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Inst. of Computer Science, Univ. of Białystok

Consequences of the Reflection Theorem

Grzegorz Bancerek
Warsaw University
Białystok

Summary. Some consequences of the reflection theorem are discussed. To formulate them the notions of elementary equivalence and subsystems, and of models for a set of formulae are introduced. Besides, the concept of cofinality of a ordinal number with second one is used. The consequences of the reflection theorem (it is sometimes called the Scott-Scarpellini lemma) are: (i) If A_ξ is a transfinite sequence as in the reflection theorem (see [10]) and $A = \bigcup_{\xi \in On} A_\xi$, then there is an increasing and continuous mapping ϕ from On into On such that for every critical number κ the set A_κ is an elementary subsystem of A ($A_\kappa \prec A$). (ii) There is an increasing continuous mapping $\phi : On \to On$ such that $\mathbf{R}_\kappa \prec V$ for each of its critical numbers κ (V is the universal class and On is the class of all ordinals belonging to V). (iii) There are ordinal numbers α cofinal with ω for which $\mathbf{R}_\alpha$ are models of ZF set theory. (iv) For each set X from universe V there is a model of ZF M which belongs to V and has X as an element.

MML Identifier: ZFREFLE1.

WWW: http://mizar.org/JFM/Vol2/zfrefle1.html

The articles [20], [19], [15], [22], [23], [13], [14], [16], [21], [4], [2], [3], [6], [1], [5], [17], [7], [9], [12], [18], [8], [11], and [10] provide the notation and terminology for this paper.

We adopt the following rules: H, S denote ZF-formulae, X, Y denote sets, and e, u denote sets.

Let M be a non empty set and let F be a subset of WFF. The predicate $M \models F$ is defined as follows:

(Def. 1) For every H such that $H \in F$ holds $M \models H$.

Let M_1, M_2 be non empty sets. The predicate $M_1 \equiv M_2$ is defined by:

(Def. 2) For every H such that $\mathrm{Free}\, H = \emptyset$ holds $M_1 \models H$ iff $M_2 \models H$.

Let us notice that the predicate $M_1 \equiv M_2$ is reflexive and symmetric. The predicate $M_1 \prec M_2$ is defined as follows:

(Def. 3) $M_1 \subseteq M_2$ and for every H and for every function v from VAR into M_1 holds $M_1, v \models H$ iff $M_2, M_2[v] \models H$.

Let us note that the predicate $M_1 \prec M_2$ is reflexive.

The set $\mathbf{Ax}_{\mathrm{ZF}}$ is defined by the condition (Def. 4).

(Def. 4) $e \in \mathbf{Ax}_{\mathrm{ZF}}$ if and only if the following conditions are satisfied:

(i) $e \in$ WFF, and

(ii) $e =$ the axiom of extensionality or $e =$ the axiom of pairs or $e =$ the axiom of unions or $e =$ the axiom of infinity or $e =$ the axiom of power sets or there exists H such that $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\, H$ and $e =$ the axiom of substitution for H.

$\mathbf{Ax}_{\mathrm{ZF}}$ is a subset of WFF.

For simplicity, we adopt the following rules: M, M_1, M_2 denote non empty sets, f denotes a function, F, F_1, F_2 denote subsets of WFF, W denotes a universal class, a, b denote ordinals of W, A, B, C denote ordinal numbers, L denotes a transfinite sequence of non empty sets from W, and p_1 denotes a transfinite sequence of ordinals of W.

The following propositions are true:

(1) $M \models \emptyset_{\mathrm{WFF}}$.

(2) If $F_1 \subseteq F_2$ and $M \models F_2$, then $M \models F_1$.

(3) If $M \models F_1$ and $M \models F_2$, then $M \models F_1 \cup F_2$.

(4) If M is a model of ZF, then $M \models \mathbf{Ax}_{\mathrm{ZF}}$.

(5) If $M \models \mathbf{Ax}_{\mathrm{ZF}}$ and M is transitive, then M is a model of ZF.

(6) There exists S such that $\mathrm{Free}\, S = \emptyset$ and for every M holds $M \models S$ iff $M \models H$.

(7) $M_1 \equiv M_2$ iff for every H holds $M_1 \models H$ iff $M_2 \models H$.

(8) $M_1 \equiv M_2$ iff for every F holds $M_1 \models F$ iff $M_2 \models F$.

(9) If $M_1 \prec M_2$, then $M_1 \equiv M_2$.

(10) If M_1 is a model of ZF and $M_1 \equiv M_2$ and M_2 is transitive, then M_2 is a model of ZF.

The scheme *NonUniqFuncEx* deals with a set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a function f such that $\mathrm{dom}\, f = \mathcal{A}$ and for every e such that $e \in \mathcal{A}$ holds $\mathcal{P}[e, f(e)]$

provided the parameters satisfy the following condition:

- For every e such that $e \in \mathcal{A}$ there exists u such that $\mathcal{P}[e, u]$.

Next we state several propositions:

(12)[1] If $\mathrm{dom}\, f \in W$ and $\mathrm{rng}\, f \subseteq W$, then $\mathrm{rng}\, f \in W$.

(13) If $X \approx Y$ or $\overline{\overline{X}} = \overline{\overline{Y}}$, then $2^X \approx 2^Y$ and $\overline{\overline{2^X}} = \overline{\overline{2^Y}}$.

(14) Let D be a non empty set and P_1 be a function from D into $(\mathrm{On}\, W)^{\mathrm{On}\, W}$. Suppose $\overline{\overline{D}} < \overline{\overline{W}}$. Then there exists p_1 such that

(i) p_1 is increasing and continuous,

(ii) $p_1(\mathbf{0}_W) = \mathbf{0}_W$,

(iii) for every a holds $p_1(\mathrm{succ}\, a) = \sup(\{p_1(a)\} \cup (\mathrm{uncurry}\, P_1)^{\circ}[{:}D, \{\mathrm{succ}\, a\}{:}])$, and

(iv) for every a such that $a \neq \mathbf{0}_W$ and a is a limit ordinal number holds $p_1(a) = \sup(p_1 \restriction a)$.

(15) For every sequence p_1 of ordinal numbers such that p_1 is increasing holds $C + p_1$ is increasing.

(16) For every sequence x_1 of ordinal numbers holds $(C + x_1) \restriction A = C + x_1 \restriction A$.

(17) For every sequence p_1 of ordinal numbers such that p_1 is increasing and continuous holds $C + p_1$ is continuous.

Let A, B be ordinal numbers. We say that A is cofinal with B if and only if:

(Def. 5) There exists a sequence x_1 of ordinal numbers such that $\mathrm{dom}\, x_1 = B$ and $\mathrm{rng}\, x_1 \subseteq A$ and x_1 is increasing and $A = \sup x_1$.

[1] The proposition (11) has been removed.

Let us note that the predicate A is cofinal with B is reflexive.

In the sequel p_2 denotes a sequence of ordinal numbers.

We now state a number of propositions:

(19)[2] If $e \in \operatorname{rng} p_2$, then e is an ordinal number.

(20) $\operatorname{rng} p_2 \subseteq \sup p_2$.

(21) If A is cofinal with B and B is cofinal with C, then A is cofinal with C.

(22) If A is cofinal with B, then $B \subseteq A$.

(23) If A is cofinal with B and B is cofinal with A, then $A = B$.

(24) If $\operatorname{dom} p_2 \neq \emptyset$ and $\operatorname{dom} p_2$ is a limit ordinal number and p_2 is increasing and A is the limit of p_2, then A is cofinal with $\operatorname{dom} p_2$.

(25) $\operatorname{succ} A$ is cofinal with $\mathbf{1}$.

(26) If A is cofinal with $\operatorname{succ} B$, then there exists C such that $A = \operatorname{succ} C$.

(27) If A is cofinal with B, then A is a limit ordinal number iff B is a limit ordinal number.

(28) If A is cofinal with $\emptyset$, then $A = \emptyset$.

(29) $\operatorname{On} W$ is not cofinal with a.

(30) If $\omega \in W$ and p_1 is increasing and continuous, then there exists b such that $a \in b$ and $p_1(b) = b$.

(31) If $\omega \in W$ and p_1 is increasing and continuous, then there exists a such that $b \in a$ and $p_1(a) = a$ and a is cofinal with ω.

(32) Suppose that

(i) $\omega \in W$,

(ii) for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$, and

(iii) for every a such that $a \neq \emptyset$ and a is a limit ordinal number holds $L(a) = \bigcup(L{\restriction}a)$.

Then there exists p_1 such that p_1 is increasing and continuous and for every a such that $p_1(a) = a$ and $\emptyset \neq a$ holds $L(a) \prec \bigcup L$.

(33) $\mathbf{R}_a \in W$.

(34) If $a \neq \emptyset$, then $\mathbf{R}_a$ is a non empty element of W.

(35) Suppose $\omega \in W$. Then there exists p_1 such that p_1 is increasing and continuous and for all a, M such that $p_1(a) = a$ and $\emptyset \neq a$ and $M = \mathbf{R}_a$ holds $M \prec W$.

(36) If $\omega \in W$, then there exist b, M such that $a \in b$ and $M = \mathbf{R}_b$ and $M \prec W$.

(37) If $\omega \in W$, then there exist a, M such that a is cofinal with ω and $M = \mathbf{R}_a$ and $M \prec W$.

(38) Suppose that

(i) $\omega \in W$,

(ii) for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$, and

(iii) for every a such that $a \neq \emptyset$ and a is a limit ordinal number holds $L(a) = \bigcup(L{\restriction}a)$.

Then there exists p_1 such that p_1 is increasing and continuous and for every a such that $p_1(a) = a$ and $\emptyset \neq a$ holds $L(a) \equiv \bigcup L$.

[2] The proposition (18) has been removed.

(39) Suppose $\omega \in W$. Then there exists p_1 such that p_1 is increasing and continuous and for all a, M such that $p_1(a) = a$ and $\emptyset \neq a$ and $M = \mathbf{R}_a$ holds $M \equiv W$.

(40) If $\omega \in W$, then there exist b, M such that $a \in b$ and $M = \mathbf{R}_b$ and $M \equiv W$.

(41) If $\omega \in W$, then there exist a, M such that a is cofinal with ω and $M = \mathbf{R}_a$ and $M \equiv W$.

(42) If $\omega \in W$, then there exist a, M such that a is cofinal with ω and $M = \mathbf{R}_a$ and M is a model of ZF.

(43) If $\omega \in W$ and $X \in W$, then there exists M such that $X \in M$ and $M \in W$ and M is a model of ZF.

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