} F$ and $F \circ X \subseteq \operatorname{dom} G$.

(5) For every function F and for every X holds $(F \restriction X) \circ X = F \circ X$.

Let us consider seq. Then $\operatorname{rng} seq$ is a subset of $\mathbb{R}$.

One can prove the following propositions:

(6) $seq_1 = seq_2 - seq_3$ if and only if for every n holds $seq_1(n) = seq_2(n) - seq_3(n)$.

(7) $\operatorname{rng}(seq \frown n) \subseteq \operatorname{rng} seq$.

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(8) If $\operatorname{rng} seq \subseteq \operatorname{dom} h$, then $seq(n) \in \operatorname{dom} h$.

(9) $x \in \operatorname{rng} seq$ if and only if there exists n such that $x = seq(n)$.

(10) $seq(n) \in \operatorname{rng} seq$.

(11) If seq_1 is a subsequence of seq, then $\operatorname{rng} seq_1 \subseteq \operatorname{rng} seq$.

(12) If seq_1 is a subsequence of seq and seq is non-zero, then seq_1 is non-zero.

(13) $(seq_1 + seq_2) \cdot Ns = seq_1 \cdot Ns + seq_2 \cdot Ns$ and $(seq_1 - seq_2) \cdot Ns = seq_1 \cdot Ns - seq_2 \cdot Ns$ and $(seq_1 \diamond seq_2) \cdot Ns = (seq_1 \cdot Ns) \diamond (seq_2 \cdot Ns)$.

(14) $(p \diamond seq) \cdot Ns = p \diamond (seq \cdot Ns)$.

(15) $(-seq) \cdot Ns = -seq \cdot Ns$ and $|seq| \cdot Ns = |seq \cdot Ns|$.

(16) If seq is non-zero, then $(seq \cdot Ns)^{-1} = seq^{-1} \cdot Ns$.

(17) If seq is non-zero, then $\frac{seq_1}{seq} \cdot Ns = \frac{seq_1 \cdot Ns}{seq \cdot Ns}$.

(18) If seq is convergent and for every n holds $seq(n) \leq 0$, then $\lim seq \leq 0$.

(19) If for every n holds $seq(n) \in Y$, then $\operatorname{rng} seq \subseteq Y$.

Let us consider h, seq. Let us assume that $\operatorname{rng} seq \subseteq \operatorname{dom} h$. The functor $h \cdot seq$ yields a sequence of real numbers and is defined by:

$h \cdot seq = (h$ **qua** a function$) \cdot seq$.

The following propositions are true:

(20) If $\operatorname{rng} seq \subseteq \operatorname{dom} h$, then $h \cdot seq = (h$ **qua** a function$) \cdot seq$.

(21) If $\operatorname{rng} seq \subseteq \operatorname{dom} h$, then $(h \cdot seq)(n) = h(seq(n))$.

(22) If $\operatorname{rng} seq \subseteq \operatorname{dom} h$, then $(h \cdot seq) \frown n = h \cdot (seq \frown n)$.

(23) Suppose $\operatorname{rng} seq \subseteq \operatorname{dom} h_1 \cap \operatorname{dom} h_2$. Then $(h_1 + h_2) \cdot seq = h_1 \cdot seq + h_2 \cdot seq$ and $(h_1 - h_2) \cdot seq = h_1 \cdot seq - h_2 \cdot seq$ and $(h_1 \diamond h_2) \cdot seq = (h_1 \cdot seq) \diamond (h_2 \cdot seq)$.

(24) If $\operatorname{rng} seq \subseteq \operatorname{dom} h$, then $(r \diamond h) \cdot seq = r \diamond (h \cdot seq)$.

(25) If $\operatorname{rng} seq \subseteq \operatorname{dom} h$, then $|h \cdot seq| = |h| \cdot seq$ and $-h \cdot seq = (-h) \cdot seq$.

(26) If $\operatorname{rng} seq \subseteq \operatorname{dom} \frac{1}{h}$, then $h \cdot seq$ is non-zero.

(27) If $\operatorname{rng} seq \subseteq \operatorname{dom} \frac{1}{h}$, then $\frac{1}{h} \cdot seq = (h \cdot seq)^{-1}$.

(28) If $\operatorname{rng} seq \subseteq \operatorname{dom} h$, then $(h \cdot seq) \cdot Ns = h \cdot (seq \cdot Ns)$.

(29) If $\operatorname{rng} seq_1 \subseteq \operatorname{dom} h$ and seq_2 is a subsequence of seq_1, then $h \cdot seq_2$ is a subsequence of $h \cdot seq_1$.

(30) If h is total, then $(h \cdot seq)(n) = h(seq(n))$.

(31) If h is total, then $h \cdot (seq \frown n) = (h \cdot seq) \frown n$.

(32) If h_1 is total and h_2 is total, then $(h_1 + h_2) \cdot seq = h_1 \cdot seq + h_2 \cdot seq$ and $(h_1 - h_2) \cdot seq = h_1 \cdot seq - h_2 \cdot seq$ and $(h_1 \diamond h_2) \cdot seq = (h_1 \cdot seq) \diamond (h_2 \cdot seq)$.

(33) If h is total, then $(r \diamond h) \cdot seq = r \diamond (h \cdot seq)$.

(34) If $\operatorname{rng} seq \subseteq \operatorname{dom}(h \restriction X)$, then $h \restriction X \cdot seq = h \cdot seq$.

(35) If $\operatorname{rng} seq \subseteq \operatorname{dom}(h \restriction X)$ but $\operatorname{rng} seq \subseteq \operatorname{dom}(h \restriction Y)$ or $X \subseteq Y$, then $h \restriction X \cdot seq = h \restriction Y \cdot seq$.

(36) If $\operatorname{rng} seq \subseteq \operatorname{dom}(h \restriction X)$, then $|h \restriction X \cdot seq| = |h| \restriction X \cdot seq$.

(37) If $\operatorname{rng} seq \subseteq \operatorname{dom}(h \restriction X)$ and $h^{-1}\{0\} = \emptyset$, then $\frac{1}{h} \restriction X \cdot seq = (h \restriction X \cdot seq)^{-1}$.

(38) If $\operatorname{rng} seq \subseteq \operatorname{dom} h$, then $h^{\circ} \operatorname{rng} seq = \operatorname{rng}(h \cdot seq)$.

(39) If $\operatorname{rng} seq \subseteq \operatorname{dom}(h_2 \cdot h_1)$, then $h_2 \cdot (h_1 \cdot seq) = (h_2 \cdot h_1) \cdot seq$.

(40) If h is one-to-one, then $(h \restriction X)^{-1} = h^{-1} \restriction (h^{\circ} X)$.

(41) If $\operatorname{rng} h$ is bounded and $\sup(\operatorname{rng} h) = \inf(\operatorname{rng} h)$, then h is a constant on $\operatorname{dom} h$.

(42) If $Y \subseteq \operatorname{dom} h$ and $h^{\circ} Y$ is bounded and $\sup(h^{\circ} Y) = \inf(h^{\circ} Y)$, then h is a constant on Y.

We now define four new predicates. Let us consider h, Y. We say that h is increasing on Y if and only if:

for all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_1) < h(r_2)$.

We say that h is decreasing on Y if and only if:

for all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_2) < h(r_1)$.

We say that h is non-decreasing on Y if and only if:

for all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_1) \leq h(r_2)$.

We say that h is non-increasing on Y if and only if:

for all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_2) \leq h(r_1)$.

Let us consider h, Y. We say that h is monotone on Y if and only if:

h is non-decreasing on Y or h is non-increasing on Y.

Next we state a number of propositions:

(43) h is increasing on Y if and only if for all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_1) < h(r_2)$.

(44) h is decreasing on Y if and only if for all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_2) < h(r_1)$.

(45) h is non-decreasing on Y if and only if for all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_1) \leq h(r_2)$.

(46) h is non-increasing on Y if and only if for all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_2) \leq h(r_1)$.

(47) h is monotone on Y if and only if h is non-decreasing on Y or h is non-increasing on Y.

(48) h is non-decreasing on Y if and only if for all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 \leq r_2$ holds $h(r_1) \leq h(r_2)$.

(49) h is non-increasing on Y if and only if for all r_1, r_2 such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 \leq r_2$ holds $h(r_2) \leq h(r_1)$.

(50) h is increasing on X if and only if $h \restriction X$ is increasing on X.

(51) h is decreasing on X if and only if $h \restriction X$ is decreasing on X.

(52) h is non-decreasing on X if and only if $h \restriction X$ is non-decreasing on X.

(53) h is non-increasing on X if and only if $h \restriction X$ is non-increasing on X.

(54) If $Y \cap \operatorname{dom} h = \emptyset$, then h is increasing on Y and h is decreasing on Y and h is non-decreasing on Y and h is non-increasing on Y and h is monotone on Y.

(55) If h is increasing on Y, then h is non-decreasing on Y.

(56) If h is decreasing on Y, then h is non-increasing on Y.

(57) If h is a constant on Y, then h is non-decreasing on Y.

(58) If h is a constant on Y, then h is non-increasing on Y.

(59) If h is non-decreasing on Y and h is non-increasing on X, then h is a constant on $Y \cap X$.

(60) If $X \subseteq Y$ and h is increasing on Y, then h is increasing on X.

(61) If $X \subseteq Y$ and h is decreasing on Y, then h is decreasing on X.

(62) If $X \subseteq Y$ and h is non-decreasing on Y, then h is non-decreasing on X.

(63) If $X \subseteq Y$ and h is non-increasing on Y, then h is non-increasing on X.

(64) If h is increasing on Y and $0 < r$, then $r \diamond h$ is increasing on Y but if $r = 0$, then $r \diamond h$ is a constant on Y but if h is increasing on Y and $r < 0$, then $r \diamond h$ is decreasing on Y.

(65) If h is decreasing on Y and $0 < r$, then $r \diamond h$ is decreasing on Y but if h is decreasing on Y and $r < 0$, then $r \diamond h$ is increasing on Y.

(66) If h is non-decreasing on Y and $0 \leq r$, then $r \diamond h$ is non-decreasing on Y but if h is non-decreasing on Y and $r \leq 0$, then $r \diamond h$ is non-increasing on Y.

(67) If h is non-increasing on Y and $0 \leq r$, then $r \diamond h$ is non-increasing on Y but if h is non-increasing on Y and $r \leq 0$, then $r \diamond h$ is non-decreasing on Y.

(68) If $r \in (X \cap Y) \cap \operatorname{dom}(h_1 + h_2)$, then $r \in X \cap \operatorname{dom} h_1$ and $r \in Y \cap \operatorname{dom} h_2$.

(69) (i) If h_1 is increasing on X and h_2 is increasing on Y, then $h_1 + h_2$ is increasing on $X \cap Y$,

(ii) if h_1 is decreasing on X and h_2 is decreasing on Y, then $h_1 + h_2$ is decreasing on $X \cap Y$,

(iii) if h_1 is non-decreasing on X and h_2 is non-decreasing on Y, then $h_1 + h_2$ is non-decreasing on $X \cap Y$,

(iv) if h_1 is non-increasing on X and h_2 is non-increasing on Y, then $h_1 + h_2$ is non-increasing on $X \cap Y$.

(70) If h_1 is increasing on X and h_2 is a constant on Y, then $h_1 + h_2$ is increasing on $X \cap Y$ but if h_1 is decreasing on X and h_2 is a constant on Y, then $h_1 + h_2$ is decreasing on $X \cap Y$.

(71) If h_1 is increasing on X and h_2 is non-decreasing on Y, then $h_1 + h_2$ is increasing on $X \cap Y$.

(72) If h_1 is non-increasing on X and h_2 is a constant on Y, then $h_1 + h_2$ is non-increasing on $X \cap Y$.

(73) If h_1 is decreasing on X and h_2 is non-increasing on Y, then $h_1 + h_2$ is decreasing on $X \cap Y$.

(74) If h_1 is non-decreasing on X and h_2 is a constant on Y, then $h_1 + h_2$ is non-decreasing on $X \cap Y$.

(75) h is increasing on $\{x\}$.

(76) h is decreasing on $\{x\}$.

(77) h is non-decreasing on $\{x\}$.

(78) h is non-increasing on $\{x\}$.

(79) id_R is increasing on R.

(80) If h is increasing on X, then $-h$ is decreasing on X.

(81) If h is non-decreasing on X, then $-h$ is non-increasing on X.

(82) If h is increasing on $[p, g]$ or h is decreasing on $[p, g]$, then $h \upharpoonright [p, g]$ is one-to-one.

(83) If h is increasing on $[p, g]$, then $(h \upharpoonright [p, g])^{-1}$ is increasing on $h \circ [p, g]$.

(84) If h is decreasing on $[p, g]$, then $(h \upharpoonright [p, g])^{-1}$ is decreasing on $h \circ [p, g]$.

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Real Function Continuity [1]

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Summary. The continuity of real functions is discussed. There is a function defined on some domain in real numbers which is continuous in a single point and on a subset of domain of the function. Main properties of real continuous functions are proved. Among them there is the Weierstraß Theorem. Algebraic features for real continuous functions are shown. Lipschitzian functions are introduced. The Lipschitz condition entails continuity.

MML Identifier: `FCONT_1`.

The papers [11], [2], [9], [8], [4], [3], [12], [1], [5], [6], [7], and [10] provide the terminology and notation for this paper. For simplicity we adopt the following rules: n is a natural number, X, X_1, Z, Z_1 are sets, s, g, r, p, x_0, x_1, x_2 are real numbers, s_1 is a sequence of real numbers, Y is a subset of $\mathbb{R}$, and f, f_1, f_2 are partial functions from $\mathbb{R}$ to $\mathbb{R}$. Let us consider f, x_0. We say that f is continuous in x_0 if and only if:

$x_0 \in \operatorname{dom} f$ and for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f(x_0) = \lim(f \cdot s_1)$.

Next we state a number of propositions:

(1) For all f, x_0 holds f is continuous in x_0 if and only if $x_0 \in \operatorname{dom} f$ and for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f(x_0) = \lim(f \cdot s_1)$.

(2) f is continuous in x_0 if and only if $x_0 \in \operatorname{dom} f$ and for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ and for every n holds $s_1(n) \neq x_0$ holds $f \cdot s_1$ is convergent and $f(x_0) = \lim(f \cdot s_1)$.

(3) f is continuous in x_0 if and only if $x_0 \in \operatorname{dom} f$ and for every r such that $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in \operatorname{dom} f$ and $|x_1 - x_0| < s$ holds $|f(x_1) - f(x_0)| < r$.

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(4) For all f, x_0 holds f is continuous in x_0 if and only if $x_0 \in \operatorname{dom} f$ and for every neighbourhood N_1 of $f(x_0)$ there exists a neighbourhood N of x_0 such that for every x_1 such that $x_1 \in \operatorname{dom} f$ and $x_1 \in N$ holds $f(x_1) \in N_1$.

(5) For all f, x_0 holds f is continuous in x_0 if and only if $x_0 \in \operatorname{dom} f$ and for every neighbourhood N_1 of $f(x_0)$ there exists a neighbourhood N of x_0 such that $f\,^\circ\, N \subseteq N_1$.

(6) If $x_0 \in \operatorname{dom} f$ and there exists a neighbourhood N of x_0 such that $\operatorname{dom} f \cap N = \{x_0\}$, then f is continuous in x_0.

(7) If f_1 is continuous in x_0 and f_2 is continuous in x_0, then $f_1 + f_2$ is continuous in x_0 and $f_1 - f_2$ is continuous in x_0 and $f_1 \diamond f_2$ is continuous in x_0.

(8) If f is continuous in x_0, then $r \diamond f$ is continuous in x_0.

(9) If f is continuous in x_0, then $|f|$ is continuous in x_0 and $-f$ is continuous in x_0.

(10) If f is continuous in x_0 and $f(x_0) \neq 0$, then $\frac{1}{f}$ is continuous in x_0.

(11) If f_1 is continuous in x_0 and $f_1(x_0) \neq 0$ and f_2 is continuous in x_0, then $\frac{f_2}{f_1}$ is continuous in x_0.

(12) If f_1 is continuous in x_0 and f_2 is continuous in $f_1(x_0)$, then $f_2 \cdot f_1$ is continuous in x_0.

Let us consider f, X. We say that f is continuous on X if and only if:

$X \subseteq \operatorname{dom} f$ and for every x_0 such that $x_0 \in X$ holds $f \restriction X$ is continuous in x_0.

One can prove the following propositions:

(13) For all f, X holds f is continuous on X if and only if $X \subseteq \operatorname{dom} f$ and for every x_0 such that $x_0 \in X$ holds $f \restriction X$ is continuous in x_0.

(14) For all X, f holds f is continuous on X if and only if $X \subseteq \operatorname{dom} f$ and for every s_1 such that $\operatorname{rng} s_1 \subseteq X$ and s_1 is convergent and $\lim s_1 \in X$ holds $f \cdot s_1$ is convergent and $f(\lim s_1) = \lim(f \cdot s_1)$.

(15) f is continuous on X if and only if $X \subseteq \operatorname{dom} f$ and for all x_0, r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in X$ and $|x_1 - x_0| < s$ holds $|f(x_1) - f(x_0)| < r$.

(16) f is continuous on X if and only if $f \restriction X$ is continuous on X.

(17) If f is continuous on X and $X_1 \subseteq X$, then f is continuous on X_1.

(18) If $x_0 \in \operatorname{dom} f$, then f is continuous on $\{x_0\}$.

(19) For all X, f_1, f_2 such that f_1 is continuous on X and f_2 is continuous on X holds $f_1 + f_2$ is continuous on X and $f_1 - f_2$ is continuous on X and $f_1 \diamond f_2$ is continuous on X.

(20) For all X, X_1, f_1, f_2 such that f_1 is continuous on X and f_2 is continuous on X_1 holds $f_1 + f_2$ is continuous on $X \cap X_1$ and $f_1 - f_2$ is continuous on $X \cap X_1$ and $f_1 \diamond f_2$ is continuous on $X \cap X_1$.

(21) For all r, X, f such that f is continuous on X holds $r \diamond f$ is continuous on X.

(22) If f is continuous on X, then $|f|$ is continuous on X and $-f$ is continuous on X.

(23) If f is continuous on X and $f^{-1}\{0\} = \emptyset$, then $\frac{1}{f}$ is continuous on X.

(24) If f is continuous on X and $(f \restriction X)^{-1}\{0\} = \emptyset$, then $\frac{1}{f}$ is continuous on X.

(25) If f_1 is continuous on X and $f_1{}^{-1}\{0\} = \emptyset$ and f_2 is continuous on X, then $\frac{f_2}{f_1}$ is continuous on X.

(26) If f_1 is continuous on X and f_2 is continuous on $f_1 {}^\circ X$, then $f_2 \cdot f_1$ is continuous on X.

(27) If f_1 is continuous on X and f_2 is continuous on X_1, then $f_2 \cdot f_1$ is continuous on $X \cap f_1{}^{-1} X_1$.

(28) If f is total and for all x_1, x_2 holds $f(x_1 + x_2) = f(x_1) + f(x_2)$ and there exists x_0 such that f is continuous in x_0, then f is continuous on $\mathbb{R}$.

(29) For every f such that $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$ holds $\operatorname{rng} f$ is compact.

(30) If $Y \subseteq \operatorname{dom} f$ and Y is compact and f is continuous on Y, then $f {}^\circ Y$ is compact.

(31) For every f such that $\operatorname{dom} f \neq \emptyset$ and $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$ there exist x_1, x_2 such that $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} f$ and $f(x_1) = \sup(\operatorname{rng} f)$ and $f(x_2) = \inf(\operatorname{rng} f)$.

(32) For all f, Y such that $Y \neq \emptyset$ and $Y \subseteq \operatorname{dom} f$ and Y is compact and f is continuous on Y there exist x_1, x_2 such that $x_1 \in Y$ and $x_2 \in Y$ and $f(x_1) = \sup(f {}^\circ Y)$ and $f(x_2) = \inf(f {}^\circ Y)$.

Let us consider f, X. We say that f is Lipschitzian on X if and only if:

$X \subseteq \operatorname{dom} f$ and there exists r such that $0 < r$ and for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ holds $|f(x_1) - f(x_2)| \leq r \cdot |x_1 - x_2|$.

One can prove the following propositions:

(33) For every f holds f is Lipschitzian on X if and only if $X \subseteq \operatorname{dom} f$ and there exists r such that $0 < r$ and for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ holds $|f(x_1) - f(x_2)| \leq r \cdot |x_1 - x_2|$.

(34) If f is Lipschitzian on X and $X_1 \subseteq X$, then f is Lipschitzian on X_1.

(35) If f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1, then $f_1 + f_2$ is Lipschitzian on $X \cap X_1$.

(36) If f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1, then $f_1 - f_2$ is Lipschitzian on $X \cap X_1$.

(37) If f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1 and f_1 is bounded on Z and f_2 is bounded on Z_1, then $f_1 \diamond f_2$ is Lipschitzian on $((X \cap Z) \cap X_1) \cap Z_1$.

(38) If f is Lipschitzian on X, then $p \diamond f$ is Lipschitzian on X.

(39) If f is Lipschitzian on X, then $-f$ is Lipschitzian on X and $|f|$ is Lipschitzian on X.

(40) If $X \subseteq \mathrm{dom}\, f$ and f is a constant on X, then f is Lipschitzian on X.

(41) id_Y is Lipschitzian on Y.

(42) If f is Lipschitzian on X, then f is continuous on X.

(43) For every f such that there exists r such that $\mathrm{rng}\, f = \{r\}$ holds f is continuous on $\mathrm{dom}\, f$.

(44) If $X \subseteq \mathrm{dom}\, f$ and f is a constant on X, then f is continuous on X.

(45) For every f such that for every x_0 such that $x_0 \in \mathrm{dom}\, f$ holds $f(x_0) = x_0$ holds f is continuous on $\mathrm{dom}\, f$.

(46) If $f = \mathrm{id}_{\mathrm{dom}\, f}$, then f is continuous on $\mathrm{dom}\, f$.

(47) If $Y \subseteq \mathrm{dom}\, f$ and $f \restriction Y = \mathrm{id}_Y$, then f is continuous on Y.

(48) If $X \subseteq \mathrm{dom}\, f$ and for every x_0 such that $x_0 \in X$ holds $f(x_0) = r \cdot x_0 + p$, then f is continuous on X.

(49) If for every x_0 such that $x_0 \in \mathrm{dom}\, f$ holds $f(x_0) = {x_0}^{\mathbf{2}}$, then f is continuous on $\mathrm{dom}\, f$.

(50) If $X \subseteq \mathrm{dom}\, f$ and for every x_0 such that $x_0 \in X$ holds $f(x_0) = {x_0}^{\mathbf{2}}$, then f is continuous on X.

(51) If for every x_0 such that $x_0 \in \mathrm{dom}\, f$ holds $f(x_0) = |x_0|$, then f is continuous on $\mathrm{dom}\, f$.

(52) If $X \subseteq \mathrm{dom}\, f$ and for every x_0 such that $x_0 \in X$ holds $f(x_0) = |x_0|$, then f is continuous on X.

(53) If $X \subseteq \mathrm{dom}\, f$ and f is monotone on X and there exist p, g such that $p \leq g$ and $f \circ X = [p, g]$, then f is continuous on X.

(54) If $p \leq g$ and $[p, g] \subseteq \mathrm{dom}\, f$ but f is increasing on $[p, g]$ or f is decreasing on $[p, g]$, then $(f \restriction [p, g])^{-1}$ is continuous on $f \circ [p, g]$.

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Real Function Uniform Continuity [1]

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Summary. The uniform continuity for real functions is introduced. More theorems concerning continuous functions are given. (See [10]) The Darboux Theorem is exposed. Algebraic features for uniformly continuous functions are presented. Various facts, e.g., a continuous function on a compact set is uniformly continuous are proved.

MML Identifier: FCONT_2.

The notation and terminology used in this paper have been introduced in the following articles: [12], [13], [3], [1], [9], [8], [4], [2], [5], [6], [7], [11], and [10]. For simplicity we adopt the following convention: X, X_1, Z, Z_1 are sets, s, g, r, p, x_1, x_2 are real numbers, Y is a subset of $\mathbb{R}$, and f, f_1, f_2 are partial functions from $\mathbb{R}$ to $\mathbb{R}$. Let us consider f, X. We say that f is uniformly continuous on X if and only if:

$X \subseteq \operatorname{dom} f$ and for every r such that $0 < r$ there exists s such that $0 < s$ and for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ and $|x_1 - x_2| < s$ holds $|f(x_1) - f(x_2)| < r$.

We now state a number of propositions:

(1) Given f, X. Then f is uniformly continuous on X if and only if $X \subseteq \operatorname{dom} f$ and for every r such that $0 < r$ there exists s such that $0 < s$ and for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ and $|x_1 - x_2| < s$ holds $|f(x_1) - f(x_2)| < r$.

(2) If f is uniformly continuous on X and $X_1 \subseteq X$, then f is uniformly continuous on X_1.

(3) If f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1, then $f_1 + f_2$ is uniformly continuous on $X \cap X_1$.

(4) If f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1, then $f_1 - f_2$ is uniformly continuous on $X \cap X_1$.

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(5) If f is uniformly continuous on X, then $p \diamond f$ is uniformly continuous on X.

(6) If f is uniformly continuous on X, then $-f$ is uniformly continuous on X.

(7) If f is uniformly continuous on X, then $|f|$ is uniformly continuous on X.

(8) If f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1 and f_1 is bounded on Z and f_2 is bounded on Z_1, then $f_1 \diamond f_2$ is uniformly continuous on $((X \cap Z) \cap X_1) \cap Z_1$.

(9) If f is uniformly continuous on X, then f is continuous on X.

(10) If f is Lipschitzian on X, then f is uniformly continuous on X.

(11) For all f, Y such that Y is compact and f is continuous on Y holds f is uniformly continuous on Y.

(12) For every f such that $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$ holds f is uniformly continuous on $\operatorname{dom} f$.

(13) If $Y \subseteq \operatorname{dom} f$ and Y is compact and f is uniformly continuous on Y, then $f \circ Y$ is compact.

(14) For all f, Y such that $Y \neq \emptyset$ and $Y \subseteq \operatorname{dom} f$ and Y is compact and f is uniformly continuous on Y there exist x_1, x_2 such that $x_1 \in Y$ and $x_2 \in Y$ and $f(x_1) = \sup(f \circ Y)$ and $f(x_2) = \inf(f \circ Y)$.

(15) If $X \subseteq \operatorname{dom} f$ and f is a constant on X, then f is uniformly continuous on X.

(16) If $p \leq g$ and f is continuous on $[p, g]$, then for every r such that $r \in [f(p), f(g)] \cup [f(g), f(p)]$ there exists s such that $s \in [p, g]$ and $r = f(s)$.

(17) If $p \leq g$ and f is continuous on $[p, g]$, then for every r such that $r \in [\inf(f \circ [p, g]), \sup(f \circ [p, g])]$ there exists s such that $s \in [p, g]$ and $r = f(s)$.

(18) If f is one-to-one and $p \leq g$ and f is continuous on $[p, g]$, then f is increasing on $[p, g]$ or f is decreasing on $[p, g]$.

(19) Suppose f is one-to-one and $p \leq g$ and f is continuous on $[p, g]$. Then $\inf(f \circ [p, g]) = f(p)$ and $\sup(f \circ [p, g]) = f(g)$ or $\inf(f \circ [p, g]) = f(g)$ and $\sup(f \circ [p, g]) = f(p)$.

(20) If $p \leq g$ and f is continuous on $[p, g]$, then $f \circ [p, g] = [\inf(f \circ [p, g]), \sup(f \circ [p, g])]$.

(21) If f is one-to-one and $p \leq g$ and f is continuous on $[p, g]$, then f^{-1} is continuous on $[\inf(f \circ [p, g]), \sup(f \circ [p, g])]$.

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Real Function Differentiability [1]

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Summary. For a real valued function defined on its domain in real numbers the differentiability in a single point and on a subset of the domain is presented. The main elements of differential calculus are developed. The algebraic properties of differential real functions are shown.

MML Identifier: `FDIFF_1`.

The terminology and notation used here have been introduced in the following articles: [11], [2], [8], [3], [4], [1], [5], [6], [7], [10], and [9]. For simplicity we follow the rules: x, x_0, r, p will be real numbers, n will be a natural number, Y will be a subset of $\mathbb{R}$, Z will be a real open subset, X will be a set, s_1 will be a sequence of real numbers, and f, f_1, f_2 will be partial functions from $\mathbb{R}$ to $\mathbb{R}$. We now state the proposition

(1) For every r holds $r \in Y$ if and only if $r \in \mathbb{R}$ if and only if $Y = \mathbb{R}$.

A sequence of real numbers is called a real sequence convergent to 0 if:
it is non-zero and it is convergent and $\lim \mathrm{it} = 0$.

The following proposition is true

(2) For every s_1 holds s_1 is a real sequence convergent to 0 if and only if s_1 is non-zero and s_1 is convergent and $\lim s_1 = 0$.

A sequence of real numbers is called a constant real sequence if:
it is constant.

We now state the proposition

(3) For every s_1 holds s_1 is a constant real sequence if and only if s_1 is constant.

In the sequel h will be a real sequence convergent to 0 and c will be a constant real sequence. A partial function from $\mathbb{R}$ to $\mathbb{R}$ is called a rest if:

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it is total and for every h holds $h^{-1} \diamond (\mathrm{it} \cdot h)$ is convergent and $\lim(h^{-1} \diamond (\mathrm{it} \cdot h)) = 0$.

One can prove the following proposition

(4) For every f holds f is a rest if and only if f is total and for every h holds $h^{-1} \diamond (f \cdot h)$ is convergent and $\lim(h^{-1} \diamond (f \cdot h)) = 0$.

A partial function from $\mathbb{R}$ to $\mathbb{R}$ is called a linear function if:

it is total and there exists r such that for every p holds $\mathrm{it}(p) = r \cdot p$.

The following proposition is true

(5) For every f holds f is a linear function if and only if f is total and there exists r such that for every p holds $f(p) = r \cdot p$.

We follow the rules: R, R_1, R_2 are rests and L, L_1, L_2 are linear functions. We now state several propositions:

(6) For all L_1, L_2 holds $L_1 + L_2$ is a linear function and $L_1 - L_2$ is a linear function.

(7) For all r, L holds $r \diamond L$ is a linear function.

(8) For all R_1, R_2 holds $R_1 + R_2$ is a rest and $R_1 - R_2$ is a rest and $R_1 \diamond R_2$ is a rest.

(9) For all r, R holds $r \diamond R$ is a rest.

(10) $L_1 \diamond L_2$ is a rest.

(11) $R \diamond L$ is a rest and $L \diamond R$ is a rest.

Let us consider f, x_0. We say that f is differentiable in x_0 if and only if:

there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\, f$ and there exist L, R such that for every x such that $x \in N$ holds $f(x) - f(x_0) = L(x - x_0) + R(x - x_0)$.

The following proposition is true

(12) For all f, x_0 holds f is differentiable in x_0 if and only if there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\, f$ and there exist L, R such that for every x such that $x \in N$ holds $f(x)-f(x_0) = L(x-x_0)+R(x-x_0)$.

Let us consider f, x_0. Let us assume that f is differentiable in x_0. The functor $f'(x_0)$ yields a real number and is defined as follows:

there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\, f$ and there exist L, R such that $f'(x_0) = L(1)$ and for every x such that $x \in N$ holds $f(x) - f(x_0) = L(x - x_0) + R(x - x_0)$.

The following proposition is true

(13) Given r, f, x_0. Suppose f is differentiable in x_0. Then $r = f'(x_0)$ if and only if there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\, f$ and there exist L, R such that $r = L(1)$ and for every x such that $x \in N$ holds $f(x) - f(x_0) = L(x - x_0) + R(x - x_0)$.

Let us consider f, X. We say that f is differentiable on X if and only if:

$X \subseteq \mathrm{dom}\, f$ and for every x such that $x \in X$ holds $f \restriction X$ is differentiable in x.

The following four propositions are true:

(14) For all f, X holds f is differentiable on X if and only if $X \subseteq \operatorname{dom} f$ and for every x such that $x \in X$ holds $f \restriction X$ is differentiable in x.

(15) If f is differentiable on X, then X is a subset of $\mathbb{R}$.

(16) f is differentiable on Z if and only if $Z \subseteq \operatorname{dom} f$ and for every x such that $x \in Z$ holds f is differentiable in x.

(17) If f is differentiable on Y, then Y is open.

Let us consider f, X. Let us assume that f is differentiable on X. The functor $f'_{\restriction X}$ yielding a partial function from $\mathbb{R}$ to $\mathbb{R}$ is defined by:

$\operatorname{dom}(f'_{\restriction X}) = X$ and for every x such that $x \in X$ holds $(f'_{\restriction X})(x) = f'(x)$.

One can prove the following two propositions:

(18) For all f, X and for every partial function F from $\mathbb{R}$ to $\mathbb{R}$ such that f is differentiable on X holds $F = f'_{\restriction X}$ if and only if $\operatorname{dom} F = X$ and for every x such that $x \in X$ holds $F(x) = f'(x)$.

(19) For all f, Z such that $Z \subseteq \operatorname{dom} f$ and there exists r such that $\operatorname{rng} f = \{r\}$ holds f is differentiable on Z and for every x such that $x \in Z$ holds $(f'_{\restriction Z})(x) = 0$.

Let us consider h, n. Then $h \frown n$ is a real sequence convergent to 0.

Let us consider c, n. Then $c \frown n$ is a constant real sequence.

Next we state a number of propositions:

(20) Given f, x_0. Let N be a neighbourhood of x_0. Suppose f is differentiable in x_0 and $N \subseteq \operatorname{dom} f$. Then for all h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq N$ holds $h^{-1} \diamond (f \cdot (h + c) - f \cdot c)$ is convergent and $f'(x_0) = \lim(h^{-1} \diamond (f \cdot (h + c) - f \cdot c))$.

(21) For all f_1, f_2, x_0 such that f_1 is differentiable in x_0 and f_2 is differentiable in x_0 holds $f_1 + f_2$ is differentiable in x_0 and $(f_1 + f_2)'(x_0) = f'_1(x_0) + f'_2(x_0)$.

(22) For all f_1, f_2, x_0 such that f_1 is differentiable in x_0 and f_2 is differentiable in x_0 holds $f_1 - f_2$ is differentiable in x_0 and $(f_1 - f_2)'(x_0) = f'_1(x_0) - f'_2(x_0)$.

(23) For all r, f, x_0 such that f is differentiable in x_0 holds $r \diamond f$ is differentiable in x_0 and $(r \diamond f)'(x_0) = r \cdot (f'(x_0))$.

(24) For all f_1, f_2, x_0 such that f_1 is differentiable in x_0 and f_2 is differentiable in x_0 holds $f_1 \diamond f_2$ is differentiable in x_0 and $(f_1 \diamond f_2)'(x_0) = f_2(x_0) \cdot (f'_1(x_0)) + f_1(x_0) \cdot (f'_2(x_0))$.

(25) For all f, Z such that $Z \subseteq \operatorname{dom} f$ and $f \restriction Z = \mathrm{id}_Z$ holds f is differentiable on Z and for every x such that $x \in Z$ holds $(f'_{\restriction Z})(x) = 1$.

(26) For all f_1, f_2, Z such that $Z \subseteq \operatorname{dom}(f_1 + f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z holds $f_1 + f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $((f_1 + f_2)'_{\restriction Z})(x) = f'_1(x) + f'_2(x)$.

(27) For all f_1, f_2, Z such that $Z \subseteq \operatorname{dom}(f_1 - f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z holds $f_1 - f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $((f_1 - f_2)'_{\restriction Z})(x) = f'_1(x) - f'_2(x)$.

(28) For all r, f, Z such that $Z \subseteq \mathrm{dom}(r \diamond f)$ and f is differentiable on Z holds $r \diamond f$ is differentiable on Z and for every x such that $x \in Z$ holds $((r \diamond f)'_{\restriction Z})(x) = r \cdot (f'(x))$.

(29) Given f_1, f_2, Z. Then if $Z \subseteq \mathrm{dom}(f_1 \diamond f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z, then $f_1 \diamond f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $((f_1 \diamond f_2)'_{\restriction Z})(x) = f_2(x) \cdot (f_1'(x)) + f_1(x) \cdot (f_2'(x))$.

(30) If $Z \subseteq \mathrm{dom}\, f$ and f is a constant on Z, then f is differentiable on Z and for every x such that $x \in Z$ holds $(f'_{\restriction Z})(x) = 0$.

(31) If $Z \subseteq \mathrm{dom}\, f$ and for every x such that $x \in Z$ holds $f(x) = r \cdot x + p$, then f is differentiable on Z and for every x such that $x \in Z$ holds $(f'_{\restriction Z})(x) = r$.

(32) If f is differentiable in x_0, then f is continuous in x_0.

(33) If f is differentiable on X, then f is continuous on X.

(34) If f is differentiable on X and $Z \subseteq X$, then f is differentiable on Z.

(35) If f is differentiable in x_0, then there exists R such that $R(0) = 0$ and R is continuous in 0.

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Average Value Theorems for Real Functions of One Variable [1]

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Summary. Three basic theorems in differential calculus of one variable functions are presented: Rolle Theorem, Lagrange Theorem and Cauchy Theorem. There are also direct conclusions.

MML Identifier: `ROLLE`.

The terminology and notation used here have been introduced in the following papers: [2], [1], [3], [4], [5], [8], [6], and [7]. We adopt the following rules: g, r, s, p, t, x, x_0, x_1 will denote real numbers and f, f_1, f_2 will denote partial functions from $\mathbb{R}$ to $\mathbb{R}$. We now state a number of propositions:

(1) For all p, g such that $p < g$ for every f such that f is continuous on $[p, g]$ and $f(p) = f(g)$ and f is differentiable on $]p, g[$ there exists x_0 such that $x_0 \in\,]p, g[$ and $f'(x_0) = 0$.

(2) Given x, t. Suppose $0 < t$. Then for every f such that f is continuous on $[x, x+t]$ and $f(x) = f(x+t)$ and f is differentiable on $]x, x+t[$ there exists s such that $0 < s$ and $s < 1$ and $f'(x + s \cdot t) = 0$.

(3) For all p, g such that $p < g$ for every f such that f is continuous on $[p, g]$ and f is differentiable on $]p, g[$ there exists x_0 such that $x_0 \in\,]p, g[$ and $f'(x_0) = \frac{f(g)-f(p)}{g-p}$.

(4) Given x, t. Suppose $0 < t$. Then for every f such that f is continuous on $[x, x+t]$ and f is differentiable on $]x, x+t[$ there exists s such that $0 < s$ and $s < 1$ and $f(x+t) = f(x) + t \cdot (f'(x + s \cdot t))$.

(5) Given p, g. Suppose $p < g$. Given f_1, f_2. Suppose f_1 is continuous on $[p, g]$ and f_1 is differentiable on $]p, g[$ and f_2 is continuous on $[p, g]$ and f_2 is differentiable on $]p, g[$. Then there exists x_0 such that $x_0 \in\,]p, g[$ and $(f_1(g) - f_1(p)) \cdot (f_2'(x_0)) = (f_2(g) - f_2(p)) \cdot (f_1'(x_0))$.

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(6) Given x, t. Suppose $0 < t$. Given f_1, f_2. Suppose f_1 is continuous on $[x, x+t]$ and f_1 is differentiable on $]x, x+t[$ and f_2 is continuous on $[x, x+t]$ and f_2 is differentiable on $]x, x+t[$ and for every x_1 such that $x_1 \in]x, x+t[$ holds $f_2'(x_1) \neq 0$. Then there exists s such that $0 < s$ and $s < 1$ and $\frac{f_1(x+t)-f_1(x)}{f_2(x+t)-f_2(x)} = \frac{f_1'(x+s\cdot t)}{f_2'(x+s\cdot t)}$.

(7) For all p, g such that $p < g$ for every f such that f is differentiable on $]p, g[$ and for every x such that $x \in]p, g[$ holds $f'(x) = 0$ holds f is a constant on $]p, g[$.

(8) Given p, g. Suppose $p < g$. Given f_1, f_2. Suppose f_1 is differentiable on $]p, g[$ and f_2 is differentiable on $]p, g[$ and for every x such that $x \in]p, g[$ holds $f_1'(x) = f_2'(x)$. Then $f_1 - f_2$ is a constant on $]p, g[$ and there exists r such that for every x such that $x \in]p, g[$ holds $f_1(x) = f_2(x) + r$.

(9) For all p, g such that $p < g$ for every f such that f is differentiable on $]p, g[$ and for every x such that $x \in]p, g[$ holds $0 < f'(x)$ holds f is increasing on $]p, g[$.

(10) For all p, g such that $p < g$ for every f such that f is differentiable on $]p, g[$ and for every x such that $x \in]p, g[$ holds $f'(x) < 0$ holds f is decreasing on $]p, g[$.

(11) For all p, g such that $p < g$ for every f such that f is differentiable on $]p, g[$ and for every x such that $x \in]p, g[$ holds $0 \leq f'(x)$ holds f is non-decreasing on $]p, g[$.

(12) For all p, g such that $p < g$ for every f such that f is differentiable on $]p, g[$ and for every x such that $x \in]p, g[$ holds $f'(x) \leq 0$ holds f is non-increasing on $]p, g[$.

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