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Families of Subsets, Subspaces and Mappings in Topological Spaces

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Summary. This article is a continuation of [8]. Some basic theorems about families of sets in a topological space have been proved. Following redefinitions have been made: singleton of a set as a family in the topological space and results of boolean operations on families as a family of the topological space. Notion of a family of complements of sets and a closed (open) family have been also introduced. Next some theorems refer to subspaces in a topological space: some facts about types in a subspace, theorems about open and closed sets and families in a subspace. A notion of restriction of a family has been also introduced and basic properties of this notion have been proved. The last part of the article is about mappings. There are proved necessary and sufficient conditions for a mapping to be continuous. A notion of homeomorphism has been defined next. Theorems about homeomorphisms of topological spaces have been also proved.

MML Identifier: TOPS_2.

The articles [7], [2], [3], [1], [5], [4], and [6] provide the notation and terminology for this paper. For simplicity we follow the rules: x will be arbitrary, T, S, V will denote topological spaces, P, Q, R will denote subsets of T, F, G will denote families of subsets of T, P_1 will denote a subset of S, and H will denote a family of subsets of S. We now state several propositions:

(1) $F \subseteq 2^{\Omega_T}$.

(2) If $x \in F$, then x is a subset of T.

(3) For every set X such that $X \subseteq F$ holds X is a family of subsets of T.

(4) $F = G$ if and only if for every P holds $P \in F$ if and only if $P \in G$.

(5) If F is a cover of T, then $F \neq \emptyset$.

Let us consider T, P. Then $\{P\}$ is a family of subsets of T.

Let us consider T, F, G. Then $F \cup G$ is a family of subsets of T. Then $F \cap G$ is a family of subsets of T. Then $F \setminus G$ is a family of subsets of T.

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The following propositions are true:

(6) $\bigcup F \setminus \bigcup G \subseteq \bigcup (F \setminus G)$.

(7) $F^{c} = (F$ **qua** a family of subsets of the carrier of $T)^{c}$.

(8) $P \in F^{c}$ if and only if $P^{c} \in F$.

(9) $(F^{c})^{c} = F$.

(10) $F \neq \emptyset$ if and only if $F^{c} \neq \emptyset$.

(11) If $F \neq \emptyset$, then $\bigcap F^{c} = (\bigcup F)^{c}$.

(12) If $F \neq \emptyset$, then $\bigcup F^{c} = (\bigcap F)^{c}$.

(13) F^{c} is finite if and only if F is finite.

We now define two new predicates. Let us consider T, F. The predicate F is open is defined by:

if $P \in F$, then P is open.

The predicate F is closed is defined by:

if $P \in F$, then P is closed.

One can prove the following propositions:

(14) F is open if and only if for every P such that $P \in F$ holds P is open.

(15) F is closed if and only if for every P such that $P \in F$ holds P is closed.

(16) F is closed if and only if F^{c} is open.

(17) F is open if and only if F^{c} is closed.

(18) If $F \subseteq G$ and G is open, then F is open.

(19) If $F \subseteq G$ and G is closed, then F is closed.

(20) If F is open and G is open, then $F \cup G$ is open.

(21) If F is open, then $F \cap G$ is open.

(22) If F is open, then $F \setminus G$ is open.

(23) If F is closed and G is closed, then $F \cup G$ is closed.

(24) If F is closed, then $F \cap G$ is closed.

(25) If F is closed, then $F \setminus G$ is closed.

(26) If F is open, then $\bigcup F$ is open.

(27) If F is open and F is finite, then $\bigcap F$ is open.

(28) If F is closed and F is finite, then $\bigcup F$ is closed.

(29) If F is closed, then $\bigcap F$ is closed.

In the sequel A will be a subspace of T. The following propositions are true:

(30) For every subset B of A holds B is a subset of T.

(31) For every family F of subsets of A holds F is a family of subsets of T.

(32) For every subset B of A holds B is open if and only if there exists C being a subset of T such that C is open and $C \cap \Omega_{A} = B$.

(33) For every subset Q of T such that Q is open for every subset P of A such that $P = Q$ holds P is open.

(34) For every subset Q of T such that Q is closed for every subset P of A such that $P = Q$ holds P is closed.

(35) If F is open, then for every family G of subsets of A such that $G = F$ holds G is open.

(36) If F is closed, then for every family G of subsets of A such that $G = F$ holds G is closed.

(37) If $P = \Omega_A$, then $T \restriction P = A$.

(38) If $P \neq \emptyset$, then $Q \cap P$ is a subset of $T \restriction P$.

Let us consider T, P, F. The functor $F \restriction P$ yields a family of subsets of $T \restriction P$ and is defined by:

for every subset Q of $T \restriction P$ holds $Q \in F \restriction P$ if and only if there exists R such that $R \in F$ and $R \cap P = Q$.

We now state a number of propositions:

(39) For every subset Q of $T \restriction P$ holds $Q \in F \restriction P$ if and only if there exists R being a subset of T such that $R \in F$ and $R \cap P = Q$.

(40) If $F \subseteq G$, then $F \restriction P \subseteq G \restriction P$.

(41) If $P \neq \emptyset$ and $Q \in F$, then $Q \cap P \in F \restriction P$.

(42) If $Q \subseteq \bigcup F$, then $Q \cap P \subseteq \bigcup(F \restriction P)$.

(43) If $P \subseteq \bigcup F$, then $P = \bigcup(F \restriction P)$.

(44) $\bigcup(F \restriction P) \subseteq \bigcup F$.

(45) If $P \subseteq \bigcup(F \restriction P)$, then $P \subseteq \bigcup F$.

(46) If $P \neq \emptyset$ and F is finite, then $F \restriction P$ is finite.

(47) If $P \neq \emptyset$ and F is open, then $F \restriction P$ is open.

(48) If $P \neq \emptyset$ and F is closed, then $F \restriction P$ is closed.

(49) For every family F of subsets of A such that F is open there exists G being a family of subsets of T such that G is open and for every subset AA of T such that $AA = \Omega_A$ holds $F = G \restriction AA$.

(50) If $P \neq \emptyset$, then there exists f being a function such that $\mathrm{dom}\, f = F$ and $\mathrm{rng}\, f = F \restriction P$ and for every x such that $x \in F$ for every Q such that $Q = x$ holds $f(x) = Q \cap P$.

In the sequel f will denote a map from T into S. We now state several propositions:

(51) $\mathrm{dom}\, f = \Omega_T$ and $\mathrm{rng}\, f \subseteq \Omega_S$.

(52) $f^{-1}(\Omega_S) = \Omega_T$.

(53) $({}^\circ f)^\circ F$ is a family of subsets of S.

(54) ${}^{-1} f^\circ H$ is a family of subsets of T.

(55) f is continuous if and only if for every P_1 such that P_1 is open holds $f^{-1} P_1$ is open.

(56) f is continuous if and only if for every P_1 holds $\overline{f^{-1} P_1} \subseteq f^{-1} \overline{P_1}$.

(57) f is continuous if and only if for every P holds $f^\circ \overline{P} \subseteq \overline{f^\circ P}$.

The arguments of the notions defined below are the following: T, S, V which are objects of the type reserved above; f which is a map from T into S; g which is a map from S into V. Then $g \cdot f$ is a map from T into V.

One can prove the following propositions:

(58) For every map f from T into S for every map g from S into V such that f is continuous and g is continuous holds $g \cdot f$ is continuous.

(59) If f is continuous and H is open, then for every F such that $F = {}^{-1} f^{\circ} H$ holds F is open.

(60) If f is continuous and H is closed, then for every F such that $F = {}^{-1} f^{\circ} H$ holds F is closed.

Let us consider T, S, f. Let us assume that $\operatorname{rng} f = \Omega_S$ and f is one-to-one. The functor f^{-1} yielding a map from S into T, is defined by:
$f^{-1} = f^{-1}$.

One can prove the following propositions:

(61) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then $f^{-1} = (f\,\mathbf{qua}\,\text{a function})^{-1}$.

(62) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then $\operatorname{dom}(f^{-1}) = \Omega_S$ and $\operatorname{rng}(f^{-1}) = \Omega_T$.

(63) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then f^{-1} is one-to-one.

(64) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then $(f^{-1})^{-1} = f$.

(65) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then $f^{-1} \cdot f = \mathrm{id}_{\operatorname{dom} f}$ and $f \cdot f^{-1} = \mathrm{id}_{\operatorname{rng} f}$.

(66) For every map f from T into S for every map g from S into V such that $\operatorname{rng} f = \Omega_S$ and f is one-to-one and $\operatorname{rng} g = \Omega_V$ and g is one-to-one holds $(g \cdot f)^{-1} = f^{-1} \cdot g^{-1}$.

(67) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then $f\,^{\circ}\,P = (f^{-1})\,^{-1}\,P$.

(68) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then $f\,^{-1}\,P_1 = f^{-1}\,^{\circ}\,P_1$.

Let us consider T, S, f. The predicate f is a homeomorphism is defined by:
$\operatorname{rng} f = \Omega_S$ and f is one-to-one and f is continuous and f^{-1} is continuous.

One can prove the following propositions:

(69) f is a homeomorphism if and only if $\operatorname{rng} f = \Omega_S$ and f is one-to-one and f is continuous and f^{-1} is continuous.

(70) If f is a homeomorphism, then f^{-1} is a homeomorphism.

(71) For every map f from T into S for every map g from S into V such that f is a homeomorphism and g is a homeomorphism holds $g \cdot f$ is a homeomorphism.

(72) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then f is a homeomorphism if and only if for every P holds P is closed if and only if $f\,^{\circ}\,P$ is closed.

(73) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then f is a homeomorphism if and only if for every P_1 holds $f\,^{-1}\,\overline{P_1} = \overline{f\,^{-1}\,P_1}$.

(74) If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then f is a homeomorphism if and only if for every P holds $f\,^{\circ}\,\overline{P} = \overline{f\,^{\circ}\,P}$.

References

[1] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[4] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[5] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[6] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[8] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

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Some Properties of Functions Modul and Signum

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Summary. The article includes definitions and theorems concerning basic properties of the following functions : $|x|$ - modul of real number, sgn x - signum of real number.

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The article [1] provides the terminology and notation for this paper. In the sequel x, y, z, t are real numbers. Let us consider x. The functor $|x|$ yielding a real number, is defined by:

$|x| = x$ if $0 \leq x$, $|x| = -x$, otherwise.

One can prove the following propositions:

(1) If $0 \leq x$, then $|x| = x$.
(2) If $0 < x$, then $|x| = x$.
(3) If $0 \not\leq x$, then $|x| = -x$.
(4) If $x < 0$, then $|x| = -x$.
(5) $0 \leq |x|$.
(6) If $x \neq 0$, then $0 < |x|$.
(7) $x = 0$ if and only if $|x| = 0$.
(8) If $|x| = x$, then $0 \leq x$.
(9) If $|x| = -x$ and $x \neq 0$, then $x < 0$.
(10) For all x, y holds $|x \cdot y| = |x| \cdot |y|$.
(11) $-|x| \leq x$ and $x \leq |x|$.
(12) $-y \leq x$ and $x \leq y$ if and only if $|x| \leq y$.
(13) $|x + y| \leq |x| + |y|$.
(14) For every x such that $x \neq 0$ holds $|x| \cdot |\frac{1}{x}| = 1$.
(15) For every x such that $x \neq 0$ holds $|\frac{1}{x}| = \frac{1}{|x|}$.

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(16) For all x, y such that $y \neq 0$ holds $|\frac{x}{y}| = \frac{|x|}{|y|}$.

(17) $|x| = |-x|$.

(18) For all x, y holds $|x| - |y| \leq |x - y|$.

(19) For all x, y holds $|x - y| \leq |x| + |y|$.

(20) For every x holds $||x|| = |x|$.

(21) If $|x| \leq z$ and $|y| \leq t$, then $|x + y| \leq z + t$.

(22) $||x| - |y|| \leq |x - y|$.

(23) $y < |x|$ if and only if $x < -y$ or $y < x$.

(24) If $0 \leq x \cdot y$, then $|x + y| = |x| + |y|$.

(25) If $|x + y| = |x| + |y|$, then $0 \leq x \cdot y$.

(26) $\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$.

Let us consider x. The functor $\operatorname{sgn} x$ yielding a real number, is defined by: $\operatorname{sgn} x = 1$ if $0 < x$, $\operatorname{sgn} x = -1$ if $x < 0$, $\operatorname{sgn} x = 0$, otherwise.

The following propositions are true:

(27) If $0 < x$, then $\operatorname{sgn} x = 1$.

(28) If $x < 0$, then $\operatorname{sgn} x = -1$.

(29) If $0 \not< x$ and $x \not< 0$, then $\operatorname{sgn} x = 0$.

(30) If $x = 0$, then $\operatorname{sgn} x = 0$.

(31) If $\operatorname{sgn} x = 1$, then $0 < x$.

(32) If $\operatorname{sgn} x = -1$, then $x < 0$.

(33) If $\operatorname{sgn} x = 0$, then $x = 0$.

(34) $x = |x| \cdot (\operatorname{sgn} x)$.

(35) $\operatorname{sgn}(x \cdot y) = (\operatorname{sgn} x) \cdot (\operatorname{sgn} y)$.

(36) $\operatorname{sgn}(\operatorname{sgn} x) = \operatorname{sgn} x$.

(37) $\operatorname{sgn}(x + y) \leq (\operatorname{sgn} x + \operatorname{sgn} y) + 1$.

(38) If $x \neq 0$, then $(\operatorname{sgn} x) \cdot (\operatorname{sgn} \frac{1}{x}) = 1$.

(39) If $x \neq 0$, then $\frac{1}{\operatorname{sgn} x} = \operatorname{sgn} \frac{1}{x}$.

(40) $(\operatorname{sgn} x + \operatorname{sgn} y) - 1 \leq \operatorname{sgn}(x + y)$.

(41) If $x \neq 0$, then $\operatorname{sgn} x = \operatorname{sgn} \frac{1}{x}$.

(42) If $y \neq 0$, then $\operatorname{sgn} \frac{x}{y} = \frac{\operatorname{sgn} x}{\operatorname{sgn} y}$.

References

[1] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

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Zermelo Theorem and Axiom of Choice

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Summary. The article is continuation of [2] and [1], and the goal of it is show that Zermelo theorem (every set has a relation which well orders it - proposition (26)) and axiom of choice (for every non-empty family of non-empty and separate sets there is set which has exactly one common element with arbitraly family member - proposition (27)) are true. It is result of the Tarski's axiom A introduced in [5] and repeated in [6]. Inclusion as a settheoretical binary relation is introduced, the correspondence of well ordering relations to ordinal numbers is shown, and basic properties of equinumerosity are presented. Some facts are based on [4].

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The terminology and notation used in this paper are introduced in the following articles: [6], [7], [8], [3], [2], and [1]. For simplicity we adopt the following convention: X, Y, Z will denote sets, a will be arbitrary, R will denote a relation, and A, B will denote ordinal numbers. Let us consider X. The functor $^{\subseteq}{}_X$ yielding a relation, is defined by:

field $^{\subseteq}{}_X = X$ and for all Y, Z such that $Y \in X$ and $Z \in X$ holds $\langle Y, Z\rangle \in {}^{\subseteq}{}_X$ if and only if $Y \subseteq Z$.

The following propositions are true:

(1) $R = {}^{\subseteq}{}_X$ if and only if field $R = X$ and for all Y, Z such that $Y \in X$ and $Z \in X$ holds $\langle Y, Z\rangle \in R$ if and only if $Y \subseteq Z$.

(2) $^{\subseteq}{}_X$ is pseudo reflexive.

(3) $^{\subseteq}{}_X$ is transitive.

(4) $^{\subseteq}{}_A$ is connected.

(5) $^{\subseteq}{}_X$ is antisymmetric.

(6) $^{\subseteq}{}_A$ is well founded.

(7) $^{\subseteq}{}_A$ is well ordering relation.

(8) If $Y \subseteq X$, then $^{\subseteq}{}_X |^2 Y = {}^{\subseteq}{}_Y$.

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(9) For all A, X such that $X \subseteq A$ holds $\subseteq_X$ is well ordering relation.

We now state several propositions:

(10) If $A \in B$, then $A = \subseteq_B\text{–Seg}(A)$.

(11) If $\subseteq_A$ and $\subseteq_B$ are isomorphic, then $A = B$.

(12) For all X, R, A, B such that R and $\subseteq_A$ are isomorphic and R and $\subseteq_B$ are isomorphic holds $A = B$.

(13) For every R such that R is well ordering relation and for every a such that $a \in \operatorname{field} R$ there exists A such that $R\,|^2\,R\text{–Seg}(a)$ and $\subseteq_A$ are isomorphic there exists A such that R and $\subseteq_A$ are isomorphic.

(14) For every R such that R is well ordering relation there exists A such that R and $\subseteq_A$ are isomorphic.

Let us consider R. Let us assume that R is well ordering relation. The functor $\overline{R}$ yields an ordinal number and is defined by:

R and $\subseteq_{\overline{R}}$ are isomorphic.

Let us consider A, R. The predicate A is an order type of R is defined by:

$A = \overline{R}$.

One can prove the following propositions:

(15) If R is well ordering relation, then for every A holds $A = \overline{R}$ if and only if R and $\subseteq_A$ are isomorphic.

(16) A is an order type of R if and only if $A = \overline{R}$.

(17) If $X \subseteq A$, then $\overline{\subseteq_X} \subseteq A$.

We follow a convention: f will be a function and x, y, z, u will be arbitrary. One can prove the following proposition

(18) $X \approx Y$ if and only if there exists Z such that for every x such that $x \in X$ there exists y such that $y \in Y$ and $\langle x, y\rangle \in Z$ and for every y such that $y \in Y$ there exists x such that $x \in X$ and $\langle x, y\rangle \in Z$ and for all x, y, z, u such that $\langle x, y\rangle \in Z$ and $\langle z, u\rangle \in Z$ holds $x = z$ if and only if $y = u$.

Let us consider X, Y. Let us note that one can characterize the predicate $X \approx Y$ by the following (equivalent) condition: there exists f such that f is one-to-one and $\operatorname{dom} f = X$ and $\operatorname{rng} f = Y$.

Next we state several propositions:

(19) $X \approx Y$ if and only if there exists f such that f is one-to-one and $\operatorname{dom} f = X$ and $\operatorname{rng} f = Y$.

(20) $X \approx X$.

(21) If $X \approx Y$, then $Y \approx X$.

(22) If $X \approx Y$ and $Y \approx Z$, then $X \approx Z$.

(23) If R is well ordering relation and $X \approx \operatorname{field} R$, then there exists R such that R well orders X.

(24) If R is well ordering relation and $X \approx Y$ and $Y \subseteq \operatorname{field} R$, then there exists R such that R well orders X.

(25) If R well orders X, then $\operatorname{field}(R\,|^2\,X) = X$ and $R\,|^2\,X$ is well ordering relation.

(26) For every X there exists R such that R well orders X.

In the sequel M will be a non-empty family of sets. We now state a proposition

(27) If for every X such that $X \in M$ holds $X \neq \emptyset$ and for all X, Y such that $X \in M$ and $Y \in M$ and $X \neq Y$ holds $X \cap Y = \emptyset$, then there exists $Choice$ being a set such that for every X such that $X \in M$ there exists x such that $Choice \cap X = \{x\}$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[2] Grzegorz Bancerek. The well ordering relations. *Formalized Mathematics*, 1(**1**):123–129, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Kazimierz Kuratowski and Andrzej Mostowski. *Teoria mnogości.* PTM, Wrocław, 1952.

[5] Alfred Tarski. Über Unerreichbare Kardinalzahlen. *Fundamenta Mathematicae*, 30:176–183, 1938.

[6] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[7] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[8] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Formalized Mathematics*, 1(**1**):85–89, 1990.

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Real Sequences and Basic Operations on Them

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Summary. Definition of real sequence and operations on sequences (multiplication of sequences and multiplication by a real number, addition, subtraction, division and absolute value of sequence) are given.

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The notation and terminology used here are introduced in the following articles: [4], [1], [3], and [2]. For simplicity we follow the rules: f will be a function, n will be a natural number, r, p will be real numbers, and x will be arbitrary. We now state a proposition

(1) x is a natural number if and only if $x \in \mathbb{N}$.

The mode sequence of real numbers, which widens to the type a function, is defined by:

$\mathrm{dom}\,\mathrm{it} = \mathbb{N}$ and $\mathrm{rng}\,\mathrm{it} \subseteq \mathbb{R}$.

In the sequel seq, seq_1, seq_2, seq_3, seq', $seq_{1'}$ are sequences of real numbers. Next we state three propositions:

(2) f is a sequence of real numbers if and only if $\mathrm{dom}\,f = \mathbb{N}$ and $\mathrm{rng}\,f \subseteq \mathbb{R}$.

(3) f is a sequence of real numbers if and only if $\mathrm{dom}\,f = \mathbb{N}$ and for every x such that $x \in \mathbb{N}$ holds $f(x)$ is a real number.

(4) f is a sequence of real numbers if and only if $\mathrm{dom}\,f = \mathbb{N}$ and for every n holds $f(n)$ is a real number.

Let us consider seq, n. Then $seq(n)$ is a real number.

Let us consider seq. The predicate seq is non-zero is defined by:

$\mathrm{rng}\,seq \subseteq \mathbb{R} \setminus \{0\}$.

One can prove the following propositions:

(5) seq is non-zero if and only if $\mathrm{rng}\,seq \subseteq \mathbb{R} \setminus \{0\}$.

(6) seq is non-zero if and only if for every x such that $x \in \mathbb{N}$ holds $seq(x) \neq 0$.

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(7) seq is non-zero if and only if for every n holds $seq(n) \neq 0$.

(8) For all seq, seq_1 such that for every x such that $x \in \mathbb{N}$ holds $seq(x) = seq_1(x)$ holds $seq = seq_1$.

(9) For all seq, seq_1 such that for every n holds $seq(n) = seq_1(n)$ holds $seq = seq_1$.

(10) For every r there exists seq such that $\operatorname{rng} seq = \{r\}$.

The scheme *ExRealSeq* concerns a unary functor $\mathcal{F}$ yielding a real number and states that:

there exists seq such that for every n holds $seq(n) = \mathcal{F}(n)$

for all values of the parameter.

We now define two new functors. Let us consider seq_1, seq_2. The functor $seq_1 + seq_2$ yields a sequence of real numbers and is defined by:

for every n holds $(seq_1 + seq_2)(n) = seq_1(n) + seq_2(n)$.

The functor $seq_1 \cdot seq_2$ yielding a sequence of real numbers, is defined by:

for every n holds $(seq_1 \cdot seq_2)(n) = seq_1(n) \cdot seq_2(n)$.

The following two propositions are true:

(11) $seq = seq_1 + seq_2$ if and only if for every n holds $seq(n) = seq_1(n) + seq_2(n)$.

(12) $seq = seq_1 \cdot seq_2$ if and only if for every n holds $seq(n) = seq_1(n) \cdot seq_2(n)$.

Let us consider r, seq. The functor $r \cdot seq$ yielding a sequence of real numbers, is defined by:

for every n holds $(r \cdot seq)(n) = r \cdot seq(n)$.

One can prove the following proposition

(13) $seq = r \cdot seq_1$ if and only if for every n holds $seq(n) = r \cdot seq_1(n)$.

Let us consider seq. The functor $-seq$ yields a sequence of real numbers and is defined by:

for every n holds $(-seq)(n) = -seq(n)$.

We now state a proposition

(14) $seq = -seq_1$ if and only if for every n holds $seq(n) = -seq_1(n)$.

Let us consider seq_1, seq_2. The functor $seq_1 - seq_2$ yields a sequence of real numbers and is defined by:

$seq_1 - seq_2 = seq_1 + (-seq_2)$.

We now state a proposition

(15) $seq = seq_1 - seq_2$ if and only if $seq = seq_1 + (-seq_2)$.

Let us consider seq. Let us assume that seq is non-zero. The functor seq^{-1} yielding a sequence of real numbers, is defined by:

for every n holds $(seq^{-1})(n) = (seq(n))^{-1}$.

One can prove the following proposition

(16) If seq is non-zero, then $seq_1 = seq^{-1}$ if and only if for every n holds $seq_1(n) = (seq(n))^{-1}$.

Let us consider seq_1, seq. Let us assume that seq is non-zero. The functor $\frac{seq_1}{seq}$ yields a sequence of real numbers and is defined by:

$\frac{seq_1}{seq} = seq_1 \cdot seq^{-1}$.

The following proposition is true

(17) If seq_2 is non-zero, then $seq = \frac{seq_1}{seq_2}$ if and only if $seq = seq_1 \cdot seq_2{}^{-1}$.

Let us consider seq. The functor $|seq|$ yielding a sequence of real numbers, is defined by:

for every n holds $|seq|(n) = |seq(n)|$.

The following propositions are true:

(18) $seq = |seq_1|$ if and only if for every n holds $seq(n) = |seq_1(n)|$.

(19) $seq_1 + seq_2 = seq_2 + seq_1$.

(20) $(seq_1 + seq_2) + seq_3 = seq_1 + (seq_2 + seq_3)$.

(21) $seq_1 \cdot seq_2 = seq_2 \cdot seq_1$.

(22) $(seq_1 \cdot seq_2) \cdot seq_3 = seq_1 \cdot (seq_2 \cdot seq_3)$.

(23) $(seq_1 + seq_2) \cdot seq_3 = seq_1 \cdot seq_3 + seq_2 \cdot seq_3$.

(24) $seq_3 \cdot (seq_1 + seq_2) = seq_3 \cdot seq_1 + seq_3 \cdot seq_2$.

(25) $-seq = (-1) \cdot seq$.

(26) $r \cdot (seq_1 \cdot seq_2) = (r \cdot seq_1) \cdot seq_2$.

(27) $r \cdot (seq_1 \cdot seq_2) = seq_1 \cdot (r \cdot seq_2)$.

(28) $(seq_1 - seq_2) \cdot seq_3 = seq_1 \cdot seq_3 - seq_2 \cdot seq_3$.

(29) $seq_3 \cdot seq_1 - seq_3 \cdot seq_2 = seq_3 \cdot (seq_1 - seq_2)$.

(30) $r \cdot (seq_1 + seq_2) = r \cdot seq_1 + r \cdot seq_2$.

(31) $(r \cdot p) \cdot seq = r \cdot (p \cdot seq)$.

(32) $r \cdot (seq_1 - seq_2) = r \cdot seq_1 - r \cdot seq_2$.

(33) If seq is non-zero, then $r \cdot \frac{seq_1}{seq} = \frac{r \cdot seq_1}{seq}$.

(34) $seq_1 - (seq_2 + seq_3) = (seq_1 - seq_2) - seq_3$.

(35) $1 \cdot seq = seq$.

(36) $-(-seq) = seq$.

(37) $seq_1 - (-seq_2) = seq_1 + seq_2$.

(38) $seq_1 - (seq_2 - seq_3) = (seq_1 - seq_2) + seq_3$.

(39) $seq_1 + (seq_2 - seq_3) = (seq_1 + seq_2) - seq_3$.

(40) $(-seq_1) \cdot seq_2 = -seq_1 \cdot seq_2$ and $seq_1 \cdot (-seq_2) = -seq_1 \cdot seq_2$.

(41) If seq is non-zero, then seq^{-1} is non-zero.

(42) If seq is non-zero, then $(seq^{-1})^{-1} = seq$.

(43) seq is non-zero and seq_1 is non-zero if and only if $seq \cdot seq_1$ is non-zero.

(44) If seq is non-zero and seq_1 is non-zero, then $seq^{-1} \cdot seq_1{}^{-1} = (seq \cdot seq_1)^{-1}$.

(45) If seq is non-zero, then $\frac{seq_1}{seq} \cdot seq = seq_1$.

(46) If seq is non-zero and seq_1 is non-zero, then $\frac{seq'}{seq} \cdot \frac{seq_1{}'}{seq_1} = \frac{seq' \cdot seq_1{}'}{seq \cdot seq_1}$.

(47) If seq is non-zero and seq_1 is non-zero, then $\frac{seq}{seq_1}$ is non-zero.

(48) If seq is non-zero and seq_1 is non-zero, then $\frac{seq}{seq_1}{}^{-1} = \frac{seq_1}{seq}$.

(49) If seq is non-zero, then $seq_2 \cdot \frac{seq_1}{seq} = \frac{seq_2 \cdot seq_1}{seq}$.

(50) If seq is non-zero and seq_1 is non-zero, then $\frac{\frac{seq_2}{seq}}{seq_1} = \frac{seq_2 \cdot seq_1}{seq}$.

(51) If seq is non-zero and seq_1 is non-zero, then $\frac{seq_2}{seq} = \frac{seq_2 \cdot seq_1}{seq \cdot seq_1}$.

(52) If $r \neq 0$ and seq is non-zero, then $r \cdot seq$ is non-zero.

(53) If seq is non-zero, then $-seq$ is non-zero.

(54) If $r \neq 0$ and seq is non-zero, then $(r \cdot seq)^{-1} = r^{-1} \cdot seq^{-1}$.

(55) If seq is non-zero, then $(-seq)^{-1} = (-1) \cdot seq^{-1}$.

(56) If seq is non-zero, then $-\frac{seq_1}{seq} = \frac{-seq_1}{seq}$ and $\frac{seq_1}{-seq} = -\frac{seq_1}{seq}$.

(57) If seq is non-zero, then $\frac{seq_1}{seq} + \frac{seq_{1'}}{seq} = \frac{seq_1 + seq_{1'}}{seq}$ and $\frac{seq_1}{seq} - \frac{seq_{1'}}{seq} = \frac{seq_1 - seq_{1'}}{seq}$.

(58) If seq is non-zero and seq' is non-zero, then $\frac{seq_1}{seq} + \frac{seq_{1'}}{seq'} = \frac{seq_1 \cdot seq' + seq_{1'} \cdot seq}{seq \cdot seq'}$ and $\frac{seq_1}{seq} - \frac{seq_{1'}}{seq'} = \frac{seq_1 \cdot seq' - seq_{1'} \cdot seq}{seq \cdot seq'}$.

(59) If seq is non-zero and seq' is non-zero and seq_1 is non-zero, then $\frac{\frac{seq_{1'}}{seq}}{\frac{seq'}{seq_1}} = \frac{seq_{1'} \cdot seq_1}{seq \cdot seq'}$.

(60) $|seq \cdot seq'| = |seq| \cdot |seq'|$.

(61) If seq is non-zero, then $|seq|$ is non-zero.

(62) If seq is non-zero, then $|seq|^{-1} = |seq^{-1}|$.

(63) If seq is non-zero, then $|\frac{seq'}{seq}| = \frac{|seq'|}{|seq|}$.

(64) $|r \cdot seq| = |r| \cdot |seq|$.

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[2] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[3] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[4] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

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Convergent Sequences and the Limit of Sequences

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Summary. The article contains definitions and same basic properties of bounded sequences (above and below), convergent sequences and the limit of sequences. In the article there are some properties of real numbers useful in the other theorems of this article.

MML Identifier: `SEQ_2`.

The terminology and notation used in this paper have been introduced in the following papers: [1], and [2]. We adopt the following rules: n, m are natural numbers, r, r_1, p, g_1, g, g' are real numbers, and seq, seq', seq_1 are sequences of real numbers. One can prove the following propositions:

(1) $(-1) \cdot (-1) = 1$.

(2) $\frac{g}{2} + \frac{g}{2} = g$ and $\frac{g}{4} + \frac{g}{4} = \frac{g}{2}$.

(3) If $0 < g$, then $0 < \frac{g}{2}$ and $0 < \frac{g}{4}$.

(4) If $0 < g$, then $\frac{g}{2} < g$.

(5) If $g \neq 0$, then $\frac{r}{g \cdot 2} + \frac{r}{g \cdot 2} = \frac{r}{g}$.

(6) If $0 < g$ and $0 < p$, then $0 < \frac{g}{p}$.

(7) If $0 \leq g$ and $0 \leq r$ and $g < g_1$ and $r < r_1$, then $g \cdot r < g_1 \cdot r_1$.

(8) If $g = -g'$, then $-g = g'$.

(9) $-g < r$ and $r < g$ if and only if $|r| < g$.

(10) If $0 < r_1$ and $r_1 < r$ and $0 < g$, then $\frac{g}{r} < \frac{g}{r_1}$.

(11) If $g \neq 0$ and $r \neq 0$, then $|g^{-1} - r^{-1}| = \frac{|g-r|}{|g| \cdot |r|}$.

We now define two new predicates. Let us consider seq. The predicate seq is bounded above is defined by:

there exists r such that for every n holds $seq(n) < r$.

The predicate seq is bounded below is defined by:

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there exists r such that for every n holds $r < seq(n)$.

We now state two propositions:

(12) seq is bounded above if and only if there exists r such that for every n holds $seq(n) < r$.

(13) seq is bounded below if and only if there exists r such that for every n holds $r < seq(n)$.

Let us consider seq. The predicate seq is bounded is defined by:

seq is bounded above and seq is bounded below.

Next we state three propositions:

(14) seq is bounded if and only if seq is bounded above and seq is bounded below.

(15) seq is bounded if and only if there exists r such that $0 < r$ and for every n holds $|seq(n)| < r$.

(16) For every n there exists r such that $0 < r$ and for every m such that $m \leq n$ holds $|seq(m)| < r$.

Let us consider seq. The predicate seq is convergent is defined by:

there exists g such that for every p such that $0 < p$ there exists n such that for every m such that $n \leq m$ holds $|seq(m) - g| < p$.

One can prove the following proposition

(17) seq is convergent if and only if there exists g such that for every p such that $0 < p$ there exists n such that for every m such that $n \leq m$ holds $|seq(m) - g| < p$.

Let us consider seq. Let us assume that seq is convergent. The functor $\lim seq$ yields a real number and is defined by:

for every p such that $0 < p$ there exists n such that for every m such that $n \leq m$ holds $|seq(m) - (\lim seq)| < p$.

The following propositions are true:

(18) If seq is convergent, then $\lim seq = g$ if and only if for every p such that $0 < p$ there exists n such that for every m such that $n \leq m$ holds $|seq(m) - g| < p$.

(19) If seq is convergent and seq' is convergent, then $seq + seq'$ is convergent.

(20) If seq is convergent and seq' is convergent, then $\lim(seq+seq') = \lim seq + \lim seq'$.

(21) If seq is convergent, then $r \cdot seq$ is convergent.

(22) If seq is convergent, then $\lim(r \cdot seq) = r \cdot (\lim seq)$.

(23) If seq is convergent, then $-seq$ is convergent.

(24) If seq is convergent, then $\lim(-seq) = -\lim seq$.

(25) If seq is convergent and seq' is convergent, then $seq - seq'$ is convergent.

(26) If seq is convergent and seq' is convergent, then $\lim(seq-seq') = \lim seq - \lim seq'$.

(27) If seq is convergent, then seq is bounded.

(28) If seq is convergent and seq' is convergent, then $seq \cdot seq'$ is convergent.

(29) If seq is convergent and seq' is convergent, then $\lim(seq \cdot seq') = (\lim seq) \cdot (\lim seq')$.

(30) If seq is convergent, then if $\lim seq \neq 0$, then there exists n such that for every m such that $n \leq m$ holds $\frac{|\lim seq|}{2} < |seq(m)|$.

(31) If seq is convergent and for every n holds $0 \leq seq(n)$, then $0 \leq \lim seq$.

(32) If seq is convergent and seq' is convergent and for every n holds $seq(n) \leq seq'(n)$, then $\lim seq \leq \lim seq'$.

(33) If seq is convergent and seq' is convergent and for every n holds $seq(n) \leq seq_1(n)$ and $seq_1(n) \leq seq'(n)$ and $\lim seq = \lim seq'$, then seq_1 is convergent.

(34) If seq is convergent and seq' is convergent and for every n holds $seq(n) \leq seq_1(n)$ and $seq_1(n) \leq seq'(n)$ and $\lim seq = \lim seq'$, then $\lim seq_1 = \lim seq$.

(35) If seq is convergent and $\lim seq \neq 0$ and seq is non-zero, then seq^{-1} is convergent.

(36) If seq is convergent and $\lim seq \neq 0$ and seq is non-zero, then $\lim seq^{-1} = (\lim seq)^{-1}$.

(37) If seq' is convergent and seq is convergent and $\lim seq \neq 0$ and seq is non-zero, then $\frac{seq'}{seq}$ is convergent.

(38) If seq' is convergent and seq is convergent and $\lim seq \neq 0$ and seq is non-zero, then $\lim \frac{seq'}{seq} = \frac{\lim seq'}{\lim seq}$.

(39) If seq is convergent and seq_1 is bounded and $\lim seq = 0$, then $seq \cdot seq_1$ is convergent.

(40) If seq is convergent and seq_1 is bounded and $\lim seq = 0$, then $\lim(seq \cdot seq_1) = 0$.

References

[1] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[2] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

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Properties of ZF Models

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Summary. The article deals with the concepts of satisfiability of ZF set theory language formulae in a model (a non-empty family of sets) and the axioms of ZF theory introduced in [6]. It is shown that the transitive model satisfies the axiom of extensionality and that it satisfies the axiom of pairs if and only if it is closed to pair operation; it satisfies the axiom of unions if and only if it is closed to union operation, ect. The conditions which are satisfied by arbitrary model of ZF set theory are also shown. Besides introduced are definable and parametrically definable functions.

MML Identifier: `ZFMODEL1`.

The notation and terminology used in this paper are introduced in the following papers: [8], [4], [1], [5], [7], [3], and [2]. For simplicity we follow a convention: x, y, z will be variables, H will be a ZF-formula, E will be a non-empty family of sets, X, Y, Z will be sets, u, v, w will be elements of E, and f, g will be functions from VAR into E. One can prove the following propositions:

(1) If E is transitive, then $E \models$ the axiom of extensionality.

(2) If E is transitive, then $E \models$ the axiom of pairs if and only if for all u, v holds $\{u, v\} \in E$.

(3) If E is transitive, then $E \models$ the axiom of pairs if and only if for all X, Y such that $X \in E$ and $Y \in E$ holds $\{X, Y\} \in E$.

(4) If E is transitive, then $E \models$ the axiom of unions if and only if for every u holds $\bigcup u \in E$.

(5) If E is transitive, then $E \models$ the axiom of unions if and only if for every X such that $X \in E$ holds $\bigcup X \in E$.

(6) If E is transitive, then $E \models$ the axiom of infinity if and only if there exists u such that $u \neq \emptyset$ and for every v such that $v \in u$ there exists w such that $v \subseteq w$ and $v \neq w$ and $w \in u$.

(7) If E is transitive, then $E \models$ the axiom of infinity if and only if there exists X such that $X \in E$ and $X \neq \emptyset$ and for every Y such that $Y \in X$ there exists Z such that $Y \subseteq Z$ and $Y \neq Z$ and $Z \in X$.

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(8) If E is transitive, then $E \models$ the axiom of power sets if and only if for every u holds $E \cap 2^u \in E$.

(9) If E is transitive, then $E \models$ the axiom of power sets if and only if for every X such that $X \in E$ holds $E \cap 2^X \in E$.

(10) If $x \notin \mathrm{Free}\, H$ and $E, f \models H$, then $E, f \models \forall_x H$.

(11) If $\{x, y\}$ misses $\mathrm{Free}\, H$ and $E, f \models H$, then $E, f \models \forall_{x,y} H$.

(12) If $\{x, y, z\}$ misses $\mathrm{Free}\, H$ and $E, f \models H$, then $E, f \models \forall_{x,y,z} H$.

The arguments of the notions defined below are the following: H, E which are objects of the type reserved above; val which is a function from VAR into E. Let us assume that $x_0 \notin \mathrm{Free}\, H$ and $E, val \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$. The functor $\mathrm{f}_H[val]$ yielding a function from E into E, is defined by:

for every g such that for every y such that $g(y) \neq val(y)$ holds $x_0 = y$ or $x_3 = y$ or $x_4 = y$ holds $E, g \models H$ if and only if $\mathrm{f}_H[val](g(x_3)) = g(x_4)$.

Next we state two propositions:

(13) Suppose $x_0 \notin \mathrm{Free}\, H$ and $E, f \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$. Let F be a function from E into E. Then $F = \mathrm{f}_H[f]$ if and only if for every g such that for every y such that $g(y) \neq f(y)$ holds $x_0 = y$ or $x_3 = y$ or $x_4 = y$ holds $E, g \models H$ if and only if $F(g(x_3)) = g(x_4)$.

(14) For all H, f, g such that for every x such that $f(x) \neq g(x)$ holds $x \notin \mathrm{Free}\, H$ and $E, f \models H$ holds $E, g \models H$.

Let us consider H, E. Let us assume that $\mathrm{Free}\, H \subseteq \{x_3, x_4\}$ and $E \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$. The functor $\mathrm{f}_H[E]$ yielding a function from E into E, is defined by:

for every g holds $E, g \models H$ if and only if $\mathrm{f}_H[E](g(x_3)) = g(x_4)$.

The following proposition is true

(15) Suppose $\mathrm{Free}\, H \subseteq \{x_3, x_4\}$ and $E \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$. Then for every function F from E into E holds $F = \mathrm{f}_H[E]$ if and only if for every g holds $E, g \models H$ if and only if $F(g(x_3)) = g(x_4)$.

We now define two new predicates. The arguments of the notions defined below are the following: F which is a function; E which is an object of the type reserved above. The predicate F is definable in E is defined by:

there exists H such that $\mathrm{Free}\, H \subseteq \{x_3, x_4\}$ and $E \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$ and $F = \mathrm{f}_H[E]$.

The predicate F is parametrically definable in E is defined by:

there exist H, f such that $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\, H$ and
$E, f \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$
and $F = \mathrm{f}_H[f]$.

One can prove the following propositions:

(16) For every function F holds F is definable in E if and only if there exists H such that $\mathrm{Free}\, H \subseteq \{x_3, x_4\}$ and $E \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$ and $F = \mathrm{f}_H[E]$.

(17) For every function F holds F is parametrically definable in E if and only if there exist H, f such that $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\, H$ and $E, f \models$

$\forall_{x_3}(\exists_{x_0}(\forall_{x_4}H \Leftrightarrow x_4{=}x_0))$ and $F = \mathrm{f}_H[f]$.

(18) For every function F such that F is definable in E holds F is parametrically definable in E.

(19) Suppose E is transitive. Then for every H such that $\{x_0, x_1, x_2\}$ misses Free H holds $E \models$ the axiom of substitution for H if and only if for all H, f such that $\{x_0, x_1, x_2\}$ misses Free H and $E, f \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4}H \Leftrightarrow x_4{=}x_0))$ for every u holds $\mathrm{f}_H[f] \circ u \in E$.

(20) If E is transitive, then for every H such that $\{x_0, x_1, x_2\}$ misses Free H holds $E \models$ the axiom of substitution for H if and only if for every function F such that F is parametrically definable in E for every X such that $X \in E$ holds $F \circ X \in E$.

(21) Suppose E is a model of ZF. Then

(i) E is transitive,

(ii) for all u, v such that for every w holds $w \in u$ if and only if $w \in v$ holds $u = v$,

(iii) for all u, v holds $\{u, v\} \in E$,

(iv) for every u holds $\bigcup u \in E$,

(v) there exists u such that $u \neq \emptyset$ and for every v such that $v \in u$ there exists w such that $v \subseteq w$ and $v \neq w$ and $w \in u$,

(vi) for every u holds $E \cap 2^u \in E$,

(vii) for all H, f such that $\{x_0, x_1, x_2\}$ misses Free H and
$E, f \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4}H \Leftrightarrow x_4{=}x_0))$
for every u holds $\mathrm{f}_H[f] \circ u \in E$.

(22) Suppose that

(i) E is transitive,

(ii) for all u, v holds $\{u, v\} \in E$,

(iii) for every u holds $\bigcup u \in E$,

(iv) there exists u such that $u \neq \emptyset$ and for every v such that $v \in u$ there exists w such that $v \subseteq w$ and $v \neq w$ and $w \in u$,

(v) for every u holds $E \cap 2^u \in E$,

(vi) for all H, f such that $\{x_0, x_1, x_2\}$ misses Free H and
$E, f \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4}H \Leftrightarrow x_4{=}x_0))$
for every u holds $\mathrm{f}_H[f] \circ u \in E$.

Then E is a model of ZF.

References

[1] Grzegorz Bancerek. A model of ZF set theory language. *Formalized Mathematics*, 1(**1**):131–145, 1990.

[2] Grzegorz Bancerek. Models and satisfiability. *Formalized Mathematics*, 1(**1**):191–199, 1990.

[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Andrzej Mostowski. *Constructible Sets with Applications.* North Holland, 1969.

[7] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.

[8] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

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Sequences of Ordinal Numbers

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Summary. In the first part of the article we introduce the following operations: On X that yields the set of all ordinals which belong to the set X, Lim X that yields the set of all limit ordinals which belong to X, and inf X and sup X that yield the minimal ordinal belonging to X and the minimal ordinal greater than all ordinals belonging to X, respectively. The second part of the article starts with schemes that can be used to justify the correctness of definitions based on the transfinite induction (see [1] or [3]). The schemes are used to define addition, product and power of ordinal numbers. The operations of limes inferior and limes superior of sequences of ordinals are defined and the concepts of limet of ordinal sequence and increasing and continuous sequence are introduced.

MML Identifier: `ORDINAL2`.

The papers [5], [2], [1], and [4] provide the terminology and notation for this paper. For simplicity we adopt the following rules: A, A_1, A_2, B, C, D will denote ordinal numbers, X, Y will denote sets, x, y will be arbitrary, and L, L_1 will denote transfinite sequences. The scheme *Ordinal_Ind* concerns a unary predicate $\mathcal{P}$ and states that:

for every A holds $\mathcal{P}[A]$

provided the parameter satisfies the following conditions:

- $\mathcal{P}[\mathbf{0}]$,
- for every A such that $\mathcal{P}[A]$ holds $\mathcal{P}[\mathrm{succ}\,A]$,
- for every A such that $A \neq \mathbf{0}$ and A is a limit ordinal number and for every B such that $B \in A$ holds $\mathcal{P}[B]$ holds $\mathcal{P}[A]$.

We now state several propositions:

(1) If $A \subseteq B$, then $\mathrm{succ}\,A \subseteq \mathrm{succ}\,B$.

(2) $\bigcup(\mathrm{succ}\,A) = A$.

(3) $\mathrm{succ}\,A \subseteq 2^A$.

(4) $\mathbf{0}$ is a limit ordinal number.

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(5) $\bigcup A \subseteq A$.

Let us consider L. The functor $\operatorname{last} L$ yielding a set, is defined by:
$\operatorname{last} L = L(\bigcup(\operatorname{dom} L))$.

Next we state two propositions:

(6) $\operatorname{last} L = L(\bigcup(\operatorname{dom} L))$.

(7) If $\operatorname{dom} L = \operatorname{succ} A$, then $\operatorname{last} L = L(A)$.

We now define two new functors. Let us consider X. The functor $\operatorname{On} X$ yields a set and is defined by:

$x \in \operatorname{On} X$ if and only if $x \in X$ and x is an ordinal number.

The functor $\operatorname{Lim} X$ yielding a set, is defined by:

$x \in \operatorname{Lim} X$ if and only if $x \in X$ and there exists A such that $x = A$ and A is a limit ordinal number.

Next we state a number of propositions:

(8) $x \in \operatorname{On} X$ if and only if $x \in X$ and x is an ordinal number.

(9) $\operatorname{On} X \subseteq X$.

(10) $\operatorname{On} A = A$.

(11) If $X \subseteq Y$, then $\operatorname{On} X \subseteq \operatorname{On} Y$.

(12) $x \in \operatorname{Lim} X$ if and only if $x \in X$ and there exists A such that $x = A$ and A is a limit ordinal number.

(13) $\operatorname{Lim} X \subseteq X$.

(14) If $X \subseteq Y$, then $\operatorname{Lim} X \subseteq \operatorname{Lim} Y$.

(15) $\operatorname{Lim} X \subseteq \operatorname{On} X$.

(16) For every D there exists A such that $D \in A$ and A is a limit ordinal number.

(17) If for every x such that $x \in X$ holds x is an ordinal number, then $\bigcap X$ is an ordinal number.

We now define four new functors. The constant $\mathbf{1}$ is an ordinal number and is defined by:

$\mathbf{1} = \operatorname{succ} \mathbf{0}$.

The constant ω is an ordinal number and is defined by:

$\mathbf{0} \in \omega$ and ω is a limit ordinal number and for every A such that $\mathbf{0} \in A$ and A is a limit ordinal number holds $\omega \subseteq A$.

Let us consider X. The functor $\inf X$ yields an ordinal number and is defined by:

$\inf X = \bigcap(\operatorname{On} X)$.

The functor $\sup X$ yielding an ordinal number, is defined by:

$\operatorname{On} X \subseteq \sup X$ and for every A such that $\operatorname{On} X \subseteq A$ holds $\sup X \subseteq A$.

We now state a number of propositions:

(18) $\mathbf{1} = \operatorname{succ} \mathbf{0}$.

(19) $\mathbf{0} \in \omega$ and ω is a limit ordinal number and for every A such that $\mathbf{0} \in A$ and A is a limit ordinal number holds $\omega \subseteq A$.

(20) $\inf X = \bigcap(\operatorname{On} X)$.

(21) $B = \sup X$ if and only if $\mathrm{On}\,X \subseteq B$ and for every A such that $\mathrm{On}\,X \subseteq A$ holds $B \subseteq A$.

(22) If $A \in X$, then $\inf X \subseteq A$.

(23) If $\mathrm{On}\,X \neq \emptyset$ and for every A such that $A \in X$ holds $D \subseteq A$, then $D \subseteq \inf X$.

(24) If $A \in X$ and $X \subseteq Y$, then $\inf Y \subseteq \inf X$.

(25) If $A \in X$, then $\inf X \in X$.

(26) $\sup A = A$.

(27) If $A \in X$, then $A \in \sup X$.

(28) If for every A such that $A \in X$ holds $A \in D$, then $\sup X \subseteq D$.

(29) If $A \in \sup X$, then there exists B such that $B \in X$ and $A \subseteq B$.

(30) If $X \subseteq Y$, then $\sup X \subseteq \sup Y$.

(31) $\sup\{A\} = \mathrm{succ}\,A$.

(32) $\inf X \subseteq \sup X$.

The scheme *TS_Lambda* concerns a constant $\mathcal{A}$ that is an ordinal number and a unary functor $\mathcal{F}$ and states that:

there exists L such that $\mathrm{dom}\,L = \mathcal{A}$ and for every A such that $A \in \mathcal{A}$ holds $L(A) = \mathcal{F}(A)$

for all values of the parameters.

The mode sequence of ordinal numbers, which widens to the type a transfinite sequence, is defined by:

there exists A such that $\mathrm{rng}\,\mathrm{it} \subseteq A$.

The following proposition is true

(33) L is a sequence of ordinal numbers if and only if there exists A such that $\mathrm{rng}\,L \subseteq A$.

Let us consider A. We see that it makes sense to consider the following mode for restricted scopes of arguments. Then all the objests of the mode transfinite sequence of elements of A are a sequence of ordinal numbers.

The arguments of the notions defined below are the following: L which is a sequence of ordinal numbers; A which is an object of the type reserved above. Then $L \restriction A$ is a sequence of ordinal numbers. Then $L(A)$ is a set.

In the sequel fi, psi are sequences of ordinal numbers. Next we state a proposition

(34) If $A \in \mathrm{dom}\,fi$, then $fi(A)$ is an ordinal number.

Now we present a number of schemes. The scheme *OS_Lambda* concerns a constant $\mathcal{A}$ that is an ordinal number and a unary functor $\mathcal{F}$ yielding an ordinal number and states that:

there exists fi such that $\mathrm{dom}\,fi = \mathcal{A}$ and for every A such that $A \in \mathcal{A}$ holds $fi(A) = \mathcal{F}(A)$

for all values of the parameters.

The scheme *TS_Uniq1* deals with a constant $\mathcal{A}$ that is an ordinal number, a constant $\mathcal{B}$, a binary functor $\mathcal{F}$, a binary functor $\mathcal{G}$, a constant $\mathcal{C}$ that is a transfinite sequence and a constant $\mathcal{D}$ that is a transfinite sequence, and states that:

$\mathcal{C} = \mathcal{D}$

provided the parameters satisfy the following conditions:

- $\operatorname{dom} \mathcal{C} = \mathcal{A}$,
- if $\mathbf{0} \in \mathcal{A}$, then $\mathcal{C}(\mathbf{0}) = \mathcal{B}$,
- for all A, x such that $\operatorname{succ} A \in \mathcal{A}$ and $x = \mathcal{C}(A)$ holds $\mathcal{C}(\operatorname{succ} A) = \mathcal{F}(A, x)$,
- for all A, L such that $A \in \mathcal{A}$ and $A \neq \mathbf{0}$ and A is a limit ordinal number and $L = \mathcal{C} \upharpoonright A$ holds $\mathcal{C}(A) = \mathcal{G}(A, L)$,
- $\operatorname{dom} \mathcal{D} = \mathcal{A}$,
- if $\mathbf{0} \in \mathcal{A}$, then $\mathcal{D}(\mathbf{0}) = \mathcal{B}$,
- for all A, x such that $\operatorname{succ} A \in \mathcal{A}$ and $x = \mathcal{D}(A)$ holds $\mathcal{D}(\operatorname{succ} A) = \mathcal{F}(A, x)$,
- for all A, L such that $A \in \mathcal{A}$ and $A \neq \mathbf{0}$ and A is a limit ordinal number and $L = \mathcal{D} \upharpoonright A$ holds $\mathcal{D}(A) = \mathcal{G}(A, L)$.

The scheme *TS_Exist1* concerns a constant $\mathcal{A}$ that is an ordinal number, a constant $\mathcal{B}$, a binary functor $\mathcal{F}$ and a binary functor $\mathcal{G}$ and states that:

there exists L such that $\operatorname{dom} L = \mathcal{A}$ but if $\mathbf{0} \in \mathcal{A}$, then $L(\mathbf{0}) = \mathcal{B}$ and for all A, x such that $\operatorname{succ} A \in \mathcal{A}$ and $x = L(A)$ holds $L(\operatorname{succ} A) = \mathcal{F}(A, x)$ and for all A, L_1 such that $A \in \mathcal{A}$ and $A \neq \mathbf{0}$ and A is a limit ordinal number and $L_1 = L \upharpoonright A$ holds $L(A) = \mathcal{G}(A, L_1)$.

for all values of the parameters.

The scheme *TS_Result* deals with a constant $\mathcal{A}$ that is a transfinite sequence, a unary functor $\mathcal{F}$, a constant $\mathcal{B}$ that is an ordinal number, a constant $\mathcal{C}$, a binary functor $\mathcal{G}$ and a binary functor $\mathcal{H}$ and states that:

for every A such that $A \in \operatorname{dom} \mathcal{A}$ holds $\mathcal{A}(A) = \mathcal{F}(A)$

provided the parameters satisfy the following conditions:

- Given A, x. Then $x = \mathcal{F}(A)$ if and only if there exists L such that $x = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} A$ and $L(\mathbf{0}) = \mathcal{C}$ and for all C, y such that $\operatorname{succ} C \in \operatorname{succ} A$ and $y = L(C)$ holds $L(\operatorname{succ} C) = \mathcal{G}(C, y)$ and for all C, L_1 such that $C \in \operatorname{succ} A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \upharpoonright C$ holds $L(C) = \mathcal{H}(C, L_1)$.
- $\operatorname{dom} \mathcal{A} = \mathcal{B}$,
- if $\mathbf{0} \in \mathcal{B}$, then $\mathcal{A}(\mathbf{0}) = \mathcal{C}$,
- for all A, y such that $\operatorname{succ} A \in \mathcal{B}$ and $y = \mathcal{A}(A)$ holds $\mathcal{A}(\operatorname{succ} A) = \mathcal{G}(A, y)$,
- for all A, L_1 such that $A \in \mathcal{B}$ and $A \neq \mathbf{0}$ and A is a limit ordinal number and $L_1 = \mathcal{A} \upharpoonright A$ holds $\mathcal{A}(A) = \mathcal{H}(A, L_1)$.

The scheme *TS_Def* deals with a constant $\mathcal{A}$ that is an ordinal number, a constant $\mathcal{B}$, a binary functor $\mathcal{F}$ and a binary functor $\mathcal{G}$ and states that:

(i) there exist x, L such that $x = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} \mathcal{A}$ and $L(\mathbf{0}) = \mathcal{B}$ and for all C, y such that $\operatorname{succ} C \in \operatorname{succ} \mathcal{A}$ and $y = L(C)$ holds $L(\operatorname{succ} C) = \mathcal{F}(C, y)$ and for all C, L_1 such that $C \in \operatorname{succ} \mathcal{A}$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \upharpoonright C$ holds $L(C) = \mathcal{G}(C, L_1)$,

(ii) for arbitrary x_1, x_2 such that there exists L such that $x_1 = \mathrm{last}\, L$ and $\mathrm{dom}\, L = \mathrm{succ}\, \mathcal{A}$ and $L(\mathbf{0}) = \mathcal{B}$ and for all C, y such that $\mathrm{succ}\, C \in \mathrm{succ}\, \mathcal{A}$ and $y = L(C)$ holds $L(\mathrm{succ}\, C) = \mathcal{F}(C, y)$ and for all C, L_1 such that $C \in \mathrm{succ}\, \mathcal{A}$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \restriction C$ holds $L(C) = \mathcal{G}(C, L_1)$ and there exists L such that $x_2 = \mathrm{last}\, L$ and $\mathrm{dom}\, L = \mathrm{succ}\, \mathcal{A}$ and $L(\mathbf{0}) = \mathcal{B}$ and for all C, y such that $\mathrm{succ}\, C \in \mathrm{succ}\, \mathcal{A}$ and $y = L(C)$ holds $L(\mathrm{succ}\, C) = \mathcal{F}(C, y)$ and for all C, L_1 such that $C \in \mathrm{succ}\, \mathcal{A}$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \restriction C$ holds $L(C) = \mathcal{G}(C, L_1)$ holds $x_1 = x_2$.
for all values of the parameters.

The scheme *TS_Result0* deals with a unary functor $\mathcal{F}$, a constant $\mathcal{A}$, a binary functor $\mathcal{G}$ and a binary functor $\mathcal{H}$ and states that:

$\mathcal{F}(\mathbf{0}) = \mathcal{A}$

provided the parameters satisfy the following condition:

- Given A, x. Then $x = \mathcal{F}(A)$ if and only if there exists L such that $x = \mathrm{last}\, L$ and $\mathrm{dom}\, L = \mathrm{succ}\, A$ and $L(\mathbf{0}) = \mathcal{A}$ and for all C, y such that $\mathrm{succ}\, C \in \mathrm{succ}\, A$ and $y = L(C)$ holds $L(\mathrm{succ}\, C) = \mathcal{G}(C, y)$ and for all C, L_1 such that $C \in \mathrm{succ}\, A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \restriction C$ holds $L(C) = \mathcal{H}(C, L_1)$.

The scheme *TS_ResultS* deals with a constant $\mathcal{A}$, a binary functor $\mathcal{F}$, a binary functor $\mathcal{G}$ and a unary functor $\mathcal{H}$ and states that:

for every A holds $\mathcal{H}(\mathrm{succ}\, A) = \mathcal{F}(A, \mathcal{H}(A))$

provided the parameters satisfy the following condition:

- Given A, x. Then $x = \mathcal{H}(A)$ if and only if there exists L such that $x = \mathrm{last}\, L$ and $\mathrm{dom}\, L = \mathrm{succ}\, A$ and $L(\mathbf{0}) = \mathcal{A}$ and for all C, y such that $\mathrm{succ}\, C \in \mathrm{succ}\, A$ and $y = L(C)$ holds $L(\mathrm{succ}\, C) = \mathcal{F}(C, y)$ and for all C, L_1 such that $C \in \mathrm{succ}\, A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \restriction C$ holds $L(C) = \mathcal{G}(C, L_1)$.

The scheme *TS_ResultL* concerns a constant $\mathcal{A}$ that is a transfinite sequence, a constant $\mathcal{B}$ that is an ordinal number, a unary functor $\mathcal{F}$, a constant $\mathcal{C}$, a binary functor $\mathcal{G}$ and a binary functor $\mathcal{H}$ and states that:

$\mathcal{F}(\mathcal{B}) = \mathcal{H}(\mathcal{B}, \mathcal{A})$

provided the parameters satisfy the following conditions:

- Given A, x. Then $x = \mathcal{F}(A)$ if and only if there exists L such that $x = \mathrm{last}\, L$ and $\mathrm{dom}\, L = \mathrm{succ}\, A$ and $L(\mathbf{0}) = \mathcal{C}$ and for all C, y such that $\mathrm{succ}\, C \in \mathrm{succ}\, A$ and $y = L(C)$ holds $L(\mathrm{succ}\, C) = \mathcal{G}(C, y)$ and for all C, L_1 such that $C \in \mathrm{succ}\, A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $L_1 = L \restriction C$ holds $L(C) = \mathcal{H}(C, L_1)$.
- $\mathcal{B} \neq \mathbf{0}$ and $\mathcal{B}$ is a limit ordinal number,
- $\mathrm{dom}\, \mathcal{A} = \mathcal{B}$,
- for every A such that $A \in \mathcal{B}$ holds $\mathcal{A}(A) = \mathcal{F}(A)$.

The scheme *OS_Exist* concerns a constant $\mathcal{A}$ that is an ordinal number, a constant $\mathcal{B}$ that is an ordinal number, a binary functor $\mathcal{F}$ yielding an ordinal number and a binary functor $\mathcal{G}$ yielding an ordinal number and states that:

there exists fi such that $\mathrm{dom}\, fi = \mathcal{A}$ but if $\mathbf{0} \in \mathcal{A}$, then $fi(\mathbf{0}) = \mathcal{B}$ and for all A, B such that $\mathrm{succ}\, A \in \mathcal{A}$ and $B = fi(A)$ holds $fi(\mathrm{succ}\, A) = \mathcal{F}(A, B)$ and

for all A, psi such that $A \in \mathcal{A}$ and $A \neq \mathbf{0}$ and A is a limit ordinal number and $psi = fi \restriction A$ holds $fi(A) = \mathcal{G}(A, psi)$.
for all values of the parameters.

The scheme *OS_Result* deals with a constant $\mathcal{A}$ that is a sequence of ordinal numbers, a unary functor $\mathcal{F}$ yielding an ordinal number, a constant $\mathcal{B}$ that is an ordinal number, a constant $\mathcal{C}$ that is an ordinal number, a binary functor $\mathcal{G}$ yielding an ordinal number and a binary functor $\mathcal{H}$ yielding an ordinal number and states that:

for every A such that $A \in \operatorname{dom} \mathcal{A}$ holds $\mathcal{A}(A) = \mathcal{F}(A)$
provided the parameters satisfy the following conditions:

- Given A, B. Then $B = \mathcal{F}(A)$ if and only if there exists fi such that $B = \operatorname{last} fi$ and $\operatorname{dom} fi = \operatorname{succ} A$ and $fi(\mathbf{0}) = \mathcal{C}$ and for all C, D such that $\operatorname{succ} C \in \operatorname{succ} A$ and $D = fi(C)$ holds $fi(\operatorname{succ} C) = \mathcal{G}(C, D)$ and for all C, psi such that $C \in \operatorname{succ} A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \mathcal{H}(C, psi)$.
- $\operatorname{dom} \mathcal{A} = \mathcal{B}$,
- if $\mathbf{0} \in \mathcal{B}$, then $\mathcal{A}(\mathbf{0}) = \mathcal{C}$,
- for all A, B such that $\operatorname{succ} A \in \mathcal{B}$ and $B = \mathcal{A}(A)$ holds $\mathcal{A}(\operatorname{succ} A) = \mathcal{G}(A, B)$,
- for all A, psi such that $A \in \mathcal{B}$ and $A \neq \mathbf{0}$ and A is a limit ordinal number and $psi = \mathcal{A} \restriction A$ holds $\mathcal{A}(A) = \mathcal{H}(A, psi)$.

The scheme *OS_Def* deals with a constant $\mathcal{A}$ that is an ordinal number, a constant $\mathcal{B}$ that is an ordinal number, a binary functor $\mathcal{F}$ yielding an ordinal number and a binary functor $\mathcal{G}$ yielding an ordinal number and states that:

(i) there exist A, fi such that $A = \operatorname{last} fi$ and $\operatorname{dom} fi = \operatorname{succ} \mathcal{A}$ and $fi(\mathbf{0}) = \mathcal{B}$ and for all C, D such that $\operatorname{succ} C \in \operatorname{succ} \mathcal{A}$ and $D = fi(C)$ holds $fi(\operatorname{succ} C) = \mathcal{F}(C, D)$ and for all C, psi such that $C \in \operatorname{succ} \mathcal{A}$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \mathcal{G}(C, psi)$,

(ii) for all A_1, A_2 such that there exists fi such that $A_1 = \operatorname{last} fi$ and $\operatorname{dom} fi = \operatorname{succ} \mathcal{A}$ and $fi(\mathbf{0}) = \mathcal{B}$ and for all C, D such that $\operatorname{succ} C \in \operatorname{succ} \mathcal{A}$ and $D = fi(C)$ holds $fi(\operatorname{succ} C) = \mathcal{F}(C, D)$ and for all C, psi such that $C \in \operatorname{succ} \mathcal{A}$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \mathcal{G}(C, psi)$ and there exists fi such that $A_2 = \operatorname{last} fi$ and $\operatorname{dom} fi = \operatorname{succ} \mathcal{A}$ and $fi(\mathbf{0}) = \mathcal{B}$ and for all C, D such that $\operatorname{succ} C \in \operatorname{succ} \mathcal{A}$ and $D = fi(C)$ holds $fi(\operatorname{succ} C) = \mathcal{F}(C, D)$ and for all C, psi such that $C \in \operatorname{succ} \mathcal{A}$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \mathcal{G}(C, psi)$ holds $A_1 = A_2$.

for all values of the parameters.

The scheme *OS_Result0* concerns a unary functor $\mathcal{F}$ yielding an ordinal number, a constant $\mathcal{A}$ that is an ordinal number, a binary functor $\mathcal{G}$ yielding an ordinal number and a binary functor $\mathcal{H}$ yielding an ordinal number and states that:

$\mathcal{F}(\mathbf{0}) = \mathcal{A}$
provided the parameters satisfy the following condition:

- Given A, B. Then $B = \mathcal{F}(A)$ if and only if there exists fi such that $B = \operatorname{last} fi$ and $\operatorname{dom} fi = \operatorname{succ} A$ and $fi(\mathbf{0}) = \mathcal{A}$ and for all C, D such that $\operatorname{succ} C \in \operatorname{succ} A$ and $D = fi(C)$ holds $fi(\operatorname{succ} C) = \mathcal{G}(C, D)$ and for all C, psi such that $C \in \operatorname{succ} A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \mathcal{H}(C, psi)$.

The scheme *OS_ResultS* deals with a constant $\mathcal{A}$ that is an ordinal number, a binary functor $\mathcal{F}$ yielding an ordinal number, a binary functor $\mathcal{G}$ yielding an ordinal number and a unary functor $\mathcal{H}$ yielding an ordinal number and states that:

for every A holds $\mathcal{H}(\operatorname{succ} A) = \mathcal{F}(A, \mathcal{H}(A))$

provided the parameters satisfy the following condition:

- Given A, B. Then $B = \mathcal{H}(A)$ if and only if there exists fi such that $B = \operatorname{last} fi$ and $\operatorname{dom} fi = \operatorname{succ} A$ and $fi(\mathbf{0}) = \mathcal{A}$ and for all C, D such that $\operatorname{succ} C \in \operatorname{succ} A$ and $D = fi(C)$ holds $fi(\operatorname{succ} C) = \mathcal{F}(C, D)$ and for all C, psi such that $C \in \operatorname{succ} A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \mathcal{G}(C, psi)$.

The scheme *OS_ResultL* deals with a constant $\mathcal{A}$ that is a sequence of ordinal numbers, a constant $\mathcal{B}$ that is an ordinal number, a unary functor $\mathcal{F}$ yielding an ordinal number, a constant $\mathcal{C}$ that is an ordinal number, a binary functor $\mathcal{G}$ yielding an ordinal number and a binary functor $\mathcal{H}$ yielding an ordinal number and states that:

$\mathcal{F}(\mathcal{B}) = \mathcal{H}(\mathcal{B}, \mathcal{A})$

provided the parameters satisfy the following conditions:

- Given A, B. Then $B = \mathcal{F}(A)$ if and only if there exists fi such that $B = \operatorname{last} fi$ and $\operatorname{dom} fi = \operatorname{succ} A$ and $fi(\mathbf{0}) = \mathcal{C}$ and for all C, D such that $\operatorname{succ} C \in \operatorname{succ} A$ and $D = fi(C)$ holds $fi(\operatorname{succ} C) = \mathcal{G}(C, D)$ and for all C, psi such that $C \in \operatorname{succ} A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \mathcal{H}(C, psi)$.
- $\mathcal{B} \neq \mathbf{0}$ and $\mathcal{B}$ is a limit ordinal number,
- $\operatorname{dom} \mathcal{A} = \mathcal{B}$,
- for every A such that $A \in \mathcal{B}$ holds $\mathcal{A}(A) = \mathcal{F}(A)$.

We now define two new functors. Let us consider L. The functor $\sup L$ yields an ordinal number and is defined by:

$\sup L = \sup(\operatorname{rng} L)$.

The functor $\inf L$ yielding an ordinal number, is defined by:

$\inf L = \inf(\operatorname{rng} L)$.

One can prove the following proposition

(35) $\sup L = \sup(\operatorname{rng} L)$ and $\inf L = \inf(\operatorname{rng} L)$.

We now define two new functors. Let us consider L. The functor $\operatorname{limsup} L$ yielding an ordinal number, is defined by:

there exists fi such that $\operatorname{limsup} L = \inf fi$ and $\operatorname{dom} fi = \operatorname{dom} L$ and for every A such that $A \in \operatorname{dom} L$ holds $fi(A) = \sup(\operatorname{rng}(L \restriction (\operatorname{dom} L \setminus A)))$.

The functor $\operatorname{liminf} L$ yields an ordinal number and is defined by:

there exists fi such that $\operatorname{liminf} L = \sup fi$ and $\operatorname{dom} fi = \operatorname{dom} L$ and for every A such that $A \in \operatorname{dom} L$ holds $fi(A) = \inf(\operatorname{rng}(L \restriction (\operatorname{dom} L \setminus A)))$.

One can prove the following propositions:

(36) $A = \limsup L$ if and only if there exists fi such that $A = \inf fi$ and $\mathrm{dom}\, fi = \mathrm{dom}\, L$ and for every B such that $B \in \mathrm{dom}\, L$ holds $fi(B) = \sup(\mathrm{rng}(L \restriction (\mathrm{dom}\, L \setminus B)))$.

(37) $A = \liminf L$ if and only if there exists fi such that $A = \sup fi$ and $\mathrm{dom}\, fi = \mathrm{dom}\, L$ and for every B such that $B \in \mathrm{dom}\, L$ holds $fi(B) = \inf(\mathrm{rng}(L \restriction (\mathrm{dom}\, L \setminus B)))$.

Let us consider A, fi. The predicate A is the limit of fi is defined by:

there exists B such that $B \in \mathrm{dom}\, fi$ and for every C such that $B \subseteq C$ and $C \in \mathrm{dom}\, fi$ holds $fi(C) = \mathbf{0}$ if $A = \mathbf{0}$, for all B, C such that $B \in A$ and $A \in C$ there exists D such that $D \in \mathrm{dom}\, fi$ and for every ordinal number E such that $D \subseteq E$ and $E \in \mathrm{dom}\, fi$ holds $B \in fi(E)$ and $fi(E) \in C$, otherwise.

One can prove the following propositions:

(38) If $A = \mathbf{0}$, then A is the limit of fi if and only if there exists B such that $B \in \mathrm{dom}\, fi$ and for every C such that $B \subseteq C$ and $C \in \mathrm{dom}\, fi$ holds $fi(C) = \mathbf{0}$.

(39) If $A \neq \mathbf{0}$, then A is the limit of fi if and only if for all B, C such that $B \in A$ and $A \in C$ there exists D such that $D \in \mathrm{dom}\, fi$ and for every ordinal number E such that $D \subseteq E$ and $E \in \mathrm{dom}\, fi$ holds $B \in fi(E)$ and $fi(E) \in C$.

Let us consider fi. Let us assume that there exists A such that A is the limit of fi. The functor $\lim fi$ yielding an ordinal number, is defined by:

$\lim fi$ is the limit of fi.

Let us consider A, fi. Let us assume that $A \in \mathrm{dom}\, fi$. The functor $\lim_A fi$ yields an ordinal number and is defined by:

$\lim_A fi = \lim fi \restriction A$.

Next we state two propositions:

(40) If A is the limit of fi, then $\lim fi = A$.

(41) If $A \in \mathrm{dom}\, fi$, then $\lim_A fi = \lim fi \restriction A$.

We now define two new predicates. Let L be a sequence of ordinal numbers. The predicate L is increasing is defined by:

for all A, B such that $A \in B$ and $B \in \mathrm{dom}\, L$ holds $L(A) \in L(B)$.

The predicate L is continuous is defined by:

for all A, B such that $A \in \mathrm{dom}\, L$ and $A \neq \mathbf{0}$ and A is a limit ordinal number and $B = L(A)$ holds B is the limit of $L \restriction A$.

We now state two propositions:

(42) fi is increasing if and only if for all A, B such that $A \in B$ and $B \in \mathrm{dom}\, fi$ holds $fi(A) \in fi(B)$.

(43) fi is continuous if and only if for all A, B such that $A \in \mathrm{dom}\, fi$ and $A \neq \mathbf{0}$ and A is a limit ordinal number and $B = fi(A)$ holds B is the limit of $fi \restriction A$.

Let us consider A, B. The functor $A+B$ yielding an ordinal number, is defined by:

there exists fi such that $A+B = \operatorname{last} fi$ and $\operatorname{dom} fi = \operatorname{succ} B$ and $fi(\mathbf{0}) = A$ and for all C, D such that $\operatorname{succ} C \in \operatorname{succ} B$ and $D = fi(C)$ holds $fi(\operatorname{succ} C) = \operatorname{succ} D$ and for all C, psi such that $C \in \operatorname{succ} B$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \sup psi$.

Let us consider A, B. The functor $A \cdot B$ yielding an ordinal number, is defined by:

there exists fi such that $A \cdot B = \operatorname{last} fi$ and $\operatorname{dom} fi = \operatorname{succ} A$ and $fi(\mathbf{0}) = \mathbf{0}$ and for all C, D such that $\operatorname{succ} C \in \operatorname{succ} A$ and $D = fi(C)$ holds $fi(\operatorname{succ} C) = D + B$ and for all C, psi such that $C \in \operatorname{succ} A$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \bigcup \sup psi$.

Let us consider A, B. The functor A^B yields an ordinal number and is defined by:

there exists fi such that $A^B = \operatorname{last} fi$ and $\operatorname{dom} fi = \operatorname{succ} B$ and $fi(\mathbf{0}) = \mathbf{1}$ and for all C, D such that $\operatorname{succ} C \in \operatorname{succ} B$ and $D = fi(C)$ holds $fi(\operatorname{succ} C) = A \cdot D$ and for all C, psi such that $C \in \operatorname{succ} B$ and $C \neq \mathbf{0}$ and C is a limit ordinal number and $psi = fi \restriction C$ holds $fi(C) = \lim psi$.

The following propositions are true:

(44) $A + \mathbf{0} = A$.

(45) $A + \operatorname{succ} B = \operatorname{succ}(A + B)$.

(46) If $B \neq \mathbf{0}$ and B is a limit ordinal number, then for every fi such that $\operatorname{dom} fi = B$ and for every C such that $C \in B$ holds $fi(C) = A + C$ holds $A + B = \sup fi$.

(47) $\mathbf{0} + A = A$.

(48) $A + \mathbf{1} = \operatorname{succ} A$.

(49) If $A \in B$, then $C + A \in C + B$.

(50) If $A \subseteq B$, then $C + A \subseteq C + B$.

(51) If $A \subseteq B$, then $A + C \subseteq B + C$.

(52) $\mathbf{0} \cdot A = \mathbf{0}$.

(53) $\operatorname{succ} B \cdot A = B \cdot A + A$.

(54) If $B \neq \mathbf{0}$ and B is a limit ordinal number, then for every fi such that $\operatorname{dom} fi = B$ and for every C such that $C \in B$ holds $fi(C) = C \cdot A$ holds $B \cdot A = \bigcup \sup fi$.

(55) $A \cdot \mathbf{0} = \mathbf{0}$.

(56) $\mathbf{1} \cdot A = A$ and $A \cdot \mathbf{1} = A$.

(57) If $C \neq \mathbf{0}$ and $A \in B$, then $A \cdot C \in B \cdot C$.

(58) If $A \subseteq B$, then $A \cdot C \subseteq B \cdot C$.

(59) If $A \subseteq B$, then $C \cdot A \subseteq C \cdot B$.

(60) $A^{\mathbf{0}} = \mathbf{1}$.

(61) $A^{\operatorname{succ} B} = A \cdot (A^B)$.

(62) If $B \neq \mathbf{0}$ and B is a limit ordinal number, then for every fi such that $\operatorname{dom} fi = B$ and for every C such that $C \in B$ holds $fi(C) = A^C$ holds $A^B = \lim fi$.

(63) $A^{\mathbf{1}} = A$ and $\mathbf{1}^A = \mathbf{1}$.

Let us consider A. The predicate A is natural is defined by:

$A \in \omega$.

One can prove the following propositions:

(64) A is natural if and only if $A \in \omega$.

(65) For every A there exist B, C such that B is a limit ordinal number and C is natural and $A = B + C$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Kazimierz Kuratowski and Andrzej Mostowski. *Teoria mnogości.* PTM, Wrocław, 1952.

[4] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[5] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

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Vectors in Real Linear Space

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Summary. In this article we introduce a notion of real linear space, operations on vectors: addition, multiplication by real number, inverse vector, substraction. The sum of finite sequence of the vectors is also defined. Theorems that belong rather to [1] or [2] are proved.

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The notation and terminology used here have been introduced in the following articles: [7], [4], [5], [3], [6], [2], and [1]. We consider RLS structures which are systems

$\langle$ vectors, a zero, an addition, a multiplication $\rangle$

where the vectors is a non-empty set, the zero is an element of the vectors, the addition is a binary operation on the vectors, and the multiplication is a function from $[:\mathbb{R},$ the vectors$:]$ into the vectors. In the sequel V will denote an RLS structure, v will denote an element of the vectors of V, and x will be arbitrary. Let us consider V. A vector of V is an element of the vectors of V.

Next we state a proposition

(1) v is a vector of V.

Let us consider V, x. The predicate $x \in V$ is defined by:

$x \in$the vectors of V.

Next we state two propositions:

(2) $x \in V$ if and only if $x \in$the vectors of V.

(3) $v \in V$.

Let us consider V. The functor 0_V yielding a vector of V, is defined by:

$0_V =$the zero of V.

In the sequel v, w will denote vectors of V and a, b will denote real numbers. Let us consider V, v, w. The functor $v + w$ yields a vector of V and is defined by:

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$v + w =$(the addition of V)($\langle v, w \rangle$).

Let us consider V, v, a. The functor $a \cdot v$ yielding a vector of V, is defined by: $a \cdot v =$(the multiplication of V)($\langle a, v \rangle$).

We now state three propositions:

(4) $0_V =$the zero of V.

(5) $v + w =$(the addition of V)($\langle v, w \rangle$).

(6) $a \cdot v =$(the multiplication of V)($\langle a, v \rangle$).

The mode real linear space, which widens to the type an RLS structure, is defined by:

(i) for all vectors v, w of it holds $v + w = w + v$,
(ii) for all vectors u, v, w of it holds $(u + v) + w = u + (v + w)$,
(iii) for every vector v of it holds $v + 0_{\rm it} = v$,
(iv) for every vector v of it there exists w being a vector of it such that $v + w = 0_{\rm it}$,
(v) for every a for all vectors v, w of it holds $a \cdot (v + w) = a \cdot v + a \cdot w$,
(vi) for all a, b for every vector v of it holds $(a + b) \cdot v = a \cdot v + b \cdot v$,
(vii) for all a, b for every vector v of it holds $(a \cdot b) \cdot v = a \cdot (b \cdot v)$,
(viii) for every vector v of it holds $1 \cdot v = v$.

Next we state a proposition

(7) Suppose that
 (i) for all vectors v, w of V holds $v + w = w + v$,
 (ii) for all vectors u, v, w of V holds $(u + v) + w = u + (v + w)$,
 (iii) for every vector v of V holds $v + 0_V = v$,
 (iv) for every vector v of V there exists w being a vector of V such that $v + w = 0_V$,
 (v) for every a for all vectors v, w of V holds $a \cdot (v + w) = a \cdot v + a \cdot w$,
 (vi) for all a, b for every vector v of V holds $(a + b) \cdot v = a \cdot v + b \cdot v$,
 (vii) for all a, b for every vector v of V holds $(a \cdot b) \cdot v = a \cdot (b \cdot v)$,
 (viii) for every vector v of V holds $1 \cdot v = v$.

 Then V is a real linear space.

We follow the rules: V denotes a real linear space and u, v, v_1, v_2, w denote vectors of V. The following propositions are true:

(8) $v + w = w + v$.

(9) $(u + v) + w = u + (v + w)$.

(10) $v + 0_V = v$ and $0_V + v = v$.

(11) There exists w such that $v + w = 0_V$.

(12) $a \cdot (v + w) = a \cdot v + a \cdot w$.

(13) $(a + b) \cdot v = a \cdot v + b \cdot v$.

(14) $(a \cdot b) \cdot v = a \cdot (b \cdot v)$.

(15) $1 \cdot v = v$.

Let us consider V, v. The functor $-v$ yields a vector of V and is defined by: $v + (-v) = 0_V$.

Let us consider V, v, w. The functor $v - w$ yields a vector of V and is defined by:

$v - w = v + (-w)$.

Next we state a number of propositions:

(16) $v + (-v) = 0_V$.
(17) If $v + w = 0_V$, then $w = -v$.
(18) $v - w = v + (-w)$.
(19) If $v + w = 0_V$, then $v = -w$.
(20) There exists w such that $v + w = u$.
(21) If $w + v_1 = u$ and $w + v_2 = u$, then $v_1 = v_2$.
(22) If $v + w = v$, then $w = 0_V$.
(23) If $a = 0$ or $v = 0_V$, then $a \cdot v = 0_V$.
(24) If $a \cdot v = 0_V$, then $a = 0$ or $v = 0_V$.
(25) $-0_V = 0_V$.
(26) $v - 0_V = v$.
(27) $0_V - v = -v$.
(28) $v - v = 0_V$.
(29) $-v = (-1) \cdot v$.
(30) $-(-v) = v$.
(31) If $-v = -w$, then $v = w$.
(32) If $v = -w$, then $-v = w$.
(33) If $v = -v$, then $v = 0_V$.
(34) If $v + v = 0_V$, then $v = 0_V$.
(35) If $v - w = 0_V$, then $v = w$.
(36) There exists w such that $v - w = u$.
(37) If $w - v_1 = u$ and $w - v_2 = u$, then $v_1 = v_2$.
(38) $a \cdot (-v) = (-a) \cdot v$.
(39) $a \cdot (-v) = -a \cdot v$.
(40) $(-a) \cdot (-v) = a \cdot v$.
(41) $v - (u + w) = (v - u) - w$.
(42) $(v + u) - w = v + (u - w)$.
(43) $v - (u - w) = (v - u) + w$.
(44) $-(v + w) = (-v) - w$.
(45) $-(v + w) = (-v) + (-w)$.
(46) $(-v) - w = (-w) - v$.
(47) $-(v - w) = (-v) + w$.
(48) $a \cdot (v - w) = a \cdot v - a \cdot w$.
(49) $(a - b) \cdot v = a \cdot v - b \cdot v$.
(50) If $a \neq 0$ and $a \cdot v = a \cdot w$, then $v = w$.
(51) If $v \neq 0_V$ and $a \cdot v = b \cdot v$, then $a = b$.

For simplicity we adopt the following convention: F, G denote finite sequences of elements of the vectors of V, f denotes a function from $\mathbb{N}$ into the vectors of V, j, k, n denote natural numbers, and p, q denote finite sequences. Let us consider V, f, j. Then $f(j)$ is a vector of V.

Let us consider V, v, u. Then $\langle v, u \rangle$ is a finite sequence of elements of the vectors of V.

Let us consider V, v, u, w. Then $\langle v, u, w \rangle$ is a finite sequence of elements of the vectors of V.

Let us consider V, F. The functor $\sum F$ yields a vector of V and is defined by:

there exists f such that $\sum F = f(\operatorname{len} F)$ and $f(0) = 0_V$ and for all j, v such that $j < \operatorname{len} F$ and $v = F(j+1)$ holds $f(j+1) = f(j) + v$.

The following propositions are true:

(52) If there exists f such that $u = f(\operatorname{len} F)$ and $f(0) = 0_V$ and for all j, v such that $j < \operatorname{len} F$ and $v = F(j+1)$ holds $f(j+1) = f(j) + v$, then $u = \sum F$.

(53) There exists f such that $\sum F = f(\operatorname{len} F)$ and $f(0) = 0_V$ and for all j, v such that $j < \operatorname{len} F$ and $v = F(j+1)$ holds $f(j+1) = f(j) + v$.

(54) If $k \in \operatorname{Seg} n$ and $\operatorname{len} F = n$, then $F(k)$ is a vector of V.

(55) If $\operatorname{len} F = \operatorname{len} G + 1$ and $G = F \restriction \operatorname{Seg}(\operatorname{len} G)$ and $v = F(\operatorname{len} F)$, then $\sum F = \sum G + v$.

(56) If $\operatorname{len} F = \operatorname{len} G$ and for all k, v such that $k \in \operatorname{Seg}(\operatorname{len} F)$ and $v = G(k)$ holds $F(k) = a \cdot v$, then $\sum F = a \cdot \sum G$.

(57) If $\operatorname{len} F = \operatorname{len} G$ and for all k, v such that $k \in \operatorname{Seg}(\operatorname{len} F)$ and $v = G(k)$ holds $F(k) = -v$, then $\sum F = -\sum G$.

(58) $\sum (F \frown G) = \sum F + \sum G$.

(59) If $\operatorname{rng} F = \operatorname{rng} G$ and F is one-to-one and G is one-to-one, then $\sum F = \sum G$.

(60) $\sum \varepsilon_{(\text{the vectors of } V)} = 0_V$.

(61) $\sum \langle v \rangle = v$.

(62) $\sum \langle v, u \rangle = v + u$.

(63) $\sum \langle v, u, w \rangle = (v + u) + w$.

(64) $a \cdot \sum \varepsilon_{(\text{the vectors of } V)} = 0_V$.

(65) $a \cdot \sum \langle v \rangle = a \cdot v$.

(66) $a \cdot \sum \langle v, u \rangle = a \cdot v + a \cdot u$.

(67) $a \cdot \sum \langle v, u, w \rangle = (a \cdot v + a \cdot u) + a \cdot w$.

(68) $-\sum \varepsilon_{(\text{the vectors of } V)} = 0_V$.

(69) $-\sum \langle v \rangle = -v$.

(70) $-\sum \langle v, u \rangle = (-v) - u$.

(71) $-\sum \langle v, u, w \rangle = ((-v) - u) - w$.

(72) $\sum \langle v, w \rangle = \sum \langle w, v \rangle$.

(73) $\sum \langle v, w \rangle = \sum \langle v \rangle + \sum \langle w \rangle$.

(74) $\sum\langle 0_V, 0_V\rangle = 0_V$.

(75) $\sum\langle 0_V, v\rangle = v$ and $\sum\langle v, 0_V\rangle = v$.

(76) $\sum\langle v, -v\rangle = 0_V$ and $\sum\langle -v, v\rangle = 0_V$.

(77) $\sum\langle v, -w\rangle = v - w$ and $\sum\langle -w, v\rangle = v - w$.

(78) $\sum\langle -v, -w\rangle = -(v + w)$ and $\sum\langle -w, -v\rangle = -(v + w)$.

(79) $\sum\langle v, v\rangle = 2 \cdot v$.

(80) $\sum\langle -v, -v\rangle = (-2) \cdot v$.

(81) $\sum\langle u, v, w\rangle = (\sum\langle u\rangle + \sum\langle v\rangle) + \sum\langle w\rangle$.

(82) $\sum\langle u, v, w\rangle = \sum\langle u, v\rangle + w$.

(83) $\sum\langle u, v, w\rangle = \sum\langle v, w\rangle + u$.

(84) $\sum\langle u, v, w\rangle = \sum\langle u, w\rangle + v$.

(85) $\sum\langle u, v, w\rangle = \sum\langle u, w, v\rangle$.

(86) $\sum\langle u, v, w\rangle = \sum\langle v, u, w\rangle$.

(87) $\sum\langle u, v, w\rangle = \sum\langle v, w, u\rangle$.

(88) $\sum\langle u, v, w\rangle = \sum\langle w, u, v\rangle$.

(89) $\sum\langle u, v, w\rangle = \sum\langle w, v, u\rangle$.

(90) $\sum\langle 0_V, 0_V, 0_V\rangle = 0_V$.

(91) $\sum\langle 0_V, 0_V, v\rangle = v$ and $\sum\langle 0_V, v, 0_V\rangle = v$ and $\sum\langle v, 0_V, 0_V\rangle = v$.

(92) $\sum\langle 0_V, u, v\rangle = u + v$ and $\sum\langle u, v, 0_V\rangle = u + v$ and $\sum\langle u, 0_V, v\rangle = u + v$.

(93) $\sum\langle v, v, v\rangle = 3 \cdot v$.

(94) If $\operatorname{len} F = 0$, then $\sum F = 0_V$.

(95) If $\operatorname{len} F = 1$, then $\sum F = F(1)$.

(96) If $\operatorname{len} F = 2$ and $v_1 = F(1)$ and $v_2 = F(2)$, then $\sum F = v_1 + v_2$.

(97) If $\operatorname{len} F = 3$ and $v_1 = F(1)$ and $v_2 = F(2)$ and $v = F(3)$, then $\sum F = (v_1 + v_2) + v$.

(98) If $j < 1$, then $j = 0$.

(99) $1 \leq k$ if and only if $k \neq 0$.

(100) $k \leq k + n$ and $k \leq n + k$.

(101) $k < k + 1$ and $k < 1 + k$.

(102) If $k \neq 0$, then $n < n + k$ and $n < k + n$.

(103) $k < k + n$ if and only if $1 \leq n$.

(104) $\operatorname{Seg} k = \operatorname{Seg}(k + 1) \setminus \{k + 1\}$.

(105) $p = (p \frown q) \restriction \operatorname{Seg}(\operatorname{len} p)$.

(106) If $\operatorname{rng} p = \operatorname{rng} q$ and p is one-to-one and q is one-to-one, then $\operatorname{len} p = \operatorname{len} q$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[7] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

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Subspaces and Cosets of Subspaces in Real Linear Space

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Summary. The following notions are introduced in the article: subspace of a real linear space, zero subspace and improper subspace, coset of a subspace. The relation of a subset of the vectors being linearly closed is also introduced. Basic theorems concerning those notions are proved in the article.

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The papers [4], [2], [6], [3], [1], and [5] provide the terminology and notation for this paper. For simplicity we follow a convention: V, X, Y are real linear spaces, u, v, v_1, v_2 are vectors of V, a is a real number, V_1, V_2, V_3 are subsets of the vectors of V, and x be arbitrary. Let us consider V, V_1. The predicate V_1 is linearly closed is defined by:

for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v + u \in V_1$ and for all a, v such that $v \in V_1$ holds $a \cdot v \in V_1$.

Next we state a number of propositions:

(1) If for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v + u \in V_1$ and for all a, v such that $v \in V_1$ holds $a \cdot v \in V_1$, then V_1 is linearly closed.

(2) If V_1 is linearly closed, then for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v + u \in V_1$.

(3) If V_1 is linearly closed, then for all a, v such that $v \in V_1$ holds $a \cdot v \in V_1$.

(4) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then $0_V \in V_1$.

(5) If V_1 is linearly closed, then for every v such that $v \in V_1$ holds $-v \in V_1$.

(6) If V_1 is linearly closed, then for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v - u \in V_1$.

(7) $\{0_V\}$ is linearly closed.

(8) If the vectors of $V = V_1$, then V_1 is linearly closed.

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(9) If V_1 is linearly closed and V_2 is linearly closed and $V_3 = \{v + u : v \in V_1 \wedge u \in V_2\}$, then V_3 is linearly closed.

(10) If V_1 is linearly closed and V_2 is linearly closed, then $V_1 \cap V_2$ is linearly closed.

Let us consider V. The mode subspace of V, which widens to the type a real linear space, is defined by:

the vectors of it $\subseteq$the vectors of V and the zero of it $=$the zero of V and the addition of it $=$(the addition of V) $\restriction$ [: the vectors of it, the vectors of it :] and the multiplication of it $=$(the multiplication of V) $\restriction$ [: $\mathbb{R}$, the vectors of it :].

Next we state a proposition

(11) If the vectors of X $\subseteq$the vectors of V and the zero of X $=$the zero of V and the addition of X $=$(the addition of V) $\restriction$ [: the vectors of X, the vectors of X :] and the multiplication of X $=$(the multiplication of V) $\restriction$ [: $\mathbb{R}$, the vectors of X :], then X is a subspace of V.

We follow a convention: W, W_1, W_2 will denote subspaces of V and w, w_1, w_2 will denote vectors of W. We now state a number of propositions:

(12) the vectors of W $\subseteq$the vectors of V.

(13) the zero of W $=$the zero of V.

(14) the addition of W $=$(the addition of V) $\restriction$ [: the vectors of W, the vectors of W :].

(15) the multiplication of W $=$(the multiplication of V) $\restriction$ [: $\mathbb{R}$, the vectors of W :].

(16) If $x \in W_1$ and W_1 is a subspace of W_2, then $x \in W_2$.

(17) If $x \in W$, then $x \in V$.

(18) w is a vector of V.

(19) $0_W = 0_V$.

(20) $0_{W_1} = 0_{W_2}$.

(21) If $w_1 = v$ and $w_2 = u$, then $w_1 + w_2 = v + u$.

(22) If $w = v$, then $a \cdot w = a \cdot v$.

(23) If $w = v$, then $-v = -w$.

(24) If $w_1 = v$ and $w_2 = u$, then $w_1 - w_2 = v - u$.

(25) $0_V \in W$.

(26) $0_{W_1} \in W_2$.

(27) $0_W \in V$.

(28) If $u \in W$ and $v \in W$, then $u + v \in W$.

(29) If $v \in W$, then $a \cdot v \in W$.

(30) If $v \in W$, then $-v \in W$.

(31) If $u \in W$ and $v \in W$, then $u - v \in W$.

In the sequel D is a non-empty set, d_1 is an element of D, A is a binary operation on D, and M is a function from [: $\mathbb{R}$, D :] into D. We now state a number of propositions:

(32) If $V_1 = D$ and $d_1 = 0_V$ and $A =$(the addition of V) $\restriction [\!: V_1, V_1 :\!]$ and $M =$(the multiplication of V) $\restriction [\!: \mathbb{R}, V_1 :\!]$, then $\langle D, d_1, A, M \rangle$ is a subspace of V.

(33) V is a subspace of V.

(34) If V is a subspace of X and X is a subspace of V, then $V = X$.

(35) If V is a subspace of X and X is a subspace of Y, then V is a subspace of Y.

(36) If the vectors of $W_1 \subseteq$the vectors of W_2, then W_1 is a subspace of W_2.

(37) If for every v such that $v \in W_1$ holds $v \in W_2$, then W_1 is a subspace of W_2.

(38) If the vectors of $W_1 =$the vectors of W_2, then $W_1 = W_2$.

(39) If for every v holds $v \in W_1$ if and only if $v \in W_2$, then $W_1 = W_2$.

(40) If the vectors of $W =$the vectors of V, then $W = V$.

(41) If for every v holds $v \in W$ if and only if $v \in V$, then $W = V$.

(42) If the vectors of $W = V_1$, then V_1 is linearly closed.

(43) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then there exists W such that $V_1 =$the vectors of W.

Let us consider V. The functor $\mathbf{0}_V$ yielding a subspace of V, is defined by: the vectors of $\mathbf{0}_V = \{0_V\}$.

Let us consider V. The functor Ω_V yielding a subspace of V, is defined by: $\Omega_V = V$.

We now state a number of propositions:

(44) the vectors of $\mathbf{0}_V = \{0_V\}$.

(45) If the vectors of $W = \{0_V\}$, then $W = \mathbf{0}_V$.

(46) $\Omega_V = V$.

(47) $\Omega_V = \mathbf{0}_V$ if and only if $V = \mathbf{0}_V$.

(48) $\mathbf{0}_W = \mathbf{0}_V$.

(49) $\mathbf{0}_{W_1} = \mathbf{0}_{W_2}$.

(50) $\mathbf{0}_W$ is a subspace of V.

(51) $\mathbf{0}_V$ is a subspace of W.

(52) $\mathbf{0}_{W_1}$ is a subspace of W_2.

(53) W is a subspace of Ω_V.

(54) V is a subspace of Ω_V.

Let us consider V, v, W. The functor $v + W$ yielding a subset of the vectors of V, is defined by:

$v + W = \{v + u : u \in W\}$.

Let us consider V, W. The mode coset of W, which widens to the type a subset of the vectors of V, is defined by:

there exists v such that it $= v + W$.

In the sequel B, C will be cosets of W. We now state a number of propositions:

(55) $v + W = \{v + u : u \in W\}$.

(56) There exists v such that $C = v + W$.
(57) If $V_1 = v + W$, then V_1 is a coset of W.
(58) $0_V \in v + W$ if and only if $v \in W$.
(59) $v \in v + W$.
(60) $0_V + W =$the vectors of W.
(61) $v + \mathbf{0}_V = \{v\}$.
(62) $v + \Omega_V =$the vectors of V.
(63) $0_V \in v + W$ if and only if $v + W =$the vectors of W.
(64) $v \in W$ if and only if $v + W =$the vectors of W.
(65) If $v \in W$, then $a \cdot v + W =$the vectors of W.
(66) If $a \neq 0$ and $a \cdot v + W =$the vectors of W, then $v \in W$.
(67) $v \in W$ if and only if $(-v) + W =$the vectors of W.
(68) $u \in W$ if and only if $v + W = (v + u) + W$.
(69) $u \in W$ if and only if $v + W = (v - u) + W$.
(70) $v \in u + W$ if and only if $u + W = v + W$.
(71) $v + W = (-v) + W$ if and only if $v \in W$.
(72) If $u \in v_1 + W$ and $u \in v_2 + W$, then $v_1 + W = v_2 + W$.
(73) If $u \in v + W$ and $u \in (-v) + W$, then $v \in W$.
(74) If $a \neq 1$ and $a \cdot v \in v + W$, then $v \in W$.
(75) If $v \in W$, then $a \cdot v \in v + W$.
(76) $-v \in v + W$ if and only if $v \in W$.
(77) $u + v \in v + W$ if and only if $u \in W$.
(78) $v - u \in v + W$ if and only if $u \in W$.
(79) $u \in v + W$ if and only if there exists v_1 such that $v_1 \in W$ and $u = v + v_1$.
(80) $u \in v + W$ if and only if there exists v_1 such that $v_1 \in W$ and $u = v - v_1$.
(81) There exists v such that $v_1 \in v + W$ and $v_2 \in v + W$ if and only if $v_1 - v_2 \in W$.
(82) If $v + W = u + W$, then there exists v_1 such that $v_1 \in W$ and $v + v_1 = u$.
(83) If $v + W = u + W$, then there exists v_1 such that $v_1 \in W$ and $v - v_1 = u$.
(84) $v + W_1 = v + W_2$ if and only if $W_1 = W_2$.
(85) If $v + W_1 = u + W_2$, then $W_1 = W_2$.

In the sequel C_1 denotes a coset of W_1 and C_2 denotes a coset of W_2. We now state a number of propositions:

(86) C is linearly closed if and only if $C =$the vectors of W.
(87) If $C_1 = C_2$, then $W_1 = W_2$.
(88) $\{v\}$ is a coset of $\mathbf{0}_V$.
(89) If V_1 is a coset of $\mathbf{0}_V$, then there exists v such that $V_1 = \{v\}$.
(90) the vectors of W is a coset of W.
(91) the vectors of V is a coset of Ω_V.
(92) If V_1 is a coset of Ω_V, then $V_1 =$the vectors of V.

(93) $0_V \in C$ if and only if $C =$the vectors of W.

(94) $u \in C$ if and only if $C = u + W$.

(95) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u+v_1 = v$.

(96) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u-v_1 = v$.

(97) There exists C such that $v_1 \in C$ and $v_2 \in C$ if and only if $v_1 - v_2 \in W$.

(98) If $u \in B$ and $u \in C$, then $B = C$.

References

[1] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[2] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[3] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[4] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[5] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[6] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

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A First Order Language

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Summary. In the paper a first order language is constructed. It includes the universal quantifier and the following propositional connectives: truth, negation, and conjunction. The variables are divided into three kinds: bound variables, fixed variables, and free variables. An infinite number of predicates for each arity is provided. Schemes of structural induction and schemes justifying definitions by structural induction have been proved. The concept of a closed formula (a formula without free occurrences of bound variables) is introduced.

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The articles [7], [8], [5], [1], [3], [4], [6], and [2] provide the notation and terminology for this paper. The following propositions are true:

(1) For all non-empty sets D_1, D_2 for every element k of D_1 holds $[\!:\{k\}, D_2:\!] \subseteq [\!:D_1, D_2:\!]$.

(2) For all non-empty sets D_1, D_2 for all elements k_1, k_2, k_3 of D_1 holds $[\!:\{k_1, k_2, k_3\}, D_2:\!] \subseteq [\!:D_1, D_2:\!]$.

In the sequel k, l denote natural numbers. The constant Var is a non-empty set and is defined by:

$\mathrm{Var} = [\!:\{4, 5, 6\}, \mathbb{N}:\!]$.

Next we state two propositions:

(3) $\mathrm{Var} = [\!:\{4, 5, 6\}, \mathbb{N}:\!]$.

(4) $\mathrm{Var} \subseteq [\!:\mathbb{N}, \mathbb{N}:\!]$.

We now define five new constructions. A variable is an element of Var.

The constant BoundVar is a non-empty subset of Var and is defined by:

$\mathrm{BoundVar} = [\!:\{4\}, \mathbb{N}:\!]$.

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The constant FixedVar is a non-empty subset of Var and is defined by:
FixedVar $= [: \{5\}, \mathbb{N} :]$.
The constant FreeVar is a non-empty subset of Var and is defined by:
FreeVar $= [: \{6\}, \mathbb{N} :]$.
The constant PredSym is a non-empty set and is defined by:
PredSym $= \{\langle k, l \rangle : 7 \leq k\}$.

The following propositions are true:

(5) For every element IT of Var holds IT is a variable.

(6) BoundVar $= [: \{4\}, \mathbb{N} :]$.

(7) FixedVar $= [: \{5\}, \mathbb{N} :]$.

(8) FreeVar $= [: \{6\}, \mathbb{N} :]$.

(9) PredSym $= \{\langle k, l \rangle : 7 \leq k\}$.

(10) PredSym $\subseteq [: \mathbb{N}, \mathbb{N} :]$.

A predicate symbol is an element of PredSym.

The following proposition is true

(11) For every element IT of PredSym holds IT is a predicate symbol.

Let P be an element of PredSym. The functor Arity(P) yielding a natural number, is defined by:
$P_{\mathbf{1}} = 7 + \text{Arity}(P)$.

Next we state a proposition

(12) For every predicate symbol P for every natural number IT holds $IT =$ Arity(P) if and only if $P_{\mathbf{1}} = 7 + IT$.

In the sequel P will denote a predicate symbol. Let us consider k. The functor PredSym$_k$ yields a non-empty subset of PredSym and is defined by:
PredSym$_k = \{P : \text{Arity}(P) = k\}$.

Next we state a proposition

(13) For every natural number k for every non-empty subset IT of PredSym holds $IT =$ PredSym$_k$ if and only if $IT = \{P : \text{Arity}(P) = k\}$.

We now define four new modes. A bound variable is an element of BoundVar.
A fixed variable is an element of FixedVar.
A free variable is an element of FreeVar.
Let us consider k. A k-ary predicate symbol is an element of PredSym$_k$.

One can prove the following four propositions:

(14) For every element IT of BoundVar holds IT is a bound variable.

(15) For every element IT of FixedVar holds IT is a fixed variable.

(16) For every element IT of FreeVar holds IT is a free variable.

(17) For every natural number k for every element IT of PredSym$_k$ holds IT is a k-ary predicate symbol.

Let k be a natural number. The mode list of variables of the length k, which widens to the type a finite sequence of elements of Var, is defined by:
len it $= k$.

One can prove the following proposition

(18) For every natural number k for every finite sequence IT of elements of Var holds IT is a list of variables of the length k if and only if $\operatorname{len} IT = k$.

Let D be a non-empty set. The predicate D is closed is defined by:

(i) D is a subset of $[:\mathbb{N}, \mathbb{N}:]^*$,

(ii) for every natural number k for every k-ary predicate symbol p for every list of variables ll of the length k holds $\langle p \rangle \frown ll \in D$,

(iii) $\langle \langle 0, 0 \rangle \rangle \in D$,

(iv) for every finite sequence p of elements of $[:\mathbb{N}, \mathbb{N}:]$ such that $p \in D$ holds $\langle \langle 1, 0 \rangle \rangle \frown p \in D$,

(v) for all finite sequences p, q of elements of $[:\mathbb{N}, \mathbb{N}:]$ such that $p \in D$ and $q \in D$ holds $(\langle \langle 2, 0 \rangle \rangle \frown p) \frown q \in D$,

(vi) for every bound variable x for every finite sequence p of elements of $[:\mathbb{N}, \mathbb{N}:]$ such that $p \in D$ holds $(\langle \langle 3, 0 \rangle \rangle \frown \langle x \rangle) \frown p \in D$.

We now state a proposition

(19) Let D be a non-empty set. Then D is closed if and only if the following conditions are satisfied:

  (i) D is a subset of $[:\mathbb{N}, \mathbb{N}:]^*$,

  (ii) for every natural number k for every k-ary predicate symbol p for every list of variables ll of the length k holds $\langle p \rangle \frown ll \in D$,

  (iii) $\langle \langle 0, 0 \rangle \rangle \in D$,

  (iv) for every finite sequence p of elements of $[:\mathbb{N}, \mathbb{N}:]$ such that $p \in D$ holds $\langle \langle 1, 0 \rangle \rangle \frown p \in D$,

  (v) for all finite sequences p, q of elements of $[:\mathbb{N}, \mathbb{N}:]$ such that $p \in D$ and $q \in D$ holds $(\langle \langle 2, 0 \rangle \rangle \frown p) \frown q \in D$,

  (vi) for every bound variable x for every finite sequence p of elements of $[:\mathbb{N}, \mathbb{N}:]$ such that $p \in D$ holds $(\langle \langle 3, 0 \rangle \rangle \frown \langle x \rangle) \frown p \in D$.

The constant WFF is a non-empty set and is defined by:

WFF is closed and for every non-empty set D such that D is closed holds $\mathrm{WFF} \subseteq D$.

Next we state two propositions:

(20) For every non-empty set IT holds $IT = \mathrm{WFF}$ if and only if IT is closed and for every non-empty set D such that D is closed holds $IT \subseteq D$.

(21) WFF is closed.

A formula is an element of WFF.

The following proposition is true

(22) For every element x of WFF holds x is a formula.

The arguments of the notions defined below are the following: P which is a predicate symbol; l which is a finite sequence of elements of Var. Let us assume that $\operatorname{Arity}(P) = \operatorname{len} l$. The functor $P \frown l$ yields an element of WFF and is defined by:

$P \frown l = \langle P \rangle \frown l$.

We now state a proposition

(23) For every natural number k for every k-ary predicate symbol p for every list of variables ll of the length k holds $p \frown ll = \langle p \rangle \frown ll$.

Let p be an element of WFF. The functor $@p$ yields a finite sequence of elements of $[\!:\mathbb{N}, \mathbb{N}:\!]$ and is defined by:

$@p = p$.

One can prove the following proposition

(24) For every element p of WFF holds $@p = p$.

We now define three new functors. The constant VERUM is a formula and is defined by:

VERUM $= \langle \langle 0, 0 \rangle \rangle$.

Let p be an element of WFF. The functor $\neg p$ yielding a formula, is defined by:

$\neg p = \langle \langle 1, 0 \rangle \rangle \frown @p$.

Let q be an element of WFF. The functor $p \wedge q$ yields a formula and is defined by:

$p \wedge q = (\langle \langle 2, 0 \rangle \rangle \frown @p) \frown @q$.

We now state three propositions:

(25) VERUM $= \langle \langle 0, 0 \rangle \rangle$.

(26) For every element p of WFF holds $\neg p = \langle \langle 1, 0 \rangle \rangle \frown @p$.

(27) For all elements p, q of WFF holds $p \wedge q = (\langle \langle 2, 0 \rangle \rangle \frown @p) \frown @q$.

The arguments of the notions defined below are the following: x which is a bound variable; p which is an element of WFF. The functor $\forall_x p$ yields a formula and is defined by:

$\forall_x p = (\langle \langle 3, 0 \rangle \rangle \frown \langle x \rangle) \frown @p$.

The following proposition is true

(28) For every bound variable x for every element p of WFF holds $\forall_x p = (\langle \langle 3, 0 \rangle \rangle \frown \langle x \rangle) \frown @p$.

The scheme *QC_Ind* deals with a unary predicate $\mathcal{P}$ and states that:

for every element F of WFF holds $\mathcal{P}[F]$

provided the parameter satisfies the following conditions:

- for every natural number k for every k-ary predicate symbol P for every list of variables ll of the length k holds $\mathcal{P}[P \frown ll]$,
- $\mathcal{P}$[VERUM],
- for every element p of WFF such that $\mathcal{P}[p]$ holds $\mathcal{P}[\neg p]$,
- for all elements p, q of WFF such that $\mathcal{P}[p]$ and $\mathcal{P}[q]$ holds $\mathcal{P}[p \wedge q]$,
- for every bound variable x for every element p of WFF such that $\mathcal{P}[p]$ holds $\mathcal{P}[\forall_x p]$.

We now define four new predicates. Let F be an element of WFF. The predicate F is atomic is defined by:

there exists k being a natural number such that there exists p being a k-ary predicate symbol such that there exists ll being a list of variables of the length k such that $F = p \frown ll$.

The predicate F is negative is defined by:

there exists p being an element of WFF such that $F = \neg p$.

The predicate F is conjunctive is defined by:

there exist p, q being elements of WFF such that $F = p \wedge q$.

The predicate F is universal is defined by:

there exists x being a bound variable such that there exists p being an element of WFF such that $F = \forall_x p$.

We now state several propositions:

(29) For every element F of WFF holds F is atomic if and only if there exists k being a natural number such that there exists p being a k-ary predicate symbol such that there exists ll being a list of variables of the length k such that $F = p \mathbin{^\frown} ll$.

(30) For every element F of WFF holds F is negative if and only if there exists p being an element of WFF such that $F = \neg p$.

(31) For every element F of WFF holds F is conjunctive if and only if there exist p, q being elements of WFF such that $F = p \wedge q$.

(32) For every element F of WFF holds F is universal if and only if there exists x being a bound variable such that there exists p being an element of WFF such that $F = \forall_x p$.

(33) For every element F of WFF holds $F =$ VERUM or F is atomic or F is negative or F is conjunctive or F is universal.

(34) For every element F of WFF holds $1 \leq \mathrm{len}(@F)$.

One can prove the following proposition

(35) For every natural number k for every k-ary predicate symbol P holds $\mathrm{Arity}(P) = k$.

In the sequel F, G are elements of WFF and s is a finite sequence. The following two propositions are true:

(36) (i) If $(@F(1))_{\mathbf{1}} = 0$, then $F =$ VERUM,

(ii) if $(@F(1))_{\mathbf{1}} = 1$, then F is negative,

(iii) if $(@F(1))_{\mathbf{1}} = 2$, then F is conjunctive,

(iv) if $(@F(1))_{\mathbf{1}} = 3$, then F is universal,

(v) if there exists k being a natural number such that $@F(1)$ is a k-ary predicate symbol, then F is atomic.

(37) If $@F = @G \mathbin{^\frown} s$, then $@F = @G$.

Let F be an element of WFF satisfying the condition: F is atomic. The functor $\mathrm{PredSym}(F)$ yielding a predicate symbol, is defined by:

there exists k being a natural number such that there exists ll being a list of variables of the length k such that there exists P being a k-ary predicate symbol such that $\mathrm{PredSym}(F) = P$ and $F = P \mathbin{^\frown} ll$.

Let F be an element of WFF satisfying the condition: F is atomic. The functor $\mathrm{Args}(F)$ yielding a finite sequence of elements of Var, is defined by:

there exists k being a natural number such that there exists P being a k-ary predicate symbol such that there exists ll being a list of variables of the length k such that $\mathrm{Args}(F) = ll$ and $F = P \mathbin{^\frown} ll$.

Next we state two propositions:

(38) For every element F of WFF such that F is atomic for every predicate symbol IT holds $IT = \mathrm{PredSym}(F)$ if and only if there exists k being a natural number such that there exists ll being a list of variables of the length k such that there exists P being a k-ary predicate symbol such that $IT = P$ and $F = P \frown ll$.

(39) For every element F of WFF such that F is atomic for every finite sequence IT of elements of Var holds $IT = \mathrm{Args}(F)$ if and only if there exists k being a natural number such that there exists P being a k-ary predicate symbol such that there exists ll being a list of variables of the length k such that $IT = ll$ and $F = P \frown ll$.

Let F be an element of WFF satisfying the condition: F is negative. The functor $\mathrm{Arg}(F)$ yields a formula and is defined by:

$F = \neg\, \mathrm{Arg}(F)$.

The following proposition is true

(40) For every element F of WFF such that F is negative for every formula IT holds $IT = \mathrm{Arg}(F)$ if and only if $F = \neg IT$.

Let F be an element of WFF satisfying the condition: F is conjunctive. The functor $\mathrm{LeftArg}(F)$ yielding a formula, is defined by:

there exists q being an element of WFF such that $F = \mathrm{LeftArg}(F) \wedge q$.

Let F be an element of WFF satisfying the condition: F is conjunctive. The functor $\mathrm{RightArg}(F)$ yields a formula and is defined by:

there exists p being an element of WFF such that $F = p \wedge \mathrm{RightArg}(F)$.

Next we state two propositions:

(41) For every element F of WFF such that F is conjunctive for every formula IT holds $IT = \mathrm{LeftArg}(F)$ if and only if there exists q being an element of WFF such that $F = IT \wedge q$.

(42) For every element F of WFF such that F is conjunctive for every formula IT holds $IT = \mathrm{RightArg}(F)$ if and only if there exists p being an element of WFF such that $F = p \wedge IT$.

We now define two new functors. Let F be an element of WFF satisfying the condition: F is universal. The functor $\mathrm{Bound}(F)$ yields a bound variable and is defined by:

there exists p being an element of WFF such that $F = \forall_{\mathrm{Bound}(F)} p$.

The functor $\mathrm{Scope}(F)$ yielding a formula, is defined by:

there exists x being a bound variable such that $F = \forall_x \mathrm{Scope}(F)$.

One can prove the following propositions:

(43) For every element F of WFF such that F is universal for every bound variable IT holds $IT = \mathrm{Bound}(F)$ if and only if there exists p being an element of WFF such that $F = \forall_{IT} p$.

(44) For every element F of WFF such that F is universal for every formula IT holds $IT = \mathrm{Scope}(F)$ if and only if there exists x being a bound variable such that $F = \forall_x IT$.

In the sequel p will be an element of WFF. We now state three propositions:

(45) If p is negative, then $\mathrm{len}(@\,\mathrm{Arg}(p)) < \mathrm{len}(@p)$.

(46) If p is conjunctive, then $\mathrm{len}(@\,\mathrm{LeftArg}(p)) < \mathrm{len}(@p)$ and $\mathrm{len}(@\,\mathrm{RightArg}(p)) < \mathrm{len}(@p)$.

(47) If p is universal, then $\mathrm{len}(@\,\mathrm{Scope}(p)) < \mathrm{len}(@p)$.

The scheme *QC_Ind2* concerns a unary predicate $\mathcal{P}$ and states that:
for every element p of WFF holds $\mathcal{P}[p]$
provided the parameter satisfies the following condition:

- for every element p of WFF holds if p is atomic, then $\mathcal{P}[p]$ but $\mathcal{P}[\mathrm{VERUM}]$ but if p is negative and $\mathcal{P}[\mathrm{Arg}(p)]$, then $\mathcal{P}[p]$ but if p is conjunctive and $\mathcal{P}[\mathrm{LeftArg}(p)]$ and $\mathcal{P}[\mathrm{RightArg}(p)]$, then $\mathcal{P}[p]$ but if p is universal and $\mathcal{P}[\mathrm{Scope}(p)]$, then $\mathcal{P}[p]$.

In the sequel F will denote an element of WFF. The following propositions are true:

(48) For every natural number k for every k-ary predicate symbol P holds $P_{\mathbf{1}} \neq 0$ and $P_{\mathbf{1}} \neq 1$ and $P_{\mathbf{1}} \neq 2$ and $P_{\mathbf{1}} \neq 3$.

(49) (i) $(@\,\mathrm{VERUM}(1))_{\mathbf{1}} = 0$,

(ii) if F is atomic, then there exists k being a natural number such that $@F(1)$ is a k-ary predicate symbol,

(iii) if F is negative, then $(@F(1))_{\mathbf{1}} = 1$,

(iv) if F is conjunctive, then $(@F(1))_{\mathbf{1}} = 2$,

(v) if F is universal, then $(@F(1))_{\mathbf{1}} = 3$.

(50) If F is atomic, then $(@F(1))_{\mathbf{1}} \neq 0$ and $(@F(1))_{\mathbf{1}} \neq 1$ and $(@F(1))_{\mathbf{1}} \neq 2$ and $(@F(1))_{\mathbf{1}} \neq 3$.

In the sequel p denotes an element of WFF. The following proposition is true

(51) (i) Neither VERUM is atomic nor VERUM is negative nor VERUM is conjunctive nor VERUM is universal,

(ii) for no p holds p is atomic and p is negative or p is atomic and p is conjunctive or p is atomic and p is universal or p is negative and p is conjunctive or p is negative and p is universal or p is conjunctive and p is universal.

The scheme *QC_Func_Ex* concerns a constant $\mathcal{A}$ that is a non-empty set, a constant $\mathcal{B}$ that is an element of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$ and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$ and states that:

there exists F being a function from WFF into $\mathcal{A}$ such that for every element p of WFF for all elements d_1, d_2 of $\mathcal{A}$ holds if $p = \mathrm{VERUM}$, then $F(p) = \mathcal{B}$ but if p is atomic, then $F(p) = \mathcal{F}(p)$ but if p is negative and $d_1 = F(\mathrm{Arg}(p))$, then $F(p) = \mathcal{G}(d_1)$ but if p is conjunctive and $d_1 = F(\mathrm{LeftArg}(p))$ and $d_2 = F(\mathrm{RightArg}(p))$, then $F(p) = \mathcal{H}(d_1, d_2)$ but if p is universal and $d_1 = F(\mathrm{Scope}(p))$, then $F(p) = \mathcal{I}(p, d_1)$.
for all values of the parameters.

In the sequel k denotes a natural number. Let ll be a finite sequence of elements of Var. The functor $\mathrm{snb}(ll)$ yields an element of 2^{BoundVar} **qua** a non-empty set and is defined by:

$\mathrm{snb}(ll) = \{ll(k) : 1 \leq k \wedge k \leq \mathrm{len}\, ll \wedge ll(k) \in \mathrm{BoundVar}\}$.

The following proposition is true

(52) For every finite sequence ll of elements of Var holds $\mathrm{snb}(ll) = \{ll(k) : 1 \leq k \wedge k \leq \mathrm{len}\, ll \wedge ll(k) \in \mathrm{BoundVar}\}$.

Let x be an element of 2^{BoundVar} **qua** a non-empty set. The functor $@x$ yields an element of 2^{BoundVar} and is defined by:

$@x = x$.

Next we state a proposition

(53) For every element x of 2^{BoundVar} **qua** a non-empty set holds $@x = x$.

Let x be an element of 2^{BoundVar}. The functor $@x$ yields an element of 2^{BoundVar} **qua** a non-empty set and is defined by:

$@x = x$.

One can prove the following proposition

(54) For every element x of 2^{BoundVar} holds $@x = x$.

Let b be a bound variable. Then $\{b\}$ is an element of 2^{BoundVar}.

Let X, Y be elements of 2^{BoundVar}. Then $X \cup Y$ is an element of 2^{BoundVar}. Then $X \setminus Y$ is an element of 2^{BoundVar}.

In the sequel k denotes a natural number. Let p be a formula. The functor $\mathrm{snb}(p)$ yields an element of 2^{BoundVar} and is defined by:

there exists F being a function from

WFF

into 2^{BoundVar} such that $\mathrm{snb}(p) = F(p)$ and for every element p of WFF holds $F(\mathrm{VERUM}) = \emptyset$ but if p is atomic, then $F(p) = \{\mathrm{Args}(p)(k) : 1 \leq k \wedge k \leq \mathrm{len}\,\mathrm{Args}(p) \wedge \mathrm{Args}(p)(k) \in \mathrm{BoundVar}\}$ but if p is negative, then $F(p) = F(\mathrm{Arg}(p))$ but if p is conjunctive, then $F(p) = @(F(\mathrm{LeftArg}(p))) \cup @(F(\mathrm{RightArg}(p)))$ but if p is universal, then $F(p) = @(F(\mathrm{Scope}(p))) \setminus \{\mathrm{Bound}(p)\}$.

We now state a proposition

(55) Let p be a formula. Let IT be an element of 2^{BoundVar}. Then $IT = \mathrm{snb}(p)$ if and only if there exists F being a function from WFF into 2^{BoundVar} such that $IT = F(p)$ and for every element p of WFF holds $F(\mathrm{VERUM}) = \emptyset$ but if p is atomic, then $F(p) = \{\mathrm{Args}(p)(k) : 1 \leq k \wedge k \leq \mathrm{len}\,\mathrm{Args}(p) \wedge \mathrm{Args}(p)(k) \in \mathrm{BoundVar}\}$ but if p is negative, then $F(p) = F(\mathrm{Arg}(p))$ but if p is conjunctive, then $F(p) = @(F(\mathrm{LeftArg}(p))) \cup @(F(\mathrm{RightArg}(p)))$ but if p is universal, then $F(p) = @(F(\mathrm{Scope}(p))) \setminus \{\mathrm{Bound}(p)\}$.

Let p be a formula. The predicate p is closed is defined by:

$\mathrm{snb}(p) = \emptyset$.

One can prove the following proposition

(56) For every formula p holds p is closed if and only if $\mathrm{snb}(p) = \emptyset$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[6] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.

[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[8] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

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Partially Ordered Sets

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Summary. In the beginning of this article we define the choice function of a non-empty set family that does not contain $\emptyset$ as introduced in [5, pages 88–89]. We define order of a set as a relation being reflexive, antisymmetric and transitive in the set, partially ordered set as structure non-emty set and order of the set, chains, lower and upper cone of a subset, initial segments of element and subset of partially ordered set. Some theorems that belong rather to [4] or [9] are proved.

MML Identifier: `ORDERS_1`.

The notation and terminology used in this paper have been introduced in the following articles: [6], [2], [3], [7], [9], [8], and [1]. We adopt the following convention: X, Y will denote sets, x, y, y_1, y_2, z will be arbitrary, and f will denote a function. In the article we present several logical schemes. The scheme *FuncExS* deals with a constant $\mathcal{A}$ that is a set and a binary predicate $\mathcal{P}$ and states that:

there exists f such that $\operatorname{dom} f = \mathcal{A}$ and for every X such that $X \in \mathcal{A}$ holds $\mathcal{P}[X, f(X)]$

provided the parameters satisfy the following conditions:

- for all X, y_1, y_2 such that $X \in \mathcal{A}$ and $\mathcal{P}[X, y_1]$ and $\mathcal{P}[X, y_2]$ holds $y_1 = y_2$,
- for every X such that $X \in \mathcal{A}$ there exists y such that $\mathcal{P}[X, y]$.

The scheme *LambdaS* concerns a constant $\mathcal{A}$ that is a set and a unary functor $\mathcal{F}$ and states that:

there exists f such that $\operatorname{dom} f = \mathcal{A}$ and for every X such that $X \in \mathcal{A}$ holds $f(X) = \mathcal{F}(X)$

for all values of the parameters.

In the sequel M will be a non-empty family of sets and F will be a function from M into $\bigcup M$. Let us consider M. Let us assume that $\emptyset \notin M$. The mode choice function of M, which widens to the type a function from M into $\bigcup M$, is defined by:

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for every X such that $X \in M$ holds $\mathrm{it}(X) \in X$.

The following proposition is true

(1) If $\emptyset \notin M$ and for every X such that $X \in M$ holds $F(X) \in X$, then F is a choice function of M.

In the sequel CF will denote a choice function of M. Next we state a proposition

(2) If $\emptyset \notin M$, then for every X such that $X \in M$ holds $CF(X) \in X$.

In the sequel D, D_1 will denote non-empty sets. Let us consider D. The functor 2^D_+ yielding a non-empty family of sets, is defined by:

$2^D_+ = 2^D \setminus \{\emptyset\}$.

Next we state several propositions:

(3) $2^D_+ = 2^D \setminus \{\emptyset\}$.

(4) $\emptyset \notin 2^D_+$.

(5) $D_1 \subseteq D$ if and only if $D_1 \in 2^D_+$.

(6) D_1 is a subset of D if and only if $D_1 \in 2^D_+$.

(7) $D \in 2^D_+$.

In the sequel P denotes a relation and R denotes a relation on X. Let us consider X. The mode order in X, which widens to the type a relation on X, is defined by:

it is reflexive in X and it is antisymmetric in X and it is transitive in X.

We now state a proposition

(8) If R is reflexive in X and R is antisymmetric in X and R is transitive in X, then R is an order in X.

In the sequel O denotes an order in X. We now state several propositions:

(9) O is reflexive in X.

(10) O is antisymmetric in X.

(11) O is transitive in X.

(12) If $x \in X$, then $\langle x, x\rangle \in O$.

(13) If $x \in X$ and $y \in X$ and $\langle x, y\rangle \in O$ and $\langle y, x\rangle \in O$, then $x = y$.

(14) If $x \in X$ and $y \in X$ and $z \in X$ and $\langle x, y\rangle \in O$ and $\langle y, z\rangle \in O$, then $\langle x, z\rangle \in O$.

We consider posets which are systems

$\langle$ a carrier, an order $\rangle$

where the carrier is a non-empty set and the order is an order in the carrier. In the sequel A will denote a poset. Let us consider A. An element of A is an element of the carrier of A.

Let us consider A. A subset of A is a subset of the carrier of A.

In the sequel a is an element of the carrier of A and S is a subset of the carrier of A. One can prove the following propositions:

(15) a is an element of A.

(16) S is a subset of A.

(17) $x \in$the carrier of A if and only if x is an element of A.

(18) $X \subseteq$the carrier of A if and only if X is a subset of A.

(19) If $x \in S$, then x is an element of A.

We follow the rules: a, a_1, a_2, a_3, b, c denote elements of A and S, T denote subsets of A. Let us consider A, a. Then $\{a\}$ is a subset of A.

Let us consider A, a_1, a_2. Then $\{a_1, a_2\}$ is a subset of A.

Let us consider A, S, T. Then $S \cup T$ is a subset of A. Then $S \cap T$ is a subset of A. Then $S \setminus T$ is a subset of A. Then $S \dot{-} T$ is a subset of A.

Let us consider A. The functor $\emptyset_A$ yielding a subset of A, is defined by:
$\emptyset_A = \emptyset$.

Let us consider A. The functor Ω_A yielding a subset of A, is defined by:
$\Omega_A =$the carrier of A.

One can prove the following propositions:

(20) $\emptyset_A = \emptyset$.

(21) $\Omega_A =$the carrier of A.

Let us consider A, a_1, a_2. The predicate $a_1 \leq a_2$ is defined by:
$\langle a_1, a_2 \rangle \in$the order of A.

Let us consider A, a_1, a_2. The predicate $a_1 < a_2$ is defined by:
$a_1 \leq a_2$ and $a_1 \neq a_2$.

One can prove the following propositions:

(22) $a_1 \leq a_2$ if and only if $\langle a_1, a_2 \rangle \in$the order of A.

(23) $a_1 < a_2$ if and only if $a_1 \leq a_2$ and $a_1 \neq a_2$.

(24) $a \leq a$.

(25) If $a_1 \leq a_2$ and $a_2 \leq a_1$, then $a_1 = a_2$.

(26) If $a_1 \leq a_2$ and $a_2 \leq a_3$, then $a_1 \leq a_3$.

(27) $a \not< a$.

(28) this conjunction is not true: $a_1 < a_2$ and $a_2 < a_1$.

(29) If $a_1 < a_2$ and $a_2 < a_3$, then $a_1 < a_3$.

(30) If $a_1 \leq a_2$, then $a_2 \not< a_1$.

(31) If $a_1 < a_2$, then $a_2 \not\leq a_1$.

(32) If $a_1 < a_2$ and $a_2 \leq a_3$ or $a_1 \leq a_2$ and $a_2 < a_3$, then $a_1 < a_3$.

Let us consider A. The mode chain of A, which widens to the type a subset of A, is defined by:
the order of A is strongly connected in it .

One can prove the following proposition

(33) If the order of A is strongly connected in S, then S is a chain of A.

In the sequel C will denote a chain of A. One can prove the following propositions:

(34) the order of A is strongly connected in C.

(35) $\{a\}$ is a chain of A.

(36) $\{a_1, a_2\}$ is a chain of A if and only if $a_1 \leq a_2$ or $a_2 \leq a_1$.

(37) If $S \subseteq C$, then S is a chain of A.

(38) There exists C such that $a_1 \in C$ and $a_2 \in C$ if and only if $a_1 \leq a_2$ or $a_2 \leq a_1$.

(39) There exists C such that $a_1 \in C$ and $a_2 \in C$ if and only if $a_1 < a_2$ if and only if $a_2 \not\leq a_1$.

(40) If the order of A well orders T, then T is a chain of A.

Let us consider A, S. The functor $\mathrm{UpperCone}\,S$ yields a subset of A and is defined by:

$\mathrm{UpperCone}\,S = \{a_1 : \bigvee_{a_2}[a_2 \in S \Rightarrow a_2 < a_1]\}$.

Let us consider A, S. The functor $\mathrm{LowerCone}\,S$ yielding a subset of A, is defined by:

$\mathrm{LowerCone}\,S = \{a_1 : \bigvee_{a_2}[a_2 \in S \Rightarrow a_1 < a_2]\}$.

The following propositions are true:

(41) $\mathrm{UpperCone}\,S = \{a_1 : \bigvee_{a_2}[a_2 \in S \Rightarrow a_2 < a_1]\}$.

(42) $\mathrm{LowerCone}\,S = \{a_1 : \bigvee_{a_2}[a_2 \in S \Rightarrow a_1 < a_2]\}$.

(43) $\mathrm{UpperCone}\,\emptyset_A =$the carrier of A.

(44) $\mathrm{UpperCone}\,\Omega_A = \emptyset$.

(45) $\mathrm{LowerCone}\,\emptyset_A =$the carrier of A.

(46) $\mathrm{LowerCone}\,\Omega_A = \emptyset$.

(47) If $a \in S$, then $a \notin \mathrm{UpperCone}\,S$.

(48) $a \notin \mathrm{UpperCone}\{a\}$.

(49) If $a \in S$, then $a \notin \mathrm{LowerCone}\,S$.

(50) $a \notin \mathrm{LowerCone}\{a\}$.

(51) $c < a$ if and only if $a \in \mathrm{UpperCone}\{c\}$.

(52) $a < c$ if and only if $a \in \mathrm{LowerCone}\{c\}$.

Let us consider A, S, a. The functor $\mathrm{InitSegm}(S, a)$ yields a subset of A and is defined by:

$\mathrm{InitSegm}(S, a) = \mathrm{LowerCone}\{a\} \cap S$.

Let us consider A, S. The mode initial segment of S, which widens to the type a subset of A, is defined by:

there exists a such that $a \in S$ and it $= \mathrm{InitSegm}(S, a)$ if $S \neq \emptyset$, it $= \emptyset$, otherwise.

The following propositions are true:

(53) $\mathrm{InitSegm}(S, a) = \mathrm{LowerCone}\{a\} \cap S$.

(54) If $S \neq \emptyset$ and there exists a such that $a \in S$ and $T = \mathrm{InitSegm}(S, a)$, then T is an initial segment of S.

(55) If $S = \emptyset$, then T is an initial segment of S if and only if $T = \emptyset$.

In the sequel I will be an initial segment of S and I_0 will be an initial segment of $\emptyset_A$. One can prove the following propositions:

(56) $x \in \mathrm{InitSegm}(S, a)$ if and only if $x \in \mathrm{LowerCone}\{a\}$ and $x \in S$.

(57) $a \in \mathrm{InitSegm}(S, b)$ if and only if $a < b$ and $a \in S$.

(58) If $S \neq \emptyset$, then there exists a such that $a \in S$ and $I = \mathrm{InitSegm}(S, a)$.

(59) If $a \in T$ and $S = \mathrm{InitSegm}(T, a)$, then S is an initial segment of T.

(60) $\mathrm{InitSegm}(\emptyset_A, a) = \emptyset$.

(61) $\mathrm{InitSegm}(S, a) \subseteq S$.

(62) $a \notin \mathrm{InitSegm}(S, a)$.

(63) $a_1 \in S$ and $a_1 < a_2$ if and only if $a_1 \in \mathrm{InitSegm}(S, a_2)$.

(64) If $a_1 < a_2$, then $\mathrm{InitSegm}(S, a_1) \subseteq \mathrm{InitSegm}(S, a_2)$.

(65) If $S \subseteq T$, then $\mathrm{InitSegm}(S, a) \subseteq \mathrm{InitSegm}(T, a)$.

(66) $I_0 = \emptyset$.

(67) $I \subseteq S$.

(68) $S \neq \emptyset$ if and only if S is not an initial segment of S.

(69) If $S \neq \emptyset$ or $T \neq \emptyset$ but S is an initial segment of T, then T is not an initial segment of S.

(70) If $a_1 < a_2$ and $a_1 \in S$ and $a_2 \in T$ and T is an initial segment of S, then $a_1 \in T$.

(71) If $a \in S$ and S is an initial segment of T, then $\mathrm{InitSegm}(S, a) = \mathrm{InitSegm}(T, a)$.

(72) If $S \subseteq T$ and the order of A well orders T and for all a_1, a_2 such that $a_2 \in S$ and $a_1 < a_2$ holds $a_1 \in S$, then $S = T$ or S is an initial segment of T.

(73) If $S \subseteq T$ and the order of A well orders T and for all a_1, a_2 such that $a_2 \in S$ and $a_1 \in T$ and $a_1 < a_2$ holds $a_1 \in S$, then $S = T$ or S is an initial segment of T.

In the sequel f will denote a choice function of $2_+^{\text{the carrier of } A}$. Let us consider A, f. The mode chain of f, which widens to the type a chain of A, is defined by:

it $\neq \emptyset$ and the order of A well orders it and for every a such that $a \in$ it holds $f(\mathrm{UpperCone}\,\mathrm{InitSegm}(\mathrm{it}, a)) = a$.

Next we state a proposition

(74) If $C \neq \emptyset$ and the order of A well orders C and for every a such that $a \in C$ holds $f(\mathrm{UpperCone}\,\mathrm{InitSegm}(C, a)) = a$, then C is a chain of f.

In the sequel fC, fC_1, fC_2 denote chains of f. Next we state a number of propositions:

(75) $fC \neq \emptyset$.

(76) the order of A well orders fC.

(77) If $a \in fC$, then $f(\mathrm{UpperCone}\,\mathrm{InitSegm}(fC, a)) = a$.

(78) $\{f(\text{the carrier of } A)\}$ is a chain of f.

(79) $f(\text{the carrier of } A) \in fC$.

(80) If $a \in fC$ and $b = f(\text{the carrier of } A)$, then $b \leq a$.

(81) If $a = f(\text{the carrier of } A)$, then $\mathrm{InitSegm}(fC, a) = \emptyset$.

(82) $fC_1 \cap fC_2 \neq \emptyset$.

(83) If $f C_1 \neq f C_2$, then $f C_1$ is an initial segment of $f C_2$ if and only if $f C_2$ is not an initial segment of $f C_1$.

(84) $f C_1 \neq f C_2$ and $f C_1 \subseteq f C_2$ if and only if $f C_1$ is an initial segment of $f C_2$.

Let us consider A, f. The functor $\mathrm{Chains}\, f$ yielding a non-empty set, is defined by:

$x \in \mathrm{Chains}\, f$ if and only if x is a chain of f.

One can prove the following propositions:

(85) If for every x holds $x \in D$ if and only if x is a chain of f, then $D = \mathrm{Chains}\, f$.

(86) $x \in \mathrm{Chains}\, f$ if and only if x is a chain of f.

(87) $\bigcup(\mathrm{Chains}\, f) \neq \emptyset$.

(88) If $f C \neq \bigcup(\mathrm{Chains}\, f)$ and $S = \bigcup(\mathrm{Chains}\, f)$, then $f C$ is an initial segment of S.

(89) $\bigcup(\mathrm{Chains}\, f)$ is a chain of f.

(90) $x \in X$ if and only if $\{x\} \in 2^X$.

(91) There exists X such that $X \neq \emptyset$ and $X \in Y$ if and only if $\bigcup Y \neq \emptyset$.

(92) P is strongly connected in X if and only if P is reflexive in X and P is connected in X.

(93) If P is reflexive in X and $Y \subseteq X$, then P is reflexive in Y.

(94) If P is antisymmetric in X and $Y \subseteq X$, then P is antisymmetric in Y.

(95) If P is transitive in X and $Y \subseteq X$, then P is transitive in Y.

(96) If P is strongly connected in X and $Y \subseteq X$, then P is strongly connected in Y.

References

[1] Grzegorz Bancerek. The well ordering relations. *Formalized Mathematics*, 1(**1**):123–129, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[4] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[5] Kazimierz Kuratowski. *Wstęp do teorii mnogości i topologii.* PWN, Warszawa, 1977.

[6] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[7] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[8] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

[9] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Formalized Mathematics*, 1(**1**):85–89, 1990.

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Recursive Definitions

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Summary. The text contains some schemes which allow elimination of defintions by recursion.

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The papers [5], [1], [3], [2], and [4] provide the notation and terminology for this paper. We follow a convention: n, m, k will denote natural numbers and x, y, z, y_1, y_2 will be arbitrary. The arguments of the notions defined below are the following: D which is a non-empty set; p which is a function from $\mathbb{N}$ into D; n which is an element of $\mathbb{N}$. Then $p(n)$ is an element of D.

The arguments of the notions defined below are the following: p which is a function from $\mathbb{N}$ into $\mathbb{N}$; n which is an element of $\mathbb{N}$. Then $p(n)$ is a natural number.

In the article we present several logical schemes. The scheme *RecEx* concerns a constant $\mathcal{A}$ and a ternary predicate $\mathcal{P}$ and states that:

there exists f being a function such that $\operatorname{dom} f = \mathbb{N}$ and $f(0) = \mathcal{A}$ and for every element n of $\mathbb{N}$ holds $\mathcal{P}[n, f(n), f(n+1)]$

provided the parameters satisfy the following conditions:

- for every natural number n for arbitrary x there exists y being any such that $\mathcal{P}[n, x, y]$,
- for every natural number n for arbitrary x, y_1, y_2 such that $\mathcal{P}[n, x, y_1]$ and $\mathcal{P}[n, x, y_2]$ holds $y_1 = y_2$.

The scheme *RecExD* deals with a constant $\mathcal{A}$ that is a non-empty set, a constant $\mathcal{B}$ that is an element of $\mathcal{A}$ and a ternary predicate $\mathcal{P}$ and states that:

there exists f being a function from $\mathbb{N}$ into $\mathcal{A}$ such that $f(0) = \mathcal{B}$ and for every element n of $\mathbb{N}$ holds $\mathcal{P}[n, f(n), f(n+1)]$

provided the parameters satisfy the following conditions:

- for every natural number n for every element x of $\mathcal{A}$ there exists y being an element of $\mathcal{A}$ such that $\mathcal{P}[n, x, y]$,
- for every natural number n for all elements x, y_1, y_2 of $\mathcal{A}$ such that $\mathcal{P}[n, x, y_1]$ and $\mathcal{P}[n, x, y_2]$ holds $y_1 = y_2$.

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The scheme *LambdaRecEx* concerns a constant $\mathcal{A}$ and a binary functor $\mathcal{F}$ and states that:

there exists f being a function such that $\operatorname{dom} f = \mathbb{N}$ and $f(0) = \mathcal{A}$ and for every element n of $\mathbb{N}$ for arbitrary x such that $x = f(n)$ holds $f(n+1) = \mathcal{F}(n, x)$

for all values of the parameters.

The scheme *LambdaRecExD* concerns a constant $\mathcal{A}$ that is a non-empty set, a constant $\mathcal{B}$ that is an element of $\mathcal{A}$ and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$ and states that:

there exists f being a function from $\mathbb{N}$ into $\mathcal{A}$ such that $f(0) = \mathcal{B}$ and for every element n of $\mathbb{N}$ for every element x of $\mathcal{A}$ such that $x = f(n)$ holds $f(n+1) = \mathcal{F}(n, x)$

for all values of the parameters.

The scheme *RecFuncExR* concerns a constant $\mathcal{A}$ that is a real number and a binary functor $\mathcal{F}$ yielding a real number and states that:

there exists f being a function from $\mathbb{N}$ into $\mathbb{R}$ such that $f(0) = \mathcal{A}$ and for every natural number n for every real number x such that $x = f(n)$ holds $f(n+1) = \mathcal{F}(n, x)$

for all values of the parameters.

The scheme *RecExN* deals with a constant $\mathcal{A}$ that is a natural number and a binary functor $\mathcal{F}$ yielding a natural number and states that:

there exists f being a function from $\mathbb{N}$ into $\mathbb{N}$ such that $f(0) = \mathcal{A}$ and for every natural number n for every natural number x such that $x = f(n)$ holds $f(n+1) = \mathcal{F}(n, x)$

for all values of the parameters.

The scheme *FinRecEx* deals with a constant $\mathcal{A}$, a constant $\mathcal{B}$ that is a natural number and a ternary predicate $\mathcal{P}$ and states that:

there exists p being a finite sequence such that $\operatorname{len} p = \mathcal{B}$ but $p(1) = \mathcal{A}$ or $\mathcal{B} = 0$ and for every n such that $1 \leq n$ and $n \leq \mathcal{B} - 1$ holds $\mathcal{P}[n, p(n), p(n+1)]$

provided the parameters satisfy the following conditions:

- for every natural number n such that $1 \leq n$ and $n \leq \mathcal{B} - 1$ for arbitrary x there exists y being any such that $\mathcal{P}[n, x, y]$,
- for every natural number n such that $1 \leq n$ and $n \leq \mathcal{B} - 1$ for arbitrary x, y_1, y_2 such that $\mathcal{P}[n, x, y_1]$ and $\mathcal{P}[n, x, y_2]$ holds $y_1 = y_2$.

The scheme *FinRecExD* deals with a constant $\mathcal{A}$ that is a non-empty set, a constant $\mathcal{B}$ that is an element of $\mathcal{A}$, a constant $\mathcal{C}$ that is a natural number and a ternary predicate $\mathcal{P}$ and states that:

there exists p being a finite sequence of elements of $\mathcal{A}$ such that $\operatorname{len} p = \mathcal{C}$ but $p(1) = \mathcal{B}$ or $\mathcal{C} = 0$ and for every n such that $1 \leq n$ and $n \leq \mathcal{C} - 1$ holds $\mathcal{P}[n, p(n), p(n+1)]$

provided the parameters satisfy the following conditions:

- for every natural number n such that $1 \leq n$ and $n \leq \mathcal{C} - 1$ for every element x of $\mathcal{A}$ there exists y being an element of $\mathcal{A}$ such that $\mathcal{P}[n, x, y]$,
- for every natural number n such that $1 \leq n$ and $n \leq \mathcal{C} - 1$ for all elements x, y_1, y_2 of $\mathcal{A}$ such that $\mathcal{P}[n, x, y_1]$ and $\mathcal{P}[n, x, y_2]$ holds

$y_1 = y_2$.

The scheme *FinRecExR* deals with a constant $\mathcal{A}$ that is a real number, a constant $\mathcal{B}$ that is a natural number and a ternary predicate $\mathcal{P}$ and states that:

there exists p being a finite sequence of elements of $\mathbb{R}$ such that $\operatorname{len} p = \mathcal{B}$ but $p(1) = \mathcal{A}$ or $\mathcal{B} = 0$ and for every n such that $1 \le n$ and $n \le \mathcal{B} - 1$ holds $\mathcal{P}[n, p(n), p(n+1)]$

provided the parameters satisfy the following conditions:

- for every natural number n such that $1 \le n$ and $n \le \mathcal{B} - 1$ for every real number x there exists y being a real number such that $\mathcal{P}[n, x, y]$,
- for every natural number n such that $1 \le n$ and $n \le \mathcal{B} - 1$ for all real numbers x, y_1, y_2 such that $\mathcal{P}[n, x, y_1]$ and $\mathcal{P}[n, x, y_2]$ holds $y_1 = y_2$.

The scheme *FinRecExN* deals with a constant $\mathcal{A}$ that is a natural number, a constant $\mathcal{B}$ that is a natural number and a ternary predicate $\mathcal{P}$ and states that:

there exists p being a finite sequence of elements of $\mathbb{N}$ such that $\operatorname{len} p = \mathcal{B}$ but $p(1) = \mathcal{A}$ or $\mathcal{B} = 0$ and for every n such that $1 \le n$ and $n \le \mathcal{B} - 1$ holds $\mathcal{P}[n, p(n), p(n+1)]$

provided the parameters satisfy the following conditions:

- for every natural number n such that $1 \le n$ and $n \le \mathcal{B} - 1$ for every natural number x there exists y being a natural number such that $\mathcal{P}[n, x, y]$,
- for every natural number n such that $1 \le n$ and $n \le \mathcal{B} - 1$ for all natural numbers x, y_1, y_2 such that $\mathcal{P}[n, x, y_1]$ and $\mathcal{P}[n, x, y_2]$ holds $y_1 = y_2$.

The scheme *SeqBinOpEx* deals with a constant $\mathcal{A}$ that is a finite sequence and a ternary predicate $\mathcal{P}$ and states that:

there exists x such that there exists p being a finite sequence such that $x = p(\operatorname{len} p)$ and $\operatorname{len} p = \operatorname{len} \mathcal{A}$ and $p(1) = \mathcal{A}(1)$ and for every k such that $1 \le k$ and $k \le \operatorname{len} \mathcal{A} - 1$ holds $\mathcal{P}[\mathcal{A}(k+1), p(k), p(k+1)]$.

provided the parameters satisfy the following conditions:

- for all k, x such that $1 \le k$ and $k \le \operatorname{len} \mathcal{A} - 1$ there exists y such that $\mathcal{P}[\mathcal{A}(k+1), x, y]$,
- for all k, x, y_1, y_2, z such that $1 \le k$ and $k \le \operatorname{len} \mathcal{A} - 1$ and $z = \mathcal{A}(k+1)$ and $\mathcal{P}[z, x, y_1]$ and $\mathcal{P}[z, x, y_2]$ holds $y_1 = y_2$.

The scheme *LambdaSeqBinOpEx* deals with a constant $\mathcal{A}$ that is a finite sequence and a binary functor $\mathcal{F}$ and states that:

there exists x such that there exists p being a finite sequence such that $x = p(\operatorname{len} p)$ and $\operatorname{len} p = \operatorname{len} \mathcal{A}$ and $p(1) = \mathcal{A}(1)$ and for all k, y, z such that $1 \le k$ and $k \le \operatorname{len} \mathcal{A} - 1$ and $y = \mathcal{A}(k+1)$ and $z = p(k)$ holds $p(k+1) = \mathcal{F}(y, z)$.

for all values of the parameters.

The scheme *RecUn* deals with a constant $\mathcal{A}$, a constant $\mathcal{B}$ that is a function, a constant $\mathcal{C}$ that is a function and a ternary predicate $\mathcal{P}$ and states that:

$\mathcal{B} = \mathcal{C}$

provided the parameters satisfy the following conditions:

- $\operatorname{dom} \mathcal{B} = \mathbb{N}$ and $\mathcal{B}(0) = \mathcal{A}$ and for every n holds $\mathcal{P}[n, \mathcal{B}(n), \mathcal{B}(n+1)]$,
- $\operatorname{dom} \mathcal{C} = \mathbb{N}$ and $\mathcal{C}(0) = \mathcal{A}$ and for every n holds $\mathcal{P}[n, \mathcal{C}(n), \mathcal{C}(n+1)]$,

- for every n for arbitrary x, y_1, y_2 such that $\mathcal{P}[n, x, y_1]$ and $\mathcal{P}[n, x, y_2]$ holds $y_1 = y_2$.

The scheme *RecUnD* deals with a constant $\mathcal{A}$ that is a non-empty set, a constant $\mathcal{B}$ that is an element of $\mathcal{A}$, a ternary predicate $\mathcal{P}$, a constant $\mathcal{C}$ that is a function from $\mathbb{N}$ into $\mathcal{A}$ and a constant $\mathcal{D}$ that is a function from $\mathbb{N}$ into $\mathcal{A}$, and states that:

$\mathcal{C} = \mathcal{D}$

provided the parameters satisfy the following conditions:

- $\mathcal{C}(0) = \mathcal{B}$ and for every n holds $\mathcal{P}[n, \mathcal{C}(n), \mathcal{C}(n + 1)]$,
- $\mathcal{D}(0) = \mathcal{B}$ and for every n holds $\mathcal{P}[n, \mathcal{D}(n), \mathcal{D}(n + 1)]$,
- for every natural number n for all elements x, y_1, y_2 of $\mathcal{A}$ such that $\mathcal{P}[n, x, y_1]$ and $\mathcal{P}[n, x, y_2]$ holds $y_1 = y_2$.

The scheme *LambdaRecUn* deals with a constant $\mathcal{A}$, a binary functor $\mathcal{F}$, a constant $\mathcal{B}$ that is a function and a constant $\mathcal{C}$ that is a function, and states that:

$\mathcal{B} = \mathcal{C}$

provided the parameters satisfy the following conditions:

- $\operatorname{dom} \mathcal{B} = \mathbb{N}$ and $\mathcal{B}(0) = \mathcal{A}$ and for every n for arbitrary y such that $y = \mathcal{B}(n)$ holds $\mathcal{B}(n + 1) = \mathcal{F}(n, y)$,
- $\operatorname{dom} \mathcal{C} = \mathbb{N}$ and $\mathcal{C}(0) = \mathcal{A}$ and for every n for arbitrary y such that $y = \mathcal{C}(n)$ holds $\mathcal{C}(n + 1) = \mathcal{F}(n, y)$.

The scheme *LambdaRecUnD* concerns a constant $\mathcal{A}$ that is a non-empty set, a constant $\mathcal{B}$ that is an element of $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a constant $\mathcal{C}$ that is a function from $\mathbb{N}$ into $\mathcal{A}$ and a constant $\mathcal{D}$ that is a function from $\mathbb{N}$ into $\mathcal{A}$, and states that:

$\mathcal{C} = \mathcal{D}$

provided the parameters satisfy the following conditions:

- $\mathcal{C}(0) = \mathcal{B}$ and for every n for every element y of $\mathcal{A}$ such that $y = \mathcal{C}(n)$ holds $\mathcal{C}(n + 1) = \mathcal{F}(n, y)$,
- $\mathcal{D}(0) = \mathcal{B}$ and for every n for every element y of $\mathcal{A}$ such that $y = \mathcal{D}(n)$ holds $\mathcal{D}(n + 1) = \mathcal{F}(n, y)$.

The scheme *LambdaRecUnR* concerns a constant $\mathcal{A}$ that is a real number, a binary functor $\mathcal{F}$, a constant $\mathcal{B}$ that is a function from $\mathbb{N}$ into $\mathbb{R}$ and a constant $\mathcal{C}$ that is a function from $\mathbb{N}$ into $\mathbb{R}$, and states that:

$\mathcal{B} = \mathcal{C}$

provided the parameters satisfy the following conditions:

- $\mathcal{B}(0) = \mathcal{A}$ and for every n for every real number y such that $y = \mathcal{B}(n)$ holds $\mathcal{B}(n + 1) = \mathcal{F}(n, y)$,
- $\mathcal{C}(0) = \mathcal{A}$ and for every n for every real number y such that $y = \mathcal{C}(n)$ holds $\mathcal{C}(n + 1) = \mathcal{F}(n, y)$.

The scheme *LambdaRecUnN* deals with a constant $\mathcal{A}$ that is a natural number, a binary functor $\mathcal{F}$ yielding a natural number, a constant $\mathcal{B}$ that is a function from $\mathbb{N}$ into $\mathbb{N}$ and a constant $\mathcal{C}$ that is a function from $\mathbb{N}$ into $\mathbb{N}$, and states that:

$\mathcal{B} = \mathcal{C}$

provided the parameters satisfy the following conditions:

- $\mathcal{B}(0) = \mathcal{A}$ and for all n, m such that $m = \mathcal{B}(n)$ holds $\mathcal{B}(n+1) = \mathcal{F}(n,m)$,
- $\mathcal{C}(0) = \mathcal{A}$ and for all n, m such that $m = \mathcal{C}(n)$ holds $\mathcal{C}(n+1) = \mathcal{F}(n,m)$.

The scheme *FinRecUn* deals with a constant $\mathcal{A}$, a constant $\mathcal{B}$ that is a natural number, a constant $\mathcal{C}$ that is a finite sequence, a constant $\mathcal{D}$ that is a finite sequence and a ternary predicate $\mathcal{P}$ and states that:

$\mathcal{C} = \mathcal{D}$

provided the parameters satisfy the following conditions:

- for every n such that $1 \leq n$ and $n \leq \mathcal{B} - 1$ for arbitrary x, y_1, y_2 such that $\mathcal{P}[n, x, y_1]$ and $\mathcal{P}[n, x, y_2]$ holds $y_1 = y_2$,
- $\operatorname{len} \mathcal{C} = \mathcal{B}$ but $\mathcal{C}(1) = \mathcal{A}$ or $\mathcal{B} = 0$ and for every n such that $1 \leq n$ and $n \leq \mathcal{B} - 1$ holds $\mathcal{P}[n, \mathcal{C}(n), \mathcal{C}(n+1)]$,
- $\operatorname{len} \mathcal{D} = \mathcal{B}$ but $\mathcal{D}(1) = \mathcal{A}$ or $\mathcal{B} = 0$ and for every n such that $1 \leq n$ and $n \leq \mathcal{B} - 1$ holds $\mathcal{P}[n, \mathcal{D}(n), \mathcal{D}(n+1)]$.

The scheme *FinRecUnD* concerns a constant $\mathcal{A}$ that is a non-empty set, a constant $\mathcal{B}$ that is an element of $\mathcal{A}$, a constant $\mathcal{C}$ that is a natural number, a constant $\mathcal{D}$ that is a finite sequence of elements of $\mathcal{A}$, a constant $\mathcal{E}$ that is a finite sequence of elements of $\mathcal{A}$ and a ternary predicate $\mathcal{P}$ and states that:

$\mathcal{D} = \mathcal{E}$

provided the parameters satisfy the following conditions:

- for every n such that $1 \leq n$ and $n \leq \mathcal{C} - 1$ for all elements x, y_1, y_2 of $\mathcal{A}$ such that $\mathcal{P}[n, x, y_1]$ and $\mathcal{P}[n, x, y_2]$ holds $y_1 = y_2$,
- $\operatorname{len} \mathcal{D} = \mathcal{C}$ but $\mathcal{D}(1) = \mathcal{B}$ or $\mathcal{C} = 0$ and for every n such that $1 \leq n$ and $n \leq \mathcal{C} - 1$ holds $\mathcal{P}[n, \mathcal{D}(n), \mathcal{D}(n+1)]$,
- $\operatorname{len} \mathcal{E} = \mathcal{C}$ but $\mathcal{E}(1) = \mathcal{B}$ or $\mathcal{C} = 0$ and for every n such that $1 \leq n$ and $n \leq \mathcal{C} - 1$ holds $\mathcal{P}[n, \mathcal{E}(n), \mathcal{E}(n+1)]$.

The scheme *FinRecUnR* deals with a constant $\mathcal{A}$ that is a real number, a constant $\mathcal{B}$ that is a natural number, a constant $\mathcal{C}$ that is a finite sequence of elements of $\mathbb{R}$, a constant $\mathcal{D}$ that is a finite sequence of elements of $\mathbb{R}$ and a ternary predicate $\mathcal{P}$ and states that:

$\mathcal{C} = \mathcal{D}$

provided the parameters satisfy the following conditions:

- for every n such that $1 \leq n$ and $n \leq \mathcal{B} - 1$ for all real numbers x, y_1, y_2 such that $\mathcal{P}[n, x, y_1]$ and $\mathcal{P}[n, x, y_2]$ holds $y_1 = y_2$,
- $\operatorname{len} \mathcal{C} = \mathcal{B}$ but $\mathcal{C}(1) = \mathcal{A}$ or $\mathcal{B} = 0$ and for every n such that $1 \leq n$ and $n \leq \mathcal{B} - 1$ holds $\mathcal{P}[n, \mathcal{C}(n), \mathcal{C}(n+1)]$,
- $\operatorname{len} \mathcal{D} = \mathcal{B}$ but $\mathcal{D}(1) = \mathcal{A}$ or $\mathcal{B} = 0$ and for every n such that $1 \leq n$ and $n \leq \mathcal{B} - 1$ holds $\mathcal{P}[n, \mathcal{D}(n), \mathcal{D}(n+1)]$.

The scheme *FinRecUnN* concerns a constant $\mathcal{A}$ that is a natural number, a constant $\mathcal{B}$ that is a natural number, a constant $\mathcal{C}$ that is a finite sequence of elements of $\mathbb{N}$, a constant $\mathcal{D}$ that is a finite sequence of elements of $\mathbb{N}$ and a ternary predicate $\mathcal{P}$ and states that:

$\mathcal{C} = \mathcal{D}$

provided the parameters satisfy the following conditions:

- for every n such that $1 \leq n$ and $n \leq \mathcal{B} - 1$ for all natural numbers x, y_1, y_2 such that $\mathcal{P}[n, x, y_1]$ and $\mathcal{P}[n, x, y_2]$ holds $y_1 = y_2$,
- $\operatorname{len} \mathcal{C} = \mathcal{B}$ but $\mathcal{C}(1) = \mathcal{A}$ or $\mathcal{B} = 0$ and for every n such that $1 \leq n$ and $n \leq \mathcal{B} - 1$ holds $\mathcal{P}[n, \mathcal{C}(n), \mathcal{C}(n + 1)]$,
- $\operatorname{len} \mathcal{D} = \mathcal{B}$ but $\mathcal{D}(1) = \mathcal{A}$ or $\mathcal{B} = 0$ and for every n such that $1 \leq n$ and $n \leq \mathcal{B} - 1$ holds $\mathcal{P}[n, \mathcal{D}(n), \mathcal{D}(n + 1)]$.

The scheme *SeqBinOpUn* deals with a constant $\mathcal{A}$ that is a finite sequence, a ternary predicate $\mathcal{P}$, a constant $\mathcal{B}$ and a constant $\mathcal{C}$ and states that:

$\mathcal{B} = \mathcal{C}$

provided the parameters satisfy the following conditions:

- for all k, x, y_1, y_2, z such that $1 \leq k$ and $k \leq \operatorname{len} \mathcal{A} - 1$ and $z = \mathcal{A}(k + 1)$ and $\mathcal{P}[z, x, y_1]$ and $\mathcal{P}[z, x, y_2]$ holds $y_1 = y_2$,
- there exists p being a finite sequence such that $\mathcal{B} = p(\operatorname{len} p)$ and $\operatorname{len} p = \operatorname{len} \mathcal{A}$ and $p(1) = \mathcal{A}(1)$ and for every k such that $1 \leq k$ and $k \leq \operatorname{len} \mathcal{A} - 1$ holds $\mathcal{P}[\mathcal{A}(k + 1), p(k), p(k + 1)]$.
- there exists p being a finite sequence such that $\mathcal{C} = p(\operatorname{len} p)$ and $\operatorname{len} p = \operatorname{len} \mathcal{A}$ and $p(1) = \mathcal{A}(1)$ and for every k such that $1 \leq k$ and $k \leq \operatorname{len} \mathcal{A} - 1$ holds $\mathcal{P}[\mathcal{A}(k + 1), p(k), p(k + 1)]$.

The scheme *LambdaSeqBinOpUn* concerns a constant $\mathcal{A}$ that is a finite sequence, a binary functor $\mathcal{F}$, a constant $\mathcal{B}$ and a constant $\mathcal{C}$ and states that:

$\mathcal{B} = \mathcal{C}$

provided the parameters satisfy the following conditions:

- there exists p being a finite sequence such that $\mathcal{B} = p(\operatorname{len} p)$ and $\operatorname{len} p = \operatorname{len} \mathcal{A}$ and $p(1) = \mathcal{A}(1)$ and for all k, y, z such that $1 \leq k$ and $k \leq \operatorname{len} \mathcal{A} - 1$ and $y = \mathcal{A}(k+1)$ and $z = p(k)$ holds $p(k+1) = \mathcal{F}(y, z)$.
- there exists p being a finite sequence such that $\mathcal{C} = p(\operatorname{len} p)$ and $\operatorname{len} p = \operatorname{len} \mathcal{A}$ and $p(1) = \mathcal{A}(1)$ and for all k, y, z such that $1 \leq k$ and $k \leq \operatorname{len} \mathcal{A} - 1$ and $y = \mathcal{A}(k+1)$ and $z = p(k)$ holds $p(k+1) = \mathcal{F}(y, z)$.

The scheme *DefRec* concerns a constant $\mathcal{A}$, a constant $\mathcal{B}$ that is a natural number and a ternary predicate $\mathcal{P}$ and states that:

(i) there exists y being any such that there exists f being a function such that $y = f(\mathcal{B})$ and $\operatorname{dom} f = \mathbb{N}$ and $f(0) = \mathcal{A}$ and for every n holds $\mathcal{P}[n, f(n), f(n+1)]$,
(ii) for arbitrary y_1, y_2 such that there exists f being a function such that $y_1 = f(\mathcal{B})$ and $\operatorname{dom} f = \mathbb{N}$ and $f(0) = \mathcal{A}$ and for every n holds $\mathcal{P}[n, f(n), f(n+1)]$ and there exists f being a function such that $y_2 = f(\mathcal{B})$ and $\operatorname{dom} f = \mathbb{N}$ and $f(0) = \mathcal{A}$ and for every n holds $\mathcal{P}[n, f(n), f(n + 1)]$ holds $y_1 = y_2$.

provided the parameters satisfy the following conditions:

- for every n, x there exists y such that $\mathcal{P}[n, x, y]$,
- for all n, x, y_1, y_2 such that $\mathcal{P}[n, x, y_1]$ and $\mathcal{P}[n, x, y_2]$ holds $y_1 = y_2$.

The scheme *LambdaDefRec* deals with a constant $\mathcal{A}$, a constant $\mathcal{B}$ that is a natural number and a binary functor $\mathcal{F}$ and states that:

(i) there exists y being any such that there exists f being a function such that $y = f(\mathcal{B})$ and $\operatorname{dom} f = \mathbb{N}$ and $f(0) = \mathcal{A}$ and for all n, x such that $x = f(n)$ holds $f(n + 1) = \mathcal{F}(n, x)$,

(ii) for arbitrary y_1, y_2 such that there exists f being a function such that $y_1 = f(\mathcal{B})$ and $\operatorname{dom} f = \mathbb{N}$ and $f(0) = \mathcal{A}$ and for all n, x such that $x = f(n)$ holds $f(n+1) = \mathcal{F}(n, x)$ and there exists f being a function such that $y_2 = f(\mathcal{B})$ and $\operatorname{dom} f = \mathbb{N}$ and $f(0) = \mathcal{A}$ and for all n, x such that $x = f(n)$ holds $f(n+1) = \mathcal{F}(n, x)$ holds $y_1 = y_2$.
for all values of the parameters.

The scheme *DefRecD* concerns a constant $\mathcal{A}$ that is a non-empty set, a constant $\mathcal{B}$ that is an element of $\mathcal{A}$, a constant $\mathcal{C}$ that is a natural number and a ternary predicate $\mathcal{P}$ and states that:
(i) there exists y being an element of $\mathcal{A}$ such that there exists f being a function from $\mathbb{N}$ into $\mathcal{A}$ such that $y = f(\mathcal{C})$ and $f(0) = \mathcal{B}$ and for every n holds $\mathcal{P}[n, f(n), f(n+1)]$,
(ii) for all elements y_1, y_2 of $\mathcal{A}$ such that there exists f being a function from $\mathbb{N}$ into $\mathcal{A}$ such that $y_1 = f(\mathcal{C})$ and $f(0) = \mathcal{B}$ and for every n holds $\mathcal{P}[n, f(n), f(n+1)]$ and there exists f being a function from $\mathbb{N}$ into $\mathcal{A}$ such that $y_2 = f(\mathcal{C})$ and $f(0) = \mathcal{B}$ and for every n holds $\mathcal{P}[n, f(n), f(n+1)]$ holds $y_1 = y_2$.
provided the parameters satisfy the following conditions:

- for every natural number n for every element x of $\mathcal{A}$ there exists y being an element of $\mathcal{A}$ such that $\mathcal{P}[n, x, y]$,
- for every natural number n for all elements x, y_1, y_2 of $\mathcal{A}$ such that $\mathcal{P}[n, x, y_1]$ and $\mathcal{P}[n, x, y_2]$ holds $y_1 = y_2$.

The scheme *LambdaDefRecD* concerns a constant $\mathcal{A}$ that is a non-empty set, a constant $\mathcal{B}$ that is an element of $\mathcal{A}$, a constant $\mathcal{C}$ that is a natural number and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$ and states that:
(i) there exists y being an element of $\mathcal{A}$ such that there exists f being a function from $\mathbb{N}$ into $\mathcal{A}$ such that $y = f(\mathcal{C})$ and $f(0) = \mathcal{B}$ and for every natural number n for every element x of $\mathcal{A}$ such that $x = f(n)$ holds $f(n+1) = \mathcal{F}(n, x)$,
(ii) for all elements y_1, y_2 of $\mathcal{A}$ such that there exists f being a function from $\mathbb{N}$ into $\mathcal{A}$ such that $y_1 = f(\mathcal{C})$ and $f(0) = \mathcal{B}$ and for every natural number n for every element x of $\mathcal{A}$ such that $x = f(n)$ holds $f(n+1) = \mathcal{F}(n, x)$ and there exists f being a function from $\mathbb{N}$ into $\mathcal{A}$ such that $y_2 = f(\mathcal{C})$ and $f(0) = \mathcal{B}$ and for every natural number n for every element x of $\mathcal{A}$ such that $x = f(n)$ holds $f(n+1) = \mathcal{F}(n, x)$ holds $y_1 = y_2$.
for all values of the parameters.

The scheme *SeqBinOpDef* concerns a constant $\mathcal{A}$ that is a finite sequence and a ternary predicate $\mathcal{P}$ and states that:
(i) there exists x such that there exists p being a finite sequence such that $x = p(\operatorname{len} p)$ and $\operatorname{len} p = \operatorname{len} \mathcal{A}$ and $p(1) = \mathcal{A}(1)$ and for every k such that $1 \leq k$ and $k \leq \operatorname{len} \mathcal{A} - 1$ holds $\mathcal{P}[\mathcal{A}(k+1), p(k), p(k+1)]$,
(ii) for all x, y such that there exists p being a finite sequence such that $x = p(\operatorname{len} p)$ and $\operatorname{len} p = \operatorname{len} \mathcal{A}$ and $p(1) = \mathcal{A}(1)$ and for every k such that $1 \leq k$ and $k \leq \operatorname{len} \mathcal{A} - 1$ holds $\mathcal{P}[\mathcal{A}(k+1), p(k), p(k+1)]$ and there exists p being a finite sequence such that $y = p(\operatorname{len} p)$ and $\operatorname{len} p = \operatorname{len} \mathcal{A}$ and $p(1) = \mathcal{A}(1)$ and for every k such that $1 \leq k$ and $k \leq \operatorname{len} \mathcal{A} - 1$ holds $\mathcal{P}[\mathcal{A}(k+1), p(k), p(k+1)]$ holds $x = y$.

provided the parameters satisfy the following conditions:

- for all k, y such that $1 \leq k$ and $k \leq \operatorname{len} \mathcal{A} - 1$ there exists z such that $\mathcal{P}[\mathcal{A}(k+1), y, z]$,
- for all k, x, y_1, y_2, z such that $1 \leq k$ and $k \leq \operatorname{len} \mathcal{A} - 1$ and $z = \mathcal{A}(k+1)$ and $\mathcal{P}[z, x, y_1]$ and $\mathcal{P}[z, x, y_2]$ holds $y_1 = y_2$.

The scheme *LambdaSeqBinOpDe* concerns a constant $\mathcal{A}$ that is a finite sequence and a binary functor $\mathcal{F}$ and states that:
(i) there exists x such that there exists p being a finite sequence such that $x = p(\operatorname{len} p)$ and $\operatorname{len} p = \operatorname{len} \mathcal{A}$ and $p(1) = \mathcal{A}(1)$ and for all k, y, z such that $1 \leq k$ and $k \leq \operatorname{len} \mathcal{A} - 1$ and $y = \mathcal{A}(k+1)$ and $z = p(k)$ holds $p(k+1) = \mathcal{F}(y, z)$,
(ii) for all x, y such that there exists p being a finite sequence such that $x = p(\operatorname{len} p)$ and $\operatorname{len} p = \operatorname{len} \mathcal{A}$ and $p(1) = \mathcal{A}(1)$ and for all k, y, z such that $1 \leq k$ and $k \leq \operatorname{len} \mathcal{A} - 1$ and $y = \mathcal{A}(k+1)$ and $z = p(k)$ holds $p(k+1) = \mathcal{F}(y, z)$ and there exists p being a finite sequence such that $y = p(\operatorname{len} p)$ and $\operatorname{len} p = \operatorname{len} \mathcal{A}$ and $p(1) = \mathcal{A}(1)$ and for all k, y, z such that $1 \leq k$ and $k \leq \operatorname{len} \mathcal{A} - 1$ and $y = \mathcal{A}(k+1)$ and $z = p(k)$ holds $p(k+1) = \mathcal{F}(y, z)$ holds $x = y$.
for all values of the parameters.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

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Binary Operations Applied to Functions

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Summary. In the article we introduce functors yielding to a binary operation its composition with an arbitrary functions on its left side, its right side or both. We prove theorems describing the basic properties of these functors. We introduce also constant functions and converse of a function. The recent concept is defined for an arbitrary function, however is meaningful in the case of functions which range is a subset of a Cartesian product of two sets. Then the converse of a function has the same domain as the function itself and assigns to an element of the domain the mirror image of the ordered pair assigned by the function. In the case of functions defined on a non-empty set we redefine the above mentioned functors and prove simplified versions of theorems proved in the general case. We prove also theorems stating relationships between introduced concepts and such properties of binary operations as commutativity or associativity.

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The notation and terminology used in this paper have been introduced in the following articles: [6], [7], [3], [4], [1], [8], [2], [5], and [9]. One can prove the following proposition

(1) For every relation R for all sets A, B such that $A \neq \emptyset$ and $B \neq \emptyset$ and $R = [\!:A, B:\!]$ holds $\operatorname{dom} R = A$ and $\operatorname{rng} R = B$.

In the sequel f, g, h will be functions and A will be a set. Next we state three propositions:

(2) $\delta_A = \langle \mathrm{id}_A, \mathrm{id}_A \rangle$.

(3) If $\operatorname{dom} f = \operatorname{dom} g$, then $\operatorname{dom}(f \cdot h) = \operatorname{dom}(g \cdot h)$.

(4) If $\operatorname{dom} f = \emptyset$ and $\operatorname{dom} g = \emptyset$, then $f = g$.

We adopt the following convention: F, f, g, h denote functions, A, B denote sets, and x, y, z are arbitrary. Let us consider f. The functor $f^{\smile}$ yields a function and is defined by:

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(i) $\mathrm{dom}(f^{\smile}) = \mathrm{dom}\, f$,
(ii) for every x such that $x \in \mathrm{dom}\, f$ holds for all y, z such that $f(x) = \langle y, z \rangle$ holds $(f^{\smile})(x) = \langle z, y \rangle$ but $f(x) = (f^{\smile})(x)$ or there exist y, z such that $f(x) = \langle y, z \rangle$.

We now state several propositions:

(5) Given f, g. Then $g = f^{\smile}$ if and only if the following conditions are satisfied:
(i) $\mathrm{dom}\, g = \mathrm{dom}\, f$,
(ii) for every x such that $x \in \mathrm{dom}\, f$ holds for all y, z such that $f(x) = \langle y, z \rangle$ holds $g(x) = \langle z, y \rangle$ but $f(x) = g(x)$ or there exist y, z such that $f(x) = \langle y, z \rangle$.
(6) $\langle f, g \rangle = \langle g, f \rangle^{\smile}$.
(7) $(f \restriction A)^{\smile} = f^{\smile} \restriction A$.
(8) $(f^{\smile})^{\smile} = f$.
(9) $(\delta_A)^{\smile} = \delta_A$.
(10) $\langle f, g \rangle \restriction A = \langle f \restriction A, g \rangle$.
(11) $\langle f, g \rangle \restriction A = \langle f, g \restriction A \rangle$.

The arguments of the notions defined below are the following: A which is a set; z which is any. The functor $A \longmapsto z$ yields a function and is defined by:

$\mathrm{graph}(A \longmapsto z) = [\!:A, \{z\}:\!]$.

The following propositions are true:

(12) $f = A \longmapsto x$ if and only if $\mathrm{graph}\, f = [\!:A, \{x\}:\!]$.
(13) If $x \in A$, then $(A \longmapsto z)(x) = z$.
(14) If $A \neq \emptyset$ and $f = A \longmapsto x$, then $\mathrm{dom}\, f = A$ and $\mathrm{rng}\, f = \{x\}$.
(15) If $\mathrm{dom}\, f = A$ and $\mathrm{rng}\, f = \{x\}$, then $f = A \longmapsto x$.
(16) $\mathrm{dom}(\emptyset \longmapsto x) = \emptyset$ and $\mathrm{rng}(\emptyset \longmapsto x) = \emptyset$.
(17) If for every z such that $z \in \mathrm{dom}\, f$ holds $f(z) = x$, then $f = \mathrm{dom}\, f \longmapsto x$.
(18) $(A \longmapsto x) \restriction B = A \cap B \longmapsto x$.
(19) $\mathrm{dom}(A \longmapsto x) = A$ and $\mathrm{rng}(A \longmapsto x) \subseteq \{x\}$.
(20) If $x \in B$, then $(A \longmapsto x)^{-1}\, B = A$.
(21) $(A \longmapsto x)^{-1}\, \{x\} = A$.
(22) If $x \notin B$, then $(A \longmapsto x)^{-1}\, B = \emptyset$.
(23) If $x \in \mathrm{dom}\, h$, then $h \cdot (A \longmapsto x) = A \longmapsto h(x)$.
(24) If $A \neq \emptyset$ and $x \in \mathrm{dom}\, h$, then $\mathrm{dom}(h \cdot (A \longmapsto x)) \neq \emptyset$.
(25) $(A \longmapsto x) \cdot h = h^{-1}\, A \longmapsto x$.
(26) $(A \longmapsto \langle x, y \rangle)^{\smile} = A \longmapsto \langle y, x \rangle$.

Let us consider F, f, g. The functor $F^{\circ}(f, g)$ yields a function and is defined by:

$F^{\circ}(f, g) = F \cdot \langle f, g \rangle$.

The following propositions are true:

(27) $F^{\circ}(f, g) = F \cdot \langle f, g \rangle$.

(28) If $x \in \mathrm{dom}(F^\circ(f,g))$, then $(F^\circ(f,g))(x) = F(f(x), g(x))$.

(29) If $f \restriction A = g \restriction A$, then $(F^\circ(f,h)) \restriction A = (F^\circ(g,h)) \restriction A$.

(30) If $f \restriction A = g \restriction A$, then $(F^\circ(h,f)) \restriction A = (F^\circ(h,g)) \restriction A$.

(31) $F^\circ(f,g) \cdot h = F^\circ(f \cdot h, g \cdot h)$.

(32) $h \cdot F^\circ(f,g) = (h \cdot F)^\circ(f,g)$.

Let us consider F, f, x. The functor $F^\circ(f,x)$ yielding a function, is defined by:

$F^\circ(f,x) = F \cdot \langle f, \mathrm{dom}\, f \longmapsto x\rangle$.

Next we state several propositions:

(33) $F^\circ(f,x) = F \cdot \langle f, \mathrm{dom}\, f \longmapsto x\rangle$.

(34) $F^\circ(f,x) = F^\circ(f, \mathrm{dom}\, f \longmapsto x)$.

(35) If $x \in \mathrm{dom}(F^\circ(f,z))$, then $(F^\circ(f,z))(x) = F(f(x), z)$.

(36) If $f \restriction A = g \restriction A$, then $(F^\circ(f,x)) \restriction A = (F^\circ(g,x)) \restriction A$.

(37) $F^\circ(f,x) \cdot h = F^\circ(f \cdot h, x)$.

(38) $h \cdot F^\circ(f,x) = (h \cdot F)^\circ(f,x)$.

(39) $F^\circ(f,x) \cdot \mathrm{id}_A = F^\circ(f \restriction A, x)$.

Let us consider F, x, g. The functor $F^\circ(x,g)$ yields a function and is defined by:

$F^\circ(x,g) = F \cdot \langle \mathrm{dom}\, g \longmapsto x, g\rangle$.

We now state several propositions:

(40) $F^\circ(x,g) = F \cdot \langle \mathrm{dom}\, g \longmapsto x, g\rangle$.

(41) $F^\circ(x,g) = F^\circ(\mathrm{dom}\, g \longmapsto x, g)$.

(42) If $x \in \mathrm{dom}(F^\circ(z,f))$, then $(F^\circ(z,f))(x) = F(z, f(x))$.

(43) If $f \restriction A = g \restriction A$, then $(F^\circ(x,f)) \restriction A = (F^\circ(x,g)) \restriction A$.

(44) $F^\circ(x,f) \cdot h = F^\circ(x, f \cdot h)$.

(45) $h \cdot F^\circ(x,f) = (h \cdot F)^\circ(x,f)$.

(46) $F^\circ(x,f) \cdot \mathrm{id}_A = F^\circ(x, f \restriction A)$.

For simplicity we follow a convention: X, Y, Z will denote non-empty sets, F will denote a binary operation on X, f, g, h will denote functions from Y into X, and x, x_1, x_2 will denote elements of X. Let us consider X. Then id_X is a function from X into X.

We now state a proposition

(47) $F^\circ(f,g)$ is a function from Y into X.

The arguments of the notions defined below are the following: X, Z which are non-empty sets; F which is a binary operation on X; f, g which are functions from Z into X. Then $F^\circ(f,g)$ is a function from Z into X.

We now state a number of propositions:

(48) For every element z of Y holds $(F^\circ(f,g))(z) = F(f(z), g(z))$.

(49) For every function h from Y into X such that for every element z of Y holds $h(z) = F(f(z), g(z))$ holds $h = F^\circ(f,g)$.

(50) For every function h from Z into Y holds $F^\circ(f,g) \cdot h = F^\circ(f \cdot h, g \cdot h)$.

(51) For every function g from X into X holds $F^{\circ}(\mathrm{id}_X, g) \cdot f = F^{\circ}(f, g \cdot f)$.

(52) For every function g from X into X holds $F^{\circ}(g, \mathrm{id}_X) \cdot f = F^{\circ}(g \cdot f, f)$.

(53) $F^{\circ}(\mathrm{id}_X, \mathrm{id}_X) \cdot f = F^{\circ}(f, f)$.

(54) For every function g from X into X holds $(F^{\circ}(\mathrm{id}_X, g))(x) = F(x, g(x))$.

(55) For every function g from X into X holds $(F^{\circ}(g, \mathrm{id}_X))(x) = F(g(x), x)$.

(56) $(F^{\circ}(\mathrm{id}_X, \mathrm{id}_X))(x) = F(x, x)$.

(57) For all A, B for arbitrary x such that $x \in B$ holds $A \longmapsto x$ is a function from A into B.

(58) For all A, X, x holds $A \longmapsto x$ is a function from A into X.

(59) $F^{\circ}(f, x)$ is a function from Y into X.

The arguments of the notions defined below are the following: X, Z which are non-empty sets; F which is a binary operation on X; f which is a function from Z into X; x which is an element of X. Then $F^{\circ}(f, x)$ is a function from Z into X.

The following propositions are true:

(60) For every element y of Y holds $(F^{\circ}(f, x))(y) = F(f(y), x)$.

(61) If for every element y of Y holds $g(y) = F(f(y), x)$, then $g = F^{\circ}(f, x)$.

(62) For every function g from Z into Y holds $F^{\circ}(f, x) \cdot g = F^{\circ}(f \cdot g, x)$.

(63) $F^{\circ}(\mathrm{id}_X, x) \cdot f = F^{\circ}(f, x)$.

(64) $(F^{\circ}(\mathrm{id}_X, x))(x) = F(x, x)$.

(65) $F^{\circ}(x, g)$ is a function from Y into X.

The arguments of the notions defined below are the following: X, Z which are non-empty sets; F which is a binary operation on X; x which is an element of X; g which is a function from Z into X. Then $F^{\circ}(x, g)$ is a function from Z into X.

The following propositions are true:

(66) For every element y of Y holds $(F^{\circ}(x, f))(y) = F(x, f(y))$.

(67) If for every element y of Y holds $g(y) = F(x, f(y))$, then $g = F^{\circ}(x, f)$.

(68) For every function g from Z into Y holds $F^{\circ}(x, f) \cdot g = F^{\circ}(x, f \cdot g)$.

(69) $F^{\circ}(x, \mathrm{id}_X) \cdot f = F^{\circ}(x, f)$.

(70) $(F^{\circ}(x, \mathrm{id}_X))(x) = F(x, x)$.

(71) For all non-empty sets X, Y, Z for every function f from X into $[\!: Y, Z :\!]$ for every element x of X holds $f^{\smile}(x) = \langle (f(x))_{\mathbf{2}}, (f(x))_{\mathbf{1}} \rangle$.

(72) For all non-empty sets X, Y, Z for every function f from X into $[\!: Y, Z :\!]$ holds $\mathrm{rng}\, f$ is a relation between Y and Z.

The arguments of the notions defined below are the following: X, Y, Z which are non-empty sets; f which is a function from X into $[\!: Y, Z :\!]$. Then $\mathrm{rng}\, f$ is a relation between Y and Z.

The arguments of the notions defined below are the following: X, Y, Z which are non-empty sets; f which is a function from X into $[\!: Y, Z :\!]$. Then $f^{\smile}$ is a function from X into $[\!: Z, Y :\!]$.

We now state a proposition

(73) For all non-empty sets X, Y, Z for every function f from X into $[\!: Y, Z :\!]$ holds $\mathrm{rng}(f^{\smile}) = (\mathrm{rng}\, f)^{\smile}$.

In the sequel y denotes an element of Y. One can prove the following propositions:

(74) If F is associative, then $F^{\circ}(F^{\circ}(x_1, f), x_2) = F^{\circ}(x_1, F^{\circ}(f, x_2))$.

(75) If F is associative, then $F^{\circ}(F^{\circ}(f, x), g) = F^{\circ}(f, F^{\circ}(x, g))$.

(76) If F is associative, then $F^{\circ}(F^{\circ}(f, g), h) = F^{\circ}(f, F^{\circ}(g, h))$.

(77) If F is associative, then $F^{\circ}(F(x_1, x_2), f) = F^{\circ}(x_1, F^{\circ}(x_2, f))$.

(78) If F is associative, then $F^{\circ}(f, F(x_1, x_2)) = F^{\circ}(F^{\circ}(f, x_1), x_2)$.

(79) If F is commutative, then $F^{\circ}(x, f) = F^{\circ}(f, x)$.

(80) If F is commutative, then $F^{\circ}(f, g) = F^{\circ}(g, f)$.

(81) If F is idempotent, then $F^{\circ}(f, f) = f$.

(82) If F is idempotent, then $(F^{\circ}(f(y), f))(y) = f(y)$.

(83) If F is idempotent, then $(F^{\circ}(f, f(y)))(y) = f(y)$.

References

[1] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.

[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Czesław Byliński. Graphs of functions. *Formalized Mathematics*, 1(**1**):169–173, 1990.

[6] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[7] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

[8] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[9] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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Abelian Groups, Fields and Vector Spaces [1]

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Summary. This text includes definitions of the Abelian group, field and vector space over a field and some elementary theorems about them.

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The articles [3], [1], and [2] provide the notation and terminology for this paper. We consider group structures which are systems

⟨ a carrier, an addition, a reverse-map, a zero ⟩

where the carrier is a non-empty set, the addition is a binary operation on the carrier, the reverse-map is a unary operation on the carrier, and the zero is an element of the carrier. In the sequel GS denotes a group structure. Let us consider GS. An element of GS is an element of the carrier of GS.

Next we state a proposition

(1) For every element x of the carrier of GS holds x is an element of GS.

We now define three new functors. Let us consider GS. The functor 0_{GS} yields an element of GS and is defined by: 0_{GS} =the zero of GS.
Let x be an element of GS. The functor $-x$ yielding an element of GS, is defined by:

$-x$ =(the reverse-map of GS)(x).

Let y be an element of GS. The functor $x + y$ yielding an element of GS, is defined by:

$x + y$ =(the addition of GS)(x, y).

Next we state three propositions:

(2) 0_{GS} =the zero of GS.

(3) For every element x of GS holds $-x$ =(the reverse-map of GS)(x).

(4) For all elements x, y of GS holds $x + y$ =(the addition of GS)(x, y).

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We now define two new functors. The constant $+_{\mathbb{R}}$ is a binary operation on $\mathbb{R}$ and is defined by:

for all elements x, y of $\mathbb{R}$ holds $+_{\mathbb{R}}(x, y) = x + y$.

The constant $-_{\mathbb{R}}$ is a unary operation on $\mathbb{R}$ and is defined by:

for every element x of $\mathbb{R}$ for every real number x' such that $x' = x$ holds $-_{\mathbb{R}}(x) = -x'$.

The constant $\mathbb{R}_{\mathrm{G}}$ is a group structure and is defined by:

$\mathbb{R}_{\mathrm{G}} = \langle \mathbb{R}, +_{\mathbb{R}}, -_{\mathbb{R}}, 0 \rangle$.

We now state two propositions:

(5) $\mathbb{R}_{\mathrm{G}} = \langle \mathbb{R}, +_{\mathbb{R}}, -_{\mathbb{R}}, 0 \rangle$.

(6) For all elements x, y, z of $\mathbb{R}_{\mathrm{G}}$ holds $x+y = y+x$ and $(x+y)+z = x+(y+z)$ and $x + 0_{\mathbb{R}_{\mathrm{G}}} = x$ and $x + (-x) = 0_{\mathbb{R}_{\mathrm{G}}}$.

The mode Abelian group, which widens to the type a group structure, is defined by:

for all elements x, y, z of it holds $x + y = y + x$ and $(x + y) + z = x + (y + z)$ and $x + 0_{\mathrm{it}} = x$ and $x + (-x) = 0_{\mathrm{it}}$.

The following proposition is true

(7) For all elements x, y, z of GS holds $x+y = y+x$ and $(x+y)+z = x+(y+z)$ and $x+0_{GS} = x$ and $x+(-x) = 0_{GS}$ if and only if GS is an Abelian group.

In the sequel G is an Abelian group and x, y, z are elements of G. We now state four propositions:

(8) $x + y = y + x$.

(9) $x + (y + z) = (x + y) + z$.

(10) $x + 0_G = x$.

(11) $x + (-x) = 0_G$.

Let us consider G, x, y. The functor $x - y$ yielding an element of G, is defined by:

$x - y = x + (-y)$.

The following propositions are true:

(12) $x - y = x + (-y)$.

(13) If $x + y = x + z$, then $y = z$ but if $x + y = z + y$, then $x = z$.

(14) $-0_G = 0_G$.

We consider field structures which are systems

$\langle$ a carrier, a multiplication, an addition, a reverse-map, a unity, a zero $\rangle$

where the carrier is a non-empty set, the multiplication, the addition are binary operations on the carrier, the reverse-map is a unary operation on the carrier, and the unity, the zero are elements of the carrier. In the sequel FS will denote a field structure. We now define five new functors. Let us consider FS. The functor 1_{FS} yields an element of the carrier of FS and is defined by:

1_{FS} =the unity of FS.

The functor 0_{FS} yields an element of the carrier of FS and is defined by:

0_{FS} =the zero of FS.

Let x be an element of the carrier of FS. The functor $-x$ yields an element of the carrier of FS and is defined by:

$-x =$(the reverse-map of $FS)(x)$.

Let y be an element of the carrier of FS. The functor $x \cdot y$ yields an element of the carrier of FS and is defined by:

$x \cdot y =$(the multiplication of $FS)(x, y)$.

The functor $x + y$ yielding an element of the carrier of FS, is defined by:

$x + y =$(the addition of $FS)(x, y)$.

One can prove the following propositions:

(15) $1_{FS} =$the unity of FS.

(16) $0_{FS} =$the zero of FS.

(17) For every element x of the carrier of FS holds $-x =$(the reverse-map of $FS)(x)$.

(18) For all elements x, y of the carrier of FS holds $x \cdot y =$(the multiplication of $FS)(x, y)$.

(19) For all elements x, y of the carrier of FS holds $x + y =$(the addition of $FS)(x, y)$.

The constant $\cdot_{\mathbb{R}}$ is a binary operation on $\mathbb{R}$ and is defined by:
for all elements x, y of $\mathbb{R}$ holds $\cdot_{\mathbb{R}}(x, y) = x \cdot y$.

The constant $\mathbb{R}_{\rm F}$ is a field structure and is defined by:
$\mathbb{R}_{\rm F} = \langle \mathbb{R}, \cdot_{\mathbb{R}}, +_{\mathbb{R}}, -_{\mathbb{R}}, 1, 0 \rangle$.

We now state two propositions:

(20) $\mathbb{R}_{\rm F} = \langle \mathbb{R}, \cdot_{\mathbb{R}}, +_{\mathbb{R}}, -_{\mathbb{R}}, 1, 0 \rangle$.

(21) Let x, y, z be elements of the carrier of $\mathbb{R}_{\rm F}$. Then

(i) $x + y = y + x$,

(ii) $(x + y) + z = x + (y + z)$,

(iii) $x + 0_{\mathbb{R}_{\rm F}} = x$,

(iv) $x + (-x) = 0_{\mathbb{R}_{\rm F}}$,

(v) $x \cdot y = y \cdot x$,

(vi) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(vii) $x \cdot (1_{\mathbb{R}_{\rm F}}) = x$,

(viii) if $x \neq 0_{\mathbb{R}_{\rm F}}$, then there exists y being an element of the carrier of $\mathbb{R}_{\rm F}$ such that $x \cdot y = 1_{\mathbb{R}_{\rm F}}$,

(ix) $0_{\mathbb{R}_{\rm F}} \neq 1_{\mathbb{R}_{\rm F}}$,

(x) $x \cdot (y + z) = x \cdot y + x \cdot z$,

(xi) $(y + z) \cdot x = y \cdot x + z \cdot x$.

The mode field, which widens to the type a field structure, is defined by:
Let x, y, z be elements of the carrier of it . Then

(i) $x + y = y + x$,

(ii) $(x + y) + z = x + (y + z)$,

(iii) $x + 0_{\rm it} = x$,

(iv) $x + (-x) = 0_{\rm it}$,

(v) $x \cdot y = y \cdot x$,

(vi) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(vii) $x \cdot (1_{\rm it}) = x$,

(viii) if $x \neq 0_{\rm it}$, then there exists y being an element of the carrier of it such that $x \cdot y = 1_{\rm it}$,

(ix) $0_{\rm it} \neq 1_{\rm it}$,

(x) $x \cdot (y + z) = x \cdot y + x \cdot z$,

(xi) $(y + z) \cdot x = y \cdot x + z \cdot x$.

We now state a proposition

(22) The following conditions are equivalent:

(i) for all elements x, y, z of the carrier of FS holds $x + y = y + x$ and $(x+y)+z = x+(y+z)$ and $x+0_{FS} = x$ and $x+(-x) = 0_{FS}$ and $x \cdot y = y \cdot x$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $x \cdot (1_{FS}) = x$ but if $x \neq 0_{FS}$, then there exists y being an element of the carrier of FS such that $x \cdot y = 1_{FS}$ and $0_{FS} \neq 1_{FS}$ and $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$,

(ii) FS is a field.

In the sequel F is a field and x, y, z are elements of the carrier of F. The following propositions are true:

(23) $x + y = y + x$.

(24) $(x + y) + z = x + (y + z)$.

(25) $x + 0_F = x$.

(26) $x + (-x) = 0_F$.

(27) $x \cdot y = y \cdot x$.

(28) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(29) $x \cdot (1_F) = x$.

(30) If $x \neq 0_F$, then there exists y such that $x \cdot y = 1_F$.

(31) $0_F \neq 1_F$.

(32) $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$.

(33) If $x \neq 0_F$ and $x \cdot y = x \cdot z$, then $y = z$.

Let us consider F, x. Let us assume that $x \neq 0_F$. The functor x^{-1} yields an element of the carrier of F and is defined by:

$x \cdot (x^{-1}) = 1_F$.

We now state a proposition

(34) If $x \neq 0_F$, then $x \cdot x^{-1} = 1_F$ and $x^{-1} \cdot x = 1_F$.

We now define two new functors. Let us consider F, x, y. The functor $x - y$ yielding an element of the carrier of F, is defined by:

$x - y = x + (-y)$.

The functor $\frac{x}{y}$ yielding an element of the carrier of F, is defined by:

$\frac{x}{y} = x \cdot y^{-1}$.

One can prove the following propositions:

(35) $x - y = x + (-y)$.

(36) $\frac{x}{y} = x \cdot y^{-1}$.

(37) If $x + y = x + z$, then $y = z$ but if $x + y = z + y$, then $x = z$.

(38) $-(x + y) = (-x) + (-y)$.

(39) $x \cdot 0_F = 0_F$ and $0_F \cdot x = 0_F$.
(40) $-(-x) = x$.
(41) $(-x) \cdot y = -x \cdot y$.
(42) $(-x) \cdot (-y) = x \cdot y$.
(43) $x \cdot (y - z) = x \cdot y - x \cdot z$.
(44) $x \cdot y = 0_F$ if and only if $x = 0_F$ or $y = 0_F$.

We consider vector space structures which are systems
⟨ scalars, a carrier, a multiplication ⟩
where the scalars is a field, the carrier is an Abelian group, and the multiplication is a function from [: the carrier of the scalars, the carrier of the carrier :] into the carrier of the carrier. In the sequel VS will denote a vector space structure. Let us consider VS. A vector of VS is an element of the carrier of VS.

One can prove the following proposition

(45) For every element x of the carrier of VS holds x is a vector of VS.

Let us consider F. The mode vector space structure over F, which widens to the type a vector space structure, is defined by:
the scalars of it $= F$.

One can prove the following proposition

(46) For every VS being a vector space structure holds VS is a vector space structure over F if and only if the scalars of $VS = F$.

In the sequel V is a vector space structure over F. The arguments of the notions defined below are the following: F, V which are objects of the type reserved above; x which is an element of the carrier of F; v which is an element of the carrier of V. The functor $x \cdot v$ yields an element of the carrier of V and is defined by:

for every element x' of the carrier of the scalars of V such that $x' = x$ holds $x \cdot v =$(the multiplication of $V)(x', v)$.

We now state a proposition

(47) For every vector space structure V over F for every element x of the carrier of F for every element v of the carrier of V for every element x' of the carrier of the scalars of V such that $x' = x$ holds $x \cdot v =$(the multiplication of $V)(x', v)$.

Let us consider F. The mode vector space over F, which widens to the type a vector space structure over F, is defined by:

Let x, y be elements of the carrier of F. Let v, w be elements of the carrier of it . Then $x \cdot (v+w) = x \cdot v + x \cdot w$ and $(x+y) \cdot v = x \cdot v + y \cdot v$ and $(x \cdot y) \cdot v = x \cdot (y \cdot v)$ and $(1_F) \cdot v = v$.

We now state a proposition

(48) The following conditions are equivalent:
(i) for all elements x, y of the carrier of F for all elements v, w of the carrier of V holds $x \cdot (v + w) = x \cdot v + x \cdot w$ and $(x + y) \cdot v = x \cdot v + y \cdot v$ and $(x \cdot y) \cdot v = x \cdot (y \cdot v)$ and $(1_F) \cdot v = v$,
(ii) V is a vector space over F.

We follow a convention: V, V_1 denote vector spaces over F, x, y denote elements of the carrier of F, and v, w denote elements of the carrier of V. Let us consider F, V. The functor Θ_V yielding an element of the carrier of V, is defined by:

$\Theta_V = 0_{\text{the carrier of } V}$.

One can prove the following propositions:

(49) $\Theta_V = 0_{\text{the carrier of } V}$.

(50) $\Theta_V + v = v$.

(51) $v + \Theta_V = v$.

(52) $v + (-v) = \Theta_V$.

(53) $(-v) + v = \Theta_V$.

(54) $-\Theta_V = \Theta_V$.

(55) $x \cdot (v + w) = x \cdot v + x \cdot w$.

(56) $(x + y) \cdot v = x \cdot v + y \cdot v$.

(57) $(x \cdot y) \cdot v = x \cdot (y \cdot v)$.

(58) $(1_F) \cdot v = v$.

(59) $0_F \cdot v = \Theta_V$ and $(-1_F) \cdot v = -v$ and $x \cdot (\Theta_V) = \Theta_V$.

(60) $x \cdot v = \Theta_V$ if and only if $x = 0_F$ or $v = \Theta_V$.

Let us consider F, V. The mode VSS of V, which widens to the type a vector space over F, is defined by: the carrier of the carrier of it $\subseteq$the carrier of the carrier of V and for all elements v, w of the carrier of it for all elements x, y of the carrier of F holds $x \cdot v + y \cdot w$ is an element of the carrier of it .

The following proposition is true

(61) the carrier of the carrier of V_1 $\subseteq$the carrier of the carrier of V and for all elements v, w of the carrier of V_1 for all elements x, y of the carrier of F holds $x \cdot v + y \cdot w$ is an element of the carrier of V_1 if and only if V_1 is a VSS of V.

In the sequel u, v, w will be elements of the carrier of V. We now state a number of propositions:

(62) $v - w = v + (-w)$.

(63) $v + w = \Theta_V$ if and only if $-v = w$.

(64) (i) $-(v + w) = (-v) - w$,

(ii) $-(-v) = v$,

(iii) $-((-v) + w) = v - w$,

(iv) $-(v - w) = (-v) + w$,

(v) $-((-v) - w) = v + w$,

(vi) $u - (v + w) = (u - v) - w$.

(65) $\Theta_V - v = -v$ and $v - \Theta_V = v$.

(66) $x + (-y) = 0_F$ if and only if $x = y$ but $x - y = 0_F$ if and only if $x = y$.

(67) If $x \neq 0_F$, then $x^{-1} \cdot (x \cdot v) = v$.

(68) $-x \cdot v = (-x) \cdot v$ and $w - x \cdot v = w + (-x) \cdot v$.

(69) $x \cdot (-v) = -x \cdot v$.

(70) $x \cdot (v - w) = x \cdot v - x \cdot w$.

(71) $v - x \cdot (y \cdot w) = v - (x \cdot y) \cdot w$.

(72) $\mathbb{R}_{\mathrm{F}}$ is a field.

(73) If $x \neq 0_F$, then $(x^{-1})^{-1} = x$.

(74) If $x \neq 0_F$, then $x^{-1} \neq 0_F$ and $-x^{-1} \neq 0_F$.

(75) For all elements x, y of $\mathbb{R}$ holds $+_{\mathbb{R}}(x, y) = x + y$.

(76) For every element x of $\mathbb{R}$ for every real number x' such that $x' = x$ holds $-_{\mathbb{R}}(x) = -x'$.

(77) For all elements x, y of $\mathbb{R}$ holds $\cdot_{\mathbb{R}}(x, y) = x \cdot y$.

(78) $1_{\mathbb{R}_{\mathrm{F}}} + 1_{\mathbb{R}_{\mathrm{F}}} \neq 0_{\mathbb{R}_{\mathrm{F}}}$.

The mode Fano field, which widens to the type a field, is defined by:
$1_{\mathrm{it}} + 1_{\mathrm{it}} \neq 0_{\mathrm{it}}$.

The following proposition is true

(79) For every field F holds F is a Fano field if and only if $1_F + 1_F \neq 0_F$.

In the sequel F will denote a field and a, b, c will denote elements of the carrier of F. One can prove the following propositions:

(80) $-(a - b) = (-a) + b$.

(81) $-(a - b) = b - a$.

(82) $0_F + a = a$.

(83) $(-a) + a = 0_F$.

(84) If $a - b = 0_F$, then $a = b$.

(85) $-0_F = 0_F$.

(86) If $-a = 0_F$, then $a = 0_F$.

(87) If $a - b = 0_F$, then $b - a = 0_F$.

(88) If $a \neq 0_F$ and $a \cdot c - b = 0_F$, then $c = b \cdot a^{-1}$ but if $a \neq 0_F$ and $b - c \cdot a = 0_F$, then $c = b \cdot a^{-1}$.

(89) $a + b = -((-a) + (-b))$.

(90) $(a + b) - (a + c) = b - c$.

References

[1] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[2] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[3] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

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Parallelity Spaces [1]

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Summary. In the monography [5] W. Szmielew introduced the parallelity planes $\langle S; \| \rangle$, where $\| \subseteq S\times S\times S\times S$. In this text we omit upper bound axiom which must be satisfied by the parallelity planes (see also E.Kusak [3]). Further we will list those theorems which remain true when we pass from the parallelity planes to the parallelity spaces. We construct a model of the parallelity space in Abelian group $\langle F\times F\times F; +_F, -_F, \mathbf{0}_F\rangle$, where F is a field.

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The papers [7], [6], [2], [1], and [4] provide the terminology and notation for this paper. We follow the rules: F will denote a field, a, b, c, f, g, h will denote elements of the carrier of F, and x, y will denote elements of [: the carrier of F, the carrier of F, the carrier of F :]. Let us consider F. The functor $+_F$ yields a binary operation on [: the carrier of F, the carrier of F, the carrier of F :] and is defined by:

$(+_F)(x,y) = \langle x_{\mathbf{1}} + y_{\mathbf{1}}, x_{\mathbf{2}} + y_{\mathbf{2}}, x_{\mathbf{3}} + y_{\mathbf{3}}\rangle$.

The following proposition is true

(1) $(+_F)(x,y) = \langle x_{\mathbf{1}} + y_{\mathbf{1}}, x_{\mathbf{2}} + y_{\mathbf{2}}, x_{\mathbf{3}} + y_{\mathbf{3}}\rangle$.

Let us consider F, x, y. The functor $x+y$ yielding an element of [: the carrier of F, the carrier of F, the carrier of F :], is defined by:

$x + y = (+_F)(x,y)$.

One can prove the following three propositions:

(2) $x + y = (+_F)(x,y)$.

(3) $x + y = \langle x_{\mathbf{1}} + y_{\mathbf{1}}, x_{\mathbf{2}} + y_{\mathbf{2}}, x_{\mathbf{3}} + y_{\mathbf{3}}\rangle$.

(4) $\langle a,b,c\rangle + \langle f,g,h\rangle = \langle a+f, b+g, c+h\rangle$.

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Let us consider F. The functor $-_F$ yielding a unary operation on $[\!:$ the carrier of F, the carrier of F, the carrier of F $:\!]$, is defined by:

$(-_F)(x) = \langle -x_{\mathbf{1}}, -x_{\mathbf{2}}, -x_{\mathbf{3}} \rangle$.

The following proposition is true

(5) $(-_F)(x) = \langle -x_{\mathbf{1}}, -x_{\mathbf{2}}, -x_{\mathbf{3}} \rangle$.

Let us consider F, x. The functor $-x$ yields an element of $[\!:$ the carrier of F, the carrier of F, the carrier of F $:\!]$ and is defined by:

$-x = (-_F)(x)$.

We now state two propositions:

(6) $(-_F)(x) = -x$.

(7) $-x = \langle -x_{\mathbf{1}}, -x_{\mathbf{2}}, -x_{\mathbf{3}} \rangle$.

In the sequel S denotes a set. Let us consider S. The mode 4-ary relation over the S, which widens to the type a set, is defined by:

it $\subseteq [\!: S, S, S, S :\!]$.

We now state a proposition

(8) For every set R holds $R \subseteq [\!: S, S, S, S :\!]$ if and only if R is a 4-ary relation over the S.

We consider parallelity structures which are systems

$\langle$ a universum, a parallelity $\rangle$

where the universum is a non-empty set and the parallelity is a 4-ary relation over the the universum. In the sequel F is a field and PS is a parallelity structure. The arguments of the notions defined below are the following: PS which is an object of the type reserved above; a, b, c, d which are elements of the universum of PS. The predicate $a, b \parallel c, d$ is defined by:

$\langle a, b, c, d \rangle \in$the parallelity of PS.

Next we state a proposition

(9) For all elements a, b, c, d of the universum of PS holds $a, b \parallel c, d$ if and only if $\langle a, b, c, d \rangle \in$the parallelity of PS.

Let us consider F. The functor $F^{\mathbf{3}}$ yields a non-empty set and is defined by:

$F^{\mathbf{3}} = [\!:$ the carrier of F, the carrier of F, the carrier of F $:\!]$.

Next we state a proposition

(10) $F^{\mathbf{3}} = [\!:$ the carrier of F, the carrier of F, the carrier of F $:\!]$.

Let us consider F. The functor $(F^{\mathbf{3}})^{\mathbf{4}}$ yields a non-empty set and is defined by:

$(F^{\mathbf{3}})^{\mathbf{4}} = [\!: F^{\mathbf{3}}, F^{\mathbf{3}}, F^{\mathbf{3}}, F^{\mathbf{3}} :\!]$.

One can prove the following proposition

(11) $(F^{\mathbf{3}})^{\mathbf{4}} = [\!: F^{\mathbf{3}}, F^{\mathbf{3}}, F^{\mathbf{3}}, F^{\mathbf{3}} :\!]$.

We adopt the following convention: x will be arbitrary and a, b, c, d, e, f, g, h will denote elements of $[\!:$ the carrier of F, the carrier of F, the carrier of F $:\!]$. Let us consider F. The functor $\mathbf{Par'}_F$ yielding a set, is defined by:

$x \in \mathbf{Par'}_F$ if and only if the following conditions are satisfied:

(i) $x \in (F^{\mathbf{3}})^{\mathbf{4}}$,

(ii) there exist a, b, c, d such that $x = \langle a, b, c, d \rangle$ and $(a_{\mathbf{1}} - b_{\mathbf{1}}) \cdot (c_{\mathbf{2}} - d_{\mathbf{2}}) - (c_{\mathbf{1}} - d_{\mathbf{1}}) \cdot (a_{\mathbf{2}} - b_{\mathbf{2}}) = 0_F$ and $(a_{\mathbf{1}} - b_{\mathbf{1}}) \cdot (c_{\mathbf{3}} - d_{\mathbf{3}}) - (c_{\mathbf{1}} - d_{\mathbf{1}}) \cdot (a_{\mathbf{3}} - b_{\mathbf{3}}) = 0_F$ and $(a_{\mathbf{2}} - b_{\mathbf{2}}) \cdot (c_{\mathbf{3}} - d_{\mathbf{3}}) - (c_{\mathbf{2}} - d_{\mathbf{2}}) \cdot (a_{\mathbf{3}} - b_{\mathbf{3}}) = 0_F$.

Next we state two propositions:

(12) (i) For every x holds $x \in \mathbf{Par'}_F$ if and only if $x \in (F^{\mathbf{3}})^{\mathbf{4}}$ and there exist a, b, c, d such that $x = \langle a, b, c, d \rangle$ and $(a_{\mathbf{1}} - b_{\mathbf{1}}) \cdot (c_{\mathbf{2}} - d_{\mathbf{2}}) - (c_{\mathbf{1}} - d_{\mathbf{1}}) \cdot (a_{\mathbf{2}} - b_{\mathbf{2}}) = 0_F$ and $(a_{\mathbf{1}} - b_{\mathbf{1}}) \cdot (c_{\mathbf{3}} - d_{\mathbf{3}}) - (c_{\mathbf{1}} - d_{\mathbf{1}}) \cdot (a_{\mathbf{3}} - b_{\mathbf{3}}) = 0_F$ and $(a_{\mathbf{2}} - b_{\mathbf{2}}) \cdot (c_{\mathbf{3}} - d_{\mathbf{3}}) - (c_{\mathbf{2}} - d_{\mathbf{2}}) \cdot (a_{\mathbf{3}} - b_{\mathbf{3}}) = 0_F$,

(ii) $\mathbf{Par'}_F$ is a set.

(13) $\mathbf{Par'}_F \subseteq [\!: F^{\mathbf{3}}, F^{\mathbf{3}}, F^{\mathbf{3}}, F^{\mathbf{3}} :\!]$.

Let us consider F. The functor $\mathbf{Par}_F$ yielding a 4-ary relation over the $F^{\mathbf{3}}$, is defined by:

$\mathbf{Par}_F = \mathbf{Par'}_F$.

We now state a proposition

(14) $\mathbf{Par}_F = \mathbf{Par'}_F$ and $\mathbf{Par}_F$ is a 4-ary relation over the $F^{\mathbf{3}}$.

Let us consider F. The functor Aff_{F^3} yields a parallelity structure and is defined by:

$\mathrm{Aff}_{F^3} = \langle F^{\mathbf{3}}, \mathbf{Par}_F \rangle$.

We now state three propositions:

(15) $\mathrm{Aff}_{F^3} = \langle F^{\mathbf{3}}, \mathbf{Par}_F \rangle$.

(16) the universum of $\mathrm{Aff}_{F^3} = F^{\mathbf{3}}$.

(17) the parallelity of $\mathrm{Aff}_{F^3} = \mathbf{Par}_F$.

In the sequel a, b, c, d, p, q, r, s denote elements of the universum of Aff_{F^3}. One can prove the following propositions:

(18) $a, b \parallel c, d$ if and only if $\langle a, b, c, d \rangle \in \mathbf{Par}_F$.

(19) $\langle a, b, c, d \rangle \in \mathbf{Par}_F$ if and only if the following conditions are satisfied:

(i) $\langle a, b, c, d \rangle \in (F^{\mathbf{3}})^{\mathbf{4}}$,

(ii) there exist e, f, g, h such that $\langle a, b, c, d \rangle = \langle e, f, g, h \rangle$ and $(e_{\mathbf{1}} - f_{\mathbf{1}}) \cdot (g_{\mathbf{2}} - h_{\mathbf{2}}) - (g_{\mathbf{1}} - h_{\mathbf{1}}) \cdot (e_{\mathbf{2}} - f_{\mathbf{2}}) = 0_F$ and $(e_{\mathbf{1}} - f_{\mathbf{1}}) \cdot (g_{\mathbf{3}} - h_{\mathbf{3}}) - (g_{\mathbf{1}} - h_{\mathbf{1}}) \cdot (e_{\mathbf{3}} - f_{\mathbf{3}}) = 0_F$ and $(e_{\mathbf{2}} - f_{\mathbf{2}}) \cdot (g_{\mathbf{3}} - h_{\mathbf{3}}) - (g_{\mathbf{2}} - h_{\mathbf{2}}) \cdot (e_{\mathbf{3}} - f_{\mathbf{3}}) = 0_F$.

(20) $a, b \parallel c, d$ if and only if the following conditions are satisfied:

(i) $\langle a, b, c, d \rangle \in (F^{\mathbf{3}})^{\mathbf{4}}$,

(ii) there exist e, f, g, h such that $\langle a, b, c, d \rangle = \langle e, f, g, h \rangle$ and $(e_{\mathbf{1}} - f_{\mathbf{1}}) \cdot (g_{\mathbf{2}} - h_{\mathbf{2}}) - (g_{\mathbf{1}} - h_{\mathbf{1}}) \cdot (e_{\mathbf{2}} - f_{\mathbf{2}}) = 0_F$ and $(e_{\mathbf{1}} - f_{\mathbf{1}}) \cdot (g_{\mathbf{3}} - h_{\mathbf{3}}) - (g_{\mathbf{1}} - h_{\mathbf{1}}) \cdot (e_{\mathbf{3}} - f_{\mathbf{3}}) = 0_F$ and $(e_{\mathbf{2}} - f_{\mathbf{2}}) \cdot (g_{\mathbf{3}} - h_{\mathbf{3}}) - (g_{\mathbf{2}} - h_{\mathbf{2}}) \cdot (e_{\mathbf{3}} - f_{\mathbf{3}}) = 0_F$.

(21) the universum of $\mathrm{Aff}_{F^3} = [\!:$ the carrier of F, the carrier of F, the carrier of $F :\!]$.

(22) $\langle a, b, c, d \rangle \in (F^{\mathbf{3}})^{\mathbf{4}}$.

(23) $a, b \parallel c, d$ if and only if there exist e, f, g, h such that $\langle a, b, c, d \rangle = \langle e, f, g, h \rangle$ and $(e_{\mathbf{1}} - f_{\mathbf{1}}) \cdot (g_{\mathbf{2}} - h_{\mathbf{2}}) - (g_{\mathbf{1}} - h_{\mathbf{1}}) \cdot (e_{\mathbf{2}} - f_{\mathbf{2}}) = 0_F$ and $(e_{\mathbf{1}} - f_{\mathbf{1}}) \cdot (g_{\mathbf{3}} - h_{\mathbf{3}}) - (g_{\mathbf{1}} - h_{\mathbf{1}}) \cdot (e_{\mathbf{3}} - f_{\mathbf{3}}) = 0_F$ and $(e_{\mathbf{2}} - f_{\mathbf{2}}) \cdot (g_{\mathbf{3}} - h_{\mathbf{3}}) - (g_{\mathbf{2}} - h_{\mathbf{2}}) \cdot (e_{\mathbf{3}} - f_{\mathbf{3}}) = 0_F$.

(24) $a, b \parallel b, a$.

(25) $a, b \parallel c, c$.

(26) If $a, b \parallel p, q$ and $a, b \parallel r, s$, then $p, q \parallel r, s$ or $a = b$.

(27) If $a, b \parallel a, c$, then $b, a \parallel b, c$.

(28) There exists d such that $a, b \parallel c, d$ and $a, c \parallel b, d$.

The mode parallelity space, which widens to the type a parallelity structure, is defined by:

Let a, b, c, d, p, q, r, s be elements of the universum of it . Then

(i) $a, b \parallel b, a$,

(ii) $a, b \parallel c, c$,

(iii) if $a, b \parallel p, q$ and $a, b \parallel r, s$, then $p, q \parallel r, s$ or $a = b$,

(iv) if $a, b \parallel a, c$, then $b, a \parallel b, c$,

(v) there exists x being an element of the universum of it such that $a, b \parallel c, x$ and $a, c \parallel b, x$.

We now state a proposition

(29) Let P be a parallelity structure. Then the following conditions are equivalent:
   (i) for all elements a, b, c, d, p, q, r, s of the universum of P holds $a, b \parallel b, a$ and $a, b \parallel c, c$ but if $a, b \parallel p, q$ and $a, b \parallel r, s$, then $p, q \parallel r, s$ or $a = b$ but if $a, b \parallel a, c$, then $b, a \parallel b, c$ and there exists x being an element of the universum of P such that $a, b \parallel c, x$ and $a, c \parallel b, x$,
   (ii) P is a parallelity space.

We follow the rules: PS denotes a parallelity space and a, b, c, d, p, q, r, s denote elements of the universum of PS. One can prove the following propositions:

(30) $a, b \parallel b, a$.

(31) $a, b \parallel c, c$.

(32) If $a, b \parallel p, q$ and $a, b \parallel r, s$, then $p, q \parallel r, s$ or $a = b$.

(33) If $a, b \parallel a, c$, then $b, a \parallel b, c$.

(34) There exists d such that $a, b \parallel c, d$ and $a, c \parallel b, d$.

(35) $a, b \parallel a, b$.

(36) If $a, b \parallel c, d$, then $c, d \parallel a, b$.

(37) $a, a \parallel b, c$.

(38) If $a, b \parallel c, d$, then $b, a \parallel c, d$.

(39) If $a, b \parallel c, d$, then $a, b \parallel d, c$.

(40) If $a, b \parallel c, d$, then $b, a \parallel c, d$ and $a, b \parallel d, c$ and $b, a \parallel d, c$ and $c, d \parallel a, b$ and $d, c \parallel a, b$ and $c, d \parallel b, a$ and $d, c \parallel b, a$.

(41) Suppose $a, b \parallel a, c$. Then $a, c \parallel a, b$ and $b, a \parallel a, c$ and $a, b \parallel c, a$ and $a, c \parallel b, a$ and $b, a \parallel c, a$ and $c, a \parallel a, b$ and $c, a \parallel b, a$ and $b, a \parallel b, c$ and $a, b \parallel b, c$ and $b, a \parallel c, b$ and $b, c \parallel b, a$ and $a, b \parallel c, b$ and $c, b \parallel b, a$ and $b, c \parallel a, b$ and $c, b \parallel a, b$ and $c, a \parallel c, b$ and $a, c \parallel c, b$ and $c, a \parallel b, c$ and $a, c \parallel b, c$ and $c, b \parallel c, a$ and $b, c \parallel c, a$ and $c, b \parallel a, c$ and $b, c \parallel a, c$.

(42) If $a = b$ or $c = d$ or $a = c$ and $b = d$ or $a = d$ and $b = c$, then $a, b \parallel c, d$.

(43) If $a \neq b$ and $p, q \parallel a, b$ and $a, b \parallel r, s$, then $p, q \parallel r, s$.

(44) If $a, b \nparallel a, c$, then $a \neq b$ and $b \neq c$ and $c \neq a$.

(45) If $a, b \nparallel c, d$, then $a \neq b$ and $c \neq d$.

(46) Suppose $a, b \nparallel c, d$. Then $b, a \nparallel c, d$ and $a, b \nparallel d, c$ and $b, a \nparallel d, c$ and $c, d \nparallel a, b$ and $d, c \nparallel a, b$ and $c, d \nparallel b, a$ and $d, c \nparallel b, a$.

(47) Suppose $a, b \nparallel a, c$. Then $a, c \nparallel a, b$ and $b, a \nparallel a, c$ and $a, b \nparallel c, a$ and $a, c \nparallel b, a$ and $b, a \nparallel c, a$ and $c, a \nparallel a, b$ and $c, a \nparallel b, a$ and $b, a \nparallel b, c$ and $a, b \nparallel b, c$ and $b, a \nparallel c, b$ and $b, c \nparallel b, a$ and $b, a \nparallel c, b$ and $c, b \nparallel b, a$ and $b, c \nparallel a, b$ and $c, b \nparallel a, b$ and $c, a \nparallel c, b$ and $a, c \nparallel c, b$ and $c, a \nparallel b, c$ and $a, c \nparallel b, c$ and $c, b \nparallel c, a$ and $b, c \nparallel c, a$ and $c, b \nparallel a, c$ and $b, c \nparallel a, c$.

(48) If $a, b \nparallel c, d$ and $a, b \parallel p, q$ and $c, d \parallel r, s$ and $p \neq q$ and $r \neq s$, then $p, q \nparallel r, s$.

(49) If $a, b \nparallel a, c$ and $a, b \parallel p, q$ and $a, c \parallel p, r$ and $b, c \parallel q, r$ and $p \neq q$, then $p, q \nparallel p, r$.

(50) If $a, b \nparallel a, c$ and $a, c \parallel p, r$ and $b, c \parallel p, r$, then $p = r$.

(51) If $p, q \nparallel p, r$ and $p, r \parallel p, s$ and $q, r \parallel q, s$, then $r = s$.

(52) If $a, b \nparallel a, c$ and $a, b \parallel p, q$ and $a, c \parallel p, r$ and $a, c \parallel p, s$ and $b, c \parallel q, r$ and $b, c \parallel q, s$, then $r = s$.

(53) If $a, b \parallel a, c$ and $a, b \parallel a, d$, then $a, b \parallel c, d$.

(54) If for all a, b holds $a = b$, then for all p, q, r, s holds $p, q \parallel r, s$.

(55) If there exist a, b such that $a \neq b$ and for every c holds $a, b \parallel a, c$, then for all p, q, r, s holds $p, q \parallel r, s$.

(56) If $a, b \nparallel a, c$ and $p \neq q$, then $p, q \nparallel p, a$ or $p, q \nparallel p, b$ or $p, q \nparallel p, c$.

References

[1] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[2] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[3] Eugeniusz Kusak. A new approach to dimension-free affine geometry. *Bull. Acad. Polon. Sci. Sér. Sci. Math.*, 27(11 – 12):875 – 882, 1979.

[4] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[5] Wanda Szmielew. *From Affine to Euclidean Geometry.* Volume 27, PWN – D.Reidel Publ. Co., Warszawa – Dordrecht, 1983.

[6] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

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Construction of a bilinear antisymmetric form in symplectic vector space [1]

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Summary. In this text we will present unpublished results by Eugeniusz Kusak. It contains an axiomatic description of the class of all spaces $\langle V;\perp_\xi\rangle$, where V is a vector space over a field F, $\xi : V \times V \to F$ is a bilinear antisymmetric form i.e. $\xi(x,y) = -\xi(y,x)$ and $x \perp_\xi y$ iff $\xi(x,y) = 0$ for x, $y \in V$. It also contains an effective construction of bilinear antisymmetric form ξ for given symplectic space $\langle V;\perp\rangle$ such that $\perp=\perp_\xi$. The basic tool used in this method is the notion of orthogonal projection $\mathrm{J}(a,b,x)$ for $a,b,x \in V$. We should stress the fact that axioms of orthogonal and symplectic spaces differ only by one axiom, namely: $x \perp y + \varepsilon z \;\&\; y \perp z + \varepsilon x \Rightarrow z \perp x + \varepsilon y$. For $\varepsilon = +1$ we get the axiom characterizing symplectic geometry. For $\varepsilon = -1$ we get the axiom on three perpendiculars characterizing orthogonal geometry - see [1].

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The terminology and notation used in this paper have been introduced in the following papers: [2], and [3]. In the sequel F will be a field. We consider symplectic structures which are systems

$\langle$ scalars, a carrier, an orthogonality $\rangle$

where the scalars is a field, the carrier is a vector space over the scalars, and the orthogonality is a relation on the carrier of the carrier of the carrier. The arguments of the notions defined below are the following: S which is a symplectic structure; a, b which are elements of the carrier of the carrier of S. The predicate $a \perp b$ is defined by:

$\langle a,b\rangle \in$the orthogonality of S.

One can prove the following proposition

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(1) For every S being a symplectic structure for all elements a, b of the carrier of the carrier of S holds $a \perp b$ if and only if $\langle a, b\rangle \in$the orthogonality of S.

The mode symplectic space, which widens to the type a symplectic structure, is defined by:

Let a, b, c, x be elements of the carrier of the carrier of it . Let l be an element of the carrier of the scalars of it . Then

(i) if $a \neq \Theta_{\text{the carrier of it}}$, then there exists y being an element of the carrier of the carrier of it such that $y \not\perp a$,

(ii) if $a \perp b$, then $l \cdot a \perp b$,

(iii) if $b \perp a$ and $c \perp a$, then $b + c \perp a$,

(iv) if $b \not\perp a$, then there exists k being an element of the carrier of the scalars of it such that $x - k \cdot b \perp a$,

(v) if $a \perp b + c$ and $b \perp c + a$, then $c \perp a + b$.

In the sequel S is a symplectic structure. We now state a proposition

(2) The following conditions are equivalent:

(i) for all elements a, b, c, x of the carrier of the carrier of S for every element l of the carrier of the scalars of S holds if $a \neq \Theta_{\text{the carrier of } S}$, then there exists y being an element of the carrier of the carrier of S such that $y \not\perp a$ but if $a \perp b$, then $l \cdot a \perp b$ but if $b \perp a$ and $c \perp a$, then $b + c \perp a$ but if $b \not\perp a$, then there exists k being an element of the carrier of the scalars of S such that $x - k \cdot b \perp a$ but if $a \perp b + c$ and $b \perp c + a$, then $c \perp a + b$,

(ii) S is a symplectic space.

We follow the rules: S is a symplectic space, a, b, c, d, a', b', p, q, x, y, z are elements of the carrier of the carrier of S, and k, l are elements of the carrier of the scalars of S. Let us consider S. The functor 0_S yields an element of the carrier of the scalars of S and is defined by:

$0_S = 0_{\text{the scalars of } S}$.

Next we state a proposition

(3) $0_S = 0_{\text{the scalars of } S}$.

Let us consider S. The functor Ω_S yielding an element of the carrier of the scalars of S, is defined by:

$\Omega_S = 1_{\text{the scalars of } S}$.

The following proposition is true

(4) $\Omega_S = 1_{\text{the scalars of } S}$.

Let us consider S. The functor Θ_S yields an element of the carrier of the carrier of S and is defined by:

$\Theta_S = \Theta_{\text{the carrier of } S}$.

The following propositions are true:

(5) $\Theta_S = \Theta_{\text{the carrier of } S}$.

(6) If $a \neq \Theta_S$, then there exists b such that $b \not\perp a$.

(7) If $a \perp b$, then $l \cdot a \perp b$.

(8) If $b \perp a$ and $c \perp a$, then $b + c \perp a$.

(9) If $b \not\perp a$, then there exists l such that $x - l \cdot b \perp a$.

(10) If $a \perp b + c$ and $b \perp c + a$, then $c \perp a + b$.
(11) $\Theta_S \perp a$.
(12) If $a \perp b$, then $b \perp a$.
(13) If $a \not\perp b$ and $c + a \perp b$, then $c \not\perp b$.
(14) If $b \not\perp a$ and $c \perp a$, then $b + c \not\perp a$.
(15) If $b \not\perp a$ and $l \neq 0_S$, then $l \cdot b \not\perp a$ and $b \not\perp l \cdot a$.
(16) If $a \perp b$, then $-a \perp b$.
(17) If $a + b \perp c$ and $a \perp c$, then $b \perp c$.
(18) If $a + b \perp c$ and $b \perp c$, then $a \perp c$.
(19) If $a \not\perp c$, then $a + b \not\perp c$ or $(\Omega_S + \Omega_S) \cdot a + b \not\perp c$.
(20) If $a' \not\perp a$ and $a' \perp b$ and $b' \not\perp b$ and $b' \perp a$, then $a' + b' \not\perp a$ and $a' + b' \not\perp b$.
(21) If $a \neq \Theta_S$ and $b \neq \Theta_S$, then there exists p such that $p \not\perp a$ and $p \not\perp b$.
(22) If $\Omega_S + \Omega_S \neq 0_S$ and $a \neq \Theta_S$ and $b \neq \Theta_S$ and $c \neq \Theta_S$, then there exists p such that $p \not\perp a$ and $p \not\perp b$ and $p \not\perp c$.
(23) If $a - b \perp d$ and $a - c \perp d$, then $b - c \perp d$.
(24) If $b \not\perp a$ and $x - k \cdot b \perp a$ and $x - l \cdot b \perp a$, then $k = l$.
(25) If $\Omega_S + \Omega_S \neq 0_S$, then $a \perp a$.

Let us consider S, a, b, x. Let us assume that $b \not\perp a$. The functor $\mathrm{J}(a, b, x)$ yields an element of the carrier of the scalars of S and is defined by:

for every element l of the carrier of the scalars of S such that $x - l \cdot b \perp a$ holds $\mathrm{J}(a, b, x) = l$.

The following propositions are true:

(26) If $b \not\perp a$ and $x - l \cdot b \perp a$, then $\mathrm{J}(a, b, x) = l$.
(27) If $b \not\perp a$, then $x - \mathrm{J}(a, b, x) \cdot b \perp a$.
(28) If $b \not\perp a$, then $\mathrm{J}(a, b, l \cdot x) = l \cdot \mathrm{J}(a, b, x)$.
(29) If $b \not\perp a$, then $\mathrm{J}(a, b, x + y) = \mathrm{J}(a, b, x) + \mathrm{J}(a, b, y)$.
(30) If $b \not\perp a$ and $l \neq 0_S$, then $\mathrm{J}(a, l \cdot b, x) = l^{-1} \cdot \mathrm{J}(a, b, x)$.
(31) If $b \not\perp a$ and $l \neq 0_S$, then $\mathrm{J}(l \cdot a, b, x) = \mathrm{J}(a, b, x)$.
(32) If $b \not\perp a$ and $p \perp a$, then $\mathrm{J}(a, b + p, c) = \mathrm{J}(a, b, c)$ and $\mathrm{J}(a, b, c + p) = \mathrm{J}(a, b, c)$.
(33) If $b \not\perp a$ and $p \perp b$ and $p \perp c$, then $\mathrm{J}(a + p, b, c) = \mathrm{J}(a, b, c)$.
(34) If $b \not\perp a$ and $c - b \perp a$, then $\mathrm{J}(a, b, c) = \Omega_S$.
(35) If $b \not\perp a$, then $\mathrm{J}(a, b, b) = \Omega_S$.
(36) If $b \not\perp a$, then $x \perp a$ if and only if $\mathrm{J}(a, b, x) = 0_S$.
(37) If $b \not\perp a$ and $q \not\perp a$, then $\mathrm{J}(a, b, p) \cdot \mathrm{J}(a, b, q)^{-1} = \mathrm{J}(a, q, p)$.
(38) If $b \not\perp a$ and $c \not\perp a$, then $\mathrm{J}(a, b, c) = \mathrm{J}(a, c, b)^{-1}$.
(39) If $b \not\perp a$ and $b \perp c + a$, then $\mathrm{J}(a, b, c) = \mathrm{J}(c, b, a)$.
(40) If $a \not\perp b$ and $c \not\perp b$, then $\mathrm{J}(c, b, a) = (-\mathrm{J}(b, a, c)^{-1}) \cdot \mathrm{J}(a, b, c)$.
(41) If $\Omega_S + \Omega_S \neq 0_S$ and $a \not\perp p$ and $a \not\perp q$ and $b \not\perp p$ and $b \not\perp q$, then $\mathrm{J}(a, p, q) \cdot \mathrm{J}(b, q, p) = \mathrm{J}(p, a, b) \cdot \mathrm{J}(q, b, a)$.

(42) If $\Omega_S + \Omega_S \neq 0_S$ and $p \not\perp a$ and $p \not\perp x$ and $q \not\perp a$ and $q \not\perp x$, then $\mathrm{J}(a, q, p) \cdot \mathrm{J}(p, a, x) = \mathrm{J}(x, q, p) \cdot \mathrm{J}(q, a, x)$.

(43) Suppose $\Omega_S + \Omega_S \neq 0_S$ and $p \not\perp a$ and $p \not\perp x$ and $q \not\perp a$ and $q \not\perp x$ and $b \not\perp a$. Then $(\mathrm{J}(a, b, p) \cdot \mathrm{J}(p, a, x)) \cdot \mathrm{J}(x, p, y) = (\mathrm{J}(a, b, q) \cdot \mathrm{J}(q, a, x)) \cdot \mathrm{J}(x, q, y)$.

(44) If $a \not\perp p$ and $x \not\perp p$ and $y \not\perp p$, then $\mathrm{J}(p, a, x) \cdot \mathrm{J}(x, p, y) = (-\mathrm{J}(p, a, y)) \cdot \mathrm{J}(y, p, x)$.

Let us consider S, x, y, a, b. Let us assume that $b \not\perp a$ and $\Omega_S + \Omega_S \neq 0_S$. The functor $x \cdot_{a,b} y$ yields an element of the carrier of the scalars of S and is defined by:

for every q such that $q \not\perp a$ and $q \not\perp x$ holds $x \cdot_{a,b} y = (\mathrm{J}(a, b, q) \cdot \mathrm{J}(q, a, x)) \cdot \mathrm{J}(x, q, y)$ if there exists p such that $p \not\perp a$ and $p \not\perp x$, $x \cdot_{a,b} y = 0_S$ if for every p holds $p \perp a$ or $p \perp x$.

One can prove the following propositions:

(45) If $\Omega_S + \Omega_S \neq 0_S$ and $b \not\perp a$ and $p \not\perp a$ and $p \not\perp x$, then $x \cdot_{a,b} y = (\mathrm{J}(a, b, p) \cdot \mathrm{J}(p, a, x)) \cdot \mathrm{J}(x, p, y)$.

(46) If $\Omega_S + \Omega_S \neq 0_S$ and $b \not\perp a$ and for every p holds $p \perp a$ or $p \perp x$, then $x \cdot_{a,b} y = 0_S$.

(47) If $\Omega_S + \Omega_S \neq 0_S$ and $b \not\perp a$ and $x = \Theta_S$, then $x \cdot_{a,b} y = 0_S$.

(48) If $\Omega_S + \Omega_S \neq 0_S$ and $b \not\perp a$, then $x \cdot_{a,b} y = 0_S$ if and only if $y \perp x$.

(49) If $\Omega_S + \Omega_S \neq 0_S$ and $b \not\perp a$, then $x \cdot_{a,b} y = -y \cdot_{a,b} x$.

(50) If $\Omega_S + \Omega_S \neq 0_S$ and $b \not\perp a$, then $x \cdot_{a,b} (l \cdot y) = l \cdot x \cdot_{a,b} y$.

(51) If $\Omega_S + \Omega_S \neq 0_S$ and $b \not\perp a$, then $x \cdot_{a,b} (y + z) = x \cdot_{a,b} y + x \cdot_{a,b} z$.

References

[1] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Construction of a bilinear symmetric form in orthogonal vector space. *Formalized Mathematics*, 1(**2**):353–356, 1990.

[2] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[3] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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Construction of a bilinear symmetric form in orthogonal vector space [1]

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Summary. In this text we present unpublished results by Eugeniusz Kusak and Wojciech Leończuk. They contain an axiomatic description of the class of all spaces $\langle V;\ \perp_\xi \rangle$, where V is a vector space over a field F, $\xi : V \times V \to F$ is a bilinear symmetric form i.e. $\xi(x,y) = \xi(y,x)$ and $x \perp_\xi y$ iff $\xi(x,y) = 0$ for $x,\ y \in V$. They also contain an effective construction of bilinear symmetric form ξ for given orthogonal space $\langle V;\ \perp \rangle$ such that $\perp = \perp_\xi$. The basic tool used in this method is the notion of orthogonal projection $\mathrm{J}(a,b,x)$ for $a,b,x \in V$. We should stress the fact that axioms of orthogonal and symplectic spaces differ only by one axiom, namely: $x \perp y + \varepsilon z \,\&\, y \perp z + \varepsilon x \Rightarrow z \perp x + \varepsilon y$. For $\varepsilon = -1$ we get the axiom on three perpendiculars characterizing orthogonal geometry. For $\varepsilon = +1$ we get the axiom characterizing symplectic geometry - see [1].

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The papers [2], and [3] provide the terminology and notation for this paper. In the sequel F will be a field. We consider orthogonality structures which are systems

⟨ scalars, a carrier, an orthogonality ⟩

where the scalars is a field, the carrier is a vector space over the scalars, and the orthogonality is a relation on the carrier of the carrier of the carrier. The arguments of the notions defined below are the following: O which is an orthogonality structure; a, b which are elements of the carrier of the carrier of O. The predicate $a \perp b$ is defined by:

$\langle a, b \rangle \in$the orthogonality of O.

The following proposition is true

(1) For every O being an orthogonality structure for all elements a, b of the carrier of the carrier of O holds $a \perp b$ if and only if $\langle a, b \rangle \in$the orthogonality of O.

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The mode orthogonality space, which widens to the type an orthogonality structure, is defined by:

Let a, b, c, d, x be elements of the carrier of the carrier of it . Let l be an element of the carrier of the scalars of it . Then

(i) if $a \neq \Theta_{\text{the carrier of it}}$ and $b \neq \Theta_{\text{the carrier of it}}$ and $c \neq \Theta_{\text{the carrier of it}}$ and $d \neq \Theta_{\text{the carrier of it}}$, then there exists p being an element of the carrier of the carrier of it such that $p \not\perp a$ and $p \not\perp b$ and $p \not\perp c$ and $p \not\perp d$,

(ii) if $a \perp b$, then $l \cdot a \perp b$,

(iii) if $b \perp a$ and $c \perp a$, then $b + c \perp a$,

(iv) if $b \not\perp a$, then there exists k being an element of the carrier of the scalars of it such that $x - k \cdot b \perp a$,

(v) if $a \perp b - c$ and $b \perp c - a$, then $c \perp a - b$.

In the sequel S will denote an orthogonality structure. Next we state a proposition

(2) The following conditions are equivalent:

(i) for all elements a, b, c, d, x of the carrier of the carrier of S for every element l of the carrier of the scalars of S holds if $a \neq \Theta_{\text{the carrier of } S}$ and $b \neq \Theta_{\text{the carrier of } S}$ and $c \neq \Theta_{\text{the carrier of } S}$ and $d \neq \Theta_{\text{the carrier of } S}$, then there exists p being an element of the carrier of the carrier of S such that $p \not\perp a$ and $p \not\perp b$ and $p \not\perp c$ and $p \not\perp d$ but if $a \perp b$, then $l \cdot a \perp b$ but if $b \perp a$ and $c \perp a$, then $b + c \perp a$ but if $b \not\perp a$, then there exists k being an element of the carrier of the scalars of S such that $x - k \cdot b \perp a$ but if $a \perp b - c$ and $b \perp c - a$, then $c \perp a - b$,

(ii) S is an orthogonality space.

We adopt the following convention: S denotes an orthogonality space, a, b, c, d, p, q, x, y, z denote elements of the carrier of the carrier of S, and k, l denote elements of the carrier of the scalars of S. Let us consider S. The functor 0_S yielding an element of the carrier of the scalars of S, is defined by:

$0_S = 0_{\text{the scalars of } S}$.

One can prove the following proposition

(3) $0_S = 0_{\text{the scalars of } S}$.

Let us consider S. The functor Ω_S yields an element of the carrier of the scalars of S and is defined by:

$\Omega_S = 1_{\text{the scalars of } S}$.

The following proposition is true

(4) $\Omega_S = 1_{\text{the scalars of } S}$.

Let us consider S. The functor Θ_S yields an element of the carrier of the carrier of S and is defined by:

$\Theta_S = \Theta_{\text{the carrier of } S}$.

One can prove the following propositions:

(5) $\Theta_S = \Theta_{\text{the carrier of } S}$.

(6) If $a \neq \Theta_S$ and $b \neq \Theta_S$ and $c \neq \Theta_S$ and $d \neq \Theta_S$, then there exists p such that $p \not\perp a$ and $p \not\perp b$ and $p \not\perp c$ and $p \not\perp d$.

(7) If $a \perp b$, then $l \cdot a \perp b$.

(8) If $b \perp a$ and $c \perp a$, then $b + c \perp a$.

(9) If $b \not\perp a$, then there exists k such that $x - k \cdot b \perp a$.

(10) If $a \perp b - c$ and $b \perp c - a$, then $c \perp a - b$.

(11) $\Theta_S \perp a$.

(12) If $a \perp b$, then $b \perp a$.

(13) If $a \not\perp b$ and $c + a \perp b$, then $c \not\perp b$.

(14) If $b \not\perp a$ and $c \perp a$, then $b + c \not\perp a$.

(15) If $b \not\perp a$ and $l \neq 0_S$, then $l \cdot b \not\perp a$ and $b \not\perp l \cdot a$.

(16) If $a \perp b$, then $-a \perp b$.

(17) If $a + b \perp c$ and $a \perp c$, then $b \perp c$.

(18) If $a + b \perp c$ and $b \perp c$, then $a \perp c$.

(19) If $a - b \perp d$ and $a - c \perp d$, then $b - c \perp d$.

(20) If $b \not\perp a$ and $x - k \cdot b \perp a$ and $x - l \cdot b \perp a$, then $k = l$.

(21) If $a \perp a$ and $b \perp b$, then $a + b \perp a - b$.

(22) If $\Omega_S + \Omega_S \neq 0_S$ and there exists a such that $a \neq \Theta_S$, then there exists b such that $b \not\perp b$.

Let us consider S, a, b, x. Let us assume that $b \not\perp a$. The functor $J(a, b, x)$ yielding an element of the carrier of the scalars of S, is defined by:

for every element l of the carrier of the scalars of S such that $x - l \cdot b \perp a$ holds $J(a, b, x) = l$.

Next we state a number of propositions:

(23) If $b \not\perp a$ and $x - l \cdot b \perp a$, then $J(a, b, x) = l$.

(24) If $b \not\perp a$, then $x - J(a, b, x) \cdot b \perp a$.

(25) If $b \not\perp a$, then $J(a, b, l \cdot x) = l \cdot J(a, b, x)$.

(26) If $b \not\perp a$, then $J(a, b, x + y) = J(a, b, x) + J(a, b, y)$.

(27) If $b \not\perp a$ and $l \neq 0_S$, then $J(a, l \cdot b, x) = l^{-1} \cdot J(a, b, x)$.

(28) If $b \not\perp a$ and $l \neq 0_S$, then $J(l \cdot a, b, x) = J(a, b, x)$.

(29) If $b \not\perp a$ and $p \perp a$, then $J(a, b + p, c) = J(a, b, c)$ and $J(a, b, c + p) = J(a, b, c)$.

(30) If $b \not\perp a$ and $p \perp b$ and $p \perp c$, then $J(a + p, b, c) = J(a, b, c)$.

(31) If $b \not\perp a$ and $c - b \perp a$, then $J(a, b, c) = \Omega_S$.

(32) If $b \not\perp a$, then $J(a, b, b) = \Omega_S$.

(33) If $b \not\perp a$, then $x \perp a$ if and only if $J(a, b, x) = 0_S$.

(34) If $b \not\perp a$ and $q \not\perp a$, then $J(a, b, p) \cdot J(a, b, q)^{-1} = J(a, q, p)$.

(35) If $b \not\perp a$ and $c \not\perp a$, then $J(a, b, c) = J(a, c, b)^{-1}$.

(36) If $b \not\perp a$ and $b \perp c + a$, then $J(a, b, c) = -J(c, b, a)$.

(37) If $a \not\perp b$ and $c \not\perp b$, then $J(c, b, a) = J(b, a, c)^{-1} \cdot J(a, b, c)$.

(38) If $p \not\perp a$ and $p \not\perp x$ and $q \not\perp a$ and $q \not\perp x$, then $J(a, q, p) \cdot J(p, a, x) = J(q, a, x) \cdot J(x, q, p)$.

(39) Suppose $p \not\perp a$ and $p \not\perp x$ and $q \not\perp a$ and $q \not\perp x$ and $b \not\perp a$. Then $(J(a,b,p) \cdot J(p,a,x)) \cdot J(x,p,y) = (J(a,b,q) \cdot J(q,a,x)) \cdot J(x,q,y)$.

(40) If $a \not\perp p$ and $x \not\perp p$ and $y \not\perp p$, then $J(p,a,x) \cdot J(x,p,y) = J(p,a,y) \cdot J(y,p,x)$.

Let us consider S, x, y, a, b. Let us assume that $b \not\perp a$. The functor $x \cdot_{a,b} y$ yielding an element of the carrier of the scalars of S, is defined by:

for every q such that $q \not\perp a$ and $q \not\perp x$ holds $x \cdot_{a,b} y = (J(a,b,q) \cdot J(q,a,x)) \cdot J(x,q,y)$ if there exists p such that $p \not\perp a$ and $p \not\perp x$, $x \cdot_{a,b} y = 0_S$ if for every p holds $p \perp a$ or $p \perp x$.

One can prove the following propositions:

(41) If $b \not\perp a$ and $p \not\perp a$ and $p \not\perp x$, then $x \cdot_{a,b} y = (J(a,b,p) \cdot J(p,a,x)) \cdot J(x,p,y)$.

(42) If $b \not\perp a$ and for every p holds $p \perp a$ or $p \perp x$, then $x \cdot_{a,b} y = 0_S$.

(43) If $b \not\perp a$ and $x = \Theta_S$, then $x \cdot_{a,b} y = 0_S$.

(44) If $b \not\perp a$, then $x \cdot_{a,b} y = 0_S$ if and only if $y \perp x$.

(45) If $b \not\perp a$, then $x \cdot_{a,b} y = y \cdot_{a,b} x$.

(46) If $b \not\perp a$, then $x \cdot_{a,b} (l \cdot y) = l \cdot x \cdot_{a,b} y$.

(47) If $b \not\perp a$, then $x \cdot_{a,b} (y + z) = x \cdot_{a,b} y + x \cdot_{a,b} z$.

References

[1] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Construction of a bilinear antisymmetric form in symplectic vector space. *Formalized Mathematics*, 1(**2**):349–352, 1990.

[2] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[3] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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Partial Functions

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Summary. In the article we define partial functions. We also define the following notions related to partial functions and functions themselves: the empty function, the restriction of a function to a partial function from a set into a set, the set of all partial functions from a set into a set, the total functions, the relation of tolerance of two functions and the set of all total functions which are tolerated by a partial function. Some simple propositions related to the introduced notions are proved. In the beginning of this article we prove some auxiliary theorems and schemas related to the articles: [1] and [2].

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The terminology and notation used in this paper are introduced in the following articles: [4], [1], [2], and [3]. We adopt the following convention: x, y, y_1, y_2, z, z_1, z_2 will be arbitrary, P, Q, X, X', X_1, X_2, Y, Y', Y_1, Y_2, V, Z will denote sets, and C, D will denote non-empty sets. One can prove the following propositions:

(1) If $P \subseteq [: X_1, Y_1 :]$ and $Q \subseteq [: X_2, Y_2 :]$, then $P \cup Q \subseteq [: X_1 \cup X_2, Y_1 \cup Y_2 :]$.

(2) For all functions f, g such that for every x such that $x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$ holds $f(x) = g(x)$ there exists h being a function such that $\mathrm{graph}\, f \cup \mathrm{graph}\, g = \mathrm{graph}\, h$.

(3) For all functions f, g, h such that $\mathrm{graph}\, f \cup \mathrm{graph}\, g = \mathrm{graph}\, h$ for every x such that $x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$ holds $f(x) = g(x)$.

(4) For arbitrary f such that $f \in Y^X$ holds f is a function from X into Y.

In the article we present several logical schemes. The scheme *LambdaC* deals with a constant $\mathcal{A}$ that is a set, a unary predicate $\mathcal{P}$, a unary functor $\mathcal{F}$ and a unary functor $\mathcal{G}$ and states that:

there exists f being a function such that $\mathrm{dom}\, f = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ holds if $\mathcal{P}[x]$, then $f(x) = \mathcal{F}(x)$ but if not $\mathcal{P}[x]$, then $f(x) = \mathcal{G}(x)$
for all values of the parameters.

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The scheme *Lambda1C* deals with a constant $\mathcal{A}$ that is a set, a constant $\mathcal{B}$ that is a set, a unary predicate $\mathcal{P}$, a unary functor $\mathcal{F}$ and a unary functor $\mathcal{G}$ and states that:

there exists f being a function from $\mathcal{A}$ into $\mathcal{B}$ such that for every x such that $x \in \mathcal{A}$ holds if $\mathcal{P}[x]$, then $f(x) = \mathcal{F}(x)$ but if not $\mathcal{P}[x]$, then $f(x) = \mathcal{G}(x)$ provided the parameters satisfy the following condition:

- for every x such that $x \in \mathcal{A}$ holds if $\mathcal{P}[x]$, then $\mathcal{F}(x) \in \mathcal{B}$ but if not $\mathcal{P}[x]$, then $\mathcal{G}(x) \in \mathcal{B}$.

The constant $\Box$ is a function and is defined by:
$\operatorname{graph} \Box = \emptyset$.

Next we state a number of propositions:

(5) For every function f such that $\operatorname{graph} f = \emptyset$ holds $\Box = f$.

(6) $\operatorname{graph} \Box = \emptyset$.

(7) $\Box = \emptyset$.

(8) For every function f such that $\operatorname{dom} f = \emptyset$ or $\operatorname{rng} f = \emptyset$ holds $\Box = f$.

(9) $\operatorname{dom} \Box = \emptyset$.

(10) $\operatorname{rng} \Box = \emptyset$.

(11) For every function f holds $f \cdot \Box = \Box$ and $\Box \cdot f = \Box$.

(12) $\mathrm{id}_{\emptyset} = \Box$.

(13) $\Box$ is one-to-one.

(14) $\Box^{-1} = \Box$.

(15) For every function f holds $f \restriction \emptyset = \Box$.

(16) $\Box \restriction X = \Box$.

(17) For every function f holds $\emptyset \restriction f = \Box$.

(18) $Y \restriction \Box = \Box$.

(19) $\Box \circ X = \emptyset$.

(20) $\Box^{-1} Y = \emptyset$.

(21) $\Box$ is a function from $\emptyset$ into Y.

(22) For every function f from $\emptyset$ into Y holds $f = \Box$.

Let us consider X, Y. The mode partial function from X to Y, which widens to the type a function, is defined by:
$\operatorname{dom} \mathrm{it} \subseteq X$ and $\operatorname{rng} \mathrm{it} \subseteq Y$.

Next we state a number of propositions:

(23) For every function f holds f is a partial function from X to Y if and only if $\operatorname{dom} f \subseteq X$ and $\operatorname{rng} f \subseteq Y$.

(24) For every function f holds f is a partial function from $\operatorname{dom} f$ to $\operatorname{rng} f$.

(25) For every function f such that $\operatorname{rng} f \subseteq Y$ holds f is a partial function from $\operatorname{dom} f$ to Y.

(26) For every partial function f from C to D such that $y \in \operatorname{rng} f$ there exists x being an element of C such that $x \in \operatorname{dom} f$ and $y = f(x)$.

(27) For every partial function f from X to Y such that $x \in \operatorname{dom} f$ holds $f(x) \in Y$.

(28) For every partial function f from X to Y such that $\operatorname{dom} f \subseteq Z$ holds f is a partial function from Z to Y.

(29) For every partial function f from X to Y such that $\operatorname{rng} f \subseteq Z$ holds f is a partial function from X to Z.

(30) For every partial function f from X to Y such that $X \subseteq Z$ holds f is a partial function from Z to Y.

(31) For every partial function f from X to Y such that $Y \subseteq Z$ holds f is a partial function from X to Z.

(32) For every partial function f from X_1 to Y_1 such that $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ holds f is a partial function from X_2 to Y_2.

(33) For every function f for every partial function g from X to Y such that $\operatorname{graph} f \subseteq \operatorname{graph} g$ holds f is a partial function from X to Y.

(34) For all partial functions f_1, f_2 from C to D such that $X = \operatorname{dom} f_1$ and $X = \operatorname{dom} f_2$ and for every element x of C such that $x \in X$ holds $f_1(x) = f_2(x)$ holds $f_1 = f_2$.

(35) For all partial functions f_1, f_2 from $[\![X, Y]\!]$ to Z such that $V = \operatorname{dom} f_1$ and $V = \operatorname{dom} f_2$ and for all x, y such that $\langle x, y \rangle \in V$ holds $f_1(\langle x, y \rangle) = f_2(\langle x, y \rangle)$ holds $f_1 = f_2$.

Now we present four schemes. The scheme *PartFuncEx* concerns a constant $\mathcal{A}$ that is a set, a constant $\mathcal{B}$ that is a set and a binary predicate $\mathcal{P}$ and states that:

there exists f being a partial function from $\mathcal{A}$ to $\mathcal{B}$ such that for every x holds $x \in \operatorname{dom} f$ if and only if $x \in \mathcal{A}$ and there exists y such that $\mathcal{P}[x, y]$ and for every x such that $x \in \operatorname{dom} f$ holds $\mathcal{P}[x, f(x)]$

provided the parameters satisfy the following conditions:

- for all x, y such that $x \in \mathcal{A}$ and $\mathcal{P}[x, y]$ holds $y \in \mathcal{B}$,
- for all x, y_1, y_2 such that $x \in \mathcal{A}$ and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$.

The scheme *LambdaR* concerns a constant $\mathcal{A}$ that is a set, a constant $\mathcal{B}$ that is a set, a unary functor $\mathcal{F}$ and a unary predicate $\mathcal{P}$ and states that:

there exists f being a partial function from $\mathcal{A}$ to $\mathcal{B}$ such that for every x holds $x \in \operatorname{dom} f$ if and only if $x \in \mathcal{A}$ and $\mathcal{P}[x]$ and for every x such that $x \in \operatorname{dom} f$ holds $f(x) = \mathcal{F}(x)$

provided the parameters satisfy the following condition:

- for every x such that $\mathcal{P}[x]$ holds $\mathcal{F}(x) \in \mathcal{B}$.

The scheme *PartFuncEx2* concerns a constant $\mathcal{A}$ that is a set, a constant $\mathcal{B}$ that is a set, a constant $\mathcal{C}$ that is a set and a ternary predicate $\mathcal{P}$ and states that:

there exists f being a partial function from $[\![\mathcal{A}, \mathcal{B}]\!]$ to $\mathcal{C}$ such that for all x, y holds $\langle x, y \rangle \in \operatorname{dom} f$ if and only if $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and there exists z such that $\mathcal{P}[x, y, z]$ and for all x, y such that $\langle x, y \rangle \in \operatorname{dom} f$ holds $\mathcal{P}[x, y, f(\langle x, y \rangle)]$.

provided the parameters satisfy the following conditions:

- for all x, y, z such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and $\mathcal{P}[x, y, z]$ holds $z \in \mathcal{C}$,

- for all x, y, z_1, z_2 such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and $\mathcal{P}[x, y, z_1]$ and $\mathcal{P}[x, y, z_2]$ holds $z_1 = z_2$.

The scheme *LambdaR2* concerns a constant $\mathcal{A}$ that is a set, a constant $\mathcal{B}$ that is a set, a constant $\mathcal{C}$ that is a set, a binary functor $\mathcal{F}$ and a binary predicate $\mathcal{P}$ and states that:

there exists f being a partial function from $[\!: \mathcal{A}, \mathcal{B} :\!]$ to $\mathcal{C}$ such that for all x, y holds $\langle x, y\rangle \in \operatorname{dom} f$ if and only if $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and $\mathcal{P}[x, y]$ and for all x, y such that $\langle x, y\rangle \in \operatorname{dom} f$ holds $f(\langle x, y\rangle) = \mathcal{F}(x, y)$

provided the parameters satisfy the following condition:

- for all x, y such that $\mathcal{P}[x, y]$ holds $\mathcal{F}(x, y) \in \mathcal{C}$.

The arguments of the notions defined below are the following: X, Y, V, Z which are objects of the type reserved above; f which is a partial function from X to Y; g which is a partial function from V to Z. Then $g \cdot f$ is a partial function from X to Z.

One can prove the following propositions:

(36) For every partial function f from X to Y holds $f \cdot \mathrm{id}_X = f$.

(37) For every partial function f from X to Y holds $\mathrm{id}_Y \cdot f = f$.

(38) For every partial function f from C to D such that for all elements x_1, x_2 of C such that $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} f$ and $f(x_1) = f(x_2)$ holds $x_1 = x_2$ holds f is one-to-one.

(39) For every partial function f from X to Y such that f is one-to-one holds f^{-1} is a partial function from Y to X.

(40) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ but f is one-to-one holds f^{-1} is a partial function from Y to X.

(41) For every function f from X into X such that f is one-to-one holds f^{-1} is a partial function from X to X.

(42) For every function f from X into D such that f is one-to-one holds f^{-1} is a partial function from D to X.

(43) For every partial function f from X to Y holds $f \restriction Z$ is a partial function from Z to Y.

(44) For every partial function f from X to Y holds $f \restriction Z$ is a partial function from X to Y.

(45) For every partial function f from X to Y holds $Z \restriction f$ is a partial function from X to Z.

(46) For every partial function f from X to Y holds $Z \restriction f$ is a partial function from X to Y.

(47) For every function f holds $(Y \restriction f) \restriction X$ is a partial function from X to Y.

(48) For every partial function f from X to Y holds $(Y' \restriction f) \restriction X'$ is a partial function from X to Y.

(49) For every partial function f from C to D such that $y \in f^\circ X$ there exists x being an element of C such that $x \in \operatorname{dom} f$ and $y = f(x)$.

(50) For every partial function f from X to Y holds $f^\circ P \subseteq Y$.

The arguments of the notions defined below are the following: X, Y which are objects of the type reserved above; f which is a partial function from X to Y; P which is an object of the type reserved above. Then $f \circ P$ is a subset of Y.

We now state two propositions:

(51) For every partial function f from X to Y holds $f \circ X = \operatorname{rng} f$.

(52) For every partial function f from X to Y holds $f^{-1} Q \subseteq X$.

The arguments of the notions defined below are the following: X, Y which are objects of the type reserved above; f which is a partial function from X to Y; Q which is an object of the type reserved above. Then $f^{-1} Q$ is a subset of X.

Next we state a number of propositions:

(53) For every partial function f from X to Y holds $f^{-1} Y = \operatorname{dom} f$.

(54) For every partial function f from $\emptyset$ to Y holds $\operatorname{dom} f = \emptyset$ and $\operatorname{rng} f = \emptyset$.

(55) For every function f such that $\operatorname{dom} f = \emptyset$ holds f is a partial function from X to Y.

(56) $\square$ is a partial function from X to Y.

(57) For every partial function f from $\emptyset$ to Y holds $f = \square$.

(58) For every partial function f_1 from $\emptyset$ to Y_1 for every partial function f_2 from $\emptyset$ to Y_2 holds $f_1 = f_2$.

(59) For every partial function f from $\emptyset$ to Y holds f is one-to-one.

(60) For every partial function f from $\emptyset$ to Y holds $f \circ P = \emptyset$.

(61) For every partial function f from $\emptyset$ to Y holds $f^{-1} Q = \emptyset$.

(62) For every partial function f from X to $\emptyset$ holds $\operatorname{dom} f = \emptyset$ and $\operatorname{rng} f = \emptyset$.

(63) For every function f such that $\operatorname{rng} f = \emptyset$ holds f is a partial function from X to Y.

(64) For every partial function f from X to $\emptyset$ holds $f = \square$.

(65) For every partial function f_1 from X_1 to $\emptyset$ for every partial function f_2 from X_2 to $\emptyset$ holds $f_1 = f_2$.

(66) For every partial function f from X to $\emptyset$ holds f is one-to-one.

(67) For every partial function f from X to $\emptyset$ holds $f \circ P = \emptyset$.

(68) For every partial function f from X to $\emptyset$ holds $f^{-1} Q = \emptyset$.

(69) For every partial function f from $\{x\}$ to Y holds $\operatorname{rng} f \subseteq \{f(x)\}$.

(70) For every partial function f from $\{x\}$ to Y holds f is one-to-one.

(71) For every partial function f from $\{x\}$ to Y holds $f \circ P \subseteq \{f(x)\}$.

(72) For every function f such that $\operatorname{dom} f = \{x\}$ and $x \in X$ and $f(x) \in Y$ holds f is a partial function from X to Y.

(73) For every partial function f from X to $\{y\}$ such that $x \in \operatorname{dom} f$ holds $f(x) = y$.

(74) For all partial functions f_1, f_2 from X to $\{y\}$ such that $\operatorname{dom} f_1 = \operatorname{dom} f_2$ holds $f_1 = f_2$.

The arguments of the notions defined below are the following: f which is a function; X, Y which are sets. The functor $f_{\restriction X \dot{\to} Y}$ yielding a partial function from X to Y, is defined by:

$f_{\restriction X \dot{\to} Y} = (Y \restriction f) \restriction X$.

We now state a number of propositions:

(75) For every function f for all X, Y holds $f_{\restriction X \dot{\to} Y} = (Y \restriction f) \restriction X$.

(76) For every function f holds $\operatorname{graph}(f_{\restriction X \dot{\to} Y}) \subseteq \operatorname{graph} f$.

(77) For every function f holds $\operatorname{dom}(f_{\restriction X \dot{\to} Y}) \subseteq \operatorname{dom} f$ and $\operatorname{rng}(f_{\restriction X \dot{\to} Y}) \subseteq \operatorname{rng} f$.

(78) For every function f holds $x \in \operatorname{dom}(f_{\restriction X \dot{\to} Y})$ if and only if $x \in \operatorname{dom} f$ and $x \in X$ and $f(x) \in Y$.

(79) For every function f such that $x \in \operatorname{dom} f$ and $x \in X$ and $f(x) \in Y$ holds $(f_{\restriction X \dot{\to} Y})(x) = f(x)$.

(80) For every function f such that $x \in \operatorname{dom}(f_{\restriction X \dot{\to} Y})$ holds $(f_{\restriction X \dot{\to} Y})(x) = f(x)$.

(81) For all functions f, g such that $\operatorname{graph} f \subseteq \operatorname{graph} g$ holds $\operatorname{graph}(f_{\restriction X \dot{\to} Y}) \subseteq \operatorname{graph}(g_{\restriction X \dot{\to} Y})$.

(82) For every function f such that $Z \subseteq X$ holds $\operatorname{graph}(f_{\restriction Z \dot{\to} Y}) \subseteq \operatorname{graph}(f_{\restriction X \dot{\to} Y})$.

(83) For every function f such that $Z \subseteq Y$ holds $\operatorname{graph}(f_{\restriction X \dot{\to} Z}) \subseteq \operatorname{graph}(f_{\restriction X \dot{\to} Y})$.

(84) For every function f such that $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ holds $\operatorname{graph}(f_{\restriction X_1 \dot{\to} Y_1}) \subseteq \operatorname{graph}(f_{\restriction X_2 \dot{\to} Y_2})$.

(85) For every function f such that $\operatorname{dom} f \subseteq X$ and $\operatorname{rng} f \subseteq Y$ holds $f = f_{\restriction X \dot{\to} Y}$.

(86) For every function f holds $f = f_{\restriction \operatorname{dom} f \dot{\to} \operatorname{rng} f}$.

(87) For every partial function f from X to Y holds $f_{\restriction X \dot{\to} Y} = f$.

(88) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds $f_{\restriction X \dot{\to} Y} = f$.

(89) For every function f from X into X holds $f_{\restriction X \dot{\to} X} = f$.

(90) For every function f from X into D holds $f_{\restriction X \dot{\to} D} = f$.

(91) $\square_{\restriction X \dot{\to} Y} = \square$.

(92) For all functions f, g holds $\operatorname{graph}((g_{\restriction Y \dot{\to} Z}) \cdot (f_{\restriction X \dot{\to} Y})) \subseteq \operatorname{graph}(g \cdot f_{\restriction X \dot{\to} Z})$.

(93) For all functions f, g such that $\operatorname{rng} f \cap \operatorname{dom} g \subseteq Y$ holds $(g_{\restriction Y \dot{\to} Z}) \cdot (f_{\restriction X \dot{\to} Y}) = g \cdot f_{\restriction X \dot{\to} Z}$.

(94) For every function f such that f is one-to-one holds $f_{\restriction X \dot{\to} Y}$ is one-to-one.

(95) For every function f such that f is one-to-one holds $(f_{\restriction X \dot{\to} Y})^{-1} = f^{-1}{}_{\restriction Y \dot{\to} X}$.

(96) For every function f holds $(f_{\restriction X \dot{\to} Y}) \restriction Z = f_{\restriction X \cap Z \dot{\to} Y}$.

(97) For every function f holds $Z \restriction (f_{\restriction X \dot{\to} Y}) = f_{\restriction X \dot{\to} Z \cap Y}$.

The arguments of the notions defined below are the following: X, Y which are objects of the type reserved above; f which is a partial function from X to Y. The predicate f is total is defined by:

dom $f = X$.

We now state a number of propositions:

(98) For every partial function f from X to Y holds f is total if and only if dom $f = X$.

(99) For every partial function f from X to Y such that f is total and $Y = \emptyset$ holds $X = \emptyset$.

(100) For every partial function f from X to Y such that dom $f = X$ holds f is a function from X into Y.

(101) For every partial function f from X to Y such that f is total holds f is a function from X into Y.

(102) For every partial function f from X to Y such that if $Y = \emptyset$, then $X = \emptyset$ but f is a function from X into Y holds f is total.

(103) For every function f from X into Y for every partial function f' from X to Y such that if $Y = \emptyset$, then $X = \emptyset$ but $f = f'$ holds f' is total.

(104) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds $f_{\restriction X \dot{\rightarrow} Y}$ is total.

(105) For every function f from X into X holds $f_{\restriction X \dot{\rightarrow} X}$ is total.

(106) For every function f from X into D holds $f_{\restriction X \dot{\rightarrow} D}$ is total.

(107) For every partial function f from X to Y such that if $Y = \emptyset$, then $X = \emptyset$ there exists g being a function from X into Y such that for every x such that $x \in \operatorname{dom} f$ holds $g(x) = f(x)$.

(108) For every partial function f from X to D there exists g being a function from X into D such that for every x such that $x \in \operatorname{dom} f$ holds $g(x) = f(x)$.

(109) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds f is a partial function from X to Y.

(110) For every function f from X into X holds f is a partial function from X to X.

(111) For every function f from X into D holds f is a partial function from X to D.

(112) For every partial function f from $\emptyset$ to Y holds f is total.

(113) For every function f such that $f_{\restriction X \dot{\rightarrow} Y}$ is total holds $X \subseteq \operatorname{dom} f$.

(114) If $\square_{\restriction X \dot{\rightarrow} Y}$ is total, then $X = \emptyset$.

(115) For every function f such that $X \subseteq \operatorname{dom} f$ and $\operatorname{rng} f \subseteq Y$ holds $f_{\restriction X \dot{\rightarrow} Y}$ is total.

(116) For every function f such that $f_{\restriction X \dot{\rightarrow} Y}$ is total holds $f \circ X \subseteq Y$.

(117) For every function f such that $X \subseteq \operatorname{dom} f$ and $f \circ X \subseteq Y$ holds $f_{\restriction X \dot{\rightarrow} Y}$ is total.

Let us consider X, Y. The functor $X \dot{\rightarrow} Y$ yielding a non-empty set, is defined by:

$x \in X \dot{\to} Y$ if and only if there exists f being a function such that $x = f$ and $\operatorname{dom} f \subseteq X$ and $\operatorname{rng} f \subseteq Y$.

We now state a number of propositions:

(118) For every non-empty set F holds $F = X \dot{\to} Y$ if and only if for every x holds $x \in F$ if and only if there exists f being a function such that $x = f$ and $\operatorname{dom} f \subseteq X$ and $\operatorname{rng} f \subseteq Y$.

(119) For every partial function f from X to Y holds $f \in X \dot{\to} Y$.

(120) For arbitrary f such that $f \in X \dot{\to} Y$ holds f is a partial function from X to Y.

(121) For every element f of $X \dot{\to} Y$ holds f is a partial function from X to Y.

(122) $\emptyset \dot{\to} Y = \{\square\}$.

(123) $X \dot{\to} \emptyset = \{\square\}$.

(124) $Y^X \subseteq X \dot{\to} Y$.

(125) If $Z \subseteq X$, then $Z \dot{\to} Y \subseteq X \dot{\to} Y$.

(126) $\emptyset \dot{\to} Y \subseteq X \dot{\to} Y$.

(127) If $Z \subseteq Y$, then $X \dot{\to} Z \subseteq X \dot{\to} Y$.

(128) If $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$, then $X_1 \dot{\to} Y_1 \subseteq X_2 \dot{\to} Y_2$.

Let f, g be functions. The predicate $f \approx g$ is defined by:
for every x such that $x \in \operatorname{dom} f \cap \operatorname{dom} g$ holds $f(x) = g(x)$.

The following propositions are true:

(129) For all functions f, g holds $f \approx g$ if and only if for every x such that $x \in \operatorname{dom} f \cap \operatorname{dom} g$ holds $f(x) = g(x)$.

(130) For all functions f, g holds $f \approx g$ if and only if there exists h being a function such that $\operatorname{graph} f \cup \operatorname{graph} g = \operatorname{graph} h$.

(131) For all functions f, g holds $f \approx g$ if and only if there exists h being a function such that $\operatorname{graph} f \subseteq \operatorname{graph} h$ and $\operatorname{graph} g \subseteq \operatorname{graph} h$.

(132) For all functions f, g such that $\operatorname{dom} f \subseteq \operatorname{dom} g$ holds $f \approx g$ if and only if for every x such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$.

(133) For all functions f, g holds $f \approx f$.

(134) For all functions f, g such that $f \approx g$ holds $g \approx f$.

(135) For all functions f, g such that $\operatorname{graph} f \subseteq \operatorname{graph} g$ holds $f \approx g$.

(136) For all functions f, g such that $\operatorname{dom} f = \operatorname{dom} g$ and $f \approx g$ holds $f = g$.

(137) For all functions f, g such that $f = g$ holds $f \approx g$.

(138) For all functions f, g such that $\operatorname{dom} f \cap \operatorname{dom} g = \emptyset$ holds $f \approx g$.

(139) For all functions f, g, h such that $\operatorname{graph} f \subseteq \operatorname{graph} h$ and $\operatorname{graph} g \subseteq \operatorname{graph} h$ holds $f \approx g$.

(140) For all partial functions f, g from X to Y for every function h such that $f \approx h$ and $\operatorname{graph} g \subseteq \operatorname{graph} f$ holds $g \approx h$.

(141) For every function f holds $\square \approx f$ and $f \approx \square$.

(142) For every function f holds $\square_{\restriction X \dot{\to} Y} \approx f$ and $f \approx \square_{\restriction X \dot{\to} Y}$.

(143) For all partial functions f, g from X to $\{y\}$ holds $f \approx g$.

(144) For every function f holds $f \restriction X \approx f$ and $f \restriction X \approx f$.

(145) For every function f holds $Y \restriction f \approx f$ and $f \approx Y \restriction f$.

(146) For every function f holds $(Y \restriction f) \restriction X \approx f$ and $f \approx (Y \restriction f) \restriction X$.

(147) For every function f holds $f_{\restriction X \dot{\rightarrow} Y} \approx f$ and $f \approx f_{\restriction X \dot{\rightarrow} Y}$.

(148) For all partial functions f, g from X to Y such that f is total and g is total and $f \approx g$ holds $f = g$.

(149) For all functions f, g from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ but $f \approx g$ holds $f = g$.

(150) For all functions f, g from X into X such that $f \approx g$ holds $f = g$.

(151) For all functions f, g from X into D such that $f \approx g$ holds $f = g$.

(152) For every partial function f from X to Y for every function g from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds $f \approx g$ if and only if for every x such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$.

(153) For every partial function f from X to X for every function g from X into X holds $f \approx g$ if and only if for every x such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$.

(154) For every partial function f from X to D for every function g from X into D holds $f \approx g$ if and only if for every x such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$.

(155) For every partial function f from X to Y such that if $Y = \emptyset$, then $X = \emptyset$ there exists g being a function from X into Y such that $f \approx g$.

(156) For every partial function f from X to X there exists g being a function from X into X such that $f \approx g$.

(157) For every partial function f from X to D there exists g being a function from X into D such that $f \approx g$.

(158) For all partial functions f, g, h from X to Y such that $f \approx h$ and $g \approx h$ and h is total holds $f \approx g$.

(159) For all partial functions f, g from X to Y for every function h from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ but $f \approx h$ and $g \approx h$ holds $f \approx g$.

(160) For all partial functions f, g from X to X for every function h from X into X such that $f \approx h$ and $g \approx h$ holds $f \approx g$.

(161) For all partial functions f, g from X to D for every function h from X into D such that $f \approx h$ and $g \approx h$ holds $f \approx g$.

(162) For all partial functions f, g from X to Y such that if $Y = \emptyset$, then $X = \emptyset$ but $f \approx g$ there exists h being a partial function from X to Y such that h is total and $f \approx h$ and $g \approx h$.

(163) For all partial functions f, g from X to Y such that if $Y = \emptyset$, then $X = \emptyset$ but $f \approx g$ there exists h being a function from X into Y such that $f \approx h$ and $g \approx h$.

The arguments of the notions defined below are the following: X, Y which are objects of the type reserved above; f which is a partial function from X to Y. The functor $\operatorname{TotFuncs} f$ yields a set and is defined by:

$x \in \mathrm{TotFuncs}\, f$ if and only if there exists g being a partial function from X to Y such that $g = x$ and g is total and $f \approx g$.

The following propositions are true:

(164) For all X, Y for every partial function f from X to Y for every Z holds $Z = \mathrm{TotFuncs}\, f$ if and only if for every x holds $x \in Z$ if and only if there exists g being a partial function from X to Y such that $g = x$ and g is total and $f \approx g$.

(165) For every partial function f from X to Y for every function g from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ but $f \approx g$ holds $g \in \mathrm{TotFuncs}\, f$.

(166) For every partial function f from X to X for every function g from X into X such that $f \approx g$ holds $g \in \mathrm{TotFuncs}\, f$.

(167) For every partial function f from X to D for every function g from X into D such that $f \approx g$ holds $g \in \mathrm{TotFuncs}\, f$.

(168) For every partial function f from X to Y for arbitrary g such that $g \in \mathrm{TotFuncs}\, f$ holds g is a partial function from X to Y.

(169) For all partial functions f, g from X to Y such that $g \in \mathrm{TotFuncs}\, f$ holds g is total.

(170) For every partial function f from X to Y for arbitrary g such that $g \in \mathrm{TotFuncs}\, f$ holds g is a function from X into Y.

(171) For every partial function f from X to Y for every function g such that $g \in \mathrm{TotFuncs}\, f$ holds $f \approx g$ and $g \approx f$.

(172) For every partial function f from X to $\emptyset$ such that $X \neq \emptyset$ holds $\mathrm{TotFuncs}\, f = \emptyset$.

(173) For every partial function f from X to Y holds $\mathrm{TotFuncs}\, f \subseteq Y^X$.

(174) For every partial function f from X to Y holds f is total if and only if $\mathrm{TotFuncs}\, f = \{f\}$.

(175) For every partial function f from $\emptyset$ to Y holds $\mathrm{TotFuncs}\, f = \{f\}$.

(176) For every partial function f from $\emptyset$ to Y holds $\mathrm{TotFuncs}\, f = \{\square\}$.

(177) $\mathrm{TotFuncs}(\square_{\restriction X \dot{\to} Y}) = Y^X$.

(178) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds $\mathrm{TotFuncs}(f_{\restriction X \dot{\to} Y}) = \{f\}$.

(179) For every function f from X into X holds $\mathrm{TotFuncs}(f_{\restriction X \dot{\to} X}) = \{f\}$.

(180) For every function f from X into D holds $\mathrm{TotFuncs}(f_{\restriction X \dot{\to} D}) = \{f\}$.

(181) For every partial function f from X to $\{y\}$ for every function g from X into $\{y\}$ holds $\mathrm{TotFuncs}\, f = \{g\}$.

(182) For all partial functions f, g from X to Y such that $\mathrm{graph}\, g \subseteq \mathrm{graph}\, f$ holds $\mathrm{TotFuncs}\, f \subseteq \mathrm{TotFuncs}\, g$.

(183) For all partial functions f, g from X to Y such that $\mathrm{dom}\, g \subseteq \mathrm{dom}\, f$ and $\mathrm{TotFuncs}\, f \subseteq \mathrm{TotFuncs}\, g$ holds $\mathrm{graph}\, g \subseteq \mathrm{graph}\, f$.

(184) For all partial functions f, g from X to Y such that $\mathrm{TotFuncs}\, f \subseteq \mathrm{TotFuncs}\, g$ and for every y holds $Y \neq \{y\}$ holds $\mathrm{graph}\, g \subseteq \mathrm{graph}\, f$.

(185) For all partial functions f, g from X to Y such that $\mathrm{TotFuncs}\, f \cap \mathrm{TotFuncs}\, g \neq \emptyset$ holds $f \approx g$.

(186) For all partial functions f, g from X to Y such that if $Y = \emptyset$, then $X = \emptyset$ but $f \approx g$ holds $\mathrm{TotFuncs}\, f \cap \mathrm{TotFuncs}\, g \neq \emptyset$.

(187) For all partial functions f, g from X to Y such that for every y holds $Y \neq \{y\}$ and $\mathrm{TotFuncs}\, f = \mathrm{TotFuncs}\, g$ holds $f = g$.

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[2] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[3] Czesław Byliński. Graphs of functions. *Formalized Mathematics*, 1(**1**):169–173, 1990.

[4] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

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Semilattice Operations on Finite Subsets

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Summary. In the article we deal with a binary operation that is associative, commutative. We define for such an operation a functor that depends on two more arguments: a finite set of indices and a function indexing elements of the domain of the operation and yields the result of applying the operation to all indexed elements. The definition has a restriction that requires that either the set of indices is non empty or the operation has the unity. We prove theorems describing some properties of the functor introduced. Most of them we prove in two versions depending on which requirement is fulfilled. In the second part we deal with the union of finite sets that enjoys mentioned above properties. We prove analogs of the theorems proved in the first part. We precede the main part of the article with auxiliary theorems related to boolean properties of sets, enumerated sets, finite subsets, and functions. We define a casting function that yields to a set the empty set typed as a finite subset of the set. We prove also two schemes of the induction on finite sets.

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The terminology and notation used in this paper have been introduced in the following articles: [5], [4], [7], [6], [2], [1], and [3]. We adopt the following rules: x, y, z will be arbitrary and X, Y, Z, X', Y' will be sets. The following propositions are true:

(1) If $\{x\} \subseteq \{y\}$, then $x = y$.
(2) $\{x, y, z\} \neq \emptyset$.
(3) $\{x\} \subseteq \{x, y, z\}$.
(4) $\{x, y\} \subseteq \{x, y, z\}$.
(5) If $X \subseteq Y \cup \{x\}$, then $x \in X$ or $X \subseteq Y$.
(6) $x \in X \cup \{y\}$ if and only if $x \in X$ or $x = y$.
(7) If $X \cup Y \subseteq Z$, then $X \subseteq Z$ and $Y \subseteq Z$.

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(8) $X \cup \{x\} \subseteq Y$ if and only if $x \in Y$ and $X \subseteq Y$.

(9) If $X' \cup Y' = X \cup Y$ and X misses X' and Y misses Y', then $X = Y'$ and $Y = X'$.

(10) If $X' \cup Y' = X \cup Y$ and Y misses X' and X misses Y', then $X = X'$ and $Y = Y'$.

(11) For all X, Y for every function f holds $f \circ (Y \setminus f^{-1} X) = f \circ Y \setminus X$.

In the sequel X, Y will denote non-empty sets and f will denote a function from X into Y. Next we state two propositions:

(12) For every element x of X holds $x \in f^{-1} \{f(x)\}$.

(13) For every element x of X holds $f \circ \{x\} = \{f(x)\}$.

The scheme *SubsetEx* deals with a constant $\mathcal{A}$ that is a non-empty set and a unary predicate $\mathcal{P}$ and states that:

there exists B being a subset of $\mathcal{A}$ such that for every element x of $\mathcal{A}$ holds $x \in B$ if and only if $\mathcal{P}[x]$

for all values of the parameters.

We now state several propositions:

(14) For every element B of $\mathrm{Fin}\, X$ for every x such that $x \in B$ holds x is an element of X.

(15) For every element A of $\mathrm{Fin}\, X$ for every set B for every function f from X into Y such that for every element x of X such that $x \in A$ holds $f(x) \in B$ holds $f \circ A \subseteq B$.

(16) For every set X for every element B of $\mathrm{Fin}\, X$ for every set A such that $A \subseteq B$ holds A is an element of $\mathrm{Fin}\, X$.

(17) For every element A of $\mathrm{Fin}\, X$ holds $f \circ A$ is an element of $\mathrm{Fin}\, Y$.

(18) For every element B of $\mathrm{Fin}\, X$ such that $B \neq \emptyset$ there exists x being an element of X such that $x \in B$.

(19) For every element A of $\mathrm{Fin}\, X$ such that $f \circ A = \emptyset$ holds $A = \emptyset$.

Let X be a set. The functor 0_X yielding an element of $\mathrm{Fin}\, X$, is defined by:

$0_X = \emptyset$.

One can prove the following proposition

(20) For every set X holds $0_X = \emptyset$.

The arguments of the notions defined below are the following: X which is a non-empty set; A which is a set; f which is a function from X into $\mathrm{Fin}\, A$; x which is an element of X. Then $f(x)$ is an element of $\mathrm{Fin}\, A$.

The scheme *FinSubFuncEx* deals with a constant $\mathcal{A}$ that is a non-empty set, a constant $\mathcal{B}$ that is an element of $\mathrm{Fin}\, \mathcal{A}$ and a binary predicate $\mathcal{P}$ and states that:

there exists f being a function from $\mathcal{A}$ into $\mathrm{Fin}\, \mathcal{A}$ such that for all elements b, a of $\mathcal{A}$ holds $a \in f(b)$ if and only if $a \in \mathcal{B}$ and $\mathcal{P}[a, b]$

for all values of the parameters.

The arguments of the notions defined below are the following: X which is a non-empty set; F which is a binary operation on X. The predicate F has a unity is defined by:

there exists x being an element of X such that x is a unity w.r.t. F.

We now state three propositions:

(21) For every non-empty set X for every binary operation F on X holds F has a unity if and only if there exists x being an element of X such that x is a unity w.r.t. F.

(22) For every non-empty set X for every binary operation F on X holds F has a unity if and only if $\mathbf{1}_F$ is a unity w.r.t. F.

(23) For every non-empty set X for every binary operation F on X such that F has a unity for every element x of X holds $F(\mathbf{1}_F, x) = x$ and $F(x, \mathbf{1}_F) = x$.

The arguments of the notions defined below are the following: X which is a non-empty set; x which is an element of X. Then $\{x\}$ is an element of $\mathrm{Fin}\, X$. Let y be an element of X. Then $\{x, y\}$ is an element of $\mathrm{Fin}\, X$. Let z be an element of X. Then $\{x, y, z\}$ is an element of $\mathrm{Fin}\, X$.

Now we present three schemes. The scheme *FinSubInd1* concerns a constant $\mathcal{A}$ that is a non-empty set and a unary predicate $\mathcal{P}$ and states that:

for every element B of $\mathrm{Fin}\, \mathcal{A}$ holds $\mathcal{P}[B]$

provided the parameters satisfy the following conditions:

- $\mathcal{P}[0_{\mathcal{A}}]$,
- for every element B' of $\mathrm{Fin}\, \mathcal{A}$ for every element b of $\mathcal{A}$ such that $\mathcal{P}[B']$ and $b \notin B'$ holds $\mathcal{P}[B' \cup \{b\}]$.

The scheme *FinSubInd2* concerns a constant $\mathcal{A}$ that is a non-empty set and a unary predicate $\mathcal{P}$ and states that:

for every element B of $\mathrm{Fin}\, \mathcal{A}$ such that $B \neq \emptyset$ holds $\mathcal{P}[B]$

provided the parameters satisfy the following conditions:

- for every element x of $\mathcal{A}$ holds $\mathcal{P}[\{x\}]$,
- for all elements B_1, B_2 of $\mathrm{Fin}\, \mathcal{A}$ such that $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$ holds if $\mathcal{P}[B_1]$ and $\mathcal{P}[B_2]$, then $\mathcal{P}[B_1 \cup B_2]$.

The scheme *FinSubInd3* concerns a constant $\mathcal{A}$ that is a non-empty set and a unary predicate $\mathcal{P}$ and states that:

for every element B of $\mathrm{Fin}\, \mathcal{A}$ holds $\mathcal{P}[B]$

provided the parameters satisfy the following conditions:

- $\mathcal{P}[0_{\mathcal{A}}]$,
- for every element B' of $\mathrm{Fin}\, \mathcal{A}$ for every element b of $\mathcal{A}$ such that $\mathcal{P}[B']$ holds $\mathcal{P}[B' \cup \{b\}]$.

The arguments of the notions defined below are the following: X which is a non-empty family of sets; Y which is a non-empty set; f which is a function from X into Y; x which is an element of X. Then $f(x)$ is an element of Y.

In the sequel C will be a non-empty set. The arguments of the notions defined below are the following: X, Y which are non-empty sets; F which is a binary operation on Y; B which is an element of $\mathrm{Fin}\, X$; f which is a function from X into Y. Let us assume that $B \neq \emptyset$ or F has a unity and F is commutative and F is associative. The functor $F\text{-}\sum_B f$ yielding an element of Y, is defined by:

there exists G being a function from $\mathrm{Fin}\, X$ into Y such that $F\text{-}\sum_B f = G(B)$ and for every element e of Y such that e is a unity w.r.t. F holds $G(\emptyset) = e$ and for every element x of X holds $G(\{x\}) = f(x)$ and for every element B' of $\operatorname{Fin} X$ such that $B' \subseteq B$ and $B' \neq \emptyset$ for every element x of X such that $x \in B \setminus B'$ holds $G(B' \cup \{x\}) = F(G(B'), f(x))$.

One can prove the following propositions:

(24) Let X, Y be non-empty sets. Let F be a binary operation on Y. Let B be an element of $\operatorname{Fin} X$. Let f be a function from X into Y. Suppose $B \neq \emptyset$ or F has a unity but F is commutative and F is associative. Let IT be an element of Y. Then $IT = F\text{-}\sum_B f$ if and only if there exists G being a function from $\operatorname{Fin} X$ into Y such that $IT = G(B)$ and for every element e of Y such that e is a unity w.r.t. F holds $G(\emptyset) = e$ and for every element x of X holds $G(\{x\}) = f(x)$ and for every element B' of $\operatorname{Fin} X$ such that $B' \subseteq B$ and $B' \neq \emptyset$ for every element x of X such that $x \in B \setminus B'$ holds $G(B' \cup \{x\}) = F(G(B'), f(x))$.

(25) Let X, Y be non-empty sets. Let F be a binary operation on Y. Let B be an element of $\operatorname{Fin} X$. Let f be a function from X into Y. Suppose $B \neq \emptyset$ or F has a unity but F is idempotent and F is commutative and F is associative. Let IT be an element of Y. Then $IT = F\text{-}\sum_B f$ if and only if there exists G being a function from $\operatorname{Fin} X$ into Y such that $IT = G(B)$ and for every element e of Y such that e is a unity w.r.t. F holds $G(\emptyset) = e$ and for every element x of X holds $G(\{x\}) = f(x)$ and for every element B' of $\operatorname{Fin} X$ such that $B' \subseteq B$ and $B' \neq \emptyset$ for every element x of X such that $x \in B$ holds $G(B' \cup \{x\}) = F(G(B'), f(x))$.

For simplicity we follow the rules: X, Y denote non-empty sets, F denotes a binary operation on Y, B denotes an element of $\operatorname{Fin} X$, and f denotes a function from X into Y. Next we state a number of propositions:

(26) If F is commutative and F is associative, then for every element b of X holds $F\text{-}\sum_{\{b\}} f = f(b)$.

(27) If F is idempotent and F is commutative and F is associative, then for all elements a, b of X holds $F\text{-}\sum_{\{a,b\}} f = F(f(a), f(b))$.

(28) If F is idempotent and F is commutative and F is associative, then for all elements a, b, c of X holds $F\text{-}\sum_{\{a,b,c\}} f = F(F(f(a), f(b)), f(c))$.

(29) If F is idempotent and F is commutative and F is associative and $B \neq \emptyset$, then for every element x of X holds $F\text{-}\sum_{B \cup \{x\}} f = F(F\text{-}\sum_B f, f(x))$.

(30) If F is idempotent and F is commutative and F is associative, then for all elements B_1, B_2 of $\operatorname{Fin} X$ such that $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$ holds $F\text{-}\sum_{B_1 \cup B_2} f = F(F\text{-}\sum_{B_1} f, F\text{-}\sum_{B_2} f)$.

(31) If F is commutative and F is associative and F is idempotent, then for every element x of X such that $x \in B$ holds $F(f(x), F\text{-}\sum_B f) = F\text{-}\sum_B f$.

(32) If F is commutative and F is associative and F is idempotent, then for all elements B, C of $\operatorname{Fin} X$ such that $B \neq \emptyset$ and $B \subseteq C$ holds $F(F\text{-}\sum_B f, F\text{-}\sum_C f) = F\text{-}\sum_C f$.

(33) If $B \neq \emptyset$ and F is commutative and F is associative and F is idempotent, then for every element a of Y such that for every element b of X such that $b \in B$ holds $f(b) = a$ holds $F\text{-}\sum_B f = a$.

(34) If F is commutative and F is associative and F is idempotent, then for every element a of Y such that $f \circ B = \{a\}$ holds $F\text{-}\sum_B f = a$.

(35) If F is commutative and F is associative and F is idempotent, then for all functions f, g from X into Y for all elements A, B of $\operatorname{Fin} X$ such that $A \neq \emptyset$ and $f \circ A = g \circ B$ holds $F\text{-}\sum_A f = F\text{-}\sum_B g$.

(36) Let F, G be binary operations on Y. Then if F is idempotent and F is commutative and F is associative and G is distributive w.r.t. F, then for every element B of $\operatorname{Fin} X$ such that $B \neq \emptyset$ for every function f from X into Y for every element a of Y holds $G(a, F\text{-}\sum_B f) = F\text{-}\sum_B(G^\circ(a, f))$.

(37) Let F, G be binary operations on Y. Then if F is idempotent and F is commutative and F is associative and G is distributive w.r.t. F, then for every element B of $\operatorname{Fin} X$ such that $B \neq \emptyset$ for every function f from X into Y for every element a of Y holds $G(F\text{-}\sum_B f, a) = F\text{-}\sum_B(G^\circ(f, a))$.

The arguments of the notions defined below are the following: A, X, Y which are non-empty sets; f which is a function from X into Y; g which is a function from Y into A. Then $g \cdot f$ is a function from X into A.

The arguments of the notions defined below are the following: X, Y which are non-empty sets; f which is a function from X into Y; A which is an element of $\operatorname{Fin} X$. Then $f \circ A$ is an element of $\operatorname{Fin} Y$.

The following propositions are true:

(38) Let A, X, Y be non-empty sets. Then for every binary operation F on A such that F is idempotent and F is commutative and F is associative for every element B of $\operatorname{Fin} X$ such that $B \neq \emptyset$ for every function f from X into Y for every function g from Y into A holds $F\text{-}\sum_{f \circ B} g = F\text{-}\sum_B(g \cdot f)$.

(39) Suppose F is commutative and F is associative and F is idempotent. Let Z be a non-empty set. Let G be a binary operation on Z. Suppose G is commutative and G is associative and G is idempotent. Let f be a function from X into Y. Then for every function g from Y into Z such that for all elements x, y of Y holds $g(F(x, y)) = G(g(x), g(y))$ for every element B of $\operatorname{Fin} X$ such that $B \neq \emptyset$ holds $g(F\text{-}\sum_B f) = G\text{-}\sum_B(g \cdot f)$.

(40) If F is commutative and F is associative and F has a unity, then for every f holds $F\text{-}\sum_{0_X} f = \mathbf{1}_F$.

(41) If F is idempotent and F is commutative and
F
is associative and F has a unity, then for every element x of X holds $F\text{-}\sum_{B \cup \{x\}} f = F(F\text{-}\sum_B f, f(x))$.

(42) If F is idempotent and F is commutative and F is associative and F has a unity, then for all elements B_1, B_2 of $\operatorname{Fin} X$ holds $F\text{-}\sum_{B_1 \cup B_2} f = F(F\text{-}\sum_{B_1} f, F\text{-}\sum_{B_2} f)$.

(43) If F is commutative and F is associative and F is idempotent and F has a unity, then for all functions f, g from X into Y for all elements A, B of $\operatorname{Fin} X$ such that $f \circ A = g \circ B$ holds $F\text{-}\sum_A f = F\text{-}\sum_B g$.

(44) For all non-empty sets A, X, Y for every binary operation F on A such

that F is idempotent and F is commutative and F is associative and F has a unity for every element B of $\mathrm{Fin}\,X$ for every function f from X into Y for every function g from Y into A holds $F\text{-}\sum_{f\circ B} g = F\text{-}\sum_B (g \cdot f)$.

(45) Suppose F is commutative and F is associative and F is idempotent and F has a unity. Let Z be a non-empty set. Let G be a binary operation on Z. Suppose G is commutative and G is associative and G is idempotent and G has a unity. Let f be a function from X into Y. Let g be a function from Y into Z. Then if $g(\mathbf{1}_F) = \mathbf{1}_G$ and for all elements x, y of Y holds $g(F(x,y)) = G(g(x), g(y))$, then for every element B of $\mathrm{Fin}\,X$ holds $g(F\text{-}\sum_B f) = G\text{-}\sum_B (g \cdot f)$.

The arguments of the notions defined below are the following: A which is a set; x which is an element of $\mathrm{Fin}\,A$. The functor $@x$ yielding an element of $\mathrm{Fin}\,A$ **qua** a non-empty set, is defined by:

$@x = x$.

The following proposition is true

(46) For every set A for every element x of $\mathrm{Fin}\,A$ holds $@x = x$.

Let A be a set. The functor $\mathrm{FinUnion}_A$ yields a binary operation on $\mathrm{Fin}\,A$ and is defined by:

for all elements x, y of $\mathrm{Fin}\,A$ holds $(\mathrm{FinUnion}_A)(x,y) = @(x \cup y)$.

In the sequel A will denote a set and x, y will denote elements of $\mathrm{Fin}\,A$. One can prove the following propositions:

(47) For every binary operation IT on $\mathrm{Fin}\,A$ holds $IT = \mathrm{FinUnion}_A$ if and only if for all elements x, y of $\mathrm{Fin}\,A$ holds $IT(x,y) = @(x \cup y)$.

(48) $\mathrm{FinUnion}_A(x,y) = x \cup y$.

(49) $\mathrm{FinUnion}_A$ is idempotent.

(50) $\mathrm{FinUnion}_A$ is commutative.

(51) $\mathrm{FinUnion}_A$ is associative.

(52) $@0_A$ is a unity w.r.t. $\mathrm{FinUnion}_A$.

(53) $\mathrm{FinUnion}_A$ has a unity.

(54) $\mathbf{1}_{\mathrm{FinUnion}_A}$ is a unity w.r.t. $\mathrm{FinUnion}_A$.

(55) $\mathbf{1}_{\mathrm{FinUnion}_A} = \emptyset$.

For simplicity we adopt the following rules: X, Y are non-empty sets, A is a set, f is a function from X into $\mathrm{Fin}\,A$, and i, j, k are elements of X. The arguments of the notions defined below are the following: X which is a non-empty set; A which is a set; B which is an element of $\mathrm{Fin}\,X$; f which is a function from X into $\mathrm{Fin}\,A$. The functor $\mathrm{FinUnion}(B,f)$ yields an element of $\mathrm{Fin}\,A$ and is defined by:

$\mathrm{FinUnion}(B,f) = \mathrm{FinUnion}_A\text{-}\sum_B f$.

The following propositions are true:

(56) $\mathrm{FinUnion}(\{i\}, f) = f(i)$.

(57) $\mathrm{FinUnion}(\{i,j\}, f) = f(i) \cup f(j)$.

(58) $\mathrm{FinUnion}(\{i,j,k\}, f) = (f(i) \cup f(j)) \cup f(k)$.

(59) $\mathrm{FinUnion}(0_X, f) = \emptyset$.

(60) For every element B of $\mathrm{Fin}\, X$ holds $\mathrm{FinUnion}(B \cup \{i\}, f) = \mathrm{FinUnion}(B, f) \cup f(i)$.

(61) For every element B of $\mathrm{Fin}\, X$ holds $\mathrm{FinUnion}(B, f) = \bigcup(f \circ B)$.

(62) For all elements B_1, B_2 of $\mathrm{Fin}\, X$ holds $\mathrm{FinUnion}(B_1 \cup B_2, f) = \mathrm{FinUnion}(B_1, f) \cup \mathrm{FinUnion}(B_2, f)$.

(63) For every element B of $\mathrm{Fin}\, X$ for every function f from X into Y for every function g from Y into $\mathrm{Fin}\, A$ holds $\mathrm{FinUnion}(f \circ B, g) = \mathrm{FinUnion}(B, g \cdot f)$.

(64) Let A, X be non-empty sets. Let Y be a set. Let G be a binary operation on A. Suppose G is commutative and G is associative and G is idempotent. Let B be an element of $\mathrm{Fin}\, X$. Then if $B \neq \emptyset$, then for every function f from X into $\mathrm{Fin}\, Y$ for every function g from $\mathrm{Fin}\, Y$ into A such that for all elements x, y of $\mathrm{Fin}\, Y$ holds $g(x \cup y) = G(g(x), g(y))$ holds $g(\mathrm{FinUnion}(B, f)) = G\text{-}\sum_B(g \cdot f)$.

(65) Let Z be a non-empty set. Let Y be a set. Let G be a binary operation on Z. Suppose G is commutative and G is associative and G is idempotent and G has a unity. Let f be a function from X into $\mathrm{Fin}\, Y$. Let g be a function from $\mathrm{Fin}\, Y$ into Z. Then if $g(0_Y) = \mathbf{1}_G$ and for all elements x, y of $\mathrm{Fin}\, Y$ holds $g(x \cup y) = G(g(x), g(y))$, then for every element B of $\mathrm{Fin}\, X$ holds $g(\mathrm{FinUnion}(B, f)) = G\text{-}\sum_B(g \cdot f)$.

Let A be a set. The functor $\mathrm{singleton}_A$ yielding a function from A into $\mathrm{Fin}\, A$, is defined by:

for arbitrary x such that $x \in A$ holds $(\mathrm{singleton}_A)(x) = \{x\}$.

The following propositions are true:

(66) For every set A for every function f from A into $\mathrm{Fin}\, A$ holds $f = \mathrm{singleton}_A$ if and only if for arbitrary x such that $x \in A$ holds $f(x) = \{x\}$.

(67) For every non-empty set A for every function f from A into $\mathrm{Fin}\, A$ holds $f = \mathrm{singleton}_A$ if and only if for every element x of A holds $f(x) = \{x\}$.

(68) For arbitrary x for every element y of X holds $x \in \mathrm{singleton}_X(y)$ if and only if $x = y$.

(69) For all elements x, y, z of X such that $x \in \mathrm{singleton}_X(z)$ and $y \in \mathrm{singleton}_X(z)$ holds $x = y$.

(70) For every element B of $\mathrm{Fin}\, X$ for arbitrary x holds $x \in \mathrm{FinUnion}(B, f)$ if and only if there exists i being an element of X such that $i \in B$ and $x \in f(i)$.

(71) For every element B of $\mathrm{Fin}\, X$ holds $\mathrm{FinUnion}(B, \mathrm{singleton}_X) = B$.

The arguments of the notions defined below are the following: X, Y which are non-empty families of sets; g which is a function from X into Y; x which is an element of X. Then $g(x)$ is an element of Y.

Next we state a proposition

(72) Let Y, Z be sets. Let f be a function from X into $\mathrm{Fin}\, Y$. Let g be a function from $\mathrm{Fin}\, Y$ into $\mathrm{Fin}\, Z$. Then if $g(0_Y) = 0_Z$ and for all elements x,

y of $\mathrm{Fin}\,Y$ holds $g(x \cup y) = g(x) \cup g(y)$, then for every element B of $\mathrm{Fin}\,X$ holds $g(\mathrm{FinUnion}(B, f)) = \mathrm{FinUnion}(B, g \cdot f)$.

References

[1] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[2] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[3] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[4] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.

[5] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[6] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Formalized Mathematics*, 1(**1**):187–190, 1990.

[7] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

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Cardinal Numbers

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Summary. We present the choice function rule in the beginning of the article. In the main part of the article we formalize the base of cardinal theory. In the first section we introduce the concept of cardinal numbers and order relations between them. We present here Cantor-Bernstein theorem and other properties of order relation of cardinals. In the second section we show that every set has cardinal number equipotence to it. We introduce notion of alephs and we deal with the concept of finite set. At the end of the article we show two schemes of cardinal induction. Some definitions are based on [9] and [11].

MML Identifier: CARD_1.

The papers [12], [10], [1], [13], [7], [4], [2], [3], [5], [6], and [8] provide the notation and terminology for this paper. For simplicity we follow the rules: A, B will be ordinal numbers, X, X_1, Y, Y_1, Z will be sets, R will be a relation, f will be a function, x, y will be arbitrary, m, n will be natural numbers, and M will be a non-empty family of sets. We now state a proposition

(1) If for every X such that $X \in M$ holds $X \neq \emptyset$, then there exists $Choice$ being a function such that $\mathrm{dom}\, Choice = M$ and for every X such that $X \in M$ holds $Choice(X) \in X$.

The mode cardinal number, which widens to the type a set, is defined by:
there exists B such that it $= B$ and for every A such that $A \approx B$ holds $B \subseteq A$.

One can prove the following proposition

(2) X is a cardinal number if and only if there exists A such that $X = A$ and for every B such that $B \approx A$ holds $A \subseteq B$.

Let M be a cardinal number. The functor $\overline{M}$ yielding an ordinal number, is defined by:

$\overline{M} = M$.

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In the sequel K, M, N will be cardinal numbers. We now state three propositions:

(3) $\overline{M} = M$.

(4) For every X there exists A such that $X \approx A$.

(5) M is an ordinal number.

We now define two new predicates. Let us consider M, N. The predicate $M \leq N$ is defined by:

$M \subseteq N$.

The predicate $M < N$ is defined by:

$M \in N$.

Next we state a number of propositions:

(6) $M \leq N$ if and only if $M \subseteq N$.

(7) $M < N$ if and only if $M \in N$.

(8) $M = N$ if and only if $M \approx N$.

(9) $M \leq M$.

(10) If $M \leq N$ and $N \leq M$, then $M = N$.

(11) If $M \leq N$ and $N \leq K$, then $M \leq K$.

(12) $M \leq N$ or $N \leq M$.

(13) $M < N$ if and only if $M \leq N$ and $M \neq N$.

(14) $M < N$ if and only if $N \nleq M$.

(15) If $M < N$, then $N \nless M$.

(16) $M < N$ or $M = N$ or $N < M$.

(17) If $M < N$ and $N < K$, then $M < K$.

(18) If $M < N$ and $N \leq K$ or $M \leq N$ and $N < K$, then $M < K$.

Let us consider X. The functor $\overline{\overline{X}}$ yields a cardinal number and is defined by:

$X \approx \overline{\overline{X}}$.

Next we state a number of propositions:

(19) $M = \overline{\overline{X}}$ if and only if $X \approx M$.

(20) $\overline{\overline{M}} = M$.

(21) $X \approx Y$ if and only if $\overline{\overline{X}} = \overline{\overline{Y}}$.

(22) If R is well ordering relation, then field $R \approx \overline{R}$.

(23) If $X \subseteq M$, then $\overline{\overline{X}} \leq M$.

(24) $\overline{\overline{A}} \subseteq A$.

(25) If $X \in M$, then $\overline{\overline{X}} < M$.

(26) $\overline{\overline{X}} \leq \overline{\overline{Y}}$ if and only if there exists f such that f is one-to-one and $\operatorname{dom} f = X$ and $\operatorname{rng} f \subseteq Y$.

(27) If $X \subseteq Y$, then $\overline{\overline{X}} \leq \overline{\overline{Y}}$.

(28) $\overline{\overline{X}} \leq \overline{\overline{Y}}$ if and only if there exists f such that $\operatorname{dom} f = Y$ and $X \subseteq \operatorname{rng} f$.

(29) $X \not\approx 2^X$.

(30) $\overline{\overline{X}} < \overline{\overline{2^X}}$.

Let us consider X. The functor X^+ yielding a cardinal number, is defined by: $\overline{\overline{X}} < X^+$ and for every M such that $\overline{\overline{X}} < M$ holds $X^+ \leq M$.

We now state several propositions:

(31) $M = X^+$ if and only if $\overline{\overline{X}} < M$ and for every N such that $\overline{\overline{X}} < N$ holds $M \leq N$.

(32) $M < M^+$.

(33) $\overline{\overline{\mathbf{0}}} < X^+$.

(34) If $\overline{\overline{X}} = \overline{\overline{Y}}$, then $X^+ = Y^+$.

(35) If $X \approx Y$, then $X^+ = Y^+$.

(36) $A \in A^+$.

In the sequel L, L_1 will be transfinite sequences. Let us consider M. The predicate M is a limit cardinal number is defined by:

for no N holds $M = N^+$.

One can prove the following proposition

(37) M is a limit cardinal number if and only if for no N holds $M = N^+$.

Let us consider A. The functor $\aleph_A$ yielding any, is defined by:

there exists L such that $\aleph_A = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} A$ and $L(\mathbf{0}) = \overline{\overline{\mathbb{N}}}$ and for all B, y such that $\operatorname{succ} B \in \operatorname{succ} A$ and $y = L(B)$ holds $L(\operatorname{succ} B) = (\bigcup\{y\})^+$ and for all B, L_1 such that $B \in \operatorname{succ} A$ and $B \neq \mathbf{0}$ and B is a limit ordinal number and $L_1 = L \restriction B$ holds $L(B) = \overline{\overline{\sup L_1}}$.

Let us consider A. Then $\aleph_A$ is a cardinal number.

The following propositions are true:

(38) $\aleph_{\mathbf{0}} = \overline{\overline{\mathbb{N}}}$.

(39) $\aleph_{\operatorname{succ} A} = {\aleph_A}^+$.

(40) If $A \neq \mathbf{0}$ and A is a limit ordinal number, then for every L such that $\operatorname{dom} L = A$ and for every B such that $B \in A$ holds $L(B) = \aleph_B$ holds $\aleph_A = \overline{\overline{\sup L}}$.

(41) $A \in B$ if and only if $\aleph_A < \aleph_B$.

(42) If $\aleph_A = \aleph_B$, then $A = B$.

(43) $A \subseteq B$ if and only if $\aleph_A \leq \aleph_B$.

(44) If $X \subseteq Y$ and $Y \subseteq Z$ and $X \approx Z$, then $X \approx Y$ and $Y \approx Z$.

(45) If $2^Y \subseteq X$, then $\overline{\overline{Y}} < \overline{\overline{X}}$ and $Y \not\approx X$.

(46) $X \approx \emptyset$ if and only if $X = \emptyset$.

(47) $\overline{\overline{\emptyset}} = \mathbf{0}$.

(48) $X \approx \{x\}$ if and only if there exists x such that $X = \{x\}$.

(49) $\overline{\overline{X}} = \overline{\overline{\{x\}}}$ if and only if there exists x such that $X = \{x\}$.

(50) $\overline{\overline{\{x\}}} = \mathbf{1}$.

Let us consider n. The functor $\overline{n}$ yielding an ordinal number, is defined by:

there exists f such that $\overline{n} = f(n)$ and $\operatorname{dom} f = \mathbb{N}$ and $f(0) = \mathbf{0}$ and for every element n of $\mathbb{N}$ for every x such that $x = f(n)$ holds $f(n+1) = \operatorname{succ}(\bigcup\{x\})$.

We now state a number of propositions:

(51) $\overline{0} = \mathbf{0}$.

(52) $\overline{n+1} = \operatorname{succ}(\overline{n})$.

(53) $\overline{n} \in \omega$.

(54) If A is natural, then there exists n such that $\overline{n} = A$.

(55) If $\overline{n} = \overline{m}$, then $n = m$.

(56) $n \leq m$ if and only if $\overline{n} \subseteq \overline{m}$.

(57) $\mathbb{N} \approx \omega$.

(58) If $X \cap X_1 = \emptyset$ and $Y \cap Y_1 = \emptyset$ and $X \approx Y$ and $X_1 \approx Y_1$, then $X \cup X_1 \approx Y \cup Y_1$.

(59) If $x \in X$ and $y \in X$, then $X \setminus \{x\} \approx X \setminus \{y\}$.

(60) If $X \subseteq \operatorname{dom} f$ and f is one-to-one, then $X \approx f \circ X$.

(61) If $X \approx Y$ and $x \in X$ and $y \in Y$, then $X \setminus \{x\} \approx Y \setminus \{y\}$.

(62) If $\operatorname{Seg} n \approx \operatorname{Seg} m$, then $n = m$.

(63) $\operatorname{Seg} n \approx \overline{n}$.

(64) If $\overline{n} \approx \overline{m}$, then $n = m$.

(65) If $A \in \omega$, then A is a cardinal number.

(66) $\overline{n} = \overline{\overline{\overline{n}}}$.

Let us consider n. The functor $\overline{\overline{n}}$ yielding a cardinal number, is defined by:
$\overline{\overline{n}} = \overline{n}$.

One can prove the following propositions:

(67) $\overline{\overline{n}} = \overline{n}$.

(68) If $X \approx Y$ or $Y \approx X$ but X is finite, then Y is finite.

(69) $\overline{n}$ is finite and $\overline{\overline{n}}$ is finite.

(70) $\overline{\overline{\operatorname{Seg} n}} = \overline{\overline{n}}$.

(71) If $\overline{\overline{n}} = \overline{\overline{m}}$, then $n = m$.

(72) $\overline{\overline{n}} \leq \overline{\overline{m}}$ if and only if $n \leq m$.

(73) $\overline{\overline{n}} < \overline{\overline{m}}$ if and only if $n < m$.

(74) If X is finite, then there exists n such that $X \approx \overline{n}$.

(75) If X is finite, then there exists n such that $X \approx \operatorname{Seg} n$.

(76) $\overline{\overline{n}}^{+} = \overline{\overline{n+1}}$.

Let us consider X. Let us assume that X is finite. The functor $\operatorname{card} X$ yields a natural number and is defined by:
$\overline{\overline{\operatorname{card} X}} = \overline{\overline{X}}$.

We now state several propositions:

(77) If X is finite, then $\operatorname{card} X = n$ if and only if $\overline{\overline{n}} = \overline{\overline{X}}$.

(78) $\operatorname{card} \emptyset = 0$.

(79) $\operatorname{card}\{x\} = 1$.

(80) If Y is finite and $X \subseteq Y$, then $\operatorname{card} X \leq \operatorname{card} Y$.

(81) If X is finite or Y is finite but $X \approx Y$, then $\operatorname{card} X = \operatorname{card} Y$.

(82) If X is finite, then X^+ is finite.

In the article we present several logical schemes. The scheme *Cardinal_Ind* concerns a unary predicate $\mathcal{P}$ and states that:

for every M holds $\mathcal{P}[M]$

provided the parameter satisfies the following conditions:

- $\mathcal{P}[\mathbf{0}]$,
- for every M such that $\mathcal{P}[M]$ holds $\mathcal{P}[M^+]$,
- for every M such that $M \neq \mathbf{0}$ and M is a limit cardinal number and for every N such that $N < M$ holds $\mathcal{P}[N]$ holds $\mathcal{P}[M]$.

The scheme *Cardinal_CompInd* concerns a unary predicate $\mathcal{P}$ and states that:

for every M holds $\mathcal{P}[M]$

provided the parameter satisfies the following condition:

- for every M such that for every N such that $N < M$ holds $\mathcal{P}[N]$ holds $\mathcal{P}[M]$.

Next we state several propositions:

(83) $\aleph_{\mathbf{0}} = \omega$.

(84) $\overline{\overline{\omega}} = \omega$ and $\overline{\overline{\mathbb{N}}} = \omega$.

(85) $\overline{\overline{\omega}}$ is a limit cardinal number.

(86) If M is finite, then there exists n such that $M = \overline{\overline{n}}$.

(87) $\operatorname{card}(\operatorname{Seg} n) = n$ and $\operatorname{card}(\overline{n}) = n$ and $\operatorname{card} \overline{\overline{n}} = n$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(**2**):281–290, 1990.

[4] Grzegorz Bancerek. The well ordering relations. *Formalized Mathematics*, 1(**1**):123–129, 1990.

[5] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Formalized Mathematics*, 1(**2**):265–267, 1990.

[6] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[9] Wojciech Guzicki and Paweł Zbierski. *Podstawy teorii mnogości.* PWN, Warszawa, 1978.

[10] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[11] Kazimierz Kuratowski and Andrzej Mostowski. *Teoria mnogości.* PTM, Wrocław, 1952.

[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[13] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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Compact Spaces

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Summary. The article contains definition of a compact space and some theorems about compact spaces. The notions of a cover of a set and a centered family are defined in the article to be used in these theorems. A set is compact in the topological space if and only if every open cover of the set has a finite subcover. This definition is equivalent, what has been shown next, to the following definition: a set is compact if and only if a subspace generated by that set is compact. Some theorems about mappings and homeomorphisms of compact spaces have been also proved. The following schemes used in proofs of theorems have been proved in the article : FuncExChoice - the scheme of choice of a function, BiFuncEx - the scheme of parallel choice of two functions and the theorem about choice of a finite counter image of a finite image.

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The articles [6], [1], [4], [3], [5], and [2] provide the terminology and notation for this paper. We follow a convention: x, y, z are arbitrary, Y, Z denote sets, and f denotes a function. In the article we present several logical schemes. The scheme *FuncExChoice* deals with a constant $\mathcal{A}$ that is a set, a constant $\mathcal{B}$ that is a set and a binary predicate $\mathcal{P}$ and states that:

there exists f being a function such that $\mathrm{dom}\, f = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x, f(x)]$

provided the parameters satisfy the following condition:

- for every x such that $x \in \mathcal{A}$ there exists y such that $y \in \mathcal{B}$ and $\mathcal{P}[x, y]$.

The scheme *BiFuncEx* deals with a constant $\mathcal{A}$ that is a set, a constant $\mathcal{B}$ that is a set, a constant $\mathcal{C}$ that is a set and a ternary predicate $\mathcal{P}$ and states that:

there exist f, g being functions such that $\mathrm{dom}\, f = \mathcal{A}$ and $\mathrm{dom}\, g = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x, f(x), g(x)]$

provided the parameters satisfy the following condition:

- if $x \in \mathcal{A}$, then there exist y, z such that $y \in \mathcal{B}$ and $z \in \mathcal{C}$ and $\mathcal{P}[x, y, z]$.

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Next we state a proposition

(1) If Z is finite and $Z \subseteq \operatorname{rng} f$, then there exists Y such that $Y \subseteq \operatorname{dom} f$ and Y is finite and $f \circ Y = Z$.

For simplicity we adopt the following convention: T, S are topological spaces, A is a subspace of T, p, q are points of T, P, Q, W, V are subsets of T, and F, G are families of subsets of T. Let us consider T, F, P. The predicate F is a cover of P is defined by:

$P \subseteq \bigcup F$.

One can prove the following proposition

(2) F is a cover of P if and only if $P \subseteq \bigcup F$.

Let us consider T, F. The predicate F is centered is defined by:

$F \neq \emptyset$ and for every G such that $G \neq \emptyset$ and $G \subseteq F$ and G is finite holds $\bigcap G \neq \emptyset$.

One can prove the following proposition

(3) F is centered if and only if $F \neq \emptyset$ and for every G such that $G \neq \emptyset$ and $G \subseteq F$ and G is finite holds $\bigcap G \neq \emptyset$.

We now define five new predicates. Let us consider T. The predicate T is compact is defined by:

for every F such that F is a cover of T and F is open there exists G such that $G \subseteq F$ and G is a cover of T and G is finite.

The predicate T is a T2 space is defined by:

for all p, q such that $p \neq q$ there exist W, V such that W is open and V is open and $p \in W$ and $q \in V$ and $W \cap V = \emptyset$.

The predicate T is a T3 space is defined by:

for every p for every P such that $P \neq \emptyset$ and P is closed and $p \notin P$ there exist W, V such that W is open and V is open and $p \in W$ and $P \subseteq V$ and $W \cap V = \emptyset$.

The predicate T is a T4 space is defined by:

for all W, V such that $W \neq \emptyset$ and $V \neq \emptyset$ and W is closed and V is closed and $W \cap V = \emptyset$ there exist P, Q such that P is open and Q is open and $W \subseteq P$ and $V \subseteq Q$ and $P \cap Q = \emptyset$.

Let us consider P. The predicate P is compact is defined by:

for every F such that F is a cover of P and F is open there exists G such that $G \subseteq F$ and G is a cover of P and G is finite.

We now state a number of propositions:

(4) T is compact if and only if for every F such that F is a cover of T and F is open there exists G such that $G \subseteq F$ and G is a cover of T and G is finite.

(5) T is a T2 space if and only if for all p, q such that $p \neq q$ there exist W, V such that W is open and V is open and $p \in W$ and $q \in V$ and $W \cap V = \emptyset$.

(6) T is a T3 space if and only if for every p for every P such that $P \neq \emptyset$ and P is closed and $p \notin P$ there exist W, V such that W is open and V is open and $p \in W$ and $P \subseteq V$ and $W \cap V = \emptyset$.

(7) T is a T4 space if and only if for all P, Q such that $P \neq \emptyset$ and $Q \neq \emptyset$ and P is closed and Q is closed and $P \cap Q = \emptyset$ there exist W, V such that W is open and V is open and $P \subseteq W$ and $Q \subseteq V$ and $W \cap V = \emptyset$.

(8) P is compact if and only if for every F such that F is a cover of P and F is open there exists G such that $G \subseteq F$ and G is a cover of P and G is finite.

(9) $\emptyset_T$ is compact.

(10) T is compact if and only if Ω_T is compact.

(11) If $Q \subseteq \Omega_A$, then Q is compact if and only if for every subset P of A such that $P = Q$ holds P is compact.

(12) If $P \neq \emptyset$, then P is compact if and only if $T \restriction P$ is compact.

(13) T is compact if and only if for every F such that F is centered and F is closed holds $\bigcap F \neq \emptyset$.

(14) T is compact if and only if for every F such that $F \neq \emptyset$ and F is closed and $\bigcap F = \emptyset$ there exists G such that $G \neq \emptyset$ and $G \subseteq F$ and G is finite and $\bigcap G = \emptyset$.

(15) For every T such that T is a T2 space for every subset A of T such that $A \neq \emptyset$ and A is compact for every p such that $p \notin A$ there exist P, Q such that P is open and Q is open and $p \in P$ and $A \subseteq Q$ and $P \cap Q = \emptyset$.

(16) If T is a T2 space and P is compact, then P is closed.

(17) If T is compact and P is closed, then P is compact.

(18) If P is compact and $Q \subseteq P$ and Q is closed, then Q is compact.

(19) If P is compact and Q is compact, then $P \cup Q$ is compact.

(20) If T is a T2 space and P is compact and Q is compact, then $P \cap Q$ is compact.

(21) If T is a T2 space and T is compact, then T is a T3 space.

(22) If T is a T2 space and T is compact, then T is a T4 space.

In the sequel f will be a map from T into S. Next we state four propositions:

(23) If T is compact and f is continuous and $\operatorname{rng} f = \Omega_S$, then S is compact.

(24) If f is continuous and $\operatorname{rng} f = \Omega_S$ and P is compact, then $f \circ P$ is compact.

(25) If T is compact and S is a T2 space and $\operatorname{rng} f = \Omega_S$ and f is continuous, then for every P such that P is closed holds $f \circ P$ is closed.

(26) If T is compact and S is a T2 space and $\operatorname{rng} f = \Omega_S$ and f is one-to-one and f is continuous, then f is a homeomorphism.

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[2] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.

[3] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[4] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[5] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[6] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

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Kuratowski - Zorn Lemma [1]

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Summary. The goal of this article is to prove Kuratowski - Zorn lemma. We prove it in a number of forms (theorems and schemes). We introduce the following notions: a relation is a quasi (or partial, or linear) order, a relation quasi (or partially, or lineary) orders a set, minimal and maximal element in a relation, inferior and superior element of a relation, a set has lower (or upper) Zorn property w.r.t. a relation. We prove basic theorems concerning those notions and theorems that relate them to the notions introduced in [6]. At the end of the article we prove some theorems that belong rather to [7], [9] or [2].

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The notation and terminology used here are introduced in the following articles: [5], [3], [7], [9], [8], [2], [4], [6], and [1]. For simplicity we follow a convention: R, P are relations, X, X_1, X_2, Y, Z are sets, O is an order in X, D, D_1 are non-empty sets, x, y are arbitrary, A is a poset, C is a chain of A, S is a subset of A, and a, b are elements of A. In the article we present several logical schemes. The scheme *RelOnDomEx* deals with a constant $\mathcal{A}$ that is a non-empty set, a constant $\mathcal{B}$ that is a non-empty set and a binary predicate $\mathcal{P}$ and states that:

there exists R being a relation between $\mathcal{A}$ and $\mathcal{B}$ such that for every element a of $\mathcal{A}$ for every element b of $\mathcal{B}$ holds $\langle a, b\rangle \in R$ if and only if $\mathcal{P}[a, b]$
for all values of the parameters.

The scheme *RelOnDomEx1* deals with a constant $\mathcal{A}$ that is a non-empty set and a binary predicate $\mathcal{P}$ and states that:

there exists R being a relation on $\mathcal{A}$ such that for all elements a, b of $\mathcal{A}$ holds $\langle a, b\rangle \in R$ if and only if $\mathcal{P}[a, b]$
for all values of the parameters.

One can prove the following propositions:

(1) $\operatorname{dom} O = X$ and $\operatorname{rng} O = X$.

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(2) $\operatorname{field} O = X$.

We now define three new predicates. Let us consider R. The predicate R is a quasi order is defined by:

R is pseudo reflexive and R is transitive.

The predicate R is a partial order is defined by:

R is pseudo reflexive and R is transitive and R is antisymmetric.

The predicate R is a linear order is defined by:

R is pseudo reflexive and R is transitive and R is antisymmetric and R is connected.

We now state a number of propositions:

(3) R is a quasi order if and only if R is pseudo reflexive and R is transitive.

(4) R is a partial order if and only if R is pseudo reflexive and R is transitive and R is antisymmetric.

(5) R is a linear order if and only if R is pseudo reflexive and R is transitive and R is antisymmetric and R is connected.

(6) If R is a quasi order, then $R^{\smile}$ is a quasi order.

(7) If R is a partial order, then $R^{\smile}$ is a partial order.

(8) If R is a linear order, then $R^{\smile}$ is a linear order.

(9) If R is well ordering relation, then R is a quasi order and R is a partial order and R is a linear order.

(10) If R is a linear order, then R is a quasi order and R is a partial order.

(11) If R is a partial order, then R is a quasi order.

(12) O is a partial order.

(13) O is a quasi order.

(14) If O is connected, then O is a linear order.

(15) If R is a quasi order, then $R\,|^2\,X$ is a quasi order.

(16) If R is a partial order, then $R\,|^2\,X$ is a partial order.

(17) If R is a linear order, then $R\,|^2\,X$ is a linear order.

(18) $\operatorname{field}((\text{the order of } A)\,|^2\,S) = S$.

(19) If (the order of A) $|^2\,S$ is a linear order, then S is a chain of A.

(20) (the order of A) $|^2\,C$ is a linear order.

(21) $\varnothing$ is a quasi order and $\varnothing$ is a partial order and $\varnothing$ is a linear order and $\varnothing$ is well ordering relation.

(22) $\triangle_X$ is a quasi order and $\triangle_X$ is a partial order.

We now define three new predicates. Let us consider R, X. The predicate R quasi orders X is defined by:

R is reflexive in X and R is transitive in X.

The predicate R partially orders X is defined by:

R is reflexive in X and R is transitive in X and R is antisymmetric in X.

The predicate R linearly orders X is defined by:

R is reflexive in X and R is transitive in X and R is antisymmetric in X and R is connected in X.

The following propositions are true:

(23) R quasi orders X if and only if R is reflexive in X and R is transitive in X.

(24) R partially orders X if and only if R is reflexive in X and R is transitive in X and R is antisymmetric in X.

(25) R linearly orders X if and only if R is reflexive in X and R is transitive in X and R is antisymmetric in X and R is connected in X.

(26) If R well orders X, then R quasi orders X and R partially orders X and R linearly orders X.

(27) If R linearly orders X, then R quasi orders X and R partially orders X.

(28) If R partially orders X, then R quasi orders X.

(29) If R is a quasi order, then R quasi orders field R.

(30) If R quasi orders Y and $X \subseteq Y$, then R quasi orders X.

(31) If R quasi orders X, then $R\,|^2\, X$ is a quasi order.

(32) If R is a partial order, then R partially orders field R.

(33) If R partially orders Y and $X \subseteq Y$, then R partially orders X.

(34) If R partially orders X, then $R\,|^2\, X$ is a partial order.

(35) If R is a linear order, then R linearly orders field R.

(36) If R linearly orders Y and $X \subseteq Y$, then R linearly orders X.

(37) If R linearly orders X, then $R\,|^2\, X$ is a linear order.

(38) If R quasi orders X, then $R^\smile$ quasi orders X.

(39) If R partially orders X, then $R^\smile$ partially orders X.

(40) If R linearly orders X, then $R^\smile$ linearly orders X.

(41) O quasi orders X.

(42) O partially orders X.

(43) If R partially orders X, then $R\,|^2\, X$ is an order in X.

(44) If R linearly orders X, then $R\,|^2\, X$ is an order in X.

(45) If R well orders X, then $R\,|^2\, X$ is an order in X.

(46) If the order of A linearly orders S, then S is a chain of A.

(47) the order of A linearly orders C.

(48) $\triangle_X$ quasi orders X and $\triangle_X$ partially orders X.

We now define two new predicates. Let us consider R, X. The predicate X has the upper Zorn property w.r.t. R is defined by:

for every Y such that $Y \subseteq X$ and $R\,|^2\, Y$ is a linear order there exists x such that $x \in X$ and for every y such that $y \in Y$ holds $\langle y, x\rangle \in R$.

The predicate X has the lower Zorn property w.r.t. R is defined by:

for every Y such that $Y \subseteq X$ and $R\,|^2\, Y$ is a linear order there exists x such that $x \in X$ and for every y such that $y \in Y$ holds $\langle x, y\rangle \in R$.

We now state several propositions:

(49) X has the upper Zorn property w.r.t. R if and only if for every Y such that $Y \subseteq X$ and $R\,|^2\,Y$ is a linear order there exists x such that $x \in X$ and for every y such that $y \in Y$ holds $\langle y, x\rangle \in R$.

(50) X has the lower Zorn property w.r.t. R if and only if for every Y such that $Y \subseteq X$ and $R\,|^2\,Y$ is a linear order there exists x such that $x \in X$ and for every y such that $y \in Y$ holds $\langle x, y\rangle \in R$.

(51) If X has the upper Zorn property w.r.t. R, then $X \neq \emptyset$.

(52) If X has the lower Zorn property w.r.t. R, then $X \neq \emptyset$.

(53) X has the upper Zorn property w.r.t. R if and only if X has the lower Zorn property w.r.t. $R^{\smile}$.

(54) X has the upper Zorn property w.r.t. $R^{\smile}$ if and only if X has the lower Zorn property w.r.t. R.

We now define four new predicates. Let us consider R, x. The predicate x is maximal in R is defined by:

$x \in \operatorname{field} R$ and for no y holds $y \in \operatorname{field} R$ and $y \neq x$ and $\langle x, y\rangle \in R$.

The predicate x is minimal in R is defined by:

$x \in \operatorname{field} R$ and for no y holds $y \in \operatorname{field} R$ and $y \neq x$ and $\langle y, x\rangle \in R$.

The predicate x is superior of R is defined by:

$x \in \operatorname{field} R$ and for every y such that $y \in \operatorname{field} R$ and $y \neq x$ holds $\langle y, x\rangle \in R$.

The predicate x is inferior of R is defined by:

$x \in \operatorname{field} R$ and for every y such that $y \in \operatorname{field} R$ and $y \neq x$ holds $\langle x, y\rangle \in R$.

Next we state a number of propositions:

(55) x is maximal in R if and only if $x \in \operatorname{field} R$ and for no y holds $y \in \operatorname{field} R$ and $y \neq x$ and $\langle x, y\rangle \in R$.

(56) x is minimal in R if and only if $x \in \operatorname{field} R$ and for no y holds $y \in \operatorname{field} R$ and $y \neq x$ and $\langle y, x\rangle \in R$.

(57) x is superior of R if and only if $x \in \operatorname{field} R$ and for every y such that $y \in \operatorname{field} R$ and $y \neq x$ holds $\langle y, x\rangle \in R$.

(58) x is inferior of R if and only if $x \in \operatorname{field} R$ and for every y such that $y \in \operatorname{field} R$ and $y \neq x$ holds $\langle x, y\rangle \in R$.

(59) If x is inferior of R and R is antisymmetric, then x is minimal in R.

(60) If x is superior of R and R is antisymmetric, then x is maximal in R.

(61) If x is minimal in R and R is connected, then x is inferior of R.

(62) If x is maximal in R and R is connected, then x is superior of R.

(63) If $x \in X$ and x is superior of R and $X \subseteq \operatorname{field} R$ and R is pseudo reflexive, then X has the upper Zorn property w.r.t. R.

(64) If $x \in X$ and x is inferior of R and $X \subseteq \operatorname{field} R$ and R is pseudo reflexive, then X has the lower Zorn property w.r.t. R.

(65) x is minimal in R if and only if x is maximal in $R^{\smile}$.

(66) x is minimal in $R^{\smile}$ if and only if x is maximal in R.

(67) x is inferior of R if and only if x is superior of $R^{\smile}$.

(68) x is inferior of $R^{\smile}$ if and only if x is superior of R.

(69) a is minimal in the order of A if and only if for every b holds $b \not< a$.

(70) a is maximal in the order of A if and only if for every b holds $a \not< b$.

(71) a is superior of the order of A if and only if for every b such that $a \neq b$ holds $b < a$.

(72) a is inferior of the order of A if and only if for every b such that $a \neq b$ holds $a < b$.

(73) If for every C there exists a such that for every b such that $b \in C$ holds $b \leq a$, then there exists a such that for every b holds $a \not< b$.

(74) If for every C there exists a such that for every b such that $b \in C$ holds $a \leq b$, then there exists a such that for every b holds $b \not< a$.

We now state several propositions:

(75) For all R, X such that R partially orders X and field $R = X$ and X has the upper Zorn property w.r.t. R there exists x such that x is maximal in R.

(76) For all R, X such that R partially orders X and field $R = X$ and X has the lower Zorn property w.r.t. R there exists x such that x is minimal in R.

(77) Given X. Suppose $X \neq \emptyset$ and for every Z such that $Z \subseteq X$ and for all X_1, X_2 such that $X_1 \in Z$ and $X_2 \in Z$ holds $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$ there exists Y such that $Y \in X$ and for every X_1 such that $X_1 \in Z$ holds $X_1 \subseteq Y$. Then there exists Y such that $Y \in X$ and for every Z such that $Z \in X$ and $Z \neq Y$ holds $Y \not\subseteq Z$.

(78) Given X. Suppose $X \neq \emptyset$ and for every Z such that $Z \subseteq X$ and for all X_1, X_2 such that $X_1 \in Z$ and $X_2 \in Z$ holds $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$ there exists Y such that $Y \in X$ and for every X_1 such that $X_1 \in Z$ holds $Y \subseteq X_1$. Then there exists Y such that $Y \in X$ and for every Z such that $Z \in X$ and $Z \neq Y$ holds $Z \not\subseteq Y$.

(79) Given X. Suppose $X \neq \emptyset$ and for every Z such that $Z \neq \emptyset$ and $Z \subseteq X$ and for all X_1, X_2 such that $X_1 \in Z$ and $X_2 \in Z$ holds $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$ holds $\bigcup Z \in X$. Then there exists Y such that $Y \in X$ and for every Z such that $Z \in X$ and $Z \neq Y$ holds $Y \not\subseteq Z$.

(80) Given X. Suppose $X \neq \emptyset$ and for every Z such that $Z \neq \emptyset$ and $Z \subseteq X$ and for all X_1, X_2 such that $X_1 \in Z$ and $X_2 \in Z$ holds $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$ holds $\bigcap Z \in X$. Then there exists Y such that $Y \in X$ and for every Z such that $Z \in X$ and $Z \neq Y$ holds $Z \not\subseteq Y$.

Now we present two schemes. The scheme *Zorn_Max* concerns a constant $\mathcal{A}$ that is a non-empty set and a binary predicate $\mathcal{P}$ and states that:

there exists x being an element of $\mathcal{A}$ such that for every element y of $\mathcal{A}$ such that $x \neq y$ holds not $\mathcal{P}[x, y]$

provided the parameters satisfy the following conditions:

- for every element x of $\mathcal{A}$ holds $\mathcal{P}[x, x]$,
- for all elements x, y of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, x]$ holds $x = y$,
- for all elements x, y, z of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$,

- for every X such that $X \subseteq \mathcal{A}$ and for all elements x, y of $\mathcal{A}$ such that $x \in X$ and $y \in X$ holds $\mathcal{P}[x, y]$ or $\mathcal{P}[y, x]$ there exists y being an element of $\mathcal{A}$ such that for every element x of $\mathcal{A}$ such that $x \in X$ holds $\mathcal{P}[x, y]$.

The scheme *Zorn_Min* deals with a constant $\mathcal{A}$ that is a non-empty set and a binary predicate $\mathcal{P}$ and states that:

there exists x being an element of $\mathcal{A}$ such that for every element y of $\mathcal{A}$ such that $x \neq y$ holds not $\mathcal{P}[y, x]$

provided the parameters satisfy the following conditions:

- for every element x of $\mathcal{A}$ holds $\mathcal{P}[x, x]$,
- for all elements x, y of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, x]$ holds $x = y$,
- for all elements x, y, z of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$,
- for every X such that $X \subseteq \mathcal{A}$ and for all elements x, y of $\mathcal{A}$ such that $x \in X$ and $y \in X$ holds $\mathcal{P}[x, y]$ or $\mathcal{P}[y, x]$ there exists y being an element of $\mathcal{A}$ such that for every element x of $\mathcal{A}$ such that $x \in X$ holds $\mathcal{P}[y, x]$.

One can prove the following propositions:

(81) If R partially orders X and $\operatorname{field} R = X$, then there exists P such that $R \subseteq P$ and P linearly orders X and $\operatorname{field} P = X$.

(82) $R \subseteq [\!:\operatorname{field} R, \operatorname{field} R\,:\!]$.

(83) If R is pseudo reflexive and $X \subseteq \operatorname{field} R$, then $\operatorname{field}(R \,|^2\, X) = X$.

(84) If R is reflexive in X, then $R \,|^2\, X$ is pseudo reflexive.

(85) If R is transitive in X, then $R \,|^2\, X$ is transitive.

(86) If R is antisymmetric in X, then $R \,|^2\, X$ is antisymmetric.

(87) If R is connected in X, then $R \,|^2\, X$ is connected.

(88) If R is connected in X and $Y \subseteq X$, then R is connected in Y.

(89) If R well orders X and $Y \subseteq X$, then R well orders Y.

(90) If R is connected, then $R^{\smile}$ is connected.

(91) If R is reflexive in X, then $R^{\smile}$ is reflexive in X.

(92) If R is transitive in X, then $R^{\smile}$ is transitive in X.

(93) If R is antisymmetric in X, then $R^{\smile}$ is antisymmetric in X.

(94) If R is connected in X, then $R^{\smile}$ is connected in X.

(95) $(R \,|^2\, X)^{\smile} = R^{\smile} \,|^2\, X$.

(96) $R \,|^2\, \emptyset = \varnothing$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[2] Grzegorz Bancerek. The well ordering relations. *Formalized Mathematics*, 1(**1**):123–129, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[5] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[6] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.

[7] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[8] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

[9] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Formalized Mathematics*, 1(**1**):85–89, 1990.

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Operations on Subspaces in Real Linear Space

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Summary. In this article the following operations on subspaces of real linear space are intoduced: sum, intersection and direct sum. Some theorems about those notions are proved. We define linear complement of a subspace. Some theorems about decomposition of a vector onto two subspaces and onto subspace and it's linear complement are proved. We also show that a set of subspaces with operations sum and intersection is a lattice. At the end of the article theorems that belong rather to [7], [6], [5] or [8] are proved.

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The notation and terminology used in this paper are introduced in the following papers: [1], [8], [4], [3], [6], [5], and [2]. For simplicity we adopt the following convention: V is a real linear space, W, W_1, W_2, W_3 are subspaces of V, u, u_1, u_2, v, v_1, v_2 are vectors of V, X, Y are sets, and x be arbitrary. Let us consider V, W_1, W_2. The functor $W_1 + W_2$ yielding a subspace of V, is defined by:

the vectors of $W_1 + W_2 = \{v + u : v \in W_1 \wedge u \in W_2\}$.

Let us consider V, W_1, W_2. The functor $W_1 \cap W_2$ yielding a subspace of V, is defined by:

the vectors of $W_1 \cap W_2 =$(the vectors of W_1)$\cap$(the vectors of W_2).

Next we state a number of propositions:

(1) the vectors of $W_1 + W_2 = \{v + u : v \in W_1 \wedge u \in W_2\}$.

(2) If the vectors of $W = \{v + u : v \in W_1 \wedge u \in W_2\}$, then $W = W_1 + W_2$.

(3) the vectors of $W_1 \cap W_2 =$(the vectors of W_1)$\cap$(the vectors of W_2).

(4) If the vectors of W =(the vectors of W_1)$\cap$(the vectors of W_2), then $W = W_1 \cap W_2$.

(5) $x \in W_1 + W_2$ if and only if there exist v_1, v_2 such that $v_1 \in W_1$ and $v_2 \in W_2$ and $x = v_1 + v_2$.

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(6) If $v \in W_1$ or $v \in W_2$, then $v \in W_1 + W_2$.

(7) $x \in W_1 \cap W_2$ if and only if $x \in W_1$ and $x \in W_2$.

(8) $W + W = W$.

(9) $W_1 + W_2 = W_2 + W_1$.

(10) $W_1 + (W_2 + W_3) = (W_1 + W_2) + W_3$.

(11) W_1 is a subspace of $W_1 + W_2$ and W_2 is a subspace of $W_1 + W_2$.

(12) W_1 is a subspace of W_2 if and only if $W_1 + W_2 = W_2$.

(13) $\mathbf{0}_V + W = W$ and $W + \mathbf{0}_V = W$.

(14) $\mathbf{0}_V + \Omega_V = V$ and $\Omega_V + \mathbf{0}_V = V$.

(15) $\Omega_V + W = V$ and $W + \Omega_V = V$.

(16) $\Omega_V + \Omega_V = V$.

(17) $W \cap W = W$.

(18) $W_1 \cap W_2 = W_2 \cap W_1$.

(19) $W_1 \cap (W_2 \cap W_3) = (W_1 \cap W_2) \cap W_3$.

(20) $W_1 \cap W_2$ is a subspace of W_1 and $W_1 \cap W_2$ is a subspace of W_2.

(21) W_1 is a subspace of W_2 if and only if $W_1 \cap W_2 = W_1$.

(22) $\mathbf{0}_V \cap W = \mathbf{0}_V$ and $W \cap \mathbf{0}_V = \mathbf{0}_V$.

(23) $\mathbf{0}_V \cap \Omega_V = \mathbf{0}_V$ and $\Omega_V \cap \mathbf{0}_V = \mathbf{0}_V$.

(24) $\Omega_V \cap W = W$ and $W \cap \Omega_V = W$.

(25) $\Omega_V \cap \Omega_V = V$.

(26) $W_1 \cap W_2$ is a subspace of $W_1 + W_2$.

(27) $W_1 \cap W_2 + W_2 = W_2$.

(28) $W_1 \cap (W_1 + W_2) = W_1$.

(29) $W_1 \cap W_2 + W_2 \cap W_3$ is a subspace of $W_2 \cap (W_1 + W_3)$.

(30) If W_1 is a subspace of W_2, then $W_2 \cap (W_1 + W_3) = W_1 \cap W_2 + W_2 \cap W_3$.

(31) $W_2 + W_1 \cap W_3$ is a subspace of $(W_1 + W_2) \cap (W_2 + W_3)$.

(32) If W_1 is a subspace of W_2, then $W_2 + W_1 \cap W_3 = (W_1 + W_2) \cap (W_2 + W_3)$.

(33) If W_1 is a subspace of W_3, then $W_1 + W_2 \cap W_3 = (W_1 + W_2) \cap W_3$.

(34) $W_1 + W_2 = W_2$ if and only if $W_1 \cap W_2 = W_1$.

(35) If W_1 is a subspace of W_2, then $W_1 + W_3$ is a subspace of $W_2 + W_3$.

(36) There exists W such that the vectors of W =(the vectors of W_1)$\cup$(the vectors of W_2) if and only if W_1 is a subspace of W_2 or W_2 is a subspace of W_1.

Let us consider V. The functor Subspaces V yielding a non-empty set, is defined by:

for every x holds $x \in$ Subspaces V if and only if x is a subspace of V.

In the sequel D will denote a non-empty set. We now state three propositions:

(37) If for every x holds $x \in D$ if and only if x is a subspace of V, then $D =$ Subspaces V.

(38) $x \in$ Subspaces V if and only if x is a subspace of V.

(39) $V \in \operatorname{Subspaces} V$.

Let us consider V, W_1, W_2. The predicate V is the direct sum of W_1 and W_2 is defined by:

$V = W_1 + W_2$ and $W_1 \cap W_2 = \mathbf{0}_V$.

Let us consider V, W. The mode linear complement of W, which widens to the type a subspace of V, is defined by:

V is the direct sum of it and W.

One can prove the following propositions:

(40) V is the direct sum of W_1 and W_2 if and only if $V = W_1 + W_2$ and $W_1 \cap W_2 = \mathbf{0}_V$.

(41) If V is the direct sum of W_1 and W_2, then W_1 is a linear complement of W_2.

(42) If V is the direct sum of W_1 and W_2, then W_2 is a linear complement of W_1.

In the sequel L denotes a linear complement of W. One can prove the following propositions:

(43) V is the direct sum of L and W and V is the direct sum of W and L.

(44) $W + L = V$ and $L + W = V$.

(45) $W \cap L = \mathbf{0}_V$ and $L \cap W = \mathbf{0}_V$.

(46) If V is the direct sum of W_1 and W_2, then V is the direct sum of W_2 and W_1.

(47) V is the direct sum of $\mathbf{0}_V$ and Ω_V and V is the direct sum of Ω_V and $\mathbf{0}_V$.

(48) W is a linear complement of L.

(49) $\mathbf{0}_V$ is a linear complement of Ω_V and Ω_V is a linear complement of $\mathbf{0}_V$.

In the sequel C is a coset of W, C_1 is a coset of W_1, and C_2 is a coset of W_2. We now state several propositions:

(50) If $C_1 \cap C_2 \neq \emptyset$, then $C_1 \cap C_2$ is a coset of $W_1 \cap W_2$.

(51) V is the direct sum of W_1 and W_2 if and only if for every C_1, C_2 there exists v such that $C_1 \cap C_2 = \{v\}$.

(52) $W_1 + W_2 = V$ if and only if for every v there exist v_1, v_2 such that $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$.

(53) If V is the direct sum of W_1 and W_2 and $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$, then $v_1 = u_1$ and $v_2 = u_2$.

(54) Suppose $V = W_1 + W_2$ and there exists v such that for all v_1, v_2, u_1, u_2 such that $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$ holds $v_1 = u_1$ and $v_2 = u_2$. Then V is the direct sum of W_1 and W_2.

In the sequel t will be an element of $[\!:$ the vectors of V, the vectors of $V :\!]$. Let us consider V, t. Then $t_{\mathbf{1}}$ is a vector of V. Then $t_{\mathbf{2}}$ is a vector of V.

Let us consider V, v, W_1, W_2. Let us assume that V is the direct sum of W_1 and W_2. The functor $v \lhd (W_1, W_2)$ yields an element of $[\!:$ the vectors of V, the vectors of $V :\!]$ and is defined by:

$v = (v \lhd (W_1, W_2))_{\mathbf{1}} + (v \lhd (W_1, W_2))_{\mathbf{2}}$ and $(v \lhd (W_1, W_2))_{\mathbf{1}} \in W_1$ and $(v \lhd (W_1, W_2))_{\mathbf{2}} \in W_2$.

We now state a number of propositions:

(55) If V is the direct sum of W_1 and W_2 and $t_{\mathbf{1}} + t_{\mathbf{2}} = v$ and $t_{\mathbf{1}} \in W_1$ and $t_{\mathbf{2}} \in W_2$, then $t = v \lhd (W_1, W_2)$.

(56) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{1}} + (v \lhd (W_1, W_2))_{\mathbf{2}} = v$.

(57) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{1}} \in W_1$.

(58) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{2}} \in W_2$.

(59) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{1}} = (v \lhd (W_2, W_1))_{\mathbf{2}}$.

(60) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{2}} = (v \lhd (W_2, W_1))_{\mathbf{1}}$.

(61) If $t_{\mathbf{1}} + t_{\mathbf{2}} = v$ and $t_{\mathbf{1}} \in W$ and $t_{\mathbf{2}} \in L$, then $t = v \lhd (W, L)$.

(62) $(v \lhd (W, L))_{\mathbf{1}} + (v \lhd (W, L))_{\mathbf{2}} = v$.

(63) $(v \lhd (W, L))_{\mathbf{1}} \in W$ and $(v \lhd (W, L))_{\mathbf{2}} \in L$.

(64) $(v \lhd (W, L))_{\mathbf{1}} = (v \lhd (L, W))_{\mathbf{2}}$.

(65) $(v \lhd (W, L))_{\mathbf{2}} = (v \lhd (L, W))_{\mathbf{1}}$.

In the sequel A_1, A_2 will be elements of Subspaces V. Let us consider V. The functor SubJoin V yields a binary operation on Subspaces V and is defined by:

for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubJoin}\, V)(A_1, A_2) = W_1 + W_2$.

Let us consider V. The functor SubMeet V yielding a binary operation on Subspaces V, is defined by:

for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubMeet}\, V)(A_1, A_2) = W_1 \cap W_2$.

In the sequel o will be a binary operation on Subspaces V. The following propositions are true:

(66) If $A_1 = W_1$ and $A_2 = W_2$, then SubJoin $V(A_1, A_2) = W_1 + W_2$.

(67) If for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $o(A_1, A_2) = W_1 + W_2$, then $o = \mathrm{SubJoin}\, V$.

(68) If $A_1 = W_1$ and $A_2 = W_2$, then SubMeet $V(A_1, A_2) = W_1 \cap W_2$.

(69) If for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $o(A_1, A_2) = W_1 \cap W_2$, then $o = \mathrm{SubMeet}\, V$.

(70) $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a lattice.

(71) $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a lower bound lattice.

(72) $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is an upper bound lattice.

(73) $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a bound lattice.

(74) $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a modular lattice.

For simplicity we adopt the following convention: l will be a bound lattice, l_0 will be a lower bound lattice, l_1 will be an upper bound lattice, a, b will be elements of the carrier of l, a_0, b_0 will be elements of the carrier of l_0, and a_1, b_1 will be elements of the carrier of l_1. One can prove the following propositions:

(75) $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a complemented lattice.

(76) If W_1 is a subspace of W_2, then $W_1 \cap W_3$ is a subspace of $W_2 \cap W_3$.

(77) If $X \subseteq Y$ and $X \neq Y$, then there exists x such that $x \in Y$ and $x \notin X$.

(78) $v = v_1 + v_2$ if and only if $v_1 = v - v_2$.

(79) If for every v holds $v \in W$, then $W = V$.

(80) There exists C such that $v \in C$.

(81) $x \in v + W$ if and only if there exists u such that $u \in W$ and $x = v + u$.

(82) l is a complemented lattice if and only if for every a there exists b such that b is a complement of a.

(83) a is a complement of b if and only if $a \sqcup b = \top_l$ and $a \sqcap b = \bot_l$.

(84) If for every a_0 holds $a_0 \sqcap b_0 = b_0$, then $b_0 = \bot_{l_0}$.

(85) If for every a_1 holds $a_1 \sqcup b_1 = b_1$, then $b_1 = \top_{l_1}$.

References

[1] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[2] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[3] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[4] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

[5] Wojciech A. Trybulec. Subspaces and cosets of subspaces in real linear space. *Formalized Mathematics*, 1(**2**):297–301, 1990.

[6] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[7] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[8] Stanisław Żukowski. Introduction to lattice theory. *Formalized Mathematics*, 1(**1**):215–222, 1990.

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σ-Fields and Probability

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Summary. This article contains definitions and theorems concerning basic properties of following objects: - a field of subsets of given nonempty set; - a sequence of subsets of given nonempty set; - a σ-field of subsets of given nonempty set and events from this σ-field; - a probability i.e. σ-additive normed measure defined on previously introduced σ-field; - a σ-field generated by family of subsets of given set; - family of Borel Sets.

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The articles [7], [1], [3], [2], [5], [4], [6], and [8] provide the notation and terminology for this paper. For simplicity we adopt the following rules: *Omega* will be a non-empty set, Y, Z, V will be sets, A, B, D will be subsets of *Omega*, f will be a function, m, n will be natural numbers, p, x, y, z will be arbitrary, r, r_1, r_2 will be real numbers, and *seq* will be a sequence of real numbers. We now state three propositions:

(1) For every x holds x is a subset of *Omega* if and only if $x \in 2^{Omega}$.

(2) For all r, r_1, r_2 such that $0 \leq r$ and $r_1 = r_2 - r$ holds $r_1 \leq r_2$.

(3) For all r, *seq* such that there exists n such that for every m such that $n \leq m$ holds $seq(m) = r$ holds *seq* is convergent and $\lim seq = r$.

Let us consider *Omega*. The mode field of subsets of *Omega*, which widens to the type a set, is defined by:

it $\subseteq 2^{Omega}$ and there exists A such that $A \in$ it but if $A \in$ it and $B \in$ it, then $A \cap B \in$ it but if $A \in$ it, then $A^{\rm c} \in$ it.

Next we state a proposition

(4) For all *Omega*, Y holds for all A, B holds $Y \subseteq 2^{Omega}$ and there exists A such that $A \in Y$ but if $A \in Y$ and $B \in Y$, then $A \cap B \in Y$ but if $A \in Y$, then $A^{\rm c} \in Y$ if and only if Y is a field of subsets of *Omega*.

In the sequel *Fld* will be a field of subsets of *Omega*. Next we state a number of propositions:

(5) $Fld \subseteq 2^{Omega}$.

(6) There exists A such that $A \in Fld$.

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(7) If $A \in Fld$ and $B \in Fld$, then $A \cap B \in Fld$.

(8) If $A \in Fld$, then $A^{\rm c} \in Fld$.

(9) If $A \in Fld$ and $B \in Fld$, then $A \cup B \in Fld$.

(10) $\emptyset \in Fld$.

(11) $Omega \in Fld$.

(12) If $A \in Fld$ and $B \in Fld$, then $A \setminus B \in Fld$.

(13) If $A \in Fld$ and $B \in Fld$, then $A \cup B = (A \setminus B) \cup B$ and $(A \setminus B) \cup B \in Fld$ and $A \setminus B$ misses B.

(14) For every $Omega$ holds $\{\emptyset, Omega\}$ is a field of subsets of $Omega$.

(15) For every $Omega$ holds 2^{Omega} is a field of subsets of $Omega$.

(16) $\{\emptyset, Omega\} \subseteq Fld$ and $Fld \subseteq 2^{Omega}$.

(17) For every x such that $x \in Fld$ holds x is a subset of $Omega$.

(18) For every $Omega$ holds for every p such that $p \in [\!:\mathbb{N}, \{Omega\}:\!]$ there exist x, y such that $\langle x, y \rangle = p$ and for all x, y, z such that $\langle x, y \rangle \in [\!:\mathbb{N}, \{Omega\}:\!]$ and $\langle x, z \rangle \in [\!:\mathbb{N}, \{Omega\}:\!]$ holds $y = z$.

(19) For every $Omega$ there exists f such that $\mathrm{dom}\, f = \mathbb{N}$ and for every n holds $f(n) = Omega$ and $f(n) \in 2^{Omega}$.

Let us consider $Omega$. The mode sequence of subsets of $Omega$, which widens to the type a function, is defined by:

$\mathrm{dom}\, \mathrm{it} = \mathbb{N}$ and for every n holds $\mathrm{it}(n) \in 2^{Omega}$.

One can prove the following proposition

(20) f is a sequence of subsets of $Omega$ if and only if $\mathrm{dom}\, f = \mathbb{N}$ and for every n holds $f(n) \in 2^{Omega}$.

In the sequel $ASeq$, $BSeq$ denote sequences of subsets of $Omega$. We now state two propositions:

(21) There exists $ASeq$ such that for every n holds $ASeq(n) = Omega$.

(22) For every A, B there exists $ASeq$ such that $ASeq(0) = A$ and for every n such that $n \neq 0$ holds $ASeq(n) = B$.

Let us consider $Omega$, $ASeq$, n. Then $ASeq(n)$ is a subset of $Omega$.

The following proposition is true

(23) For all $ASeq$, V such that $V = \bigcup(\mathrm{rng}\, ASeq)$ holds V is a subset of $Omega$.

Let us consider $Omega$, $ASeq$. The functor $\mathrm{Union}\, ASeq$ yields a set and is defined by:

$\mathrm{Union}\, ASeq = \bigcup(\mathrm{rng}\, ASeq)$.

We now state a proposition

(24) For all $ASeq$, V holds $V = \mathrm{Union}\, ASeq$ if and only if $V = \bigcup(\mathrm{rng}\, ASeq)$.

Let us consider $Omega$, $ASeq$. Then $\mathrm{Union}\, ASeq$ is a subset of $Omega$.

We now state two propositions:

(25) For all x, $ASeq$ holds $x \in \mathrm{Union}\, ASeq$ if and only if there exists n such that $x \in ASeq(n)$.

(26) For every $ASeq$ there exists $BSeq$ such that for every n holds $BSeq(n) = (ASeq(n))^{\mathrm{c}}$.

Let us consider $Omega$, $ASeq$. The functor Complement $ASeq$ yields a sequence of subsets of $Omega$ and is defined by:

for every n holds (Complement $ASeq)(n) = (ASeq(n))^{\mathrm{c}}$.

One can prove the following proposition

(27) For all $ASeq$, $BSeq$ holds $BSeq =$ Complement $ASeq$ if and only if for every n holds $BSeq(n) = (ASeq(n))^{\mathrm{c}}$.

Let us consider $Omega$, $ASeq$. The functor Intersection $ASeq$ yields a subset of $Omega$ and is defined by:

Intersection $ASeq =$ (Union(Complement $ASeq$))$^{\mathrm{c}}$.

One can prove the following propositions:

(28) For all $ASeq$, A holds $A =$ Intersection $ASeq$ if and only if $A =$ (Union(Complement $ASeq$))$^{\mathrm{c}}$.

(29) For all $ASeq$, x holds $x \in$ Intersection $ASeq$ if and only if for every n holds $x \in ASeq(n)$.

(30) For all A, B, $ASeq$ such that $ASeq(0) = A$ and for every n such that $n \neq 0$ holds $ASeq(n) = B$ holds Intersection $ASeq = A \cap B$.

(31) For every $ASeq$ holds Complement(Complement $ASeq$) $= ASeq$.

We now define two new predicates. Let us consider $Omega$, $ASeq$. The predicate $ASeq$ is nonincreasing is defined by:

for all n, m such that $n \leq m$ holds $ASeq(m) \subseteq ASeq(n)$.

The predicate $ASeq$ is nondecreasing is defined by:

for all n, m such that $n \leq m$ holds $ASeq(n) \subseteq ASeq(m)$.

The following two propositions are true:

(32) For all $Omega$, $ASeq$ holds $ASeq$ is nonincreasing if and only if for all n, m such that $n \leq m$ holds $ASeq(m) \subseteq ASeq(n)$.

(33) For all $Omega$, $ASeq$ holds $ASeq$ is nondecreasing if and only if for all n, m such that $n \leq m$ holds $ASeq(n) \subseteq ASeq(m)$.

Let us consider $Omega$. The mode σ-field of subsets of $Omega$, which widens to the type a set, is defined by:

it $\subseteq 2^{Omega}$ and there exists A such that $A \in$ it and for every $ASeq$ such that for every n holds $ASeq(n) \in$ it holds Intersection $ASeq \in$ it and for every A such that $A \in$ it holds $A^{\mathrm{c}} \in$ it.

We now state two propositions:

(34) For all $Omega$, Y holds Y is a σ-field of subsets of $Omega$ if and only if $Y \subseteq 2^{Omega}$ and there exists A such that $A \in Y$ and for every $ASeq$ such that for every n holds $ASeq(n) \in Y$ holds Intersection $ASeq \in Y$ and for every A such that $A \in Y$ holds $A^{\mathrm{c}} \in Y$.

(35) For all $Omega$, Y such that Y is a σ-field of subsets of $Omega$ holds Y is a field of subsets of $Omega$.

In the sequel $Sigma$ is a σ-field of subsets of $Omega$. Next we state several propositions:

(36) $Sigma \subseteq 2^{Omega}$.

(37) For every x such that $x \in Sigma$ holds x is a subset of $Omega$.

(38) There exists A such that $A \in Sigma$.

(39) For all A, B such that $A \in Sigma$ and $B \in Sigma$ holds $A \cap B \in Sigma$.

(40) For every A such that $A \in Sigma$ holds $A^{c} \in Sigma$.

(41) For all A, B such that $A \in Sigma$ and $B \in Sigma$ holds $A \cup B \in Sigma$.

(42) $\emptyset \in Sigma$.

(43) $Omega \in Sigma$.

(44) For all A, B such that $A \in Sigma$ and $B \in Sigma$ holds $A \setminus B \in Sigma$.

Let us consider $Omega$, $Sigma$. The mode sequence of subsets of $Sigma$, which widens to the type a sequence of subsets of $Omega$, is defined by:

for every n holds $\mathrm{it}(n) \in Sigma$.

We now state two propositions:

(45) $ASeq$ is a sequence of subsets of $Sigma$ if and only if for every n holds $ASeq(n) \in Sigma$.

(46) For all $Omega$, $Sigma$ for every sequence $ASeq$ of subsets of $Sigma$ holds $\mathrm{Union}\, ASeq \in Sigma$.

Let us consider $Omega$, $Sigma$. The mode event of $Sigma$, which widens to the type a subset of $Omega$, is defined by:

$\mathrm{it} \in Sigma$.

The following propositions are true:

(47) For all $Sigma$, A holds A is an event of $Sigma$ if and only if $A \in Sigma$.

(48) For all $Sigma$, x such that $x \in Sigma$ holds x is an event of $Sigma$.

(49) For all events A, B of $Sigma$ holds $A \cap B$ is an event of $Sigma$.

(50) For every event A of $Sigma$ holds A^{c} is an event of $Sigma$.

(51) For all events A, B of $Sigma$ holds $A \cup B$ is an event of $Sigma$.

(52) For all $Omega$, $Sigma$ holds $\emptyset$ is an event of $Sigma$.

(53) For all $Omega$, $Sigma$ holds $Omega$ is an event of $Sigma$.

(54) For all events A, B of $Sigma$ holds $A \setminus B$ is an event of $Sigma$.

We now define two new functors. Let us consider $Omega$, $Sigma$. The functor Ω_{Sigma} yields an event of $Sigma$ and is defined by:

$\Omega_{Sigma} = Omega$.

The functor $\emptyset_{Sigma}$ yielding an event of $Sigma$, is defined by:

$\emptyset_{Sigma} = \emptyset$.

Next we state two propositions:

(55) For all $Omega$, $Sigma$ holds $\Omega_{Sigma} = Omega$.

(56) For all $Omega$, $Sigma$ holds $\emptyset_{Sigma} = \emptyset$.

The arguments of the notions defined below are the following: $Omega$, $Sigma$ which are objects of the type reserved above; A, B which are events of $Sigma$. Then $A \cap B$ is an event of $Sigma$. Then $A \cup B$ is an event of $Sigma$. Then $A \setminus B$ is an event of $Sigma$.

We now state two propositions:

(57) For all *Omega*, *Sigma*, *ASeq* holds *ASeq* is a sequence of subsets of *Sigma* if and only if for every n holds $ASeq(n)$ is an event of *Sigma*.

(58) For all *Omega*, *Sigma*, *ASeq* such that *ASeq* is a sequence of subsets of *Sigma* holds Union *ASeq* is an event of *Sigma*.

In the sequel *Sigma* is a σ-field of subsets of *Omega*, A, B are events of *Sigma*, and *ASeq* is a sequence of subsets of *Sigma*. Next we state a proposition

(59) For every *Omega*, *Sigma*, p there exists f such that $\operatorname{dom} f = Sigma$ and for every D such that $D \in Sigma$ holds if $p \in D$, then $f(D) = 1$ but if $p \notin D$, then $f(D) = 0$.

In the sequel P is a function from *Sigma* into $\mathbb{R}$. The following three propositions are true:

(60) For every *Omega*, *Sigma*, p there exists P such that for every D such that $D \in Sigma$ holds if $p \in D$, then $P(D) = 1$ but if $p \notin D$, then $P(D) = 0$.

(61) For every P holds $\operatorname{dom} P = Sigma$ and $\operatorname{rng} P \subseteq \mathbb{R}$.

(62) For all *Sigma*, *ASeq*, P holds $P \cdot ASeq$ is a sequence of real numbers.

Let us consider *Omega*, *Sigma*, *ASeq*, P. Then $P \cdot ASeq$ is a sequence of real numbers.

Let us consider *Omega*, *Sigma*, P, A. Then $P(A)$ is a real number.

Let us consider *Omega*, *Sigma*. The mode probability on *Sigma*, which widens to the type a function from *Sigma* into $\mathbb{R}$, is defined by:

(i) for every A holds $0 \leq \mathrm{it}(A)$,
(ii) $\mathrm{it}(Omega) = 1$,
(iii) for all A, B such that A misses B holds $\mathrm{it}(A \cup B) = \mathrm{it}(A) + \mathrm{it}(B)$,
(iv) for every *ASeq* such that *ASeq* is nonincreasing holds $\mathrm{it} \cdot ASeq$ is convergent and $\lim(\mathrm{it} \cdot ASeq) = \mathrm{it}(\text{Intersection}\, ASeq)$.

Next we state a proposition

(63) Let P be a function from *Sigma* into $\mathbb{R}$. Then P is a probability on *Sigma* if and only if the following conditions are satisfied:
  (i) for every A holds $0 \leq P(A)$,
  (ii) $P(Omega) = 1$,
  (iii) for all A, B such that A misses B holds $P(A \cup B) = P(A) + P(B)$,
  (iv) for every *ASeq* such that *ASeq* is nonincreasing holds $P \cdot ASeq$ is convergent and $\lim(P \cdot ASeq) = P(\text{Intersection}\, ASeq)$.

In the sequel P will be a probability on *Sigma*. One can prove the following propositions:

(64) $P(\emptyset) = 0$.

(65) $P(\emptyset_{Sigma}) = 0$.

(66) $P(\Omega_{Sigma}) = 1$.

(67) For all P, A holds $P(\Omega_{Sigma} \setminus A) + P(A) = 1$.

(68) For all P, A holds $P(\Omega_{Sigma} \setminus A) = 1 - P(A)$.

(69) For all P, A, B such that $A \subseteq B$ holds $P(B \setminus A) = P(B) - P(A)$.

(70) For all P, A, B such that $A \subseteq B$ holds $P(A) \leq P(B)$.

(71) For all P, A holds $P(A) \leq 1$.

(72) For all P, A, B holds $P(A \cup B) = P(A) + P(B \setminus A)$.

(73) For all P, A, B holds $P(A \cup B) = P(A) + P(B \setminus A \cap B)$.

(74) For all P, A, B holds $P(A \cup B) = (P(A) + P(B)) - P(A \cap B)$.

(75) For all P, A, B holds $P(A \cup B) \leq P(A) + P(B)$.

In the sequel D denotes a subset of $\mathbb{R}$ and S denotes a subset of 2^{Omega}. Next we state a proposition

(76) 2^{Omega} is a σ-field of subsets of $Omega$.

The arguments of the notions defined below are the following: $Omega$ which is an object of the type reserved above; X which is a subset of 2^{Omega}. The functor σX yields a σ-field of subsets of $Omega$ and is defined by:

$X \subseteq \sigma X$ and for every Z such that $X \subseteq Z$ and Z is a σ-field of subsets of $Omega$ holds $\sigma X \subseteq Z$.

Next we state a proposition

(77) For all S, $Sigma$ holds $Sigma = \sigma S$ if and only if $S \subseteq Sigma$ and for every Z such that $S \subseteq Z$ and Z is a σ-field of subsets of $Omega$ holds $Sigma \subseteq Z$.

Let us consider r. The functor $\mathrm{HL}(r)$ yielding a subset of $\mathbb{R}$, is defined by:
$\mathrm{HL}(r) = \{r_1 : r_1 < r\}$.

Next we state a proposition

(78) For all r, D holds $D = \mathrm{HL}(r)$ if and only if $D = \{r_1 : r_1 < r\}$.

The constant Halflines is a subset of $2^{\mathbb{R}}$ and is defined by:
Halflines $= \{D : \bigwedge_r D = \mathrm{HL}(r)\}$.

The following proposition is true

(79) For every subset Z of $2^{\mathbb{R}}$ holds $Z =$ Halflines if and only if $Z = \{D : \bigwedge_r D = \mathrm{HL}(r)\}$.

The constant the Borel sets is a σ-field of subsets of $\mathbb{R}$ and is defined by:
the Borel sets $= \sigma$ Halflines.

One can prove the following proposition

(80) For every σ-field Z of subsets of $\mathbb{R}$ holds $Z =$ the Borel sets if and only if $Z = \sigma$ Halflines.

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[2] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[3] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[4] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.

[5] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[6] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[8] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

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Introduction to Categories and Functors

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Summary. The category is introduced as an ordered 5-tuple of the form $\langle O, M, dom, cod, \cdot, id\rangle$ where O (objects) and M (morphisms) are arbitrary nonempty sets, dom and cod map M onto O and assign to a morphism domain and codomain, $\cdot$ is a partial binary map from $M \times M$ to M (composition of morphisms), id applied to an object yields the identity morphism. We define the basic notions of the category theory such as hom, monic, epi, invertible. We next define functors, the composition of functors, faithfulness and fullness of functors, isomorphism between categories and the identity functor.

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The papers [5], [1], [3], [2], and [4] provide the terminology and notation for this paper. In the sequel a, b, c, o, m, x are arbitrary. Let us consider x. Then $\{x\}$ is a non-empty set.

Next we state several propositions:

(1) x is an element of $\{x\}$.

(2) For every element x of $\{a\}$ holds $x = a$.

(3) For every set X for all non-empty sets C, D for every function f from C into D for every element c of C such that $c \in X$ holds $(f \restriction X)(c) = f(c)$.

(4) For all sets X, Y, Z for every non-empty set D for every function f from X into D such that $Y \subseteq X$ and $f\,^\circ\, Y \subseteq Z$ holds $f \restriction Y$ is a function from Y into Z.

(5) For every function f from $\{a\}$ into $\{b\}$ for every element x of $\{a\}$ holds $f(x) = b$.

The arguments of the notions defined below are the following: A which is a non-empty set; b which is an object of the type reserved above. of the type reserved above. Then $A \longmapsto b$ is a function from A into $\{b\}$.

Let us consider a, b, c. The functor $\langle a, b\rangle \longmapsto c$ yields a partial function from $[\!:\{a\}, \{b\}:\!]$ to $\{c\}$ and is defined by:

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$\langle a, b\rangle \longmapsto c = \{\langle a, b\rangle\} \longmapsto c$.

One can prove the following propositions:

(6) $\langle a, b\rangle \longmapsto c = \{\langle a, b\rangle\} \longmapsto c$.

(7) $\mathrm{dom}(\langle a, b\rangle \longmapsto c) = \{\langle a, b\rangle\}$ and $\mathrm{dom}(\langle a, b\rangle \longmapsto c) = [\!:\{a\}, \{b\}:\!]$.

(8) $(\langle a, b\rangle \longmapsto c)(\langle a, b\rangle) = c$.

(9) For every element x of $\{a\}$ for every element y of $\{b\}$ holds $(\langle a, b\rangle \longmapsto c)(\langle x, y\rangle) = c$.

Let D be a non-empty set. Then id_D is a function from D into D.

We consider category structures which are systems

$\langle$ objects, morphisms, a dom-map, a cod-map, a composition, an id-map $\rangle$

where the objects, the morphisms are non-empty sets, the dom-map, the cod-map are functions from the morphisms into the objects, the composition is a partial function from $[\!:$the morphisms, the morphisms$:\!]$ to the morphisms, and the id-map is a function from the objects into the morphisms. In the sequel C denotes a category structure. We now define two new modes. Let us consider C. An object of C is an element of the objects of C.

A morphism of C is an element of the morphisms of C.

We now state two propositions:

(10) For every element a of the objects of C holds a is an object of C.

(11) For every element f of the morphisms of C holds f is a morphism of C.

We adopt the following convention: a, b, c, d are objects of C and f, g are morphisms of C. We now define two new functors. Let us consider C, f. The functor $\mathrm{dom}\, f$ yields an object of C and is defined by:

$\mathrm{dom}\, f =$(the dom-map of C)(f).

The functor $\mathrm{cod}\, f$ yielding an object of C, is defined by:

$\mathrm{cod}\, f =$(the cod-map of C)(f).

We now state two propositions:

(12) $\mathrm{dom}\, f =$(the dom-map of C)(f).

(13) $\mathrm{cod}\, f =$(the cod-map of C)(f).

Let us consider C, f, g. Let us assume that $\langle g, f\rangle \in$ dom(the composition of C). The functor $g \cdot f$ yielding a morphism of C, is defined by:

$g \cdot f =$(the composition of C)($\langle g, f\rangle$).

Next we state a proposition

(14) If $\langle g, f\rangle \in$ dom(the composition of C), then $g \cdot f =$(the composition of C)($\langle g, f\rangle$).

Let us consider C, a. The functor id_a yields a morphism of C and is defined by:

$\mathrm{id}_a =$(the id-map of C)(a).

One can prove the following proposition

(15) $\mathrm{id}_a =$(the id-map of C)(a).

Let us consider C, a, b. The functor $\mathrm{hom}(a, b)$ yielding sets of morphisms of C, is defined by:

$\hom(a, b) = \{f : \operatorname{dom} f = a \wedge \operatorname{cod} f = b\}$.

We now state four propositions:

(16) $\hom(a, b) = \{f : \operatorname{dom} f = a \wedge \operatorname{cod} f = b\}$.

(17) If $\hom(a, b) \neq \emptyset$, then there exists f such that $f \in \hom(a, b)$.

(18) $f \in \hom(a, b)$ if and only if $\operatorname{dom} f = a$ and $\operatorname{cod} f = b$.

(19) $\hom(\operatorname{dom} f, \operatorname{cod} f) \neq \emptyset$.

Let us consider C, a, b. Let us assume that $\hom(a, b) \neq \emptyset$. The mode morphism from a to b, which widens to the type a morphism of C, is defined by:

it $\in \hom(a, b)$.

Next we state several propositions:

(20) If $\hom(a, b) \neq \emptyset$, then for every morphism f of C holds f is a morphism from a to b if and only if $f \in \hom(a, b)$.

(21) For arbitrary f such that $f \in \hom(a, b)$ holds f is a morphism from a to b.

(22) For every morphism f of C holds f is a morphism from $\operatorname{dom} f$ to $\operatorname{cod} f$.

(23) For every morphism f from a to b such that $\hom(a, b) \neq \emptyset$ holds $\operatorname{dom} f = a$ and $\operatorname{cod} f = b$.

(24) For every morphism f from a to b for every morphism h from c to d such that $\hom(a, b) \neq \emptyset$ and $\hom(c, d) \neq \emptyset$ and $f = h$ holds $a = c$ and $b = d$.

(25) For every morphism f from a to b such that $\hom(a, b) = \{f\}$ for every morphism g from a to b holds $f = g$.

(26) For every morphism f from a to b such that $\hom(a, b) \neq \emptyset$ and for every morphism g from a to b holds $f = g$ holds $\hom(a, b) = \{f\}$.

(27) For every morphism f from a to b such that $\hom(a, b) \approx \hom(c, d)$ and $\hom(a, b) = \{f\}$ there exists h being a morphism from c to d such that $\hom(c, d) = \{h\}$.

The mode category, which widens to the type a category structure, is defined by:

(i) for all elements f, g of the morphisms of it holds $\langle g, f\rangle \in$ dom(the composition of it) if and only if (the dom-map of it)(g) =(the cod-map of it)(f),

(ii) for all elements f, g of the morphisms of it such that (the dom-map of it)(g) =(the cod-map of it)(f) holds (the dom-map of it)((the composition of it)($\langle g, f\rangle$)) =(the dom-map of it)(f) and (the cod-map of it)((the composition of it)($\langle g, f\rangle$)) =(the cod-map of it)(g),

(iii) for all elements f, g, h of the morphisms of it such that (the dom-map of it)(h) =(the cod-map of it)(g) and (the dom-map of it)(g) =(the cod-map of it)(f) holds (the composition of it)($\langle h$,(the composition of it)($\langle g, f\rangle$)$\rangle$) =(the composition of it)($\langle$(the composition of it)($\langle h, g\rangle$)$, f\rangle$),

(iv) for every element b of the objects of it holds (the dom-map of it)((the id-map of it)(b)) $= b$ and (the cod-map of it)((the id-map of it)(b)) $= b$ and for every element f of the morphisms of it such that (the cod-map of it)(f) $= b$ holds (the composition of it)($\langle$(the id-map of it)(b)$, f\rangle$) $= f$ and for every element g of the morphisms of it such that (the dom-map of it)$(g) = b$ holds (the composition of it)$(\langle g,$(the id-map of it)$(b)\rangle) = g$.

The following three propositions are true:

(28) Let C be a category structure. Then C is a category if and only if the following conditions are satisfied:

(i) for all elements f, g of the morphisms of C holds $\langle g, f\rangle \in$ dom(the composition of C) if and only if (the dom-map of C)$(g) =$(the cod-map of C)(f),

(ii) for all elements f, g of the morphisms of C such that (the dom-map of C)$(g) =$(the cod-map of C)(f) holds (the dom-map of C)((the composition of C)$(\langle g, f\rangle)) =$(the dom-map of C)(f) and (the cod-map of C)((the composition of C)$(\langle g, f\rangle)) =$(the cod-map of C)(g),

(iii) for all elements f, g, h of the morphisms of C such that (the dom-map of C)$(h) =$(the cod-map of C)(g) and (the dom-map of C)$(g) =$(the cod-map of C)(f) holds (the composition of C)$(\langle h,$(the composition of C)$(\langle g, f\rangle)\rangle) =$(the composition of C)$(\langle$(the composition of C)$(\langle h, g\rangle), f\rangle)$,

(iv) for every element b of the objects of C holds (the dom-map of C)((the id-map of C)$(b)) = b$ and (the cod-map of C)((the id-map of C)$(b)) = b$ and for every element f of the morphisms of C such that (the cod-map of C)$(f) = b$ holds (the composition of C)$(\langle$(the id-map of C)$(b), f\rangle) = f$ and for every element g of the morphisms of C such that (the dom-map of C)$(g) = b$ holds (the composition of C)$(\langle g,$(the id-map of C)$(b)\rangle) = g$.

(29) Let C be a category structure. Suppose that

(i) for all morphisms f, g of C holds $\langle g, f\rangle \in$ dom(the composition of C) if and only if $\operatorname{dom} g = \operatorname{cod} f$,

(ii) for all morphisms f, g of C such that $\operatorname{dom} g = \operatorname{cod} f$ holds $\operatorname{dom}(g \cdot f) = \operatorname{dom} f$ and $\operatorname{cod}(g \cdot f) = \operatorname{cod} g$,

(iii) for all morphisms f, g, h of C such that $\operatorname{dom} h = \operatorname{cod} g$ and $\operatorname{dom} g = \operatorname{cod} f$ holds $h \cdot (g \cdot f) = (h \cdot g) \cdot f$,

(iv) for every object b of C holds $\operatorname{dom}(\mathrm{id}_b) = b$ and $\operatorname{cod}(\mathrm{id}_b) = b$ and for every morphism f of C such that $\operatorname{cod} f = b$ holds $\mathrm{id}_b \cdot f = f$ and for every morphism g of C such that $\operatorname{dom} g = b$ holds $g \cdot \mathrm{id}_b = g$.

Then C is a category.

(30) Let O, M be non-empty sets. Let d, c be functions from M into O. Let p be a partial function from $[\!:M, M:\!]$ to M. Let i be a function from O into M. Let C be a category structure. Suppose C. Then C is a category if and only if the following conditions are satisfied:

(i) for all elements f, g of M holds $\langle g, f\rangle \in \operatorname{dom} p$ if and only if $d(g) = c(f)$,

(ii) for all elements f, g of M such that $d(g) = c(f)$ holds $d(p(\langle g, f\rangle)) = d(f)$ and $c(p(\langle g, f\rangle)) = c(g)$,

(iii) for all elements f, g, h of M such that $d(h) = c(g)$ and $d(g) = c(f)$ holds $p(\langle h, p(\langle g, f\rangle)\rangle) = p(\langle p(\langle h, g\rangle), f\rangle)$,

(iv) for every element b of O holds $d(i(b)) = b$ and $c(i(b)) = b$ and for every element f of M such that $c(f) = b$ holds $p(\langle i(b), f\rangle) = f$ and for every element g of M such that $d(g) = b$ holds $p(\langle g, i(b)\rangle) = g$.

Let us consider o, m. The functor $\dot{\circlearrowleft}(o,m)$ yielding a category, is defined by:
$\dot{\circlearrowleft}(o,m) = \langle\{o\},\{m\},\{m\} \longmapsto o,\{m\} \longmapsto o,\langle m,m\rangle \longmapsto m,\{o\} \longmapsto m\rangle$.

One can prove the following propositions:

(31) $\dot{\circlearrowleft}(o,m) = \langle\{o\},\{m\},\{m\} \longmapsto o,\{m\} \longmapsto o,\langle m,m\rangle \longmapsto m,\{o\} \longmapsto m\rangle$.

(32) o is an object of $\dot{\circlearrowleft}(o,m)$.

(33) m is a morphism of $\dot{\circlearrowleft}(o,m)$.

(34) For every object a of $\dot{\circlearrowleft}(o,m)$ holds $a = o$.

(35) For every morphism f of $\dot{\circlearrowleft}(o,m)$ holds $f = m$.

(36) For all objects a, b of $\dot{\circlearrowleft}(o,m)$ for every morphism f of $\dot{\circlearrowleft}(o,m)$ holds $f \in \hom(a,b)$.

(37) For all objects a, b of $\dot{\circlearrowleft}(o,m)$ for every morphism f of $\dot{\circlearrowleft}(o,m)$ holds f is a morphism from a to b.

(38) For all objects a, b of $\dot{\circlearrowleft}(o,m)$ holds $\hom(a,b) \neq \emptyset$.

(39) For all objects a, b, c, d of $\dot{\circlearrowleft}(o,m)$ for every morphism f from a to b for every morphism g from c to d holds $f = g$.

We adopt the following rules: B, C, D will be categories, a, b, c, d will be objects of C, and f, f_1, f_2, g, g_1, g_2 will be morphisms of C. Next we state several propositions:

(40) $\mathrm{dom}\, g = \mathrm{cod}\, f$ if and only if $\langle g, f\rangle \in \mathrm{dom}(\text{the composition of } C)$.

(41) If $\mathrm{dom}\, g = \mathrm{cod}\, f$, then $g \cdot f =(\text{the composition of } C)(\langle g, f\rangle)$.

(42) For all morphisms f, g of C such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $\mathrm{dom}(g \cdot f) = \mathrm{dom}\, f$ and $\mathrm{cod}(g \cdot f) = \mathrm{cod}\, g$.

(43) For all morphisms f, g, h of C such that $\mathrm{dom}\, h = \mathrm{cod}\, g$ and $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $h \cdot (g \cdot f) = (h \cdot g) \cdot f$.

(44) $\mathrm{dom}(\mathrm{id}_b) = b$ and $\mathrm{cod}(\mathrm{id}_b) = b$.

(45) If $\mathrm{id}_a = \mathrm{id}_b$, then $a = b$.

(46) For every morphism f of C such that $\mathrm{cod}\, f = b$ holds $\mathrm{id}_b \cdot f = f$.

(47) For every morphism g of C such that $\mathrm{dom}\, g = b$ holds $g \cdot \mathrm{id}_b = g$.

Let us consider C, g. The predicate g is monic is defined by:
for all f_1, f_2 such that $\mathrm{dom}\, f_1 = \mathrm{dom}\, f_2$ and $\mathrm{cod}\, f_1 = \mathrm{dom}\, g$ and $\mathrm{cod}\, f_2 = \mathrm{dom}\, g$ and $g \cdot f_1 = g \cdot f_2$ holds $f_1 = f_2$.

The following proposition is true

(48) g is monic if and only if for all f_1, f_2 such that $\mathrm{dom}\, f_1 = \mathrm{dom}\, f_2$ and $\mathrm{cod}\, f_1 = \mathrm{dom}\, g$ and $\mathrm{cod}\, f_2 = \mathrm{dom}\, g$ and $g \cdot f_1 = g \cdot f_2$ holds $f_1 = f_2$.

Let us consider C, f. The predicate f is epi is defined by:
for all g_1, g_2 such that $\mathrm{dom}\, g_1 = \mathrm{cod}\, f$ and $\mathrm{dom}\, g_2 = \mathrm{cod}\, f$ and $\mathrm{cod}\, g_1 = \mathrm{cod}\, g_2$ and $g_1 \cdot f = g_2 \cdot f$ holds $g_1 = g_2$.

One can prove the following proposition

(49) f is epi if and only if for all g_1, g_2 such that $\mathrm{dom}\, g_1 = \mathrm{cod}\, f$ and $\mathrm{dom}\, g_2 = \mathrm{cod}\, f$ and $\mathrm{cod}\, g_1 = \mathrm{cod}\, g_2$ and $g_1 \cdot f = g_2 \cdot f$ holds $g_1 = g_2$.

Let us consider C, f. The predicate f is invertible is defined by:

there exists g such that $\operatorname{dom} g = \operatorname{cod} f$ and $\operatorname{cod} g = \operatorname{dom} f$ and $f \cdot g = \mathrm{id}_{\operatorname{cod} f}$ and $g \cdot f = \mathrm{id}_{\operatorname{dom} f}$.

The following proposition is true

(50) f is invertible if and only if there exists g such that $\operatorname{dom} g = \operatorname{cod} f$ and $\operatorname{cod} g = \operatorname{dom} f$ and $f \cdot g = \mathrm{id}_{\operatorname{cod} f}$ and $g \cdot f = \mathrm{id}_{\operatorname{dom} f}$.

In the sequel f will denote a morphism from a to b, f' will denote a morphism from b to a, g will denote a morphism from b to c, and h will denote a morphism from c to d. Next we state two propositions:

(51) If $\hom(a, b) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$, then $g \cdot f \in \hom(a, c)$.

(52) If $\hom(a, b) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$, then $\hom(a, c) \neq \emptyset$.

Let us consider C, a, b, c, f, g. Let us assume that $\hom(a, b) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$. The functor $g \cdot f$ yields a morphism from a to c and is defined by:

$g \cdot f = g \cdot f$.

One can prove the following propositions:

(53) If $\hom(a, b) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$, then $g \cdot f = g \cdot (f\, \textbf{qua}\, \text{a morphism of } C)$.

(54) If $\hom(a, b) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$ and $\hom(c, d) \neq \emptyset$, then $(h \cdot g) \cdot f = h \cdot (g \cdot f)$.

(55) $\mathrm{id}_a \in \hom(a, a)$.

(56) $\hom(a, a) \neq \emptyset$.

Let us consider C, a. Then id_a is a morphism from a to a.

The following propositions are true:

(57) If $\hom(a, b) \neq \emptyset$, then $\mathrm{id}_b \cdot f = f$.

(58) If $\hom(b, c) \neq \emptyset$, then $g \cdot \mathrm{id}_b = g$.

(59) $\mathrm{id}_a \cdot \mathrm{id}_a = \mathrm{id}_a$.

(60) If $\hom(b, c) \neq \emptyset$, then g is monic if and only if for every a for all morphisms f_1, f_2 from a to b such that $\hom(a, b) \neq \emptyset$ and $g \cdot f_1 = g \cdot f_2$ holds $f_1 = f_2$.

(61) If $\hom(b, c) \neq \emptyset$ and $\hom(c, d) \neq \emptyset$ and g is monic and h is monic, then $h \cdot g$ is monic.

(62) If $\hom(b, c) \neq \emptyset$ and $\hom(c, d) \neq \emptyset$ and $h \cdot g$ is monic, then g is monic.

(63) For every morphism h from a to b for every morphism g from b to a such that $\hom(a, b) \neq \emptyset$ and $\hom(b, a) \neq \emptyset$ and $h \cdot g = \mathrm{id}_b$ holds g is monic.

(64) id_b is monic.

(65) If $\hom(a, b) \neq \emptyset$, then f is epi if and only if for every c for all morphisms g_1, g_2 from b to c such that $\hom(b, c) \neq \emptyset$ and $g_1 \cdot f = g_2 \cdot f$ holds $g_1 = g_2$.

(66) If $\hom(a, b) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$ and f is epi and g is epi, then $g \cdot f$ is epi.

(67) If $\hom(a, b) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$ and $g \cdot f$ is epi, then g is epi.

(68) For every morphism h from a to b for every morphism g from b to a such that $\hom(a, b) \neq \emptyset$ and $\hom(b, a) \neq \emptyset$ and $h \cdot g = \mathrm{id}_b$ holds h is epi.

(69) id_b is epi.

(70) If $\hom(a,b) \neq \emptyset$, then f is invertible if and only if $\hom(b,a) \neq \emptyset$ and there exists g being a morphism from b to a such that $f \cdot g = \mathrm{id}_b$ and $g \cdot f = \mathrm{id}_a$.

(71) If $\hom(a,b) \neq \emptyset$ and $\hom(b,a) \neq \emptyset$, then for all morphisms g_1, g_2 from b to a such that $f \cdot g_1 = \mathrm{id}_b$ and $g_2 \cdot f = \mathrm{id}_a$ holds $g_1 = g_2$.

Let us consider C, a, b, f. Let us assume that $\hom(a,b) \neq \emptyset$ and f is invertible. The functor f^{-1} yielding a morphism from b to a, is defined by:

$f \cdot (f^{-1}) = \mathrm{id}_b$ and $(f^{-1}) \cdot f = \mathrm{id}_a$.

We now state several propositions:

(72) If $\hom(a,b) \neq \emptyset$ and f is invertible, then for every morphism g from b to a holds $g = f^{-1}$ if and only if $f \cdot g = \mathrm{id}_b$ and $g \cdot f = \mathrm{id}_a$.

(73) If $\hom(a,b) \neq \emptyset$ and f is invertible, then f is monic and f is epi.

(74) id_a is invertible.

(75) If $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$ and f is invertible and g is invertible, then $g \cdot f$ is invertible.

(76) If $\hom(a,b) \neq \emptyset$ and f is invertible, then f^{-1} is invertible.

(77) If $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$ and f is invertible and g is invertible, then $(g \cdot f)^{-1} = f^{-1} \cdot g^{-1}$.

We now define three new predicates. Let us consider C, a. The predicate a is a terminal object is defined by:

$\hom(b,a) \neq \emptyset$ and there exists f being a morphism from b to a such that for every morphism g from b to a holds $f = g$.

The predicate a is an initial object is defined by:

$\hom(a,b) \neq \emptyset$ and there exists f being a morphism from a to b such that for every morphism g from a to b holds $f = g$.

Let us consider b. The predicate a and b are isomorphic is defined by:

$\hom(a,b) \neq \emptyset$ and there exists f such that f is invertible.

We now state a number of propositions:

(78) a is a terminal object if and only if for every b holds $\hom(b,a) \neq \emptyset$ and there exists f being a morphism from b to a such that for every morphism g from b to a holds $f = g$.

(79) a is an initial object if and only if for every b holds $\hom(a,b) \neq \emptyset$ and there exists f being a morphism from a to b such that for every morphism g from a to b holds $f = g$.

(80) a and b are isomorphic if and only if $\hom(a,b) \neq \emptyset$ and there exists f such that f is invertible.

(81) a and b are isomorphic if and only if $\hom(a,b) \neq \emptyset$ and $\hom(b,a) \neq \emptyset$ and there exist f, f' such that $f \cdot f' = \mathrm{id}_b$ and $f' \cdot f = \mathrm{id}_a$.

(82) a is an initial object if and only if for every b there exists f being a morphism from a to b such that $\hom(a,b) = \{f\}$.

(83) If a is an initial object, then for every morphism h from a to a holds $\mathrm{id}_a = h$.

(84) If a is an initial object and b is an initial object, then a and b are isomorphic.

(85) If a is an initial object and a and b are isomorphic, then b is an initial object.

(86) b is a terminal object if and only if for every a there exists f being a morphism from a to b such that $\hom(a, b) = \{f\}$.

(87) If a is a terminal object, then for every morphism h from a to a holds $\mathrm{id}_a = h$.

(88) If a is a terminal object and b is a terminal object, then a and b are isomorphic.

(89) If b is a terminal object and a and b are isomorphic, then a is a terminal object.

(90) If $\hom(a, b) \neq \emptyset$ and a is a terminal object, then f is monic.

(91) a and a are isomorphic.

(92) If a and b are isomorphic, then b and a are isomorphic.

(93) If a and b are isomorphic and b and c are isomorphic, then a and c are isomorphic.

Let us consider C, D. The mode functor from C to D, which widens to the type a function from the morphisms of C into the morphisms of D, is defined by:

(i) for every element c of the objects of C there exists d being an element of the objects of D such that it((the id-map of C)(c)) =(the id-map of D)(d),

(ii) for every element f of the morphisms of C holds it((the id-map of C)((the dom-map of C)(f))) =(the id-map of D)((the dom-map of D)(it(f))) and it((the id-map of C)((the cod-map of C)(f))) =(the id-map of D)((the cod-map of D)(it(f))),

(iii) for all elements f, g of the morphisms of C such that $\langle g, f\rangle \in$ dom(the composition of C) holds it((the composition of C)($\langle g, f\rangle$)) =(the composition of D)($\langle \mathrm{it}(g), \mathrm{it}(f)\rangle$).

We now state two propositions:

(94) Let C, D be categories. Let T be a function from the morphisms of C into the morphisms of D. Then T is a functor from C to D if and only if the following conditions are satisfied:

  (i) for every element c of the objects of C there exists d being an element of the objects of D such that T((the id-map of C)(c)) =(the id-map of D)(d),

  (ii) for every element f of the morphisms of C holds T((the id-map of C)(
(the dom-map of C)(f))) =(the id-map of D)((the dom-map of D)($T(f)$)) and T((the id-map of C)((the cod-map of C)(f))) =(the id-map of D)((the cod-map of D)($T(f)$)),

  (iii) for all elements f, g of the morphisms of C such that $\langle g, f\rangle \in$ dom(the composition of C) holds T((the composition of C)($\langle g, f\rangle$)) =(the composition of D)($\langle T(g), T(f)\rangle$).

(95) For all functors F_1, F_2 from C to D such that for every morphism f of C holds $F_1(f) = F_2(f)$ holds $F_1 = F_2$.

The arguments of the notions defined below are the following: C, D which are categories; F which is a function from the objects of C into the objects of D; c which is an object of C. Then $F(c)$ is an object of D.

The following propositions are true:

(96) Let T be a function from the morphisms of C into the morphisms of D. Suppose that

(i) for every object c of C there exists d being an object of D such that $T(\mathrm{id}_c) = \mathrm{id}_d$,

(ii) for every morphism f of C holds $T(\mathrm{id}_{\mathrm{dom}\, f}) = \mathrm{id}_{\mathrm{dom}(T(f))}$ and $T(\mathrm{id}_{\mathrm{cod}\, f}) = \mathrm{id}_{\mathrm{cod}(T(f))}$,

(iii) for all morphisms f, g of C such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $T(g \cdot f) = T(g) \cdot T(f)$.

Then T is a functor from C to D.

(97) For every functor T from C to D for every object c of C there exists d being an object of D such that $T(\mathrm{id}_c) = \mathrm{id}_d$.

(98) For every functor T from C to D for every morphism f of C holds $T(\mathrm{id}_{\mathrm{dom}\, f}) = \mathrm{id}_{\mathrm{dom}(T(f))}$ and $T(\mathrm{id}_{\mathrm{cod}\, f}) = \mathrm{id}_{\mathrm{cod}(T(f))}$.

(99) For every functor T from C to D for all morphisms f, g of C such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $\mathrm{dom}(T(g)) = \mathrm{cod}(T(f))$ and $T(g \cdot f) = T(g) \cdot T(f)$.

(100) Let T be a function from the morphisms of C into the morphisms of D. Let F be a function from the objects of C into the objects of D. Suppose that

(i) for every object c of C holds $T(\mathrm{id}_c) = \mathrm{id}_{F(c)}$,

(ii) for every morphism f of C holds $F(\mathrm{dom}\, f) = \mathrm{dom}(T(f))$ and $F(\mathrm{cod}\, f) = \mathrm{cod}(T(f))$,

(iii) for all morphisms f, g of C such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $T(g \cdot f) = T(g) \cdot T(f)$.

Then T is a functor from C to D.

The arguments of the notions defined below are the following: C, D which are objects of the type reserved above; F which is a function from the morphisms of C into the morphisms of D. Let us assume that for every element c of the objects of C there exists d being an element of the objects of D such that $F(($the id-map of $C)(c)) =($the id-map of $D)(d)$. The functor $\mathrm{Obj}\, F$ yielding a function from the objects of C into the objects of D, is defined by:

for every element c of the objects of C for every element d of the objects of D such that $F(($the id-map of $C)(c)) =($the id-map of $D)(d)$ holds $(\mathrm{Obj}\, F)(c) = d$.

Next we state several propositions:

(101) Let C, D be categories. Let T be a function from the morphisms of C into the morphisms of D. Suppose for every element c of the objects of C there exists d being an element of the objects of D such that $T(($the id-map of $C)(c)) =($the id-map of $D)(d)$. Then for every function F from the objects of C into the objects of D holds $F = \mathrm{Obj}\, T$ if and only if for

every element c of the objects of C for every element d of the objects of D such that T((the id-map of C)(c)) =(the id-map of D)(d) holds $F(c) = d$.

(102) For every function T from the morphisms of C into the morphisms of D such that for every object c of C there exists d being an object of D such that $T(\mathrm{id}_c) = \mathrm{id}_d$ for every object c of C for every object d of D such that $T(\mathrm{id}_c) = \mathrm{id}_d$ holds $(\mathrm{Obj}\, T)(c) = d$.

(103) For every functor T from C to D for every object c of C for every object d of D such that $T(\mathrm{id}_c) = \mathrm{id}_d$ holds $(\mathrm{Obj}\, T)(c) = d$.

(104) For every functor T from C to D for every object c of C holds $T(\mathrm{id}_c) = \mathrm{id}_{(\mathrm{Obj}\, T)(c)}$.

(105) For every functor T from C to D for every morphism f of C holds $(\mathrm{Obj}\, T)(\mathrm{dom}\, f) = \mathrm{dom}(T(f))$ and $(\mathrm{Obj}\, T)(\mathrm{cod}\, f) = \mathrm{cod}(T(f))$.

The arguments of the notions defined below are the following: C, D which are categories; T which is a functor from C to D; c which is an object of C. The functor $T(c)$ yielding an object of D, is defined by:

$T(c) = (\mathrm{Obj}\, T)(c)$.

We now state several propositions:

(106) For every functor T from C to D for every object c of C holds $T(c) = (\mathrm{Obj}\, T)(c)$.

(107) For every functor T from C to D for every object c of C for every object d of D such that $T(\mathrm{id}_c) = \mathrm{id}_d$ holds $T(c) = d$.

(108) For every functor T from C to D for every object c of C holds $T(\mathrm{id}_c) = \mathrm{id}_{T(c)}$.

(109) For every functor T from C to D for every morphism f of C holds $T(\mathrm{dom}\, f) = \mathrm{dom}(T(f))$ and $T(\mathrm{cod}\, f) = \mathrm{cod}(T(f))$.

(110) For every functor T from B to C for every functor S from C to D holds $S \cdot T$ is a functor from B to D.

The arguments of the notions defined below are the following: B, C, D which are objects of the type reserved above; T which is a functor from B to C; S which is a functor from C to D. Then $S \cdot T$ is a functor from B to D.

One can prove the following three propositions:

(111) $\mathrm{id}_{\text{the morphisms of } C}$ is a functor from C to C.

(112) For every functor T from B to C for every functor S from C to D for every object b of B holds $(\mathrm{Obj}(S \cdot T))(b) = (\mathrm{Obj}\, S)((\mathrm{Obj}\, T)(b))$.

(113) For every functor T from B to C for every functor S from C to D for every object b of B holds $(S \cdot T)(b) = S(T(b))$.

Let us consider C. The functor id_C yielding a functor from C to C, is defined by:

$\mathrm{id}_C = \mathrm{id}_{\text{the morphisms of } C}$.

The following propositions are true:

(114) $\mathrm{id}_C = \mathrm{id}_{\text{the morphisms of } C}$.

(115) For every morphism f of C holds $\mathrm{id}_C(f) = f$.

(116) For every object c of C holds $(\mathrm{Obj}\,\mathrm{id}_C)(c) = c$.

(117) $\mathrm{Obj}\,\mathrm{id}_C = \mathrm{id}_{\text{the objects of } C}$.

(118) For every object c of C holds $\mathrm{id}_C(c) = c$.

We now define three new predicates. The arguments of the notions defined below are the following: C, D which are categories; T which is a functor from C to D. The predicate T is an isomorphism is defined by:

T is one-to-one and $\mathrm{rng}\,T =$the morphisms of D and $\mathrm{rng}(\mathrm{Obj}\,T) =$the objects of D.

The predicate T is full is defined by:

for all objects c, c' of C such that $\hom(T(c), T(c')) \neq \emptyset$ for every morphism g from $T(c)$ to $T(c')$ holds $\hom(c, c') \neq \emptyset$ and there exists f being a morphism from c to c' such that $g = T(f)$.

The predicate T is faithful is defined by:

for all objects c, c' of C such that $\hom(c, c') \neq \emptyset$ for all morphisms f_1, f_2 from c to c' such that $T(f_1) = T(f_2)$ holds $f_1 = f_2$.

One can prove the following propositions:

(119) For every functor T from C to D holds T is an isomorphism if and only if T is one-to-one and $\mathrm{rng}\,T =$the morphisms of D and $\mathrm{rng}(\mathrm{Obj}\,T) =$the objects of D.

(120) For every functor T from C to D holds T is full if and only if for all objects c, c' of C such that $\hom(T(c), T(c')) \neq \emptyset$ for every morphism g from $T(c)$ to $T(c')$ holds $\hom(c, c') \neq \emptyset$ and there exists f being a morphism from c to c' such that $g = T(f)$.

(121) For every functor T from C to D holds T is faithful if and only if for all objects c, c' of C such that $\hom(c, c') \neq \emptyset$ for all morphisms f_1, f_2 from c to c' such that $T(f_1) = T(f_2)$ holds $f_1 = f_2$.

(122) id_C is an isomorphism.

(123) For every functor T from C to D for all objects c, c' of C for arbitrary f such that $f \in \hom(c, c')$ holds $T(f) \in \hom(T(c), T(c'))$.

(124) For every functor T from C to D for all objects c,
c'
of C such that $\hom(c, c') \neq \emptyset$ for every morphism f from c to c' holds $T(f) \in \hom(T(c), T(c'))$.

(125) For every functor T from C to D for all objects c, c' of C such that $\hom(c, c') \neq \emptyset$ for every morphism f from c to c' holds $T(f)$ is a morphism from $T(c)$ to $T(c')$.

(126) For every functor T from C to D for all objects c, c' of C such that $\hom(c, c') \neq \emptyset$ holds $\hom(T(c), T(c')) \neq \emptyset$.

(127) For every functor T from B to C for every functor S from C to D such that T is full and S is full holds $S \cdot T$ is full.

(128) For every functor T from B to C for every functor S from C to D such that T is faithful and S is faithful holds $S \cdot T$ is faithful.

(129) For every functor T from C to D for all objects c, c' of C holds $T \circ \hom(c, c') \subseteq \hom(T(c), T(c'))$.

The arguments of the notions defined below are the following: C, D which are categories; T which is a functor from C to D; c, c' which are objects of C. The functor $T_{c,c'}$ yielding a function from $\hom(c, c')$ into $\hom(T(c), T(c'))$, is defined by:

$T_{c,c'} = T \restriction \hom(c, c')$.

One can prove the following four propositions:

(130) For every functor T from C to D for all objects c, c' of C holds $T_{c,c'} = T \restriction \hom(c, c')$.

(131) For every functor T from C to D for all objects c, c' of C such that $\hom(c, c') \neq \emptyset$ for every morphism f from c to c' holds $T_{c,c'}(f) = T(f)$.

(132) For every functor T from C to D holds T is full if and only if for all objects c, c' of C holds $\operatorname{rng} T_{c,c'} = \hom(T(c), T(c'))$.

(133) For every functor T from C to D holds T is faithful if and only if for all objects c, c' of C holds $T_{c,c'}$ is one-to-one.

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[2] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[3] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[4] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[5] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

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Introduction to Trees

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Summary. The article consists of two parts: the first one deals with the concept of the prefixes of a finite sequence, the second one introduces and deals with the concept of tree. Besides some auxiliary propositions concerning finite sequences are presented. The trees are introduced as non-empty sets of finite sequences of natural numbers which are closed on prefixes and on sequences of less numbers (i.e. if $\langle n_1, n_2, \ldots, n_k \rangle$ is a vertex (element) of a tree and $m_i \leq n_i$ for $i = 1, 2, \ldots, k$, then $\langle m_1, m_2, \ldots, m_k \rangle$ also is). Finite trees, elementary trees with n leaves, the leaves and the subtrees of a tree, the inserting of a tree into another tree, with a node used for detemining the place of insertion, antichains of prefixes, and height and width of finite trees are introduced.

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The notation and terminology used in this paper have been introduced in the following papers: [8], [7], [2], [5], [4], [6], [3], and [1]. For simplicity we adopt the following rules: D is a non-empty set, X is a set, x, y are arbitrary, k, n are natural numbers, and p, q, r are finite sequences of elements of $\mathbb{N}$. We now state several propositions:

(1) For all finite sequences p, q such that $q = p \restriction \operatorname{Seg} n$ holds $\operatorname{len} q \leq n$.

(2) For all finite sequences p, q such that $q = p \restriction \operatorname{Seg} n$ holds $\operatorname{len} q \leq \operatorname{len} p$.

(3) For all finite sequences p, r such that $r = p \restriction \operatorname{Seg} n$ there exists q being a finite sequence such that $p = r \frown q$.

(4) $\varepsilon \neq \langle x \rangle$.

(5) For all finite sequences p, q such that $p = p \frown q$ or $p = q \frown p$ holds $q = \varepsilon$.

(6) For all finite sequences p, q such that $p \frown q = \langle x \rangle$ holds $p = \langle x \rangle$ and $q = \varepsilon$ or $p = \varepsilon$ and $q = \langle x \rangle$.

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Let p, q be finite sequences. The predicate $p \preceq q$ is defined by:
there exists n such that $p = q \restriction \operatorname{Seg} n$.

We now state a number of propositions:

(7) For all finite sequences p, q holds $p \preceq q$ if and only if there exists n such that $p = q \restriction \operatorname{Seg} n$.

(8) For all finite sequences p, q holds $p \preceq q$ if and only if there exists r being a finite sequence such that $q = p \frown r$.

(9) For all finite sequences p, q such that $p \preceq q$ holds $\operatorname{len} p \leq \operatorname{len} q$.

(10) For every finite sequence p holds $\varepsilon \preceq p$ and $\varepsilon_D \preceq p$.

(11) For every finite sequence p such that $p \preceq \varepsilon$ holds $p = \varepsilon$.

(12) For every finite sequence p holds $p \preceq p$.

(13) For all finite sequences p, q such that $p \preceq q$ and $q \preceq p$ holds $p = q$.

(14) For all finite sequences p, q, r such that $p \preceq q$ and $q \preceq r$ holds $p \preceq r$.

(15) For all finite sequences p, q such that $p \preceq q$ and $\operatorname{len} p = \operatorname{len} q$ holds $p = q$.

(16) $\langle x \rangle \preceq \langle y \rangle$ if and only if $x = y$.

We now define two new predicates. Let p, q be finite sequences. The predicate p and q are comparable is defined by:
$p \preceq q$ or $q \preceq p$.
The predicate $p \prec q$ is defined by:
$p \preceq q$ and $p \neq q$.

One can prove the following propositions:

(17) For all finite sequences p, q holds p and q are comparable if and only if $p \preceq q$ or $q \preceq p$.

(18) For all finite sequences p, q holds $p \prec q$ if and only if $p \preceq q$ and $p \neq q$.

(19) For all finite sequences p, q such that p and q are comparable and $\operatorname{len} p = \operatorname{len} q$ holds $p = q$.

(20) For all finite sequences p, q holds $p \prec q$ or $p = q$ or $q \prec p$ if and only if p and q are comparable.

(21) For every finite sequence p holds p and p are comparable.

In the sequel p_1, p_2 will be finite sequences. Next we state a number of propositions:

(22) If p_1 and p_2 are comparable, then p_2 and p_1 are comparable.

(23) $\langle x \rangle$ and $\langle y \rangle$ are comparable if and only if $x = y$.

(24) For all finite sequences p, q such that $p \prec q$ holds $\operatorname{len} p < \operatorname{len} q$.

(25) For no finite sequence p holds $p \prec \varepsilon$ or $p \prec \varepsilon_D$.

(26) For no finite sequences p, q holds $p \prec q$ and $q \prec p$.

(27) For all finite sequences p, q, r such that $p \prec q$ and $q \prec r$ or $p \prec q$ and $q \preceq r$ or $p \preceq q$ and $q \prec r$ holds $p \prec r$.

(28) If $p_1 \preceq p_2$, then $p_2 \not\prec p_1$.

(29) If $p_1 \prec p_2$, then $p_2 \not\preceq p_1$.

(30) If $p_1 \frown \langle x \rangle \preceq p_2$, then $p_1 \prec p_2$.

(31) If $p_1 \preceq p_2$, then $p_1 \prec p_2 \frown \langle x \rangle$.

(32) If $p_1 \prec p_2 \frown \langle x \rangle$, then $p_1 \preceq p_2$.

(33) If $\varepsilon \prec p_2$ or $\varepsilon \neq p_2$, then $p_1 \prec p_1 \frown p_2$.

Let p be a finite sequence. The functor $\mathrm{Seg}_{\preceq}(p)$ yielding a set, is defined by:

$x \in \mathrm{Seg}_{\preceq}(p)$ if and only if there exists q being a finite sequence such that $x = q$ and $q \prec p$.

The following propositions are true:

(34) For every finite sequence p holds $X = \mathrm{Seg}_{\preceq}(p)$ if and only if for every x holds $x \in X$ if and only if there exists q being a finite sequence such that $x = q$ and $q \prec p$.

(35) For every finite sequence p such that $x \in \mathrm{Seg}_{\preceq}(p)$ holds x is a finite sequence.

(36) For all finite sequences p, q holds $p \in \mathrm{Seg}_{\preceq}(q)$ if and only if $p \prec q$.

(37) For all finite sequences p, q such that $p \in \mathrm{Seg}_{\preceq}(q)$ holds $\mathrm{len}\, p < \mathrm{len}\, q$.

(38) For all finite sequences p, q, r such that $q \frown r \in \mathrm{Seg}_{\preceq}(p)$ holds $q \in \mathrm{Seg}_{\preceq}(p)$.

(39) $\mathrm{Seg}_{\preceq}(\varepsilon) = \emptyset$.

(40) $\mathrm{Seg}_{\preceq}(\langle x \rangle) = \{\varepsilon\}$.

(41) For all finite sequences p, q such that $p \preceq q$ holds $\mathrm{Seg}_{\preceq}(p) \subseteq \mathrm{Seg}_{\preceq}(q)$.

(42) For all finite sequences p, q, r such that $q \in \mathrm{Seg}_{\preceq}(p)$ and $r \in \mathrm{Seg}_{\preceq}(p)$ holds q and r are comparable.

The mode tree, which widens to the type a non-empty set, is defined by:

it $\subseteq \mathbb{N}^*$ and for every p such that $p \in$ it holds $\mathrm{Seg}_{\preceq}(p) \subseteq$ it and for all p, k, n such that $p \frown \langle k \rangle \in$ it and $n \leq k$ holds $p \frown \langle n \rangle \in$ it.

Next we state a proposition

(43) D is a tree if and only if $D \subseteq \mathbb{N}^*$ and for every p such that $p \in D$ holds $\mathrm{Seg}_{\preceq}(p) \subseteq D$ and for all p, k, n such that $p \frown \langle k \rangle \in D$ and $n \leq k$ holds $p \frown \langle n \rangle \in D$.

In the sequel T, T_1 denote trees. The following proposition is true

(44) If $x \in T$, then x is a finite sequence of elements of $\mathbb{N}$.

Let us consider T. We see that it makes sense to consider the following mode for restricted scopes of arguments. Then all the objests of the mode element of T are a finite sequence of elements of $\mathbb{N}$.

The following propositions are true:

(45) For all finite sequences p, q such that $p \in T$ and $q \preceq p$ holds $q \in T$.

(46) For every finite sequence r such that $q \frown r \in T$ holds $q \in T$.

(47) $\varepsilon \in T$ and $\varepsilon_{\mathbb{N}} \in T$.

(48) $\{\varepsilon\}$ is a tree.

(49) $T \cup T_1$ is a tree.

(50) $T \cap T_1$ is a tree.

The mode finite tree, which widens to the type a tree, is defined by:

it is finite.

The following proposition is true

(51) T is a finite tree if and only if T is finite.

In the sequel fT, fT_1 will be finite trees. Next we state two propositions:

(52) $fT \cup fT_1$ is a finite tree.

(53) $fT \cap T$ is a finite tree and $T \cap fT$ is a finite tree.

Let us consider n. The functor elementary tree of n yielding a finite tree, is defined by:

elementary tree of $n = \{\langle k \rangle : k < n\} \cup \{\varepsilon\}$.

The following propositions are true:

(54) $fT =$ elementary tree of n if and only if $fT = \{\langle k \rangle : k < n\} \cup \{\varepsilon\}$.

(55) If $k < n$, then $\langle k \rangle \in$ elementary tree of n.

(56) elementary tree of $0 = \{\varepsilon\}$.

(57) If $p \in$ elementary tree of n, then $p = \varepsilon$ or there exists k such that $k < n$ and $p = \langle k \rangle$.

We now define two new functors. Let us consider T. The functor Leaves T yields a subset of T and is defined by:

$p \in$ Leaves T if and only if $p \in T$ and for no q holds $q \in T$ and $p \prec q$.

Let us consider p. Let us assume that $p \in T$. The functor $T \restriction p$ yields a tree and is defined by:

$q \in T \restriction p$ if and only if $p \frown q \in T$.

We now state three propositions:

(58) For every subset X of T holds $X =$ Leaves T if and only if for every p holds $p \in X$ if and only if $p \in T$ and for no q holds $q \in T$ and $p \prec q$.

(59) If $p \in T$, then $T_1 = T \restriction p$ if and only if for every q holds $q \in T_1$ if and only if $p \frown q \in T$.

(60) $T \restriction \varepsilon_{\mathbb{N}} = T$.

The arguments of the notions defined below are the following: T which is a finite tree; p which is an element of T. Then $T \restriction p$ is a finite tree.

Let us consider T. Let us assume that Leaves $T \neq \emptyset$. The mode leaf of T, which widens to the type an element of T, is defined by:

it $\in$ Leaves T.

We now state a proposition

(61) If Leaves $T \neq \emptyset$, then for every element p of T holds p is a leaf of T if and only if $p \in$ Leaves T.

Let us consider T. The mode subtree of T, which widens to the type a tree, is defined by:

there exists p being an element of T such that it $= T \restriction p$.

One can prove the following proposition

(62) T_1 is a subtree of T if and only if there exists p being an element of T such that $T_1 = T \restriction p$.

In the sequel t is an element of T. Let us consider T, p, T_1. Let us assume that $p \in T$. The functor $T(p/T_1)$ yields a tree and is defined by:

$q \in T(p/T_1)$ if and only if $q \in T$ and $p \not\preceq q$ or there exists r such that $r \in T_1$ and $q = p \frown r$.

In the sequel T_2 is a tree. Next we state four propositions:

(63) If $p \in T_1$, then $T = T_1(p/T_2)$ if and only if for every q holds $q \in T$ if and only if $q \in T_1$ and $p \not\preceq q$ or there exists r such that $r \in T_2$ and $q = p \frown r$.

(64) If $p \in T$, then $T(p/T_1) = \{t_1 : p \not\preceq t_1\} \cup \{p \frown s : s = s\}$.

(65) If $p \in T$ and $q \in T_1$, then $p \frown q \in T(p/T_1)$.

(66) If $p \in T$, then $T_1 = (T(p/T_1)) \upharpoonright p$.

The arguments of the notions defined below are the following: T which is a finite tree; t which is an element of T; T_1 which is a finite tree. Then $T(t/T_1)$ is a finite tree.

In the sequel w will denote a finite sequence. The following two propositions are true:

(67) For every finite sequence p holds $\mathrm{Seg}_{\preceq}(p) \approx \mathrm{Seg}(\mathrm{len}\, p)$.

(68) For every finite sequence p holds $\mathrm{card}(\mathrm{Seg}_{\preceq}(p)) = \mathrm{len}\, p$.

The mode antichain of prefixes, which widens to the type a set, is defined by:

for every x such that $x \in$ it holds x is a finite sequence and for all p_1, p_2 such that $p_1 \in$ it and $p_2 \in$ it and $p_1 \neq p_2$ holds p_1 and p_2 are not comparable.

Next we state three propositions:

(69) X is an antichain of prefixes if and only if for every x such that $x \in X$ holds x is a finite sequence and for all p_1, p_2 such that $p_1 \in X$ and $p_2 \in X$ and $p_1 \neq p_2$ holds p_1 and p_2 are not comparable.

(70) $\{w\}$ is an antichain of prefixes.

(71) If p_1 and p_2 are not comparable, then $\{p_1, p_2\}$ is an antichain of prefixes.

Let us consider T. The mode antichain of prefixes of T, which widens to the type an antichain of prefixes, is defined by:

it $\subseteq T$.

We now state a proposition

(72) For every antichain S of prefixes holds S is an antichain of prefixes of T if and only if $S \subseteq T$.

In the sequel t_1, t_2 will be elements of T. The following three propositions are true:

(73) $\emptyset$ is an antichain of prefixes of T and $\{\varepsilon\}$ is an antichain of prefixes of T.

(74) $\{t\}$ is an antichain of prefixes of T.

(75) If t_1 and t_2 are not comparable, then $\{t_1, t_2\}$ is an antichain of prefixes of T.

We now define two new functors. Let T be a finite tree. The functor height T yields a natural number and is defined by:

there exists p such that $p \in T$ and $\mathrm{len}\, p = \mathrm{height}\, T$ and for every p such that $p \in T$ holds $\mathrm{len}\, p \leq \mathrm{height}\, T$.

The functor width T yielding a natural number, is defined by:

there exists X being an antichain of prefixes of T such that $\operatorname{width} T = \operatorname{card} X$ and for every antichain Y of prefixes of T holds $\operatorname{card} Y \leq \operatorname{card} X$.

We now state three propositions:

(76) For every finite tree T for every n holds $n = \operatorname{height} T$ if and only if there exists p such that $p \in T$ and $\operatorname{len} p = n$ and for every p such that $p \in T$ holds $\operatorname{len} p \leq n$.

(77) For every finite tree T for every n holds $n = \operatorname{width} T$ if and only if there exists X being an antichain of prefixes of T such that $n = \operatorname{card} X$ and for every antichain Y of prefixes of T holds $\operatorname{card} Y \leq \operatorname{card} X$.

(78) $1 \leq \operatorname{width} fT$.

The following propositions are true:

(79) height(elementary tree of 0) = 0.

(80) If height $fT = 0$, then fT = elementary tree of 0.

(81) height(elementary tree of$(n+1)$) = 1.

(82) width(elementary tree of 0) = 1.

(83) width(elementary tree of$(n+1)$) = $n+1$.

(84) For every element t of fT holds $\operatorname{height}(fT \restriction t) \leq \operatorname{height} fT$.

(85) For every element t of fT such that $t \neq \varepsilon$ holds $\operatorname{height}(fT \restriction t) < \operatorname{height} fT$.

The scheme *Tree_Ind* deals with a unary predicate $\mathcal{P}$ and states that:
for every fT holds $\mathcal{P}[fT]$
provided the parameter satisfies the following condition:

- for every fT such that for every n such that $\langle n \rangle \in fT$ holds $\mathcal{P}[fT \restriction \langle n \rangle]$ holds $\mathcal{P}[fT]$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[3] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Formalized Mathematics*, 1(**2**):265–267, 1990.

[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[6] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[7] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[8] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

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