FORMALIZED MATHEMATICS
Vol.1, No.**3**, May–August 1990
Université Catholique de Louvain

Zermelo's Theorem [1]

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Summary. The article contains direct proof of Zermelo's theorem about the existence of a well ordering for any set and the lemma the proof depends on.

MML Identifier: `WELLSET1`.

The articles [4], [3], [5], [2], and [1] provide the notation and terminology for this paper. For simplicity we follow the rules: a, x, y will be arbitrary, B, D, N, X, Y will denote sets, R, S, T will denote relations, F will denote a function, and W will denote a relation. We now state several propositions:

(1) $x \in \operatorname{field} R$ if and only if there exists y such that $\langle x, y\rangle \in R$ or $\langle y, x\rangle \in R$.

(2) $R \cup S$ is a relation.

(3) If $X \neq \emptyset$ and $Y \neq \emptyset$ and $W = [\!:X, Y:\!]$, then $\operatorname{field} W = X \cup Y$.

(4) If $y = R$, then y is a relation.

(5) For all a, T holds $x \in T{-}\operatorname{Seg}(a)$ if and only if $x \neq a$ and $\langle x, a\rangle \in T$.

In the article we present several logical schemes. The scheme *R_Separation* deals with a set $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists B such that for every relation R holds $R \in B$ if and only if $R \in \mathcal{A}$ and $\mathcal{P}[R]$

for all values of the parameters.

The scheme *S_Separation* deals with a set $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists B such that for every set X holds $X \in B$ if and only if $X \in \mathcal{A}$ and $\mathcal{P}[X]$

for all values of the parameters.

The following four propositions are true:

(6) For all x, y, W such that $x \in \operatorname{field} W$ and $y \in \operatorname{field} W$ and W is well ordering relation holds if $x \notin W{-}\operatorname{Seg}(y)$, then $\langle y, x\rangle \in W$.

[1]Supported by RPBP III.24 C9

ISSN 0777–4028

(7) For all x, y, W such that $x \in \operatorname{field} W$ and $y \in \operatorname{field} W$ and W is well ordering relation holds if $x \in W{-}\mathrm{Seg}(y)$, then $\langle y, x\rangle \notin W$.

(8) Given F, D. Suppose for every X such that $X \in D$ holds $F(X) \notin X$ and $F(X) \in \bigcup D$. Then there exists R such that $\operatorname{field} R \subseteq \bigcup D$ and R is well ordering relation and $\operatorname{field} R \notin D$ and for every y such that $y \in \operatorname{field} R$ holds $R{-}\mathrm{Seg}(y) \in D$ and $F(R{-}\mathrm{Seg}(y)) = y$.

(9) For every N there exists R such that R is well ordering relation and $\operatorname{field} R = N$.

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Received October 27, 1989

FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Group and Field Definitions

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Summary. The article contains exactly the same definitions of group and field as those in [3]. These definitions were prepared without the help of the definitions and properties of *Nat* and *Real* modes icluded in the MML. This is the first of a series of articles in which we are going to introduce the concept of the set of real numbers in a elementary axiomatic way.

MML Identifier: `REALSET1`.

The terminology and notation used here are introduced in the following papers: [4], [1], and [2]. Let x be arbitrary. The functor single(x) yields a set and is defined as follows:

single(x) $= \{x\}$.

One can prove the following proposition

(1) For arbitrary x holds single(x) $= \{x\}$.

Let X, Y be sets. The functor $X\#Y$ yields a set and is defined by:

$X\#Y = [\!:X, Y:\!]$.

We now state several propositions:

(2) For all sets X, Y holds $X\#Y = [\!:X, Y:\!]$.

(3) For arbitrary z and for every set A holds $z \in A\#A$ if and only if there exist arbitrary x, y such that $x \in A$ and $y \in A$ and $z = \langle x, y \rangle$.

(4) For every set X and for every subset A of X holds $A\#A \subseteq X\#X$.

(5) For every set X such that $X = \emptyset$ holds $X\#X = \emptyset$.

(6) For every set X such that $X\#X = \emptyset$ holds $X = \emptyset$.

(7) For every set X holds $X\#X = \emptyset$ if and only if $X = \emptyset$.

Let X be a set. A binary operation of X is a function from $X\#X$ into X.

The following propositions are true:

[1]Supported by RPBP III.24 C9

ISSN 0777–4028

(8) For every set X and for every function F from $X\#X$ into X holds F is a binary operation of X.

(9) For every set X and for every function F holds F is a function from $X\#X$ into X if and only if F is a binary operation of X.

(10) For every set X and for every function F from $X\#X$ into X and for arbitrary x such that $x \in X\#X$ holds $F(x) \in X$.

(11) For every set X and for every binary operation F of X there exists a subset A of X such that for arbitrary x such that $x \in A\#A$ holds $F(x) \in A$.

Let X be a set, and let F be a binary operation of X, and let A be a subset of X. We say that F is in A if and only if:

for arbitrary x such that $x \in A\#A$ holds $F(x) \in A$.

Next we state a proposition

(12) For every set X and for every binary operation F of X and for every subset A of X holds F is in A if and only if for arbitrary x such that $x \in A\#A$ holds $F(x) \in A$.

Let X be a set, and let F be a binary operation of X. A subset of X is said to be a set closed w.r.t. F if:

for arbitrary x such that $x \in \mathrm{it}\#\mathrm{it}$ holds $F(x) \in \mathrm{it}$.

The following propositions are true:

(13) For every set X and for every binary operation F of X and for every subset A of X holds A is a set closed w.r.t. F if and only if for arbitrary x such that $x \in A\#A$ holds $F(x) \in A$.

(14) For every set X and for every binary operation F of X and for every set A closed w.r.t. F holds $F \upharpoonright (A\#A)$ is a binary operation of A.

Let X be a set, and let F be a binary operation of X, and let A be a set closed w.r.t. F. The functor $F \upharpoonright A$ yielding a binary operation of A, is defined by:

$F \upharpoonright A = F \upharpoonright (A\#A)$.

The following propositions are true:

(15) For every set X and for every binary operation F of X and for every set A closed w.r.t. F holds $F \upharpoonright A = F \upharpoonright (A\#A)$.

(16) For every set X and for every binary operation F of X and for every subset A of X such that A is a set closed w.r.t. F holds $F \upharpoonright (A\#A)$ is a binary operation of A.

(17) For every set X and for every binary operation F of X and for every set A closed w.r.t. F holds $F \upharpoonright A$ is a binary operation of A.

We consider group structures which are systems

$\langle$ a carrier, an addition, a zero $\rangle$

where the carrier is a non-empty set, the addition is a binary operation of the carrier, and the zero is an element of the carrier. Let A be a non-empty set, and let og be a binary operation of A, and let ng be an element of A. The functor group(A, og, ng) yielding a group structure, is defined as follows:

$A =$ the carrier of group(A, og, ng) and $og =$ the addition of group(A, og, ng) and $ng =$ the zero of group(A, og, ng).

The following propositions are true:

(18) For every non-empty set A and for every binary operation og of A and for every element ng of A and for every GR being a group structure holds $GR =$ group(A, og, ng) if and only if $A =$ the carrier of GR and $og =$ the addition of GR and $ng =$ the zero of GR.

(19) For every non-empty set A and for every binary operation og of A and for every element ng of A holds group(A, og, ng) is a group structure and $A =$ the carrier of group(A, og, ng) and $og =$ the addition of group(A, og, ng) and $ng =$ the zero of group(A, og, ng).

A group structure is called a group if:

there exists a non-empty set A and there exists a binary operation og of A and there exists an element ng of A such that it $=$ group(A, og, ng) and for all elements a, b, c of A holds $og(\langle og(\langle a, b\rangle), c\rangle) = og(\langle a, og(\langle b, c\rangle)\rangle)$ and for every element a of A holds $og(\langle a, ng\rangle) = a$ and $og(\langle ng, a\rangle) = a$ and for every element a of A there exists an element b of A such that $og(\langle a, b\rangle) = ng$ and $og(\langle b, a\rangle) = ng$ and for all elements a, b of A holds $og(\langle a, b\rangle) = og(\langle b, a\rangle)$.

Let D be a group. The carrier of D yields a non-empty set and is defined as follows:

there exists a binary operation od of the carrier of D and there exists an element nd of the carrier of D such that $D =$ group(the carrier of D, od, nd).

The following two propositions are true:

(20) For every group D and for every non-empty set A holds
$A =$ the carrier of D
if and only if there exists a binary operation od of A and there exists an element nd of A such that $D =$ group(A, od, nd).

(21) For every group D holds the carrier of D is a non-empty set and there exists a binary operation od of the carrier of D and there exists an element nd of the carrier of D such that $D =$ group(the carrier of D, od, nd).

Let D be a group. The functor $+_D$ yielding a binary operation of the carrier of D, is defined as follows:

there exists an element nd of the carrier of D such that
$D =$ group(the carrier of $D, +_D, nd$) .

The following propositions are true:

(22) For every group D and for every binary operation od of the carrier of D holds $od = +_D$ if and only if there exists an element nd of the carrier of D such that $D =$ group(the carrier of D, od, nd).

(23) For every group D holds $+_D$ is a binary operation of the carrier of D and there exists an element nd of the carrier of D such that
$D =$ group(the carrier of $D, +_D, nd$) .

Let D be a group. The functor $\mathbf{0}_D$ yielding an element of the carrier of D, is defined by:

$D = \mathrm{group}(\text{the carrier of } D, +_D, \mathbf{0}_D)$.

Next we state a number of propositions:

(24) For every group D and for every element ng of the carrier of D holds $ng = \mathbf{0}_D$ if and only if $D = \mathrm{group}(\text{the carrier of } D, +_D, ng)$.

(25) For every group D holds $\mathbf{0}_D$ is an element of the carrier of D and $D = \mathrm{group}(\text{the carrier of } D, +_D, \mathbf{0}_D)$.

(26) For every group D holds $D = \mathrm{group}(\text{the carrier of } D, +_D, \mathbf{0}_D)$.

(27) For every group D and for every non-empty set A and for every binary operation og of A and for every element ng of A such that $D = \mathrm{group}(A, og, ng)$ holds the carrier of $D = A$ and $+_D = og$ and $\mathbf{0}_D = ng$.

(28) For every group D and for all elements a, b, c of the carrier of D holds $+_D(\langle +_D(\langle a, b\rangle), c\rangle) = +_D(\langle a, +_D(\langle b, c\rangle)\rangle)$.

(29) For every group D and for every element a of the carrier of D holds $+_D(\langle a, \mathbf{0}_D\rangle) = a$ and $+_D(\langle \mathbf{0}_D, a\rangle) = a$.

(30) For every group D and for every element a of the carrier of D there exists an element b of the carrier of D such that $+_D(\langle a, b\rangle) = \mathbf{0}_D$ and $+_D(\langle b, a\rangle) = \mathbf{0}_D$.

(31) For every group D and for all elements a, b of the carrier of D holds $+_D(\langle a, b\rangle) = +_D(\langle b, a\rangle)$.

(32) There exist arbitrary x, y such that $x \neq y$.

(33) There exists a non-empty set A such that for every element z of A holds $A \setminus \mathrm{single}(z)$ is a non-empty set.

A non-empty set is said to be an at least 2-elements set if:

for every element x of it holds it $\setminus \mathrm{single}(x)$ is a non-empty set.

We now state two propositions:

(34) For every non-empty set A holds A is an at least 2-elements set if and only if for every element x of A holds $A \setminus \mathrm{single}(x)$ is a non-empty set.

(35) For every non-empty set A such that for every element x of A holds $A \setminus \mathrm{single}(x)$ is a non-empty set holds A is an at least 2-elements set.

We consider field structures which are systems

$\langle$ a carrier, an addition, a multiplication, a zero, a unit $\rangle$

where the carrier is an at least 2-elements set, the addition is a binary operation of the carrier, the multiplication is a binary operation of the carrier, the zero is an element of the carrier, and the unit is an element of the carrier. Let A be an at least 2-elements set, and let od, om be binary operations of A, and let nd be an element of A, and let nm be an element of $A \setminus \mathrm{single}(nd)$. The functor $\mathrm{field}(A, od, om, nd, nm)$ yielding a field structure, is defined as follows:

$A =$ the carrier of $\mathrm{field}(A, od, om, nd, nm)$ and $od =$ the addition of $\mathrm{field}(A, od, om, nd, nm)$ and $om =$ the multiplication of $\mathrm{field}(A, od, om, nd, nm)$ and

nd = the zero of field(A, od, om, nd, nm) and nm = the unit of field(A, od, om, nd, nm).

We now state two propositions:

(36) Let A be an at least 2-elements set. Let od, om be binary operations of A. Then for every element nd of A and for every element nm of $A \setminus$ single(nd) and for every F being a field structure holds $F =$ field(A, od, om, nd, nm) if and only if $A =$ the carrier of F and $od =$ the addition of F and $om =$ the multiplication of F and $nd =$ the zero of F and $nm =$ the unit of F.

(37) Let A be an at least 2-elements set. Let od, om be binary operations of A. Let nd be an element of A. Let nm be an element of $A \setminus$ single(nd). Then

(i) field(A, od, om, nd, nm) is a field structure,
(ii) $A =$ the carrier of field(A, od, om, nd, nm),
(iii) $od =$ the addition of field(A, od, om, nd, nm),
(iv) $om =$ the multiplication of field(A, od, om, nd, nm),
(v) $nd =$ the zero of field(A, od, om, nd, nm),
(vi) $nm =$ the unit of field(A, od, om, nd, nm).

Let X be an at least 2-elements set, and let F be a binary operation of X, and let x be an element of X. We say that F is binary operation preserving x if and only if:

$X \setminus$ single(x) is a set closed w.r.t. F and $F \restriction ((X \setminus \text{single}(x)) \# (X \setminus \text{single}(x)))$ is a binary operation of $X \setminus$ single(x).

Next we state two propositions:

(38) For every at least 2-elements set X and for every binary operation F of X and for every element x of X holds F is binary operation preserving x if and only if $X \setminus$ single(x) is a set closed w.r.t. F and $F \restriction ((X \setminus \text{single}(x)) \# (X \setminus \text{single}(x)))$ is a binary operation of $X \setminus$ single(x).

(39) For every set X and for every subset A of X there exists a binary operation F of X such that for arbitrary x such that $x \in A \# A$ holds $F(x) \in A$.

Let X be a set, and let A be a subset of X. A binary operation of X is said to be a binary operation of X preserving A if:

for arbitrary x such that $x \in A \# A$ holds it$(x) \in A$.

One can prove the following two propositions:

(40) For every set X and for every subset A of X and for every binary operation F of X holds F is a binary operation of X preserving A if and only if for arbitrary x such that $x \in A \# A$ holds $F(x) \in A$.

(41) For every set X and for every subset A of X and for every binary operation F of X preserving A holds $F \restriction (A \# A)$ is a binary operation of A.

Let X be a set, and let A be a subset of X, and let F be a binary operation of X preserving A. The functor $F \restriction A$ yielding a binary operation of A, is defined as follows:

$F \restriction A = F \restriction (A\#A)$.

We now state two propositions:

(42) For every set X and for every subset A of X and for every binary operation F of X preserving A holds $F \restriction A = F \restriction (A\#A)$.

(43) For every at least 2-elements set A and for every element x of A there exists a binary operation F of A such that for arbitrary y such that $y \in (A \setminus \mathrm{single}(x))\#(A \setminus \mathrm{single}(x))$ holds $F(y) \in A \setminus \mathrm{single}(x)$.

Let A be an at least 2-elements set, and let x be an element of A. A binary operation of A is called a binary operation of A preserving $A \setminus \{x\}$ if:

for arbitrary y such that $y \in (A \setminus \mathrm{single}(x))\#(A \setminus \mathrm{single}(x))$ holds $\mathrm{it}(y) \in A \setminus \mathrm{single}(x)$.

One can prove the following two propositions:

(44) For every at least 2-elements set A and for every element x of A and for every binary operation F of A holds F is a binary operation of A preserving $A \setminus \{x\}$ if and only if for arbitrary y such that $y \in (A \setminus \mathrm{single}(x))\#(A \setminus \mathrm{single}(x))$ holds $F(y) \in A \setminus \mathrm{single}(x)$.

(45) For every at least 2-elements set A and for every element x of A and for every binary operation F of A preserving $A \setminus \{x\}$ holds $F \restriction ((A \setminus \mathrm{single}(x))\#(A \setminus \mathrm{single}(x)))$ is a binary operation of $A \setminus \mathrm{single}(x)$.

Let A be an at least 2-elements set, and let x be an element of A, and let F be a binary operation of A preserving $A \setminus \{x\}$. The functor $F \restriction_x A$ yields a binary operation of $A \setminus \mathrm{single}(x)$ and is defined as follows:

$F \restriction_x A = F \restriction ((A \setminus \mathrm{single}(x))\#(A \setminus \mathrm{single}(x)))$.

One can prove the following proposition

(46) For every at least 2-elements set A and for every element x of A and for every binary operation F of A preserving $A \setminus \{x\}$ holds $F \restriction_x A = F \restriction ((A \setminus \mathrm{single}(x))\#(A \setminus \mathrm{single}(x)))$.

A field structure is said to be a field if:

there exists an at least 2-elements set A and there exists a binary operation od of A and there exists an element nd of A and there exists a binary operation om of A preserving $A \setminus \{nd\}$ and there exists an element nm of $A \setminus \mathrm{single}(nd)$ such that $\mathrm{it} = \mathrm{field}(A, od, om, nd, nm)$ and $\mathrm{group}(A, od, nd)$ is a group and for every non-empty set B and for every binary operation P of B and for every element e of B such that $B = A \setminus \mathrm{single}(nd)$ and $e = nm$ and $P = om \restriction_{nd} A$ holds $\mathrm{group}(B, P, e)$ is a group and for all elements x, y, z of A holds $om(\langle x, od(\langle y, z\rangle)\rangle) = od(\langle om(\langle x, y\rangle), om(\langle x, z\rangle)\rangle)$.

We now state two propositions:

(47) Let F be a group structure. Then F is a group if and only if there exists a non-empty set A and there exists a binary operation og of A and there exists an element ng of A such that $F = \mathrm{group}(A, og, ng)$ and for all elements a, b, c of A holds $og(\langle og(\langle a, b\rangle), c\rangle) = og(\langle a, og(\langle b, c\rangle)\rangle)$ and for every element a of A holds $og(\langle a, ng\rangle) = a$ and $og(\langle ng, a\rangle) = a$

and for every element a of A there exists an element b of A such that $og(\langle a, b\rangle) = ng$ and $og(\langle b, a\rangle) = ng$ and for all elements a, b of A holds $og(\langle a, b\rangle) = og(\langle b, a\rangle)$.

(48) Let F be a field structure. Then F is a field if and only if there exists an at least 2-elements set A and there exists a binary operation od of A and there exists an element nd of A and there exists a binary operation om of A preserving $A \setminus \{nd\}$ and there exists an element nm of $A \setminus \mathrm{single}(nd)$ such that $F = \mathrm{field}(A, od, om, nd, nm)$ and $\mathrm{group}(A, od, nd)$ is a group and for every non-empty set B and for every binary operation P of B and for every element e of B such that $B = A \setminus \mathrm{single}(nd)$ and $e = nm$ and $P = om \restriction_{nd} A$ holds $\mathrm{group}(B, P, e)$ is a group and for all elements x, y, z of A holds $om(\langle x, od(\langle y, z\rangle)\rangle) = od(\langle om(\langle x, y\rangle), om(\langle x, z\rangle)\rangle)$.

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Received October 27, 1989

FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Equivalence Relations and Classes of Abstraction [1]

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Summary. In this article we deal with the notion of equivalence relation. The main properties of equivalence relations are proved. Then we define the classes of abstraction determined by an equivalence relation. Finally, the connections between a partition of a set and an equivalence relation are presented. We introduce the following notation of modes: *Equivalence Relation, a partition.*

MML Identifier: `EQREL_1`.

The notation and terminology used in this paper are introduced in the following articles: [6], [7], [9], [8], [5], [3], [2], [4], and [1]. For simplicity we adopt the following rules: x, y, z are arbitrary, i, j are natural numbers, X, Y are sets, A, B are subsets of X, R, R_1, R_2 are relations on X, and $SFXX$ is a family of subsets of $[\!: X, X :\!]$. The following two propositions are true:

(1) If $i < j$, then $j - i$ is a natural number.

(2) For every Y such that $Y \subseteq [\!: X, X :\!]$ holds Y is a relation on X.

Let us consider X. The functor ∇_X yielding a relation on X, is defined as follows:

$\nabla_X = [\!: X, X :\!]$.

We now state a proposition

(3) $\nabla_X = [\!: X, X :\!]$.

Let us consider X, R_1, R_2. Then $R_1 \cap R_2$ is a relation on X. Then $R_1 \cup R_2$ is a relation on X.

Next we state a proposition

(4) $\triangle_X$ is reflexive in X and $\triangle_X$ is symmetric in X and $\triangle_X$ is transitive in X.

[1] Supported by RPBP III.24 C8

ISSN 0777–4028

Let us consider X. A relation on X is called an equivalence relation of X if: it is reflexive in X and it is symmetric in X and it is transitive in X.

The following three propositions are true:

(5) R is an equivalence relation of X if and only if R is reflexive in X and R is symmetric in X and R is transitive in X.

(6) $\triangle_X$ is an equivalence relation of X.

(7) ∇_X is an equivalence relation of X.

Let us consider X. Then $\triangle_X$ is an equivalence relation of X. Then ∇_X is an equivalence relation of X.

In the sequel EqR, EqR_1, EqR_2 will be equivalence relations of X. We now state several propositions:

(8) EqR is reflexive in X.

(9) EqR is symmetric in X.

(10) EqR is transitive in X.

(11) If $x \in X$, then $\langle x, x \rangle \in EqR$.

(12) If $\langle x, y \rangle \in EqR$, then $\langle y, x \rangle \in EqR$.

(13) If $\langle x, y \rangle \in EqR$ and $\langle y, z \rangle \in EqR$, then $\langle x, z \rangle \in EqR$.

(14) If there exists x such that $x \in X$, then $EqR \neq \varnothing$.

(15) $\operatorname{field} EqR = X$.

(16) R is an equivalence relation of X if and only if R is pseudo reflexive and R is symmetric and R is transitive and $\operatorname{field} R = X$.

Let us consider X, EqR_1, EqR_2. Then $EqR_1 \cap EqR_2$ is an equivalence relation of X.

We now state four propositions:

(17) $\triangle_X \cap EqR = \triangle_X$.

(18) $(\nabla_X) \cap R = R$.

(19) For every $SFXX$ such that $SFXX \neq \emptyset$ and for every Y such that $Y \in SFXX$ holds Y is an equivalence relation of X holds $\bigcap SFXX$ is an equivalence relation of X.

(20) For every R there exists EqR such that $R \subseteq EqR$ and for every EqR_2 such that $R \subseteq EqR_2$ holds $EqR \subseteq EqR_2$.

Let us consider X, EqR_1, EqR_2. The functor $EqR_1 \sqcup EqR_2$ yielding an equivalence relation of X, is defined by:

$EqR_1 \cup EqR_2 \subseteq EqR_1 \sqcup EqR_2$ and for every EqR such that $EqR_1 \cup EqR_2 \subseteq EqR$ holds $EqR_1 \sqcup EqR_2 \subseteq EqR$.

Next we state several propositions:

(21) For every equivalence relation R of X holds $R = EqR_1 \sqcup EqR_2$ if and only if $EqR_1 \cup EqR_2 \subseteq R$ and for every EqR such that $EqR_1 \cup EqR_2 \subseteq EqR$ holds $R \subseteq EqR$.

(22) $EqR \sqcup EqR = EqR$.

(23) $EqR_1 \sqcup EqR_2 = EqR_2 \sqcup EqR_1$.

(24) $EqR_1 \cap (EqR_1 \sqcup EqR_2) = EqR_1$.

(25) $EqR_1 \sqcup (EqR_1 \cap EqR_2) = EqR_1$.

The scheme *Ex_Eq_Rel* concerns a set $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists an equivalence relation EqR of $\mathcal{A}$ such that for all x, y holds $\langle x, y\rangle \in EqR$ if and only if $x \in \mathcal{A}$ and $y \in \mathcal{A}$ and $\mathcal{P}[x, y]$

provided the parameters satisfy the following conditions:

- for every x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x, x]$,
- for all x, y such that $\mathcal{P}[x, y]$ holds $\mathcal{P}[y, x]$,
- for all x, y, z such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$.

Let us consider X, EqR, x. The functor $[x]_{EqR}$ yielding a subset of X, is defined by:

$[x]_{EqR} = EqR^{\circ} \{x\}$.

We now state a number of propositions:

(26) $[x]_{EqR} = EqR^{\circ} \{x\}$.

(27) $y \in [x]_{EqR}$ if and only if $\langle y, x\rangle \in EqR$.

(28) For every x such that $x \in X$ holds $x \in [x]_{EqR}$.

(29) For every x such that $x \in X$ there exists y such that $x \in [y]_{EqR}$.

(30) If $y \in [x]_{EqR}$ and $z \in [x]_{EqR}$, then $\langle y, z\rangle \in EqR$.

(31) For every x such that $x \in X$ holds $y \in [x]_{EqR}$ if and only if $[x]_{EqR} = [y]_{EqR}$.

(32) For all x, y such that $x \in X$ and $y \in X$ holds $[x]_{EqR} = [y]_{EqR}$ or $[x]_{EqR}$ misses $[y]_{EqR}$.

(33) For every x such that $x \in X$ holds $[x]_{\triangle_X} = \{x\}$.

(34) For every x such that $x \in X$ holds $[x]_{\nabla_X} = X$.

(35) If there exists x such that $[x]_{EqR} = X$, then $EqR = \nabla_X$.

(36) Suppose $x \in X$. Then $\langle x, y\rangle \in EqR_1 \sqcup EqR_2$ if and only if there exists a finite sequence f such that $1 \leq \operatorname{len} f$ and $x = f(1)$ and $y = f(\operatorname{len} f)$ and for every i such that $1 \leq i$ and $i < \operatorname{len} f$ holds $\langle f(i), f(i+1)\rangle \in EqR_1 \cup EqR_2$.

(37) For every equivalence relation E of X such that $E = EqR_1 \cup EqR_2$ for every x such that $x \in X$ holds $[x]_E = [x]_{EqR_1}$ or $[x]_E = [x]_{EqR_2}$.

(38) If $EqR_1 \cup EqR_2 = \nabla_X$, then $EqR_1 = \nabla_X$ or $EqR_2 = \nabla_X$.

Let us consider X, EqR. The functor Classes EqR yields a family of subsets of X and is defined as follows:

$A \in \text{Classes}\, EqR$ if and only if there exists x such that $x \in X$ and $A = [x]_{EqR}$.

The following two propositions are true:

(39) $A \in \text{Classes}\, EqR$ if and only if there exists x such that $x \in X$ and $A = [x]_{EqR}$.

(40) If $X = \emptyset$, then $\text{Classes}\, EqR = \emptyset$.

Let us consider X. A family of subsets of X is said to be a partition of X if:

$\bigcup$it $= X$ and for every A such that $A \in$ it holds $A \neq \emptyset$ and for every B such that $B \in$ it holds $A = B$ or A misses B if $X \neq \emptyset$, it $= \emptyset$, otherwise.

We now state several propositions:

(41) If $X \neq \emptyset$, then for every family F of subsets of X holds F is a partition of X if and only if $\bigcup F = X$ and for every A such that $A \in F$ holds $A \neq \emptyset$ and for every B such that $B \in F$ holds $A = B$ or A misses B.

(42) Classes EqR is a partition of X.

(43) For every partition P of X there exists EqR such that $P =$ Classes EqR.

(44) For every x such that $x \in X$ holds $\langle x, y \rangle \in EqR$ if and only if $[x]_{EqR} = [y]_{EqR}$.

(45) If $x \in$ Classes EqR, then there exists an element y of X such that $x = [y]_{EqR}$.

(46) For every x such that $x \in X$ holds $[x]_{EqR} \in$ Classes EqR.

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Received November 16, 1989

FORMALIZED MATHEMATICS
Vol.1, No.**3**, May–August 1990
Université Catholique de Louvain

Some Properties of Real Numbers [1]

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Summary. We define the following operations on real numbers: $max(x,y)$, $min(x,y)$, x^2, $\sqrt{x}$. We prove basic properties of introduced operations. A number of auxiliary theorems absent in [1] and [2] is proved.

MML Identifier: `SQUARE_1`.

The terminology and notation used here are introduced in the papers [1] and [2]. In the sequel a, b, x, y, z will be real numbers. Next we state a number of propositions:

(1) $1 < 2$.

(2) If $1 < x$, then $\frac{1}{x} < 1$.

(3) $\frac{1}{2} < 1$.

(4) $2^{-1} < 1$.

(5) $2 \cdot a = a + a$.

(6) $a = (a - x) + x$.

(7) $a = (a + x) - x$.

(8) If $x - y = 0$, then $x = y$.

(9) $x \leq y$ if and only if $z + x \leq z + y$.

(10) $a \leq a + 1$.

(11) If $x < y$, then $0 < y - x$.

(12) If $x \leq y$, then $0 \leq y - x$.

(13) $1^{-1} = 1$.

(14) $\frac{x}{1} = x$.

(15) $\frac{x+x}{2} = x$.

(16) If $x \neq 0$, then $\frac{1}{\frac{1}{x}} = x$.

[1] Supported by RPBP III.24 C1

ISSN 0777–4028

(17) If $y \neq 0$ and $z \neq 0$, then $\frac{x}{y \cdot z} = \frac{\frac{x}{y}}{z}$.

(18) If $z \neq 0$, then $x \cdot \frac{y}{z} = \frac{x \cdot y}{z}$.

(19) If $0 \leq x$ and $0 \leq y$, then $0 \leq x \cdot y$.

(20) If $x \leq 0$ and $y \leq 0$, then $0 \leq x \cdot y$.

(21) If $0 < x$ and $0 < y$, then $0 < x \cdot y$.

(22) If $x < 0$ and $y < 0$, then $0 < x \cdot y$.

(23) If $0 \leq x$ and $y \leq 0$, then $x \cdot y \leq 0$ and $y \cdot x \leq 0$.

(24) If $0 < x$ and $y < 0$, then $x \cdot y < 0$ and $y \cdot x < 0$.

(25) If $0 \leq x \cdot y$, then $0 \leq x$ and $0 \leq y$ or $x \leq 0$ and $y \leq 0$.

(26) If $0 < x \cdot y$, then $0 < x$ and $0 < y$ or $x < 0$ and $y < 0$.

(27) If $0 \leq a$ and $0 < b$, then $0 \leq \frac{a}{b}$.

(28) If $0 \leq x$, then $y - x \leq y$.

(29) If $0 < x$, then $y - x < y$.

(30) If $x \leq y$, then $z - y \leq z - x$.

The scheme *RealContinuity* deals with two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

there exists z such that for all x, y such that $\mathcal{P}[x]$ and $\mathcal{Q}[y]$ holds $x \leq z$ and $z \leq y$

provided the following requirements are met:

- there exists x such that $\mathcal{P}[x]$,
- there exists x such that $\mathcal{Q}[x]$,
- for all x, y such that $\mathcal{P}[x]$ and $\mathcal{Q}[y]$ holds $x \leq y$.

We now define two new functors. Let us consider x, y. The functor $\min(x, y)$ yields a real number and is defined by:

$\min(x, y) = x$ if $x \leq y$, $\min(x, y) = y$, otherwise.

The functor $\max(x, y)$ yielding a real number, is defined as follows:

$\max(x, y) = x$ if $y \leq x$, $\max(x, y) = y$, otherwise.

We now state a number of propositions:

(31) If $x \leq y$, then $z = x$ if and only if $z = \min(x, y)$ but $x \leq y$ or $z = y$ if and only if $z = \min(x, y)$.

(32) If $y \leq x$, then $\min(x, y) = y$.

(33) If $y \not\leq x$, then $\min(x, y) = x$.

(34) $\min(x, y) = \frac{(x+y) - |x-y|}{2}$.

(35) $\min(x, y) \leq x$ and $\min(y, x) \leq x$.

(36) $\min(x, x) = x$.

(37) $\min(x, y) = \min(y, x)$.

(38) $\min(x, y) = x$ or $\min(x, y) = y$.

(39) $x \leq y$ and $x \leq z$ if and only if $x \leq \min(y, z)$.

(40) $\min(x, \min(y, z)) = \min(\min(x, y), z)$.

(41) If $z < x$ and $z < y$, then $z < \min(x, y)$.

(42) If $y \leq x$, then $z = x$ if and only if $z = \max(x, y)$ but $y \leq x$ or $z = y$ if and only if $z = \max(x, y)$.

(43) If $x \leq y$, then $\max(x, y) = y$.

(44) If $x \not\leq y$, then $\max(x, y) = x$.

(45) $\max(x, y) = \frac{(x+y)+|x-y|}{2}$.

(46) $x \leq \max(x, y)$ and $x \leq \max(y, x)$.

(47) $\max(x, x) = x$.

(48) $\max(x, y) = \max(y, x)$.

(49) $\max(x, y) = x$ or $\max(x, y) = y$.

(50) $y \leq x$ and $z \leq x$ if and only if $\max(y, z) \leq x$.

(51) $\max(x, \max(y, z)) = \max(\max(x, y), z)$.

(52) If $0 < x$ and $0 < y$, then $0 < \max(x, y)$.

(53) $\min(x, y) + \max(x, y) = x + y$.

(54) $\max(x, \min(x, y)) = x$ and $\max(\min(x, y), x) = x$ and $\max(\min(y, x), x) = x$ and $\max(x, \min(y, x)) = x$.

(55) $\min(x, \max(x, y)) = x$ and $\min(\max(x, y), x) = x$ and $\min(\max(y, x), x) = x$ and $\min(x, \max(y, x)) = x$.

(56) $\min(x, \max(y, z)) = \max(\min(x, y), \min(x, z))$ and $\min(\max(y, z), x) = \max(\min(y, x), \min(z, x))$.

(57) $\max(x, \min(y, z)) = \min(\max(x, y), \max(x, z))$ and $\max(\min(y, z), x) = \min(\max(y, x), \max(z, x))$.

Let us consider x. The functor $x^{\mathbf{2}}$ yields an element of $\mathbb{R}$ and is defined by: $x^{\mathbf{2}} = x \cdot x$.

The following proposition is true

(58) $x^{\mathbf{2}} = x \cdot x$.

Let us consider a. Then $a^{\mathbf{2}}$ is a real number.

The following propositions are true:

(59) $1^{\mathbf{2}} = 1$.

(60) $0^{\mathbf{2}} = 0$.

(61) $a^{\mathbf{2}} = (-a)^{\mathbf{2}}$.

(62) $|a|^{\mathbf{2}} = a^{\mathbf{2}}$.

(63) $(a + b)^{\mathbf{2}} = (a^{\mathbf{2}} + (2 \cdot a) \cdot b) + b^{\mathbf{2}}$.

(64) $(a - b)^{\mathbf{2}} = (a^{\mathbf{2}} - (2 \cdot a) \cdot b) + b^{\mathbf{2}}$.

(65) $(a + 1)^{\mathbf{2}} = (a^{\mathbf{2}} + 2 \cdot a) + 1$.

(66) $(a - 1)^{\mathbf{2}} = (a^{\mathbf{2}} - 2 \cdot a) + 1$.

(67) $(a - b) \cdot (a + b) = a^{\mathbf{2}} - b^{\mathbf{2}}$ and $(a + b) \cdot (a - b) = a^{\mathbf{2}} - b^{\mathbf{2}}$.

(68) $(a \cdot b)^{\mathbf{2}} = a^{\mathbf{2}} \cdot b^{\mathbf{2}}$.

(69) If $0 \neq b$, then $\frac{a}{b}^{\mathbf{2}} = \frac{a^{\mathbf{2}}}{b^{\mathbf{2}}}$.

(70) If $a^{\mathbf{2}} - b^{\mathbf{2}} \neq 0$, then $\frac{1}{a+b} = \frac{a-b}{a^{\mathbf{2}}-b^{\mathbf{2}}}$.

(71) If $a^{\mathbf{2}} - b^{\mathbf{2}} \neq 0$, then $\frac{1}{a-b} = \frac{a+b}{a^{\mathbf{2}}-b^{\mathbf{2}}}$.

(72) $0 \leq a^{\mathbf{2}}$.

(73) If $a^{\mathbf{2}} = 0$, then $a = 0$.

(74) If $0 \neq a$, then $0 < a^{\mathbf{2}}$.

(75) If $0 < a$ and $a < 1$, then $a^{\mathbf{2}} < a$.

(76) If $1 < a$, then $a < a^{\mathbf{2}}$.

(77) If $0 \leq x$ and $x \leq y$, then $x^{\mathbf{2}} \leq y^{\mathbf{2}}$.

(78) If $0 \leq x$ and $x < y$, then $x^{\mathbf{2}} < y^{\mathbf{2}}$.

(79) If $0 \leq x$ and $0 \leq y$ and $x^{\mathbf{2}} \leq y^{\mathbf{2}}$, then $x \leq y$.

(80) If $0 \leq x$ and $0 \leq y$ and $x^{\mathbf{2}} < y^{\mathbf{2}}$, then $x < y$.

Let us consider a. Let us assume that $0 \leq a$. The functor $\sqrt{a}$ yielding a real number, is defined by:

$0 \leq \sqrt{a}$ and $\sqrt{a}^{\mathbf{2}} = a$.

We now state a number of propositions:

(81) If $0 \leq a$, then for every b holds $b = \sqrt{a}$ if and only if $0 \leq b$ and $b^{\mathbf{2}} = a$.

(82) $\sqrt{0} = 0$.

(83) $\sqrt{1} = 1$.

(84) $1 < \sqrt{2}$.

(85) $\sqrt{4} = 2$.

(86) $\sqrt{2} < 2$.

(87) If $0 \leq a$, then $0 \leq \sqrt{a}$.

(88) If $0 \leq a$, then $\sqrt{a}^{\mathbf{2}} = a$.

(89) If $0 \leq a$, then $\sqrt{a^{\mathbf{2}}} = a$.

(90) If $a \leq 0$, then $\sqrt{a^{\mathbf{2}}} = -a$.

(91) $\sqrt{a^{\mathbf{2}}} = |a|$.

(92) If $0 \leq a$ and $\sqrt{a} = 0$, then $a = 0$.

(93) If $0 < a$, then $0 < \sqrt{a}$.

(94) If $0 \leq x$ and $x \leq y$, then $\sqrt{x} \leq \sqrt{y}$.

(95) If $0 \leq x$ and $x < y$, then $\sqrt{x} < \sqrt{y}$.

(96) If $0 \leq x$ and $0 \leq y$ and $\sqrt{x} = \sqrt{y}$, then $x = y$.

(97) If $0 \leq a$ and $0 \leq b$, then $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$.

(98) If $0 \leq a \cdot b$, then $\sqrt{a \cdot b} = \sqrt{|a|} \cdot \sqrt{|b|}$.

(99) If $0 \leq a$ and $0 < b$, then $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

(100) If $0 < \frac{a}{b}$ and $b \neq 0$, then $\sqrt{\frac{a}{b}} = \frac{\sqrt{|a|}}{\sqrt{|b|}}$.

(101) If $0 < a$, then $\sqrt{\frac{1}{a}} = \frac{1}{\sqrt{a}}$.

(102) If $0 < a$, then $\frac{\sqrt{a}}{a} = \frac{1}{\sqrt{a}}$.

(103) If $0 < a$, then $\frac{a}{\sqrt{a}} = \sqrt{a}$.

(104) If $0 \leq a$ and $0 \leq b$, then $(\sqrt{a} - \sqrt{b}) \cdot (\sqrt{a} + \sqrt{b}) = a - b$.

(105) If $0 \leq a$ and $0 \leq b$ and $a \neq b$, then $\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$.

(106) If $0 \leq b$ and $b < a$, then $\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$.

(107) If $0 \leq a$ and $0 \leq b$ and $a \neq b$, then $\frac{1}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}+\sqrt{b}}{a-b}$.

(108) If $0 \leq b$ and $b < a$, then $\frac{1}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}+\sqrt{b}}{a-b}$.

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Received November 16, 1989

FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Connectives and Subformulae of the First Order Language

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Summary. In the article the development of the first order language defined in [5] is continued. The following connectives are introduced: implication ($\Rightarrow$), disjunction ($\lor$), and equivalence ($\Leftrightarrow$). We introduce also the existential quantifier ($\exists$) and FALSUM. Some theorems on disjunctive, conditional, biconditional and existential formulae are proved and their selector functors are introduced. The second part of the article deals with notions of subformula, proper subformula and immediate constituent of a QC-formula.

MML Identifier: `QC_LANG2`.

The papers [7], [6], [3], [4], [1], [2], and [5] provide the terminology and notation for this paper. We adopt the following convention: x, y, z will be bound variables and p, q, p_1, p_2, q_1 will be elements of WFF. One can prove the following propositions:

(1) If $\neg p = \neg q$, then $p = q$.

(2) $\mathrm{Arg}(\neg p) = p$.

(3) If $p \land q = p_1 \land q_1$, then $p = p_1$ and $q = q_1$.

(4) If p is conjunctive, then $p = \mathrm{LeftArg}(p) \land \mathrm{RightArg}(p)$.

(5) $\mathrm{LeftArg}(p \land q) = p$ and $\mathrm{RightArg}(p \land q) = q$.

(6) If $\forall_x p = \forall_y q$, then $x = y$ and $p = q$.

(7) If p is universal, then $p = \forall_{\mathrm{Bound}(p)} \mathrm{Scope}(p)$.

(8) $\mathrm{Bound}(\forall_x p) = x$ and $\mathrm{Scope}(\forall_x p) = p$.

We now define three new functors. The formula FALSUM is defined as follows:

FALSUM $= \neg$ VERUM.

[1]Partially supported by RPBP III.24.C1

ISSN 0777–4028

Let p, q be elements of WFF. The functor $p \Rightarrow q$ yields a formula and is defined by:

$p \Rightarrow q = \neg(p \wedge \neg q)$.

The functor $p \vee q$ yields a formula and is defined as follows:

$p \vee q = \neg(\neg p \wedge \neg q)$.

Let p, q be elements of WFF. The functor $p \Leftrightarrow q$ yielding a formula, is defined as follows:

$p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$.

Let x be a bound variable, and let p be an element of WFF. The functor $\exists_x p$ yielding a formula, is defined as follows:

$\exists_x p = \neg(\forall_x \neg p)$.

The following propositions are true:

(9) FALSUM $= \neg$ VERUM.

(10) $p \Rightarrow q = \neg(p \wedge \neg q)$.

(11) $p \vee q = \neg(\neg p \wedge \neg q)$.

(12) $p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$.

(13) FALSUM is negative and Arg(FALSUM) = VERUM.

(14) $p \vee q = \neg p \Rightarrow q$.

(15) $\exists_x p = \neg(\forall_x \neg p)$.

(16) If $p \vee q = p_1 \vee q_1$, then $p = p_1$ and $q = q_1$.

(17) If $p \Rightarrow q = p_1 \Rightarrow q_1$, then $p = p_1$ and $q = q_1$.

(18) If $p \Leftrightarrow q = p_1 \Leftrightarrow q_1$, then $p = p_1$ and $q = q_1$.

(19) If $\exists_x p = \exists_y q$, then $x = y$ and $p = q$.

We now define two new functors. Let x, y be bound variables, and let p be an element of WFF. The functor $\forall_{x,y} p$ yielding a formula, is defined by:

$\forall_{x,y} p = \forall_x (\forall_y p)$.

The functor $\exists_{x,y} p$ yields a formula and is defined by:

$\exists_{x,y} p = \exists_x (\exists_y p)$.

Next we state several propositions:

(20) $\forall_{x,y} p = \forall_x (\forall_y p)$ and $\exists_{x,y} p = \exists_x (\exists_y p)$.

(21) For all bound variables x_1, x_2, y_1, y_2 such that $\forall_{x_1,y_1} p_1 = \forall_{x_2,y_2} p_2$ holds $x_1 = x_2$ and $y_1 = y_2$ and $p_1 = p_2$.

(22) If $\forall_{x,y} p = \forall_z q$, then $x = z$ and $\forall_y p = q$.

(23) For all bound variables x_1, x_2, y_1, y_2 such that $\exists_{x_1,y_1} p_1 = \exists_{x_2,y_2} p_2$ holds $x_1 = x_2$ and $y_1 = y_2$ and $p_1 = p_2$.

(24) If $\exists_{x,y} p = \exists_z q$, then $x = z$ and $\exists_y p = q$.

(25) $\forall_{x,y} p$ is universal and Bound($\forall_{x,y} p$) $= x$ and Scope($\forall_{x,y} p$) $= \forall_y p$.

We now define two new functors. Let x, y, z be bound variables, and let p be an element of WFF. The functor $\forall_{x,y,z} p$ yields a formula and is defined by:

$\forall_{x,y,z} p = \forall_x (\forall_{y,z} p)$.

The functor $\exists_{x,y,z} p$ yields a formula and is defined by:

$\exists_{x,y,z}p = \exists_x(\exists_{y,z}p)$.

The following propositions are true:

(26) $\forall_{x,y,z}p = \forall_x(\forall_{y,z}p)$ and $\exists_{x,y,z}p = \exists_x(\exists_{y,z}p)$.

(27) For all bound variables x_1, x_2, y_1, y_2, z_1, z_2 such that $\forall_{x_1,y_1,z_1}p_1 = \forall_{x_2,y_2,z_2}p_2$ holds $x_1 = x_2$ and $y_1 = y_2$ and $z_1 = z_2$ and $p_1 = p_2$.

In the sequel s, t will be bound variables. We now state several propositions:

(28) If $\forall_{x,y,z}p = \forall_t q$, then $x = t$ and $\forall_{y,z}p = q$.

(29) If $\forall_{x,y,z}p = \forall_{t,s}q$, then $x = t$ and $y = s$ and $\forall_z p = q$.

(30) For all bound variables x_1, x_2, y_1, y_2, z_1, z_2 such that $\exists_{x_1,y_1,z_1}p_1 = \exists_{x_2,y_2,z_2}p_2$ holds $x_1 = x_2$ and $y_1 = y_2$ and $z_1 = z_2$ and $p_1 = p_2$.

(31) If $\exists_{x,y,z}p = \exists_t q$, then $x = t$ and $\exists_{y,z}p = q$.

(32) If $\exists_{x,y,z}p = \exists_{t,s}q$, then $x = t$ and $y = s$ and $\exists_z p = q$.

(33) $\forall_{x,y,z}p$ is universal and $\mathrm{Bound}(\forall_{x,y,z}p) = x$ and $\mathrm{Scope}(\forall_{x,y,z}p) = \forall_{y,z}p$.

We now define four new predicates. Let H be an element of WFF. We say that H is disjunctive if and only if:

there exist elements p, q of WFF such that $H = p \vee q$.

We say that H is conditional if and only if:

there exist elements p, q of WFF such that $H = p \Rightarrow q$.

We say that H is biconditional if and only if:

there exist elements p, q of WFF such that $H = p \Leftrightarrow q$.

We say that H is existential if and only if:

there exists a bound variable x and there exists an element p of WFF such that $H = \exists_x p$.

We now state several propositions:

(34) For every element H of WFF holds H is disjunctive if and only if there exist elements p, q of WFF such that $H = p \vee q$.

(35) For every element H of WFF holds H is conditional if and only if there exist elements p, q of WFF such that $H = p \Rightarrow q$.

(36) For every element H of WFF holds H is biconditional if and only if there exist elements p, q of WFF such that $H = p \Leftrightarrow q$.

(37) For every element H of WFF holds H is existential if and only if there exists a bound variable x and there exists an element p of WFF such that $H = \exists_x p$.

(38) $\exists_{x,y}p$ is existential and $\exists_{x,y,z}p$ is existential.

We now define four new functors. Let H be an element of WFF. The functor $\mathrm{LeftDisj}(H)$ yields a formula and is defined by:

$\mathrm{LeftDisj}(H) = \mathrm{Arg}(\mathrm{LeftArg}(\mathrm{Arg}(H)))$.

The functor $\mathrm{RightDisj}(H)$ yielding a formula, is defined as follows:

$\mathrm{RightDisj}(H) = \mathrm{Arg}(\mathrm{RightArg}(\mathrm{Arg}(H)))$.

The functor $\mathrm{Antecedent}(H)$ yields a formula and is defined by:

$\mathrm{Antecedent}(H) = \mathrm{LeftArg}(\mathrm{Arg}(H))$.

The functor $\mathrm{Consequent}(H)$ yields a formula and is defined by:

$\mathrm{Consequent}(H) = \mathrm{Arg}(\mathrm{RightArg}(\mathrm{Arg}(H)))$.

We now define two new functors. Let H be an element of WFF. The functor $\mathrm{LeftSide}(H)$ yields a formula and is defined by:

$\mathrm{LeftSide}(H) = \mathrm{Antecedent}(\mathrm{LeftArg}(H))$.

The functor $\mathrm{RightSide}(H)$ yielding a formula, is defined as follows:

$\mathrm{RightSide}(H) = \mathrm{Consequent}(\mathrm{LeftArg}(H))$.

The following propositions are true:

(39) For every element H of WFF holds $\mathrm{LeftDisj}(H) = \mathrm{Arg}(\mathrm{LeftArg}(\mathrm{Arg}(H)))$.

(40) For every element H of WFF holds $\mathrm{RightDisj}(H) = \mathrm{Arg}(\mathrm{RightArg}(\mathrm{Arg}(H)))$.

(41) For every element H of WFF holds $\mathrm{Antecedent}(H) = \mathrm{LeftArg}(\mathrm{Arg}(H))$.

(42) For every element H of WFF holds $\mathrm{Consequent}(H) = \mathrm{Arg}(\mathrm{RightArg}(\mathrm{Arg}(H)))$.

(43) For every element H of WFF holds $\mathrm{LeftSide}(H) = \mathrm{Antecedent}(\mathrm{LeftArg}(H))$.

(44) For every element H of WFF holds $\mathrm{RightSide}(H) = \mathrm{Consequent}(\mathrm{LeftArg}(H))$.

In the sequel F, G, H will be elements of WFF. We now state a number of propositions:

(45) $\mathrm{LeftDisj}(F \vee G) = F$ and $\mathrm{RightDisj}(F \vee G) = G$ and $\mathrm{Arg}(F \vee G) = \neg F \wedge \neg G$.

(46) $\mathrm{Antecedent}(F \Rightarrow G) = F$ and $\mathrm{Consequent}(F \Rightarrow G) = G$ and $\mathrm{Arg}(F \Rightarrow G) = F \wedge \neg G$.

(47) $\mathrm{LeftSide}(F \Leftrightarrow G) = F$ and $\mathrm{RightSide}(F \Leftrightarrow G) = G$ and $\mathrm{LeftArg}(F \Leftrightarrow G) = F \Rightarrow G$ and $\mathrm{RightArg}(F \Leftrightarrow G) = G \Rightarrow F$.

(48) $\mathrm{Arg}(\exists_x H) = \forall_x \neg H$.

(49) If H is disjunctive, then H is conditional and H is negative and $\mathrm{Arg}(H)$ is conjunctive and $\mathrm{LeftArg}(\mathrm{Arg}(H))$ is negative and $\mathrm{RightArg}(\mathrm{Arg}(H))$ is negative.

(50) If H is conditional, then H is negative and $\mathrm{Arg}(H)$ is conjunctive and $\mathrm{RightArg}(\mathrm{Arg}(H))$ is negative.

(51) If H is biconditional, then H is conjunctive and $\mathrm{LeftArg}(H)$ is conditional and $\mathrm{RightArg}(H)$ is conditional.

(52) If H is existential, then H is negative and $\mathrm{Arg}(H)$ is universal and $\mathrm{Scope}(\mathrm{Arg}(H))$ is negative.

(53) If H is disjunctive, then $H = \mathrm{LeftDisj}(H) \vee \mathrm{RightDisj}(H)$.

(54) If H is conditional, then $H = \mathrm{Antecedent}(H) \Rightarrow \mathrm{Consequent}(H)$.

(55) If H is biconditional, then $H = \mathrm{LeftSide}(H) \Leftrightarrow \mathrm{RightSide}(H)$.

(56) If H is existential, then $H = \exists_{\mathrm{Bound}(\mathrm{Arg}(H))} \mathrm{Arg}(\mathrm{Scope}(\mathrm{Arg}(H)))$.

Let G, H be elements of WFF. We say that G is an immediate constituent of H if and only if:

$H = \neg G$ or there exists an element F of WFF such that $H = G \wedge F$ or $H = F \wedge G$ or there exists a bound variable x such that $H = \forall_x G$.

For simplicity we adopt the following convention: x is a bound variable, k, n are natural numbers, P is a k-ary predicate symbol, and V is a list of variables of the length k. One can prove the following propositions:

(57) G is an immediate constituent of H if and only if $H = \neg G$ or there exists F such that $H = G \wedge F$ or $H = F \wedge G$ or there exists x such that $H = \forall_x G$.

(58) H is not an immediate constituent of VERUM.

(59) H is not an immediate constituent of $P[V]$.

(60) F is an immediate constituent of $\neg H$ if and only if $F = H$.

(61) H is an immediate constituent of FALSUM if and only if $H =$ VERUM.

(62) F is an immediate constituent of $G \wedge H$ if and only if $F = G$ or $F = H$.

(63) F is an immediate constituent of $\forall_x H$ if and only if $F = H$.

(64) If H is atomic, then F is not an immediate constituent of H.

(65) If H is negative, then F is an immediate constituent of H if and only if $F = \mathrm{Arg}(H)$.

(66) If H is conjunctive, then F is an immediate constituent of H if and only if $F = \mathrm{LeftArg}(H)$ or $F = \mathrm{RightArg}(H)$.

(67) If H is universal, then F is an immediate constituent of H if and only if $F = \mathrm{Scope}(H)$.

In the sequel L denotes a finite sequence. Let us consider G, H. We say that G is a subformula of H if and only if:

there exist n, L such that $1 \leq n$ and $\mathrm{len}\, L = n$ and $L(1) = G$ and $L(n) = H$ and for every k such that $1 \leq k$ and $k < n$ there exist elements G_1, H_1 of WFF such that $L(k) = G_1$ and $L(k+1) = H_1$ and G_1 is an immediate constituent of H_1.

We now state two propositions:

(68) G is a subformula of H if and only if there exist n, L such that $1 \leq n$ and $\mathrm{len}\, L = n$ and $L(1) = G$ and $L(n) = H$ and for every k such that $1 \leq k$ and $k < n$ there exist elements G_1, H_1 of WFF such that $L(k) = G_1$ and $L(k+1) = H_1$ and G_1 is an immediate constituent of H_1.

(69) H is a subformula of H.

Let us consider H, F. We say that H is a proper subformula of F if and only if:

H is a subformula of F and $H \neq F$.

One can prove the following propositions:

(70) H is a proper subformula of F if and only if H is a subformula of F and $H \neq F$.

(71) If H is an immediate constituent of F, then $\mathrm{len}(@H) < \mathrm{len}(@F)$.

(72) If H is an immediate constituent of F, then H is a subformula of F.

(73) If H is an immediate constituent of F, then H is a proper subformula of F.

(74) If H is a proper subformula of F, then $\mathrm{len}(@H) < \mathrm{len}(@F)$.

(75) If H is a proper subformula of F, then there exists G such that G is an immediate constituent of F.

(76) If F is a proper subformula of G and G is a proper subformula of H, then F is a proper subformula of H.

(77) If F is a subformula of G and G is a subformula of H, then F is a subformula of H.

(78) If G is a subformula of H and H is a subformula of G, then $G = H$.

(79) It is not true that: G is a proper subformula of H and H is a subformula of G.

(80) It is not true that: G is a proper subformula of H and H is a proper subformula of G.

(81) It is not true that: G is a subformula of H and H is an immediate constituent of G.

(82) It is not true that: G is a proper subformula of H and H is an immediate constituent of G.

(83) Suppose F is a proper subformula of G and G is a subformula of H or F is a subformula of G and G is a proper subformula of H or F is a subformula of G and G is an immediate constituent of H or F is an immediate constituent of G and G is a subformula of H or F is a proper subformula of G and G is an immediate constituent of H or F is an immediate constituent of G and G is a proper subformula of H. Then F is a proper subformula of H.

(84) F is not a proper subformula of VERUM.

(85) F is not a proper subformula of $P[V]$.

(86) F is a subformula of H if and only if F is a proper subformula of $\neg H$.

(87) If $\neg F$ is a subformula of H, then F is a proper subformula of H.

(88) F is a proper subformula of FALSUM if and only if F is a subformula of VERUM.

(89) F is a subformula of G or F is a subformula of H if and only if F is a proper subformula of $G \wedge H$.

(90) If $F \wedge G$ is a subformula of H, then F is a proper subformula of H and G is a proper subformula of H.

(91) F is a subformula of H if and only if F is a proper subformula of $\forall_x H$.

(92) If $\forall_x H$ is a subformula of F, then H is a proper subformula of F.

(93) $F \wedge \neg G$ is a proper subformula of $F \Rightarrow G$ and F is a proper subformula of $F \Rightarrow G$ and $\neg G$ is a proper subformula of $F \Rightarrow G$ and G is a proper subformula of $F \Rightarrow G$.

(94) $\neg F \wedge \neg G$ is a proper subformula of $F \vee G$ and $\neg F$ is a proper subformula of $F \vee G$ and $\neg G$ is a proper subformula of $F \vee G$ and F is a proper subformula of $F \vee G$ and G is a proper subformula of $F \vee G$.

(95) If H is atomic, then F is not a proper subformula of H.

(96) If H is negative, then $\mathrm{Arg}(H)$ is a proper subformula of H.

(97) If H is conjunctive, then $\mathrm{LeftArg}(H)$ is a proper subformula of H and $\mathrm{RightArg}(H)$ is a proper subformula of H.

(98) If H is universal, then $\mathrm{Scope}(H)$ is a proper subformula of H.

(99) H is a subformula of VERUM if and only if $H = \mathrm{VERUM}$.

(100) H is a subformula of $P[V]$ if and only if $H = P[V]$.

(101) H is a subformula of FALSUM if and only if $H = \mathrm{FALSUM}$ or $H = \mathrm{VERUM}$.

Let us consider H. The functor $\mathrm{Subformulae}\, H$ yields a set and is defined by:

for arbitrary a holds $a \in \mathrm{Subformulae}\, H$ if and only if there exists F such that $F = a$ and F is a subformula of H.

Next we state a number of propositions:

(102) For arbitrary a holds $a \in \mathrm{Subformulae}\, H$ if and only if there exists F such that $F = a$ and F is a subformula of H.

(103) If $G \in \mathrm{Subformulae}\, H$, then G is a subformula of H.

(104) If F is a subformula of H, then $\mathrm{Subformulae}\, F \subseteq \mathrm{Subformulae}\, H$.

(105) If $G \in \mathrm{Subformulae}\, H$, then $\mathrm{Subformulae}\, G \subseteq \mathrm{Subformulae}\, H$.

(106) $H \in \mathrm{Subformulae}\, H$.

(107) $\mathrm{Subformulae}\, \mathrm{VERUM} = \{\mathrm{VERUM}\}$.

(108) $\mathrm{Subformulae}(P[V]) = \{P[V]\}$.

(109) $\mathrm{Subformulae}\, \mathrm{FALSUM} = \{\mathrm{VERUM}, \mathrm{FALSUM}\}$.

(110) $\mathrm{Subformulae}\, \neg H = \mathrm{Subformulae}\, H \cup \{\neg H\}$.

(111) $\mathrm{Subformulae}\, H \wedge F = (\mathrm{Subformulae}\, H \cup \mathrm{Subformulae}\, F) \cup \{H \wedge F\}$.

(112) $\mathrm{Subformulae}\, \forall_x H = \mathrm{Subformulae}\, H \cup \{\forall_x H\}$.

(113) $\mathrm{Subformulae}\, F \Rightarrow G = (\mathrm{Subformulae}\, F \cup \mathrm{Subformulae}\, G) \cup \{\neg G, F \wedge \neg G, F \Rightarrow G\}$.

(114) $\mathrm{Subformulae}\, F \vee G = (\mathrm{Subformulae}\, F \cup \mathrm{Subformulae}\, G) \cup \{\neg G, \neg F, \neg F \wedge \neg G, F \vee G\}$.

(115) $\mathrm{Subformulae}\, F \Leftrightarrow G = (\mathrm{Subformulae}\, F \cup \mathrm{Subformulae}\, G) \cup \{\neg G, F \wedge \neg G, F \Rightarrow G, \neg F, G \wedge \neg F, G \Rightarrow F, F \Leftrightarrow G\}$.

(116) $H = \mathrm{VERUM}$ or H is atomic if and only if $\mathrm{Subformulae}\, H = \{H\}$.

(117) If H is negative, then $\mathrm{Subformulae}\, H = \mathrm{Subformulae}\, \mathrm{Arg}(H) \cup \{H\}$.

(118) If H is conjunctive, then $\mathrm{Subformulae}\, H = (\mathrm{Subformulae}\, \mathrm{LeftArg}(H) \cup \mathrm{Subformulae}\, \mathrm{RightArg}(H)) \cup \{H\}$.

(119) If H is universal, then $\mathrm{Subformulae}\, H = \mathrm{Subformulae}\, \mathrm{Scope}(H) \cup \{H\}$.

(120) If H is an immediate constituent of G or H is a proper subformula of G or H is a subformula of G but $G \in \mathrm{Subformulae}\, F$, then $H \in \mathrm{Subformulae}\, F$.

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Received November 23, 1989

FORMALIZED MATHEMATICS
Vol.1, No.**3**, May–August 1990
Université Catholique de Louvain

Variables in Formulae of the First Order Language [1]

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Summary. We develop the first order language defined in [5]. We continue the work done in the article [1]. We prove some schemes of defining by structural induction. We deal with notions of closed subformulae and of still not bound variables in a formula. We introduce the concept of the set of all free variables and the set of all fixed variables occurring in a formula.

MML Identifier: `QC_LANG3`.

The notation and terminology used in this paper have been introduced in the following articles: [6], [3], [4], [2], [5], and [1]. For simplicity we follow the rules: i, j, k are natural numbers, x is a bound variable, a is a free variable, p, q are elements of WFF, l is a finite sequence of elements of Var, P is a predicate symbol, and V is a non-empty subset of Var. Let F be a function from WFF into WFF, and let us consider p. Then $F(p)$ is an element of WFF.

In the article we present several logical schemes. The scheme *QC_Func_Uniq* deals with a non-empty set $\mathcal{A}$, a function $\mathcal{B}$ from WFF into $\mathcal{A}$, a function $\mathcal{C}$ from WFF into $\mathcal{A}$, an element $\mathcal{D}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{B} = \mathcal{C}$

provided the following conditions are satisfied:

- Given p. Let d_1, d_2 be elements of $\mathcal{A}$. Then
  - (i) if $p =$ VERUM, then $\mathcal{B}(p) = \mathcal{D}$,
  - (ii) if p is atomic, then $\mathcal{B}(p) = \mathcal{F}(p)$,
  - (iii) if p is negative and $d_1 = \mathcal{B}(\mathrm{Arg}(p))$, then $\mathcal{B}(p) = \mathcal{G}(d_1)$,
  - (iv) if p is conjunctive and $d_1 = \mathcal{B}(\mathrm{LeftArg}(p))$ and

[1]Partially supported by RPBP III.24.C1

ISSN 0777–4028

$d_2 = \mathcal{B}(\mathrm{RightArg}(p))$,
then $\mathcal{B}(p) = \mathcal{H}(d_1, d_2)$,

(v) if p is universal and $d_1 = \mathcal{B}(\mathrm{Scope}(p))$, then $\mathcal{B}(p) = \mathcal{I}(p, d_1)$,

- Given p. Let d_1, d_2 be elements of $\mathcal{A}$. Then
  (i) if $p = \mathrm{VERUM}$, then $\mathcal{C}(p) = \mathcal{D}$,
  (ii) if p is atomic, then $\mathcal{C}(p) = \mathcal{F}(p)$,
  (iii) if p is negative and $d_1 = \mathcal{C}(\mathrm{Arg}(p))$, then $\mathcal{C}(p) = \mathcal{G}(d_1)$,
  (iv) if p is conjunctive and $d_1 = \mathcal{C}(\mathrm{LeftArg}(p))$ and
  $d_2 = \mathcal{C}(\mathrm{RightArg}(p))$,
  then $\mathcal{C}(p) = \mathcal{H}(d_1, d_2)$,
  (v) if p is universal and $d_1 = \mathcal{C}(\mathrm{Scope}(p))$, then $\mathcal{C}(p) = \mathcal{I}(p, d_1)$.

The scheme *QC_Def_D* deals with a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, an element $\mathcal{C}$ of WFF, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$ and states that:

(i) there exists an element d of $\mathcal{A}$ and there exists a function F from WFF into $\mathcal{A}$ such that $d = F(\mathcal{C})$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$,

(ii) for all elements x_1, x_2 of $\mathcal{A}$ such that there exists a function F from WFF into $\mathcal{A}$ such that $x_1 = F(\mathcal{C})$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$ and there exists a function F from WFF into $\mathcal{A}$ such that $x_2 = F(\mathcal{C})$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$ holds $x_1 = x_2$

for all values of the parameters.

The scheme *QC_D_Result'VERU* deals with a non-empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{F}(\mathrm{VERUM}) = \mathcal{B}$

provided the parameters fulfill the following condition:

- Let p be a formula. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{F}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{G}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{H}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{I}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{J}(p, d_1)$.

The scheme *QC_D_Result'atom* concerns a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a formula $\mathcal{C}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{F}(\mathcal{C}) = \mathcal{G}(\mathcal{C})$

provided the following conditions are fulfilled:

- Let p be a formula. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{F}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{G}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{H}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{I}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{J}(p, d_1)$,
- $\mathcal{C}$ is atomic.

The scheme *QC_D_Result'nega* deals with a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a formula $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and a unary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{J}(\mathcal{C}) = \mathcal{G}(\mathcal{J}(\mathrm{Arg}(\mathcal{C})))$

provided the following requirements are met:

- Let p be a formula. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{J}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$,
- $\mathcal{C}$ is negative.

The scheme *QC_D_Result'conj* concerns a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$, and a formula $\mathcal{C}$ and states that:

for all elements d_1, d_2 of $\mathcal{A}$ such that $d_1 = \mathcal{J}(\mathrm{LeftArg}(\mathcal{C}))$ and $d_2 = \mathcal{J}(\mathrm{RightArg}(\mathcal{C}))$
holds $\mathcal{J}(\mathcal{C}) = \mathcal{H}(d_1, d_2)$

provided the parameters satisfy the following conditions:

- Let p be a formula. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{J}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$,
- $\mathcal{C}$ is conjunctive.

The scheme *QC_D_Result'univ* deals with a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a formula $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and a unary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{J}(\mathcal{C}) = \mathcal{I}(\mathcal{C}, \mathcal{J}(\mathrm{Scope}(\mathcal{C})))$

provided the following requirements are fulfilled:

- Let p be a formula. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{J}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$,
- $\mathcal{C}$ is universal.

Let us consider V. The functor $\emptyset_V$ yields an element of 2^V **qua** a non-empty set and is defined as follows:

$\emptyset_V = \emptyset$.

Next we state three propositions:

(1) $\emptyset_V = \emptyset$.

(2) For every k-ary predicate symbol P holds P is a predicate symbol.

(3) P is a $\mathrm{Arity}(P)$-ary predicate symbol.

Let us consider l, V. The functor $\mathrm{variables}_V(l)$ yielding an element of 2^V, is defined by:

$\mathrm{variables}_V(l) = \{l(k) : 1 \leq k \wedge k \leq \mathrm{len}\, l \wedge l(k) \in V\}$.

One can prove the following propositions:

(4) $\mathrm{variables}_V(l) = \{l(k) : 1 \leq k \wedge k \leq \mathrm{len}\, l \wedge l(k) \in V\}$.

(5) $\mathrm{variables}_V(l) \subseteq V$.

(6) $\mathrm{snb}(l) = \mathrm{variables}_{\mathrm{BoundVar}}(l)$.

(7) $\mathrm{snb}(\mathrm{VERUM}) = \emptyset$.

(8) For every formula p such that p is atomic holds $\mathrm{snb}(p) = \mathrm{snb}(\mathrm{Args}(p))$.

(9) For every k-ary predicate symbol P and for every list of variables l of the length k holds $\mathrm{snb}(P[l]) = \mathrm{snb}(l)$.

(10) For every formula p such that p is negative holds $\operatorname{snb}(p) = \operatorname{snb}(\operatorname{Arg}(p))$.

(11) For every formula p holds $\operatorname{snb}(\neg p) = \operatorname{snb}(p)$.

(12) $\operatorname{snb}(\mathrm{FALSUM}) = \emptyset$.

(13) For every formula p such that p is conjunctive holds $\operatorname{snb}(p) = \operatorname{snb}(\operatorname{LeftArg}(p)) \cup \operatorname{snb}(\operatorname{RightArg}(p))$.

(14) For all formulae p, q holds $\operatorname{snb}(p \wedge q) = \operatorname{snb}(p) \cup \operatorname{snb}(q)$.

(15) For every formula p such that p is universal holds $\operatorname{snb}(p) = \operatorname{snb}(\operatorname{Scope}(p)) \setminus \{\operatorname{Bound}(p)\}$.

(16) For every formula p holds $\operatorname{snb}(\forall_x p) = \operatorname{snb}(p) \setminus \{x\}$.

(17) For every formula p such that p is disjunctive holds $\operatorname{snb}(p) = \operatorname{snb}(\operatorname{LeftDisj}(p)) \cup \operatorname{snb}(\operatorname{RightDisj}(p))$.

(18) For all formulae p, q holds $\operatorname{snb}(p \vee q) = \operatorname{snb}(p) \cup \operatorname{snb}(q)$.

(19) For every formula p such that p is conditional holds $\operatorname{snb}(p) = \operatorname{snb}(\operatorname{Antecedent}(p)) \cup \operatorname{snb}(\operatorname{Consequent}(p))$.

(20) For all formulae p, q holds $\operatorname{snb}(p \Rightarrow q) = \operatorname{snb}(p) \cup \operatorname{snb}(q)$.

(21) For every formula p such that p is biconditional holds $\operatorname{snb}(p) = \operatorname{snb}(\operatorname{LeftSide}(p)) \cup \operatorname{snb}(\operatorname{RightSide}(p))$.

(22) For all formulae p, q holds $\operatorname{snb}(p \Leftrightarrow q) = \operatorname{snb}(p) \cup \operatorname{snb}(q)$.

(23) For every formula p holds $\operatorname{snb}(\exists_x p) = \operatorname{snb}(p) \setminus \{x\}$.

(24) VERUM is closed and FALSUM is closed.

(25) For every formula p holds p is closed if and only if $\neg p$ is closed.

(26) For all formulae p, q holds p is closed and q is closed if and only if $p \wedge q$ is closed.

(27) For every formula p holds $\forall_x p$ is closed if and only if $\operatorname{snb}(p) \subseteq \{x\}$.

(28) For every formula p such that p is closed holds $\forall_x p$ is closed.

(29) For all formulae p, q holds p is closed and q is closed if and only if $p \vee q$ is closed.

(30) For all formulae p, q holds p is closed and q is closed if and only if $p \Rightarrow q$ is closed.

(31) For all formulae p, q holds p is closed and q is closed if and only if $p \Leftrightarrow q$ is closed.

(32) For every formula p holds $\exists_x p$ is closed if and only if $\operatorname{snb}(p) \subseteq \{x\}$.

(33) For every formula p such that p is closed holds $\exists_x p$ is closed.

Let us consider V, and let F be a function from WFF into 2^V, and let us consider p. Then $F(p)$ is an element of 2^V.

Let us consider k. The functor x_k yielding a bound variable, is defined as follows:

$x_k = \langle 4, k \rangle$.

One can prove the following propositions:

(34) $x_k = \langle 4, k \rangle$.

(35) If $x_i = x_j$, then $i = j$.

(36) There exists i such that $x_i = x$.

Let us consider k. The functor $\mathbf{a}_k$ yields a free variable and is defined as follows:

$\mathbf{a}_k = \langle 6, k \rangle$.

One can prove the following propositions:

(37) $\mathbf{a}_k = \langle 6, k \rangle$.

(38) If $\mathbf{a}_i = \mathbf{a}_j$, then $i = j$.

(39) There exists i such that $\mathbf{a}_i = a$.

(40) For every element c of FixedVar and for every element a of FreeVar holds $c \neq a$.

(41) For every element c of FixedVar and for every element x of BoundVar holds $c \neq x$.

(42) For every element a of FreeVar and for every element x of BoundVar holds $a \neq x$.

Let us consider V, and let V_1, V_2 be elements of 2^V. Then $V_1 \cup V_2$ is an element of 2^V.

Let D be a non-empty family of sets, and let d be an element of D. The functor $@d$ yields an element of D **qua** a non-empty set and is defined as follows:

$@d = d$.

One can prove the following proposition

(43) For every non-empty family D of sets and for every element d of D holds $@d = d$.

Let D be a non-empty family of sets, and let d be an element of D **qua** a non-empty set. The functor $@d$ yielding an element of D, is defined as follows:

$@d = d$.

We now state a proposition

(44) For every non-empty family D of sets and for every element d of D **qua** a non-empty set holds $@d = d$.

Now we present several schemes. The scheme *QC_Def_SETD* deals with a non-empty family $\mathcal{A}$ of sets, an element $\mathcal{B}$ of $\mathcal{A}$, an element $\mathcal{C}$ of WFF, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$ and states that:

(i) there exists an element d of $\mathcal{A}$ and there exists a function F from WFF into $\mathcal{A}$ such that $d = F(\mathcal{C})$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$,

(ii) for all elements x_1, x_2 of $\mathcal{A}$ such that there exists a function F from WFF into $\mathcal{A}$ such that $x_1 = F(\mathcal{C})$ and for every element p of WFF and for all

elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$ and there exists a function F from WFF into $\mathcal{A}$ such that $x_2 = F(\mathcal{C})$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$ holds $x_1 = x_2$
for all values of the parameters.

The scheme *QC_SETD_Result'V* concerns a non-empty family $\mathcal{A}$ of sets, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{F}(\mathrm{VERUM}) = \mathcal{B}$

provided the parameters meet the following requirement:

- Let p be an element of WFF. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{F}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{G}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{H}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{I}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{J}(p, d_1)$.

The scheme *QC_SETD_Result'a* concerns a non-empty family $\mathcal{A}$ of sets, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, an element $\mathcal{C}$ of WFF, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{F}(\mathcal{C}) = \mathcal{G}(\mathcal{C})$

provided the parameters fulfill the following requirements:

- Let p be an element of WFF. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{F}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{G}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{H}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{I}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{J}(p, d_1)$,
- $\mathcal{C}$ is atomic.

The scheme *QC_SETD_Result'n* deals with a non-empty family $\mathcal{A}$ of sets, an element $\mathcal{B}$ of $\mathcal{A}$, an element $\mathcal{C}$ of WFF, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and a unary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{J}(\mathcal{C}) = \mathcal{G}(\mathcal{J}(\mathrm{Arg}(\mathcal{C})))$

provided the following requirements are met:

- Let p be an element of WFF. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{J}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$,
- $\mathcal{C}$ is negative.

The scheme *QC_SETD_Result'c* deals with a non-empty family $\mathcal{A}$ of sets, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$, and an element $\mathcal{C}$ of WFF and states that:

for all elements d_1, d_2 of $\mathcal{A}$ such that $d_1 = \mathcal{J}(\mathrm{LeftArg}(\mathcal{C}))$ and $d_2 = \mathcal{J}(\mathrm{RightArg}(\mathcal{C}))$
holds $\mathcal{J}(\mathcal{C}) = \mathcal{H}(d_1, d_2)$

provided the parameters fulfill the following conditions:

- Let p be an element of WFF. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{J}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$,
- $\mathcal{C}$ is conjunctive.

The scheme *QC_SETD_Result'u* deals with a non-empty family $\mathcal{A}$ of sets, an element $\mathcal{B}$ of $\mathcal{A}$, an element $\mathcal{C}$ of WFF, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and a unary functor $\mathcal{J}$ yielding an element of $\mathcal{A}$ and states that:

$\mathcal{J}(\mathcal{C}) = \mathcal{I}(\mathcal{C}, \mathcal{J}(\mathrm{Scope}(\mathcal{C})))$

provided the parameters meet the following requirements:

- Let p be an element of WFF. Let d be an element of $\mathcal{A}$. Then $d = \mathcal{J}(p)$ if and only if there exists a function F from WFF into $\mathcal{A}$ such that $d = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and

$d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$,

- $\mathcal{C}$ is universal.

Let us consider V, p. The functor $\mathrm{Vars}_V(p)$ yielding an element of 2^V, is defined as follows:

there exists a function F from WFF into 2^V such that $\mathrm{Vars}_V(p) = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of 2^V holds if $p =$ VERUM, then $F(p) = @(\emptyset_V)$ but if p is atomic, then $F(p) = \mathrm{variables}_V(\mathrm{Args}(p))$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = d_1$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = d_1 \cup d_2$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = d_1$.

We now state a number of propositions:

(45) Let X be an element of 2^V. Then $X = \mathrm{Vars}_V(p)$ if and only if there exists a function F from WFF into 2^V such that $X = F(p)$ and for every element p of WFF and for all elements d_1, d_2 of 2^V holds if $p =$ VERUM, then $F(p) = @(\emptyset_V)$ but if p is atomic, then $F(p) = \mathrm{variables}_V(\mathrm{Args}(p))$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = d_1$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = d_1 \cup d_2$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = d_1$.

(46) $\mathrm{Vars}_V(\mathrm{VERUM}) = \emptyset$.

(47) If p is atomic, then $\mathrm{Vars}_V(p) = \mathrm{variables}_V(\mathrm{Args}(p))$ and $\mathrm{Vars}_V(p) = \{\mathrm{Args}(p)(k) : 1 \leq k \wedge k \leq \mathrm{len}\,\mathrm{Args}(p) \wedge \mathrm{Args}(p)(k) \in V\}$.

(48) For every k-ary predicate symbol P and for every list of variables l of the length k holds $\mathrm{Vars}_V(P[l]) = \mathrm{variables}_V(l)$ and $\mathrm{Vars}_V(P[l]) = \{l(i) : 1 \leq i \wedge i \leq \mathrm{len}\,l \wedge l(i) \in V\}$.

(49) If p is negative, then $\mathrm{Vars}_V(p) = \mathrm{Vars}_V(\mathrm{Arg}(p))$.

(50) $\mathrm{Vars}_V(\neg p) = \mathrm{Vars}_V(p)$.

(51) $\mathrm{Vars}_V(\mathrm{FALSUM}) = \emptyset$.

(52) If p is conjunctive, then
$$\mathrm{Vars}_V(p) = \mathrm{Vars}_V(\mathrm{LeftArg}(p)) \cup \mathrm{Vars}_V(\mathrm{RightArg}(p)) .$$

(53) $\mathrm{Vars}_V(p \wedge q) = \mathrm{Vars}_V(p) \cup \mathrm{Vars}_V(q)$.

(54) If p is universal, then $\mathrm{Vars}_V(p) = \mathrm{Vars}_V(\mathrm{Scope}(p))$.

(55) $\mathrm{Vars}_V(\forall_x p) = \mathrm{Vars}_V(p)$.

(56) If p is disjunctive, then
$$\mathrm{Vars}_V(p) = \mathrm{Vars}_V(\mathrm{LeftDisj}(p)) \cup \mathrm{Vars}_V(\mathrm{RightDisj}(p)) .$$

(57) $\mathrm{Vars}_V(p \vee q) = \mathrm{Vars}_V(p) \cup \mathrm{Vars}_V(q)$.

(58) If p is conditional, then
$$\mathrm{Vars}_V(p) = \mathrm{Vars}_V(\mathrm{Antecedent}(p)) \cup \mathrm{Vars}_V(\mathrm{Consequent}(p)) .$$

(59) $\mathrm{Vars}_V(p \Rightarrow q) = \mathrm{Vars}_V(p) \cup \mathrm{Vars}_V(q)$.

(60) If p is biconditional, then
$$\mathrm{Vars}_V(p) = \mathrm{Vars}_V(\mathrm{LeftSide}(p)) \cup \mathrm{Vars}_V(\mathrm{RightSide}(p)) .$$

(61) $\mathrm{Vars}_V(p \Leftrightarrow q) = \mathrm{Vars}_V(p) \cup \mathrm{Vars}_V(q)$.

(62) If p is existential, then $\mathrm{Vars}_V(p) = \mathrm{Vars}_V(\mathrm{Arg}(\mathrm{Scope}(\mathrm{Arg}(p))))$.

(63) $\mathrm{Vars}_V(\exists_x p) = \mathrm{Vars}_V(p)$.

Let us consider p. The functor $\mathrm{Free}\, p$ yielding an element of 2^{FreeVar}, is defined as follows:

$\mathrm{Free}\, p = \mathrm{Vars}_{\mathrm{FreeVar}}(p)$.

One can prove the following propositions:

(64) $\mathrm{Free}\, p = \mathrm{Vars}_{\mathrm{FreeVar}}(p)$.

(65) $\mathrm{Free}\, \mathrm{VERUM} = \emptyset$.

(66) For every k-ary predicate symbol P and for every list of variables l of the length k holds $\mathrm{Free}(P[l]) = \{l(i) : 1 \leq i \wedge i \leq \mathrm{len}\, l \wedge l(i) \in \mathrm{FreeVar}\}$.

(67) $\mathrm{Free}\, \neg p = \mathrm{Free}\, p$.

(68) $\mathrm{Free}\, \mathrm{FALSUM} = \emptyset$.

(69) $\mathrm{Free}\, p \wedge q = \mathrm{Free}\, p \cup \mathrm{Free}\, q$.

(70) $\mathrm{Free}\, \forall_x p = \mathrm{Free}\, p$.

(71) $\mathrm{Free}\, p \vee q = \mathrm{Free}\, p \cup \mathrm{Free}\, q$.

(72) $\mathrm{Free}\, p \Rightarrow q = \mathrm{Free}\, p \cup \mathrm{Free}\, q$.

(73) $\mathrm{Free}\, p \Leftrightarrow q = \mathrm{Free}\, p \cup \mathrm{Free}\, q$.

(74) $\mathrm{Free}\, \exists_x p = \mathrm{Free}\, p$.

Let us consider p. The functor $\mathrm{Fixed}\, p$ yielding an element of 2^{FixedVar}, is defined as follows:

$\mathrm{Fixed}\, p = \mathrm{Vars}_{\mathrm{FixedVar}}(p)$.

Next we state a number of propositions:

(75) $\mathrm{Fixed}\, p = \mathrm{Vars}_{\mathrm{FixedVar}}(p)$.

(76) $\mathrm{Fixed}\, \mathrm{VERUM} = \emptyset$.

(77) For every k-ary predicate symbol P and for every list of variables l of the length k holds $\mathrm{Fixed}(P[l]) = \{l(i) : 1 \leq i \wedge i \leq \mathrm{len}\, l \wedge l(i) \in \mathrm{FixedVar}\}$.

(78) $\mathrm{Fixed}\, \neg p = \mathrm{Fixed}\, p$.

(79) $\mathrm{Fixed}\, \mathrm{FALSUM} = \emptyset$.

(80) $\mathrm{Fixed}\, p \wedge q = \mathrm{Fixed}\, p \cup \mathrm{Fixed}\, q$.

(81) $\mathrm{Fixed}(\forall_x p) = \mathrm{Fixed}\, p$.

(82) $\mathrm{Fixed}\, p \vee q = \mathrm{Fixed}\, p \cup \mathrm{Fixed}\, q$.

(83) $\mathrm{Fixed}\, p \Rightarrow q = \mathrm{Fixed}\, p \cup \mathrm{Fixed}\, q$.

(84) $\mathrm{Fixed}\, p \Leftrightarrow q = \mathrm{Fixed}\, p \cup \mathrm{Fixed}\, q$.

(85) $\mathrm{Fixed}(\exists_x p) = \mathrm{Fixed}\, p$.

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Received November 23, 1989

FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Monotone Real Sequences. Subsequences

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Summary. The article contains definitions of constant, increasing, decreasing, non decreasing, non increasing sequences, the definition of a subsequence and their basic properties.

MML Identifier: SEQM_3.

The articles [2], [4], [3], [1], and [5] provide the terminology and notation for this paper. We adopt the following convention: n, m, k will be natural numbers, r will be a real number, and seq, seq_1, seq_2 will be sequences of real numbers. We now define five new predicates. Let us consider seq. We say that seq is increasing if and only if:

for every n holds $seq(n) < seq(n+1)$.

We say that seq is decreasing if and only if:

for every n holds $seq(n+1) < seq(n)$.

We say that seq is non-decreasing if and only if:

for every n holds $seq(n) \leq seq(n+1)$.

We say that seq is non-increasing if and only if:

for every n holds $seq(n+1) \leq seq(n)$.

We say that seq is constant if and only if:

there exists r such that for every n holds $seq(n) = r$.

Let us consider seq. We say that seq is monotone if and only if:

seq is non-decreasing or seq is non-increasing.

We now state a number of propositions:

(1) seq is increasing if and only if for every n holds $seq(n) < seq(n+1)$.

(2) seq is decreasing if and only if for every n holds $seq(n+1) < seq(n)$.

(3) seq is non-decreasing if and only if for every n holds $seq(n) \leq seq(n+1)$.

(4) seq is non-increasing if and only if for every n holds $seq(n+1) \leq seq(n)$.

[1]Supported by RPBP III.24

ISSN 0777–4028

(5) seq is constant if and only if there exists r such that for every n holds $seq(n) = r$.

(6) seq is monotone if and only if seq is non-decreasing or seq is non-increasing.

(7) seq is increasing if and only if for all n, m such that $n < m$ holds $seq(n) < seq(m)$.

(8) seq is increasing if and only if for all n, k holds $seq(n) < seq((n+1)+k)$.

(9) seq is decreasing if and only if for all n, k holds $seq((n+1)+k) < seq(n)$.

(10) seq is decreasing if and only if for all n, m such that $n < m$ holds $seq(m) < seq(n)$.

(11) seq is non-decreasing if and only if for all n, k holds $seq(n) \leq seq(n+k)$.

(12) seq is non-decreasing if and only if for all n, m such that $n \leq m$ holds $seq(n) \leq seq(m)$.

(13) seq is non-increasing if and only if for all n, k holds $seq(n+k) \leq seq(n)$.

(14) seq is non-increasing if and only if for all n, m such that $n \leq m$ holds $seq(m) \leq seq(n)$.

(15) seq is constant if and only if there exists r such that $\operatorname{rng} seq = \{r\}$.

(16) seq is constant if and only if for every n holds $seq(n) = seq(n+1)$.

(17) seq is constant if and only if for all n, k holds $seq(n) = seq(n+k)$.

(18) seq is constant if and only if for all n, m holds $seq(n) = seq(m)$.

(19) If seq is increasing, then for every n such that $0 < n$ holds $seq(0) < seq(n)$.

(20) If seq is decreasing, then for every n such that $0 < n$ holds $seq(n) < seq(0)$.

(21) If seq is non-decreasing, then for every n holds $seq(0) \leq seq(n)$.

(22) If seq is non-increasing, then for every n holds $seq(n) \leq seq(0)$.

(23) If seq is increasing, then seq is non-decreasing.

(24) If seq is decreasing, then seq is non-increasing.

(25) If seq is constant, then seq is non-decreasing.

(26) If seq is constant, then seq is non-increasing.

(27) If seq is non-decreasing and seq is non-increasing, then seq is constant.

A sequence of real numbers is said to be an increasing sequence of naturals if:

$\operatorname{rng} \mathrm{it} \subseteq \mathbb{N}$ and for every n holds $\mathrm{it}(n) < \mathrm{it}(n+1)$.

Let us consider seq, k. The functor $seq \frown k$ yielding a sequence of real numbers, is defined as follows:

for every n holds $(seq \frown k)(n) = seq(n+k)$.

In the sequel $Nseq$, $Nseq_1$ will be increasing sequences of naturals. Next we state four propositions:

(28) seq is an increasing sequence of naturals if and only if $\operatorname{rng} seq \subseteq \mathbb{N}$ and for every n holds $seq(n) < seq(n+1)$.

(29) *seq* is an increasing sequence of naturals if and only if *seq* is increasing and for every n holds $seq(n)$ is a natural number.

(30) $seq_1 = seq \frown k$ if and only if for every n holds $seq_1(n) = seq(n + k)$.

(31) For every n holds $(seq \cdot Nseq)(n) = seq(Nseq(n))$.

Let us consider $Nseq$, n. Then $Nseq(n)$ is a natural number.

Let us consider $Nseq$, seq. Then $seq \cdot Nseq$ is a sequence of real numbers.

Let us consider $Nseq$, $Nseq_1$. Then $Nseq_1 \cdot Nseq$ is an increasing sequence of naturals.

Let us consider $Nseq$, k. Then $Nseq \frown k$ is an increasing sequence of naturals.

Let us consider seq, seq_1. We say that seq is a subsequence of seq_1 if and only if:

there exists $Nseq$ such that $seq = seq_1 \cdot Nseq$.

Next we state a number of propositions:

(32) seq is a subsequence of seq_1 if and only if there exists $Nseq$ such that $seq = seq_1 \cdot Nseq$.

(33) For every n holds $n \leq Nseq(n)$.

(34) $seq \frown 0 = seq$.

(35) $(seq \frown k) \frown m = (seq \frown m) \frown k$.

(36) $(seq \frown k) \frown m = seq \frown (k + m)$.

(37) $(seq + seq_1) \frown k = seq \frown k + seq_1 \frown k$.

(38) $(-seq) \frown k = -seq \frown k$.

(39) $(seq - seq_1) \frown k = seq \frown k - seq_1 \frown k$.

(40) If seq is non-zero, then $seq \frown k$ is non-zero.

(41) If seq is non-zero, then $seq^{-1} \frown k = (seq \frown k)^{-1}$.

(42) $(seq \cdot seq_1) \frown k = (seq \frown k) \cdot (seq_1 \frown k)$.

(43) If seq_1 is non-zero, then $\frac{seq}{seq_1} \frown k = \frac{seq \frown k}{seq_1 \frown k}$.

(44) $(r \cdot seq) \frown k = r \cdot (seq \frown k)$.

(45) $(seq \cdot Nseq) \frown k = seq \cdot (Nseq \frown k)$.

(46) seq is a subsequence of seq.

(47) $seq \frown k$ is a subsequence of seq.

(48) If seq is a subsequence of seq_1 and seq_1 is a subsequence of seq_2, then seq is a subsequence of seq_2.

(49) If seq is increasing and seq_1 is a subsequence of seq, then seq_1 is increasing.

(50) If seq is decreasing and seq_1 is a subsequence of seq, then seq_1 is decreasing.

(51) If seq is non-decreasing and seq_1 is a subsequence of seq, then seq_1 is non-decreasing.

(52) If seq is non-increasing and seq_1 is a subsequence of seq, then seq_1 is non-increasing.

(53) If seq is monotone and seq_1 is a subsequence of seq, then seq_1 is monotone.

(54) If seq is constant and seq_1 is a subsequence of seq, then seq_1 is constant.

(55) If seq is constant and seq_1 is a subsequence of seq, then $seq = seq_1$.

(56) If seq is upper bounded and seq_1 is a subsequence of seq, then seq_1 is upper bounded.

(57) If seq is lower bounded and seq_1 is a subsequence of seq, then seq_1 is lower bounded.

(58) If seq is bounded and seq_1 is a subsequence of seq, then seq_1 is bounded.

(59) If seq is increasing and $0 < r$, then $r \cdot seq$ is increasing but if seq is increasing and $0 = r$, then $r \cdot seq$ is constant but if seq is increasing and $r < 0$, then $r \cdot seq$ is decreasing.

(60) If seq is decreasing and $0 < r$, then $r \cdot seq$ is decreasing but if seq is decreasing and $0 = r$, then $r \cdot seq$ is constant but if seq is decreasing and $r < 0$, then $r \cdot seq$ is increasing.

(61) If seq is non-decreasing and $0 \leq r$, then $r \cdot seq$ is non-decreasing but if seq is non-decreasing and $r \leq 0$, then $r \cdot seq$ is non-increasing.

(62) If seq is non-increasing and $0 \leq r$, then $r \cdot seq$ is non-increasing but if seq is non-increasing and $r \leq 0$, then $r \cdot seq$ is non-decreasing.

(63) If seq is increasing and seq_1 is increasing, then $seq + seq_1$ is increasing but if seq is decreasing and seq_1 is decreasing, then $seq+seq_1$ is decreasing but if seq is non-decreasing and seq_1 is non-decreasing, then $seq + seq_1$ is non-decreasing but if seq is non-increasing and seq_1 is non-increasing, then $seq + seq_1$ is non-increasing.

(64) If seq is increasing and seq_1 is constant, then $seq + seq_1$ is increasing but if seq is decreasing and seq_1 is constant, then $seq + seq_1$ is decreasing but if seq is non-decreasing and seq_1 is constant, then $seq + seq_1$ is non-decreasing but if seq is non-increasing and seq_1 is constant, then $seq+seq_1$ is non-increasing.

(65) If seq is constant, then for every r holds $r \cdot seq$ is constant and $-seq$ is constant and $|seq|$ is constant.

(66) If seq is constant and seq_1 is constant, then $seq \cdot seq_1$ is constant and $seq + seq_1$ is constant.

(67) If seq is constant and seq_1 is constant, then $seq - seq_1$ is constant.

(68) If seq is upper bounded and $0 < r$, then $r \cdot seq$ is upper bounded but if seq is upper bounded and $0 = r$, then $r \cdot seq$ is bounded but if seq is upper bounded and $r < 0$, then $r \cdot seq$ is lower bounded.

(69) If seq is lower bounded and $0 < r$, then $r \cdot seq$ is lower bounded but if seq is lower bounded and $0 = r$, then $r \cdot seq$ is bounded but if seq is lower bounded and $r < 0$, then $r \cdot seq$ is upper bounded.

(70) If seq is bounded, then for every r holds $r \cdot seq$ is bounded and $-seq$ is bounded and $|seq|$ is bounded.

(71) If seq is upper bounded and seq_1 is upper bounded, then $seq + seq_1$ is upper bounded but if seq is lower bounded and seq_1 is lower bounded, then $seq+seq_1$ is lower bounded but if seq is bounded and seq_1 is bounded, then $seq + seq_1$ is bounded.

(72) If seq is bounded and seq_1 is bounded, then $seq \cdot seq_1$ is bounded and $seq - seq_1$ is bounded.

(73) If seq is constant, then seq is bounded.

(74) If seq is constant, then for every r holds $r \cdot seq$ is bounded and $-seq$ is bounded and $|seq|$ is bounded.

(75) If seq is upper bounded and seq_1 is constant, then $seq + seq_1$ is upper bounded but if seq is lower bounded and seq_1 is constant, then $seq + seq_1$ is lower bounded but if seq is bounded and seq_1 is constant, then $seq+seq_1$ is bounded.

(76) If seq is upper bounded and seq_1 is constant, then $seq - seq_1$ is upper bounded but if seq is lower bounded and seq_1 is constant, then $seq - seq_1$ is lower bounded but if seq is bounded and seq_1 is constant, then $seq-seq_1$ is bounded and $seq_1 - seq$ is bounded and $seq \cdot seq_1$ is bounded.

(77) If seq is upper bounded and seq_1 is non-increasing, then $seq + seq_1$ is upper bounded.

(78) If seq is lower bounded and seq_1 is non-decreasing, then $seq + seq_1$ is lower bounded.

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Received November 23, 1989

FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Convergent Real Sequences. Upper and Lower Bound of Sets of Real Numbers

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Summary. The article contains theorems about convergent sequences and the limit of sequences occurring in [3] such as Bolzano-Weirrstrass theorem, Cauchy theorem and others. Bounded sets of real numbers and lower and upper bound of subset of real numbers are defined.

MML Identifier: SEQ_4.

The papers [7], [2], [5], [3], [1], [4], [8], and [6] provide the notation and terminology for this paper. For simplicity we follow a convention: n, k, m will denote natural numbers, r, r_1, p, g, g_1, g_2, s will denote real numbers, seq, seq_1 will denote sequences of real numbers, $Nseq$ will denote an increasing sequence of naturals, and X, Y will denote subsets of $\mathbb{R}$. One can prove the following propositions:

(1) If $0 < r_1$ and $r_1 \leq r$ and $0 < g$, then $\frac{g}{r} \leq \frac{g}{r_1}$.

(2) If $r < p$, then $0 < p - r$.

(3) $r - (r - s) = s$ and $r + (s - r) = s$ and $(r + s) - r = s$.

(4) If $0 < s$, then $0 < \frac{s}{3}$.

(5) $(\frac{s}{3} + \frac{s}{3}) + \frac{s}{3} = s$.

(6) If $0 < g$ and $0 < r$ and $g \leq g_1$ and $r < r_1$, then $g \cdot r < g_1 \cdot r_1$ and $r \cdot g < r_1 \cdot g_1$.

(7) If $0 < g$ and $0 < r$ and $g \leq g_1$ and $r \leq r_1$, then $g \cdot r \leq g_1 \cdot r_1$ and $r \cdot g \leq r_1 \cdot g_1$.

(8) Given X, Y. Then if there exists r such that $r \in X$ and there exists r such that $r \in Y$ and for all r, p such that $r \in X$ and $p \in Y$ holds $r < p$, then there exists g such that for all r, p such that $r \in X$ and $p \in Y$ holds $r \leq g$ and $g \leq p$.

[1]Supported by RPBP III.24

ISSN 0777–4028

(9) If $0 < p$ and there exists r such that $r \in X$ and for every r such that $r \in X$ holds $r + p \in X$, then for every g there exists r such that $r \in X$ and $g < r$.

(10) For every r there exists n such that $r < n$.

We now define two new predicates. Let us consider X. Let us assume that there exists r such that $r \in X$. We say that X is upper bounded if and only if:

there exists p such that for every r such that $r \in X$ holds $r \leq p$.

We say that X is lower bounded if and only if:

there exists p such that for every r such that $r \in X$ holds $p \leq r$.

Let us consider X. Let us assume that there exists r such that $r \in X$. We say that X is bounded if and only if:

X is lower bounded and X is upper bounded.

We now state several propositions:

(11) If there exists r such that $r \in X$, then X is upper bounded if and only if there exists p such that for every r such that $r \in X$ holds $r \leq p$.

(12) If there exists r such that $r \in X$, then X is lower bounded if and only if there exists p such that for every r such that $r \in X$ holds $p \leq r$.

(13) If there exists r such that $r \in X$, then X is bounded if and only if X is upper bounded and X is lower bounded.

(14) If there exists r such that $r \in X$, then X is bounded if and only if there exists s such that $0 < s$ and for every r such that $r \in X$ holds $|r| < s$.

(15) If $X = \{r\}$, then X is bounded.

(16) If there exists r such that $r \in X$ and X is upper bounded, then there exists g such that for every r such that $r \in X$ holds $r \leq g$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $g - s < r$.

(17) Suppose that
 (i) for every r such that $r \in X$ holds $r \leq g_1$,
 (ii) for every s such that $0 < s$ there exists r such that $r \in X$ and $g_1 - s < r$,
 (iii) for every r such that $r \in X$ holds $r \leq g_2$,
 (iv) for every s such that $0 < s$ there exists r such that $r \in X$ and $g_2 - s < r$.

 Then $g_1 = g_2$.

(18) If there exists r such that $r \in X$ and X is lower bounded, then there exists g such that for every r such that $r \in X$ holds $g \leq r$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $r < g + s$.

(19) Suppose that
 (i) for every r such that $r \in X$ holds $g_1 \leq r$,
 (ii) for every s such that $0 < s$ there exists r such that $r \in X$ and $r < g_1 + s$,
 (iii) for every r such that $r \in X$ holds $g_2 \leq r$,
 (iv) for every s such that $0 < s$ there exists r such that $r \in X$ and $r < g_2 + s$.

 Then $g_1 = g_2$.

Let us consider X. Let us assume that there exists r such that $r \in X$ and X is upper bounded. The functor $\sup X$ yielding a real number, is defined as follows:

for every r such that $r \in X$ holds $r \leq \sup X$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $(\sup X) - s < r$.

Let us consider X. Let us assume that there exists r such that $r \in X$ and X is lower bounded. The functor $\inf X$ yields a real number and is defined by:

for every r such that $r \in X$ holds $\inf X \leq r$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $r < (\inf X) + s$.

One can prove the following propositions:

(20) If there exists r such that $r \in X$ and X is upper bounded, then $\sup X = g$ if and only if for every r such that $r \in X$ holds $r \leq g$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $g - s < r$.

(21) If there exists r such that $r \in X$ and X is lower bounded, then $\inf X = g$ if and only if for every r such that $r \in X$ holds $g \leq r$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $r < g + s$.

(22) If $X = \{r\}$, then $\inf X = r$ and $\sup X = r$.

(23) If $X = \{r\}$, then $\inf X = \sup X$.

(24) If X is bounded and there exists r such that $r \in X$, then $\inf X \leq \sup X$.

(25) If X is bounded and there exists r such that $r \in X$, then there exist r, p such that $r \in X$ and $p \in X$ and $p \neq r$ if and only if $\inf X < \sup X$.

The scheme *SepNat* concerns a unary predicate $\mathcal{P}$, and states that:

there exists a X being sets of natural numbers such that for every n holds $n \in X$ if and only if $\mathcal{P}[n]$

for all values of the parameter.

We now state a number of propositions:

(26) If seq is convergent, then $|seq|$ is convergent.

(27) If seq is convergent, then $\lim |seq| = |\lim seq|$.

(28) If $|seq|$ is convergent and $\lim |seq| = 0$, then seq is convergent and $\lim seq = 0$.

(29) If seq_1 is a subsequence of seq and seq is convergent, then seq_1 is convergent.

(30) If seq_1 is a subsequence of seq and seq is convergent, then $\lim seq_1 = \lim seq$.

(31) If seq is convergent and there exists k such that for every n such that $k \leq n$ holds $seq_1(n) = seq(n)$, then seq_1 is convergent.

(32) If seq is convergent and there exists k such that for every n such that $k \leq n$ holds $seq_1(n) = seq(n)$, then $\lim seq = \lim seq_1$.

(33) If seq is convergent, then $seq \frown k$ is convergent and $\lim(seq \frown k) = \lim seq$.

(34) If seq is convergent and there exists k such that $seq_1 = seq \frown k$, then seq_1 is convergent and $\lim seq_1 = \lim seq$.

(35) If seq is convergent and there exists k such that $seq = seq_1 \frown k$, then seq_1 is convergent.

(36) If seq is convergent and there exists k such that $seq = seq_1 \frown k$, then $\lim seq_1 = \lim seq$.

(37) If seq is convergent and $\lim seq \neq 0$, then there exists k such that $seq \frown k$ is non-zero.

(38) If seq is convergent and $\lim seq \neq 0$, then there exists seq_1 such that seq_1 is a subsequence of seq and seq_1 is non-zero.

(39) If seq is constant, then seq is convergent.

(40) If seq is constant and $r \in \operatorname{rng} seq$ or seq is constant and there exists n such that $seq(n) = r$, then $\lim seq = r$.

(41) If seq is constant, then for every n holds $\lim seq = seq(n)$.

(42) If seq is convergent and $\lim seq \neq 0$, then for every seq_1 such that seq_1 is a subsequence of seq and seq_1 is non-zero holds $\lim {seq_1}^{-1} = (\lim seq)^{-1}$.

(43) For all r, seq such that $0 < r$ and for every n holds $seq(n) = \frac{1}{n+r}$ holds seq is convergent.

(44) For all r, seq such that $0 < r$ and for every n holds $seq(n) = \frac{1}{n+r}$ holds $\lim seq = 0$.

(45) If for every n holds $seq(n) = \frac{1}{n+1}$, then seq is convergent and $\lim seq = 0$.

(46) If $0 < r$ and for every n holds $seq(n) = \frac{g}{n+r}$, then seq is convergent and $\lim seq = 0$.

(47) For all r, seq such that $0 < r$ and for every n holds $seq(n) = \frac{1}{n \cdot n+r}$ holds seq is convergent.

(48) For all r, seq such that $0 < r$ and for every n holds $seq(n) = \frac{1}{n \cdot n+r}$ holds $\lim seq = 0$.

(49) If for every n holds $seq(n) = \frac{1}{n \cdot n+1}$, then seq is convergent and $\lim seq = 0$.

(50) If $0 < r$ and for every n holds $seq(n) = \frac{g}{n \cdot n+r}$, then seq is convergent and $\lim seq = 0$.

(51) If seq is non-decreasing and seq is upper bounded, then seq is convergent.

(52) If seq is non-increasing and seq is lower bounded, then seq is convergent.

(53) If seq is monotone and seq is bounded, then seq is convergent.

(54) If seq is upper bounded and seq is non-decreasing, then for every n holds $seq(n) \leq \lim seq$.

(55) If seq is lower bounded and seq is non-increasing, then for every n holds $\lim seq \leq seq(n)$.

(56) For every seq there exists $Nseq$ such that $seq \cdot Nseq$ is monotone.

(57) If seq is bounded, then there exists seq_1 such that seq_1 is a subsequence of seq and seq_1 is convergent.

(58) seq is convergent if and only if for every s such that $0 < s$ there exists n such that for every m such that $n \leq m$ holds $|seq(m) - seq(n)| < s$.

(59) Suppose seq is constant and seq_1 is convergent. Then $\lim(seq + seq_1) = seq(0) + \lim seq_1$ and $\lim(seq - seq_1) = seq(0) - \lim seq_1$ and $\lim(seq_1 -$

$seq) = \lim seq_1 - seq(0)$ and $\lim(seq \cdot seq_1) = seq(0) \cdot (\lim seq_1)$.

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Received November 23, 1989

FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Midpoint algebras

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Summary. In this article basic properties of midpoint algebras are proved. We define a congruence relation $\equiv$ on bound vectors and free vectors as the equivalence classes of $\equiv$.

MML Identifier: `MIDSP_1`.

The notation and terminology used in this paper are introduced in the following articles: [5], [1], [2], [3], [4], and [6]. We consider midpoint algebra structures which are systems

$\langle$ points, a midpoint operation $\rangle$

where the points is a non-empty set and the midpoint operation is a binary operation on the points. In the sequel MS is a midpoint algebra structure and a, b are elements of the points of MS. Let us consider MS, a, b. The functor $a \oplus b$ yielding an element of the points of MS, is defined by:

$a \oplus b =$ (the midpoint operation of MS)(a, b).

We now state a proposition

(1) $a \oplus b =$ (the midpoint operation of MS)(a, b).

Let x be arbitrary. Then $\{x\}$ is a non-empty set.

zo is a binary operation on $\{0\}$.

One can prove the following propositions:

(2) zo is a function from $[\!:\{0\}, \{0\}:\!]$ into $\{0\}$.

(3) For all elements x, y of $\{0\}$ holds $zo(x, y) = 0$.

The midpoint algebra structure EX is defined by:

$\mathrm{EX} = \langle\{0\}, zo\rangle$.

The following propositions are true:

(4) $\mathrm{EX} = \langle\{0\}, zo\rangle$.

(5) The points of $\mathrm{EX} = \{0\}$.

[1]Supported by RPBP.III-24.C6.

ISSN 0777–4028

(6) The midpoint operation of EX $= zo$.

(7) For every element a of the points of EX holds $a = 0$.

(8) For all elements a, b of the points of EX holds $a \oplus b = zo(a, b)$.

(9) For all elements a, b of the points of EX holds $a \oplus b = 0$.

(10) For all elements a, b, c, d of the points of EX holds $a \oplus a = a$ and $a \oplus b = b \oplus a$ and $(a \oplus b) \oplus (c \oplus d) = (a \oplus c) \oplus (b \oplus d)$ and there exists an element x of the points of EX such that $x \oplus a = b$.

A midpoint algebra structure is called a midpoint algebra if:
for all elements a, b, c, d of the points of it holds $a \oplus a = a$ and $a \oplus b = b \oplus a$ and $(a \oplus b) \oplus (c \oplus d) = (a \oplus c) \oplus (b \oplus d)$ and there exists an element x of the points of it such that $x \oplus a = b$.

We follow the rules: M denotes a midpoint algebra and a, b, c, d, a', b', c', d', x, y, x' denote elements of the points of M. Next we state several propositions:

(11) $a \oplus a = a$.

(12) $a \oplus b = b \oplus a$.

(13) $(a \oplus b) \oplus (c \oplus d) = (a \oplus c) \oplus (b \oplus d)$.

(14) There exists x such that $x \oplus a = b$.

(15) $(a \oplus b) \oplus c = (a \oplus c) \oplus (b \oplus c)$.

(16) $a \oplus (b \oplus c) = (a \oplus b) \oplus (a \oplus c)$.

(17) If $a \oplus b = a$, then $a = b$.

(18) If $x \oplus a = x' \oplus a$, then $x = x'$.

(19) If $a \oplus x = a \oplus x'$, then $x = x'$.

Let us consider M, a, b, c, d. The predicate $a, b \equiv c, d$ is defined by:
$a \oplus d = b \oplus c$.

The following propositions are true:

(20) $a, b \equiv c, d$ if and only if $a \oplus d = b \oplus c$.

(21) $a, a \equiv b, b$.

(22) If $a, b \equiv c, d$, then $c, d \equiv a, b$.

(23) If $a, a \equiv b, c$, then $b = c$.

(24) If $a, b \equiv c, c$, then $a = b$.

(25) $a, b \equiv a, b$.

(26) There exists d such that $a, b \equiv c, d$.

(27) If $a, b \equiv c, d$ and $a, b \equiv c, d'$, then $d = d'$.

(28) If $x, y \equiv a, b$ and $x, y \equiv c, d$, then $a, b \equiv c, d$.

(29) If $a, b \equiv a', b'$ and $b, c \equiv b', c'$, then $a, c \equiv a', c'$.

In the sequel p, q, r will denote elements of $[\!:$ the points of M, the points of $M :\!]$. Let us consider M, p. Then $p_{\mathbf{1}}$ is an element of the points of M.

Let us consider M, p. Then $p_{\mathbf{2}}$ is an element of the points of M.

Let us consider M, p, q. The predicate $p \equiv q$ is defined as follows:
$p_{\mathbf{1}}, p_{\mathbf{2}} \equiv q_{\mathbf{1}}, q_{\mathbf{2}}$.

One can prove the following proposition

(30) $p \equiv q$ if and only if $p_{\mathbf{1}}, p_{\mathbf{2}} \equiv q_{\mathbf{1}}, q_{\mathbf{2}}$.

Let us consider M, a, b. Then $\langle a, b\rangle$ is an element of $[\!:$ the points of $M,$ the points of $M:\!]$.

One can prove the following propositions:

(31) If $a, b \equiv c, d$, then $\langle a, b\rangle \equiv \langle c, d\rangle$.

(32) If $\langle a, b\rangle \equiv \langle c, d\rangle$, then $a, b \equiv c, d$.

(33) $p \equiv p$.

(34) If $p \equiv q$, then $q \equiv p$.

(35) If $p \equiv q$ and $p \equiv r$, then $q \equiv r$.

(36) If $p \equiv r$ and $q \equiv r$, then $p \equiv q$.

(37) If $p \equiv q$ and $q \equiv r$, then $p \equiv r$.

(38) If $p \equiv q$, then $r \equiv p$ if and only if $r \equiv q$.

(39) For every p holds $\{q : q \equiv p\}$is a non-empty subset of $[\!:$ the points of $M,$ the points of $M:\!]$.

Let us consider M, p. The functor $p^{\smile}$ yields a non-empty subset of $[\!:$ the points of $M,$ the points of $M:\!]$ and is defined as follows:

$p^{\smile} = \{q : q \equiv p\}$.

The following propositions are true:

(40) For every p holds $p^{\smile} = \{q : q \equiv p\}$ and $p^{\smile}$ is a non-empty subset of $[\!:$ the points of $M,$ the points of $M:\!]$.

(41) For every p holds $r \in p^{\smile}$ if and only if $r \equiv p$.

(42) If $p \equiv q$, then $p^{\smile} = q^{\smile}$.

(43) If $p^{\smile} = q^{\smile}$, then $p \equiv q$.

(44) If $\langle a, b\rangle^{\smile} = \langle c, d\rangle^{\smile}$, then $a \oplus d = b \oplus c$.

(45) $p \in p^{\smile}$.

Let us consider M. A non-empty subset of $[\!:$ the points of $M,$ the points of $M:\!]$ is said to be a vector of M if:

there exists p such that it $= p^{\smile}$.

The following proposition is true

(46) For every non-empty subset X of $[\!:$ the points of $M,$ the points of $M:\!]$ holds X is a vector of M if and only if there exists p such that $X = p^{\smile}$.

In the sequel u, v, w, w' denote vectors of M. The following proposition is true

(47) $p^{\smile}$ is a vector of M.

Let us consider M, p. Then $p^{\smile}$ is a vector of M.

We now state a proposition

(48) There exists u such that for every p holds $p \in u$ if and only if $p_{\mathbf{1}} = p_{\mathbf{2}}$.

Let us consider M. The functor I_M yielding a vector of M, is defined by:

$\mathrm{I}_M = \{p : p_{\mathbf{1}} = p_{\mathbf{2}}\}$.

Next we state four propositions:

(49) $\mathrm{I}_M = \{p : p_{\mathbf{1}} = p_{\mathbf{2}}\}$.

(50) $\mathrm{I}_M = \langle b, b\rangle^{\smile}$.

(51) There exist w, p, q such that $u = p^{\smile}$ and $v = q^{\smile}$ and $p_{\mathbf{2}} = q_{\mathbf{1}}$ and $w = \langle p_{\mathbf{1}}, q_{\mathbf{2}}\rangle^{\smile}$.

(52) Suppose that

(i) there exist p, q such that $u = p^{\smile}$ and $v = q^{\smile}$ and $p_{\mathbf{2}} = q_{\mathbf{1}}$ and $w = \langle p_{\mathbf{1}}, q_{\mathbf{2}}\rangle^{\smile}$,

(ii) there exist p, q such that $u = p^{\smile}$ and $v = q^{\smile}$ and $p_{\mathbf{2}} = q_{\mathbf{1}}$ and $w' = \langle p_{\mathbf{1}}, q_{\mathbf{2}}\rangle^{\smile}$.

Then $w = w'$.

Let us consider M, u, v. The functor $u+v$ yields a vector of M and is defined by:

there exist p, q such that $u = p^{\smile}$ and $v = q^{\smile}$ and $p_{\mathbf{2}} = q_{\mathbf{1}}$ and $u + v = \langle p_{\mathbf{1}}, q_{\mathbf{2}}\rangle^{\smile}$.

We now state a proposition

(53) There exists b such that $u = \langle a, b\rangle^{\smile}$.

Let us consider M, a, b. The functor $\overrightarrow{[a, b]}$ yields a vector of M and is defined by:

$\overrightarrow{[a, b]} = \langle a, b\rangle^{\smile}$.

Next we state a number of propositions:

(54) $\overrightarrow{[a, b]} = \langle a, b\rangle^{\smile}$.

(55) There exists b such that $u = \overrightarrow{[a, b]}$.

(56) If $\langle a, b\rangle \equiv \langle c, d\rangle$, then $\overrightarrow{[a, b]} = \overrightarrow{[c, d]}$.

(57) If $\overrightarrow{[a, b]} = \overrightarrow{[c, d]}$, then $a \oplus d = b \oplus c$.

(58) $\mathrm{I}_M = \overrightarrow{[b, b]}$.

(59) If $\overrightarrow{[a, b]} = \overrightarrow{[a, c]}$, then $b = c$.

(60) $\overrightarrow{[a, b]} + \overrightarrow{[b, c]} = \overrightarrow{[a, c]}$.

(61) $\langle a, a \oplus b\rangle \equiv \langle a \oplus b, b\rangle$.

(62) $\overrightarrow{[a, a \oplus b]} + \overrightarrow{[a, a \oplus b]} = \overrightarrow{[a, b]}$.

(63) $(u + v) + w = u + (v + w)$.

(64) $u + \mathrm{I}_M = u$.

(65) There exists v such that $u + v = \mathrm{I}_M$.

(66) $u + v = v + u$.

(67) If $u + v = u + w$, then $v = w$.

Let us consider M, u. The functor $-u$ yields a vector of M and is defined by:

$u + (-u) = \mathrm{I}_M$.

We now state a proposition

(68) $u + (-u) = \mathrm{I}_M$.

In the sequel X denotes an element of $2^{[\text{the points of } M, \text{the points of } M]}$. Let us consider M. The functor setvect M yields a set and is defined as follows:

setvect $M = \{X : X \text{ is a vector of } M\}$.

Next we state a proposition

(69) setvect $M = \{X : X \text{ is a vector of } M\}$.

In the sequel x is arbitrary. One can prove the following two propositions:

(70) u is an element of $2^{[\text{the points of } M, \text{the points of } M]}$.

(71) x is a vector of M if and only if $x \in$ setvect M.

Let us consider M. Then setvect M is a non-empty set.

The following proposition is true

(72) x is a vector of M if and only if x is an element of setvect M.

In the sequel u_1, v_1, w_1, W, W_1, W_2, T will denote elements of setvect M. Let us consider M, u_1, v_1. The functor $u_1 + v_1$ yields an element of setvect M and is defined as follows:

for all u, v such that $u_1 = u$ and $v_1 = v$ holds $u_1 + v_1 = u + v$.

One can prove the following propositions:

(73) If $u_1 = u$ and $v_1 = v$, then $u_1 + v_1 = u + v$.

(74) $u_1 + v_1 = v_1 + u_1$.

(75) $(u_1 + v_1) + w_1 = u_1 + (v_1 + w_1)$.

Let us consider M. The functor addvect M yields a binary operation on setvect M and is defined as follows:

for all u_1, v_1 holds (addvect M)$(u_1, v_1) = u_1 + v_1$.

The following three propositions are true:

(76) (addvect M)$(u_1, v_1) = u_1 + v_1$.

(77) For every W there exists T such that $W + T = \mathrm{I}_M$.

(78) For all W, W_1, W_2 such that $W + W_1 = \mathrm{I}_M$ and $W + W_2 = \mathrm{I}_M$ holds $W_1 = W_2$.

Let us consider M. The functor complvect M yielding a unary operation on setvect M, is defined by:

for every W holds $W +$ (complvect M)$(W) = \mathrm{I}_M$.

One can prove the following proposition

(79) $W +$ (complvect M)$(W) = \mathrm{I}_M$.

Let us consider M. The functor zerovect M yields an element of setvect M and is defined as follows:

zerovect $M = \mathrm{I}_M$.

The following proposition is true

(80) zerovect $M = \mathrm{I}_M$.

Let us consider M. The functor vectgroup M yielding a group structure, is defined by:

vectgroup $M = \langle$setvect M, addvect M, complvect M, zerovect $M\rangle$.

Next we state several propositions:

(81) vectgroup $M = \langle$setvect M, addvect M, complvect M, zerovect $M\rangle$.

(82) The carrier of vectgroup M = setvect M.

(83) The addition of vectgroup M = addvect M.

(84) The reverse-map of vectgroup M = complvect M.

(85) The zero of vectgroup M = zerovect M.

(86) vectgroup M is an Abelian group.

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Received November 26, 1989

FORMALIZED MATHEMATICS
Vol.1, No.**3**, May–August 1990
Université Catholique de Louvain

The Fundamental Logic Structure in Quantum Mechanics

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Summary. In this article we present the logical structure given by four axioms of Mackey [3] in the set of propositions of Quantum Mechanics. The equivalence relation (PropRel(Q)) in the set of propositions (Prop Q) for given Quantum Mechanics Q is considered. The main text for this article is [6] where the structure of quotient space and the properties of equivalence relations, classes and partitions are studied.

MML Identifier: `QMAX_1`.

The articles [10], [1], [4], [2], [9], [8], [7], [5], and [6] provide the notation and terminology for this paper. In the sequel x will be arbitrary, X will be a non-empty set, and X_1 will be a set. Let us consider X, and let S be a σ-field of subsets of X. The functor probabilities S yields a non-empty set and is defined by:

$x \in$ probabilities S if and only if x is a probability on S.

We now state a proposition

(1) For every σ-field S of subsets of X holds $x \in$ probabilities S if and only if x is a probability on S.

We consider quantum mechanics structures which are systems

$\langle$ observables, states, a probability $\rangle$

where the observables, the states are non-empty sets and the probability is a function from $[\!:$ the observables, the states $:\!]$ into probabilities the Borel sets. In the sequel Q denotes a quantum mechanics structure. We now define two new functors. Let us consider Q. The functor Obs Q yields a non-empty set and is defined by:

Obs Q = the observables of Q.

The functor Sts Q yields a non-empty set and is defined by:

Sts Q = the states of Q.

The following propositions are true:

ISSN 0777–4028

(2) Obs Q = the observables of Q.

(3) Sts Q = the states of Q.

We adopt the following convention: A_1, A_2 will denote elements of Obs Q, s, s_1, s_2 will denote elements of Sts Q, and E will denote an event of the Borel sets. Let us consider Q, A_1, s. The functor $\mathrm{Meas}(A_1, s)$ yielding a probability on the Borel sets, is defined as follows:

$\mathrm{Meas}(A_1, s) =$ (the probability of Q)($\langle A_1, s\rangle$).

One can prove the following proposition

(4) $\mathrm{Meas}(A_1, s) =$ (the probability of Q)($\langle A_1, s\rangle$).

A quantum mechanics structure is said to be a quantum mechanics if:

(i) for all elements A_1, A_2 of Obs it such that for every element s of Sts it holds $\mathrm{Meas}(A_1, s) = \mathrm{Meas}(A_2, s)$ holds $A_1 = A_2$,

(ii) for all elements s_1, s_2 of Sts it such that for every element A of Obs it holds $\mathrm{Meas}(A, s_1) = \mathrm{Meas}(A, s_2)$ holds $s_1 = s_2$,

(iii) for every elements s_1, s_2 of Sts it there exists an element s of Sts it such that for every element A of Obs it and for every E there exists a real number t such that $0 \leq t$ and $t \leq 1$ and $\mathrm{Meas}(A, s)(E) = t \cdot \mathrm{Meas}(A, s_1)(E) + (1 - t) \cdot \mathrm{Meas}(A, s_2)(E)$.

Next we state a proposition

(5) Q is a quantum mechanics if and only if the following conditions are satisfied:

(i) for all A_1, A_2 such that for every s holds $\mathrm{Meas}(A_1, s) = \mathrm{Meas}(A_2, s)$ holds $A_1 = A_2$,

(ii) for all s_1, s_2 such that for every A_1 holds $\mathrm{Meas}(A_1, s_1) = \mathrm{Meas}(A_1, s_2)$ holds $s_1 = s_2$,

(iii) for every s_1, s_2 there exists s such that for every A_1, E there exists a real number t such that $0 \leq t$ and $t \leq 1$ and $\mathrm{Meas}(A_1, s)(E) = t \cdot \mathrm{Meas}(A_1, s_1)(E) + (1 - t) \cdot \mathrm{Meas}(A_1, s_2)(E)$.

We follow the rules: Q denotes a quantum mechanics, A, A_1, A_2 denote elements of Obs Q, and s, s_1, s_2 denote elements of Sts Q. We now state three propositions:

(6) If for every s holds $\mathrm{Meas}(A_1, s) = \mathrm{Meas}(A_2, s)$, then $A_1 = A_2$.

(7) If for every A holds $\mathrm{Meas}(A, s_1) = \mathrm{Meas}(A, s_2)$, then $s_1 = s_2$.

(8) For every s_1, s_2 there exists s such that for every A, E there exists a real number t such that $0 \leq t$ and $t \leq 1$ and $\mathrm{Meas}(A, s)(E) = t \cdot \mathrm{Meas}(A, s_1)(E) + (1 - t) \cdot \mathrm{Meas}(A, s_2)(E)$.

We consider POI structures which are systems

$\langle$ a carrier, an ordering, an involution $\rangle$

where the carrier is a set, the ordering is a relation on the carrier, and the involution is a function from the carrier into the carrier. In the sequel x_1 will denote an element of X_1, Ord will denote a relation on X_1, and Inv will denote a function from X_1 into X_1. Let us consider X_1. A POI structure is said to be a poset with involution over X_1 if:

the carrier of it $= X_1$.

One can prove the following proposition

(9) For every poset W with involution over X_1 holds the carrier of $W = X_1$.

Let us consider X_1, Ord, Inv. The functor $\mathrm{LOG}(Ord, Inv)$ yielding a poset with involution over X_1, is defined by:

$\mathrm{LOG}(Ord, Inv) = \langle X_1, Ord, Inv \rangle$.

Next we state a proposition

(10) $\mathrm{LOG}(Ord, Inv) = \langle X_1, Ord, Inv \rangle$.

Let us consider X_1, Inv. We say that Inv is an involution in X_1 if and only if:

$Inv(Inv(x_1)) = x_1$.

We now state a proposition

(11) Inv is an involution in X_1 if and only if for every x_1 holds $Inv(Inv(x_1)) = x_1$.

Let us consider X_1, and let W be a poset with involution over X_1. We say that W is a quantum logic on X_1 if and only if:

there exists a relation Ord on X_1 and there exists a function Inv from X_1 into X_1 such that $W = \mathrm{LOG}(Ord, Inv)$ and Ord partially orders X_1 and Inv is an involution in X_1 and for all elements x, y of X_1 such that $\langle x, y \rangle \in Ord$ holds $\langle Inv(y), Inv(x) \rangle \in Ord$.

Next we state a proposition

(12) Let W be a poset with involution over X_1. Then W is a quantum logic on X_1 if and only if there exists a relation Ord on X_1 and there exists a function Inv from X_1 into X_1 such that $W = \mathrm{LOG}(Ord, Inv)$ and Ord partially orders X_1 and Inv is an involution in X_1 and for all elements x, y of X_1 such that $\langle x, y \rangle \in Ord$ holds $\langle Inv(y), Inv(x) \rangle \in Ord$.

Let us consider Q. The functor $\mathrm{Prop}\, Q$ yielding a non-empty set, is defined by:

$\mathrm{Prop}\, Q = [\!:\mathrm{Obs}\, Q, \text{the Borel sets}:\!]$.

The following proposition is true

(13) $\mathrm{Prop}\, Q = [\!:\mathrm{Obs}\, Q, \text{the Borel sets}:\!]$.

In the sequel p, q, r, p_1, q_1 are elements of $\mathrm{Prop}\, Q$. Let us consider Q, p. Then $p_{\mathbf{1}}$ is an element of $\mathrm{Obs}\, Q$. Then $p_{\mathbf{2}}$ is an event of the Borel sets.

The following propositions are true:

(14) $p = \langle p_{\mathbf{1}}, p_{\mathbf{2}} \rangle$.

(15) $(E^{\mathrm{c}})^{\mathrm{c}} = E$.

(16) For every E such that $E = {p_{\mathbf{2}}}^{\mathrm{c}}$ holds $\mathrm{Meas}(p_{\mathbf{1}}, s)(p_{\mathbf{2}}) = 1 - \mathrm{Meas}(p_{\mathbf{1}}, s)(E)$.

Let us consider Q, p. The functor $\neg p$ yields an element of $\mathrm{Prop}\, Q$ and is defined as follows:

$\neg p = \langle p_{\mathbf{1}}, {p_{\mathbf{2}}}^{\mathrm{c}} \rangle$.

The following proposition is true

(17) $\neg p = \langle p_{\mathbf{1}}, p_{\mathbf{2}}{}^{c} \rangle$.

Let us consider Q, p, q. The predicate $p \vdash q$ is defined by:
for every s holds $\mathrm{Meas}(p_{\mathbf{1}}, s)(p_{\mathbf{2}}) \leq \mathrm{Meas}(q_{\mathbf{1}}, s)(q_{\mathbf{2}})$.

We now state a proposition

(18) $p \vdash q$ if and only if for every s holds $\mathrm{Meas}(p_{\mathbf{1}}, s)(p_{\mathbf{2}}) \leq \mathrm{Meas}(q_{\mathbf{1}}, s)(q_{\mathbf{2}})$.

Let us consider Q, p, q. The predicate $p \equiv q$ is defined as follows:
$p \vdash q$ and $q \vdash p$.

One can prove the following propositions:

(19) $p \equiv q$ if and only if $p \vdash q$ and $q \vdash p$.

(20) $p \equiv q$ if and only if for every s holds $\mathrm{Meas}(p_{\mathbf{1}}, s)(p_{\mathbf{2}}) = \mathrm{Meas}(q_{\mathbf{1}}, s)(q_{\mathbf{2}})$.

(21) $p \vdash p$.

(22) If $p \vdash q$ and $q \vdash r$, then $p \vdash r$.

(23) $p \equiv p$.

(24) If $p \equiv q$, then $q \equiv p$.

(25) If $p \equiv q$ and $q \equiv r$, then $p \equiv r$.

(26) $(\neg p)_{\mathbf{1}} = p_{\mathbf{1}}$ and $(\neg p)_{\mathbf{2}} = p_{\mathbf{2}}{}^{c}$.

(27) $\neg(\neg p) = p$.

(28) If $p \vdash q$, then $\neg q \vdash \neg p$.

Let us consider Q. The functor $\mathrm{PropRel}\, Q$ yields an equivalence relation of $\mathrm{Prop}\, Q$ and is defined as follows:
$\langle p, q \rangle \in \mathrm{PropRel}\, Q$ if and only if $p \equiv q$.

We now state a proposition

(29) $\langle p, q \rangle \in \mathrm{PropRel}\, Q$ if and only if $p \equiv q$.

In the sequel B, C will denote subsets of $\mathrm{Prop}\, Q$. Next we state a proposition

(30) For all B, C such that $B \in \mathrm{Classes}(\mathrm{PropRel}\, Q)$ and $C \in \mathrm{Classes}(\mathrm{PropRel}\, Q)$
for all elements a, b, c, d of $\mathrm{Prop}\, Q$ such that $a \in B$ and $b \in B$ and $c \in C$ and $d \in C$ and $a \vdash c$ holds $b \vdash d$.

Let us consider Q. The functor $\mathrm{OrdRel}\, Q$ yielding a relation on $\mathrm{Classes}(\mathrm{PropRel}\, Q)$,
is defined as follows:
$\langle B, C \rangle \in \mathrm{OrdRel}\, Q$ if and only if $B \in \mathrm{Classes}(\mathrm{PropRel}\, Q)$ and $C \in \mathrm{Classes}(\mathrm{PropRel}\, Q)$
and for all p, q such that $p \in B$ and $q \in C$ holds $p \vdash q$.

Next we state four propositions:

(31) $\langle B, C \rangle \in \mathrm{OrdRel}\, Q$ if and only if $B \in \mathrm{Classes}(\mathrm{PropRel}\, Q)$ and $C \in \mathrm{Classes}(\mathrm{PropRel}\, Q)$ and for all p, q such that $p \in B$ and $q \in C$ holds $p \vdash q$.

(32) $p \vdash q$ if and only if $\langle [p]_{\mathrm{PropRel}\, Q}, [q]_{\mathrm{PropRel}\, Q} \rangle \in \mathrm{OrdRel}\, Q$.

(33) For all B, C such that $B \in \mathrm{Classes}(\mathrm{PropRel}\, Q)$ and $C \in \mathrm{Classes}(\mathrm{PropRel}\, Q)$ for all p_1, q_1 such that $p_1 \in B$ and $q_1 \in B$ and $\neg p_1 \in C$ holds $\neg q_1 \in C$.

(34) For all B, C such that $B \in \mathrm{Classes}(\mathrm{PropRel}\, Q)$ and $C \in \mathrm{Classes}(\mathrm{PropRel}\, Q)$ for all p, q such that $\neg p \in C$ and $\neg q \in C$ and $p \in B$ holds $q \in B$.

Let us consider Q. The functor $\mathrm{InvRel}\, Q$ yielding a function from $\mathrm{Classes}(\mathrm{PropRel}\, Q)$ into $\mathrm{Classes}(\mathrm{PropRel}\, Q)$, is defined by:

$(\mathrm{InvRel}\, Q)([p]_{\mathrm{PropRel}\, Q}) = [\neg p]_{\mathrm{PropRel}\, Q}$.

One can prove the following two propositions:

(35) $(\mathrm{InvRel}\, Q)([p]_{\mathrm{PropRel}\, Q}) = [\neg p]_{\mathrm{PropRel}\, Q}$.

(36) For every Q holds $\mathrm{LOG}(\mathrm{OrdRel}\, Q, \mathrm{InvRel}\, Q)$ is a quantum logic on $\mathrm{Classes}(\mathrm{PropRel}\, Q)$.

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Received December 18, 1989

FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Function Domains and Frænkel Operator

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Summary. We deal with a non–empty set of functions and a non–empty set of functions from a set A to a non–empty set B. In the case when B is a non–empty set, B^A is redefined. It yields a non–empty set of functions from A to B. An element of such a set is redefined as a function from A to B. Some theorems concerning these concepts are proved, as well as a number of schemes dealing with infinity and the Axiom of Choice. The article contains a number of schemes allowing for simple logical transformations related to terms constructed with the Frænkel Operator.

MML Identifier: `FRAENKEL`.

The articles [5], [4], [6], [1], [2], and [3] provide the notation and terminology for this paper. In the sequel A, B will be non-empty sets. We now state a proposition

(1) For arbitrary x holds $\{x\}$ is a non-empty set.

In the article we present several logical schemes. The scheme *Fraenkel5'* deals with a non-empty set $\mathcal{A}$, a unary functor $\mathcal{F}$, and two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

$\{\mathcal{F}(v') : \mathcal{P}[v']\} \subseteq \{\mathcal{F}(u') : \mathcal{Q}[u']\}$

provided the parameters enjoy the following property:

- for every element v of $\mathcal{A}$ such that $\mathcal{P}[v]$ holds $\mathcal{Q}[v]$.

The scheme *Fraenkel5"* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, and two binary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

$\{\mathcal{F}(u_1, v_1) : \mathcal{P}[u_1, v_1]\} \subseteq \{\mathcal{F}(u_2, v_2) : \mathcal{Q}[u_2, v_2]\}$

provided the following condition is fulfilled:

- for every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ such that $\mathcal{P}[u, v]$ holds $\mathcal{Q}[u, v]$.

The scheme *Fraenkel6'* deals with a non-empty set $\mathcal{A}$, a unary functor $\mathcal{F}$, and two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

[1]Supported by RPBP III.24 C1

ISSN 0777–4028

$\{\mathcal{F}(v_1) : \mathcal{P}[v_1]\} = \{\mathcal{F}(v_2) : \mathcal{Q}[v_2]\}$
provided the following requirement is fulfilled:

- for every element v of $\mathcal{A}$ holds $\mathcal{P}[v]$ if and only if $\mathcal{Q}[v]$.

The scheme *Fraenkel6"* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, and two binary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:
$\{\mathcal{F}(u_1, v_1) : \mathcal{P}[u_1, v_1]\} = \{\mathcal{F}(u_2, v_2) : \mathcal{Q}[u_2, v_2]\}$
provided the parameters fulfill the following requirement:

- for every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{P}[u, v]$ if and only if $\mathcal{Q}[u, v]$.

The scheme *FraenkelF'* concerns a non-empty set $\mathcal{A}$, a unary functor $\mathcal{F}$, a unary functor $\mathcal{G}$, and a unary predicate $\mathcal{P}$, and states that:
$\{\mathcal{F}(v_1) : \mathcal{P}[v_1]\} = \{\mathcal{G}(v_2) : \mathcal{P}[v_2]\}$
provided the following requirement is met:

- for every element v of $\mathcal{A}$ holds $\mathcal{F}(v) = \mathcal{G}(v)$.

The scheme *FraenkelF'R* concerns a non-empty set $\mathcal{A}$, a unary functor $\mathcal{F}$, a unary functor $\mathcal{G}$, and a unary predicate $\mathcal{P}$, and states that:
$\{\mathcal{F}(v_1) : \mathcal{P}[v_1]\} = \{\mathcal{G}(v_2) : \mathcal{P}[v_2]\}$
provided the parameters fulfill the following condition:

- for every element v of $\mathcal{A}$ such that $\mathcal{P}[v]$ holds $\mathcal{F}(v) = \mathcal{G}(v)$.

The scheme *FraenkelF"* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, a binary functor $\mathcal{G}$, and a binary predicate $\mathcal{P}$, and states that:
$\{\mathcal{F}(u_1, v_1) : \mathcal{P}[u_1, v_1]\} = \{\mathcal{G}(u_2, v_2) : \mathcal{P}[u_2, v_2]\}$
provided the parameters meet the following requirement:

- for every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{F}(u, v) = \mathcal{G}(u, v)$.

The scheme *FraenkelF6"C* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, and a binary predicate $\mathcal{P}$, and states that:
$\{\mathcal{F}(u_1, v_1) : \mathcal{P}[u_1, v_1]\} = \{\mathcal{F}(v_2, u_2) : \mathcal{P}[u_2, v_2]\}$
provided the following requirement is met:

- for every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{F}(u, v) = \mathcal{F}(v, u)$.

The scheme *FraenkelF6"* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, and two binary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:
$\{\mathcal{F}(u_1, v_1) : \mathcal{P}[u_1, v_1]\} = \{\mathcal{F}(v_2, u_2) : \mathcal{Q}[u_2, v_2]\}$
provided the parameters meet the following requirements:

- for every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{P}[u, v]$ if and only if $\mathcal{Q}[u, v]$,
- for every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{F}(u, v) = \mathcal{F}(v, u)$.

The following propositions are true:

(2) For all non-empty sets A, B and for every function F from A into B and for every set X and for every element x of A such that $x \in X$ holds $(F \restriction X)(x) = F(x)$.

(3) For all non-empty sets A, B and for all functions F, G from A into B and for every set X such that $F \restriction X = G \restriction X$ for every element x of A such that $x \in X$ holds $F(x) = G(x)$.

(4) For every function f from A into B holds $f \in B^A$.

(5) For all sets A, B holds $B^A \subseteq 2^{[: A, B :]}$.

(6) For all sets X, Y such that $Y^X \neq \emptyset$ and $X \subseteq A$ and $Y \subseteq B$ for every element f of Y^X holds f is a partial function from A to B.

Now we present a number of schemes. The scheme *RelevantArgs* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a set $\mathcal{C}$, a function $\mathcal{D}$ from $\mathcal{A}$ into $\mathcal{B}$, a function $\mathcal{E}$ from $\mathcal{A}$ into $\mathcal{B}$, and two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

$\{\mathcal{D}(u') : \mathcal{P}[u'] \wedge u' \in \mathcal{C}\} = \{\mathcal{E}(v') : \mathcal{Q}[v'] \wedge v' \in \mathcal{C}\}$

provided the following requirements are met:

- $\mathcal{D} \restriction \mathcal{C} = \mathcal{E} \restriction \mathcal{C}$,
- for every element u of $\mathcal{A}$ such that $u \in \mathcal{C}$ holds $\mathcal{P}[u]$ if and only if $\mathcal{Q}[u]$.

The scheme *Fr_Set0* deals with a non-empty set $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\{xx : \mathcal{P}[xx]\} \subseteq \mathcal{A}$

for all values of the parameters.

The scheme *Gen1"* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

for every element s of $\mathcal{A}$ and for every element t of $\mathcal{B}$ such that $\mathcal{P}[s,t]$ holds $\mathcal{Q}[\mathcal{F}(s,t)]$

provided the parameters meet the following requirement:

- for arbitrary s_t such that $s_t \in \{\mathcal{F}(s_1,t_1) : \mathcal{P}[s_1,t_1]\}$ holds $\mathcal{Q}[s_t]$.

The scheme *Gen1"A* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

for arbitrary s_t such that $s_t \in \{\mathcal{F}(s_1,t_1) : \mathcal{P}[s_1,t_1]\}$ holds $\mathcal{Q}[s_t]$

provided the following requirement is met:

- for every element s of $\mathcal{A}$ and for every element t of $\mathcal{B}$ such that $\mathcal{P}[s,t]$ holds $\mathcal{Q}[\mathcal{F}(s,t)]$.

The scheme *Gen2"* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a non-empty set $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

$\{s_t : s_t \in \{\mathcal{F}(s_1,t_1) : \mathcal{P}[s_1,t_1]\} \wedge \mathcal{Q}[s_t]\} = \{\mathcal{F}(s_2,t_2) : \mathcal{P}[s_2,t_2] \wedge \mathcal{Q}[\mathcal{F}(s_2,t_2)]\}$

for all values of the parameters.

The scheme *Gen3'* concerns a non-empty set $\mathcal{A}$, a unary functor $\mathcal{F}$, and two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

$\{\mathcal{F}(s) : s \in \{s_1 : \mathcal{Q}[s_1]\} \wedge \mathcal{P}[s]\} = \{\mathcal{F}(s_2) : \mathcal{Q}[s_2] \wedge \mathcal{P}[s_2]\}$

for all values of the parameters.

The scheme *Gen3"* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(s,t) : s \in \{s_1 : \mathcal{Q}[s_1]\} \wedge \mathcal{P}[s,t]\} = \{\mathcal{F}(s_2,t_2) : \mathcal{Q}[s_2] \wedge \mathcal{P}[s_2,t_2]\}$

for all values of the parameters.

The scheme *Gen4"* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, and two binary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

$\{\mathcal{F}(s,t) : \mathcal{P}[s,t]\} \subseteq \{\mathcal{F}(s_1,t_1) : \mathcal{Q}[s_1,t_1]\}$

provided the following condition is satisfied:

- for every element s of $\mathcal{A}$ and for every element t of $\mathcal{B}$ such that $\mathcal{P}[s,t]$ there exists an element s' of $\mathcal{A}$ such that $\mathcal{Q}[s',t]$ and $\mathcal{F}(s,t) = \mathcal{F}(s',t)$.

The scheme *FrSet_1* concerns a non-empty set $\mathcal{A}$, a set $\mathcal{B}$, a unary functor $\mathcal{F}$, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(y) : \mathcal{F}(y) \in \mathcal{B} \wedge \mathcal{P}[y]\} \subseteq \mathcal{B}$

for all values of the parameters.

The scheme *FrSet_2* deals with a non-empty set $\mathcal{A}$, a set $\mathcal{B}$, a unary functor $\mathcal{F}$, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(y) : \mathcal{P}[y] \wedge \mathcal{F}(y) \notin \mathcal{B}\}$ misses $\mathcal{B}$

for all values of the parameters.

The scheme *FrEqua1* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, an element $\mathcal{C}$ of $\mathcal{B}$, and two binary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

$\{\mathcal{F}(s,t) : \mathcal{Q}[s,t]\} = \{\mathcal{F}(s',\mathcal{C}) : \mathcal{P}[s',\mathcal{C}]\}$

provided the parameters meet the following requirement:

- for every element s of $\mathcal{A}$ and for every element t of $\mathcal{B}$ holds $\mathcal{Q}[s,t]$ if and only if $t = \mathcal{C}$ and $\mathcal{P}[s,t]$.

The scheme *FrEqua2* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a binary functor $\mathcal{F}$, an element $\mathcal{C}$ of $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(s,t) : t = \mathcal{C} \wedge \mathcal{P}[s,t]\} = \{\mathcal{F}(s',\mathcal{C}) : \mathcal{P}[s',\mathcal{C}]\}$

for all values of the parameters.

A non-empty set is said to be a non-empty set of functions if:

for every element x of it holds x is a function.

Next we state two propositions:

(7) A is a non-empty set of functions if and only if for every element x of A holds x is a function.

(8) For every function f holds $\{f\}$ is a non-empty set of functions.

Let A be a set, and let B be a non-empty set. A non-empty set of functions is called a non-empty set of functions from A to B if:

for every element x of it holds x is a function from A into B.

Next we state three propositions:

(9) For every set A and for every non-empty set B and for every non-empty set C of functions holds C is a non-empty set of functions from A to B if and only if for every element x of C holds x is a function from A into B.

(10) For every function f from A into B holds $\{f\}$ is a non-empty set of functions from A to B.

(11) For every set A and for every non-empty set B holds B^A is a non-empty set of functions from A to B.

Let A be a set, and let B be a non-empty set. Then B^A is a non-empty set of functions from A to B. Let F be a non-empty set of functions from A to B. We see that it makes sense to consider the following mode for restricted scopes of arguments. Then all the objests of the mode element of F are a function from A into B.

In the sequel *phi* will be an element of B^A. The following propositions are true:

(12) For every function f from A into B holds f is an element of B^A.

(13) For every element f of B^A holds $\mathrm{dom}\, f = A$ and $\mathrm{rng}\, f \subseteq B$.

(14) For all sets X, Y such that $Y^X \neq \emptyset$ and $X \subseteq A$ and $Y \subseteq B$ for every element f of Y^X there exists an element *phi* of B^A such that $phi \restriction X = f$.

(15) For every set X and for every *phi* holds $phi \restriction X = phi \restriction (A \cap X)$.

Now we present four schemes. The scheme *FraenkelFin* deals with a non-empty set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ and states that:

$\{\mathcal{F}(w) : w \in \mathcal{B}\}$ is finite

provided the parameters meet the following requirement:

- $\mathcal{B}$ is finite.

The scheme *CartFin* deals with a non-empty set $\mathcal{A}$, a set $\mathcal{B}$, a set $\mathcal{C}$, and a binary functor $\mathcal{F}$ and states that:

$\{\mathcal{F}(u', v') : u' \in \mathcal{B} \wedge v' \in \mathcal{C}\}$ is finite

provided the parameters fulfill the following requirements:

- $\mathcal{B}$ is finite,
- $\mathcal{C}$ is finite.

The scheme *Finiteness* deals with a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathrm{Fin}\, \mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

for every element x of $\mathcal{A}$ such that $x \in \mathcal{B}$ there exists an element y of $\mathcal{A}$ such that $y \in \mathcal{B}$ and $\mathcal{P}[y, x]$ and for every element z of $\mathcal{A}$ such that $z \in \mathcal{B}$ and $\mathcal{P}[z, y]$ holds $\mathcal{P}[y, z]$

provided the following requirements are fulfilled:

- for every element x of $\mathcal{A}$ holds $\mathcal{P}[x, x]$,
- for all elements x, y, z of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$.

The scheme *Fin_Im* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\mathrm{Fin}\, \mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists an element c_1 of $\mathrm{Fin}\, \mathcal{A}$ such that for every element t of $\mathcal{A}$ holds $t \in c_1$ if and only if there exists an element t' of $\mathcal{B}$ such that $t' \in \mathcal{C}$ and $t = \mathcal{F}(t')$ and $\mathcal{P}[t, t']$

for all values of the parameters.

The following proposition is true

(16) For all sets A, B such that A is finite and B is finite holds B^A is finite.

Now we present three schemes. The scheme *ImFin* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a set $\mathcal{C}$, a set $\mathcal{D}$, and a unary functor $\mathcal{F}$ and states that:

$\{\mathcal{F}(phi') : phi' \circ \mathcal{C} \subseteq \mathcal{D}\}$ is finite

provided the parameters fulfill the following conditions:

- $\mathcal{C}$ is finite,
- $\mathcal{D}$ is finite,
- for all elements phi, psi of $\mathcal{B}^{\mathcal{A}}$ such that $phi \restriction \mathcal{C} = psi \restriction \mathcal{C}$ holds $\mathcal{F}(phi) = \mathcal{F}(psi)$.

The scheme *FunctChoice* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\operatorname{Fin}\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a function ff from $\mathcal{A}$ into $\mathcal{B}$ such that for every element t of $\mathcal{A}$ such that $t \in \mathcal{C}$ holds $\mathcal{P}[t, ff(t)]$

provided the parameters fulfill the following condition:

- for every element t of $\mathcal{A}$ such that $t \in \mathcal{C}$ there exists an element ff of $\mathcal{B}$ such that $\mathcal{P}[t, ff]$.

The scheme *FuncsChoice* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\operatorname{Fin}\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists an element ff of $\mathcal{B}^{\mathcal{A}}$ such that for every element t of $\mathcal{A}$ such that $t \in \mathcal{C}$ holds $\mathcal{P}[t, ff(t)]$

provided the parameters meet the following requirement:

- for every element t of $\mathcal{A}$ such that $t \in \mathcal{C}$ there exists an element ff of $\mathcal{B}$ such that $\mathcal{P}[t, ff]$.

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Received February 7, 1990

FORMALIZED MATHEMATICS
Vol.1, No.**3**, May–August 1990
Université Catholique de Louvain

Integers

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Summary. In the article the following concepts were introduced: the set of integers ($\mathbb{Z}$) and its elements (integers), congruences ($i_1 \equiv i_2 (\bmod\, i_3)$), the ceiling and floor functors ($\lceil x \rceil$ and $\lfloor x \rfloor$), also the fraction part of a real number (frac), the integer division ($\div$) and remainder of integer division (mod). The following schemes were also included: the separation scheme (*SepInt*), the schemes of integer induction (*Int_Ind_Down*, *Int_Ind_Up*, *Int_Ind_Full*), the minimum (*Int_Min*) and maximum (*Int_Max*) schemes (the existence of minimum and maximum integers enjoying a given property).

MML Identifier: `INT_1`.

The papers [2], and [1] provide the notation and terminology for this paper. For simplicity we follow a convention: x is arbitrary, k, n_1, n_2 denote natural numbers, r, r_1, r_2 denote real numbers, and D denotes a non-empty set. The following propositions are true:

(1) $(r + r_1) - r_2 = (r - r_2) + r_1$.

(2) $(-r_1) + r_2 = r_2 - r_1$.

(3) $r_1 = ((-r_2) + r_1) + r_2$ and $r_1 = r_2 + ((-r_2) + r_1)$ and $r_1 = r_2 + (r_1 - r_2)$ and $r_1 = (r_2 + r_1) - r_2$.

(4) $(r_1 - r_2) + r_2 = r_1$ and $(r_1 + r_2) - r_2 = r_1$.

(5) $r_1 \leq r_2$ if and only if $r_1 < r_2$ or $r_1 = r_2$.

The non-empty set $\mathbb{Z}$ is defined by:
$x \in \mathbb{Z}$ if and only if there exists k such that $x = k$ or $x = -k$.

One can prove the following proposition

(6) For every x holds $x \in D$ if and only if there exists k such that $x = k$ or $x = -k$ if and only if $D = \mathbb{Z}$.

A real number is called an integer if:
it is an element of $\mathbb{Z}$.

The following propositions are true:

ISSN 0777–4028

(7) r is an integer if and only if r is an element of $\mathbb{Z}$.

(8) r is an integer if and only if there exists k such that $r = k$ or $r = -k$.

(9) If x is a natural number, then x is an integer.

(10) 0 is an integer and 1 is an integer.

(11) If $x \in \mathbb{Z}$, then $x \in \mathbb{R}$.

(12) x is an integer if and only if $x \in \mathbb{Z}$.

(13) x is an integer if and only if x is an element of $\mathbb{Z}$.

(14) $\mathbb{N} \subseteq \mathbb{Z}$.

(15) $\mathbb{Z} \subseteq \mathbb{R}$.

In the sequel i_0, i_1, i_2, i_3, i_4, i_5 are integers. Let i_1, i_2 be integers. Then $i_1 + i_2$ is an integer. Then $i_1 \cdot i_2$ is an integer.

Let i_0 be an integer. Then $-i_0$ is an integer.

Let i_1, i_2 be integers. Then $i_1 - i_2$ is an integer.

Let n be a natural number. Then $-n$ is an integer. Let i_1 be an integer. Then $n + i_1$ is an integer. Then $n \cdot i_1$ is an integer. Then $n - i_1$ is an integer.

Let i_1 be an integer, and let n be a natural number. Then $i_1 + n$ is an integer. Then $i_1 \cdot n$ is an integer. Then $i_1 - n$ is an integer.

Let us consider n_1, n_2. Then $n_1 - n_2$ is an integer.

We now state a number of propositions:

(16) If $0 \leq i_0$, then i_0 is a natural number.

(17) If r is an integer, then $r + 1$ is an integer and $r - 1$ is an integer.

(18) If $i_2 \leq i_1$, then $i_1 - i_2$ is a natural number.

(19) If $i_1 + k = i_2$ or $k + i_1 = i_2$, then $i_1 \leq i_2$.

(20) If $i_0 < i_1$, then $i_0 + 1 \leq i_1$ and $1 + i_0 \leq i_1$.

(21) If $i_1 < 0$, then $i_1 \leq -1$.

(22) $i_1 \cdot i_2 = 1$ if and only if $i_1 = 1$ and $i_2 = 1$ or $i_1 = -1$ and $i_2 = -1$.

(23) $i_1 \cdot i_2 = -1$ if and only if $i_1 = -1$ and $i_2 = 1$ or $i_1 = 1$ and $i_2 = -1$.

(24) If $i_0 \neq 0$, then $i_1 \neq i_1 + i_0$.

(25) $i_1 < i_1 + 1$.

(26) $i_1 - 1 < i_1$.

(27) For no i_0 holds for every i_1 holds $i_0 < i_1$.

(28) For no i_0 holds for every i_1 holds $i_1 < i_0$.

In the article we present several logical schemes. The scheme *SepInt* deals with a unary predicate $\mathcal{P}$, and states that:

there exists a subset X of $\mathbb{Z}$ such that for every integer x holds $x \in X$ if and only if $\mathcal{P}[x]$

for all values of the parameter.

The scheme *Int_Ind_Up* concerns an integer $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

for every i_0 such that $\mathcal{A} \leq i_0$ holds $\mathcal{P}[i_0]$

provided the following conditions are fulfilled:

- $\mathcal{P}[\mathcal{A}]$,
- for every i_2 such that $\mathcal{A} \leq i_2$ holds if $\mathcal{P}[i_2]$, then $\mathcal{P}[i_2 + 1]$.

The scheme *Int_Ind_Down* deals with an integer $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

for every i_0 such that $i_0 \leq \mathcal{A}$ holds $\mathcal{P}[i_0]$

provided the parameters fulfill the following conditions:

- $\mathcal{P}[\mathcal{A}]$,
- for every i_2 such that $i_2 \leq \mathcal{A}$ holds if $\mathcal{P}[i_2]$, then $\mathcal{P}[i_2 - 1]$.

The scheme *Int_Ind_Full* deals with an integer $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

for every i_0 holds $\mathcal{P}[i_0]$

provided the following requirements are fulfilled:

- $\mathcal{P}[\mathcal{A}]$,
- for every i_2 such that $\mathcal{P}[i_2]$ holds $\mathcal{P}[i_2 - 1]$ and $\mathcal{P}[i_2 + 1]$.

The scheme *Int_Min* concerns an integer $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists i_0 such that $\mathcal{P}[i_0]$ and for every i_1 such that $\mathcal{P}[i_1]$ holds $i_0 \leq i_1$

provided the following conditions are satisfied:

- for every i_1 such that $\mathcal{P}[i_1]$ holds $\mathcal{A} \leq i_1$,
- there exists i_1 such that $\mathcal{P}[i_1]$.

The scheme *Int_Max* deals with an integer $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists i_0 such that $\mathcal{P}[i_0]$ and for every i_1 such that $\mathcal{P}[i_1]$ holds $i_1 \leq i_0$

provided the parameters satisfy the following conditions:

- for every i_1 such that $\mathcal{P}[i_1]$ holds $i_1 \leq \mathcal{A}$,
- there exists i_1 such that $\mathcal{P}[i_1]$.

Let us consider r. Then $\operatorname{sgn} r$ is an integer.

We now state two propositions:

(29) $\operatorname{sgn} r = 1$ or $\operatorname{sgn} r = -1$ or $\operatorname{sgn} r = 0$.

(30) $|r| = r$ or $|r| = -r$.

Let us consider i_0. Then $|i_0|$ is an integer.

Let i_1, i_2, i_3 be integers. The predicate $i_1 \equiv i_2 (\operatorname{mod} i_3)$ is defined by:

there exists i_4 such that $i_3 \cdot i_4 = i_1 - i_2$.

We now state a number of propositions:

(31) $i_1 \equiv i_2 (\operatorname{mod} i_3)$ if and only if there exists an integer i_4 such that $i_3 \cdot i_4 = i_1 - i_2$.

(32) $i_1 \equiv i_1 (\operatorname{mod} i_2)$.

(33) If $i_2 = 0$, then $i_1 \equiv i_2 (\operatorname{mod} i_1)$ and $i_2 \equiv i_1 (\operatorname{mod} i_1)$.

(34) If $i_3 = 1$, then $i_1 \equiv i_2 (\operatorname{mod} i_3)$.

(35) If $i_1 \equiv i_2 (\operatorname{mod} i_3)$, then $i_2 \equiv i_1 (\operatorname{mod} i_3)$.

(36) If $i_1 \equiv i_2 (\operatorname{mod} i_5)$ and $i_2 \equiv i_3 (\operatorname{mod} i_5)$, then $i_1 \equiv i_3 (\operatorname{mod} i_5)$.

(37) If $i_1 \equiv i_2 (\operatorname{mod} i_5)$ and $i_3 \equiv i_4 (\operatorname{mod} i_5)$, then $i_1 + i_3 \equiv i_2 + i_4 (\operatorname{mod} i_5)$.

(38) If $i_1 \equiv i_2 (\operatorname{mod} i_5)$ and $i_3 \equiv i_4 (\operatorname{mod} i_5)$, then $i_1 - i_3 \equiv i_2 - i_4 (\operatorname{mod} i_5)$.

(39) If $i_1 \equiv i_2 (\operatorname{mod} i_5)$ and $i_3 \equiv i_4 (\operatorname{mod} i_5)$, then $i_1 \cdot i_3 \equiv i_2 \cdot i_4 (\operatorname{mod} i_5)$.

(40) $i_1 + i_2 \equiv i_3 (\operatorname{mod} i_5)$ if and only if $i_1 \equiv i_3 - i_2 (\operatorname{mod} i_5)$.

(41) If $i_4 \cdot i_5 = i_3$, then if $i_1 \equiv i_2 (\operatorname{mod} i_3)$, then $i_1 \equiv i_2 (\operatorname{mod} i_4)$.

(42) $i_1 \equiv i_2 (\operatorname{mod} i_5)$ if and only if $i_1 + i_5 \equiv i_2 (\operatorname{mod} i_5)$.

(43) $i_1 \equiv i_2 (\operatorname{mod} i_5)$ if and only if $i_1 - i_5 \equiv i_2 (\operatorname{mod} i_5)$.

(44) If $i_1 \leq r$ and $r - 1 < i_1$ and $i_2 \leq r$ and $r - 1 < i_2$, then $i_1 = i_2$.

(45) If $r \leq i_1$ and $i_1 < r + 1$ and $r \leq i_2$ and $i_2 < r + 1$, then $i_1 = i_2$.

Let us consider r. The functor $\lfloor r \rfloor$ yielding an integer, is defined as follows: $\lfloor r \rfloor \leq r$ and $r - 1 < \lfloor r \rfloor$.

The following propositions are true:

(46) $i_0 \leq r$ and $r - 1 < i_0$ if and only if $\lfloor r \rfloor = i_0$.

(47) $\lfloor r \rfloor = r$ if and only if r is an integer.

(48) $\lfloor r \rfloor < r$ if and only if r is not an integer.

(49) $\lfloor r \rfloor \leq r$.

(50) $\lfloor r \rfloor - 1 < r$ and $\lfloor r \rfloor < r + 1$.

(51) $\lfloor r \rfloor + i_0 = \lfloor r + i_0 \rfloor$.

(52) $r \leq \lfloor r \rfloor + 1$.

Let us consider r. The functor $\lceil r \rceil$ yields an integer and is defined as follows: $r \leq \lceil r \rceil$ and $\lceil r \rceil < r + 1$.

We now state a number of propositions:

(53) $r \leq i_0$ and $i_0 < r + 1$ if and only if $\lceil r \rceil = i_0$.

(54) $\lceil r \rceil = r$ if and only if r is an integer.

(55) $r < \lceil r \rceil$ if and only if r is not an integer.

(56) $r \leq \lceil r \rceil$.

(57) $r - 1 < \lceil r \rceil$ and $r < \lceil r \rceil + 1$.

(58) $\lceil r \rceil + i_0 = \lceil r + i_0 \rceil$.

(59) $\lfloor r \rfloor = \lceil r \rceil$ if and only if r is an integer.

(60) $\lfloor r \rfloor < \lceil r \rceil$ if and only if r is not an integer.

(61) $\lfloor r \rfloor \leq \lceil r \rceil$.

(62) $\lfloor \lceil r \rceil \rfloor = \lceil r \rceil$.

(63) $\lfloor \lfloor r \rfloor \rfloor = \lfloor r \rfloor$.

(64) $\lceil \lceil r \rceil \rceil = \lceil r \rceil$.

(65) $\lceil \lfloor r \rfloor \rceil = \lfloor r \rfloor$.

(66) $\lfloor r \rfloor = \lceil r \rceil$ if and only if $\lfloor r \rfloor + 1 \neq \lceil r \rceil$.

Let us consider r. The functor $\operatorname{frac} r$ yielding a real number, is defined by: $\operatorname{frac} r = r - \lfloor r \rfloor$.

One can prove the following propositions:

(67) $\operatorname{frac} r = r - \lfloor r \rfloor$.

(68) $r = \lfloor r \rfloor + \operatorname{frac} r$.

(69) $\operatorname{frac} r < 1$ and $0 \leq \operatorname{frac} r$.

(70) $\lfloor \operatorname{frac} r \rfloor = 0$.

(71) $\operatorname{frac} r = 0$ if and only if r is an integer.

(72) $0 < \operatorname{frac} r$ if and only if r is not an integer.

Let i_1, i_2 be integers. The functor $i_1 \div i_2$ yields an integer and is defined by:
$i_1 \div i_2 = \lfloor \frac{i_1}{i_2} \rfloor$.

One can prove the following proposition

(73) $i_1 \div i_2 = \lfloor \frac{i_1}{i_2} \rfloor$.

Let i_1, i_2 be integers. The functor $i_1 \bmod i_2$ yielding an integer, is defined as follows:
$i_1 \bmod i_2 = i_1 - (i_1 \div i_2) \cdot i_2$.

Next we state a proposition

(74) $i_1 \bmod i_2 = i_1 - (i_1 \div i_2) \cdot i_2$.

Let i_1, i_2 be integers. The predicate $i_1 \mid i_2$ is defined as follows:
there exists i_3 such that $i_2 = i_1 \cdot i_3$.

The following proposition is true

(75) $i_1 \mid i_2$ if and only if there exists i_3 such that $i_1 \cdot i_3 = i_2$.

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Received February 7, 1990

FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

The Complex Numbers

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Summary. We define the set $\mathbb{C}$ of complex numbers as the set of all ordered pairs $z = \langle a, b \rangle$ where a and b are real numbers and where addition and multiplication are defined. We define the real and imaginary parts of z and denote this by $a = \Re(z)$, $b = \Im(z)$. These definitions satisfy all the axioms for a field. $0_{\mathbb{C}} = 0 + 0i$ and $1_{\mathbb{C}} = 1 + 0i$ are identities for addition and multiplication respectively, and there are multiplicative inverses for each non zero element in $\mathbb{C}$. The difference and division of complex numbers are also defined. We do not interpret the set of all real numbers $\mathbb{R}$ as a subset of $\mathbb{C}$. From here on we do not abandon the ordered pair notation for complex numbers. For example: $i^2 = (0+1i)^2 = -1 + 0i \neq -1$. We conclude this article by introducing two operations on $\mathbb{C}$ which are not field operations. We define the absolute value of z denoted by $|z|$ and the conjugate of z denoted by z^*.

MML Identifier: `COMPLEX1`.

The articles [1], [3], [2], and [4] provide the notation and terminology for this paper. In the sequel a, b, a_1, b_1, a_2, b_2 denote real numbers. The following two propositions are true:

(1) If $a \neq 0$, then $\frac{0}{a} = 0$.

(2) $a^{\mathbf{2}} + b^{\mathbf{2}} = 0$ if and only if $a = 0$ and $b = 0$.

The non-empty set $\mathbb{C}$ is defined as follows:

$\mathbb{C} = [\!:\mathbb{R}, \mathbb{R}:\!]$.

One can prove the following proposition

(3) $\mathbb{C} = [\!:\mathbb{R}, \mathbb{R}:\!]$.

In the sequel z, z_1, z_2, z_3, z_4 will denote elements of $\mathbb{C}$. We now define two new functors. Let us consider z. The functor $\Re(z)$ yielding a real number, is defined by:

$\Re(z) = z_{\mathbf{1}}$.

[1]Supported by RPBP.III-24.C1.

ISSN 0777–4028

The functor $\Im(z)$ yielding a real number, is defined as follows:

$\Im(z) = z_{\mathbf{2}}$.

We now state two propositions:

(4) $\Re(z) = z_{\mathbf{1}}$.

(5) $\Im(z) = z_{\mathbf{2}}$.

Let x, y be elements of $\mathbb{R}$. The functor $x + yi$ yields an element of $\mathbb{C}$ and is defined as follows:

$x + yi = \langle x, y \rangle$.

Next we state several propositions:

(6) For all elements x, y of $\mathbb{R}$ holds $x + yi = \langle x, y \rangle$.

(7) $\Re(a + bi) = a$ and $\Im(a + bi) = b$.

(8) $\Re(z) + \Im(z)i = z$.

(9) If $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$, then $z_1 = z_2$.

(10) If $a_1 + b_1 i = a_2 + b_2 i$, then $a_1 = a_2$ and $b_1 = b_2$.

Let us consider z_1, z_2. Let us note that one can characterize the predicate $z_1 = z_2$ by the following (equivalent) condition: $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$.

We now define three new functors. The element $0_{\mathbb{C}}$ of $\mathbb{C}$ is defined as follows:

$0_{\mathbb{C}} = 0 + 0i$.

The element $1_{\mathbb{C}}$ of $\mathbb{C}$ is defined by:

$1_{\mathbb{C}} = 1 + 0i$.

The element i of $\mathbb{C}$ is defined as follows:

$i = 0 + 1i$.

The following propositions are true:

(11) $0_{\mathbb{C}} = 0 + 0i$.

(12) $\Re(0_{\mathbb{C}}) = 0$ and $\Im(0_{\mathbb{C}}) = 0$.

(13) $z = 0_{\mathbb{C}}$ if and only if $\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}} = 0$.

(14) $1_{\mathbb{C}} = 1 + 0i$.

(15) $\Re(1_{\mathbb{C}}) = 1$ and $\Im(1_{\mathbb{C}}) = 0$.

(16) $i = 0 + 1i$.

(17) $\Re(i) = 0$ and $\Im(i) = 1$.

Let us consider z_1, z_2. The functor $z_1 + z_2$ yields an element of $\mathbb{C}$ and is defined as follows:

$z_1 + z_2 = \Re(z_1) + \Re(z_2) + \Im(z_1) + \Im(z_2)i$.

We now state several propositions:

(18) $z_1 + z_2 = \Re(z_1) + \Re(z_2) + \Im(z_1) + \Im(z_2)i$.

(19) $\Re(z_1 + z_2) = \Re(z_1) + \Re(z_2)$ and $\Im(z_1 + z_2) = \Im(z_1) + \Im(z_2)$.

(20) $z_1 + z_2 = z_2 + z_1$.

(21) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.

(22) $0_{\mathbb{C}} + z = z$ and $z + 0_{\mathbb{C}} = z$.

Let us consider z_1, z_2. The functor $z_1 \cdot z_2$ yielding an element of $\mathbb{C}$, is defined as follows:

$z_1 \cdot z_2 = \Re(z_1) \cdot \Re(z_2) - \Im(z_1) \cdot \Im(z_2) + \Re(z_1) \cdot \Im(z_2) + \Re(z_2) \cdot \Im(z_1)i$.

Next we state a number of propositions:

(23) $z_1 \cdot z_2 = \Re(z_1) \cdot \Re(z_2) - \Im(z_1) \cdot \Im(z_2) + \Re(z_1) \cdot \Im(z_2) + \Re(z_2) \cdot \Im(z_1)i$.

(24) $\Re(z_1 \cdot z_2) = \Re(z_1) \cdot \Re(z_2) - \Im(z_1) \cdot \Im(z_2)$ and $\Im(z_1 \cdot z_2) = \Re(z_1) \cdot \Im(z_2) + \Re(z_2) \cdot \Im(z_1)$.

(25) $z_1 \cdot z_2 = z_2 \cdot z_1$.

(26) $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$.

(27) $z \cdot (z_1 + z_2) = z \cdot z_1 + z \cdot z_2$ and $(z_1 + z_2) \cdot z = z_1 \cdot z + z_2 \cdot z$.

(28) $0_{\mathbb{C}} \cdot z = 0_{\mathbb{C}}$ and $z \cdot 0_{\mathbb{C}} = 0_{\mathbb{C}}$.

(29) $1_{\mathbb{C}} \cdot z = z$ and $z \cdot 1_{\mathbb{C}} = z$.

(30) If $\Im(z_1) = 0$ and $\Im(z_2) = 0$, then $\Re(z_1 \cdot z_2) = \Re(z_1) \cdot \Re(z_2)$ and $\Im(z_1 \cdot z_2) = 0$.

(31) If $\Re(z_1) = 0$ and $\Re(z_2) = 0$, then $\Re(z_1 \cdot z_2) = -\Im(z_1) \cdot \Im(z_2)$ and $\Im(z_1 \cdot z_2) = 0$.

(32) $\Re(z \cdot z) = \Re(z)^{\mathbf{2}} - \Im(z)^{\mathbf{2}}$ and $\Im(z \cdot z) = 2 \cdot (\Re(z) \cdot \Im(z))$.

Let us consider z. The functor $-z$ yielding an element of $\mathbb{C}$, is defined by:

$-z = -\Re(z) + -\Im(z)i$.

One can prove the following propositions:

(33) $-z = -\Re(z) + -\Im(z)i$.

(34) $\Re(-z) = -\Re(z)$ and $\Im(-z) = -\Im(z)$.

(35) $-0_{\mathbb{C}} = 0_{\mathbb{C}}$.

(36) If $-z = 0_{\mathbb{C}}$, then $z = 0_{\mathbb{C}}$.

(37) $i \cdot i = -1_{\mathbb{C}}$.

(38) $z + (-z) = 0_{\mathbb{C}}$ and $(-z) + z = 0_{\mathbb{C}}$.

(39) If $z_1 + z_2 = 0_{\mathbb{C}}$, then $z_2 = -z_1$ and $z_1 = -z_2$.

(40) $-(-z) = z$.

(41) If $-z_1 = -z_2$, then $z_1 = z_2$.

(42) If $z_1 + z = z_2 + z$ or $z_1 + z = z + z_2$, then $z_1 = z_2$.

(43) $-(z_1 + z_2) = (-z_1) + (-z_2)$.

(44) $(-z_1) \cdot z_2 = -z_1 \cdot z_2$ and $z_1 \cdot (-z_2) = -z_1 \cdot z_2$.

(45) $(-z_1) \cdot (-z_2) = z_1 \cdot z_2$.

(46) $-z = (-1_{\mathbb{C}}) \cdot z$.

Let us consider z_1, z_2. The functor $z_1 - z_2$ yields an element of $\mathbb{C}$ and is defined by:

$z_1 - z_2 = \Re(z_1) - \Re(z_2) + \Im(z_1) - \Im(z_2)i$.

We now state a number of propositions:

(47) $z_1 - z_2 = \Re(z_1) - \Re(z_2) + \Im(z_1) - \Im(z_2)i$.

(48) $\Re(z_1 - z_2) = \Re(z_1) - \Re(z_2)$ and $\Im(z_1 - z_2) = \Im(z_1) - \Im(z_2)$.

(49) $z_1 - z_2 = z_1 + (-z_2)$.
(50) If $z_1 - z_2 = 0_{\mathbb{C}}$, then $z_1 = z_2$.
(51) $z - z = 0_{\mathbb{C}}$.
(52) $z - 0_{\mathbb{C}} = z$.
(53) $0_{\mathbb{C}} - z = -z$.
(54) $z_1 - (-z_2) = z_1 + z_2$.
(55) $-(z_1 - z_2) = (-z_1) + z_2$.
(56) $-(z_1 - z_2) = z_2 - z_1$.
(57) $z_1 + (z_2 - z_3) = (z_1 + z_2) - z_3$.
(58) $z_1 - (z_2 - z_3) = (z_1 - z_2) + z_3$.
(59) $(z_1 - z_2) - z_3 = z_1 - (z_2 + z_3)$.
(60) $z_1 = (z_1 + z) - z$.
(61) $z_1 = (z_1 - z) + z$.
(62) $z \cdot (z_1 - z_2) = z \cdot z_1 - z \cdot z_2$ and $(z_1 - z_2) \cdot z = z_1 \cdot z - z_2 \cdot z$.

Let us consider z. The functor z^{-1} yields an element of $\mathbb{C}$ and is defined by: $z^{-1} = \frac{\Re(z)}{\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}} + \frac{-\Im(z)}{\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}} i$.

Next we state a number of propositions:

(63) $z^{-1} = \frac{\Re(z)}{\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}} + \frac{-\Im(z)}{\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}} i$.
(64) $\Re(z^{-1}) = \frac{\Re(z)}{\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}}$ and $\Im(z^{-1}) = \frac{-\Im(z)}{\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}}$.
(65) If $z \neq 0_{\mathbb{C}}$, then $z \cdot z^{-1} = 1_{\mathbb{C}}$ and $z^{-1} \cdot z = 1_{\mathbb{C}}$.
(66) If $z_1 \cdot z_2 = 0_{\mathbb{C}}$, then $z_1 = 0_{\mathbb{C}}$ or $z_2 = 0_{\mathbb{C}}$.
(67) If $z \neq 0_{\mathbb{C}}$, then $z^{-1} \neq 0_{\mathbb{C}}$.
(68) If $z_1 \neq 0_{\mathbb{C}}$ and $z_2 \neq 0_{\mathbb{C}}$ and ${z_1}^{-1} = {z_2}^{-1}$, then $z_1 = z_2$.
(69) If $z_2 \neq 0_{\mathbb{C}}$ but $z_1 \cdot z_2 = 1_{\mathbb{C}}$ or $z_2 \cdot z_1 = 1_{\mathbb{C}}$, then $z_1 = {z_2}^{-1}$.
(70) If $z_2 \neq 0_{\mathbb{C}}$ but $z_1 \cdot z_2 = z_3$ or $z_2 \cdot z_1 = z_3$, then $z_1 = z_3 \cdot {z_2}^{-1}$ and $z_1 = {z_2}^{-1} \cdot z_3$.
(71) ${1_{\mathbb{C}}}^{-1} = 1_{\mathbb{C}}$.
(72) $i^{-1} = -i$.
(73) If $z_1 \neq 0_{\mathbb{C}}$ and $z_2 \neq 0_{\mathbb{C}}$, then $(z_1 \cdot z_2)^{-1} = {z_1}^{-1} \cdot {z_2}^{-1}$.
(74) If $z \neq 0_{\mathbb{C}}$, then $(z^{-1})^{-1} = z$.
(75) If $z \neq 0_{\mathbb{C}}$, then $(-z)^{-1} = -z^{-1}$.
(76) If $z \neq 0_{\mathbb{C}}$ but $z_1 \cdot z = z_2 \cdot z$ or $z_1 \cdot z = z \cdot z_2$, then $z_1 = z_2$.
(77) If $z_1 \neq 0_{\mathbb{C}}$ and $z_2 \neq 0_{\mathbb{C}}$, then ${z_1}^{-1} + {z_2}^{-1} = (z_1 + z_2) \cdot (z_1 \cdot z_2)^{-1}$.
(78) If $z_1 \neq 0_{\mathbb{C}}$ and $z_2 \neq 0_{\mathbb{C}}$, then ${z_1}^{-1} - {z_2}^{-1} = (z_2 - z_1) \cdot (z_1 \cdot z_2)^{-1}$.
(79) If $\Re(z) \neq 0$ and $\Im(z) = 0$, then $\Re(z^{-1}) = \Re(z)^{-1}$ and $\Im(z^{-1}) = 0$.
(80) If $\Re(z) = 0$ and $\Im(z) \neq 0$, then $\Re(z^{-1}) = 0$ and $\Im(z^{-1}) = -\Im(z)^{-1}$.

Let us consider z_1, z_2. The functor $\frac{z_1}{z_2}$ yields an element of $\mathbb{C}$ and is defined by:

$\frac{z_1}{z_2} = \frac{\Re(z_1)\cdot\Re(z_2)+\Im(z_1)\cdot\Im(z_2)}{\Re(z_2)^{\mathbf{2}}+\Im(z_2)^{\mathbf{2}}} + \frac{\Re(z_2)\cdot\Im(z_1)-\Re(z_1)\cdot\Im(z_2)}{\Re(z_2)^{\mathbf{2}}+\Im(z_2)^{\mathbf{2}}}i.$

Next we state a number of propositions:

(81) $\frac{z_1}{z_2} = \frac{\Re(z_1)\cdot\Re(z_2)+\Im(z_1)\cdot\Im(z_2)}{\Re(z_2)^{\mathbf{2}}+\Im(z_2)^{\mathbf{2}}} + \frac{\Re(z_2)\cdot\Im(z_1)-\Re(z_1)\cdot\Im(z_2)}{\Re(z_2)^{\mathbf{2}}+\Im(z_2)^{\mathbf{2}}}i.$

(82) $\Re(\frac{z_1}{z_2}) = \frac{\Re(z_1)\cdot\Re(z_2)+\Im(z_1)\cdot\Im(z_2)}{\Re(z_2)^{\mathbf{2}}+\Im(z_2)^{\mathbf{2}}}$ and $\Im(\frac{z_1}{z_2}) = \frac{\Re(z_2)\cdot\Im(z_1)-\Re(z_1)\cdot\Im(z_2)}{\Re(z_2)^{\mathbf{2}}+\Im(z_2)^{\mathbf{2}}}$.

(83) If $z_2 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z_2} = z_1 \cdot {z_2}^{-1}$.

(84) If $z \neq 0_{\mathbb{C}}$, then $z^{-1} = \frac{1_{\mathbb{C}}}{z}$.

(85) $\frac{z}{1_{\mathbb{C}}} = z$.

(86) If $z \neq 0_{\mathbb{C}}$, then $\frac{z}{z} = 1_{\mathbb{C}}$.

(87) If $z \neq 0_{\mathbb{C}}$, then $\frac{0_{\mathbb{C}}}{z} = 0_{\mathbb{C}}$.

(88) If $z_2 \neq 0_{\mathbb{C}}$ and $\frac{z_1}{z_2} = 0_{\mathbb{C}}$, then $z_1 = 0_{\mathbb{C}}$.

(89) If $z_2 \neq 0_{\mathbb{C}}$ and $z_4 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z_2} \cdot \frac{z_3}{z_4} = \frac{z_1 \cdot z_3}{z_2 \cdot z_4}$.

(90) If $z_2 \neq 0_{\mathbb{C}}$, then $z \cdot \frac{z_1}{z_2} = \frac{z \cdot z_1}{z_2}$.

(91) If $z_2 \neq 0_{\mathbb{C}}$ and $\frac{z_1}{z_2} = 1_{\mathbb{C}}$, then $z_1 = z_2$.

(92) If $z \neq 0_{\mathbb{C}}$, then $z_1 = \frac{z_1 \cdot z}{z}$.

(93) If $z_1 \neq 0_{\mathbb{C}}$ and $z_2 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z_2}^{-1} = \frac{z_2}{z_1}$.

(94) If $z_1 \neq 0_{\mathbb{C}}$ and $z_2 \neq 0_{\mathbb{C}}$, then $\frac{{z_1}^{-1}}{{z_2}^{-1}} = \frac{z_2}{z_1}$.

(95) If $z_2 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{{z_2}^{-1}} = z_1 \cdot z_2$.

(96) If $z_1 \neq 0_{\mathbb{C}}$ and $z_2 \neq 0_{\mathbb{C}}$, then $\frac{{z_1}^{-1}}{z_2} = (z_1 \cdot z_2)^{-1}$.

(97) If $z_1 \neq 0_{\mathbb{C}}$ and $z_2 \neq 0_{\mathbb{C}}$, then ${z_1}^{-1} \cdot \frac{z}{z_2} = \frac{z}{z_1 \cdot z_2}$.

(98) If $z \neq 0_{\mathbb{C}}$ and $z_2 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z_2} = \frac{z_1 \cdot z}{z_2 \cdot z}$ and $\frac{z_1}{z_2} = \frac{z \cdot z_1}{z \cdot z_2}$.

(99) If $z_2 \neq 0_{\mathbb{C}}$ and $z_3 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z_2 \cdot z_3} = \frac{\frac{z_1}{z_2}}{z_3}$.

(100) If $z_2 \neq 0_{\mathbb{C}}$ and $z_3 \neq 0_{\mathbb{C}}$, then $\frac{z_1 \cdot z_3}{z_2} = \frac{z_1}{\frac{z_2}{z_3}}$.

(101) If $z_2 \neq 0_{\mathbb{C}}$ and $z_3 \neq 0_{\mathbb{C}}$ and $z_4 \neq 0_{\mathbb{C}}$, then $\frac{\frac{z_1}{z_2}}{\frac{z_3}{z_4}} = \frac{z_1 \cdot z_4}{z_2 \cdot z_3}$.

(102) If $z_2 \neq 0_{\mathbb{C}}$ and $z_4 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z_2} + \frac{z_3}{z_4} = \frac{z_1 \cdot z_4 + z_3 \cdot z_2}{z_2 \cdot z_4}$.

(103) If $z \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z} + \frac{z_2}{z} = \frac{z_1 + z_2}{z}$.

(104) If $z_2 \neq 0_{\mathbb{C}}$, then $-\frac{z_1}{z_2} = \frac{-z_1}{z_2}$ and $-\frac{z_1}{z_2} = \frac{z_1}{-z_2}$.

(105) If $z_2 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z_2} = \frac{-z_1}{-z_2}$.

(106) If $z_2 \neq 0_{\mathbb{C}}$ and $z_4 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z_2} - \frac{z_3}{z_4} = \frac{z_1 \cdot z_4 - z_3 \cdot z_2}{z_2 \cdot z_4}$.

(107) If $z \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z} - \frac{z_2}{z} = \frac{z_1 - z_2}{z}$.

(108) If $z_2 \neq 0_{\mathbb{C}}$ but $z_1 \cdot z_2 = z_3$ or $z_2 \cdot z_1 = z_3$, then $z_1 = \frac{z_3}{z_2}$.

(109) If $\Im(z_1) = 0$ and $\Im(z_2) = 0$ and $\Re(z_2) \neq 0$, then $\Re(\frac{z_1}{z_2}) = \frac{\Re(z_1)}{\Re(z_2)}$ and $\Im(\frac{z_1}{z_2}) = 0$.

(110) If $\Re(z_1) = 0$ and $\Re(z_2) = 0$ and $\Im(z_2) \neq 0$, then $\Re(\frac{z_1}{z_2}) = \frac{\Im(z_1)}{\Im(z_2)}$ and $\Im(\frac{z_1}{z_2}) = 0$.

Let us consider z. The functor z^* yielding an element of $\mathbb{C}$, is defined as follows:

$z^* = \Re(z) + -\Im(z)i$.

The following propositions are true:

(111) $z^* = \Re(z) + -\Im(z)i$.

(112) $\Re(z^*) = \Re(z)$ and $\Im(z^*) = -\Im(z)$.

(113) $0_{\mathbb{C}}{}^* = 0_{\mathbb{C}}$.

(114) If $z^* = 0_{\mathbb{C}}$, then $z = 0_{\mathbb{C}}$.

(115) $1_{\mathbb{C}}{}^* = 1_{\mathbb{C}}$.

(116) $i^* = -i$.

(117) $z^{**} = z$.

(118) $(z_1 + z_2)^* = z_1{}^* + z_2{}^*$.

(119) $(-z)^* = -z^*$.

(120) $(z_1 - z_2)^* = z_1{}^* - z_2{}^*$.

(121) $(z_1 \cdot z_2)^* = z_1{}^* \cdot z_2{}^*$.

(122) If $z \neq 0_{\mathbb{C}}$, then $(z^{-1})^* = z^{*-1}$.

(123) If $z_2 \neq 0_{\mathbb{C}}$, then $\frac{z_1}{z_2}{}^* = \frac{z_1{}^*}{z_2{}^*}$.

(124) If $\Im(z) = 0$, then $z^* = z$.

(125) If $\Re(z) = 0$, then $z^* = -z$.

(126) $\Re(z \cdot z^*) = \Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}$ and $\Im(z \cdot z^*) = 0$.

(127) $\Re(z + z^*) = 2 \cdot \Re(z)$ and $\Im(z + z^*) = 0$.

(128) $\Re(z - z^*) = 0$ and $\Im(z - z^*) = 2 \cdot \Im(z)$.

Let us consider z. The functor $|z|$ yielding a real number, is defined as follows:

$|z| = \sqrt{\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}}$.

One can prove the following propositions:

(129) $|z| = \sqrt{\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}}$.

(130) $|0_{\mathbb{C}}| = 0$.

(131) If $|z| = 0$, then $z = 0_{\mathbb{C}}$.

(132) $0 \leq |z|$.

(133) $z \neq 0_{\mathbb{C}}$ if and only if $0 < |z|$.

(134) $|1_{\mathbb{C}}| = 1$.

(135) $|i| = 1$.

(136) If $\Im(z) = 0$, then $|z| = |\Re(z)|$.

(137) If $\Re(z) = 0$, then $|z| = |\Im(z)|$.

(138) $|-z| = |z|$.

(139) $|z^*| = |z|$.

(140) $\Re(z) \leq |z|$.
(141) $\Im(z) \leq |z|$.
(142) $|z_1 + z_2| \leq |z_1| + |z_2|$.
(143) $|z_1 - z_2| \leq |z_1| + |z_2|$.
(144) $|z_1| - |z_2| \leq |z_1 + z_2|$.
(145) $|z_1| - |z_2| \leq |z_1 - z_2|$.
(146) $|z_1 - z_2| = |z_2 - z_1|$.
(147) $|z_1 - z_2| = 0$ if and only if $z_1 = z_2$.
(148) $z_1 \neq z_2$ if and only if $0 < |z_1 - z_2|$.
(149) $|z_1 - z_2| \leq |z_1 - z| + |z - z_2|$.
(150) $||z_1| - |z_2|| \leq |z_1 - z_2|$.
(151) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.
(152) If $z \neq 0_{\mathbb{C}}$, then $|z^{-1}| = |z|^{-1}$.
(153) If $z_2 \neq 0_{\mathbb{C}}$, then $\frac{|z_1|}{|z_2|} = |\frac{z_1}{z_2}|$.
(154) $|z \cdot z| = \Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}$.
(155) $|z \cdot z| = |z \cdot z^*|$.

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Received March 1, 1990

FORMALIZED MATHEMATICS
Vol.1, No.**3**, May–August 1990
Université Catholique de Louvain

Ordinal Arithmetics

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Summary. At the beginning the article contains some auxiliary theorems concerning the constructors defined in papers [1] and [2]. Next simple properties of addition and multiplication of ordinals are shown, e.g. associativity of addition. Addition and multiplication of a transfinite sequence of ordinals and a ordinal are also introduced here. The goal of the article is the proof that the distributivity of multiplication wrt addition and the associativity of multiplication hold. Additionally new binary functors of ordinals are introduced: subtraction, exact division, and remainder and some of their basic properties are presented.

MML Identifier: `ORDINAL3`.

The notation and terminology used here are introduced in the following papers: [5], [3], [1], [4], and [2]. For simplicity we adopt the following convention: fi, psi denote sequences of ordinal numbers, A, B, C, D denote ordinal numbers, X, Y denote sets, and x is arbitrary. We now state a number of propositions:

(1) $X \subseteq \operatorname{succ} X$.

(2) If $\operatorname{succ} X \subseteq Y$, then $X \subseteq Y$.

(3) If $\operatorname{succ} A \subseteq B$, then $A \in B$.

(4) $A \subseteq B$ if and only if $\operatorname{succ} A \subseteq \operatorname{succ} B$.

(5) $A \in B$ if and only if $\operatorname{succ} A \in \operatorname{succ} B$.

(6) If $X \subseteq A$, then $\bigcup X$ is an ordinal number.

(7) $\bigcup(\operatorname{On} X)$ is an ordinal number.

(8) If $X \subseteq A$, then $\operatorname{On} X = X$.

(9) $\operatorname{On}\{A\} = \{A\}$.

(10) If $A \neq \mathbf{0}$, then $\mathbf{0} \in A$.

(11) $\inf A = \mathbf{0}$.

(12) $\inf\{A\} = A$.

(13) If $X \subseteq A$, then $\bigcap X$ is an ordinal number.

ISSN 0777–4028

Let us consider x. Let us assume that x is an ordinal number. The functor x (as an ordinal) yielding an ordinal number, is defined as follows:

x (as an ordinal) $= x$.

The following proposition is true

(14) If x is an ordinal number, then x (as an ordinal) $= x$.

Let us consider A, B. Then $A \cup B$ is an ordinal number. Then $A \cap B$ is an ordinal number.

We now state a number of propositions:

(15) $A \cup B = A$ or $A \cup B = B$.

(16) $A \cap B = A$ or $A \cap B = B$.

(17) If $A \in \mathbf{1}$, then $A = \mathbf{0}$.

(18) $\mathbf{1} = \{\mathbf{0}\}$.

(19) If $A \subseteq \mathbf{1}$, then $A = \mathbf{0}$ or $A = \mathbf{1}$.

(20) If $A \subseteq B$ or $A \in B$ but $C \in D$, then $A + C \in B + D$.

(21) If $A \subseteq B$ and $C \subseteq D$, then $A + C \subseteq B + D$.

(22) If $A \in B$ but $C \subseteq D$ and $D \neq \mathbf{0}$ or $C \in D$, then $A \cdot C \in B \cdot D$.

(23) If $A \subseteq B$ and $C \subseteq D$, then $A \cdot C \subseteq B \cdot D$.

(24) If $B + C = B + D$, then $C = D$.

(25) If $B + C \in B + D$, then $C \in D$.

(26) If $B + C \subseteq B + D$, then $C \subseteq D$.

(27) $A \subseteq A + B$ and $B \subseteq A + B$.

(28) If $A \in B$, then $A \in B + C$ and $A \in C + B$.

(29) If $A + B = \mathbf{0}$, then $A = \mathbf{0}$ and $B = \mathbf{0}$.

(30) If $A \subseteq B$, then there exists C such that $B = A + C$.

(31) If $A \in B$, then there exists C such that $B = A + C$ and $C \neq \mathbf{0}$.

(32) If $A \neq \mathbf{0}$ and A is a limit ordinal number, then $B + A$ is a limit ordinal number.

(33) $(A + B) + C = A + (B + C)$.

(34) If $A \cdot B = \mathbf{0}$, then $A = \mathbf{0}$ or $B = \mathbf{0}$.

(35) If $A \in B$ and $C \neq \mathbf{0}$, then $A \in B \cdot C$ and $A \in C \cdot B$.

(36) If $B \cdot A = C \cdot A$ and $A \neq \mathbf{0}$, then $B = C$.

(37) If $B \cdot A \in C \cdot A$, then $B \in C$.

(38) If $B \cdot A \subseteq C \cdot A$ and $A \neq \mathbf{0}$, then $B \subseteq C$.

(39) If $B \neq \mathbf{0}$, then $A \subseteq A \cdot B$ and $A \subseteq B \cdot A$.

(40) If $A \in B$ and $C \neq \mathbf{0}$, then $A \in B \cdot C$ and $A \in C \cdot B$.

(41) If $A \cdot B = \mathbf{1}$, then $A = \mathbf{1}$ and $B = \mathbf{1}$.

(42) If $A \in B + C$, then $A \in B$ or there exists D such that $D \in C$ and $A = B + D$.

We now define four new functors. Let us consider C, fi. The functor $C + fi$ yields a sequence of ordinal numbers and is defined by:

$\mathrm{dom}(C + fi) = \mathrm{dom}\, fi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $(C + fi)(A) = C + (fi(A))$ (as an ordinal).

The functor $fi + C$ yields a sequence of ordinal numbers and is defined by:

$\mathrm{dom}(fi + C) = \mathrm{dom}\, fi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $(fi + C)(A) = (fi(A))$ (as an ordinal) $+ C$.

The functor $C \cdot fi$ yields a sequence of ordinal numbers and is defined as follows:

$\mathrm{dom}(C \cdot fi) = \mathrm{dom}\, fi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $(C \cdot fi)(A) = C \cdot (fi(A))$ (as an ordinal).

The functor $fi \cdot C$ yields a sequence of ordinal numbers and is defined by:

$\mathrm{dom}(fi \cdot C) = \mathrm{dom}\, fi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $(fi \cdot C)(A) = (fi(A))$ (as an ordinal) $\cdot\, C$.

The following propositions are true:

(43) $psi = C + fi$ if and only if $\mathrm{dom}\, psi = \mathrm{dom}\, fi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $psi(A) = C + (fi(A))$ (as an ordinal).

(44) $psi = fi + C$ if and only if $\mathrm{dom}\, psi = \mathrm{dom}\, fi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $psi(A) = (fi(A))$ (as an ordinal) $+ C$.

(45) $psi = C \cdot fi$ if and only if $\mathrm{dom}\, psi = \mathrm{dom}\, fi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $psi(A) = C \cdot (fi(A))$ (as an ordinal).

(46) $psi = fi \cdot C$ if and only if $\mathrm{dom}\, psi = \mathrm{dom}\, fi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $psi(A) = (fi(A))$ (as an ordinal) $\cdot\, C$.

(47) If $\mathbf{0} \neq \mathrm{dom}\, fi$ and $\mathrm{dom}\, fi = \mathrm{dom}\, psi$ and for all A, B such that $A \in \mathrm{dom}\, fi$ and $B = fi(A)$ holds $psi(A) = C + B$, then $\sup psi = C + \sup fi$.

(48) If A is a limit ordinal number, then $A \cdot B$ is a limit ordinal number.

(49) If $A \in B \cdot C$ and B is a limit ordinal number, then there exists D such that $D \in B$ and $A \in D \cdot C$.

(50) If $\mathbf{0} \neq \mathrm{dom}\, fi$ and $\mathrm{dom}\, fi = \mathrm{dom}\, psi$ and $C \neq \mathbf{0}$ and $\sup fi$ is a limit ordinal number and for all A, B such that $A \in \mathrm{dom}\, fi$ and $B = fi(A)$ holds $psi(A) = B \cdot C$, then $\sup psi = \sup fi \cdot C$.

(51) If $\mathbf{0} \neq \mathrm{dom}\, fi$, then $\sup(C + fi) = C + \sup fi$.

(52) If $\mathbf{0} \neq \mathrm{dom}\, fi$ and $C \neq \mathbf{0}$ and $\sup fi$ is a limit ordinal number, then $\sup(fi \cdot C) = \sup fi \cdot C$.

(53) If $B \neq \mathbf{0}$, then $\bigcup(A + B) = A + \bigcup B$.

(54) $(A + B) \cdot C = A \cdot C + B \cdot C$.

(55) If $A \neq \mathbf{0}$, then there exist C, D such that $B = C \cdot A + D$ and $D \in A$.

(56) For all ordinal numbers C_1, D_1, C_2, D_2 such that $C_1 \cdot A + D_1 = C_2 \cdot A + D_2$ and $D_1 \in A$ and $D_2 \in A$ holds $C_1 = C_2$ and $D_1 = D_2$.

(57) If $\mathbf{1} \in B$ and $A \neq \mathbf{0}$ and A is a limit ordinal number, then for every fi such that $\mathrm{dom}\, fi = A$ and for every C such that $C \in A$ holds $fi(C) = C \cdot B$ holds $A \cdot B = \sup fi$.

(58) $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

We now define two new functors. Let us consider A, B. The functor $A - B$ yields an ordinal number and is defined as follows:

$A = B + (A - B)$ if $B \subseteq A$, $A - B = \mathbf{0}$, otherwise.

The functor $A \div B$ yielding an ordinal number, is defined by:

there exists C such that $A = (A \div B) \cdot B + C$ and $C \in B$ if $B \neq \mathbf{0}$, $A \div B = \mathbf{0}$, otherwise.

Let us consider A, B. The functor $A \bmod B$ yielding an ordinal number, is defined by:

$A \bmod B = A - (A \div B) \cdot B$.

The following propositions are true:

(59) If $A \subseteq B$, then $B = A + (B - A)$.

(60) If $A \in B$, then $B = A + (B - A)$.

(61) If $A \not\subseteq B$, then $B - A = \mathbf{0}$.

(62) If $B \neq \mathbf{0}$, then there exists C such that $A = (A \div B) \cdot B + C$ and $C \in B$.

(63) $A \div \mathbf{0} = \mathbf{0}$.

(64) $A \bmod B = A - (A \div B) \cdot B$.

(65) $(A + B) - A = B$.

(66) If $A \in B$ but $C \subseteq A$ or $C \in A$, then $A - C \in B - C$.

(67) $A - A = \mathbf{0}$.

(68) If $A \in B$, then $B - A \neq \mathbf{0}$ and $\mathbf{0} \in B - A$.

(69) $A - \mathbf{0} = A$ and $\mathbf{0} - A = \mathbf{0}$.

(70) $A - (B + C) = (A - B) - C$.

(71) If $A \subseteq B$, then $C - B \subseteq C - A$.

(72) If $A \subseteq B$, then $A - C \subseteq B - C$.

(73) If $C \neq \mathbf{0}$ and $A \in B + C$, then $A - B \in C$.

(74) If $A + B \in C$, then $B \in C - A$.

(75) $A \subseteq B + (A - B)$.

(76) $A \cdot C - B \cdot C = (A - B) \cdot C$.

(77) $(A \div B) \cdot B \subseteq A$.

(78) $A = (A \div B) \cdot B + (A \bmod B)$.

(79) If $A = B \cdot C + D$ and $D \in C$, then $B = A \div C$ and $D = A \bmod C$.

(80) If $A \in B \cdot C$, then $A \div C \in B$ and $A \bmod C \in C$.

(81) If $B \neq \mathbf{0}$, then $A \cdot B \div B = A$.

(82) $A \cdot B \bmod B = \mathbf{0}$.

(83) $\mathbf{0} \div A = \mathbf{0}$ and $\mathbf{0} \bmod A = \mathbf{0}$ and $A \bmod \mathbf{0} = A$.

(84) $A \div \mathbf{1} = A$ and $A \bmod \mathbf{1} = \mathbf{0}$.

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Received March 1, 1990

FORMALIZED MATHEMATICS
Vol.1, No.**3**, May–August 1990
Université Catholique de Louvain

The Modification of a Function by a Function and the Iteration of the Composition of a Function

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Summary. In the article we introduce some operation on functions. We define the natural ordering relation on functions. The fact that a function f is less than a function g we denote by $f \leq g$ and we define by graph$f \subseteq$ graph$f$. In the sequel we define the modifications of a function f by a function g denoted $f{+}{\cdot}g$ and the n-th iteration of the composition of a function f denoted by f^n. We prove some propositions related to the introduced notions.

MML Identifier: `FUNCT_4`.

The papers [7], [1], [2], [3], [4], [5], and [6] provide the terminology and notation for this paper. For simplicity we adopt the following rules: a, b, x, x', y, y', z will be arbitrary, X, X', Y, Y', Z, Z' will be sets, D, D' will be non-empty sets, and f, g, h will be functions. We now state several propositions:

(1) If for every z such that $z \in Z$ there exist x, y such that $z = \langle x, y \rangle$, then there exist X, Y such that $Z \subseteq [\!: X, Y :\!]$.

(2) If $\operatorname{rng} f \cap \operatorname{dom} g = \emptyset$, then $g \cdot f = \square$.

(3) $g \cdot f = g \restriction \operatorname{rng} f \cdot f$.

(4) $\square = \emptyset \longmapsto a$.

(5) $\operatorname{graph}(\operatorname{id}_X) \subseteq \operatorname{graph}(\operatorname{id}_Y)$ if and only if $X \subseteq Y$.

(6) If $X \subseteq Y$, then $\operatorname{graph}(X \longmapsto a) \subseteq \operatorname{graph}(Y \longmapsto a)$.

(7) If $\operatorname{graph}(X \longmapsto a) \subseteq \operatorname{graph}(Y \longmapsto b)$, then $X \subseteq Y$.

(8) If $X \neq \emptyset$ and $\operatorname{graph}(X \longmapsto a) \subseteq \operatorname{graph}(Y \longmapsto b)$, then $a = b$.

[1]Supported by RPBP.III-24.C1.

ISSN 0777–4028

(9) If $x \in \operatorname{dom} f$, then $\operatorname{graph}(\{x\} \longmapsto f(x)) \subseteq \operatorname{graph} f$.

Let us consider f, g. The predicate $f \leq g$ is defined as follows: $\operatorname{graph} f \subseteq \operatorname{graph} g$.

We now state a number of propositions:

(10) For all f, g holds $f \leq g$ if and only if $\operatorname{graph} f \subseteq \operatorname{graph} g$.

(11) $f \leq g$ if and only if $\operatorname{dom} f \subseteq \operatorname{dom} g$ and for every x such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$.

(12) If $f \leq g$, then $f \approx g$.

(13) If $f \leq g$, then $\operatorname{dom} f \subseteq \operatorname{dom} g$ and $\operatorname{rng} f \subseteq \operatorname{rng} g$.

(14) If $f \leq g$ and $\operatorname{dom} f = \operatorname{dom} g$, then $f = g$.

(15) $\square \leq f$.

(16) $f \leq f$.

(17) If $f \leq g$ and $g \leq h$, then $f \leq h$.

(18) $f \leq g$ and $g \leq f$ if and only if $f = g$.

(19) $\operatorname{id}_X \leq \operatorname{id}_Y$ if and only if $X \subseteq Y$.

(20) If $X \subseteq Y$, then $X \longmapsto a \leq Y \longmapsto a$.

(21) If $X \longmapsto a \leq Y \longmapsto b$, then $X \subseteq Y$.

(22) If $X \neq \emptyset$ and $X \longmapsto a \leq Y \longmapsto b$, then $a = b$.

(23) If $x \in \operatorname{dom} f$, then $\{x\} \longmapsto f(x) \leq f$.

(24) If $f \leq g$ and g is one-to-one, then f is one-to-one.

(25) If $f \leq g$, then $g \restriction \operatorname{dom} f = f$.

(26) If $f \leq g$ and g is one-to-one, then $\operatorname{rng} f \restriction g = f$.

(27) $f \restriction X \leq f$.

(28) If $X \subseteq Y$, then $f \restriction X \leq f \restriction Y$.

(29) If $X \subseteq Y$, then $X \restriction f \leq Y \restriction f$.

(30) $Y \restriction f \leq f$.

(31) $(Y \restriction f) \restriction X \leq f$.

(32) $f_{\restriction X \dot{\to} Y} \leq f$.

(33) If $f \leq g$, then $f \cdot h \leq g \cdot h$.

(34) If $f \leq g$, then $h \cdot f \leq h \cdot g$.

(35) For all functions f_1, f_2, g_1, g_2 such that $f_1 \leq g_1$ and $f_2 \leq g_2$ holds $f_1 \cdot f_2 \leq g_1 \cdot g_2$.

(36) If $f \leq g$, then $f \restriction X \leq g \restriction X$.

(37) If $f \leq g$, then $Y \restriction f \leq Y \restriction g$.

(38) If $f \leq g$, then $(Y \restriction f) \restriction X \leq (Y \restriction g) \restriction X$.

(39) If $f \leq g$, then $f_{\restriction X \dot{\to} Y} \leq g_{\restriction X \dot{\to} Y}$.

(40) If $f \leq h$ and $g \leq h$, then $f \approx g$.

Let us consider f, g. The functor $f +\cdot g$ yields a function and is defined by:

$\mathrm{dom}(f +\cdot g) = \mathrm{dom}\, f \cup \mathrm{dom}\, g$ and for every x such that $x \in \mathrm{dom}\, f \cup \mathrm{dom}\, g$ holds if $x \in \mathrm{dom}\, g$, then $(f +\cdot g)(x) = g(x)$ but if $x \notin \mathrm{dom}\, g$, then $(f +\cdot g)(x) = f(x)$.

We now state a number of propositions:

(41) Let f, g, h be functions. Then $h = f +\cdot g$ if and only if the following conditions are satisfied:

(i) $\mathrm{dom}\, h = \mathrm{dom}\, f \cup \mathrm{dom}\, g$,

(ii) for every x such that $x \in \mathrm{dom}\, f \cup \mathrm{dom}\, g$ holds if $x \in \mathrm{dom}\, g$, then $h(x) = g(x)$ but if $x \notin \mathrm{dom}\, g$, then $h(x) = f(x)$.

(42) If $x \in \mathrm{dom}(f +\cdot g)$ and $x \notin \mathrm{dom}\, g$, then $(f +\cdot g)(x) = f(x)$.

(43) $x \in \mathrm{dom}(f +\cdot g)$ if and only if $x \in \mathrm{dom}\, f$ or $x \in \mathrm{dom}\, g$.

(44) If $x \in \mathrm{dom}\, g$, then $(f +\cdot g)(x) = g(x)$.

(45) If $x \in \mathrm{dom}\, f \setminus \mathrm{dom}\, g$, then $(f +\cdot g)(x) = f(x)$.

(46) If $f \approx g$ and $x \in \mathrm{dom}\, f$, then $(f +\cdot g)(x) = f(x)$.

(47) If $\mathrm{dom}\, f \cap \mathrm{dom}\, g = \emptyset$ and $x \in \mathrm{dom}\, f$, then $(f +\cdot g)(x) = f(x)$.

(48) $\mathrm{rng}(f +\cdot g) \subseteq \mathrm{rng}\, f \cup \mathrm{rng}\, g$.

(49) $\mathrm{rng}\, g \subseteq \mathrm{rng}(f +\cdot g)$.

(50) If $\mathrm{dom}\, f \subseteq \mathrm{dom}\, g$, then $f +\cdot g = g$.

(51) If $\mathrm{dom}\, f = \mathrm{dom}\, g$, then $f +\cdot g = g$.

(52) $f +\cdot f = f$.

(53) $\square +\cdot f = f$.

(54) $f +\cdot \square = f$.

(55) $\mathrm{id}_X +\cdot \mathrm{id}_Y = \mathrm{id}_{X \cup Y}$.

(56) $(f +\cdot g) \restriction \mathrm{dom}\, g = g$.

(57) $\mathrm{graph}((f +\cdot g) \restriction (\mathrm{dom}\, f \setminus \mathrm{dom}\, g)) \subseteq \mathrm{graph}\, f$.

(58) $(f +\cdot g) \restriction (\mathrm{dom}\, f \setminus \mathrm{dom}\, g) \leq f$.

(59) $\mathrm{graph}\, g \subseteq \mathrm{graph}(f +\cdot g)$.

(60) $g \leq f +\cdot g$.

(61) If $f \approx g +\cdot h$, then $f \restriction (\mathrm{dom}\, f \setminus \mathrm{dom}\, h) \approx g$.

(62) If $f \approx g +\cdot h$, then $f \approx h$.

(63) $f \approx g$ if and only if $\mathrm{graph}\, f \subseteq \mathrm{graph}(f +\cdot g)$.

(64) $f \approx g$ if and only if $f \leq f +\cdot g$.

(65) $\mathrm{graph}(f +\cdot g) \subseteq \mathrm{graph}\, f \cup \mathrm{graph}\, g$.

(66) $f \approx g$ if and only if $\mathrm{graph}\, f \cup \mathrm{graph}\, g = \mathrm{graph}(f +\cdot g)$.

(67) If $\mathrm{dom}\, f \cap \mathrm{dom}\, g = \emptyset$, then $\mathrm{graph}\, f \cup \mathrm{graph}\, g = \mathrm{graph}(f +\cdot g)$.

(68) If $\mathrm{dom}\, f \cap \mathrm{dom}\, g = \emptyset$, then $\mathrm{graph}\, f \subseteq \mathrm{graph}(f +\cdot g)$.

(69) If $\mathrm{dom}\, f \cap \mathrm{dom}\, g = \emptyset$, then $f \leq f +\cdot g$.

(70) If $\mathrm{dom}\, f \cap \mathrm{dom}\, g = \emptyset$, then $(f +\cdot g) \restriction \mathrm{dom}\, f = f$.

(71) $f \approx g$ if and only if $f +\cdot g = g +\cdot f$.

(72) If $\mathrm{dom}\, f \cap \mathrm{dom}\, g = \emptyset$, then $f +\cdot g = g +\cdot f$.

(73) For all partial functions f, g from X to Y such that g is total holds $f +\cdot g = g$.

(74) For all functions f, g from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds $f +\cdot g = g$.

(75) For all functions f, g from X into X holds $f +\cdot g = g$.

(76) For all functions f, g from X into D holds $f +\cdot g = g$.

(77) For all partial functions f, g from X to Y holds $f +\cdot g$ is a partial function from X to Y.

Let us consider f. The functor $\curvearrowleft f$ yields a function and is defined by:

for every x holds $x \in \operatorname{dom}(\curvearrowleft f)$ if and only if there exist y, z such that $x = \langle z, y \rangle$ and $\langle y, z \rangle \in \operatorname{dom} f$ and for all y, z such that $\langle y, z \rangle \in \operatorname{dom} f$ holds $(\curvearrowleft f)(\langle z, y \rangle) = f(\langle y, z \rangle)$.

We now state a number of propositions:

(78) Let f, h be functions. Then $h = \curvearrowleft f$ if and only if for every z holds $z \in \operatorname{dom} h$ if and only if there exist x, y such that $z = \langle y, x \rangle$ and $\langle x, y \rangle \in \operatorname{dom} f$ and for all x, y such that $\langle x, y \rangle \in \operatorname{dom} f$ holds $h(\langle y, x \rangle) = f(\langle x, y \rangle)$.

(79) $\operatorname{rng}(\curvearrowleft f) \subseteq \operatorname{rng} f$.

(80) $\langle x, y \rangle \in \operatorname{dom} f$ if and only if $\langle y, x \rangle \in \operatorname{dom}(\curvearrowleft f)$.

(81) If $\langle y, x \rangle \in \operatorname{dom}(\curvearrowleft f)$, then $\curvearrowleft f(\langle y, x \rangle) = f(\langle x, y \rangle)$.

(82) There exist X, Y such that $\operatorname{dom}(\curvearrowleft f) \subseteq [\!: X, Y :\!]$.

(83) If $\operatorname{dom} f \subseteq [\!: X, Y :\!]$, then $\operatorname{dom}(\curvearrowleft f) \subseteq [\!: Y, X :\!]$.

(84) If $\operatorname{dom} f = [\!: X, Y :\!]$, then $\operatorname{dom}(\curvearrowleft f) = [\!: Y, X :\!]$.

(85) If $\operatorname{dom} f \subseteq [\!: X, Y :\!]$, then $\operatorname{rng}(\curvearrowleft f) = \operatorname{rng} f$.

(86) If $\operatorname{dom} f = [\!: X, Y :\!]$, then $\operatorname{rng}(\curvearrowleft f) = \operatorname{rng} f$.

(87) For every partial function f from $[\!: X, Y :\!]$ to Z holds $\curvearrowleft f$ is a partial function from $[\!: Y, X :\!]$ to Z.

(88) For every function f from $[\!: X, Y :\!]$ into Z such that $Z \neq \emptyset$ holds $\curvearrowleft f$ is a function from $[\!: Y, X :\!]$ into Z.

(89) For every function f from $[\!: X, Y :\!]$ into D holds $\curvearrowleft f$ is a function from $[\!: Y, X :\!]$ into D.

(90) $\operatorname{graph}(\curvearrowleft(\curvearrowleft f)) \subseteq \operatorname{graph} f$.

(91) If $\operatorname{dom} f \subseteq [\!: X, Y :\!]$, then $\curvearrowleft(\curvearrowleft f) = f$.

(92) If $\operatorname{dom} f = [\!: X, Y :\!]$, then $\curvearrowleft(\curvearrowleft f) = f$.

(93) For every partial function f from $[\!: X, Y :\!]$ to Z holds $\curvearrowleft(\curvearrowleft f) = f$.

(94) For every function f from $[\!: X, Y :\!]$ into Z such that $Z \neq \emptyset$ holds $\curvearrowleft(\curvearrowleft f) = f$.

(95) For every function f from $[\!: X, Y :\!]$ into D holds $\curvearrowleft(\curvearrowleft f) = f$.

Let us consider f, g. The functor $|\!: f, g :\!|$ yielding a function, is defined as follows:

(i) for every z holds $z \in \operatorname{dom} |\!: f, g :\!|$ if and only if there exist x, y, x', y' such that $z = \langle \langle x, x' \rangle, \langle y, y' \rangle \rangle$ and $\langle x, y \rangle \in \operatorname{dom} f$ and $\langle x', y' \rangle \in \operatorname{dom} g$,

(ii) for all x, y, x', y' such that $\langle x,y\rangle \in \operatorname{dom} f$ and $\langle x',y'\rangle \in \operatorname{dom} g$ holds $|{:}f, g{:}|(\langle\langle x,x'\rangle,\langle y,y'\rangle\rangle) = \langle f(\langle x,y\rangle), g(\langle x',y'\rangle)\rangle$.

The following propositions are true:

(96) Given f, g, h. Then $h = |{:}f, g{:}|$ if and only if the following conditions are satisfied:

(i) for every z holds $z \in \operatorname{dom} h$ if and only if there exist x, y, x', y' such that $z = \langle\langle x,x'\rangle,\langle y,y'\rangle\rangle$ and $\langle x,y\rangle \in \operatorname{dom} f$ and $\langle x',y'\rangle \in \operatorname{dom} g$,

(ii) for all x, y, x', y' such that $\langle x,y\rangle \in \operatorname{dom} f$ and $\langle x',y'\rangle \in \operatorname{dom} g$ holds $h(\langle\langle x,x'\rangle,\langle y,y'\rangle\rangle) = \langle f(\langle x,y\rangle), g(\langle x',y'\rangle)\rangle$.

(97) $\langle\langle x,x'\rangle,\langle y,y'\rangle\rangle \in \operatorname{dom}|{:}f, g{:}|$ if and only if $\langle x,y\rangle \in \operatorname{dom} f$ and $\langle x',y'\rangle \in \operatorname{dom} g$.

(98) If $\langle\langle x,x'\rangle,\langle y,y'\rangle\rangle \in \operatorname{dom}|{:}f, g{:}|$, then $|{:}f, g{:}|(\langle\langle x,x'\rangle,\langle y,y'\rangle\rangle) = \langle f(\langle x,y\rangle), g(\langle x',y'\rangle)\rangle$.

(99) $\operatorname{rng}|{:}f, g{:}| \subseteq [{:}\operatorname{rng} f, \operatorname{rng} g{:}]$.

(100) If $\operatorname{dom} f \subseteq [{:}X, Y{:}]$ and $\operatorname{dom} g \subseteq [{:}X', Y'{:}]$, then $\operatorname{dom}|{:}f, g{:}| \subseteq [{:}[{:}X, X'{:}], [{:}Y, Y'{:}]{:}]$.

(101) If $\operatorname{dom} f = [{:}X, Y{:}]$ and $\operatorname{dom} g = [{:}X', Y'{:}]$, then $\operatorname{dom}|{:}f, g{:}| = [{:}[{:}X, X'{:}], [{:}Y, Y'{:}]{:}]$.

(102) For every partial function f from $[{:}X, Y{:}]$ to Z and for every partial function g from $[{:}X', Y'{:}]$ to Z' holds $|{:}f, g{:}|$ is a partial function from $[{:}[{:}X, X'{:}], [{:}Y, Y'{:}]{:}]$ to $[{:}Z, Z'{:}]$.

(103) For every function f from $[{:}X, Y{:}]$ into Z and for every function g from $[{:}X', Y'{:}]$ into Z' such that $Z \neq \emptyset$ and $Z' \neq \emptyset$ holds $|{:}f, g{:}|$ is a function from $[{:}[{:}X, X'{:}], [{:}Y, Y'{:}]{:}]$ into $[{:}Z, Z'{:}]$.

(104) For every function f from $[{:}X, Y{:}]$ into D and for every function g from $[{:}X', Y'{:}]$ into D' holds $|{:}f, g{:}|$ is a function from $[{:}[{:}X, X'{:}], [{:}Y, Y'{:}]{:}]$ into $[{:}D, D'{:}]$.

Let f be a function, and let n be an element of $\mathbb{N}$. The functor f^n yields a function and is defined as follows:

there exists a function p from $\mathbb{N}$ into $(\operatorname{dom} f \cup \operatorname{rng} f) \dot{\rightarrow} (\operatorname{dom} f \cup \operatorname{rng} f)$ such that $f^n = p(n)$ and $p(0) = \mathrm{id}_{\operatorname{dom} f \cup \operatorname{rng} f}$ and for every element k of $\mathbb{N}$ there exists a function g such that $g = p(k)$ and $p(k+1) = g \cdot f$.

One can prove the following proposition

(105) Let f be a function. Let n be an element of $\mathbb{N}$. Suppose $\operatorname{rng} f \subseteq \operatorname{dom} f$. Let h be a function. Then $h = f^n$ if and only if there exists a function p from $\mathbb{N}$ into $(\operatorname{dom} f \cup \operatorname{rng} f) \dot{\rightarrow} (\operatorname{dom} f \cup \operatorname{rng} f)$ such that $h = p(n)$ and $p(0) = \mathrm{id}_{\operatorname{dom} f \cup \operatorname{rng} f}$ and for every element k of $\mathbb{N}$ there exists a function g such that $g = p(k)$ and $p(k+1) = g \cdot f$.

In the sequel m, n will be natural numbers. Next we state a number of propositions:

(106) $f^0 = \mathrm{id}_{\operatorname{dom} f \cup \operatorname{rng} f}$.

(107) $f^{n+1} = (f^n) \cdot f$.

(108) $f^1 = f$.

(109) $f^{n+1} = f \cdot (f^n)$.

(110) $\mathrm{dom}(f^n) \subseteq \mathrm{dom}\, f \cup \mathrm{rng}\, f$ and $\mathrm{rng}(f^n) \subseteq \mathrm{dom}\, f \cup \mathrm{rng}\, f$.

(111) If $n \neq 0$, then $\mathrm{dom}(f^n) \subseteq \mathrm{dom}\, f$ and $\mathrm{rng}(f^n) \subseteq \mathrm{rng}\, f$.

(112) If $\mathrm{rng}\, f \subseteq \mathrm{dom}\, f$, then $\mathrm{dom}(f^n) = \mathrm{dom}\, f$ and $\mathrm{rng}(f^n) \subseteq \mathrm{dom}\, f$.

(113) $(f^n) \cdot \mathrm{id}_{\mathrm{dom}\, f \cup \mathrm{rng}\, f} = f^n$.

(114) $\mathrm{id}_{\mathrm{dom}\, f \cup \mathrm{rng}\, f} \cdot (f^n) = f^n$.

(115) $(f^n) \cdot (f^m) = f^{n+m}$.

(116) If $n \neq 0$, then $(f^m)^n = f^{m \cdot n}$.

(117) If $\mathrm{rng}\, f \subseteq \mathrm{dom}\, f$, then $(f^m)^n = f^{m \cdot n}$.

(118) $\square^n = \square$.

(119) ${\mathrm{id}_X}^n = \mathrm{id}_X$.

(120) If $\mathrm{rng}\, f \cap \mathrm{dom}\, f = \emptyset$, then $f^2 = \square$.

(121) For every function f from X into X holds f^n is a function from X into X.

(122) For every function f from X into X holds $f^0 = \mathrm{id}_X$.

(123) For every function f from X into X holds $(f^m)^n = f^{m \cdot n}$.

(124) For every partial function f from X to X holds f^n is a partial function from X to X.

(125) If $n \neq 0$ and $a \in X$ and $f = X \longmapsto a$, then $f^n = f$.

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Received March 1, 1990

FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Finite Sequences and Tuples of Elements of a Non-empty Sets

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Summary. The first part of the article is a continuation of [2]. Next, we define the identity sequence of natural numbers and the constant sequences. The main part of this article is the definition of tuples. The element of a set of all sequences of the length n of D is called a tuple of a non-empty set D and it is denoted by element of D^n. Also some basic facts about tuples of a non-empty set are proved.

MML Identifier: `FINSEQ_2`.

The notation and terminology used here have been introduced in the following articles: [9], [8], [6], [1], [10], [4], [5], [2], [3], and [7]. For simplicity we adopt the following rules: i, j, l denote natural numbers, a, b, x_1, x_2, x_3 are arbitrary, D, D', E denote non-empty sets, d, d_1, d_2, d_3 denote elements of D, d', d'_1, d'_2, d'_3 denote elements of D', and p, q, r denote finite sequences. Next we state a number of propositions:

(1) $\min(i, j)$ is a natural number and $\max(i, j)$ is a natural number.

(2) If $l = \min(i, j)$, then $\mathrm{Seg}\, i \cap \mathrm{Seg}\, j = \mathrm{Seg}\, l$.

(3) If $i \leq j$, then $\max(0, i - j) = 0$.

(4) If $j \leq i$, then $\max(0, i - j) = i - j$.

(5) $\max(0, i - j)$ is a natural number.

(6) $\min(0, i) = 0$ and $\min(i, 0) = 0$ and $\max(0, i) = i$ and $\max(i, 0) = i$.

(7) If $i \neq 0$, then $\mathrm{Seg}\, i$ is a non-empty subset of $\mathbb{N}$.

(8) If $i \in \mathrm{Seg}(l + 1)$, then $i \in \mathrm{Seg}\, l$ or $i = l + 1$.

(9) If $i \in \mathrm{Seg}\, l$, then $i \in \mathrm{Seg}(l + j)$.

(10) If $\mathrm{len}\, p = i$ and $\mathrm{len}\, q = i$ and for every j such that $j \in \mathrm{Seg}\, i$ holds $p(j) = q(j)$, then $p = q$.

[1]Supported by RPBP.III-24.C1.

ISSN 0777–4028

(11) If $b \in \operatorname{rng} p$, then there exists i such that $i \in \operatorname{Seg}(\operatorname{len} p)$ and $p(i) = b$.

(12) If $i \in \operatorname{Seg}(\operatorname{len} p)$, then $p(i) \in \operatorname{rng} p$.

(13) For every finite sequence p of elements of D such that $i \in \operatorname{Seg}(\operatorname{len} p)$ holds $p(i) \in D$.

(14) If for every i such that $i \in \operatorname{Seg}(\operatorname{len} p)$ holds $p(i) \in D$, then p is a finite sequence of elements of D.

(15) $\langle d_1, d_2 \rangle$ is a finite sequence of elements of D.

(16) $\langle d_1, d_2, d_3 \rangle$ is a finite sequence of elements of D.

(17) If $i \in \operatorname{Seg}(\operatorname{len} p)$, then $(p \frown q)(i) = p(i)$.

(18) If $i \in \operatorname{Seg}(\operatorname{len} p)$, then $i \in \operatorname{Seg}(\operatorname{len}(p \frown q))$.

(19) $\operatorname{len}(p \frown \langle a \rangle) = \operatorname{len} p + 1$.

(20) If $p \frown \langle a \rangle = q \frown \langle b \rangle$, then $p = q$ and $a = b$.

(21) If $\operatorname{len} p = i + 1$, then there exist q, a such that $p = q \frown \langle a \rangle$.

(22) For every finite sequence p of elements of D such that $\operatorname{len} p \neq 0$ there exists a finite sequence q of elements of D and there exists d such that $p = q \frown \langle d \rangle$.

(23) If $q = p \restriction \operatorname{Seg} i$ and $\operatorname{len} p \leq i$, then $p = q$.

(24) If $q = p \restriction \operatorname{Seg} i$, then $\operatorname{len} q = \min(i, \operatorname{len} p)$.

(25) If $\operatorname{len} r = i + j$, then there exist p, q such that $\operatorname{len} p = i$ and $\operatorname{len} q = j$ and $r = p \frown q$.

(26) For every finite sequence r of elements of D such that $\operatorname{len} r = i + j$ there exist finite sequences p, q of elements of D such that $\operatorname{len} p = i$ and $\operatorname{len} q = j$ and $r = p \frown q$.

In the article we present several logical schemes. The scheme *SeqLambdaD* concerns a natural number $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$ and states that:

there exists a finite sequence z of elements of $\mathcal{B}$ such that $\operatorname{len} z = \mathcal{A}$ and for every j such that $j \in \operatorname{Seg} \mathcal{A}$ holds $z(j) = \mathcal{F}(j)$

for all values of the parameters.

The scheme *IndSeqD* deals with a non-empty set $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

for every finite sequence p of elements of $\mathcal{A}$ holds $\mathcal{P}[p]$

provided the parameters meet the following requirements:

- $\mathcal{P}[\varepsilon_{\mathcal{A}}]$,
- for every finite sequence p of elements of $\mathcal{A}$ and for every element x of $\mathcal{A}$ such that $\mathcal{P}[p]$ holds $\mathcal{P}[p \frown \langle x \rangle]$.

We now state a number of propositions:

(27) For every non-empty subset D' of D and for every finite sequence p of elements of D' holds p is a finite sequence of elements of D.

(28) For every function f from $\operatorname{Seg} i$ into D holds f is a finite sequence of elements of D.

(29) p is a function from $\operatorname{Seg}(\operatorname{len} p)$ into $\operatorname{rng} p$.

(30) For every finite sequence p of elements of D holds p is a function from $\mathrm{Seg}(\mathrm{len}\, p)$ into D.

(31) For every function f from $\mathbb{N}$ into D holds $f \restriction \mathrm{Seg}\, i$ is a finite sequence of elements of D.

(32) For every function f from $\mathbb{N}$ into D such that $q = f \restriction \mathrm{Seg}\, i$ holds $\mathrm{len}\, q = i$.

(33) For every function f such that $\mathrm{rng}\, p \subseteq \mathrm{dom}\, f$ and $q = f \cdot p$ holds $\mathrm{len}\, q = \mathrm{len}\, p$.

(34) If $D = \mathrm{Seg}\, i$, then for every finite sequence p and for every finite sequence q of elements of D such that $i \leq \mathrm{len}\, p$ holds $p \cdot q$ is a finite sequence.

(35) If $D = \mathrm{Seg}\, i$, then for every finite sequence p of elements of D' and for every finite sequence q of elements of D such that $i \leq \mathrm{len}\, p$ holds $p \cdot q$ is a finite sequence of elements of D'.

(36) For every finite sequence p of elements of D and for every function f from D into D' holds $f \cdot p$ is a finite sequence of elements of D'.

(37) For every finite sequence p of elements of D and for every function f from D into D' such that $q = f \cdot p$ holds $\mathrm{len}\, q = \mathrm{len}\, p$.

(38) For every function f from D into D' holds $f \cdot \varepsilon_D = \varepsilon_{D'}$.

(39) For every finite sequence p of elements of D and for every function f from D into D' such that $p = \langle x_1 \rangle$ holds $f \cdot p = \langle f(x_1) \rangle$.

(40) For every finite sequence p of elements of D and for every function f from D into D' such that $p = \langle x_1, x_2 \rangle$ holds $f \cdot p = \langle f(x_1), f(x_2) \rangle$.

(41) For every finite sequence p of elements of D and for every function f from D into D' such that $p = \langle x_1, x_2, x_3 \rangle$ holds $f \cdot p = \langle f(x_1), f(x_2), f(x_3) \rangle$.

(42) For every function f from $\mathrm{Seg}\, i$ into $\mathrm{Seg}\, j$ such that if $j = 0$, then $i = 0$ but $j \leq \mathrm{len}\, p$ holds $p \cdot f$ is a finite sequence.

(43) For every function f from $\mathrm{Seg}\, i$ into $\mathrm{Seg}\, i$ such that $i \leq \mathrm{len}\, p$ holds $p \cdot f$ is a finite sequence.

(44) For every function f from $\mathrm{Seg}(\mathrm{len}\, p)$ into $\mathrm{Seg}(\mathrm{len}\, p)$ holds $p \cdot f$ is a finite sequence.

(45) For every function f from $\mathrm{Seg}\, i$ into $\mathrm{Seg}\, i$ such that $\mathrm{rng}\, f = \mathrm{Seg}\, i$ and $i \leq \mathrm{len}\, p$ and $q = p \cdot f$ holds $\mathrm{len}\, q = i$.

(46) For every function f from $\mathrm{Seg}(\mathrm{len}\, p)$ into $\mathrm{Seg}(\mathrm{len}\, p)$ such that $\mathrm{rng}\, f = \mathrm{Seg}(\mathrm{len}\, p)$ and $q = p \cdot f$ holds $\mathrm{len}\, q = \mathrm{len}\, p$.

(47) For every permutation f of $\mathrm{Seg}\, i$ such that $i \leq \mathrm{len}\, p$ and $q = p \cdot f$ holds $\mathrm{len}\, q = i$.

(48) For every permutation f of $\mathrm{Seg}(\mathrm{len}\, p)$ such that $q = p \cdot f$ holds $\mathrm{len}\, q = \mathrm{len}\, p$.

(49) For every finite sequence p of elements of D and for every function f from $\mathrm{Seg}\, i$ into $\mathrm{Seg}\, j$ such that if $j = 0$, then $i = 0$ but $j \leq \mathrm{len}\, p$ holds $p \cdot f$ is a finite sequence of elements of D.

(50) For every finite sequence p of elements of D and for every function f from $\mathrm{Seg}\, i$ into $\mathrm{Seg}\, i$ such that $i \leq \mathrm{len}\, p$ holds $p \cdot f$ is a finite sequence of elements of D.

(51) For every finite sequence p of elements of D and for every function f from $\mathrm{Seg}(\mathrm{len}\, p)$ into $\mathrm{Seg}(\mathrm{len}\, p)$ holds $p \cdot f$ is a finite sequence of elements of D.

(52) $\mathrm{id}_{\mathrm{Seg}\, i}$ is a finite sequence of elements of $\mathbb{N}$.

Let us consider i. The functor id_i yielding a finite sequence, is defined as follows:

$\mathrm{id}_i = \mathrm{id}_{\mathrm{Seg}\, i}$.

One can prove the following propositions:

(53) $\mathrm{id}_i = \mathrm{id}_{\mathrm{Seg}\, i}$.

(54) $\mathrm{dom}(\mathrm{id}_i) = \mathrm{Seg}\, i$.

(55) $\mathrm{len}(\mathrm{id}_i) = i$.

(56) If $j \in \mathrm{Seg}\, i$, then $\mathrm{id}_i(j) = j$.

(57) If $i \neq 0$, then for every element k of $\mathrm{Seg}\, i$ holds $\mathrm{id}_i(k) = k$.

(58) $\mathrm{id}_0 = \varepsilon$.

(59) $\mathrm{id}_1 = \langle 1 \rangle$.

(60) $\mathrm{id}_{i+1} = \mathrm{id}_i {}^\frown \langle i+1 \rangle$.

(61) $\mathrm{id}_2 = \langle 1, 2 \rangle$.

(62) $\mathrm{id}_3 = \langle 1, 2, 3 \rangle$.

(63) $p \cdot \mathrm{id}_i = p \restriction \mathrm{Seg}\, i$.

(64) If $\mathrm{len}\, p \leq i$, then $p \cdot \mathrm{id}_i = p$.

(65) id_i is a permutation of $\mathrm{Seg}\, i$.

(66) $\mathrm{Seg}\, i \longmapsto a$ is a finite sequence.

Let us consider i, a. The functor $i \longmapsto a$ yielding a finite sequence, is defined as follows:

$i \longmapsto a = \mathrm{Seg}\, i \longmapsto a$.

We now state a number of propositions:

(67) $i \longmapsto a = \mathrm{Seg}\, i \longmapsto a$.

(68) $\mathrm{dom}(i \longmapsto a) = \mathrm{Seg}\, i$.

(69) $\mathrm{len}(i \longmapsto a) = i$.

(70) If $j \in \mathrm{Seg}\, i$, then $(i \longmapsto a)(j) = a$.

(71) If $i \neq 0$, then for every element k of $\mathrm{Seg}\, i$ holds $(i \longmapsto d)(k) = d$.

(72) $0 \longmapsto a = \varepsilon$.

(73) $1 \longmapsto a = \langle a \rangle$.

(74) $i + 1 \longmapsto a = (i \longmapsto a) {}^\frown \langle a \rangle$.

(75) $2 \longmapsto a = \langle a, a \rangle$.

(76) $3 \longmapsto a = \langle a, a, a \rangle$.

(77) $i \longmapsto d$ is a finite sequence of elements of D.

(78) For every function F such that $[: \operatorname{rng} p, \operatorname{rng} q :] \subseteq \operatorname{dom} F$ holds $F^\circ(p,q)$ is a finite sequence.

(79) For every function F such that $[: \operatorname{rng} p, \operatorname{rng} q :] \subseteq \operatorname{dom} F$ and $r = F^\circ(p,q)$ holds $\operatorname{len} r = \min(\operatorname{len} p, \operatorname{len} q)$.

(80) For every function F such that $[: \{a\}, \operatorname{rng} p :] \subseteq \operatorname{dom} F$ holds $F^\circ(a,p)$ is a finite sequence.

(81) For every function F such that $[: \{a\}, \operatorname{rng} p :] \subseteq \operatorname{dom} F$ and $r = F^\circ(a,p)$ holds $\operatorname{len} r = \operatorname{len} p$.

(82) For every function F such that $[: \operatorname{rng} p, \{a\} :] \subseteq \operatorname{dom} F$ holds $F^\circ(p,a)$ is a finite sequence.

(83) For every function F such that $[: \operatorname{rng} p, \{a\} :] \subseteq \operatorname{dom} F$ and $r = F^\circ(p,a)$ holds $\operatorname{len} r = \operatorname{len} p$.

(84) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D and for every finite sequence q of elements of D' holds $F^\circ(p,q)$ is a finite sequence of elements of E.

(85) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D and for every finite sequence q of elements of D' such that $r = F^\circ(p,q)$ holds $\operatorname{len} r = \min(\operatorname{len} p, \operatorname{len} q)$.

(86) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D and for every finite sequence q of elements of D' such that $\operatorname{len} p = \operatorname{len} q$ and $r = F^\circ(p,q)$ holds $\operatorname{len} r = \operatorname{len} p$ and $\operatorname{len} r = \operatorname{len} q$.

(87) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D and for every finite sequence p' of elements of D' holds $F^\circ(\varepsilon_D, p') = \varepsilon_E$ and $F^\circ(p, \varepsilon_{D'}) = \varepsilon_E$.

(88) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D and for every finite sequence q of elements of D' such that $p = \langle d_1 \rangle$ and $q = \langle d'_1 \rangle$ holds $F^\circ(p,q) = \langle F(d_1, d'_1) \rangle$.

(89) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D and for every finite sequence q of elements of D' such that $p = \langle d_1, d_2 \rangle$ and $q = \langle d'_1, d'_2 \rangle$ holds $F^\circ(p,q) = \langle F(d_1, d'_1), F(d_2, d'_2) \rangle$.

(90) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D and for every finite sequence q of elements of D' such that $p = \langle d_1, d_2, d_3 \rangle$ and $q = \langle d'_1, d'_2, d'_3 \rangle$ holds $F^\circ(p,q) = \langle F(d_1, d'_1), F(d_2, d'_2), F(d_3, d'_3) \rangle$.

(91) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D' holds $F^\circ(d,p)$ is a finite sequence of elements of E.

(92) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D' such that $r = F^\circ(d,p)$ holds $\operatorname{len} r = \operatorname{len} p$.

(93) For every function F from $[: D, D' :]$ into E holds $F^\circ(d, \varepsilon_{D'}) = \varepsilon_E$.

(94) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D' such that $p = \langle d'_1 \rangle$ holds $F^\circ(d,p) = \langle F(d, d'_1) \rangle$.

(95) For every function F from $[: D, D' :]$ into E and for every finite sequence

p of elements of D' such that $p = \langle d'_1, d'_2 \rangle$ holds $F^{\circ}(d,p) = \langle F(d,d'_1), F(d,d'_2) \rangle$.

(96) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D' such that $p = \langle d'_1, d'_2, d'_3 \rangle$ holds $F^{\circ}(d,p) = \langle F(d,d'_1), F(d,d'_2), F(d,d'_3) \rangle$.

(97) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D holds $F^{\circ}(p,d')$ is a finite sequence of elements of E.

(98) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D such that $r = F^{\circ}(p,d')$ holds $\operatorname{len} r = \operatorname{len} p$.

(99) For every function F from $[: D, D' :]$ into E holds $F^{\circ}(\varepsilon_D, d') = \varepsilon_E$.

(100) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D such that $p = \langle d_1 \rangle$ holds $F^{\circ}(p,d') = \langle F(d_1,d') \rangle$.

(101) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D such that $p = \langle d_1, d_2 \rangle$ holds $F^{\circ}(p,d') = \langle F(d_1,d'), F(d_2,d') \rangle$.

(102) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D such that $p = \langle d_1, d_2, d_3 \rangle$ holds $F^{\circ}(p,d') = \langle F(d_1,d'), F(d_2,d'), F(d_3,d') \rangle$.

Let us consider D. A non-empty set is said to be a non-empty set of finite sequences of D if:

if $a \in$ it, then a is a finite sequence of elements of D.

We now state two propositions:

(103) For all D, D' holds D' is a non-empty set of finite sequences of D if and only if for every a such that $a \in D'$ holds a is a finite sequence of elements of D.

(104) D^* is a non-empty set of finite sequences of D.

Let us consider D. Then D^* is a non-empty set of finite sequences of D.

Next we state two propositions:

(105) For every non-empty set D' of finite sequences of D holds $D' \subseteq D^*$.

(106) For every non-empty set S of finite sequences of D and for every element s of S holds s is a finite sequence of elements of D.

Let us consider D, and let S be a non-empty set of finite sequences of D. We see that it makes sense to consider the following mode for restricted scopes of arguments. Then all the objests of the mode element of S are a finite sequence of elements of D.

One can prove the following proposition

(107) For every non-empty subset D' of D and for every non-empty set S of finite sequences of D' holds S is a non-empty set of finite sequences of D.

In the sequel s is an element of D^*. Let us consider i, D. The functor D^i yielding a non-empty set of finite sequences of D, is defined as follows:

$D^i = \{s : \operatorname{len} s = i\}$.

Next we state a number of propositions:

(108) $D^i = \{s : \operatorname{len} s = i\}$.

(109) For every element z of D^i holds $\operatorname{len} z = i$.

(110) For every finite sequence z of elements of D holds z is an element of $D^{\operatorname{len} z}$.

(111) $D^i = D^{\operatorname{Seg} i}$.

(112) $D^0 = \{\varepsilon_D\}$.

(113) For every element z of D^0 holds $z = \varepsilon_D$.

(114) ε_D is an element of D^0.

(115) For every element z of D^0 and for every element t of D^i holds $z \frown t = t$ and $t \frown z = t$.

(116) $D^1 = \{\langle d \rangle\}$.

(117) For every element z of D^1 there exists d such that $z = \langle d \rangle$.

(118) $\langle d \rangle$ is an element of D^1.

(119) $D^2 = \{\langle d_1, d_2 \rangle\}$.

(120) For every element z of D^2 there exist d_1, d_2 such that $z = \langle d_1, d_2 \rangle$.

(121) $\langle d_1, d_2 \rangle$ is an element of D^2.

(122) $D^3 = \{\langle d_1, d_2, d_3 \rangle\}$.

(123) For every element z of D^3 there exist d_1, d_2, d_3 such that $z = \langle d_1, d_2, d_3 \rangle$.

(124) $\langle d_1, d_2, d_3 \rangle$ is an element of D^3.

(125) $D^{i+j} = \{z \frown t\}$.

(126) For every element s of D^{i+j} there exists an element z of D^i and there exists an element t of D^j such that $s = z \frown t$.

(127) For every element z of D^i and for every element t of D^j holds $z \frown t$ is an element of D^{i+j}.

(128) $D^* = \bigcup\{D^i\}$.

(129) For every non-empty subset D' of D and for every element z of ${D'}^i$ holds z is an element of D^i.

(130) If $D^i = D^j$, then $i = j$.

(131) id_i is an element of $\mathbb{N}^i$.

(132) $i \longmapsto d$ is an element of D^i.

(133) For every element z of D^i and for every function f from D into D' holds $f \cdot z$ is an element of ${D'}^i$.

(134) For every element z of D^i and for every function f from $\operatorname{Seg} i$ into $\operatorname{Seg} i$ such that $\operatorname{rng} f = \operatorname{Seg} i$ holds $z \cdot f$ is an element of D^i.

(135) For every element z of D^i and for every permutation f of $\operatorname{Seg} i$ holds $z \cdot f$ is an element of D^i.

(136) For every element z of D^i and for every d holds $(z \frown \langle d \rangle)(i+1) = d$.

(137) For every element z of D^{i+1} there exists an element t of D^i and there exists d such that $z = t \frown \langle d \rangle$.

(138) For every element z of D^i holds $z \cdot \mathrm{id}_i = z$.

(139) For all elements z_1, z_2 of D^i such that for every j such that $j \in \mathrm{Seg}\, i$ holds $z_1(j) = z_2(j)$ holds $z_1 = z_2$.

(140) For every function F from $[: D, D' :]$ into E and for every element z_1 of D^i and for every element z_2 of D'^i holds $F^\circ(z_1, z_2)$ is an element of E^i.

(141) For every function F from $[: D, D' :]$ into E and for every element z of D'^i holds $F^\circ(d, z)$ is an element of E^i.

(142) For every function F from $[: D, D' :]$ into E and for every element z of D^i holds $F^\circ(z, d')$ is an element of E^i.

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Received March 1, 1990

FORMALIZED MATHEMATICS
Vol.1, No.**3**, May–August 1990
Université Catholique de Louvain

Curried and Uncurried Functions

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Summary. In the article following functors are introduced: the projections of subsets of the Cartesian product, the functor which for every function $f : X \times Y \to Z$ gives some curried function $(X \to (Y \to Z))$, and the functor which from curried functions makes uncurried functions. Some of their properties and some properties of the set of all functions from a set into a set are also shown.

MML Identifier: `FUNCT_5`.

The papers [8], [3], [2], [4], [9], [1], [6], [7], and [5] provide the terminology and notation for this paper. We follow a convention: X, Y, Z, X_1, X_2, Y_1, Y_2 are sets, f, g, f_1, f_2 are functions, and x, y, z, t are arbitrary. The scheme *LambdaFS* deals with a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ and states that:

there exists f such that $\operatorname{dom} f = \mathcal{A}$ and for every g such that $g \in \mathcal{A}$ holds $f(g) = \mathcal{F}(g)$

for all values of the parameters.

We now state a proposition

(1) $\curvearrowleft\square = \square$.

We now define two new functors. Let us consider X. The functor $\pi_1(X)$ yields a set and is defined as follows:

$x \in \pi_1(X)$ if and only if there exists y such that $\langle x, y \rangle \in X$.

The functor $\pi_2(X)$ yields a set and is defined as follows:

$y \in \pi_2(X)$ if and only if there exists x such that $\langle x, y \rangle \in X$.

The following propositions are true:

(2) $Z = \pi_1(X)$ if and only if for every x holds $x \in Z$ if and only if there exists y such that $\langle x, y \rangle \in X$.

(3) $Z = \pi_2(X)$ if and only if for every y holds $y \in Z$ if and only if there exists x such that $\langle x, y \rangle \in X$.

(4) If $\langle x, y \rangle \in X$, then $x \in \pi_1(X)$ and $y \in \pi_2(X)$.

ISSN 0777–4028

(5) If $X \subseteq Y$, then $\pi_1(X) \subseteq \pi_1(Y)$ and $\pi_2(X) \subseteq \pi_2(Y)$.

(6) $\pi_1(X \cup Y) = \pi_1(X) \cup \pi_1(Y)$ and $\pi_2(X \cup Y) = \pi_2(X) \cup \pi_2(Y)$.

(7) $\pi_1(X \cap Y) \subseteq \pi_1(X) \cap \pi_1(Y)$ and $\pi_2(X \cap Y) \subseteq \pi_2(X) \cap \pi_2(Y)$.

(8) $\pi_1(X) \setminus \pi_1(Y) \subseteq \pi_1(X \setminus Y)$ and $\pi_2(X) \setminus \pi_2(Y) \subseteq \pi_2(X \setminus Y)$.

(9) $\pi_1(X) \dot{-} \pi_1(Y) \subseteq \pi_1(X \dot{-} Y)$ and $\pi_2(X) \dot{-} \pi_2(Y) \subseteq \pi_2(X \dot{-} Y)$.

(10) $\pi_1(\emptyset) = \emptyset$ and $\pi_2(\emptyset) = \emptyset$.

(11) If $Y \neq \emptyset$ or $[\!: X, Y :\!] \neq \emptyset$ or $[\!: Y, X :\!] \neq \emptyset$, then $\pi_1([\!: X, Y :\!]) = X$ and $\pi_2([\!: Y, X :\!]) = X$.

(12) $\pi_1([\!: X, Y :\!]) \subseteq X$ and $\pi_2([\!: X, Y :\!]) \subseteq Y$.

(13) If $Z \subseteq [\!: X, Y :\!]$, then $\pi_1(Z) \subseteq X$ and $\pi_2(Z) \subseteq Y$.

(14) $\pi_1([\!: X, \{x\} :\!]) = X$ and $\pi_2([\!: \{x\}, X :\!]) = X$ and $\pi_1([\!: X, \{x, y\} :\!]) = X$ and $\pi_2([\!: \{x, y\}, X :\!]) = X$.

(15) $\pi_1(\{\langle x, y\rangle\}) = \{x\}$ and $\pi_2(\{\langle x, y\rangle\}) = \{y\}$.

(16) $\pi_1(\{\langle x, y\rangle, \langle z, t\rangle\}) = \{x, z\}$ and $\pi_2(\{\langle x, y\rangle, \langle z, t\rangle\}) = \{y, t\}$.

(17) If for no x, y holds $\langle x, y\rangle \in X$, then $\pi_1(X) = \emptyset$ and $\pi_2(X) = \emptyset$.

(18) If $\pi_1(X) = \emptyset$ or $\pi_2(X) = \emptyset$, then for no x, y holds $\langle x, y\rangle \in X$.

(19) $\pi_1(X) = \emptyset$ if and only if $\pi_2(X) = \emptyset$.

(20) $\pi_1(\mathrm{dom}\, f) = \pi_2(\mathrm{dom}(\curvearrowleft f))$ and $\pi_2(\mathrm{dom}\, f) = \pi_1(\mathrm{dom}(\curvearrowleft f))$.

(21) $\pi_1(\mathrm{graph}\, f) = \mathrm{dom}\, f$ and $\pi_2(\mathrm{graph}\, f) = \mathrm{rng}\, f$.

We now define two new functors. Let us consider f. The functor curry f yielding a function, is defined by:

(i) $\mathrm{dom}(\mathrm{curry}\, f) = \pi_1(\mathrm{dom}\, f)$,

(ii) for every x such that $x \in \pi_1(\mathrm{dom}\, f)$ there exists g such that $(\mathrm{curry}\, f)(x) = g$ and $\mathrm{dom}\, g = \pi_2(\mathrm{dom}\, f \cap [\!: \{x\}, \pi_2(\mathrm{dom}\, f) :\!])$ and for every y such that $y \in \mathrm{dom}\, g$ holds $g(y) = f(\langle x, y\rangle)$.

The functor uncurry f yields a function and is defined as follows:

(i) for every t holds $t \in \mathrm{dom}(\mathrm{uncurry}\, f)$ if and only if there exist x, g, y such that $t = \langle x, y\rangle$ and $x \in \mathrm{dom}\, f$ and $g = f(x)$ and $y \in \mathrm{dom}\, g$,

(ii) for all x, g such that $x \in \mathrm{dom}(\mathrm{uncurry}\, f)$ and $g = f(x_{\mathbf{1}})$ holds $(\mathrm{uncurry}\, f)(x) = g(x_{\mathbf{2}})$.

We now define two new functors. Let us consider f. The functor curry$'$ f yields a function and is defined as follows:

$\mathrm{curry}'\, f = \mathrm{curry}(\curvearrowleft f)$.

The functor uncurry$'$ f yielding a function, is defined by:

$\mathrm{uncurry}'\, f = \curvearrowleft(\mathrm{uncurry}\, f)$.

The following propositions are true:

(22) Let F be a function. Then $F = \mathrm{curry}\, f$ if and only if the following conditions are satisfied:

(i) $\mathrm{dom}\, F = \pi_1(\mathrm{dom}\, f)$,

(ii) for every x such that $x \in \pi_1(\mathrm{dom}\, f)$ there exists g such that $F(x) = g$ and $\mathrm{dom}\, g = \pi_2(\mathrm{dom}\, f \cap [\!: \{x\}, \pi_2(\mathrm{dom}\, f) :\!])$ and for every y such that $y \in \mathrm{dom}\, g$ holds $g(y) = f(\langle x, y\rangle)$.

(23) $\operatorname{curry}' f = \operatorname{curry}(\curvearrowleft f)$.

(24) Let F be a function. Then $F = \operatorname{uncurry} f$ if and only if the following conditions are satisfied:

(i) for every t holds $t \in \operatorname{dom} F$ if and only if there exist x, g, y such that $t = \langle x, y \rangle$ and $x \in \operatorname{dom} f$ and $g = f(x)$ and $y \in \operatorname{dom} g$,

(ii) for all x, g such that $x \in \operatorname{dom} F$ and $g = f(x_{\mathbf{1}})$ holds $F(x) = g(x_{\mathbf{2}})$.

(25) $\operatorname{uncurry}' f = \curvearrowleft(\operatorname{uncurry} f)$.

(26) If $\langle x, y \rangle \in \operatorname{dom} f$, then $x \in \operatorname{dom}(\operatorname{curry} f)$ and $\operatorname{curry} f(x)$ is a function.

(27) If $\langle x, y \rangle \in \operatorname{dom} f$ and $g = \operatorname{curry} f(x)$, then $y \in \operatorname{dom} g$ and $g(y) = f(\langle x, y \rangle)$.

(28) If $\langle x, y \rangle \in \operatorname{dom} f$, then $y \in \operatorname{dom}(\operatorname{curry}' f)$ and $\operatorname{curry}' f(y)$ is a function.

(29) If $\langle x, y \rangle \in \operatorname{dom} f$ and $g = \operatorname{curry}' f(y)$, then $x \in \operatorname{dom} g$ and $g(x) = f(\langle x, y \rangle)$.

(30) $\operatorname{dom}(\operatorname{curry}' f) = \pi_2(\operatorname{dom} f)$.

(31) If $[\!: X, Y :\!] \neq \emptyset$ and $\operatorname{dom} f = [\!: X, Y :\!]$, then $\operatorname{dom}(\operatorname{curry} f) = X$ and $\operatorname{dom}(\operatorname{curry}' f) = Y$.

(32) If $\operatorname{dom} f \subseteq [\!: X, Y :\!]$, then $\operatorname{dom}(\operatorname{curry} f) \subseteq X$ and $\operatorname{dom}(\operatorname{curry}' f) \subseteq Y$.

(33) If $\operatorname{rng} f \subseteq Y^X$, then $\operatorname{dom}(\operatorname{uncurry} f) = [\!: \operatorname{dom} f, X :\!]$ and $\operatorname{dom}(\operatorname{uncurry}' f) = [\!: X, \operatorname{dom} f :\!]$.

(34) If for no x, y holds $\langle x, y \rangle \in \operatorname{dom} f$, then $\operatorname{curry} f = \square$ and $\operatorname{curry}' f = \square$.

(35) If for no x holds $x \in \operatorname{dom} f$ and $f(x)$ is a function, then $\operatorname{uncurry} f = \square$ and $\operatorname{uncurry}' f = \square$.

(36) Suppose $[\!: X, Y :\!] \neq \emptyset$ and $\operatorname{dom} f = [\!: X, Y :\!]$ and $x \in X$. Then there exists g such that $\operatorname{curry} f(x) = g$ and $\operatorname{dom} g = Y$ and $\operatorname{rng} g \subseteq \operatorname{rng} f$ and for every y such that $y \in Y$ holds $g(y) = f(\langle x, y \rangle)$.

(37) If $x \in \operatorname{dom}(\operatorname{curry} f)$, then $\operatorname{curry} f(x)$ is a function.

(38) Suppose $x \in \operatorname{dom}(\operatorname{curry} f)$ and $g = \operatorname{curry} f(x)$. Then

(i) $\operatorname{dom} g = \pi_2(\operatorname{dom} f \cap [\!: \{x\}, \pi_2(\operatorname{dom} f) :\!])$,

(ii) $\operatorname{dom} g \subseteq \pi_2(\operatorname{dom} f)$,

(iii) $\operatorname{rng} g \subseteq \operatorname{rng} f$,

(iv) for every y such that $y \in \operatorname{dom} g$ holds $g(y) = f(\langle x, y \rangle)$ and $\langle x, y \rangle \in \operatorname{dom} f$.

(39) Suppose $[\!: X, Y :\!] \neq \emptyset$ and $\operatorname{dom} f = [\!: X, Y :\!]$ and $y \in Y$. Then there exists g such that $\operatorname{curry}' f(y) = g$ and $\operatorname{dom} g = X$ and $\operatorname{rng} g \subseteq \operatorname{rng} f$ and for every x such that $x \in X$ holds $g(x) = f(\langle x, y \rangle)$.

(40) If $x \in \operatorname{dom}(\operatorname{curry}' f)$, then $\operatorname{curry}' f(x)$ is a function.

(41) Suppose $x \in \operatorname{dom}(\operatorname{curry}' f)$ and $g = \operatorname{curry}' f(x)$. Then

(i) $\operatorname{dom} g = \pi_1(\operatorname{dom} f \cap [\!: \pi_1(\operatorname{dom} f), \{x\} :\!])$,

(ii) $\operatorname{dom} g \subseteq \pi_1(\operatorname{dom} f)$,

(iii) $\operatorname{rng} g \subseteq \operatorname{rng} f$,

(iv) for every y such that $y \in \operatorname{dom} g$ holds $g(y) = f(\langle y, x \rangle)$ and $\langle y, x \rangle \in \operatorname{dom} f$.

(42) If $\mathrm{dom}\, f = [\!:X, Y:\!]$, then $\mathrm{rng}(\mathrm{curry}\, f) \subseteq (\mathrm{rng}\, f)^Y$ and $\mathrm{rng}(\mathrm{curry}'\, f) \subseteq (\mathrm{rng}\, f)^X$.

(43) $\mathrm{rng}(\mathrm{curry}\, f) \subseteq \pi_2(\mathrm{dom}\, f) \dot{\rightarrow} (\mathrm{rng}\, f)$ and $\mathrm{rng}(\mathrm{curry}'\, f) \subseteq \pi_1(\mathrm{dom}\, f) \dot{\rightarrow} (\mathrm{rng}\, f)$.

(44) If $\mathrm{rng}\, f \subseteq X \dot{\rightarrow} Y$, then $\mathrm{dom}(\mathrm{uncurry}\, f) \subseteq [\!:\mathrm{dom}\, f, X:\!]$ and $\mathrm{dom}(\mathrm{uncurry}'\, f) \subseteq [\!:X, \mathrm{dom}\, f:\!]$.

(45) If $x \in \mathrm{dom}\, f$ and $g = f(x)$ and $y \in \mathrm{dom}\, g$, then $\langle x, y\rangle \in \mathrm{dom}(\mathrm{uncurry}\, f)$ and $\mathrm{uncurry}\, f(\langle x, y\rangle) = g(y)$ and $g(y) \in \mathrm{rng}(\mathrm{uncurry}\, f)$.

(46) If $x \in \mathrm{dom}\, f$ and $g = f(x)$ and $y \in \mathrm{dom}\, g$, then $\langle y, x\rangle \in \mathrm{dom}(\mathrm{uncurry}'\, f)$ and $\mathrm{uncurry}'\, f(\langle y, x\rangle) = g(y)$ and $g(y) \in \mathrm{rng}(\mathrm{uncurry}'\, f)$.

(47) If $\mathrm{rng}\, f \subseteq X \dot{\rightarrow} Y$, then $\mathrm{rng}(\mathrm{uncurry}\, f) \subseteq Y$ and $\mathrm{rng}(\mathrm{uncurry}'\, f) \subseteq Y$.

(48) If $\mathrm{rng}\, f \subseteq Y^X$, then $\mathrm{rng}(\mathrm{uncurry}\, f) \subseteq Y$ and $\mathrm{rng}(\mathrm{uncurry}'\, f) \subseteq Y$.

(49) $\mathrm{curry}\, \square = \square$ and $\mathrm{curry}'\, \square = \square$.

(50) $\mathrm{uncurry}\, \square = \square$ and $\mathrm{uncurry}'\, \square = \square$.

(51) If $\mathrm{dom}\, f_1 = [\!:X, Y:\!]$ and $\mathrm{dom}\, f_2 = [\!:X, Y:\!]$ and $\mathrm{curry}\, f_1 = \mathrm{curry}\, f_2$, then $f_1 = f_2$.

(52) If $\mathrm{dom}\, f_1 = [\!:X, Y:\!]$ and $\mathrm{dom}\, f_2 = [\!:X, Y:\!]$ and $\mathrm{curry}'\, f_1 = \mathrm{curry}'\, f_2$, then $f_1 = f_2$.

(53) If $\mathrm{rng}\, f_1 \subseteq Y^X$ and $\mathrm{rng}\, f_2 \subseteq Y^X$ and $X \neq \emptyset$ and $\mathrm{uncurry}\, f_1 = \mathrm{uncurry}\, f_2$, then $f_1 = f_2$.

(54) If $\mathrm{rng}\, f_1 \subseteq Y^X$ and $\mathrm{rng}\, f_2 \subseteq Y^X$ and $X \neq \emptyset$ and $\mathrm{uncurry}'\, f_1 = \mathrm{uncurry}'\, f_2$, then $f_1 = f_2$.

(55) If $\mathrm{rng}\, f \subseteq Y^X$ and $X \neq \emptyset$, then $\mathrm{curry}(\mathrm{uncurry}\, f) = f$ and $\mathrm{curry}'(\mathrm{uncurry}'\, f) = f$.

(56) If $\mathrm{dom}\, f = [\!:X, Y:\!]$, then $\mathrm{uncurry}(\mathrm{curry}\, f) = f$ and $\mathrm{uncurry}'(\mathrm{curry}'\, f) = f$.

(57) If $\mathrm{dom}\, f \subseteq [\!:X, Y:\!]$, then $\mathrm{uncurry}(\mathrm{curry}\, f) = f$ and $\mathrm{uncurry}'(\mathrm{curry}'\, f) = f$.

(58) If $\mathrm{rng}\, f \subseteq X \dot{\rightarrow} Y$ and $\square \notin \mathrm{rng}\, f$, then $\mathrm{curry}(\mathrm{uncurry}\, f) = f$ and $\mathrm{curry}'(\mathrm{uncurry}'\, f) = f$.

(59) If $\mathrm{dom}\, f_1 \subseteq [\!:X, Y:\!]$ and $\mathrm{dom}\, f_2 \subseteq [\!:X, Y:\!]$ and $\mathrm{curry}\, f_1 = \mathrm{curry}\, f_2$, then $f_1 = f_2$.

(60) If $\mathrm{dom}\, f_1 \subseteq [\!:X, Y:\!]$ and $\mathrm{dom}\, f_2 \subseteq [\!:X, Y:\!]$ and $\mathrm{curry}'\, f_1 = \mathrm{curry}'\, f_2$, then $f_1 = f_2$.

(61) If $\mathrm{rng}\, f_1 \subseteq X \dot{\rightarrow} Y$ and $\mathrm{rng}\, f_2 \subseteq X \dot{\rightarrow} Y$ and $\square \notin \mathrm{rng}\, f_1$ and $\square \notin \mathrm{rng}\, f_2$ and $\mathrm{uncurry}\, f_1 = \mathrm{uncurry}\, f_2$, then $f_1 = f_2$.

(62) If $\mathrm{rng}\, f_1 \subseteq X \dot{\rightarrow} Y$ and $\mathrm{rng}\, f_2 \subseteq X \dot{\rightarrow} Y$ and $\square \notin \mathrm{rng}\, f_1$ and $\square \notin \mathrm{rng}\, f_2$ and $\mathrm{uncurry}'\, f_1 = \mathrm{uncurry}'\, f_2$, then $f_1 = f_2$.

(63) If $X \subseteq Y$, then $X^Z \subseteq Y^Z$.

(64) $X^{\emptyset} = \{\square\}$.

(65) $X \approx X^{\{x\}}$ and $\overline{\overline{X}} = \overline{\overline{X^{\{x\}}}}$.

(66) $\{x\}^X = \{X \longmapsto x\}$.

(67) If $X_1 \approx Y_1$ and $X_2 \approx Y_2$, then ${X_2}^{X_1} \approx {Y_2}^{Y_1}$ and $\overline{\overline{{X_2}^{X_1}}} = \overline{\overline{{Y_2}^{Y_1}}}$.

(68) If $\overline{\overline{X_1}} = \overline{\overline{Y_1}}$ and $\overline{\overline{X_2}} = \overline{\overline{Y_2}}$, then $\overline{\overline{{X_2}^{X_1}}} = \overline{\overline{{Y_2}^{Y_1}}}$.

(69) If $X_1 \cap X_2 = \emptyset$, then $X^{X_1 \cup X_2} \approx [\!: X^{X_1}, X^{X_2} :\!]$ and $\overline{\overline{X^{X_1 \cup X_2}}} = \overline{\overline{[\!: X^{X_1}, X^{X_2} :\!]}}$.

(70) $Z^{[\!: X, Y :\!]} \approx (Z^Y)^X$ and $\overline{\overline{Z^{[\!: X, Y :\!]}}} = \overline{\overline{(Z^Y)^X}}$.

(71) $[\!: X, Y :\!]^Z \approx [\!: X^Z, Y^Z :\!]$ and $\overline{\overline{[\!: X, Y :\!]^Z}} = \overline{\overline{[\!: X^Z, Y^Z :\!]}}$.

(72) If $x \neq y$, then $\{x, y\}^X \approx 2^X$ and $\overline{\overline{\{x, y\}^X}} = \overline{\overline{2^X}}$.

(73) If $x \neq y$, then $X^{\{x,y\}} \approx [\!: X, X :\!]$ and $\overline{\overline{X^{\{x,y\}}}} = \overline{\overline{[\!: X, X :\!]}}$.

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Received March 6, 1990

FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Cardinal Arithmetics

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Summary. In the article addition, multiplication and power operation of cardinals are introduced. Presented are some properties of equipotence of Cartesian products, basic cardinal arithmetics laws (transformativity, associativity, distributivity), and some facts about finite sets.

MML Identifier: CARD_2.

The articles [12], [11], [7], [8], [3], [4], [5], [10], [2], [6], [9], and [1] provide the terminology and notation for this paper. For simplicity we follow a convention: A, B denote ordinal numbers, K, M, N denote cardinal numbers, x, x_1, x_2, y, y_1, y_2 are arbitrary, X, Y, Z, X_1, X_2, Y_1, Y_2 denote sets, and f denotes a function. Let us consider x. The functor $[x]$ yielding a set, is defined by:

$[x] = x$.

Next we state several propositions:

(1) $[x] = x$.

(2) $\overline{\overline{X}} \leq \overline{\overline{Y}}$ if and only if there exists f such that $X = f^\circ Y$ or $X \subseteq f^\circ Y$.

(3) $\overline{\overline{f^\circ X}} \leq \overline{\overline{X}}$.

(4) If $\overline{\overline{X}} < \overline{\overline{Y}}$, then $Y \setminus X \neq \emptyset$.

(5) If $x \in X$ and $X \approx Y$, then $Y \neq \emptyset$ and there exists x such that $x \in Y$.

(6) $2^X \approx 2^{\overline{\overline{X}}}$ and $\overline{\overline{2^X}} = \overline{\overline{2^{\overline{\overline{X}}}}}$.

(7) If $Z \in Y^X$, then $Z \approx X$ and $\overline{\overline{Z}} = \overline{\overline{X}}$.

We now define three new functors. Let us consider M, N. The functor $M + N$ yielding a cardinal number, is defined as follows:

$M + N = \overline{\overline{\mathrm{ord}(M) + \mathrm{ord}(N)}}$.

The functor $M \cdot N$ yielding a cardinal number, is defined by:

$M \cdot N = \overline{\overline{[: M, N :]}}$.

The functor M^N yielding a cardinal number, is defined by:

ISSN 0777–4028

$M^N = \overline{\overline{M^N}}$.

Next we state a number of propositions:

(8) $M + N = \overline{\overline{\mathrm{ord}(M) + \mathrm{ord}(N)}}$.

(9) $M \cdot N = \overline{\overline{[\!: M, N :\!]}}$.

(10) $M^N = \overline{\overline{M^N}}$.

(11) $[\!: X, Y :\!] \approx [\!: Y, X :\!]$ and $\overline{\overline{[\!: X, Y :\!]}} = \overline{\overline{[\!: Y, X :\!]}}$.

(12) $[\!: [\!: X, Y :\!], Z :\!] \approx [\!: X, [\!: Y, Z :\!] :\!]$ and $\overline{\overline{[\!: [\!: X, Y :\!], Z :\!]}} = \overline{\overline{[\!: X, [\!: Y, Z :\!] :\!]}}$.

(13) $X \approx [\!: X, \{x\} :\!]$ and $\overline{\overline{X}} = \overline{\overline{[\!: X, \{x\} :\!]}}$.

(14) (i) $[\!: X, Y :\!] \approx [\!: \overline{\overline{X}}, Y :\!]$,

(ii) $[\!: X, Y :\!] \approx [\!: X, \overline{\overline{Y}} :\!]$,

(iii) $[\!: X, Y :\!] \approx [\!: \overline{\overline{X}}, \overline{\overline{Y}} :\!]$,

(iv) $\overline{\overline{[\!: X, Y :\!]}} = \overline{\overline{[\!: \overline{\overline{X}}, Y :\!]}}$,

(v) $\overline{\overline{[\!: X, Y :\!]}} = \overline{\overline{[\!: X, \overline{\overline{Y}} :\!]}}$,

(vi) $\overline{\overline{[\!: X, Y :\!]}} = \overline{\overline{[\!: \overline{\overline{X}}, \overline{\overline{Y}} :\!]}}$.

(15) If $X_1 \approx Y_1$ and $X_2 \approx Y_2$, then $[\!: X_1, X_2 :\!] \approx [\!: Y_1, Y_2 :\!]$ and $\overline{\overline{[\!: X_1, X_2 :\!]}} = \overline{\overline{[\!: Y_1, Y_2 :\!]}}$.

(16) If $x_1 \neq x_2$, then $A + B \approx [\!: A, \{x_1\} :\!] \cup [\!: B, \{x_2\} :\!]$ and $\overline{\overline{A + B}} = \overline{\overline{[\!: A, \{x_1\} :\!] \cup [\!: B, \{x_2\} :\!]}}$.

(17) If $x_1 \neq x_2$, then $K + M \approx [\!: K, \{x_1\} :\!] \cup [\!: M, \{x_2\} :\!]$ and $K + M = \overline{\overline{[\!: K, \{x_1\} :\!] \cup [\!: M, \{x_2\} :\!]}}$.

(18) $A \cdot B \approx [\!: A, B :\!]$ and $\overline{\overline{A \cdot B}} = \overline{\overline{[\!: A, B :\!]}}$.

We now define three new functors. The cardinal number $\overline{\mathbf{0}}$ is defined by:

$\overline{\mathbf{0}} = \overline{\overline{\mathbf{0}}}$.

The cardinal number $\overline{\mathbf{1}}$ is defined as follows:

$\overline{\mathbf{1}} = \overline{\overline{\mathbf{1}}}$.

The cardinal number $\overline{\mathbf{2}}$ is defined as follows:

$\overline{\mathbf{2}} = \overline{\overline{\mathrm{succ}\,\mathbf{1}}}$.

The following propositions are true:

(19) $\overline{\mathbf{0}} = \overline{\overline{\mathbf{0}}}$ and $\overline{\mathbf{1}} = \overline{\overline{\mathbf{1}}}$ and $\overline{\mathbf{2}} = \overline{\overline{\mathrm{succ}\,\mathbf{1}}}$.

(20) $\overline{\mathbf{0}} = \mathbf{0}$ and $\overline{\mathbf{0}} = \emptyset$ and $\overline{\mathbf{1}} = \mathbf{1}$.

(21) $\overline{\mathbf{0}} = \overline{\overline{0}}$ and $\overline{\mathbf{1}} = \overline{\overline{1}}$ and $\overline{\mathbf{2}} = \overline{\overline{2}}$.

(22) $\overline{\mathbf{2}} = \{\mathbf{0}, \mathbf{1}\}$ and $\overline{\mathbf{2}} = \mathrm{succ}\,\mathbf{1}$.

(23) Suppose $X_1 \approx Y_1$ and $X_2 \approx Y_2$ and $x_1 \neq x_2$ and $y_1 \neq y_2$. Then $[\!: X_1, \{x_1\} :\!] \cup [\!: X_2, \{x_2\} :\!] \approx [\!: Y_1, \{y_1\} :\!] \cup [\!: Y_2, \{y_2\} :\!]$ and $\overline{\overline{[\!: X_1, \{x_1\} :\!] \cup [\!: X_2, \{x_2\} :\!]}} = \overline{\overline{[\!: Y_1, \{y_1\} :\!] \cup [\!: Y_2, \{y_2\} :\!]}}$.

(24) $\overline{\overline{A + B}} = \overline{\overline{A}} + \overline{\overline{B}}$.

(25) $\overline{\overline{A \cdot B}} = \overline{\overline{A}} \cdot \overline{\overline{B}}$.

(26) $[\!: X, \{0\} :\!] \cup [\!: Y, \{1\} :\!] \approx [\!: Y, \{0\} :\!] \cup [\!: X, \{1\} :\!]$ and $\overline{\overline{[\!: X, \{0\} :\!] \cup [\!: Y, \{1\} :\!]}} = \overline{\overline{[\!: Y, \{0\} :\!] \cup [\!: X, \{1\} :\!]}}$.

(27) $[\!: X_1, X_2 :\!] \cup [\!: Y_1, Y_2 :\!] \approx [\!: X_2, X_1 :\!] \cup [\!: Y_2, Y_1 :\!]$ and $\overline{\overline{[\!: X_1, X_2 :\!] \cup [\!: Y_1, Y_2 :\!]}} = \overline{\overline{[\!: X_2, X_1 :\!] \cup [\!: Y_2, Y_1 :\!]}}$.

(28) If $x \neq y$, then $\overline{\overline{X}} + \overline{\overline{Y}} = \overline{\overline{[\!: X, \{x\} :\!] \cup [\!: Y, \{y\} :\!]}}$.

(29) $M + \overline{\mathbf{0}} = M$ and $\overline{\mathbf{0}} + M = M$.

(30) $M + N = N + M$.

(31) $(K + M) + N = K + (M + N)$.

(32) $K \cdot \overline{\mathbf{0}} = \overline{\mathbf{0}}$ and $\overline{\mathbf{0}} \cdot K = \overline{\mathbf{0}}$.

(33) $K \cdot \overline{\mathbf{1}} = K$ and $\overline{\mathbf{1}} \cdot K = K$.

(34) $K \cdot M = M \cdot K$.

(35) $(K \cdot M) \cdot N = K \cdot (M \cdot N)$.

(36) $\overline{\mathbf{2}} \cdot K = K + K$ and $K \cdot \overline{\mathbf{2}} = K + K$.

(37) $K \cdot (M + N) = K \cdot M + K \cdot N$ and $(M + N) \cdot K = M \cdot K + N \cdot K$.

(38) $K^{\overline{\mathbf{0}}} = \overline{\mathbf{1}}$.

(39) If $K \neq \overline{\mathbf{0}}$, then $\overline{\mathbf{0}}^K = \overline{\mathbf{0}}$.

(40) $K^{\overline{\mathbf{1}}} = K$ and $\overline{\mathbf{1}}^K = \overline{\mathbf{1}}$.

(41) $K^{M+N} = (K^M) \cdot (K^N)$.

(42) $(K \cdot M)^N = (K^N) \cdot (M^N)$.

(43) $K^{M \cdot N} = (K^M)^N$.

(44) $\overline{\mathbf{2}}^{\overline{\overline{X}}} = \overline{\overline{2^X}}$.

(45) $K^{\overline{\mathbf{2}}} = K \cdot K$.

(46) $(K + M)^{\overline{\mathbf{2}}} = (K \cdot K + (\overline{\mathbf{2}} \cdot K) \cdot M) + M \cdot M$.

(47) $\overline{\overline{X \cup Y}} \leq \overline{\overline{X}} + \overline{\overline{Y}}$.

(48) If $X \cap Y = \emptyset$, then $\overline{\overline{X \cup Y}} = \overline{\overline{X}} + \overline{\overline{Y}}$.

In the sequel m, n will denote natural numbers. Next we state a number of propositions:

(49) $\mathrm{ord}(n + m) = \mathrm{ord}(n) + \mathrm{ord}(m)$.

(50) $\mathrm{ord}(n \cdot m) = \mathrm{ord}(n) \cdot \mathrm{ord}(m)$.

(51) $\overline{\overline{n + m}} = \overline{\overline{n}} + \overline{\overline{m}}$.

(52) $\overline{\overline{n \cdot m}} = \overline{\overline{n}} \cdot \overline{\overline{m}}$.

(53) If X is finite and Y is finite and $X \cap Y = \emptyset$, then $\mathrm{card}(X \cup Y) = \mathrm{card}\, X + \mathrm{card}\, Y$.

(54) If X is finite and $x \notin X$, then $\mathrm{card}(X \cup \{x\}) = \mathrm{card}\, X + 1$.

(55) If X is finite and Y is finite, then $\mathrm{card}\, X = \mathrm{card}\, Y$ if and only if $X \approx Y$.

(56) If X is finite and Y is finite, then $\overline{\overline{X}} = \overline{\overline{Y}}$ if and only if $\mathrm{card}\, X = \mathrm{card}\, Y$.

(57) If X is finite and Y is finite, then $\overline{\overline{X}} \leq \overline{\overline{Y}}$ if and only if $\operatorname{card} X \leq \operatorname{card} Y$.

(58) If X is finite and Y is finite, then $\overline{\overline{X}} < \overline{\overline{Y}}$ if and only if $\operatorname{card} X < \operatorname{card} Y$.

(59) If X is finite, then $X = \emptyset$ if and only if $\operatorname{card} X = 0$.

(60) If X is finite, then $\operatorname{card} X = 1$ if and only if there exists x such that $X = \{x\}$.

(61) If X is finite, then $X \approx \operatorname{ord}(\operatorname{card} X)$ and $X \approx \overline{\overline{\operatorname{card} X}}$ and $X \approx \operatorname{Seg}(\operatorname{card} X)$.

(62) If X is finite and Y is finite, then $\operatorname{card}(X \cup Y) \leq \operatorname{card} X + \operatorname{card} Y$.

(63) If $Y \subseteq X$ and X is finite, then $\operatorname{card}(X \setminus Y) = \operatorname{card} X - \operatorname{card} Y$.

(64) If X is finite and Y is finite, then $\operatorname{card}(X \cup Y) = (\operatorname{card} X + \operatorname{card} Y) - \operatorname{card}(X \cap Y)$.

(65) If X is finite and Y is finite, then $\operatorname{card}[\!: X, Y :\!] = \operatorname{card} X \cdot \operatorname{card} Y$.

(66) If $X \subseteq Y$ and Y is finite, then $\operatorname{card} X \leq \operatorname{card} Y$.

(67) If $X \subseteq Y$ and $X \neq Y$ and Y is finite, then $\operatorname{card} X < \operatorname{card} Y$ and $\overline{\overline{X}} < \overline{\overline{Y}}$.

(68) If $\overline{\overline{X}} \leq \overline{\overline{Y}}$ or $\overline{\overline{X}} < \overline{\overline{Y}}$ but Y is finite, then X is finite.

In the sequel x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 are arbitrary. One can prove the following propositions:

(69) $\operatorname{card}\{x_1, x_2\} \leq 2$.

(70) $\operatorname{card}\{x_1, x_2, x_3\} \leq 3$.

(71) $\operatorname{card}\{x_1, x_2, x_3, x_4\} \leq 4$.

(72) $\operatorname{card}\{x_1, x_2, x_3, x_4, x_5\} \leq 5$.

(73) $\operatorname{card}\{x_1, x_2, x_3, x_4, x_5, x_6\} \leq 6$.

(74) $\operatorname{card}\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\} \leq 7$.

(75) $\operatorname{card}\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\} \leq 8$.

(76) If $x_1 \neq x_2$, then $\operatorname{card}\{x_1, x_2\} = 2$.

(77) If $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_2 \neq x_3$, then $\operatorname{card}\{x_1, x_2, x_3\} = 3$.

(78) If $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_2 \neq x_3$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_3 \neq x_4$, then $\operatorname{card}\{x_1, x_2, x_3, x_4\} = 4$.

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Received March 6, 1990

FORMALIZED MATHEMATICS
Vol.1, No.**3**, May–August 1990
Université Catholique de Louvain

Fano-Desargues Parallelity Spaces [1]

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Summary. This article is the second part of Parallelity Space. It contain definition of a Fano-Desargues space, axioms of a Fano-Desargues parallelity space, definition of the relations: collinearity, parallelogram and directed congruence and some basic facts concerned with them.

MML Identifier: `PARSP_2`.

The papers [2], and [1] provide the notation and terminology for this paper. In the sequel F will denote a field. We now state a proposition

(1) Aff_{F^3} is a parallelity space.

We follow the rules: a, b, c, d, p, q, r will denote elements of the universum of Aff_{F^3}, e, f, g, h will denote elements of $[\!:$ the carrier of F, the carrier of F, the carrier of $F\,:\!]$, and K, L will denote elements of the carrier of F. One can prove the following propositions:

(2) $a, b \parallel c, d$ if and only if there exist e, f, g, h such that $\langle a, b, c, d \rangle = \langle e, f, g, h \rangle$ but there exists K such that $K \cdot (e_{\mathbf{1}} - f_{\mathbf{1}}) = g_{\mathbf{1}} - h_{\mathbf{1}}$ and $K \cdot (e_{\mathbf{2}} - f_{\mathbf{2}}) = g_{\mathbf{2}} - h_{\mathbf{2}}$ and $K \cdot (e_{\mathbf{3}} - f_{\mathbf{3}}) = g_{\mathbf{3}} - h_{\mathbf{3}}$ or $e_{\mathbf{1}} - f_{\mathbf{1}} = 0_F$ and $e_{\mathbf{2}} - f_{\mathbf{2}} = 0_F$ and $e_{\mathbf{3}} - f_{\mathbf{3}} = 0_F$.

(3) If $a, b \nparallel a, c$ and $\langle a, b, a, c \rangle = \langle e, f, e, g \rangle$, then $e \neq f$ and $e \neq g$ and $f \neq g$.

(4) Suppose that
 (i) $a, b \nparallel a, c$,
 (ii) $\langle a, b, a, c \rangle = \langle e, f, e, g \rangle$,
 (iii) $K \cdot (e_{\mathbf{1}} - f_{\mathbf{1}}) = L \cdot (e_{\mathbf{1}} - g_{\mathbf{1}})$,
 (iv) $K \cdot (e_{\mathbf{2}} - f_{\mathbf{2}}) = L \cdot (e_{\mathbf{2}} - g_{\mathbf{2}})$,
 (v) $K \cdot (e_{\mathbf{3}} - f_{\mathbf{3}}) = L \cdot (e_{\mathbf{3}} - g_{\mathbf{3}})$.

 Then $K = 0_F$ and $L = 0_F$.

[1]Supported by RPBP III.24

ISSN 0777–4028

(5) Suppose $a, b \nparallel a, c$ and $a, b \parallel c, d$ and $a, c \parallel b, d$ and $\langle a, b, c, d\rangle = \langle e, f, g, h\rangle$. Then $h_{\mathbf{1}} = (f_{\mathbf{1}} + g_{\mathbf{1}}) - e_{\mathbf{1}}$ and $h_{\mathbf{2}} = (f_{\mathbf{2}} + g_{\mathbf{2}}) - e_{\mathbf{2}}$ and $h_{\mathbf{3}} = (f_{\mathbf{3}} + g_{\mathbf{3}}) - e_{\mathbf{3}}$.

(6) There exist a, b, c such that $a, b \nparallel a, c$.

(7) If $1_F + 1_F \neq 0_F$ and $b, c \parallel a, d$ and $a, b \parallel c, d$ and $a, c \parallel b, d$, then $a, b \parallel a, c$.

(8) If $a, p \nparallel a, b$ and $a, p \nparallel a, c$ and $a, p \parallel b, q$ and $a, p \parallel c, r$ and $a, b \parallel p, q$ and $a, c \parallel p, r$, then $b, c \parallel q, r$.

A parallelity space is called a Fano-Desarques space if:

(i) there exist elements a, b, c of the universum of it such that $a, b \nparallel a, c$,

(ii) for all elements a, b, c, d of the universum of it such that $b, c \parallel a, d$ and $a, b \parallel c, d$ and $a, c \parallel b, d$ holds $a, b \parallel a, c$,

(iii) for all elements a, b, c, p, q, r of the universum of it such that $a, p \nparallel a, b$ and $a, p \nparallel a, c$ and $a, p \parallel b, q$ and $a, p \parallel c, r$ and $a, b \parallel p, q$ and $a, c \parallel p, r$ holds $b, c \parallel q, r$.

We now state a proposition

(9) Let Fd be a parallelity space. Then the following conditions are equivalent:

  (i) there exist elements a, b, c of the universum of Fd such that $a, b \nparallel a, c$ and for all elements a, b, c, d of the universum of Fd such that $b, c \parallel a, d$ and $a, b \parallel c, d$ and $a, c \parallel b, d$ holds $a, b \parallel a, c$ and for all elements a, b, c, p, q, r of the universum of Fd such that $a, p \nparallel a, b$ and $a, p \nparallel a, c$ and $a, p \parallel b, q$ and $a, p \parallel c, r$ and $a, b \parallel p, q$ and $a, c \parallel p, r$ holds $b, c \parallel q, r$,

  (ii) Fd is a Fano-Desarques space.

We adopt the following convention: $FdSp$ is a Fano-Desarques space and a, b, c, d, p, q, r, s, o, x, y are elements of the universum of $FdSp$. The following propositions are true:

(10) There exist a, b, c such that $a, b \nparallel a, c$.

(11) If $b, c \parallel a, d$ and $a, b \parallel c, d$ and $a, c \parallel b, d$, then $a, b \parallel a, c$.

(12) If $a, p \nparallel a, b$ and $a, p \nparallel a, c$ and $a, p \parallel b, q$ and $a, p \parallel c, r$ and $a, b \parallel p, q$ and $a, c \parallel p, r$, then $b, c \parallel q, r$.

(13) If $p \neq q$, then there exists r such that $p, q \nparallel p, r$.

Let us consider $FdSp$, a, b, c. The predicate $\mathbf{L}(a, b, c)$ is defined as follows: $a, b \parallel a, c$.

The following propositions are true:

(14) $\mathbf{L}(a, b, c)$ if and only if $a, b \parallel a, c$.

(15) If $\mathbf{L}(a, b, c)$, then $\mathbf{L}(a, c, b)$ and $\mathbf{L}(c, b, a)$ and $\mathbf{L}(b, a, c)$ and $\mathbf{L}(b, c, a)$ and $\mathbf{L}(c, a, b)$.

(16) If not $\mathbf{L}(a, b, c)$, then not $\mathbf{L}(a, c, b)$ and not $\mathbf{L}(c, b, a)$ and not $\mathbf{L}(b, a, c)$ and not $\mathbf{L}(b, c, a)$ and not $\mathbf{L}(c, a, b)$.

(17) If not $\mathbf{L}(a, b, c)$ and $a, b \parallel p, q$ and $a, c \parallel p, r$ and $p \neq q$ and $p \neq r$, then not $\mathbf{L}(p, q, r)$.

(18) If $a = b$ or $b = c$ or $c = a$, then $\mathbf{L}(a, b, c)$.

(19) If $a \neq b$ and $\mathbf{L}(a, b, p)$ and $\mathbf{L}(a, b, q)$ and $\mathbf{L}(a, b, r)$, then $\mathbf{L}(p, q, r)$.

(20) If $p \neq q$, then there exists r such that not $\mathbf{L}(p, q, r)$.

(21) If $\mathbf{L}(a, b, c)$ and $\mathbf{L}(a, b, d)$, then $a, b \parallel c, d$.

(22) If not $\mathbf{L}(a, b, c)$ and $a, b \parallel c, d$, then not $\mathbf{L}(a, b, d)$.

(23) If not $\mathbf{L}(a, b, c)$ and $a, b \parallel c, d$ and $c \neq d$, then not $\mathbf{L}(a, b, x)$ or not $\mathbf{L}(c, d, x)$.

(24) If not $\mathbf{L}(o, a, b)$, then not $\mathbf{L}(o, a, x)$ or not $\mathbf{L}(o, b, x)$ or $o = x$.

(25) If $o \neq a$ and $o \neq b$ and $\mathbf{L}(o, a, b)$ and $\mathbf{L}(o, a, p)$ and $\mathbf{L}(o, b, q)$, then $a, b \parallel p, q$.

(26) If $a, b \nparallel c, d$ and $\mathbf{L}(a, b, p)$ and $\mathbf{L}(a, b, q)$ and $\mathbf{L}(c, d, p)$ and $\mathbf{L}(c, d, q)$, then $p = q$.

(27) If $a \neq b$ and $\mathbf{L}(a, b, c)$ and $a, b \parallel c, d$, then $a, c \parallel b, d$.

(28) If $a \neq b$ and $\mathbf{L}(a, b, c)$ and $a, b \parallel c, d$, then $c, b \parallel c, d$.

(29) If not $\mathbf{L}(o, a, c)$ and $\mathbf{L}(o, a, b)$ and $\mathbf{L}(o, c, p)$ and $\mathbf{L}(o, c, q)$ and $a, c \parallel b, p$ and $a, c \parallel b, q$, then $p = q$.

(30) If $a \neq b$ and $\mathbf{L}(a, b, c)$ and $\mathbf{L}(a, b, d)$, then $\mathbf{L}(a, c, d)$.

(31) If $\mathbf{L}(a, b, c)$ and $\mathbf{L}(a, c, d)$ and $a \neq c$, then $\mathbf{L}(b, c, d)$.

(32) $\mathbf{L}(a, b, c)$ if and only if $a, b \parallel a, c$.

Let us consider $FdSp$, a, b, c, d. The predicate $\mathbf{P}(a, b, c, d)$ is defined by: not $\mathbf{L}(a, b, c)$ and $a, b \parallel c, d$ and $a, c \parallel b, d$.

Next we state a number of propositions:

(33) $\mathbf{P}(a, b, c, d)$ if and only if not $\mathbf{L}(a, b, c)$ and $a, b \parallel c, d$ and $a, c \parallel b, d$.

(34) If $\mathbf{P}(a, b, c, d)$, then $a \neq b$ and $b \neq c$ and $c \neq a$ and $a \neq d$ and $b \neq d$ and $c \neq d$.

(35) If $\mathbf{P}(a, b, c, d)$, then not $\mathbf{L}(a, b, c)$ and not $\mathbf{L}(b, a, d)$ and not $\mathbf{L}(c, d, a)$ and not $\mathbf{L}(d, c, b)$.

(36) Suppose $\mathbf{P}(a, b, c, d)$. Then not $\mathbf{L}(a, b, c)$ and not $\mathbf{L}(b, a, d)$ and not $\mathbf{L}(c, d, a)$ and not $\mathbf{L}(d, c, b)$ and not $\mathbf{L}(a, c, b)$ and not $\mathbf{L}(b, a, c)$ and not $\mathbf{L}(b, c, a)$ and not $\mathbf{L}(c, a, b)$ and not $\mathbf{L}(c, b, a)$ and not $\mathbf{L}(b, d, a)$ and not $\mathbf{L}(a, b, d)$ and not $\mathbf{L}(a, d, b)$ and not $\mathbf{L}(d, a, b)$ and not $\mathbf{L}(d, b, a)$ and not $\mathbf{L}(c, a, d)$ and not $\mathbf{L}(a, c, d)$ and not $\mathbf{L}(a, d, c)$ and not $\mathbf{L}(d, a, c)$ and not $\mathbf{L}(d, c, a)$ and not $\mathbf{L}(d, b, c)$ and not $\mathbf{L}(b, c, d)$ and not $\mathbf{L}(b, d, c)$ and not $\mathbf{L}(c, b, d)$ and not $\mathbf{L}(c, d, b)$.

(37) If $\mathbf{P}(a, b, c, d)$, then not $\mathbf{L}(a, b, x)$ or not $\mathbf{L}(c, d, x)$.

(38) If $\mathbf{P}(a, b, c, d)$, then $\mathbf{P}(a, c, b, d)$.

(39) If $\mathbf{P}(a, b, c, d)$, then $\mathbf{P}(c, d, a, b)$.

(40) If $\mathbf{P}(a, b, c, d)$, then $\mathbf{P}(b, a, d, c)$.

(41) If $\mathbf{P}(a, b, c, d)$, then $\mathbf{P}(a, c, b, d)$ and $\mathbf{P}(c, d, a, b)$ and $\mathbf{P}(b, a, d, c)$ and $\mathbf{P}(c, a, d, b)$ and $\mathbf{P}(d, b, c, a)$ and $\mathbf{P}(b, d, a, c)$ and $\mathbf{P}(d, c, b, a)$.

(42) If not $\mathbf{L}(a,b,c)$, then there exists d such that $\mathbf{P}(a,b,c,d)$.

(43) If $\mathbf{P}(a,b,c,p)$ and $\mathbf{P}(a,b,c,q)$, then $p=q$.

(44) If $\mathbf{P}(a,b,c,d)$, then $a,d \nparallel b,c$.

(45) If $\mathbf{P}(a,b,c,d)$, then not $\mathbf{P}(a,b,d,c)$.

(46) If $a \neq b$, then there exists c such that $\mathbf{L}(a,b,c)$ and $c \neq a$ and $c \neq b$.

(47) If $\mathbf{P}(a,p,b,q)$ and $\mathbf{P}(a,p,c,r)$, then $b,c \parallel q,r$.

(48) If not $\mathbf{L}(b,q,c)$ and $\mathbf{P}(a,p,b,q)$ and $\mathbf{P}(a,p,c,r)$, then $\mathbf{P}(b,q,c,r)$.

(49) If $\mathbf{L}(a,b,c)$ and $b \neq c$ and $\mathbf{P}(a,p,b,q)$ and $\mathbf{P}(a,p,c,r)$, then $\mathbf{P}(b,q,c,r)$.

(50) If $\mathbf{P}(a,p,b,q)$ and $\mathbf{P}(a,p,c,r)$ and $\mathbf{P}(b,q,d,s)$, then $c,d \parallel r,s$.

(51) If $a \neq b$, then there exist c, d such that $\mathbf{P}(a,b,c,d)$.

(52) If $a \neq d$, then there exist b, c such that $\mathbf{P}(a,b,c,d)$.

(53) $\mathbf{P}(a,b,c,d)$ if and only if not $\mathbf{L}(a,b,c)$ and $a,b \parallel c,d$ and $a,c \parallel b,d$.

Let us consider $FdSp$, a, b, r, s. The predicate $a,b \Rrightarrow r,s$ is defined as follows:

$a=b$ and $r=s$ or there exist p, q such that $\mathbf{P}(p,q,a,b)$ and $\mathbf{P}(p,q,r,s)$.

One can prove the following propositions:

(54) $a,b \Rrightarrow r,s$ if and only if $a=b$ and $r=s$ or there exist p, q such that $\mathbf{P}(p,q,a,b)$ and $\mathbf{P}(p,q,r,s)$.

(55) If $a,a \Rrightarrow b,c$, then $b=c$.

(56) If $a,b \Rrightarrow c,c$, then $a=b$.

(57) If $a,b \Rrightarrow b,a$, then $a=b$.

(58) If $a,b \Rrightarrow c,d$, then $a,b \parallel c,d$.

(59) If $a,b \Rrightarrow c,d$, then $a,c \parallel b,d$.

(60) If $a,b \Rrightarrow c,d$ and not $\mathbf{L}(a,b,c)$, then $\mathbf{P}(a,b,c,d)$.

(61) If $\mathbf{P}(a,b,c,d)$, then $a,b \Rrightarrow c,d$.

(62) If $a,b \Rrightarrow c,d$ and $\mathbf{L}(a,b,c)$ and $\mathbf{P}(r,s,a,b)$, then $\mathbf{P}(r,s,c,d)$.

(63) If $a,b \Rrightarrow c,x$ and $a,b \Rrightarrow c,y$, then $x=y$.

(64) There exists d such that $a,b \Rrightarrow c,d$.

(65) $a,a \Rrightarrow b,b$.

(66) $a,b \Rrightarrow a,b$.

(67) If $r,s \Rrightarrow a,b$ and $r,s \Rrightarrow c,d$, then $a,b \Rrightarrow c,d$.

(68) If $a,b \Rrightarrow c,d$, then $c,d \Rrightarrow a,b$.

(69) If $a,b \Rrightarrow c,d$, then $b,a \Rrightarrow d,c$.

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Received March 23, 1990

FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Real Functions Spaces [1]

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Summary. This abstract contains a construction of the domain of functions defined in an arbitrary nonempty set, with values being real numbers. In every such set of functions we introduce several algebraic operations, which yield in this set the structures of a real linear space, of a ring, and of a real algebra. Formal definitions of such concepts are given.

MML Identifier: `FUNCSDOM`.

The notation and terminology used in this paper are introduced in the following papers: [3], [9], [11], [2], [7], [12], [6], [1], [10], [4], [5], and [8]. We adopt the following convention: x_1, x_2, z are arbitrary and A, B denote non-empty sets. Let us consider A, B, and let F be a binary operation on B^A, and let f, g be elements of B^A. Then $F(f,g)$ is an element of B^A.

Let A, B, C, D be non-empty sets, and let F be a function from $[: C, D :]$ into B^A, and let cd be an element of $[: C, D :]$. Then $F(cd)$ is an element of B^A.

Let A, B be non-empty sets, and let f be a function from A into B. The functor $@f$ yields an element of B^A and is defined by:

$@f = f$.

We now state a proposition

(1) For all functions f, g from A into B holds $@f = g$ if and only if $f = g$.

In the sequel f, g, h denote elements of $\mathbb{R}^A$. Let A, B be non-empty sets, and let x be an element of B^A. The functor $\downharpoonleft x$ yields an element of B^A **qua** a non-empty set and is defined as follows:

$\downharpoonleft x = x$.

We now state a proposition

(2) For all elements f, g of B^A holds $\downharpoonleft f = g$ if and only if $f = g$.

[1]Supported by RPBP.III-24.C2

ISSN 0777–4028

Let us consider A, B, and let f be an element of B^A **qua** a non-empty set. The functor $f\downharpoonleft$ yielding an element of B^A, is defined by:

$f\downharpoonleft = f$.

We now state two propositions:

(3) For all elements f, g of B^A **qua** non-empty sets holds $f\downharpoonleft = g$ if and only if $f = g$.

(4) $f = (\downharpoonright f)\downharpoonleft$.

Let X, Z be non-empty sets, and let F be a binary operation on X, and let f, g be functions from Z into X. Then $F^\circ(f,g)$ is an element of X^Z.

Let X, Z be non-empty sets, and let F be a binary operation on X, and let a be an element of X, and let f be a function from Z into X. Then $F^\circ(a,f)$ is an element of X^Z.

Let us consider A. The functor $+_{\mathbb{R}^A}$ yields a binary operation on $\mathbb{R}^A$ and is defined by:

for all elements f, g of $\mathbb{R}^A$ holds $+_{\mathbb{R}^A}(f,g) = +_{\mathbb{R}}{}^\circ(f,g)$.

We now state a proposition

(5) For every binary operation F on $\mathbb{R}^A$ holds $F = +_{\mathbb{R}^A}$ if and only if for all elements f, g of $\mathbb{R}^A$ holds $F(f,g) = +_{\mathbb{R}}{}^\circ(f,g)$.

Let us consider A. The functor $\cdot_{\mathbb{R}^A}$ yields a binary operation on $\mathbb{R}^A$ and is defined as follows:

for all elements f, g of $\mathbb{R}^A$ holds $\cdot_{\mathbb{R}^A}(f,g) = \cdot_{\mathbb{R}}{}^\circ(f,g)$.

Next we state a proposition

(6) For every binary operation F on $\mathbb{R}^A$ holds $F = \cdot_{\mathbb{R}^A}$ if and only if for all elements f, g of $\mathbb{R}^A$ holds $F(f,g) = \cdot_{\mathbb{R}}{}^\circ(f,g)$.

Let us consider A, and let a be a real number, and let f be an element of $\mathbb{R}^A$. Then $\langle a, f\rangle$ is an element of $[\!: \mathbb{R}, \mathbb{R}^A :\!]$.

Let us consider A. The functor $\cdot^{\mathbb{R}}_{\mathbb{R}^A}$ yielding a function from $[\!: \mathbb{R}, \mathbb{R}^A :\!]$ into $\mathbb{R}^A$, is defined as follows:

for every real number a and for every element f of $\mathbb{R}^A$ and for every element x of A holds $(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle))(x) = a \cdot f(x)$.

The following proposition is true

(7) For every function F from $[\!: \mathbb{R}, \mathbb{R}^A :\!]$ into $\mathbb{R}^A$ holds $F = \cdot^{\mathbb{R}}_{\mathbb{R}^A}$ if and only if for every real number a and for every element f of $\mathbb{R}^A$ and for every element x of A holds $(F(\langle a, f\rangle))(x) = a \cdot f(x)$.

Let us consider A. The functor $\mathbf{0}_{\mathbb{R}^A}$ yields an element of $\mathbb{R}^A$ and is defined by:

$\mathbf{0}_{\mathbb{R}^A} = A \longmapsto 0$.

The following proposition is true

(8) For every element f of $\mathbb{R}^A$ holds $f = \mathbf{0}_{\mathbb{R}^A}$ if and only if $f = A \longmapsto 0$.

Let us consider A. The functor $\mathbf{1}_{\mathbb{R}^A}$ yields an element of $\mathbb{R}^A$ and is defined by:

$\mathbf{1}_{\mathbb{R}^A} = A \longmapsto 1$.

We now state several propositions:

(9) For every element f of $\mathbb{R}^A$ holds $f = \mathbf{1}_{\mathbb{R}^A}$ if and only if $f = A \longmapsto 1$.

(10) $h = +_{\mathbb{R}^A}(f, g)$ if and only if for every element x of A holds $h(x) = f(x) + g(x)$.

(11) $h = \cdot_{\mathbb{R}^A}(f, g)$ if and only if for every element x of A holds $h(x) = f(x) \cdot g(x)$.

(12) For every element x of A holds $\mathbf{1}_{\mathbb{R}^A}(x) = 1$.

(13) For every element x of A holds $\mathbf{0}_{\mathbb{R}^A}(x) = 0$.

(14) $\mathbf{0}_{\mathbb{R}^A} \neq \mathbf{1}_{\mathbb{R}^A}$.

In the sequel a, b are real numbers. The following proposition is true

(15) $h = \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle)$ if and only if for every element x of A holds $h(x) = a \cdot f(x)$.

One can prove the following propositions:

(16) $+_{\mathbb{R}^A}(f, g) = +_{\mathbb{R}^A}(g, f)$.

(17) $+_{\mathbb{R}^A}(f, +_{\mathbb{R}^A}(g, h)) = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(f, g), h)$.

(18) $\cdot_{\mathbb{R}^A}(f, g) = \cdot_{\mathbb{R}^A}(g, f)$.

(19) $\cdot_{\mathbb{R}^A}(f, \cdot_{\mathbb{R}^A}(g, h)) = \cdot_{\mathbb{R}^A}(\cdot_{\mathbb{R}^A}(f, g), h)$.

(20) $\cdot_{\mathbb{R}^A}(\mathbf{1}_{\mathbb{R}^A}, f) = f$.

(21) $+_{\mathbb{R}^A}(\mathbf{0}_{\mathbb{R}^A}, f) = f$.

(22) $+_{\mathbb{R}^A}(f, \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle -1, f \rangle)) = \mathbf{0}_{\mathbb{R}^A}$.

(23) $\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle 1, f \rangle) = f$.

(24) $\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, f \rangle) \rangle) = \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a \cdot b, f \rangle)$.

(25) $+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, f \rangle)) = \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a + b, f \rangle)$.

(26) $\cdot_{\mathbb{R}^A}(f, +_{\mathbb{R}^A}(g, h)) = +_{\mathbb{R}^A}(\cdot_{\mathbb{R}^A}(f, g), \cdot_{\mathbb{R}^A}(f, h))$.

(27) $\cdot_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle), g) = \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, \cdot_{\mathbb{R}^A}(f, g) \rangle)$.

(28) Suppose $x_1 \in A$ and $x_2 \in A$ and $x_1 \neq x_2$. Then there exist f, g such that for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ but if $z \neq x_1$, then $f(z) = 0$ and for every z such that $z \in A$ holds if $z = x_1$, then $g(z) = 0$ but if $z \neq x_1$, then $g(z) = 1$.

(29) Suppose that
 (i) $x_1 \in A$,
 (ii) $x_2 \in A$,
 (iii) $x_1 \neq x_2$,
 (iv) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ but if $z \neq x_1$, then $f(z) = 0$,
 (v) for every z such that $z \in A$ holds if $z = x_1$, then $g(z) = 0$ but if $z \neq x_1$, then $g(z) = 1$.

 Then for all a, b such that $+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g \rangle)) = \mathbf{0}_{\mathbb{R}^A}$ holds $a = 0$ and $b = 0$.

(30) If $x_1 \in A$ and $x_2 \in A$ and $x_1 \neq x_2$, then there exist f, g such that for all a, b such that $+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g \rangle)) = \mathbf{0}_{\mathbb{R}^A}$ holds $a = 0$ and $b = 0$.

(31) Suppose that
(i) $A = \{x_1, x_2\}$,
(ii) $x_1 \neq x_2$,
(iii) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ but if $z \neq x_1$, then $f(z) = 0$,
(iv) for every z such that $z \in A$ holds if $z = x_1$, then $g(z) = 0$ but if $z \neq x_1$, then $g(z) = 1$.
Then for every h there exist a, b such that
$h = +_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle))$.

(32) If $A = \{x_1, x_2\}$ and $x_1 \neq x_2$, then there exist f, g such that for every h there exist a, b such that $h = +_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle))$.

(33) Suppose $A = \{x_1, x_2\}$ and $x_1 \neq x_2$. Then there exist f, g such that for all a, b such that $+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)) = \mathbf{0}_{\mathbb{R}^A}$ holds $a = 0$ and $b = 0$ and for every h there exist a, b such that $h = +_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle))$.

(34) $\langle \mathbb{R}^A, \rfloor \mathbf{0}_{\mathbb{R}^A}, +_{\mathbb{R}^A}, \cdot^{\mathbb{R}}_{\mathbb{R}^A}\rangle$ is a real linear space.

Let us consider A. The functor $\mathbb{R}^A_{\mathbb{R}}$ yields a real linear space and is defined by:

$\mathbb{R}^A_{\mathbb{R}} = \langle \mathbb{R}^A, \rfloor \mathbf{0}_{\mathbb{R}^A}, +_{\mathbb{R}^A}, \cdot^{\mathbb{R}}_{\mathbb{R}^A}\rangle$.

We now state two propositions:

(35) $\mathbb{R}^A_{\mathbb{R}} = \langle \mathbb{R}^A, \rfloor \mathbf{0}_{\mathbb{R}^A}, +_{\mathbb{R}^A}, \cdot^{\mathbb{R}}_{\mathbb{R}^A}\rangle$.

(36) $\mathbb{R}^A_{\mathbb{R}}$ is a real linear space.

In the sequel V will denote a real linear space and u, v, w will denote vectors of V. The following proposition is true

(37) There exists V and there exist u, v such that for all a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$ and for every w there exist a, b such that $w = a \cdot u + b \cdot v$.

We consider ring structures which are systems

⟨ a carrier, a multiplication, an addition, a unity, a zero ⟩

where the carrier is a non-empty set, the multiplication, the addition are binary operations on the carrier, and the unity, the zero are elements of the carrier. In the sequel RS will be a ring structure. We now define four new functors. Let us consider RS. The functor 1_{RS} yields an element of the carrier of RS and is defined as follows:

1_{RS} = the unity of RS.

The functor 0_{RS} yields an element of the carrier of RS and is defined as follows:

0_{RS} = the zero of RS.

Let x, y be elements of the carrier of RS. The functor $x \cdot y$ yielding an element of the carrier of RS, is defined by:

$x \cdot y =$ (the multiplication of RS)(x, y).

The functor $x + y$ yielding an element of the carrier of RS, is defined by:

$x + y =$ (the addition of RS)(x, y).

In the sequel x, y denote elements of the carrier of RS. One can prove the following four propositions:

(38) (the multiplication of $RS)(x, y) = x \cdot y$.

(39) (the addition of $RS)(x, y) = x + y$.

(40) $1_{RS} =$ the unity of RS.

(41) $0_{RS} =$ the zero of RS.

Let us consider A. The functor $\mathrm{RRing}\, A$ yielding a ring structure, is defined by:

$\mathrm{RRing}\, A = \langle \mathbb{R}^A, \cdot_{\mathbb{R}^A}, +_{\mathbb{R}^A}, \downharpoonleft \mathbf{1}_{\mathbb{R}^A}, \downharpoonleft \mathbf{0}_{\mathbb{R}^A} \rangle$.

Next we state a proposition

(42) Let x, y, z be elements of the carrier of $\mathrm{RRing}\, A$. Then

(i) $x + y = y + x$,

(ii) $(x + y) + z = x + (y + z)$,

(iii) $x + 0_{\mathrm{RRing}\, A} = x$,

(iv) there exists an element t of the carrier of $\mathrm{RRing}\, A$ such that $x + t = 0_{\mathrm{RRing}\, A}$,

(v) $x \cdot y = y \cdot x$,

(vi) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(vii) $x \cdot (1_{\mathrm{RRing}\, A}) = x$,

(viii) $x \cdot (y + z) = x \cdot y + x \cdot z$.

A ring structure is said to be a ring if:

Let x, y, z be elements of the carrier of it . Then

(i) $x + y = y + x$,

(ii) $(x + y) + z = x + (y + z)$,

(iii) $x + 0_{\mathrm{it}} = x$,

(iv) there exists an element t of the carrier of it such that $x + t = 0_{\mathrm{it}}$,

(v) $x \cdot y = y \cdot x$,

(vi) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(vii) $x \cdot (1_{\mathrm{it}}) = x$,

(viii) $x \cdot (y + z) = x \cdot y + x \cdot z$.

One can prove the following proposition

(43) $\mathrm{RRing}\, A$ is a ring.

We consider algebra structures which are systems

⟨ a carrier, a multiplication, an addition, a multiplication$_1$, a unity, a zero ⟩

where the carrier is a non-empty set, the multiplication, the addition are binary operations on the carrier, the multiplication$_1$ is a function from $[\!:\mathbb{R},$ the carrier$:\!]$ into the carrier, and the unity, the zero are elements of the carrier. In the sequel AlS denotes an algebra structure. We now define four new functors. Let us consider AlS. The functor 1_{AlS} yielding an element of the carrier of AlS, is defined as follows:

$1_{AlS} =$ the unity of AlS.

The functor 0_{AlS} yielding an element of the carrier of AlS, is defined by:

$0_{AlS} =$ the zero of AlS.

Let x, y be elements of the carrier of AlS. The functor $x \cdot y$ yields an element of the carrier of AlS and is defined by:

$x \cdot y =$ (the multiplication of $AlS)(x, y)$.

The functor $x+y$ yielding an element of the carrier of AlS, is defined as follows:

$x + y =$ (the addition of $AlS)(x, y)$.

Let us consider AlS, and let x be an element of the carrier of AlS, and let a be a real number. The functor $a \cdot x$ yields an element of the carrier of AlS and is defined as follows:

$a \cdot x =$ (the multiplication$_1$ of $AlS)(\langle\!\langle a, x\rangle\!\rangle)$.

In the sequel x, y are elements of the carrier of AlS. Next we state several propositions:

(44) (the multiplication of $AlS)(x, y) = x \cdot y$.

(45) (the addition of $AlS)(x, y) = x + y$.

(46) (the multiplication$_1$ of $AlS)(\langle\!\langle a, x\rangle\!\rangle) = a \cdot x$.

(47) $0_{AlS} =$ the zero of AlS.

(48) $1_{AlS} =$ the unity of AlS.

Let us consider A. The functor RAlgebra A yielding an algebra structure, is defined as follows:

RAlgebra $A = \langle \mathbb{R}^A, \cdot_{\mathbb{R}^A}, +_{\mathbb{R}^A}, \cdot^{\mathbb{R}}_{\mathbb{R}^A}, \downharpoonleft \mathbf{1}_{\mathbb{R}^A}, \downharpoonleft \mathbf{0}_{\mathbb{R}^A}\rangle$.

The following proposition is true

(49) Let x, y, z be elements of the carrier of RAlgebra A. Given a, b. Then

(i) $x + y = y + x$,

(ii) $(x + y) + z = x + (y + z)$,

(iii) $x + 0_{\mathrm{RAlgebra}\,A} = x$,

(iv) there exists an element t of the carrier of RAlgebra A such that $x+t = 0_{\mathrm{RAlgebra}\,A}$,

(v) $x \cdot y = y \cdot x$,

(vi) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(vii) $x \cdot (1_{\mathrm{RAlgebra}\,A}) = x$,

(viii) $x \cdot (y + z) = x \cdot y + x \cdot z$,

(ix) $a \cdot (x \cdot y) = (a \cdot x) \cdot y$,

(x) $a \cdot (x + y) = a \cdot x + a \cdot y$,

(xi) $(a + b) \cdot x = a \cdot x + b \cdot x$,

(xii) $(a \cdot b) \cdot x = a \cdot (b \cdot x)$.

An algebra structure is said to be an algebra if:

Let x, y, z be elements of the carrier of it . Given a, b. Then

(i) $x + y = y + x$,

(ii) $(x + y) + z = x + (y + z)$,

(iii) $x + 0_{\mathrm{it}} = x$,

(iv) there exists an element t of the carrier of it such that $x + t = 0_{\mathrm{it}}$,

(v) $x \cdot y = y \cdot x$,

(vi) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(vii) $x \cdot (1_{\mathrm{it}}) = x$,

(viii) $x \cdot (y + z) = x \cdot y + x \cdot z$,

(ix) $a \cdot (x \cdot y) = (a \cdot x) \cdot y$,

(x) $a \cdot (x + y) = a \cdot x + a \cdot y$,
(xi) $(a + b) \cdot x = a \cdot x + b \cdot x$,
(xii) $(a \cdot b) \cdot x = a \cdot (b \cdot x)$.

The following proposition is true

(50) RAlgebra A is an algebra.

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Received March 23, 1990

FORMALIZED MATHEMATICS
Vol.1, No.**3**, May–August 1990
Université Catholique de Louvain

Tarski's Classes and Ranks

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Summary. In the article the Tarski's classes (non-empty families of sets satisfying Tarski's axiom A given in [9]) and the rank sets are introduced and some of their properties are shown. The transitive closure and the rank of a set is given here too.

MML Identifier: `CLASSES1`.

The terminology and notation used here have been introduced in the following articles: [9], [8], [7], [3], [4], [6], [5], [2], and [1]. For simplicity we adopt the following rules: W, X, Y, Z will denote sets, D will denote a non-empty set, f will denote a function, and x, y will be arbitrary. Let B be a set. We say that B is a Tarski-Class if and only if:

for all X, Y such that $X \in B$ and $Y \subseteq X$ holds $Y \in B$ and for every X such that $X \in B$ holds $2^X \in B$ and for every X such that $X \subseteq B$ holds $X \approx B$ or $X \in B$.

Let A, B be sets. We say that B is Tarski-Class of A if and only if:

$A \in B$ and B is a Tarski-Class.

Let A be a set. The functor $\mathbf{T}(A)$ yielding a non-empty family of sets, is defined as follows:

$\mathbf{T}(A)$ is Tarski-Class of A and for every D such that D is Tarski-Class of A holds $\mathbf{T}(A) \subseteq D$.

We now state several propositions:

(1) W is a Tarski-Class if and only if for all X, Y such that $X \in W$ and $Y \subseteq X$ holds $Y \in W$ and for every X such that $X \in W$ holds $2^X \in W$ and for every X such that $X \subseteq W$ holds $X \approx W$ or $X \in W$.

(2) W is a Tarski-Class if and only if for all X, Y such that $X \in W$ and $Y \subseteq X$ holds $Y \in W$ and for every X such that $X \in W$ holds $2^X \in W$ and for every X such that $X \subseteq W$ and $\overline{\overline{X}} < \overline{\overline{W}}$ holds $X \in W$.

(3) X is Tarski-Class of Y if and only if $Y \in X$ and X is a Tarski-Class.

ISSN 0777–4028

(4) For every non-empty family W of sets holds $W = \mathbf{T}(X)$ if and only if W is Tarski-Class of X and for every D such that D is Tarski-Class of X holds $W \subseteq D$.

(5) $X \in \mathbf{T}(X)$.

(6) If $Y \in \mathbf{T}(X)$ and $Z \subseteq Y$, then $Z \in \mathbf{T}(X)$.

(7) If $Y \in \mathbf{T}(X)$, then $2^Y \in \mathbf{T}(X)$.

(8) If $Y \subseteq \mathbf{T}(X)$, then $Y \approx \mathbf{T}(X)$ or $Y \in \mathbf{T}(X)$.

(9) If $Y \subseteq \mathbf{T}(X)$ and $\overline{\overline{Y}} < \overline{\overline{\mathbf{T}(X)}}$, then $Y \in \mathbf{T}(X)$.

We follow a convention: u, v will denote elements of $\mathbf{T}(X)$, A, B, C will denote ordinal numbers, and L, L_1 will denote transfinite sequences. Let us consider X, A. The functor $\mathbf{T}_A(X)$ is defined as follows:

there exists L such that $\mathbf{T}_A(X) = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} A$ and $L(\mathbf{0}) = \{X\}$ and for all C, y such that $\operatorname{succ} C \in \operatorname{succ} A$ and $y = L(C)$ holds $L(\operatorname{succ} C) = (\{u : \bigvee_v [v \in [y] \wedge u \subseteq v]\} \cup \{2^v : v \in [y]\}) \cup 2^{[y]} \cap \mathbf{T}(X)$ and for all C, L_1 such that $C \in \operatorname{succ} A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \restriction C$ holds $L(C) = \bigcup(\operatorname{rng} L_1) \cap \mathbf{T}(X)$.

Let us consider X, A. Then $\mathbf{T}_A(X)$ is a subset of $\mathbf{T}(X)$.

Next we state a number of propositions:

(10) $\mathbf{T}_{\mathbf{0}}(X) = \{X\}$.

(11) $\mathbf{T}_{\operatorname{succ} A}(X) = (\{u : \bigvee_v [v \in \mathbf{T}_A(X) \wedge u \subseteq v]\} \cup \{2^v : v \in \mathbf{T}_A(X)\}) \cup 2^{\mathbf{T}_A(X)} \cap \mathbf{T}(X)$.

(12) If $A \neq \mathbf{0}$ and A is a limit ordinal number, then $\mathbf{T}_A(X) = \{u : \bigvee_B [B \in A \wedge u \in \mathbf{T}_B(X)]\}$.

(13) $Y \in \mathbf{T}_{\operatorname{succ} A}(X)$ if and only if $Y \subseteq \mathbf{T}_A(X)$ and $Y \in \mathbf{T}(X)$ or there exists Z such that $Z \in \mathbf{T}_A(X)$ but $Y \subseteq Z$ or $Y = 2^Z$.

(14) If $Y \subseteq Z$ and $Z \in \mathbf{T}_A(X)$, then $Y \in \mathbf{T}_{\operatorname{succ} A}(X)$.

(15) If $Y \in \mathbf{T}_A(X)$, then $2^Y \in \mathbf{T}_{\operatorname{succ} A}(X)$.

(16) If $A \neq \mathbf{0}$ and A is a limit ordinal number, then $x \in \mathbf{T}_A(X)$ if and only if there exists B such that $B \in A$ and $x \in \mathbf{T}_B(X)$.

(17) If $A \neq \mathbf{0}$ and A is a limit ordinal number and $Y \in \mathbf{T}_A(X)$ but $Z \subseteq Y$ or $Z = 2^Y$, then $Z \in \mathbf{T}_A(X)$.

(18) $\mathbf{T}_A(X) \subseteq \mathbf{T}_{\operatorname{succ} A}(X)$.

(19) If $A \subseteq B$, then $\mathbf{T}_A(X) \subseteq \mathbf{T}_B(X)$.

(20) There exists A such that $\mathbf{T}_A(X) = \mathbf{T}_{\operatorname{succ} A}(X)$.

(21) If $\mathbf{T}_A(X) = \mathbf{T}_{\operatorname{succ} A}(X)$, then $\mathbf{T}_A(X) = \mathbf{T}(X)$.

(22) There exists A such that $\mathbf{T}_A(X) = \mathbf{T}(X)$.

(23) There exists A such that $\mathbf{T}_A(X) = \mathbf{T}(X)$ and for every B such that $B \in A$ holds $\mathbf{T}_B(X) \neq \mathbf{T}(X)$.

(24) If $Y \neq X$ and $Y \in \mathbf{T}(X)$, then there exists A such that $Y \notin \mathbf{T}_A(X)$ and $Y \in \mathbf{T}_{\operatorname{succ} A}(X)$.

(25) If X is transitive, then for every A such that $A \neq \mathbf{0}$ holds $\mathbf{T}_A(X)$ is transitive.

(26) $\mathbf{T_0}(X) \in \mathbf{T_1}(X)$ and $\mathbf{T_0}(X) \neq \mathbf{T_1}(X)$.

(27) If X is transitive, then $\mathbf{T}(X)$ is transitive.

(28) If $Y \in \mathbf{T}(X)$, then $\overline{\overline{Y}} < \overline{\overline{\mathbf{T}(X)}}$.

(29) If $Y \in \mathbf{T}(X)$, then $Y \not\approx \mathbf{T}(X)$.

(30) If $x \in \mathbf{T}(X)$ and $y \in \mathbf{T}(X)$, then $\{x\} \in \mathbf{T}(X)$ and $\{x, y\} \in \mathbf{T}(X)$.

(31) If $x \in \mathbf{T}(X)$ and $y \in \mathbf{T}(X)$, then $\langle x, y \rangle \in \mathbf{T}(X)$.

(32) If $Y \subseteq \mathbf{T}(X)$ and $Z \subseteq \mathbf{T}(X)$, then $[\!: Y, Z :\!] \subseteq \mathbf{T}(X)$.

Let us consider A. The functor $\mathbf{R}_A$ is defined as follows:

there exists L such that $\mathbf{R}_A = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} A$ and $L(\mathbf{0}) = \emptyset$ and for all C, y such that $\operatorname{succ} C \in \operatorname{succ} A$ and $y = L(C)$ holds $L(\operatorname{succ} C) = 2^{[y]}$ and for all C, L_1 such that $C \in \operatorname{succ} A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \restriction C$ holds $L(C) = \bigcup(\operatorname{rng} L_1)$.

Let us consider A. Then $\mathbf{R}_A$ is a set.

One can prove the following propositions:

(33) $\mathbf{R_0} = \emptyset$.

(34) $\mathbf{R}_{\operatorname{succ} A} = 2^{\mathbf{R}_A}$.

(35) If $A \neq \mathbf{0}$ and A is a limit ordinal number, then for every x holds $x \in \mathbf{R}_A$ if and only if there exists B such that $B \in A$ and $x \in \mathbf{R}_B$.

(36) $X \subseteq \mathbf{R}_A$ if and only if $X \in \mathbf{R}_{\operatorname{succ} A}$.

(37) $\mathbf{R}_A$ is transitive.

(38) If $X \in \mathbf{R}_A$, then $X \subseteq \mathbf{R}_A$.

(39) $\mathbf{R}_A \subseteq \mathbf{R}_{\operatorname{succ} A}$.

(40) $\bigcup \mathbf{R}_A \subseteq \mathbf{R}_A$.

(41) If $X \in \mathbf{R}_A$, then $\bigcup X \in \mathbf{R}_A$.

(42) $A \in B$ if and only if $\mathbf{R}_A \in \mathbf{R}_B$.

(43) $A \subseteq B$ if and only if $\mathbf{R}_A \subseteq \mathbf{R}_B$.

(44) $A \subseteq \mathbf{R}_A$.

(45) For all A, X such that $X \in \mathbf{R}_A$ holds $X \not\approx \mathbf{R}_A$ and $\overline{\overline{X}} < \overline{\overline{\mathbf{R}_A}}$.

(46) $X \subseteq \mathbf{R}_A$ if and only if $2^X \subseteq \mathbf{R}_{\operatorname{succ} A}$.

(47) If $X \subseteq Y$ and $Y \in \mathbf{R}_A$, then $X \in \mathbf{R}_A$.

(48) $X \in \mathbf{R}_A$ if and only if $2^X \in \mathbf{R}_{\operatorname{succ} A}$.

(49) $x \in \mathbf{R}_A$ if and only if $\{x\} \in \mathbf{R}_{\operatorname{succ} A}$.

(50) $x \in \mathbf{R}_A$ and $y \in \mathbf{R}_A$ if and only if $\{x, y\} \in \mathbf{R}_{\operatorname{succ} A}$.

(51) $x \in \mathbf{R}_A$ and $y \in \mathbf{R}_A$ if and only if $\langle x, y \rangle \in \mathbf{R}_{\operatorname{succ}(\operatorname{succ} A)}$.

(52) If X is transitive and $\mathbf{R}_A \cap \mathbf{T}(X) = \mathbf{R}_{\operatorname{succ} A} \cap \mathbf{T}(X)$, then $\mathbf{T}(X) \subseteq \mathbf{R}_A$.

(53) If X is transitive, then there exists A such that $\mathbf{T}(X) \subseteq \mathbf{R}_A$.

(54) If X is transitive, then $\bigcup X \subseteq X$.

(55) If X is transitive and Y is transitive, then $X \cup Y$ is transitive.

(56) If X is transitive and Y is transitive, then $X \cap Y$ is transitive.

In the sequel k, n denote natural numbers. Let us consider X. The functor $X^{*\in}$ yielding a set, is defined by:

$x \in X^{*\in}$ if and only if there exist f, n, Y such that $x \in Y$ and $Y = f(n)$ and $\mathrm{dom}\, f = \mathbb{N}$ and $f(0) = X$ and for all k, y such that $y = f(k)$ holds $f(k+1) = \bigcup[y]$.

Next we state a number of propositions:

(57) $Z = X^{*\in}$ if and only if for every x holds $x \in Z$ if and only if there exist f, n, Y such that $x \in Y$ and $Y = f(n)$ and $\mathrm{dom}\, f = \mathbb{N}$ and $f(0) = X$ and for all k, y such that $y = f(k)$ holds $f(k+1) = \bigcup[y]$.

(58) $X^{*\in}$ is transitive.

(59) $X \subseteq X^{*\in}$.

(60) If $X \subseteq Y$ and Y is transitive, then $X^{*\in} \subseteq Y$.

(61) If for every Z such that $X \subseteq Z$ and Z is transitive holds $Y \subseteq Z$ and $X \subseteq Y$ and Y is transitive, then $X^{*\in} = Y$.

(62) If X is transitive, then $X^{*\in} = X$.

(63) $\emptyset^{*\in} = \emptyset$.

(64) $A^{*\in} = A$.

(65) If $X \subseteq Y$, then $X^{*\in} \subseteq Y^{*\in}$.

(66) $(X^{*\in})^{*\in} = X^{*\in}$.

(67) $(X \cup Y)^{*\in} = X^{*\in} \cup Y^{*\in}$.

(68) $(X \cap Y)^{*\in} \subseteq X^{*\in} \cap Y^{*\in}$.

(69) There exists A such that $X \subseteq \mathbf{R}_A$.

Let us consider X. The functor $\mathrm{rk}(X)$ yielding an ordinal number, is defined by:

$X \subseteq \mathbf{R}_{\mathrm{rk}(X)}$ and for every B such that $X \subseteq \mathbf{R}_B$ holds $\mathrm{rk}(X) \subseteq B$.

We now state a number of propositions:

(70) $A = \mathrm{rk}(X)$ if and only if $X \subseteq \mathbf{R}_A$ and for every B such that $X \subseteq \mathbf{R}_B$ holds $A \subseteq B$.

(71) $\mathrm{rk}(2^X) = \mathrm{succ}\, \mathrm{rk}(X)$.

(72) $\mathrm{rk}(\mathbf{R}_A) = A$.

(73) $X \subseteq \mathbf{R}_A$ if and only if $\mathrm{rk}(X) \subseteq A$.

(74) $X \in \mathbf{R}_A$ if and only if $\mathrm{rk}(X) \in A$.

(75) If $X \subseteq Y$, then $\mathrm{rk}(X) \subseteq \mathrm{rk}(Y)$.

(76) If $X \in Y$, then $\mathrm{rk}(X) \in \mathrm{rk}(Y)$.

(77) $\mathrm{rk}(X) \subseteq A$ if and only if for every Y such that $Y \in X$ holds $\mathrm{rk}(Y) \in A$.

(78) $A \subseteq \mathrm{rk}(X)$ if and only if for every B such that $B \in A$ there exists Y such that $Y \in X$ and $B \subseteq \mathrm{rk}(Y)$.

(79) $\mathrm{rk}(X) = \mathbf{0}$ if and only if $X = \emptyset$.

(80) If $\mathrm{rk}(X) = \mathrm{succ}\, A$, then there exists Y such that $Y \in X$ and $\mathrm{rk}(Y) = A$.

(81) $\mathrm{rk}(A) = A$.

(82) $\mathrm{rk}(\mathbf{T}(X)) \neq \mathbf{0}$ and $\mathrm{rk}(\mathbf{T}(X))$ is a limit ordinal number.

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Received March 23, 1990

FORMALIZED MATHEMATICS
Vol.1, No.**3**, May–August 1990
Université Catholique de Louvain

Non-contiguous Substrings and One-to-one Finite Sequences

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Summary. This text is a continuation of [3]. We prove a number of theorems concerning both notions introduced there and one-to-one finite sequences. We introduce a function that removes from a string elements of the string that belongs to a given set.

MML Identifier: `FINSEQ_3`.

The notation and terminology used here have been introduced in the following articles: [9], [8], [5], [3], [4], [7], [6], [1], [2], and [10]. For simplicity we follow a convention: p, q, r are finite sequences, u, v, x, y, z are arbitrary, i, j, k, l, m, n are natural numbers, A, X, Y are sets, and D is a non-empty set. The following propositions are true:

(1) $\mathrm{Seg}\,3 = \{1, 2, 3\}$.

(2) $\mathrm{Seg}\,4 = \{1, 2, 3, 4\}$.

(3) $\mathrm{Seg}\,5 = \{1, 2, 3, 4, 5\}$.

(4) $\mathrm{Seg}\,6 = \{1, 2, 3, 4, 5, 6\}$.

(5) $\mathrm{Seg}\,7 = \{1, 2, 3, 4, 5, 6, 7\}$.

(6) $\mathrm{Seg}\,8 = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

(7) $\mathrm{Seg}\,k = \emptyset$ if and only if $k \notin \mathrm{Seg}\,k$.

(8) $0 \notin \mathrm{Seg}\,k$.

(9) $k + 1 \notin \mathrm{Seg}\,k$.

(10) If $k \neq 0$, then $k \in \mathrm{Seg}(k + n)$.

(11) If $k + n \in \mathrm{Seg}\,k$, then $n = 0$.

(12) If $k \in \mathrm{Seg}\,n$ and $k < n$, then $k + 1 \in \mathrm{Seg}\,n$.

(13) If $k \in \mathrm{Seg}\,n$ and $m < k$, then $k - m \in \mathrm{Seg}\,n$.

(14) $k - n \in \mathrm{Seg}\,k$ if and only if $n < k$.

(15) $\mathrm{Seg}\,k$ misses $\{k + 1\}$.

ISSN 0777–4028

(16) $\operatorname{Seg}(k+1) \setminus \operatorname{Seg} k = \{k+1\}$.
(17) $\operatorname{Seg} k \neq \operatorname{Seg}(k+1)$.
(18) If $\operatorname{Seg} k = \operatorname{Seg}(k+n)$, then $n = 0$.
(19) $\operatorname{Seg} k \subseteq \operatorname{Seg}(k+n)$.
(20) $\operatorname{Seg} k \subseteq \operatorname{Seg} n$ or $\operatorname{Seg} n \subseteq \operatorname{Seg} k$.
(21) If $\operatorname{Seg} k = \emptyset$, then $k = 0$.
(22) If $\operatorname{Seg} k = \{y\}$, then $k = 1$ and $y = 1$.
(23) If $\operatorname{Seg} k = \{x, y\}$ and $x \neq y$, then $k = 2$ and $\{x, y\} = \{1, 2\}$.
(24) If $x \in \operatorname{dom} p$, then $x \in \operatorname{dom}(p \frown q)$.
(25) If $x \in \operatorname{dom} p$, then x is a natural number.
(26) If $x \in \operatorname{dom} p$, then $x \neq 0$.
(27) $n \in \operatorname{dom} p$ if and only if $1 \leq n$ and $n \leq \operatorname{len} p$.
(28) $n \in \operatorname{dom} p$ if and only if $n - 1$ is a natural number and $\operatorname{len} p - n$ is a natural number.
(29) $\operatorname{dom}\langle x, y \rangle = \operatorname{Seg} 2$.
(30) $\operatorname{dom}\langle x, y, z \rangle = \operatorname{Seg} 3$.
(31) $\operatorname{len} p = \operatorname{len} q$ if and only if $\operatorname{dom} p = \operatorname{dom} q$.
(32) $\operatorname{len} p \leq \operatorname{len} q$ if and only if $\operatorname{dom} p \subseteq \operatorname{dom} q$.
(33) If $x \in \operatorname{rng} p$, then $1 \in \operatorname{dom} p$.
(34) If $\operatorname{rng} p \neq \emptyset$, then $1 \in \operatorname{dom} p$.
(35) $\operatorname{rng}\langle x, y \rangle = \{x, y\}$.
(36) $\operatorname{rng}\langle x, y, z \rangle = \{x, y, z\}$.
(37) $\varepsilon = \square$.
(38) $\varepsilon \neq \langle x, y \rangle$.
(39) $\varepsilon \neq \langle x, y, z \rangle$.
(40) $\langle x \rangle \neq \langle y, z \rangle$.
(41) $\langle u \rangle \neq \langle x, y, z \rangle$.
(42) $\langle u, v \rangle \neq \langle x, y, z \rangle$.
(43) If $\operatorname{len} r = \operatorname{len} p + \operatorname{len} q$ and for every k such that $k \in \operatorname{dom} p$ holds $r(k) = p(k)$ and for every k such that $k \in \operatorname{dom} q$ holds $r(\operatorname{len} p + k) = q(k)$, then $r = p \frown q$.
(44) If $A \subseteq \operatorname{Seg} k$, then $\operatorname{len}(\operatorname{Sgm} A) = \operatorname{card} A$.
(45) If $A \subseteq \operatorname{Seg} k$, then $\operatorname{dom}(\operatorname{Sgm} A) = \operatorname{Seg}(\operatorname{card} A)$.
(46) If $X \subseteq \operatorname{Seg} i$ and $k < l$ and $1 \leq n$ and $m \leq \operatorname{len}(\operatorname{Sgm} X)$ and $\operatorname{Sgm} X(m) = k$ and $\operatorname{Sgm} X(n) = l$, then $m < n$.
(47) If $X \subseteq \operatorname{Seg} i$ and $k \leq l$ and $1 \leq n$ and $m \leq \operatorname{len}(\operatorname{Sgm} X)$ and $\operatorname{Sgm} X(m) = k$ and $\operatorname{Sgm} X(n) = l$, then $m \leq n$.
(48) If $X \subseteq \operatorname{Seg} i$ and $Y \subseteq \operatorname{Seg} j$, then for all m, n such that $m \in X$ and $n \in Y$ holds $m < n$ if and only if $\operatorname{Sgm}(X \cup Y) = \operatorname{Sgm} X \frown \operatorname{Sgm} Y$.
(49) $\operatorname{Sgm} \emptyset = \varepsilon$.

(50) If $0 \neq n$, then $\mathrm{Sgm}\{n\} = \langle n \rangle$.
(51) If $0 < n$ and $n < m$, then $\mathrm{Sgm}\{n, m\} = \langle n, m \rangle$.
(52) $\mathrm{len}(\mathrm{Sgm}(\mathrm{Seg}\, k)) = k$.
(53) $\mathrm{Sgm}(\mathrm{Seg}(k + n)) \restriction \mathrm{Seg}\, k = \mathrm{Sgm}(\mathrm{Seg}\, k)$.
(54) $\mathrm{Sgm}(\mathrm{Seg}\, k) = \mathrm{id}_k$.
(55) $p \restriction \mathrm{Seg}\, n = p$ if and only if $\mathrm{len}\, p \leq n$.
(56) $\mathrm{id}_{n+k} \restriction \mathrm{Seg}\, n = \mathrm{id}_n$.
(57) $\mathrm{id}_n \restriction \mathrm{Seg}\, m = \mathrm{id}_m$ if and only if $m \leq n$.
(58) $\mathrm{id}_n \restriction \mathrm{Seg}\, m = \mathrm{id}_n$ if and only if $n \leq m$.
(59) If $\mathrm{len}\, p = k + l$ and $q = p \restriction \mathrm{Seg}\, k$, then $\mathrm{len}\, q = k$.
(60) If $\mathrm{len}\, p = k + l$ and $q = p \restriction \mathrm{Seg}\, k$, then $\mathrm{dom}\, q = \mathrm{Seg}\, k$.
(61) If $\mathrm{len}\, p = k + 1$ and $q = p \restriction \mathrm{Seg}\, k$, then $p = q \frown \langle p(k + 1) \rangle$.
(62) $p \restriction X$ is a finite sequence if and only if there exists k such that $X \cap \mathrm{dom}\, p = \mathrm{Seg}\, k$.
(63) $\mathrm{card}((p \frown q)^{-1} A) = \mathrm{card}(p^{-1} A) + \mathrm{card}(q^{-1} A)$.
(64) $p^{-1} A \subseteq (p \frown q)^{-1} A$.

Let us consider p, A. The functor $p - A$ yields a finite sequence and is defined by:

$p - A = p \cdot \mathrm{Sgm}(\mathrm{Seg}(\mathrm{len}\, p) \setminus p^{-1} A)$.

The following propositions are true:

(65) $p - A = p \cdot \mathrm{Sgm}(\mathrm{Seg}(\mathrm{len}\, p) \setminus p^{-1} A)$.
(66) $\mathrm{len}(p - A) = \mathrm{len}\, p - \mathrm{card}(p^{-1} A)$.
(67) $\mathrm{len}(p - A) \leq \mathrm{len}\, p$.
(68) If $\mathrm{len}(p - A) = \mathrm{len}\, p$, then A misses $\mathrm{rng}\, p$.
(69) If $n = \mathrm{len}\, p - \mathrm{card}(p^{-1} A)$, then $\mathrm{dom}(p - A) = \mathrm{Seg}\, n$.
(70) $\mathrm{dom}(p - A) \subseteq \mathrm{dom}\, p$.
(71) If $\mathrm{dom}(p - A) = \mathrm{dom}\, p$, then A misses $\mathrm{rng}\, p$.
(72) $\mathrm{rng}(p - A) = \mathrm{rng}\, p \setminus A$.
(73) $\mathrm{rng}(p - A) \subseteq \mathrm{rng}\, p$.
(74) If $\mathrm{rng}(p - A) = \mathrm{rng}\, p$, then A misses $\mathrm{rng}\, p$.
(75) $p - A = \varepsilon$ if and only if $\mathrm{rng}\, p \subseteq A$.
(76) $p - A = p$ if and only if A misses $\mathrm{rng}\, p$.
(77) $p - \{x\} = p$ if and only if $x \notin \mathrm{rng}\, p$.
(78) $p - \emptyset = p$.
(79) $p - \mathrm{rng}\, p = \varepsilon$.
(80) $p \frown q - A = (p - A) \frown (q - A)$.
(81) $\varepsilon - A = \varepsilon$.
(82) $\langle x \rangle - A = \langle x \rangle$ if and only if $x \notin A$.
(83) $\langle x \rangle - A = \varepsilon$ if and only if $x \in A$.

(84) $\langle x, y\rangle - A = \varepsilon$ if and only if $x \in A$ and $y \in A$.

(85) If $x \in A$ and $y \notin A$, then $\langle x, y\rangle - A = \langle y\rangle$.

(86) If $\langle x, y\rangle - A = \langle y\rangle$ and $x \neq y$, then $x \in A$ and $y \notin A$.

(87) If $x \notin A$ and $y \in A$, then $\langle x, y\rangle - A = \langle x\rangle$.

(88) If $\langle x, y\rangle - A = \langle x\rangle$ and $x \neq y$, then $x \notin A$ and $y \in A$.

(89) $\langle x, y\rangle - A = \langle x, y\rangle$ if and only if $x \notin A$ and $y \notin A$.

(90) If $\operatorname{len} p = k + 1$ and $q = p \restriction \operatorname{Seg} k$, then $p(k + 1) \in A$ if and only if $p - A = q - A$.

(91) If $\operatorname{len} p = k + 1$ and $q = p \restriction \operatorname{Seg} k$, then $p(k + 1) \notin A$ if and only if $p - A = (q - A) \frown \langle p(k + 1)\rangle$.

(92) If $n \in \operatorname{dom} p$, then $p(n) \in A$ or $(p - A)(n - \operatorname{card}\{k : k \in \operatorname{dom} p \wedge k \leq n \wedge p(k) \in A\}) = p(n)$.

(93) If p is a finite sequence of elements of D, then $p - A$ is a finite sequence of elements of D.

(94) If p is one-to-one, then $p - A$ is one-to-one.

(95) If p is one-to-one, then $\operatorname{len}(p - A) = \operatorname{len} p - \operatorname{card}(A \cap \operatorname{rng} p)$.

(96) If p is one-to-one and $A \subseteq \operatorname{rng} p$, then $\operatorname{len}(p - A) = \operatorname{len} p - \operatorname{card} A$.

(97) If p is one-to-one and $x \in \operatorname{rng} p$, then $\operatorname{len}(p - \{x\}) = \operatorname{len} p - 1$.

(98) $\operatorname{rng} p$ misses $\operatorname{rng} q$ and p is one-to-one and q is one-to-one if and only if $p \frown q$ is one-to-one.

(99) If $A \subseteq \operatorname{Seg} k$, then $\operatorname{Sgm} A$ is one-to-one.

(100) id_n is one-to-one.

(101) ε is one-to-one.

(102) $\langle x\rangle$ is one-to-one.

(103) $x \neq y$ if and only if $\langle x, y\rangle$ is one-to-one.

(104) $x \neq y$ and $y \neq z$ and $z \neq x$ if and only if $\langle x, y, z\rangle$ is one-to-one.

(105) If p is one-to-one and $\operatorname{rng} p = \{x\}$, then $\operatorname{len} p = 1$.

(106) If p is one-to-one and $\operatorname{rng} p = \{x\}$, then $p = \langle x\rangle$.

(107) If p is one-to-one and $\operatorname{rng} p = \{x, y\}$ and $x \neq y$, then $\operatorname{len} p = 2$.

(108) If p is one-to-one and $\operatorname{rng} p = \{x, y\}$ and $x \neq y$, then $p = \langle x, y\rangle$ or $p = \langle y, x\rangle$.

(109) If p is one-to-one and $\operatorname{rng} p = \{x, y, z\}$ and $\langle x, y, z\rangle$ is one-to-one, then $\operatorname{len} p = 3$.

(110) If p is one-to-one and $\operatorname{rng} p = \{x, y, z\}$ and $x \neq y$ and $y \neq z$ and $x \neq z$, then $\operatorname{len} p = 3$.

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Received April 8, 1990

FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Pigeon Hole Principle

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Summary. We introduce the notion of a predicate that states that a function is one-to-one at a given element of it's domain (i.e. counter image of image of the element is equal to its singleton). We also introduce some rather technical functors concerning finite sequences: the lowest index of the given element of the range of the finite sequence, the substring preceding (and succeeding) the first occurrence of given element of the range. At the end of the article we prove the pigeon hole principle.

MML Identifier: `FINSEQ_4`.

The notation and terminology used here are introduced in the following papers: [8], [4], [3], [6], [7], [1], [5], [9], [2], and [10]. For simplicity we adopt the following convention: f is a function, p, q are finite sequences, x, y, z are arbitrary, i, k, n are natural numbers, and A, B are sets. Let us consider f, x. We say that f is one-to-one at x if and only if:

$f^{-1}(f^{\circ}\{x\}) = \{x\}$.

We now state several propositions:

(1) f is one-to-one at x if and only if $f^{-1}(f^{\circ}\{x\}) = \{x\}$.

(2) If f is one-to-one at x, then $x \in \operatorname{dom} f$.

(3) f is one-to-one at x if and only if $x \in \operatorname{dom} f$ and $f^{-1}\{f(x)\} = \{x\}$.

(4) f is one-to-one at x if and only if $x \in \operatorname{dom} f$ and for every z such that $z \in \operatorname{dom} f$ and $x \neq z$ holds $f(x) \neq f(z)$.

(5) For every x such that $x \in \operatorname{dom} f$ holds f is one-to-one at x if and only if f is one-to-one.

Let us consider f, y. We say that f yields y just once if and only if:

$f^{-1}\{y\}$ is finite and $\operatorname{card}(f^{-1}\{y\}) = 1$.

Next we state several propositions:

(6) f yields y just once if and only if $f^{-1}\{y\}$ is finite and $\operatorname{card}(f^{-1}\{y\}) = 1$.

(7) If f yields y just once, then $y \in \operatorname{rng} f$.

(8) f yields y just once if and only if there exists x such that $\{x\} = f^{-1}\{y\}$.

ISSN 0777–4028

(9) f yields y just once if and only if there exists x such that $x \in \operatorname{dom} f$ and $y = f(x)$ and for every z such that $z \in \operatorname{dom} f$ and $z \neq x$ holds $f(z) \neq y$.

(10) f is one-to-one if and only if for every y such that $y \in \operatorname{rng} f$ holds f yields y just once.

(11) f is one-to-one at x if and only if $x \in \operatorname{dom} f$ and f yields $f(x)$ just once.

Let us consider f, y. Let us assume that f yields y just once. The functor $f^{-1}(y)$ is defined as follows:

$f^{-1}(y) \in \operatorname{dom} f$ and $f(f^{-1}(y)) = y$.

One can prove the following propositions:

(12) If f yields y just once and $x \in \operatorname{dom} f$ and $f(x) = y$, then $x = f^{-1}(y)$.

(13) If f yields y just once, then $f^{-1}(y) \in \operatorname{dom} f$.

(14) If f yields y just once, then $f(f^{-1}(y)) = y$.

(15) If f yields y just once, then for every x such that $x \in \operatorname{dom} f$ and $x \neq f^{-1}(y)$ holds $f(x) \neq y$.

(16) If f yields y just once, then $f^{\circ} \{f^{-1}(y)\} = \{y\}$.

(17) If f yields y just once, then $f^{-1} \{y\} = \{f^{-1}(y)\}$.

(18) If f is one-to-one and $y \in \operatorname{rng} f$, then $f^{-1}(y) = f^{-1}(y)$.

(19) If $x \in \operatorname{dom} f$ and f yields $f(x)$ just once, then $f^{-1}(f(x)) = x$.

(20) If f is one-to-one at x, then $f^{-1}(f(x)) = x$.

(21) If f yields y just once, then f is one-to-one at $f^{-1}(y)$.

We adopt the following convention: D will be a non-empty set, d, d_1, d_2, d_3 will be elements of D, and P will be a finite sequence of elements of D. Let us consider D, d_1, d_2. Then $\langle d_1, d_2 \rangle$ is a finite sequence of elements of D.

Let us consider D, d_1, d_2, d_3. Then $\langle d_1, d_2, d_3 \rangle$ is a finite sequence of elements of D.

Let us consider D, P, i. Let us assume that $i \in \operatorname{dom} P$. The functor $\pi_i P$ yielding an element of D, is defined as follows:

$\pi_i P = P(i)$.

Next we state several propositions:

(22) If $i \in \operatorname{dom} P$, then $\pi_i P = P(i)$.

(23) If $i \in \operatorname{Seg}(\operatorname{len} P)$, then $\pi_i P = P(i)$.

(24) If $1 \leq i$ and $i \leq \operatorname{len} P$, then $\pi_i P = P(i)$.

(25) $\pi_1 \langle d \rangle = d$.

(26) $\pi_1 \langle d_1, d_2 \rangle = d_1$ and $\pi_2 \langle d_1, d_2 \rangle = d_2$.

(27) $\pi_1 \langle d_1, d_2, d_3 \rangle = d_1$ and $\pi_2 \langle d_1, d_2, d_3 \rangle = d_2$ and $\pi_3 \langle d_1, d_2, d_3 \rangle = d_3$.

Let us consider p, x. Let us assume that $x \in \operatorname{rng} p$. The functor $x \looparrowleft p$ yields a natural number and is defined by:

$x \looparrowleft p = \operatorname{Sgm}(p^{-1} \{x\})(1)$.

Next we state a number of propositions:

(28) If $x \in \operatorname{rng} p$, then $x \looparrowleft p = \operatorname{Sgm}(p^{-1} \{x\})(1)$.

(29) If $x \in \operatorname{rng} p$, then $p(x \looparrowleft p) = x$.

(30) If $x \in \operatorname{rng} p$, then $x \looparrowleft p \in \operatorname{dom} p$.

(31) If $x \in \operatorname{rng} p$, then $1 \leq x \looparrowleft p$ and $x \looparrowleft p \leq \operatorname{len} p$.

(32) If $x \in \operatorname{rng} p$, then $x \looparrowleft p - 1$ is a natural number and $\operatorname{len} p - x \looparrowleft p$ is a natural number.

(33) If $x \in \operatorname{rng} p$, then $x \looparrowleft p \in p^{-1} \{x\}$.

(34) If $x \in \operatorname{rng} p$, then for every k such that $k \in \operatorname{dom} p$ and $k < x \looparrowleft p$ holds $p(k) \neq x$.

(35) If p yields x just once, then $p^{-1}(x) = x \looparrowleft p$.

(36) If p yields x just once, then for every k such that $k \in \operatorname{dom} p$ and $k \neq x \looparrowleft p$ holds $p(k) \neq x$.

(37) If $x \in \operatorname{rng} p$ and for every k such that $k \in \operatorname{dom} p$ and $k \neq x \looparrowleft p$ holds $p(k) \neq x$, then p yields x just once.

(38) p yields x just once if and only if $x \in \operatorname{rng} p$ and $\{x \looparrowleft p\} = p^{-1} \{x\}$.

(39) If p is one-to-one and $x \in \operatorname{rng} p$, then $\{x \looparrowleft p\} = p^{-1} \{x\}$.

(40) p yields x just once if and only if $\operatorname{len}(p - \{x\}) = \operatorname{len} p - 1$.

(41) If p yields x just once, then for every k such that $k \in \operatorname{dom}(p - \{x\})$ holds if $k < x \looparrowleft p$, then $(p - \{x\})(k) = p(k)$ but if $x \looparrowleft p \leq k$, then $(p - \{x\})(k) = p(k+1)$.

(42) Suppose p is one-to-one and $x \in \operatorname{rng} p$. Then for every k such that $k \in \operatorname{dom}(p - \{x\})$ holds $(p - \{x\})(k) = p(k)$ if and only if $k < x \looparrowleft p$ but $(p - \{x\})(k) = p(k+1)$ if and only if $x \looparrowleft p \leq k$.

Let us consider p, x. Let us assume that $x \in \operatorname{rng} p$. The functor $p \leftarrow x$ yields a finite sequence and is defined as follows:

there exists n such that $n = x \looparrowleft p - 1$ and $p \leftarrow x = p \restriction \operatorname{Seg} n$.

One can prove the following propositions:

(43) If $x \in \operatorname{rng} p$, then there exists n such that $n = x \looparrowleft p - 1$ and $p \restriction \operatorname{Seg} n = p \leftarrow x$.

(44) If $x \in \operatorname{rng} p$ and there exists n such that $n = x \looparrowleft p - 1$ and $p \restriction \operatorname{Seg} n = q$, then $q = p \leftarrow x$.

(45) If $x \in \operatorname{rng} p$ and $n = x \looparrowleft p - 1$, then $p \restriction \operatorname{Seg} n = p \leftarrow x$.

(46) If $x \in \operatorname{rng} p$, then $\operatorname{len}(p \leftarrow x) = x \looparrowleft p - 1$.

(47) If $x \in \operatorname{rng} p$ and $n = x \looparrowleft p - 1$, then $\operatorname{dom}(p \leftarrow x) = \operatorname{Seg} n$.

(48) If $x \in \operatorname{rng} p$ and $k \in \operatorname{dom}(p \leftarrow x)$, then $p(k) = (p \leftarrow x)(k)$.

(49) If $x \in \operatorname{rng} p$, then $x \notin \operatorname{rng}(p \leftarrow x)$.

(50) If $x \in \operatorname{rng} p$, then $\operatorname{rng}(p \leftarrow x)$ misses $\{x\}$.

(51) If $x \in \operatorname{rng} p$, then $\operatorname{rng}(p \leftarrow x) \subseteq \operatorname{rng} p$.

(52) If $x \in \operatorname{rng} p$, then $x \looparrowleft p = 1$ if and only if $p \leftarrow x = \varepsilon$.

(53) If $x \in \operatorname{rng} p$ and p is a finite sequence of elements of D, then $p \leftarrow x$ is a finite sequence of elements of D.

Let us consider p, x. Let us assume that $x \in \operatorname{rng} p$. The functor $p \rightarrow x$ yields a finite sequence and is defined as follows:

$\operatorname{len}(p \rightarrow x) = \operatorname{len} p - x \hookleftarrow p$ and for every k such that $k \in \operatorname{dom}(p \rightarrow x)$ holds $(p \rightarrow x)(k) = p(k + x \hookleftarrow p)$.

One can prove the following propositions:

(54) If $x \in \operatorname{rng} p$ and $\operatorname{len} q = \operatorname{len} p - x \hookleftarrow p$ and for every k such that $k \in \operatorname{dom} q$ holds $q(k) = p(k + x \hookleftarrow p)$, then $q = p \rightarrow x$.

(55) If $x \in \operatorname{rng} p$, then $\operatorname{len}(p \rightarrow x) = \operatorname{len} p - x \hookleftarrow p$.

(56) If $x \in \operatorname{rng} p$, then for every k such that $k \in \operatorname{dom}(p \rightarrow x)$ holds $(p \rightarrow x)(k) = p(k + x \hookleftarrow p)$.

(57) If $x \in \operatorname{rng} p$ and $n = \operatorname{len} p - x \hookleftarrow p$, then $\operatorname{dom}(p \rightarrow x) = \operatorname{Seg} n$.

(58) If $x \in \operatorname{rng} p$ and $n \in \operatorname{dom}(p \rightarrow x)$, then $n + x \hookleftarrow p \in \operatorname{dom} p$.

(59) If $x \in \operatorname{rng} p$, then $\operatorname{rng}(p \rightarrow x) \subseteq \operatorname{rng} p$.

(60) p yields x just once if and only if $x \in \operatorname{rng} p$ and $x \notin \operatorname{rng}(p \rightarrow x)$.

(61) If $x \in \operatorname{rng} p$ and p is one-to-one, then $x \notin \operatorname{rng}(p \rightarrow x)$.

(62) p yields x just once if and only if $x \in \operatorname{rng} p$ and $\operatorname{rng}(p \rightarrow x)$ misses $\{x\}$.

(63) If $x \in \operatorname{rng} p$ and p is one-to-one, then $\operatorname{rng}(p \rightarrow x)$ misses $\{x\}$.

(64) If $x \in \operatorname{rng} p$, then $x \hookleftarrow p = \operatorname{len} p$ if and only if $p \rightarrow x = \varepsilon$.

(65) If $x \in \operatorname{rng} p$ and p is a finite sequence of elements of D, then $p \rightarrow x$ is a finite sequence of elements of D.

(66) If $x \in \operatorname{rng} p$, then $p = ((p \leftarrow x) \frown \langle x \rangle) \frown (p \rightarrow x)$.

(67) If $x \in \operatorname{rng} p$ and p is one-to-one, then $p \leftarrow x$ is one-to-one.

(68) If $x \in \operatorname{rng} p$ and p is one-to-one, then $p \rightarrow x$ is one-to-one.

(69) p yields x just once if and only if $x \in \operatorname{rng} p$ and $p - \{x\} = (p \leftarrow x) \frown (p \rightarrow x)$.

(70) If $x \in \operatorname{rng} p$ and p is one-to-one, then $p - \{x\} = (p \leftarrow x) \frown (p \rightarrow x)$.

(71) If $x \in \operatorname{rng} p$ and $p - \{x\}$ is one-to-one and $p - \{x\} = (p \leftarrow x) \frown (p \rightarrow x)$, then p is one-to-one.

(72) If $x \in \operatorname{rng} p$ and p is one-to-one, then $\operatorname{rng}(p \leftarrow x)$ misses $\operatorname{rng}(p \rightarrow x)$.

(73) If A is finite, then there exists p such that $\operatorname{rng} p = A$ and p is one-to-one.

(74) If $\operatorname{rng} p \subseteq \operatorname{dom} p$ and p is one-to-one, then $\operatorname{rng} p = \operatorname{dom} p$.

(75) If $\operatorname{rng} p = \operatorname{dom} p$, then p is one-to-one.

(76) If $\operatorname{rng} p = \operatorname{rng} q$ and $\operatorname{len} p = \operatorname{len} q$ and q is one-to-one, then p is one-to-one.

(77) p is one-to-one if and only if $\operatorname{card}(\operatorname{rng} p) = \operatorname{len} p$.

In the sequel f denotes a function from A into B. The following propositions are true:

(78) If $\operatorname{card} A = \operatorname{card} B$ and A is finite and B is finite and f is one-to-one, then $\operatorname{rng} f = B$.

(79) If $\operatorname{card} A = \operatorname{card} B$ and A is finite and B is finite and $\operatorname{rng} f = B$, then f is one-to-one.

(80) If $\overline{\overline{B}} < \overline{\overline{A}}$ and $B \neq \emptyset$, then there exist x, y such that $x \in A$ and $y \in A$ and $x \neq y$ and $f(x) = f(y)$.

(81) If $\overline{\overline{A}} < \overline{\overline{B}}$, then there exists x such that $x \in B$ and for every y such that $y \in A$ holds $f(y) \neq x$.

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Received April 8, 1990

FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Linear Combinations in Real Linear Space

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Summary. The article is continuation of [14]. At the beginning we prove some theorems concerning sums of finite sequence of vectors. We introduce the following notions: sum of finite subset of vectors, linear combination, carrier of linear combination, linear combination of elements of a given set of vectors, sum of linear combination. We also show that the set of linear combinations is a real linear space. At the end of article we prove some auxiliary theorems that should be proved in [16], [5], [7], [1] or [8].

MML Identifier: RLVECT_2.

The papers [16], [7], [5], [3], [6], [14], [8], [13], [15], [11], [9], [10], [4], [12], and [2] provide the notation and terminology for this paper. In the article we present several logical schemes. The scheme *LambdaSep1* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\mathcal{A}$, an element $\mathcal{D}$ of $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$ and states that:

there exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that $f(\mathcal{C}) = \mathcal{D}$ and for every element x of $\mathcal{A}$ such that $x \neq \mathcal{C}$ holds $f(x) = \mathcal{F}(x)$
for all values of the parameters.

The scheme *LambdaSep2* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\mathcal{A}$, an element $\mathcal{D}$ of $\mathcal{A}$, an element $\mathcal{E}$ of $\mathcal{B}$, an element $\mathcal{F}$ of $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$ and states that:

there exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that $f(\mathcal{C}) = \mathcal{E}$ and $f(\mathcal{D}) = \mathcal{F}$ and for every element x of $\mathcal{A}$ such that $x \neq \mathcal{C}$ and $x \neq \mathcal{D}$ holds $f(x) = \mathcal{F}(x)$
provided the following condition is satisfied:

- $\mathcal{C} \neq \mathcal{D}$.

Let D be a non-empty set. Then $\emptyset_D$ is a subset of D.

For simplicity we follow the rules: X, Y are sets, x is arbitrary, i, k, n are natural numbers, S is an RLS structure, V is a real linear space, u, v, v_1, v_2, v_3 are vectors of V, a, b, r are real numbers, F, G, H are finite sequences of elements of the vectors of V, A, B are subsets of the vectors of V, and f is a

ISSN 0777–4028

function from the vectors of V into $\mathbb{R}$. Let us consider S, and let v be an element of the vectors of S. The functor $@v$ yielding a vector of S, is defined as follows:

$@v = v$.

One can prove the following proposition

(1) For every element v of the vectors of V holds $v = @v$.

Let us consider S, x. Let us assume that $x \in S$. The functor x^S yielding a vector of S, is defined as follows:

$x^S = x$.

The following propositions are true:

(2) If $x \in S$, then $x^S = x$.

(3) For every vector v of S holds $v^S = v$.

(4) If $\operatorname{len} F = \operatorname{len} G$ and $\operatorname{len} F = \operatorname{len} H$ and for every k such that $k \in \operatorname{Seg}(\operatorname{len} F)$ holds $H(k) = @(\pi_k F) + @(\pi_k G)$, then $\sum H = \sum F + \sum G$.

(5) If $\operatorname{len} F = \operatorname{len} G$ and for every k such that $k \in \operatorname{Seg}(\operatorname{len} F)$ holds $G(k) = a \cdot @(\pi_k F)$, then $\sum G = a \cdot \sum F$.

(6) If $\operatorname{len} F = \operatorname{len} G$ and for every k such that $k \in \operatorname{Seg}(\operatorname{len} F)$ holds $G(k) = -@(\pi_k F)$, then $\sum G = -\sum F$.

(7) If $\operatorname{len} F = \operatorname{len} G$ and $\operatorname{len} F = \operatorname{len} H$ and for every k such that $k \in \operatorname{Seg}(\operatorname{len} F)$ holds $H(k) = @(\pi_k F) - @(\pi_k G)$, then $\sum H = \sum F - \sum G$.

(8) For all F, G and for every permutation f of $\operatorname{dom} F$ such that $\operatorname{len} F = \operatorname{len} G$ and for every i such that $i \in \operatorname{dom} G$ holds $G(i) = F(f(i))$ holds $\sum F = \sum G$.

(9) For every permutation f of $\operatorname{dom} F$ such that $G = F \cdot f$ holds $\sum F = \sum G$.

Let us consider V. A subset of the vectors of V is called a finite subset of V if:

it is finite.

One can prove the following proposition

(10) A is a finite subset of V if and only if A is finite.

In the sequel S, T will be finite subsets of V. Let us consider V, S, T. Then $S \cup T$ is a finite subset of V. Then $S \cap T$ is a finite subset of V. Then $S \setminus T$ is a finite subset of V. Then $S \dot{-} T$ is a finite subset of V.

Let us consider V. The functor 0_V yielding a finite subset of V, is defined by:

$0_V = \emptyset$.

One can prove the following proposition

(11) $0_V = \emptyset$.

Let us consider V, T. The functor $\sum T$ yields a vector of V and is defined as follows:

there exists F such that $\operatorname{rng} F = T$ and F is one-to-one and $\sum T = \sum F$.

One can prove the following propositions:

(12) There exists F such that $\operatorname{rng} F = T$ and F is one-to-one and $\sum T = \sum F$.

(13) If $\operatorname{rng} F = T$ and F is one-to-one and $v = \sum F$, then $v = \sum T$.

Let us consider V, v. Then $\{v\}$ is a finite subset of V.

Let us consider V, v_1, v_2. Then $\{v_1, v_2\}$ is a finite subset of V.

Let us consider V, v_1, v_2, v_3. Then $\{v_1, v_2, v_3\}$ is a finite subset of V.

One can prove the following propositions:

(14) $\sum(0_V) = 0_V$.

(15) $\sum\{v\} = v$.

(16) If $v_1 \neq v_2$, then $\sum\{v_1, v_2\} = v_1 + v_2$.

(17) If $v_1 \neq v_2$ and $v_2 \neq v_3$ and $v_1 \neq v_3$, then $\sum\{v_1, v_2, v_3\} = (v_1 + v_2) + v_3$.

(18) If T misses S, then $\sum(T \cup S) = \sum T + \sum S$.

(19) $\sum(T \cup S) = (\sum T + \sum S) - \sum(T \cap S)$.

(20) $\sum(T \cap S) = (\sum T + \sum S) - \sum(T \cup S)$.

(21) $\sum(T \setminus S) = \sum(T \cup S) - \sum S$.

(22) $\sum(T \setminus S) = \sum T - \sum(T \cap S)$.

(23) $\sum(T \dot{-} S) = \sum(T \cup S) - \sum(T \cap S)$.

(24) $\sum(T \dot{-} S) = \sum(T \setminus S) + \sum(S \setminus T)$.

Let us consider V. An element of $\mathbb{R}^{\text{the vectors of } V}$ is called a linear combination of V if:

there exists T such that for every v such that $v \notin T$ holds $\text{it}(v) = 0$.

In the sequel K, L, L_1, L_2, L_3 will be linear combinations of V. Next we state a proposition

(25) There exists T such that for every v such that $v \notin T$ holds $L(v) = 0$.

In the sequel E denotes an element of $\mathbb{R}^{\text{the vectors of } V}$. We now state a proposition

(26) If there exists T such that for every v such that $v \notin T$ holds $E(v) = 0$, then E is a linear combination of V.

Let us consider V, L. The functor $\operatorname{support} L$ yields a finite subset of V and is defined as follows:

$\operatorname{support} L = \{v : L(v) \neq 0\}$.

We now state two propositions:

(27) $\operatorname{support} L = \{v : L(v) \neq 0\}$.

(28) $L(v) = 0$ if and only if $v \notin \operatorname{support} L$.

Let us consider V. The functor $\mathbf{0}_{\mathrm{LC}_V}$ yields a linear combination of V and is defined as follows:

$\operatorname{support} \mathbf{0}_{\mathrm{LC}_V} = \emptyset$.

The following propositions are true:

(29) $L = \mathbf{0}_{\mathrm{LC}_V}$ if and only if $\operatorname{support} L = \emptyset$.

(30) $\mathbf{0}_{\mathrm{LC}_V}(v) = 0$.

Let us consider V, A. A linear combination of V is said to be a linear combination of A if:

support it $\subseteq A$.

One can prove the following proposition

(31) If support $L \subseteq A$, then L is a linear combination of A.

In the sequel l is a linear combination of A. The following propositions are true:

(32) support $l \subseteq A$.

(33) If $A \subseteq B$, then l is a linear combination of B.

(34) $\mathbf{0}_{\mathrm{LC}_V}$ is a linear combination of A.

(35) For every linear combination l of $\emptyset_{\text{the vectors of } V}$ holds $l = \mathbf{0}_{\mathrm{LC}_V}$.

(36) L is a linear combination of support L.

Let us consider V, F, f. The functor $f \cdot F$ yields a finite sequence of elements of the vectors of V and is defined as follows:

$\mathrm{len}(f \cdot F) = \mathrm{len}\, F$ and for every i such that $i \in \mathrm{dom}(f \cdot F)$ holds $(f \cdot F)(i) = f(@(\pi_i F)) \cdot @(\pi_i F)$.

Next we state several propositions:

(37) $\mathrm{len}(f \cdot F) = \mathrm{len}\, F$.

(38) For every i such that $i \in \mathrm{dom}(f \cdot F)$ holds $(f \cdot F)(i) = f(@(\pi_i F)) \cdot @(\pi_i F)$.

(39) If $\mathrm{len}\, G = \mathrm{len}\, F$ and for every i such that $i \in \mathrm{dom}\, G$ holds $G(i) = f(@(\pi_i F)) \cdot @(\pi_i F)$, then $G = f \cdot F$.

(40) If $i \in \mathrm{dom}\, F$ and $v = F(i)$, then $(f \cdot F)(i) = f(v) \cdot v$.

(41) $f \cdot \varepsilon_{\text{the vectors of } V} = \varepsilon_{\text{the vectors of } V}$.

(42) $f \cdot \langle v \rangle = \langle f(v) \cdot v \rangle$.

(43) $f \cdot \langle v_1, v_2 \rangle = \langle f(v_1) \cdot v_1, f(v_2) \cdot v_2 \rangle$.

(44) $f \cdot \langle v_1, v_2, v_3 \rangle = \langle f(v_1) \cdot v_1, f(v_2) \cdot v_2, f(v_3) \cdot v_3 \rangle$.

Let us consider V, L. The functor $\sum L$ yields a vector of V and is defined by:

there exists F such that F is one-to-one and $\mathrm{rng}\, F = \mathrm{support}\, L$ and $\sum L = \sum (L \cdot F)$.

The following propositions are true:

(45) There exists F such that F is one-to-one and $\mathrm{rng}\, F = \mathrm{support}\, L$ and $\sum L = \sum (L \cdot F)$.

(46) If F is one-to-one and $\mathrm{rng}\, F = \mathrm{support}\, L$ and $u = \sum (L \cdot F)$, then $u = \sum L$.

(47) $A \neq \emptyset$ and A is linearly closed if and only if for every l holds $\sum l \in A$.

(48) $\sum \mathbf{0}_{\mathrm{LC}_V} = 0_V$.

(49) For every linear combination l of $\emptyset_{\text{the vectors of } V}$ holds $\sum l = 0_V$.

(50) For every linear combination l of $\{v\}$ holds $\sum l = l(v) \cdot v$.

(51) If $v_1 \neq v_2$, then for every linear combination l of $\{v_1, v_2\}$ holds $\sum l = l(v_1) \cdot v_1 + l(v_2) \cdot v_2$.

(52) If support $L = \emptyset$, then $\sum L = 0_V$.

(53) If support $L = \{v\}$, then $\sum L = L(v) \cdot v$.

(54) If support $L = \{v_1, v_2\}$ and $v_1 \neq v_2$, then $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2$.

Let us consider V, L_1, L_2. Let us note that one can characterize the predicate $L_1 = L_2$ by the following (equivalent) condition: for every v holds $L_1(v) = L_2(v)$.

One can prove the following proposition

(55) If for every v holds $L_1(v) = L_2(v)$, then $L_1 = L_2$.

Let us consider V, L_1, L_2. The functor $L_1 + L_2$ yields a linear combination of V and is defined as follows:

for every v holds $(L_1 + L_2)(v) = L_1(v) + L_2(v)$.

The following propositions are true:

(56) If for every v holds $L(v) = L_1(v) + L_2(v)$, then $L = L_1 + L_2$.

(57) $(L_1 + L_2)(v) = L_1(v) + L_2(v)$.

(58) $\mathrm{support}(L_1 + L_2) \subseteq \mathrm{support}\, L_1 \cup \mathrm{support}\, L_2$.

(59) If L_1 is a linear combination of A and L_2 is a linear combination of A, then $L_1 + L_2$ is a linear combination of A.

(60) $L_1 + L_2 = L_2 + L_1$.

(61) $L_1 + (L_2 + L_3) = (L_1 + L_2) + L_3$.

(62) $L + \mathbf{0}_{\mathrm{LC}_V} = L$ and $\mathbf{0}_{\mathrm{LC}_V} + L = L$.

Let us consider V, a, L. The functor $a \cdot L$ yielding a linear combination of V, is defined by:

for every v holds $(a \cdot L)(v) = a \cdot L(v)$.

The following propositions are true:

(63) If for every v holds $K(v) = a \cdot L(v)$, then $K = a \cdot L$.

(64) $(a \cdot L)(v) = a \cdot L(v)$.

(65) If $a \neq 0$, then $\mathrm{support}(a \cdot L) = \mathrm{support}\, L$.

(66) $0 \cdot L = \mathbf{0}_{\mathrm{LC}_V}$.

(67) If L is a linear combination of A, then $a \cdot L$ is a linear combination of A.

(68) $(a + b) \cdot L = a \cdot L + b \cdot L$.

(69) $a \cdot (L_1 + L_2) = a \cdot L_1 + a \cdot L_2$.

(70) $a \cdot (b \cdot L) = (a \cdot b) \cdot L$.

(71) $1 \cdot L = L$.

Let us consider V, L. The functor $-L$ yielding a linear combination of V, is defined as follows:

$-L = (-1) \cdot L$.

Next we state several propositions:

(72) $-L = (-1) \cdot L$.

(73) $(-L)(v) = -L(v)$.

(74) If $L_1 + L_2 = \mathbf{0}_{\mathrm{LC}_V}$, then $L_2 = -L_1$.

(75) $\operatorname{support}(-L) = \operatorname{support} L$.

(76) If L is a linear combination of A, then $-L$ is a linear combination of A.

(77) $-(-L) = L$.

Let us consider V, L_1, L_2. The functor $L_1 - L_2$ yields a linear combination of V and is defined by:

$L_1 - L_2 = L_1 + (-L_2)$.

The following propositions are true:

(78) $L_1 - L_2 = L_1 + (-L_2)$.

(79) $(L_1 - L_2)(v) = L_1(v) - L_2(v)$.

(80) $\operatorname{support}(L_1 - L_2) \subseteq \operatorname{support} L_1 \cup \operatorname{support} L_2$.

(81) If L_1 is a linear combination of A and L_2 is a linear combination of A, then $L_1 - L_2$ is a linear combination of A.

(82) $L - L = \mathbf{0}_{\mathrm{LC}_V}$.

Let us consider V. The functor LC_V yields a non-empty set and is defined by:

$x \in \mathrm{LC}_V$ if and only if x is a linear combination of V.

In the sequel D denotes a non-empty set and e, e_1, e_2 denote elements of LC_V. The following propositions are true:

(83) If for every x holds $x \in D$ if and only if x is a linear combination of V, then $D = \mathrm{LC}_V$.

(84) $L \in \mathrm{LC}_V$.

Let us consider V, e. The functor $@e$ yields a linear combination of V and is defined by:

$@e = e$.

The following proposition is true

(85) $@e = e$.

Let us consider V, L. The functor $@L$ yields an element of LC_V and is defined as follows:

$@L = L$.

Next we state a proposition

(86) $@L = L$.

Let us consider V. The functor $+_{\mathrm{LC}_V}$ yields a binary operation on LC_V and is defined by:

for all e_1, e_2 holds $+_{\mathrm{LC}_V}(e_1, e_2) = @e_1 + @e_2$.

In the sequel o is a binary operation on LC_V. Next we state two propositions:

(87) If for all e_1, e_2 holds $o(e_1, e_2) = @e_1 + @e_2$, then $o = +_{\mathrm{LC}_V}$.

(88) $+_{\mathrm{LC}_V}(e_1, e_2) = @e_1 + @e_2$.

Let us consider V. The functor $\cdot_{\mathrm{LC}_V}$ yields a function from $[\!:\mathbb{R}, \mathrm{LC}_V:\!]$ into LC_V and is defined as follows:

for all a, e holds $\cdot_{\mathrm{LC}_V}(\langle a, e\rangle) = a \cdot @e$.

In the sequel g denotes a function from $[\!: \mathbb{R}, \mathrm{LC}_V :\!]$ into LC_V. We now state two propositions:

(89) If for all a, e holds $g(\langle a, e\rangle) = a \cdot @e$, then $g = \cdot_{\mathrm{LC}_V}$.

(90) $\cdot_{\mathrm{LC}_V}(\langle a, e\rangle) = a \cdot @e$.

Let us consider V. The functor $\mathbb{LC}_V$ yielding a real linear space, is defined as follows:

$\mathbb{LC}_V = \langle \mathrm{LC}_V, @\mathbf{0}_{\mathrm{LC}_V}, +_{\mathrm{LC}_V}, \cdot_{\mathrm{LC}_V}\rangle$.

Next we state several propositions:

(91) $\mathbb{LC}_V = \langle \mathrm{LC}_V, @\mathbf{0}_{\mathrm{LC}_V}, +_{\mathrm{LC}_V}, \cdot_{\mathrm{LC}_V}\rangle$.

(92) The vectors of $\mathbb{LC}_V = \mathrm{LC}_V$.

(93) The zero of $\mathbb{LC}_V = \mathbf{0}_{\mathrm{LC}_V}$.

(94) The addition of $\mathbb{LC}_V = +_{\mathrm{LC}_V}$.

(95) The multiplication$_1$ of $\mathbb{LC}_V = \cdot_{\mathrm{LC}_V}$.

(96) ${L_1}^{\mathbb{LC}_V} + {L_2}^{\mathbb{LC}_V} = L_1 + L_2$.

(97) $a \cdot L^{\mathbb{LC}_V} = a \cdot L$.

(98) $-L^{\mathbb{LC}_V} = -L$.

(99) ${L_1}^{\mathbb{LC}_V} - {L_2}^{\mathbb{LC}_V} = L_1 - L_2$.

Let us consider V, A. The functor $\mathbb{LC}_A$ yielding a subspace of $\mathbb{LC}_V$, is defined by:

the vectors of $\mathbb{LC}_A = \{l\}$.

In the sequel W denotes a subspace of $\mathbb{LC}_V$. Next we state two propositions:

(100) If the vectors of $W = \{l\}$, then $W = \mathbb{LC}_A$.

(101) The vectors of $\mathbb{LC}_A = \{l\}$.

We now state several propositions:

(102) $X \setminus Y$ misses $Y \setminus X$.

(103) If $k < n$, then $n - 1$ is a natural number.

(104) $-1 \neq 0$.

(105) $(-1) \cdot r = -r$.

(106) $r - 1 < r$.

(107) If X is finite and Y is finite, then $X \dot{-} Y$ is finite.

(108) For every function f such that $f^{-1}\, X = f^{-1}\, Y$ and $X \subseteq \operatorname{rng} f$ and $Y \subseteq \operatorname{rng} f$ holds $X = Y$.

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Received April 8, 1990

FORMALIZED MATHEMATICS
Vol.1, No.**3**, May–August 1990
Université Catholique de Louvain

König's Theorem

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Summary. In the article the sum and product of any number of cardinals are introduced and their relationships to addition, multiplication and to other concepts are shown. Then the König's theorem is proved. The theorem that the cardinal of union of increasing family of sets of power less than some cardinal **m** is not greater than **m**, is given too.

MML Identifier: `CARD_3`.

The papers [12], [6], [7], [3], [14], [13], [4], [2], [11], [9], [8], [10], [1], and [5] provide the terminology and notation for this paper. For simplicity we adopt the following rules: A, B are ordinal numbers, K, M, N are cardinal numbers, x, y, z are arbitrary, X, Y, Z, Z_1, Z_2 are sets, n is a natural number, and f, g are functions. A function is said to be a function yielding cardinal numbers if:

for every x such that $x \in \operatorname{dom} \mathrm{it}$ holds $\mathrm{it}(x)$ is a cardinal number.

Next we state a proposition

(1) f is a function yielding cardinal numbers if and only if for every x such that $x \in \operatorname{dom} f$ holds $f(x)$ is a cardinal number.

In the sequel ff denotes a function yielding cardinal numbers. Let us consider ff, X. Then $ff \restriction X$ is a function yielding cardinal numbers.

Let us consider ff, x. Then $ff(x)$ is a set.

Let us consider X, K. Then $X \longmapsto K$ is a function yielding cardinal numbers.

The following propositions are true:

(2) $ff \restriction X$ is a function yielding cardinal numbers and $X \longmapsto K$ is a function yielding cardinal numbers.

(3) $\square$ is a function yielding cardinal numbers.

The scheme *CF_Lambda* concerns a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a cardinal number and states that:

ISSN 0777–4028

there exists ff such that $\operatorname{dom} ff = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ holds $ff(x) = \mathcal{F}(x)$
for all values of the parameters.

We now define four new functors. Let us consider f. The functor $\overline{\overline{f}}$ yields a function yielding cardinal numbers and is defined as follows:

$\operatorname{dom} \overline{\overline{f}} = \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} f$ holds $\overline{\overline{f}}(x) = \overline{\overline{[f(x)]}}$.
The functor $\operatorname{disjoin} f$ yielding a function, is defined as follows:
$\operatorname{dom}(\operatorname{disjoin} f) = \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} f$ holds
$(\operatorname{disjoin} f)(x) = [\![[f(x)], \{x\}]\!]$.
The functor $\bigcup f$ yields a set and is defined by:
$\bigcup f = \bigcup(\operatorname{rng} f)$.
The functor $\prod f$ yielding a set, is defined by:
$x \in \prod f$ if and only if there exists g such that $x = g$ and $\operatorname{dom} g = \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} f$ holds $g(x) \in [f(x)]$.

We now state a number of propositions:

(4) $ff = \overline{\overline{f}}$ if and only if $\operatorname{dom} ff = \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} f$ holds $ff(x) = \overline{\overline{[f(x)]}}$.

(5) $g = \operatorname{disjoin} f$ if and only if $\operatorname{dom} g = \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} f$ holds $g(x) = [\![[f(x)], \{x\}]\!]$.

(6) $\bigcup f = \bigcup(\operatorname{rng} f)$.

(7) $X = \prod f$ if and only if for every x holds $x \in X$ if and only if there exists g such that $x = g$ and $\operatorname{dom} g = \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} f$ holds $g(x) \in [f(x)]$.

(8) $\overline{\overline{ff}} = ff$.

(9) $\overline{\overline{\square}} = \square$.

(10) $\overline{\overline{X \longmapsto Y}} = X \longmapsto \overline{\overline{Y}}$.

(11) $\operatorname{disjoin} \square = \square$.

(12) $\operatorname{disjoin}(\{x\} \longmapsto X) = \{x\} \longmapsto [\![X, \{x\}]\!]$.

(13) If $x \in \operatorname{dom} f$ and $y \in \operatorname{dom} f$ and $x \neq y$, then
$[\operatorname{disjoin} f(x)] \cap [\operatorname{disjoin} f(y)] = \emptyset$.

(14) $\bigcup \square = \emptyset$.

(15) $\bigcup(X \longmapsto Y) \subseteq Y$.

(16) If $X \neq \emptyset$, then $\bigcup(X \longmapsto Y) = Y$.

(17) $\bigcup(\{x\} \longmapsto Y) = Y$.

(18) $g \in \prod f$ if and only if $\operatorname{dom} g = \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} f$ holds $g(x) \in [f(x)]$.

(19) $\prod \square = \{\square\}$.

(20) $Y^X = \prod(X \longmapsto Y)$.

Let us consider x, X. The functor $\pi_x X$ yields a set and is defined by:
$y \in \pi_x X$ if and only if there exists f such that $f \in X$ and $y = f(x)$.

Next we state a number of propositions:

(21) $Y = \pi_x X$ if and only if for every y holds $y \in Y$ if and only if there exists f such that $f \in X$ and $y = f(x)$.

(22) If $x \in \operatorname{dom} f$ and $\prod f \neq \emptyset$, then $\pi_x(\prod f) = f(x)$.

(23) If $f \in X$, then $f(x) \in \pi_x X$.

(24) $\pi_x \emptyset = \emptyset$.

(25) $\pi_x\{g\} = \{g(x)\}$.

(26) $\pi_x\{f, g\} = \{f(x), g(x)\}$.

(27) $\pi_x(X \cup Y) = \pi_x X \cup \pi_x Y$.

(28) $\pi_x(X \cap Y) \subseteq \pi_x X \cap \pi_x Y$.

(29) $\pi_x X \setminus \pi_x Y \subseteq \pi_x(X \setminus Y)$.

(30) $\pi_x X \dot{-} \pi_x Y \subseteq \pi_x(X \dot{-} Y)$.

(31) $\overline{\overline{\pi_x X}} \leq \overline{\overline{X}}$.

(32) If $x \in \bigcup(\operatorname{disjoin} f)$, then there exist y, z such that $x = \langle y, z \rangle$.

(33) $x \in \bigcup(\operatorname{disjoin} f)$ if and only if $x_{\mathbf{2}} \in \operatorname{dom} f$ and $x_{\mathbf{1}} \in [f(x_{\mathbf{2}})]$ and $x = \langle x_{\mathbf{1}}, x_{\mathbf{2}} \rangle$.

(34) If $f \leq g$, then $\operatorname{disjoin} f \leq \operatorname{disjoin} g$.

(35) If $f \leq g$, then $\bigcup f \subseteq \bigcup g$.

(36) $\bigcup(\operatorname{disjoin}(Y \longmapsto X)) = [\!: X, Y :\!]$.

(37) $\prod f = \emptyset$ if and only if $\emptyset \in \operatorname{rng} f$.

(38) If $\operatorname{dom} f = \operatorname{dom} g$ and for every x such that $x \in \operatorname{dom} f$ holds $[f(x)] \subseteq [g(x)]$, then $\prod f \subseteq \prod g$.

In the sequel F, G will denote functions yielding cardinal numbers. The following two propositions are true:

(39) For every x such that $x \in \operatorname{dom} F$ holds $\overline{\overline{F(x)}} = F(x)$.

(40) For every x such that $x \in \operatorname{dom} F$ holds $\overline{\overline{[\operatorname{disjoin} F(x)]}} = F(x)$.

We now define two new functors. Let us consider F. The functor $\sum F$ yields a cardinal number and is defined as follows:

$\sum F = \overline{\overline{\bigcup(\operatorname{disjoin} F)}}$.

The functor $\prod F$ yielding a cardinal number, is defined as follows:

$\prod F = \overline{\overline{\prod F}}$.

The following propositions are true:

(41) $\sum F = \overline{\overline{\bigcup(\operatorname{disjoin} F)}}$.

(42) $\prod F = \overline{\overline{\prod F}}$.

(43) If $\operatorname{dom} F = \operatorname{dom} G$ and for every x such that $x \in \operatorname{dom} F$ holds $F(x) \subseteq G(x)$, then $\sum F \leq \sum G$.

(44) $\emptyset \in \operatorname{rng} F$ if and only if $\prod F = \overline{\mathbf{0}}$.

(45) If $\operatorname{dom} F = \operatorname{dom} G$ and for every x such that $x \in \operatorname{dom} F$ holds $F(x) \subseteq G(x)$, then $\prod F \leq \prod G$.

(46) If $F \leq G$, then $\sum F \leq \sum G$.
(47) If $F \leq G$ and $\overline{\mathbf{0}} \notin \operatorname{rng} G$, then $\prod F \leq \prod G$.
(48) $\sum(\emptyset \longmapsto K) = \overline{\mathbf{0}}$.
(49) $\prod(\emptyset \longmapsto K) = \overline{\mathbf{1}}$.
(50) $\sum(\{x\} \longmapsto K) = K$.
(51) $\prod(\{x\} \longmapsto K) = K$.
(52) $\sum(M \longmapsto N) = M \cdot N$.
(53) $\prod(N \longmapsto M) = M^N$.
(54) $\overline{\overline{\bigcup f}} \leq \sum \overline{\overline{f}}$.
(55) $\overline{\overline{\bigcup F}} \leq \sum F$.
(56) If $\operatorname{dom} F = \operatorname{dom} G$ and for every x such that $x \in \operatorname{dom} F$ holds $F(x) \in G(x)$, then $\sum F < \prod G$.

Now we present three schemes. The scheme *FinRegularity* deals with a set $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists x such that $x \in \mathcal{A}$ and for every y such that $y \in \mathcal{A}$ and $y \neq x$ holds not $\mathcal{P}[y, x]$

provided the following conditions are fulfilled:

- $\mathcal{A}$ is finite and $\mathcal{A} \neq \emptyset$,
- for all x, y such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, x]$ holds $x = y$,
- for all x, y, z such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$.

The scheme *MaxFinSetElem* concerns a set $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists x such that $x \in \mathcal{A}$ and for every y such that $y \in \mathcal{A}$ holds $\mathcal{P}[x, y]$

provided the following requirements are fulfilled:

- $\mathcal{A}$ is finite and $\mathcal{A} \neq \emptyset$,
- for all x, y holds $\mathcal{P}[x, y]$ or $\mathcal{P}[y, x]$,
- for all x, y, z such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$.

The scheme *FuncSeparation* deals with a set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and a binary predicate $\mathcal{P}$, and states that:

there exists f such that $\operatorname{dom} f = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ for every y holds $y \in [f(x)]$ if and only if $y \in \mathcal{F}(x)$ and $\mathcal{P}[x, y]$

for all values of the parameters.

We now state several propositions:

(57) $\mathbf{R}_{\mathrm{ord}(n)}$ is finite.
(58) If X is finite, then $\overline{\overline{X}} < \overline{\overline{\omega}}$.
(59) If $\overline{\overline{A}} < \overline{\overline{B}}$, then $A \in B$.
(60) If $\overline{\overline{A}} < M$, then $A \in M$.
(61) Suppose for all Z_1, Z_2 such that $Z_1 \in X$ and $Z_2 \in X$ holds $Z_1 \subseteq Z_2$ or $Z_2 \subseteq Z_1$. Then there exists Y such that $Y \subseteq X$ and $\bigcup Y = \bigcup X$ and for every Z such that $Z \subseteq Y$ and $Z \neq \emptyset$ there exists Z_1 such that $Z_1 \in Z$ and for every Z_2 such that $Z_2 \in Z$ holds $Z_1 \subseteq Z_2$.

(62) If for every Z such that $Z \in X$ holds $\overline{\overline{Z}} < M$ and for all Z_1, Z_2 such that $Z_1 \in X$ and $Z_2 \in X$ holds $Z_1 \subseteq Z_2$ or $Z_2 \subseteq Z_1$, then $\overline{\overline{\bigcup X}} \leq M$.

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Received April 10, 1990

FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Universal Classes

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Summary. In the article we have shown that there exist universal classes, i.e. there are sets which are closed w.r.t. basic set theory operations.

MML Identifier: `CLASSES2`.

The articles [11], [8], [4], [7], [10], [9], [5], [2], [1], [6], and [3] provide the terminology and notation for this paper. For simplicity we adopt the following convention: m is a cardinal number, A, B, C are ordinal numbers, x, y are arbitrary, and X, Y, W are sets. One can prove the following propositions:

(1) If W is a Tarski-Class and $X \in W$, then $X \not\approx W$ and $\overline{\overline{X}} < \overline{\overline{W}}$.

(2) If W is a Tarski-Class and $X \subseteq W$ and $\overline{\overline{X}} < \overline{\overline{W}}$, then $X \in W$.

(3) If W is a Tarski-Class and $x \in W$ and $y \in W$, then $\{x\} \in W$ and $\{x, y\} \in W$.

(4) If W is a Tarski-Class and $x \in W$ and $y \in W$, then $\langle x, y \rangle \in W$.

(5) If W is a Tarski-Class and $X \in W$, then $\mathbf{T}(X) \subseteq W$.

The scheme TC deals with a unary predicate $\mathcal{P}$, and states that:
for every X holds $\mathcal{P}[\mathbf{T}(X)]$
provided the parameter fulfills the following condition:

- for every X such that X is a Tarski-Class holds $\mathcal{P}[X]$.

Next we state a number of propositions:

(6) If W is a Tarski-Class and $A \in W$, then $\operatorname{succ} A \in W$ and $A \subseteq W$.

(7) If $A \in \mathbf{T}(W)$, then $\operatorname{succ} A \in \mathbf{T}(W)$ and $A \subseteq \mathbf{T}(W)$.

(8) If W is a Tarski-Class and X is transitive and $X \in W$, then $X \subseteq W$.

(9) If X is transitive and $X \in \mathbf{T}(W)$, then $X \subseteq \mathbf{T}(W)$.

(10) If W is a Tarski-Class, then $\mathrm{On}\, W = \overline{\overline{W}}$.

(11) $\mathrm{On}\, \mathbf{T}(W) = \overline{\overline{\mathbf{T}(W)}}$.

ISSN 0777–4028

(12) If W is a Tarski-Class and $X \in W$, then $\overline{\overline{X}} \in W$.

(13) If $X \in \mathbf{T}(W)$, then $\overline{\overline{X}} \in \mathbf{T}(W)$.

(14) If W is a Tarski-Class and $x \in \mathrm{ord}(\overline{\overline{W}})$, then $x \in W$.

(15) If $x \in \mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})$, then $x \in \mathbf{T}(W)$.

(16) If W is a Tarski-Class and $m < \overline{\overline{W}}$, then $m \in W$.

(17) If $m < \overline{\overline{\mathbf{T}(W)}}$, then $m \in \mathbf{T}(W)$.

(18) If W is a Tarski-Class and $m \in W$, then $m \subseteq W$.

(19) If $m \in \mathbf{T}(W)$, then $m \subseteq \mathbf{T}(W)$.

(20) If W is a Tarski-Class, then $\mathrm{ord}(\overline{\overline{W}})$ is a limit ordinal number.

(21) If W is a Tarski-Class and $W \neq \emptyset$, then $\overline{\overline{W}} \neq \overline{\mathbf{0}}$ and $\mathrm{ord}(\overline{\overline{W}}) \neq \mathbf{0}$ and $\mathrm{ord}(\overline{\overline{W}})$ is a limit ordinal number.

(22) $\overline{\overline{\mathbf{T}(W)}} \neq \overline{\mathbf{0}}$ and $\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}}) \neq \mathbf{0}$ and $\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})$ is a limit ordinal number.

In the sequel L, L_1 are transfinite sequences. We now state a number of propositions:

(23) If W is a Tarski-Class but $X \in W$ and W is transitive or $X \in W$ and $X \subseteq W$ or $\overline{\overline{X}} < \overline{\overline{W}}$ and $X \subseteq W$, then $W^X \subseteq W$.

(24) If $X \in \mathbf{T}(W)$ and W is transitive or $X \in \mathbf{T}(W)$ and $X \subseteq \mathbf{T}(W)$ or $\overline{\overline{X}} < \overline{\overline{\mathbf{T}(W)}}$ and $X \subseteq \mathbf{T}(W)$, then $\mathbf{T}(W)^X \subseteq \mathbf{T}(W)$.

(25) If $\mathrm{dom}\,L$ is a limit ordinal number and for every A such that $A \in \mathrm{dom}\,L$ holds $L(A) = \mathbf{R}_A$, then $\mathbf{R}_{\mathrm{dom}\,L} = \bigcup L$.

(26) If W is a Tarski-Class and $A \in \mathrm{On}\,W$, then $\overline{\overline{\mathbf{R}_A}} < \overline{\overline{W}}$ and $\mathbf{R}_A \in W$.

(27) If $A \in \mathrm{On}\,\mathbf{T}(W)$, then $\overline{\overline{\mathbf{R}_A}} < \overline{\overline{\mathbf{T}(W)}}$ and $\mathbf{R}_A \in \mathbf{T}(W)$.

(28) If W is a Tarski-Class, then $\mathbf{R}_{\mathrm{ord}(\overline{\overline{W}})} \subseteq W$.

(29) $\mathbf{R}_{\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})} \subseteq \mathbf{T}(W)$.

(30) If W is a Tarski-Class and W is transitive and $X \in W$, then $\mathrm{rk}(X) \in W$.

(31) If W is a Tarski-Class and W is transitive, then $W \subseteq \mathbf{R}_{\mathrm{ord}(\overline{\overline{W}})}$.

(32) If W is a Tarski-Class and W is transitive, then $\mathbf{R}_{\mathrm{ord}(\overline{\overline{W}})} = W$.

(33) If W is a Tarski-Class and $A \in \mathrm{On}\,W$, then $\overline{\overline{\mathbf{R}_A}} \leq \overline{\overline{W}}$.

(34) If $A \in \mathrm{On}\,\mathbf{T}(W)$, then $\overline{\overline{\mathbf{R}_A}} \leq \overline{\overline{\mathbf{T}(W)}}$.

(35) If W is a Tarski-Class, then $\overline{\overline{W}} = \overline{\overline{\mathbf{R}_{\mathrm{ord}(\overline{\overline{W}})}}}$.

(36) $\overline{\overline{\mathbf{T}(W)}} = \overline{\overline{\mathbf{R}_{\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})}}}$.

(37) If W is a Tarski-Class and $X \subseteq \mathbf{R}_{\mathrm{ord}(\overline{\overline{W}})}$, then $X \approx \mathbf{R}_{\mathrm{ord}(\overline{\overline{W}})}$ or $X \in \mathbf{R}_{\mathrm{ord}(\overline{\overline{W}})}$.

(38) If $X \subseteq \mathbf{R}_{\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})}$, then $X \approx \mathbf{R}_{\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})}$ or $X \in \mathbf{R}_{\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})}$.

(39) If W is a Tarski-Class, then $\mathbf{R}_{\mathrm{ord}(\overline{\overline{W}})}$ is a Tarski-Class.

(40) $\mathbf{R}_{\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})}$ is a Tarski-Class.

(41) If X is transitive and $A \in \mathrm{rk}(X)$, then there exists Y such that $Y \in X$ and $\mathrm{rk}(Y) = A$.

(42) If X is transitive, then $\overline{\overline{\mathrm{rk}(X)}} \leq \overline{\overline{X}}$.

(43) If W is a Tarski-Class and X is transitive and $X \in W$, then $X \in \mathbf{R}_{\mathrm{ord}(\overline{\overline{W}})}$.

(44) If X is transitive and $X \in \mathbf{T}(W)$, then $X \in \mathbf{R}_{\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})}$.

(45) If W is transitive, then $\mathbf{R}_{\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})}$ is Tarski-Class of W.

(46) If W is transitive, then $\mathbf{R}_{\mathrm{ord}(\overline{\overline{\mathbf{T}(W)}})} = \mathbf{T}(W)$.

A non-empty family of sets is called a universal class if:
it is transitive and it is a Tarski-Class.

In the sequel M denotes a non-empty family of sets. The following proposition is true

(47) For every M holds M is a universal class if and only if M is transitive and M is a Tarski-Class.

In the sequel U_1, U_2, U_3, $Universum$ will be universal classes. We now state several propositions:

(48) If $X \in Universum$, then $X \subseteq Universum$.

(49) If $X \in Universum$ and $Y \subseteq X$, then $Y \in Universum$.

(50) $\mathrm{On}\, Universum$ is an ordinal number.

(51) If X is transitive, then $\mathbf{T}(X)$ is a universal class.

(52) $\mathbf{T}(Universum)$ is a universal class.

Let us consider $Universum$. Then $\mathrm{On}\, Universum$ is an ordinal number. Then $\mathbf{T}(Universum)$ is a universal class.

Next we state a proposition

(53) $\mathbf{T}(A)$ is a universal class.

Let us consider A. Then $\mathbf{T}(A)$ is a universal class.

Next we state a number of propositions:

(54) $Universum = \mathbf{R}_{\mathrm{On}\, Universum}$.

(55) $\mathrm{On}\, Universum \neq \mathbf{0}$ and $\mathrm{On}\, Universum$ is a limit ordinal number.

(56) $U_1 \in U_2$ or $U_1 = U_2$ or $U_2 \in U_1$.

(57) $U_1 \subseteq U_2$ or $U_2 \in U_1$.

(58) $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.

(59) If $U_1 \in U_2$ and $U_2 \in U_3$, then $U_1 \in U_3$.

(60) If $U_1 \subseteq U_2$ and $U_2 \in U_3$, then $U_1 \in U_3$.

(61) $U_1 \cup U_2$ is a universal class and $U_1 \cap U_2$ is a universal class.

(62) $\emptyset \in Universum$.

(63) If $x \in Universum$, then $\{x\} \in Universum$.

(64) If $x \in Universum$ and $y \in Universum$, then $\{x, y\} \in Universum$ and $\langle x, y\rangle \in Universum$.

(65) If $X \in Universum$, then $2^X \in Universum$ and $\bigcup X \in Universum$ and $\bigcap X \in Universum$.

(66) If $X \in Universum$ and $Y \in Universum$, then $X \cup Y \in Universum$ and $X \cap Y \in Universum$ and $X \setminus Y \in Universum$ and $X \dot{-} Y \in Universum$.

(67) If $X \in Universum$ and $Y \in Universum$, then $[\!:X, Y:\!] \in Universum$ and $Y^X \in Universum$.

In the sequel u, v are elements of $Universum$. Let us consider $Universum$, u. Then $\{u\}$ is an element of $Universum$. Then 2^u is an element of $Universum$. Then $\bigcup u$ is an element of $Universum$. Then $\bigcap u$ is an element of $Universum$. Let us consider v. Then $\{u, v\}$ is an element of $Universum$. Then $\langle u, v\rangle$ is an element of $Universum$. Then $u \cup v$ is an element of $Universum$. Then $u \cap v$ is an element of $Universum$. Then $u \setminus v$ is an element of $Universum$. Then $u \dot{-} v$ is an element of $Universum$. Then $[\!:u, v:\!]$ is an element of $Universum$. Then v^u is an element of $Universum$.

The universal class $\mathbf{U}_0$ is defined as follows:
$\mathbf{U}_0 = \mathbf{T}(\mathbf{0})$.

We now state four propositions:

(68) $\mathbf{U}_0 = \mathbf{T}(\mathbf{0})$.

(69) $\overline{\overline{\mathbf{R}_\omega}} = \overline{\overline{\omega}}$.

(70) $\mathbf{R}_\omega$ is a Tarski-Class.

(71) $\mathbf{U}_0 = \mathbf{R}_\omega$.

The universal class $\mathbf{U}_1$ is defined by:
$\mathbf{U}_1 = \mathbf{T}(\mathbf{U}_0)$.

The following proposition is true

(72) $\mathbf{U}_1 = \mathbf{T}(\mathbf{U}_0)$.

We now define three new constructions. A set of a finite rank is an element of $\mathbf{U}_0$.

A *Set* is an element of $\mathbf{U}_1$.

Let us consider A. The functor $\mathbf{U}_A$ is defined as follows:

there exists L such that $\mathbf{U}_A = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} A$ and $L(\mathbf{0}) = \mathbf{U}_0$ and for all C, y such that $\operatorname{succ} C \in \operatorname{succ} A$ and $y = L(C)$ holds $L(\operatorname{succ} C) = \mathbf{T}([y])$ and for all C, L_1 such that $C \in \operatorname{succ} A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \restriction C$ holds $L(C) = \mathbf{T}(\bigcup L_1)$.

The following two propositions are true:

(73) For every element u of $\mathbf{U}_0$ holds u is a set of a finite rank.

(74) For every element u of $\mathbf{U}_1$ holds u is a *Set*.

Let u be a set of a finite rank. Then $\{u\}$ is a set of a finite rank. Then 2^u is a set of a finite rank. Then $\bigcup u$ is a set of a finite rank. Then $\bigcap u$ is a set of a finite rank. Let v be a set of a finite rank. Then $\{u, v\}$ is a set of a finite rank. Then $\langle u, v \rangle$ is a set of a finite rank. Then $u \cup v$ is a set of a finite rank. Then $u \cap v$ is a set of a finite rank. Then $u \setminus v$ is a set of a finite rank. Then $u \dot{-} v$ is a set of a finite rank. Then $[\!: u, v :\!]$ is a set of a finite rank. Then v^u is a set of a finite rank.

Let u be a *Set*. Then $\{u\}$ is a *Set*. Then 2^u is a *Set*. Then $\bigcup u$ is a *Set*. Then $\bigcap u$ is a *Set*. Let v be a *Set*. Then $\{u, v\}$ is a *Set*. Then $\langle u, v \rangle$ is a *Set*. Then $u \cup v$ is a *Set*. Then $u \cap v$ is a *Set*. Then $u \setminus v$ is a *Set*. Then $u \dot{-} v$ is a *Set*. Then $[\!: u, v :\!]$ is a *Set*. Then v^u is a *Set*.

Let us consider A. Then $\mathbf{U}_A$ is a universal class.

We now state several propositions:

(75) $\mathbf{U}_{\mathbf{0}} = \mathbf{U}_0$.

(76) $\mathbf{U}_{\mathrm{succ}\, A} = \mathbf{T}(\mathbf{U}_A)$.

(77) $\mathbf{U}_{\mathbf{1}} = \mathbf{U}_1$.

(78) If $A \neq \mathbf{0}$ and A is a limit ordinal number and $\operatorname{dom} L = A$ and for every B such that $B \in A$ holds $L(B) = \mathbf{U}_B$, then $\mathbf{U}_A = \mathbf{T}(\bigcup L)$.

(79) $\mathbf{U}_0 \subseteq \mathit{Universum}$ and $\mathbf{T}(\mathbf{0}) \subseteq \mathit{Universum}$ and $\mathbf{U}_{\mathbf{0}} \subseteq \mathit{Universum}$.

(80) $A \in B$ if and only if $\mathbf{U}_A \in \mathbf{U}_B$.

(81) If $\mathbf{U}_A = \mathbf{U}_B$, then $A = B$.

(82) $A \subseteq B$ if and only if $\mathbf{U}_A \subseteq \mathbf{U}_B$.

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Received April 10, 1990

FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Analytical Ordered Affine Spaces [1]

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Summary. In the article with a given arbitrary real linear space we correlate the (ordered) affine space defined in terms of a directed parallelity of segments. The abstract contains a construction of the ordered affine structure associated with a vector space; this is a structure of the type which frequently occurs in geometry and consists of the set of points and a binary relation on segments. For suitable underlying vector spaces we prove that the corresponding affine structures are ordered affine spaces or ordered affine planes, i.e. that they satisfy appropriate axioms. A formal definition of an arbitrary ordered affine space and an arbitrary ordered affine plane is given.

MML Identifier: `ANALOAF`.

The notation and terminology used here have been introduced in the following articles: [4], [3], [2], [1], and [5]. We adopt the following rules: V will denote a real linear space, p, q, u, v, w, y will denote vectors of V, and a, b will denote real numbers. Let us consider V, u, v, w, y. The predicate $u, v \upuparrows w, y$ is defined by:

$u = v$ or $w = y$ or there exist a, b such that $0 < a$ and $0 < b$ and $a \cdot (v - u) = b \cdot (y - w)$.

Next we state a number of propositions:

(1) $u, v \upuparrows w, y$ if and only if $u = v$ or $w = y$ or there exist a, b such that $0 < a$ and $0 < b$ and $a \cdot (v - u) = b \cdot (y - w)$.

(2) If $0 < a$ and $0 < b$, then $0 < a + b$.

(3) If $a \neq b$, then $0 < a - b$ or $0 < b - a$.

(4) $(w - v) + (v - u) = w - u$.

(5) $-(u - v) = v - u$.

(6) $w - (u - v) = w + (v - u)$.

(7) $(w - u) + u = w$.

[1]Supported by RPBP.III.24-C6

ISSN 0777–4028

(8) $(w + u) - u = w$.

(9) If $y + u = v + w$, then $y - w = v - u$.

(10) $a \cdot (u - v) = -a \cdot (v - u)$.

(11) $(a - b) \cdot (u - v) = (b - a) \cdot (v - u)$.

(12) If $a \neq 0$ and $a \cdot u = v$, then $u = a^{-1} \cdot v$.

(13) If $a \neq 0$ and $a \cdot u = v$, then $u = a^{-1} \cdot v$ but if $a \neq 0$ and $u = a^{-1} \cdot v$, then $a \cdot u = v$.

(14) If $u = v$ or $w = y$, then $u, v \upuparrows w, y$.

(15) If $a \cdot (v - u) = b \cdot (y - w)$ and $0 < a$ and $0 < b$, then $u, v \upuparrows w, y$.

(16) If $u, v \upuparrows w, y$ and $u \neq v$ and $w \neq y$, then there exist a, b such that $a \cdot (v - u) = b \cdot (y - w)$ and $0 < a$ and $0 < b$.

(17) $u, v \upuparrows u, v$.

(18) $u, v \upuparrows w, w$ and $u, u \upuparrows v, w$.

(19) If $u, v \upuparrows v, u$, then $u = v$.

(20) If $p \neq q$ and $p, q \upuparrows u, v$ and $p, q \upuparrows w, y$, then $u, v \upuparrows w, y$.

(21) If $u, v \upuparrows w, y$, then $v, u \upuparrows y, w$ and $w, y \upuparrows u, v$.

(22) If $u, v \upuparrows v, w$, then $u, v \upuparrows u, w$.

(23) If $u, v \upuparrows u, w$, then $u, v \upuparrows v, w$ or $u, w \upuparrows w, v$.

(24) If $v - u = y - w$, then $u, v \upuparrows w, y$.

(25) If $y = (v + w) - u$, then $u, v \upuparrows w, y$ and $u, w \upuparrows v, y$.

(26) If there exist p, q such that $p \neq q$, then for every u, v, w there exists y such that $u, v \upuparrows w, y$ and $u, w \upuparrows v, y$ and $v \neq y$.

(27) If $p \neq v$ and $v, p \upuparrows p, w$, then there exists y such that $u, p \upuparrows p, y$ and $u, v \upuparrows w, y$.

(28) If for all a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$, then $u \neq v$ and $u \neq 0_V$ and $v \neq 0_V$.

(29) If there exist u, v such that for all a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$, then there exist u, v, w, y such that $u, v \not\upuparrows w, y$ and $u, v \not\upuparrows y, w$.

(30) If $a - b = 0$, then $a = b$.

Next we state a proposition

(31) Suppose there exist p, q such that for every w there exist a, b such that $a \cdot p + b \cdot q = w$. Then for all u, v, w, y such that $u, v \not\upuparrows w, y$ and $u, v \not\upuparrows y, w$ there exists a vector z of V such that $u, v \upuparrows u, z$ or $u, v \upuparrows z, u$ but $w, y \upuparrows w, z$ or $w, y \upuparrows z, w$.

We consider affine structures which are systems

$\langle$ points, a congruence $\rangle$

where the points is a non-empty set and the congruence is a relation on $[\!:$ the points, the points $:\!]$. We adopt the following convention: AS will denote an affine structure and a, b, c, d will denote elements of the points of AS. Let us consider AS, a, b, c, d. The predicate $a, b \upuparrows c, d$ is defined by:

$\langle\langle a,b\rangle,\langle c,d\rangle\rangle \in$ the congruence of AS.

We now state a proposition

(32) $a,b \upuparrows c,d$ if and only if $\langle\langle a,b\rangle,\langle c,d\rangle\rangle \in$ the congruence of AS.

In the sequel x, z are arbitrary. Let us consider V. The functor $\upuparrows_V$ yields a relation on [: the vectors of V, the vectors of V :] and is defined as follows:

$\langle x,z\rangle \in \upuparrows_V$ if and only if there exist u, v, w, y such that $x = \langle u,v\rangle$ and $z = \langle w,y\rangle$ and $u,v \upuparrows w,y$.

One can prove the following proposition

(33) $\langle\langle u,v\rangle,\langle w,y\rangle\rangle \in \upuparrows_V$ if and only if $u,v \upuparrows w,y$.

Let us consider V. The functor OASpace V yields an affine structure and is defined as follows:

OASpace $V = \langle$ the vectors of $V, \upuparrows_V\rangle$.

Next we state three propositions:

(34) OASpace $V = \langle$ the vectors of $V, \upuparrows_V\rangle$.

(35) Suppose there exist u, v such that for all real numbers a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$. Then

(i) there exist elements a, b of the points of OASpace V such that $a \neq b$,

(ii) for all elements a, b, c, d, p, q, r, s of the points of OASpace V holds $a,b \upuparrows c,c$ but if $a,b \upuparrows b,a$, then $a = b$ but if $a \neq b$ and $a,b \upuparrows p,q$ and $a,b \upuparrows r,s$, then $p,q \upuparrows r,s$ but if $a,b \upuparrows c,d$, then $b,a \upuparrows d,c$ but if $a,b \upuparrows b,c$, then $a,b \upuparrows a,c$ but if $a,b \upuparrows a,c$, then $a,b \upuparrows b,c$ or $a,c \upuparrows c,b$,

(iii) there exist elements a, b, c, d of the points of OASpace V such that $a,b \not\upuparrows c,d$ and $a,b \not\upuparrows d,c$,

(iv) for every elements a, b, c of the points of OASpace V there exists an element d of the points of OASpace V such that $a,b \upuparrows c,d$ and $a,c \upuparrows b,d$ and $b \neq d$,

(v) for all elements p, a, b, c of the points of OASpace V such that $p \neq b$ and $b,p \upuparrows p,c$ there exists an element d of the points of OASpace V such that $a,p \upuparrows p,d$ and $a,b \upuparrows c,d$.

(36) Suppose there exist vectors p, q of V such that for every vector w of V there exist real numbers a, b such that $a \cdot p + b \cdot q = w$. Let a, b, c, d be elements of the points of OASpace V. Then if $a,b \not\upuparrows c,d$ and $a,b \not\upuparrows d,c$, then there exists an element t of the points of OASpace V such that $a,b \upuparrows a,t$ or $a,b \upuparrows t,a$ but $c,d \upuparrows c,t$ or $c,d \upuparrows t,c$.

An affine structure is called an ordered affine space if:

(i) there exist elements a, b of the points of it such that $a \neq b$,

(ii) for all elements a, b, c, d, p, q, r, s of the points of it holds $a,b \upuparrows c,c$ but if $a,b \upuparrows b,a$, then $a = b$ but if $a \neq b$ and $a,b \upuparrows p,q$ and $a,b \upuparrows r,s$, then $p,q \upuparrows r,s$ but if $a,b \upuparrows c,d$, then $b,a \upuparrows d,c$ but if $a,b \upuparrows b,c$, then $a,b \upuparrows a,c$ but if $a,b \upuparrows a,c$, then $a,b \upuparrows b,c$ or $a,c \upuparrows c,b$,

(iii) there exist elements a, b, c, d of the points of it such that $a,b \not\upuparrows c,d$ and $a,b \not\upuparrows d,c$,

(iv) for every elements a, b, c of the points of it there exists an element d of the points of it such that $a, b \upuparrows c, d$ and $a, c \upuparrows b, d$ and $b \neq d$,
(v) for all elements p, a, b, c of the points of it such that $p \neq b$ and $b, p \upuparrows p, c$ there exists an element d of the points of it such that $a, p \upuparrows p, d$ and $a, b \upuparrows c, d$.

One can prove the following propositions:

(37) The following conditions are equivalent:
 (i) there exist elements a, b of the points of AS such that $a \neq b$ and for all elements a, b, c, d, p, q, r, s of the points of AS holds $a, b \upuparrows c, c$ but if $a, b \upuparrows b, a$, then $a = b$ but if $a \neq b$ and $a, b \upuparrows p, q$ and $a, b \upuparrows r, s$, then $p, q \upuparrows r, s$ but if $a, b \upuparrows c, d$, then $b, a \upuparrows d, c$ but if $a, b \upuparrows b, c$, then $a, b \upuparrows a, c$ but if $a, b \upuparrows a, c$, then $a, b \upuparrows b, c$ or $a, c \upuparrows c, b$ and there exist elements a, b, c, d of the points of AS such that $a, b \not\upuparrows c, d$ and $a, b \not\upuparrows d, c$ and for every elements a, b, c of the points of AS there exists an element d of the points of AS such that $a, b \upuparrows c, d$ and $a, c \upuparrows b, d$ and $b \neq d$ and for all elements p, a, b, c of the points of AS such that $p \neq b$ and $b, p \upuparrows p, c$ there exists an element d of the points of AS such that $a, p \upuparrows p, d$ and $a, b \upuparrows c, d$,
 (ii) AS is an ordered affine space.

(38) If there exist u, v such that for all real numbers a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$, then OASpace V is an ordered affine space.

We adopt the following rules: A will denote an ordered affine space and a, b, c, d, p, q, r, s will denote elements of the points of A. We now state a number of propositions:

(39) There exist a, b such that $a \neq b$.
(40) $a, b \upuparrows c, c$.
(41) If $a, b \upuparrows b, a$, then $a = b$.
(42) If $a \neq b$ and $a, b \upuparrows p, q$ and $a, b \upuparrows r, s$, then $p, q \upuparrows r, s$.
(43) If $a, b \upuparrows c, d$, then $b, a \upuparrows d, c$.
(44) If $a, b \upuparrows b, c$, then $a, b \upuparrows a, c$.
(45) If $a, b \upuparrows a, c$, then $a, b \upuparrows b, c$ or $a, c \upuparrows c, b$.
(46) There exist a, b, c, d such that $a, b \not\upuparrows c, d$ and $a, b \not\upuparrows d, c$.
(47) There exists d such that $a, b \upuparrows c, d$ and $a, c \upuparrows b, d$ and $b \neq d$.
(48) If $p \neq b$ and $b, p \upuparrows p, c$, then there exists d such that $a, p \upuparrows p, d$ and $a, b \upuparrows c, d$.

An ordered affine space is said to be an ordered affine plane if:

Let a, b, c, d be elements of the points of it . Then if $a, b \not\upuparrows c, d$ and $a, b \not\upuparrows d, c$, then there exists an element p of the points of it such that $a, b \upuparrows a, p$ or $a, b \upuparrows p, a$ but $c, d \upuparrows c, p$ or $c, d \upuparrows p, c$.

We now state three propositions:

(49) The following conditions are equivalent:
 (i) for all elements a, b, c, d of the points of A such that $a, b \not\upuparrows c, d$ and $a, b \not\upuparrows d, c$ there exists an element p of the points of A such that $a, b \upuparrows a, p$ or $a, b \upuparrows p, a$ but $c, d \upuparrows c, p$ or $c, d \upuparrows p, c$,

(ii) A is an ordered affine plane.

(50) The following conditions are equivalent:

(i) there exist elements a, b of the points of AS such that $a \neq b$ and for all elements a, b, c, d, p, q, r, s of the points of AS holds $a, b \upuparrows c, c$ but if $a, b \upuparrows b, a$, then $a = b$ but if $a \neq b$ and $a, b \upuparrows p, q$ and $a, b \upuparrows r, s$, then $p, q \upuparrows r, s$ but if $a, b \upuparrows c, d$, then $b, a \upuparrows d, c$ but if $a, b \upuparrows b, c$, then $a, b \upuparrows a, c$ but if $a, b \upuparrows a, c$, then $a, b \upuparrows b, c$ or $a, c \upuparrows c, b$ and there exist elements a, b, c, d of the points of AS such that $a, b \not\upuparrows c, d$ and $a, b \not\upuparrows d, c$ and for every elements a, b, c of the points of AS there exists an element d of the points of AS such that $a, b \upuparrows c, d$ and $a, c \upuparrows b, d$ and $b \neq d$ and for all elements p, a, b, c of the points of AS such that $p \neq b$ and $b, p \upuparrows p, c$ there exists an element d of the points of AS such that $a, p \upuparrows p, d$ and $a, b \upuparrows c, d$ and for all elements a, b, c, d of the points of AS such that $a, b \not\upuparrows c, d$ and $a, b \not\upuparrows d, c$ there exists an element p of the points of AS such that $a, b \upuparrows a, p$ or $a, b \upuparrows p, a$ but $c, d \upuparrows c, p$ or $c, d \upuparrows p, c$,

(ii) AS is an ordered affine plane.

(51) If there exist u, v such that for all real numbers a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$ and for every w there exist real numbers a, b such that $w = a \cdot u + b \cdot v$, then OASpace V is an ordered affine plane.

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Received April 11, 1990

FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Metric Spaces [1]

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Summary. In this paper we define the metric spaces. Two examples of metric spaces are given. We define the discrete metric and the metric on the real axis. Moreover the open ball, the close ball and the sphere in metric spaces are introduced. We also prove some theorems concerning these concepts.

MML Identifier: METRIC_1.

The papers [3], [7], [2], [1], [5], [6], and [4] provide the notation and terminology for this paper. We consider metric structures which are systems

$\langle$ a carrier, a distance $\rangle$

where the carrier is a non-empty set and the distance is a function from [: the carrier, the carrier :] into $\mathbb{R}$. In the sequel M will be a metric structure. Let us consider M. A point of M is an element of the carrier of M.

Next we state a proposition

(1) For every element x of the carrier of M holds x is a point of M.

Let us consider M, and let a, b be elements of the carrier of M. The functor $\rho(a,b)$ yielding a real number, is defined by:

$\rho(a,b) =$ (the distance of $M)(a,b)$.

We now state a proposition

(2) For all elements x, y of the carrier of M holds $\rho(x,y) =$ (the distance of $M)(x,y)$.

[1]Supported by RPBP.III.24-B.5

ISSN 0777–4028

In the sequel x will be arbitrary. Let us consider x. Then $\{x\}$ is a non-empty set.

The function $\{[\emptyset, \emptyset]\} \mapsto 0$ from $[\!:\{\emptyset\}, \{\emptyset\}:\!]$ into $\mathbb{R}$ is defined by:

$\{[\emptyset, \emptyset]\} \mapsto 0 = [\!:\{\emptyset\}, \{\emptyset\}:\!] \longmapsto 0$.

Next we state a proposition

(3) $\{[\emptyset, \emptyset]\} \mapsto 0 = [\!:\{\emptyset\}, \{\emptyset\}:\!] \longmapsto 0$.

A metric structure is said to be a metric space if:

for all elements a, b, c of the carrier of it holds $\rho(a, b) = 0$ if and only if $a = b$ but $\rho(a, b) = \rho(b, a)$ and $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$.

We now state three propositions:

(4) For every M being a metric structure holds M is a metric space if and only if for all elements a, b, c of the carrier of M holds $\rho(a, b) = 0$ if and only if $a = b$ but $\rho(a, b) = \rho(b, a)$ and $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$.

(5) For every metric space M and for all elements a, b of the carrier of M holds $\rho(a, b) = \rho(b, a)$.

(6) For every metric space M and for all elements a, b, c of the carrier of M holds $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$.

In the sequel PM denotes a metric space and p_1, p_2 denote elements of the carrier of PM. Next we state a proposition

(7) $0 \leq \rho(p_1, p_2)$.

Let A be a non-empty set. The discrete metric of A yielding a function from $[\!:A, A:\!]$ into $\mathbb{R}$, is defined by:

for all elements x, y of A holds (the discrete metric of A)$(x, x) = 0$ but if $x \neq y$, then (the discrete metric of A)$(x, y) = 1$.

In the sequel A denotes a non-empty set and x, y denote elements of A. Next we state two propositions:

(8) (The discrete metric of A)$(x, x) = 0$.

(9) If $x \neq y$, then (the discrete metric of A)$(x, y) = 1$.

Let A be a non-empty set. The discrete space on A yielding a metric space, is defined as follows:

the discrete space on $A = \langle A$, the discrete metric of $A\rangle$.

In the sequel x will be an element of $\mathbb{R}$. Let us consider x. The functor $@x$ yielding a real number, is defined by:

$@x = x$.

Next we state a proposition

(10) $x = @x$.

The function $\rho_{\mathbb{R}}$ from $[\!:\mathbb{R}, \mathbb{R}:\!]$ into $\mathbb{R}$ is defined as follows:

for all elements x, y of $\mathbb{R}$ holds $\rho_{\mathbb{R}}(x, y) = |@x - @y|$.

Next we state several propositions:

(11) For every function F from $[\!:\mathbb{R}, \mathbb{R}:\!]$ into $\mathbb{R}$ holds $F = \rho_{\mathbb{R}}$ if and only if for all elements x, y of $\mathbb{R}$ holds $F(x, y) = |@x - @y|$.

(12) For all real numbers x, y holds $\rho_{\mathbb{R}}(x, y) = |x - y|$.

(13) For all elements x, y of $\mathbb{R}$ holds $\rho_{\mathbb{R}}(x, y) = 0$ if and only if $x = y$.

(14) For all elements x, y of $\mathbb{R}$ holds $\rho_{\mathbb{R}}(x, y) = \rho_{\mathbb{R}}(y, x)$.

(15) For all elements x, y, z of $\mathbb{R}$ holds $\rho_{\mathbb{R}}(x, y) \leq \rho_{\mathbb{R}}(x, z) + \rho_{\mathbb{R}}(z, y)$.

The metric space of real numbers a metric space is defined as follows:
the metric space of real numbers $= \langle \mathbb{R}, \rho_{\mathbb{R}} \rangle$.

Let M be a metric structure, and let p be an element of the carrier of M, and let r be a real number. The functor $\mathrm{Ball}(p, r)$ yielding a subset of the carrier of M, is defined as follows:
$\mathrm{Ball}(p, r) = \{q : \rho(p, q) < r\}$.

We now state a proposition

(16) For every M being a metric structure and for every element p of the carrier of M and for every real number r holds $\mathrm{Ball}(p, r) = \{q : \rho(p, q) < r\}$.

Let M be a metric structure, and let p be an element of the carrier of M, and let r be a real number. The functor $\overline{\mathrm{Ball}}(p, r)$ yields a subset of the carrier of M and is defined as follows:
$\overline{\mathrm{Ball}}(p, r) = \{q : \rho(p, q) \leq r\}$.

We now state a proposition

(17) For every M being a metric structure and for every element p of the carrier of M and for every real number r holds $\overline{\mathrm{Ball}}(p, r) = \{q : \rho(p, q) \leq r\}$.

Let M be a metric structure, and let p be an element of the carrier of M, and let r be a real number. The functor $\mathrm{Sphere}(p, r)$ yielding a subset of the carrier of M, is defined by:
$\mathrm{Sphere}(p, r) = \{q : \rho(p, q) = r\}$.

Next we state several propositions:

(18) For every M being a metric structure and for every element p of the carrier of M and for every real number r holds $\mathrm{Sphere}(p, r) = \{q : \rho(p, q) = r\}$.

(19) For every M being a metric structure and for all elements p, x of the carrier of M and for every real number r holds $x \in \mathrm{Ball}(p, r)$ if and only if $\rho(p, x) < r$.

(20) For every M being a metric structure and for all elements p, x of the carrier of M and for every real number r holds $x \in \overline{\mathrm{Ball}}(p, r)$ if and only if $\rho(p, x) \leq r$.

(21) For every M being a metric structure and for all elements p, x of the carrier of M and for every real number r holds $x \in \mathrm{Sphere}(p, r)$ if and only if $\rho(p, x) = r$.

(22) For every M being a metric structure and for every element p of the carrier of M and for every real number r holds $\mathrm{Ball}(p, r) \subseteq \overline{\mathrm{Ball}}(p, r)$.

(23) For every M being a metric structure and for every element p of the carrier of M and for every real number r holds $\mathrm{Sphere}(p,r) \subseteq \overline{\mathrm{Ball}}(p,r)$.

(24) For every M being a metric structure and for every element p of the carrier of M and for every real number r holds $\mathrm{Sphere}(p,r) \cup \mathrm{Ball}(p,r) = \overline{\mathrm{Ball}}(p,r)$.

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Received May 3, 1990

FORMALIZED MATHEMATICS
Vol.1, No.**3**, May–August 1990
Université Catholique de Louvain

Ordered Affine Spaces Defined in Terms of Directed Parallelity - part I [1]

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Summary. In the article we consider several geometrical relations in given arbitrary ordered affine space defined in terms of directed parallelity. In particular we introduce the notions of the nondirected parallelity of segments, of collinearity, and the betweenness relation determined by the given relation of directed parallelity. The obtained structures satisfy commonly accepted axioms for affine spaces. At the end of the article we introduce a formal definition of affine space and affine plane (defined in terms of parallelity of segments).

MML Identifier: `DIRAF`.

The notation and terminology used in this paper are introduced in the articles [2] and [1]. In the sequel X is a non-empty set. Let us consider X, and let R be a relation on $[: X, X :]$. The functor $\lambda(R)$ yielding a relation on $[: X, X :]$, is defined as follows:

for all elements a, b, c, d of X holds $\langle\langle a,b\rangle,\langle c,d\rangle\rangle \in \lambda(R)$ if and only if $\langle\langle a,b\rangle,\langle c,d\rangle\rangle \in R$ or $\langle\langle a,b\rangle,\langle d,c\rangle\rangle \in R$.

One can prove the following two propositions:

(1) For all relations R, R' on $[: X, X :]$ holds $R' = \lambda(R)$ if and only if for all elements a, b, c, d of X holds $\langle\langle a,b\rangle,\langle c,d\rangle\rangle \in R'$ if and only if $\langle\langle a,b\rangle,\langle c,d\rangle\rangle \in R$ or $\langle\langle a,b\rangle,\langle d,c\rangle\rangle \in R$.

(2) For every relation R on $[: X, X :]$ and for all elements a, b, c, d of X holds $\langle\langle a,b\rangle,\langle c,d\rangle\rangle \in \lambda(R)$ if and only if $\langle\langle a,b\rangle,\langle c,d\rangle\rangle \in R$ or $\langle\langle a,b\rangle,\langle d,c\rangle\rangle \in R$.

[1]Supported by RPBP.III.24-C.2
[2]Supported by RPBP.III-24.C2.
[3]Supported by RPBP.III-24.C2.

ISSN 0777–4028

Let S be an affine structure. The functor $\Lambda(S)$ yielding an affine structure, is defined as follows:

$\Lambda(S) = \langle$ the points of $S, \lambda($ the congruence of $S)\rangle$.

One can prove the following proposition

(3) For all S, S' being affine structures holds $\Lambda(S) = S'$ if and only if $S' = \langle$ the points of $S, \lambda($ the congruence of $S)\rangle$.

We adopt the following convention: S will be an ordered affine space and a, b, c, d, x, y, z, t, u, w will be elements of the points of S. The following propositions are true:

(4) $x, y \upuparrows x, y$.

(5) If $x, y \upuparrows z, t$, then $y, x \upuparrows t, z$ and $z, t \upuparrows x, y$ and $t, z \upuparrows y, x$.

(6) If $z \neq t$ and $x, y \upuparrows z, t$ and $z, t \upuparrows u, w$, then $x, y \upuparrows u, w$.

(7) $x, x \upuparrows y, z$ and $y, z \upuparrows x, x$.

(8) If $x, y \upuparrows z, t$ and $x, y \upuparrows t, z$, then $x = y$ or $z = t$.

(9) $x, y \upuparrows x, z$ if and only if $x, y \upuparrows y, z$ or $x, z \upuparrows z, y$.

Let us consider S, a, b, c. The predicate $\mathbf{B}(a, b, c)$ is defined as follows:

$a, b \upuparrows b, c$.

The following propositions are true:

(10) $\mathbf{B}(a, b, c)$ if and only if $a, b \upuparrows b, c$.

(11) $x, y \upuparrows x, z$ if and only if $\mathbf{B}(x, y, z)$ or $\mathbf{B}(x, z, y)$.

(12) If $\mathbf{B}(a, b, a)$, then $a = b$.

(13) If $\mathbf{B}(a, b, c)$, then $\mathbf{B}(c, b, a)$.

(14) $\mathbf{B}(x, x, y)$ and $\mathbf{B}(x, y, y)$.

(15) If $\mathbf{B}(a, b, c)$ and $\mathbf{B}(a, c, d)$, then $\mathbf{B}(b, c, d)$.

(16) If $b \neq c$ and $\mathbf{B}(a, b, c)$ and $\mathbf{B}(b, c, d)$, then $\mathbf{B}(a, c, d)$.

(17) There exists z such that $\mathbf{B}(x, y, z)$ and $y \neq z$.

(18) If $\mathbf{B}(x, y, z)$ and $\mathbf{B}(y, x, z)$, then $x = y$.

(19) If $x \neq y$ and $\mathbf{B}(x, y, z)$ and $\mathbf{B}(x, y, t)$, then $\mathbf{B}(y, z, t)$ or $\mathbf{B}(y, t, z)$.

(20) If $x \neq y$ and $\mathbf{B}(x, y, z)$ and $\mathbf{B}(x, y, t)$, then $\mathbf{B}(x, z, t)$ or $\mathbf{B}(x, t, z)$.

(21) If $\mathbf{B}(x, y, t)$ and $\mathbf{B}(x, z, t)$, then $\mathbf{B}(x, y, z)$ or $\mathbf{B}(x, z, y)$.

Let us consider S, a, b, c, d. The predicate $a, b \parallel c, d$ is defined as follows:

$a, b \upuparrows c, d$ or $a, b \upuparrows d, c$.

One can prove the following propositions:

(22) $a, b \parallel c, d$ if and only if $a, b \upuparrows c, d$ or $a, b \upuparrows d, c$.

(23) $a, b \parallel c, d$ if and only if $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in \lambda($ the congruence of $S)$.

(24) $x, y \parallel y, x$ and $x, y \parallel x, y$.

(25) $x, y \parallel z, z$ and $z, z \parallel x, y$.

(26) If $x, y \parallel x, z$, then $y, x \parallel y, z$.

(27) If $x, y \parallel z, t$, then $x, y \parallel t, z$ and $y, x \parallel z, t$ and $y, x \parallel t, z$ and $z, t \parallel x, y$ and $z, t \parallel y, x$ and $t, z \parallel x, y$ and $t, z \parallel y, x$.

(28) Suppose that

(i) $a \neq b$,

(ii) $a, b \parallel x, y$ and $a, b \parallel z, t$ or $a, b \parallel x, y$ and $z, t \parallel a, b$ or $x, y \parallel a, b$ and $z, t \parallel a, b$ or $x, y \parallel a, b$ and $a, b \parallel z, t$.

Then $x, y \parallel z, t$.

(29) There exist x, y, z such that $x, y \nparallel x, z$.

(30) There exists t such that $x, z \parallel y, t$ and $y \neq t$.

(31) There exists t such that $x, y \parallel z, t$ and $x, z \parallel y, t$.

(32) If $z, x \parallel x, t$ and $x \neq z$, then there exists u such that $y, x \parallel x, u$ and $y, z \parallel t, u$.

Let us consider S, a, b, c. The predicate $\mathbf{L}(a, b, c)$ is defined as follows:
$a, b \parallel a, c$.

One can prove the following propositions:

(33) $\mathbf{L}(a, b, c)$ if and only if $a, b \parallel a, c$.

(34) If $\mathbf{B}(a, b, c)$, then $\mathbf{L}(a, b, c)$.

(35) If $\mathbf{L}(a, b, c)$, then $\mathbf{B}(a, b, c)$ or $\mathbf{B}(b, a, c)$ or $\mathbf{B}(a, c, b)$.

(36) If $\mathbf{L}(x, y, z)$, then $\mathbf{L}(x, z, y)$ and $\mathbf{L}(y, x, z)$ and $\mathbf{L}(y, z, x)$ and $\mathbf{L}(z, x, y)$ and $\mathbf{L}(z, y, x)$.

(37) $\mathbf{L}(x, x, y)$ and $\mathbf{L}(x, y, y)$ and $\mathbf{L}(x, y, x)$.

(38) If $x \neq y$ and $\mathbf{L}(x, y, z)$ and $\mathbf{L}(x, y, t)$ and $\mathbf{L}(x, y, u)$, then $\mathbf{L}(z, t, u)$.

(39) If $x \neq y$ and $\mathbf{L}(x, y, z)$ and $x, y \parallel z, t$, then $\mathbf{L}(x, y, t)$.

(40) If $\mathbf{L}(x, y, z)$ and $\mathbf{L}(x, y, t)$, then $x, y \parallel z, t$.

(41) If $u \neq z$ and $\mathbf{L}(x, y, u)$ and $\mathbf{L}(x, y, z)$ and $\mathbf{L}(u, z, w)$, then $\mathbf{L}(x, y, w)$.

(42) There exist x, y, z such that not $\mathbf{L}(x, y, z)$.

(43) If $x \neq y$, then there exists z such that not $\mathbf{L}(x, y, z)$.

In the sequel AS will denote an affine structure. Let us consider AS, and let a, b, c, d be elements of the points of AS. The predicate $a, b \parallel c, d$ is defined as follows:

$\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the congruence of AS.

The following propositions are true:

(44) For all elements a, b, c, d of the points of AS holds $a, b \parallel c, d$ if and only if $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the congruence of AS.

(45) If $AS = \Lambda(S)$, then for all elements a, b, c, d of the points of S and for all elements a', b', c', d' of the points of AS such that $a = a'$ and $b = b'$ and $c = c'$ and $d = d'$ holds $a', b' \parallel c', d'$ if and only if $a, b \parallel c, d$.

(46) Suppose $AS = \Lambda(S)$. Then

(i) there exist elements x, y of the points of AS such that $x \neq y$,

(ii) for all elements x, y, z, t, u, w of the points of AS holds $x, y \parallel y, x$ and $x, y \parallel z, z$ but if $x \neq y$ and $x, y \parallel z, t$ and $x, y \parallel u, w$, then $z, t \parallel u, w$ but if $x, y \parallel x, z$, then $y, x \parallel y, z$,

(iii) there exist elements x, y, z of the points of AS such that $x, y \nparallel x, z$,

(iv) for every elements x, y, z of the points of AS there exists an element t of the points of AS such that $x, z \parallel y, t$ and $y \neq t$,

(v) for every elements x, y, z of the points of AS there exists an element t of the points of AS such that $x, y \parallel z, t$ and $x, z \parallel y, t$,

(vi) for all elements x, y, z, t of the points of AS such that $z, x \parallel x, t$ and $x \neq z$ there exists an element u of the points of AS such that $y, x \parallel x, u$ and $y, z \parallel t, u$.

An affine structure is said to be an affine space if:

(i) there exist elements x, y of the points of it such that $x \neq y$,

(ii) for all elements x, y, z, t, u, w of the points of it holds $x, y \parallel y, x$ and $x, y \parallel z, z$ but if $x \neq y$ and $x, y \parallel z, t$ and $x, y \parallel u, w$, then $z, t \parallel u, w$ but if $x, y \parallel x, z$, then $y, x \parallel y, z$,

(iii) there exist elements x, y, z of the points of it such that $x, y \nparallel x, z$,

(iv) for every elements x, y, z of the points of it there exists an element t of the points of it such that $x, z \parallel y, t$ and $y \neq t$,

(v) for every elements x, y, z of the points of it there exists an element t of the points of it such that $x, y \parallel z, t$ and $x, z \parallel y, t$,

(vi) for all elements x, y, z, t of the points of it such that $z, x \parallel x, t$ and $x \neq z$ there exists an element u of the points of it such that $y, x \parallel x, u$ and $y, z \parallel t, u$.

The following three propositions are true:

(47) Let AS be an affine space. Then

(i) there exist elements x, y of the points of AS such that $x \neq y$,

(ii) for all elements x, y, z, t, u, w of the points of AS holds $x, y \parallel y, x$ and $x, y \parallel z, z$ but if $x \neq y$ and $x, y \parallel z, t$ and $x, y \parallel u, w$, then $z, t \parallel u, w$ but if $x, y \parallel x, z$, then $y, x \parallel y, z$,

(iii) there exist elements x, y, z of the points of AS such that $x, y \nparallel x, z$,

(iv) for every elements x, y, z of the points of AS there exists an element t of the points of AS such that $x, z \parallel y, t$ and $y \neq t$,

(v) for every elements x, y, z of the points of AS there exists an element t of the points of AS such that $x, y \parallel z, t$ and $x, z \parallel y, t$,

(vi) for all elements x, y, z, t of the points of AS such that $z, x \parallel x, t$ and $x \neq z$ there exists an element u of the points of AS such that $y, x \parallel x, u$ and $y, z \parallel t, u$.

(48) $\Lambda(S)$ is an affine space.

(49) The following conditions are equivalent:

(i) there exist elements x, y of the points of AS such that $x \neq y$ and for all elements x, y, z, t, u, w of the points of AS holds $x, y \parallel y, x$ and $x, y \parallel z, z$ but if $x \neq y$ and $x, y \parallel z, t$ and $x, y \parallel u, w$, then $z, t \parallel u, w$ but if $x, y \parallel x, z$, then $y, x \parallel y, z$ and there exist elements x, y, z of the points of AS such that $x, y \nparallel x, z$ and for every elements x, y, z of the points of AS there exists an element t of the points of AS such that $x, z \parallel y, t$ and $y \neq t$ and for every elements x, y, z of the points of AS there exists an element t of the points of AS such that $x, y \parallel z, t$ and $x, z \parallel y, t$ and for all elements x, y, z, t of the points of AS such that $z, x \parallel x, t$ and $x \neq z$

there exists an element u of the points of AS such that $y, x \parallel x, u$ and $y, z \parallel t, u$,

(ii) AS is an affine space.

We follow the rules: S will be an ordered affine plane and x, y, z, t, u will be elements of the points of S. We now state two propositions:

(50) If $x, y \nparallel z, t$, then there exists u such that $x, y \parallel x, u$ and $z, t \parallel z, u$.

(51) If $AS = \Lambda(S)$, then for all elements x, y, z, t of the points of AS such that $x, y \nparallel z, t$ there exists an element u of the points of AS such that $x, y \parallel x, u$ and $z, t \parallel z, u$.

An affine space is said to be an affine plane if:

for all elements x, y, z, t of the points of it such that $x, y \nparallel z, t$ there exists an element u of the points of it such that $x, y \parallel x, u$ and $z, t \parallel z, u$.

In the sequel ASP will denote an affine space. Next we state three propositions:

(52) ASP is an affine plane if and only if for all elements x, y, z, t of the points of ASP such that $x, y \nparallel z, t$ there exists an element u of the points of ASP such that $x, y \parallel x, u$ and $z, t \parallel z, u$.

(53) $\Lambda(S)$ is an affine plane.

(54) AS is an affine plane if and only if the following conditions are satisfied:

(i) there exist elements x, y of the points of AS such that $x \neq y$,

(ii) for all elements x, y, z, t, u, w of the points of AS holds $x, y \parallel y, x$ and $x, y \parallel z, z$ but if $x \neq y$ and $x, y \parallel z, t$ and $x, y \parallel u, w$, then $z, t \parallel u, w$ but if $x, y \parallel x, z$, then $y, x \parallel y, z$,

(iii) there exist elements x, y, z of the points of AS such that $x, y \nparallel x, z$,

(iv) for every elements x, y, z of the points of AS there exists an element t of the points of AS such that $x, z \parallel y, t$ and $y \neq t$,

(v) for every elements x, y, z of the points of AS there exists an element t of the points of AS such that $x, y \parallel z, t$ and $x, z \parallel y, t$,

(vi) for all elements x, y, z, t of the points of AS such that $z, x \parallel x, t$ and $x \neq z$ there exists an element u of the points of AS such that $y, x \parallel x, u$ and $y, z \parallel t, u$,

(vii) for all elements x, y, z, t of the points of AS such that $x, y \nparallel z, t$ there exists an element u of the points of AS such that $x, y \parallel x, u$ and $z, t \parallel z, u$.

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Received May 4, 1990

FORMALIZED MATHEMATICS
Vol.1, No.**3**, May–August 1990
Université Catholique de Louvain

Parallelity and Lines in Affine Spaces [1]

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Summary. In the article we introduce basic notions concerning affine spaces and investigate their fundamental properties. We define the function which to every nondegenerate pair of points assigns the line joining them and we extend the relation of parallelity to a relation between segments and lines, and between lines.

MML Identifier: `AFF_1`.

The papers [3], [1], and [2] provide the notation and terminology for this paper. We adopt the following convention: AS will be an affine space and a, a', b, b', c, d, o, p, q, x, y, z, t, u, w will be elements of the points of AS. One can prove the following propositions:

(1) There exist elements x, y of the points of AS such that $x \neq y$.

(2) $x, y \parallel y, x$ and $x, y \parallel z, z$.

(3) If $x \neq y$ and $x, y \parallel z, t$ and $x, y \parallel u, w$, then $z, t \parallel u, w$.

(4) If $x, y \parallel x, z$, then $y, x \parallel y, z$.

(5) There exist x, y, z such that $x, y \nparallel x, z$.

(6) There exists t such that $x, z \parallel y, t$ and $y \neq t$.

(7) There exists t such that $x, y \parallel z, t$ and $x, z \parallel y, t$.

(8) If $z, x \parallel x, t$ and $x \neq z$, then there exists u such that $y, x \parallel x, u$ and $y, z \parallel t, u$.

Let us consider AS, a, b, c. The predicate $\mathbf{L}(a, b, c)$ is defined as follows: $a, b \parallel a, c$.

The following propositions are true:

(9) $\mathbf{L}(a, b, c)$ if and only if $a, b \parallel a, c$.

(10) For every a there exists b such that $a \neq b$.

(11) $x, y \parallel y, x$ and $x, y \parallel x, y$.

[1]Supported by RPBP.III.24-C.2

ISSN 0777–4028

(12) $x, y \parallel z, z$ and $z, z \parallel x, y$.

(13) If $x, y \parallel z, t$, then $x, y \parallel t, z$ and $y, x \parallel z, t$ and $y, x \parallel t, z$ and $z, t \parallel x, y$ and $z, t \parallel y, x$ and $t, z \parallel x, y$ and $t, z \parallel y, x$.

(14) Suppose that

(i) $a \neq b$,

(ii) $a, b \parallel x, y$ and $a, b \parallel z, t$ or $a, b \parallel x, y$ and $z, t \parallel a, b$ or $x, y \parallel a, b$ and $z, t \parallel a, b$ or $x, y \parallel a, b$ and $a, b \parallel z, t$.

Then $x, y \parallel z, t$.

(15) If $\mathbf{L}(x, y, z)$, then $\mathbf{L}(x, z, y)$ and $\mathbf{L}(y, x, z)$ and $\mathbf{L}(y, z, x)$ and $\mathbf{L}(z, x, y)$ and $\mathbf{L}(z, y, x)$.

(16) $\mathbf{L}(x, x, y)$ and $\mathbf{L}(x, y, y)$ and $\mathbf{L}(x, y, x)$.

(17) If $x \neq y$ and $\mathbf{L}(x, y, z)$ and $\mathbf{L}(x, y, t)$ and $\mathbf{L}(x, y, u)$, then $\mathbf{L}(z, t, u)$.

(18) If $x \neq y$ and $\mathbf{L}(x, y, z)$ and $x, y \parallel z, t$, then $\mathbf{L}(x, y, t)$.

(19) If $\mathbf{L}(x, y, z)$ and $\mathbf{L}(x, y, t)$, then $x, y \parallel z, t$.

(20) If $u \neq z$ and $\mathbf{L}(x, y, u)$ and $\mathbf{L}(x, y, z)$ and $\mathbf{L}(u, z, w)$, then $\mathbf{L}(x, y, w)$.

(21) There exist x, y, z such that not $\mathbf{L}(x, y, z)$.

(22) If $x \neq y$, then there exists z such that not $\mathbf{L}(x, y, z)$.

(23) If not $\mathbf{L}(o, a, b)$ and $\mathbf{L}(o, b, b')$ and $a, b \parallel a, b'$, then $b = b'$.

Let us consider AS, a, b. The functor $\mathrm{Line}(a, b)$ yielding a subset of the points of AS, is defined as follows:

for every x holds $x \in \mathrm{Line}(a, b)$ if and only if $\mathbf{L}(a, b, x)$.

In the sequel A, C, D, K are subsets of the points of AS. We now state several propositions:

(24) $A = \mathrm{Line}(a, b)$ if and only if for every x holds $x \in A$ if and only if $\mathbf{L}(a, b, x)$.

(25) $\mathrm{Line}(a, b) = \mathrm{Line}(b, a)$.

(26) $a \in \mathrm{Line}(a, b)$ and $b \in \mathrm{Line}(a, b)$.

(27) If $c \in \mathrm{Line}(a, b)$ and $d \in \mathrm{Line}(a, b)$ and $c \neq d$, then $\mathrm{Line}(c, d) \subseteq \mathrm{Line}(a, b)$.

(28) If $c \in \mathrm{Line}(a, b)$ and $d \in \mathrm{Line}(a, b)$ and $a \neq b$, then $\mathrm{Line}(a, b) \subseteq \mathrm{Line}(c, d)$.

Let us consider AS, A. We say that A is a line if and only if:

there exist a, b such that $a \neq b$ and $A = \mathrm{Line}(a, b)$.

One can prove the following propositions:

(29) A is a line if and only if there exist a, b such that $a \neq b$ and $A = \mathrm{Line}(a, b)$.

(30) For all a, b, A, C such that A is a line and C is a line and $a \in A$ and $b \in A$ and $a \in C$ and $b \in C$ holds $a = b$ or $A = C$.

(31) If A is a line, then there exist a, b such that $a \in A$ and $b \in A$ and $a \neq b$.

(32) If A is a line and $a \in A$, then there exists b such that $a \neq b$ and $b \in A$.

(33) $\mathbf{L}(a, b, c)$ if and only if there exists A such that A is a line and $a \in A$ and $b \in A$ and $c \in A$.

Let us consider AS, a, b, A. The predicate $a, b \parallel A$ is defined by: there exist c, d such that $c \neq d$ and $A = \mathrm{Line}(c, d)$ and $a, b \parallel c, d$.

The following proposition is true

(34) $a, b \parallel A$ if and only if there exist c, d such that $c \neq d$ and $A = \mathrm{Line}(c, d)$ and $a, b \parallel c, d$.

Let us consider AS, A, C. The predicate $A \parallel C$ is defined as follows: there exist a, b such that $A = \mathrm{Line}(a, b)$ and $a \neq b$ and $a, b \parallel C$.

We now state a number of propositions:

(35) $A \parallel C$ if and only if there exist a, b such that $A = \mathrm{Line}(a, b)$ and $a \neq b$ and $a, b \parallel C$.

(36) If $c \in \mathrm{Line}(a, b)$ and $a \neq b$, then $d \in \mathrm{Line}(a, b)$ if and only if $a, b \parallel c, d$.

(37) If A is a line and $a \in A$, then $b \in A$ if and only if $a, b \parallel A$.

(38) $a \neq b$ and $A = \mathrm{Line}(a, b)$ if and only if A is a line and $a \in A$ and $b \in A$ and $a \neq b$.

(39) If A is a line and $a \in A$ and $b \in A$ and $a \neq b$ and $\mathbf{L}(a, b, x)$, then $x \in A$.

(40) If there exist a, b such that $a, b \parallel A$, then A is a line.

(41) If $c \in A$ and $d \in A$ and A is a line and $c \neq d$, then $a, b \parallel A$ if and only if $a, b \parallel c, d$.

(42) If $c \neq d$ and $a, b \parallel c, d$, then $a, b \parallel \mathrm{Line}(c, d)$.

(43) If $a \neq b$, then $a, b \parallel \mathrm{Line}(a, b)$.

(44) If A is a line, then $a, b \parallel A$ if and only if there exist c, d such that $c \neq d$ and $c \in A$ and $d \in A$ and $a, b \parallel c, d$.

(45) If A is a line and $a, b \parallel A$ and $c, d \parallel A$, then $a, b \parallel c, d$.

(46) If $a, b \parallel A$ and $a, b \parallel p, q$ and $a \neq b$, then $p, q \parallel A$.

(47) If A is a line, then $a, a \parallel A$.

(48) If $a, b \parallel A$, then $b, a \parallel A$.

(49) If $a, b \parallel A$ and $a \notin A$, then $b \notin A$.

(50) If $A \parallel C$, then A is a line and C is a line.

(51) $A \parallel C$ if and only if there exist a, b, c, d such that $a \neq b$ and $c \neq d$ and $a, b \parallel c, d$ and $A = \mathrm{Line}(a, b)$ and $C = \mathrm{Line}(c, d)$.

(52) If A is a line and C is a line and $a \in A$ and $b \in A$ and $c \in C$ and $d \in C$ and $a \neq b$ and $c \neq d$, then $A \parallel C$ if and only if $a, b \parallel c, d$.

(53) If $a \in A$ and $b \in A$ and $c \in C$ and $d \in C$ and $A \parallel C$, then $a, b \parallel c, d$.

(54) If $a \in A$ and $b \in A$ and $A \parallel C$, then $a, b \parallel C$.

(55) If A is a line, then $A \parallel A$.

(56) If $A \parallel C$, then $C \parallel A$.

(57) If $a, b \parallel A$ and $A \parallel C$, then $a, b \parallel C$.

(58) If $A \parallel C$ and $C \parallel D$ or $A \parallel C$ and $D \parallel C$ or $C \parallel A$ and $C \parallel D$ or $C \parallel A$ and $D \parallel C$, then $A \parallel D$.

(59) If $A \parallel C$ and $p \in A$ and $p \in C$, then $A = C$.

(60) If $x \in K$ and $a \notin K$ and $a, b \parallel K$, then $a = b$ or not $\mathbf{L}(x, a, b)$.

(61) If $a, b \parallel K$ and $a', b' \parallel K$ and $\mathbf{L}(p, a, a')$ and $\mathbf{L}(p, b, b')$ and $p \in K$ and $a \notin K$ and $a = b$, then $a' = b'$.

(62) If A is a line and $a \in A$ and $b \in A$ and $c \in A$ and $a \neq b$ and $a, b \parallel c, d$, then $d \in A$.

(63) For all a, A such that A is a line there exists C such that $a \in C$ and $A \parallel C$.

(64) If $A \parallel C$ and $A \parallel D$ and $p \in C$ and $p \in D$, then $C = D$.

(65) If A is a line and $a \in A$ and $b \in A$ and $c \in A$ and $d \in A$, then $a, b \parallel c, d$.

(66) If A is a line and $a \in A$ and $b \in A$, then $a, b \parallel A$.

(67) If $a, b \parallel A$ and $a, b \parallel C$ and $a \neq b$, then $A \parallel C$.

(68) If not $\mathbf{L}(o, a, b)$ and $\mathbf{L}(o, a, a')$ and $\mathbf{L}(o, b, b')$ and $a, b \parallel a', b'$ and $a' = b'$, then $a' = o$ and $b' = o$.

(69) If not $\mathbf{L}(o, a, b)$ and $\mathbf{L}(o, a, a')$ and $\mathbf{L}(o, b, b')$ and $a, b \parallel a', b'$ and $a' = o$, then $b' = o$.

(70) If not $\mathbf{L}(o, a, b)$ and $\mathbf{L}(o, a, a')$ and $\mathbf{L}(o, b, b')$ and $\mathbf{L}(o, b, x)$ and $a, b \parallel a', b'$ and $a, b \parallel a', x$, then $b' = x$.

(71) For all a, b, A such that A is a line and $a \in A$ and $b \in A$ and $a \neq b$ holds $A = \mathrm{Line}(a, b)$.

We adopt the following convention: AP will be an affine plane, a, b, c, d, x, p will be elements of the points of AP, and A, C will be subsets of the points of AP. One can prove the following three propositions:

(72) If A is a line and C is a line and $A \nparallel C$, then there exists x such that $x \in A$ and $x \in C$.

(73) If A is a line and $a, b \nparallel A$, then there exists x such that $x \in A$ and $\mathbf{L}(a, b, x)$.

(74) If $a, b \nparallel c, d$, then there exists p such that $\mathbf{L}(a, b, p)$ and $\mathbf{L}(c, d, p)$.

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Received May 4, 1990

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