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Properties of ZF Models

Grzegorz Bancerek
Warsaw University
Białystok

Summary. The article deals with the concepts of satisfiability of ZF set theory language formulae in a model (a non-empty family of sets) and the axioms of ZF theory introduced in [8]. It is shown that the transitive model satisfies the axiom of extensionality and that it satisfies the axiom of pairs if and only if it is closed to pair operation; it satisfies the axiom of unions if and only if it is closed to union operation, etc. The conditions which are satisfied by arbitrary model of ZF set theory are also shown. Besides introduced are definable and parametrically definable functions.

MML Identifier: ZFMODEL1.

WWW: http://mizar.org/JFM/Vol1/zfmodel1.html

The articles [10], [9], [7], [11], [12], [5], [4], [1], [13], [6], [3], and [2] provide the notation and terminology for this paper.

For simplicity, we follow the rules: x, y, z are variables, H is a ZF-formula, E is a non empty set, X, Y, Z are sets, u, v, w are elements of E, and f, g are functions from VAR into E.

Next we state a number of propositions:

(1) If E is transitive, then $E \models$ the axiom of extensionality.

(2) If E is transitive, then $E \models$ the axiom of pairs iff for all u, v holds $\{u,v\} \in E$.

(3) If E is transitive, then $E \models$ the axiom of pairs iff for all X, Y such that $X \in E$ and $Y \in E$ holds $\{X,Y\} \in E$.

(4) If E is transitive, then $E \models$ the axiom of unions iff for every u holds $\bigcup u \in E$.

(5) If E is transitive, then $E \models$ the axiom of unions iff for every X such that $X \in E$ holds $\bigcup X \in E$.

(6) Suppose E is transitive. Then $E \models$ the axiom of infinity if and only if there exists u such that $u \neq \emptyset$ and for every v such that $v \in u$ there exists w such that $v \subset w$ and $w \in u$.

(7) Suppose E is transitive. Then $E \models$ the axiom of infinity if and only if there exists X such that $X \in E$ and $X \neq \emptyset$ and for every Y such that $Y \in X$ there exists Z such that $Y \subset Z$ and $Z \in X$.

(8) If E is transitive, then $E \models$ the axiom of power sets iff for every u holds $E \cap 2^u \in E$.

(9) If E is transitive, then $E \models$ the axiom of power sets iff for every X such that $X \in E$ holds $E \cap 2^X \in E$.

(10) If $x \notin \mathrm{Free}\, H$ and $E, f \models H$, then $E, f \models \forall_x H$.

(11) If $\{x, y\}$ misses Free$H$ and $E, f \models H$, then $E, f \models \forall_{x,y} H$.

(12) If $\{x, y, z\}$ misses Free$H$ and $E, f \models H$, then $E, f \models \forall_{x,y,z} H$.

Let us consider H, E and let v_1 be a function from VAR into E. Let us assume that $\mathrm{x}_0 \notin$ Free$H$ and $E, v_1 \models \forall_{\mathrm{x}_3} \exists_{\mathrm{x}_0} \forall_{\mathrm{x}_4} (H \Leftrightarrow \mathrm{x}_4\text{=}(\mathrm{x}_0))$. The functor $\mathrm{f'}_H[v_1]$ yielding a function from E into E is defined by the condition (Def. 1).

(Def. 1) Let given g. Suppose that for every y such that $g(y) \neq v_1(y)$ holds $\mathrm{x}_0 = y$ or $\mathrm{x}_3 = y$ or $\mathrm{x}_4 = y$. Then $E, g \models H$ if and only if $\mathrm{f'}_H[v_1](g(\mathrm{x}_3)) = g(\mathrm{x}_4)$.

We now state the proposition

(14)[1] For all H, f, g such that for every x such that $f(x) \neq g(x)$ holds $x \notin$ Free$H$ and $E, f \models H$ holds $E, g \models H$.

Let us consider H, E. Let us assume that Free$H \subseteq \{\mathrm{x}_3, \mathrm{x}_4\}$ and $E \models \forall_{\mathrm{x}_3} \exists_{\mathrm{x}_0} \forall_{\mathrm{x}_4} (H \Leftrightarrow \mathrm{x}_4\text{=}(\mathrm{x}_0))$. The functor $\mathrm{f}_H[E]$ yielding a function from E into E is defined by:

(Def. 2) For every g holds $E, g \models H$ iff $\mathrm{f}_H[E](g(\mathrm{x}_3)) = g(\mathrm{x}_4)$.

Let F be a function and let us consider E. We say that F is definable in E if and only if:

(Def. 3) There exists H such that Free$H \subseteq \{\mathrm{x}_3, \mathrm{x}_4\}$ and $E \models \forall_{\mathrm{x}_3} \exists_{\mathrm{x}_0} \forall_{\mathrm{x}_4} (H \Leftrightarrow \mathrm{x}_4\text{=}(\mathrm{x}_0))$ and $F = \mathrm{f}_H[E]$.

We say that F is parametrically definable in E if and only if:

(Def. 4) There exist H, f such that $\{\mathrm{x}_0, \mathrm{x}_1, \mathrm{x}_2\}$ misses Free$H$ and $E, f \models \forall_{\mathrm{x}_3} \exists_{\mathrm{x}_0} \forall_{\mathrm{x}_4} (H \Leftrightarrow \mathrm{x}_4\text{=}(\mathrm{x}_0))$ and $F = \mathrm{f'}_H[f]$.

One can prove the following propositions:

(18)[2] For every function F such that F is definable in E holds F is parametrically definable in E.

(19) Suppose E is transitive. Then the following statements are equivalent

(i) for every H such that $\{\mathrm{x}_0, \mathrm{x}_1, \mathrm{x}_2\}$ misses Free$H$ holds $E \models$ the axiom of substitution for H,

(ii) for all H, f such that $\{\mathrm{x}_0, \mathrm{x}_1, \mathrm{x}_2\}$ misses Free$H$ and $E, f \models \forall_{\mathrm{x}_3} \exists_{\mathrm{x}_0} \forall_{\mathrm{x}_4} (H \Leftrightarrow \mathrm{x}_4\text{=}(\mathrm{x}_0))$ and for every u holds $\mathrm{f'}_H[f]^\circ u \in E$.

(20) Suppose E is transitive. Then the following statements are equivalent

(i) for every H such that $\{\mathrm{x}_0, \mathrm{x}_1, \mathrm{x}_2\}$ misses Free$H$ holds $E \models$ the axiom of substitution for H,

(ii) for every function F such that F is parametrically definable in E and for every X such that $X \in E$ holds $F^\circ X \in E$.

(21) Suppose E is a model of ZF. Then E is transitive and for all u, v such that for every w holds $w \in u$ iff $w \in v$ holds $u = v$ and for all u, v holds $\{u, v\} \in E$ and for every u holds $\bigcup u \in E$ and there exists u such that $u \neq \emptyset$ and for every v such that $v \in u$ there exists w such that $v \subset w$ and $w \in u$ and for every u holds $E \cap 2^u \in E$ and for all H, f such that $\{\mathrm{x}_0, \mathrm{x}_1, \mathrm{x}_2\}$ misses Free$H$ and $E, f \models \forall_{\mathrm{x}_3} \exists_{\mathrm{x}_0} \forall_{\mathrm{x}_4} (H \Leftrightarrow \mathrm{x}_4\text{=}(\mathrm{x}_0))$ and for every u holds $\mathrm{f'}_H[f]^\circ u \in E$.

[1] The proposition (13) has been removed.

[2] The propositions (15)–(17) have been removed.

(22) Suppose that

(i) E is transitive,

(ii) for all u, v holds $\{u,v\} \in E$,

(iii) for every u holds $\bigcup u \in E$,

(iv) there exists u such that $u \neq \emptyset$ and for every v such that $v \in u$ there exists w such that $v \subset w$ and $w \in u$,

(v) for every u holds $E \cap 2^u \in E$, and

(vi) for all H, f such that $\{\mathrm{x}_0, \mathrm{x}_1, \mathrm{x}_2\}$ misses Free$H$ and $E, f \models \forall_{\mathrm{x}_3} \exists_{\mathrm{x}_0} \forall_{\mathrm{x}_4} (H \Leftrightarrow \mathrm{x}_4\text{=}(\mathrm{x}_0))$ and for every u holds $\mathrm{f'}_H[f]^\circ u \in E$.

Then E is a model of ZF.

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