JOURNAL OF FORMALIZED MATHEMATICS
Volume 1, Released 1989, Published 2003
Inst. of Computer Science, Univ. of Białystok

Some Basic Properties of Sets

Czesław Byliński
Warsaw University
Białystok

Summary. In this article some basic theorems about singletons, pairs, power sets, unions of families of sets, and the cartesian product of two sets are proved.

MML Identifier: ZFMISC_1.

WWW: http://mizar.org/JFM/Vol1/zfmisc_1.html

The article [1] provides the notation and terminology for this paper.

1. Cartesian product of sets

In this paper x, x_1, x_2, y, y_1, y_2, z, A, B, X, X_1, X_2, X_3, X_4, Y, Y_1, Y_2, Z, N, M denote sets.

Let x, y be sets. One can verify that $\langle x, y\rangle$ is non empty.

Let us consider X. The functor 2^X is defined by:

(Def. 1) $Z \in 2^X$ iff $Z \subseteq X$.

Let us consider X_1, X_2. The functor $[: X_1, X_2 :]$ is defined as follows:

(Def. 2) $z \in [: X_1, X_2 :]$ iff there exist x, y such that $x \in X_1$ and $y \in X_2$ and $z = \langle x, y\rangle$.

Let us consider X_1, X_2, X_3. The functor $[: X_1, X_2, X_3 :]$ is defined by:

(Def. 3) $[: X_1, X_2, X_3 :] = [: [: X_1, X_2 :], X_3 :]$.

Let us consider X_1, X_2, X_3, X_4. The functor $[: X_1, X_2, X_3, X_4 :]$ is defined by:

(Def. 4) $[: X_1, X_2, X_3, X_4 :] = [: [: X_1, X_2, X_3 :], X_4 :]$.

2. Basic properties of sets

We now state a number of propositions:

(1) $2^\emptyset = \{\emptyset\}$.

(2) $\bigcup \emptyset = \emptyset$.

(6)[1] If $\{x\} \subseteq \{y\}$, then $x = y$.

(8)[2] If $\{x\} = \{y_1, y_2\}$, then $x = y_1$.

[1] The propositions (3)–(5) have been removed.
[2] The proposition (7) has been removed.

(9) If $\{x\} = \{y_1, y_2\}$, then $y_1 = y_2$.

(10) If $\{x_1, x_2\} = \{y_1, y_2\}$, then $x_1 = y_1$ or $x_1 = y_2$.

(12)[3] $\{x\} \subseteq \{x, y\}$.

(13) If $\{x\} \cup \{y\} = \{x\}$, then $x = y$.

(14) $\{x\} \cup \{x, y\} = \{x, y\}$.

(16)[4] If $\{x\}$ misses $\{y\}$, then $x \neq y$.

(17) If $x \neq y$, then $\{x\}$ misses $\{y\}$.

(18) If $\{x\} \cap \{y\} = \{x\}$, then $x = y$.

(19) $\{x\} \cap \{x, y\} = \{x\}$.

(20) $\{x\} \setminus \{y\} = \{x\}$ iff $x \neq y$.

(21) If $\{x\} \setminus \{y\} = \emptyset$, then $x = y$.

(22) $\{x\} \setminus \{x, y\} = \emptyset$.

(23) If $x \neq y$, then $\{x, y\} \setminus \{y\} = \{x\}$.

(24) If $\{x\} \subseteq \{y\}$, then $x = y$.

(25) If $\{z\} \subseteq \{x, y\}$, then $z = x$ or $z = y$.

(26) If $\{x, y\} \subseteq \{z\}$, then $x = z$.

(27) If $\{x, y\} \subseteq \{z\}$, then $\{x, y\} = \{z\}$.

(28) If $\{x_1, x_2\} \subseteq \{y_1, y_2\}$, then $x_1 = y_1$ or $x_1 = y_2$.

(29) If $x \neq y$, then $\{x\} \dot{-} \{y\} = \{x, y\}$.

(30) $2^{\{x\}} = \{\emptyset, \{x\}\}$.

(31) $\bigcup\{x\} = x$.

(32) $\bigcup\{\{x\}, \{y\}\} = \{x, y\}$.

(33) If $\langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle$, then $x_1 = y_1$ and $x_2 = y_2$.

(34) $\langle x, y \rangle \in [: \{x_1\}, \{y_1\} :]$ iff $x = x_1$ and $y = y_1$.

(35) $[: \{x\}, \{y\} :] = \{\langle x, y \rangle\}$.

(36) $[: \{x\}, \{y, z\} :] = \{\langle x, y \rangle, \langle x, z \rangle\}$ and $[: \{x, y\}, \{z\} :] = \{\langle x, z \rangle, \langle y, z \rangle\}$.

(37) $\{x\} \subseteq X$ iff $x \in X$.

(38) $\{x_1, x_2\} \subseteq Z$ iff $x_1 \in Z$ and $x_2 \in Z$.

(39) $Y \subseteq \{x\}$ iff $Y = \emptyset$ or $Y = \{x\}$.

(40) If $Y \subseteq X$ and $x \notin Y$, then $Y \subseteq X \setminus \{x\}$.

(41) If $X \neq \{x\}$ and $X \neq \emptyset$, then there exists y such that $y \in X$ and $y \neq x$.

(42) $Z \subseteq \{x_1, x_2\}$ iff $Z = \emptyset$ or $Z = \{x_1\}$ or $Z = \{x_2\}$ or $Z = \{x_1, x_2\}$.

(43) If $\{z\} = X \cup Y$, then $X = \{z\}$ and $Y = \{z\}$ or $X = \emptyset$ and $Y = \{z\}$ or $X = \{z\}$ and $Y = \emptyset$.

---

[3] The proposition (11) has been removed.

[4] The proposition (15) has been removed.

(44) If $\{z\} = X \cup Y$ and $X \neq Y$, then $X = \emptyset$ or $Y = \emptyset$.

(45) If $\{x\} \cup X \subseteq X$, then $x \in X$.

(46) If $x \in X$, then $\{x\} \cup X = X$.

(47) If $\{x,y\} \cup Z \subseteq Z$, then $x \in Z$.

(48) If $x \in Z$ and $y \in Z$, then $\{x,y\} \cup Z = Z$.

(49) $\{x\} \cup X \neq \emptyset$.

(50) $\{x,y\} \cup X \neq \emptyset$.

(51) If $X \cap \{x\} = \{x\}$, then $x \in X$.

(52) If $x \in X$, then $X \cap \{x\} = \{x\}$.

(53) If $x \in Z$ and $y \in Z$, then $\{x,y\} \cap Z = \{x,y\}$.

(54) If $\{x\}$ misses X, then $x \notin X$.

(55) If $\{x,y\}$ misses Z, then $x \notin Z$.

(56) If $x \notin X$, then $\{x\}$ misses X.

(57) If $x \notin Z$ and $y \notin Z$, then $\{x,y\}$ misses Z.

(58) $\{x\}$ misses X or $\{x\} \cap X = \{x\}$.

(59) If $\{x,y\} \cap X = \{x\}$, then $y \notin X$ or $x = y$.

(60) If $x \in X$ and if $y \notin X$ or $x = y$, then $\{x,y\} \cap X = \{x\}$.

(63)[5] If $\{x,y\} \cap X = \{x,y\}$, then $x \in X$.

(64) $z \in X \setminus \{x\}$ iff $z \in X$ and $z \neq x$.

(65) $X \setminus \{x\} = X$ iff $x \notin X$.

(66) If $X \setminus \{x\} = \emptyset$, then $X = \emptyset$ or $X = \{x\}$.

(67) $\{x\} \setminus X = \{x\}$ iff $x \notin X$.

(68) $\{x\} \setminus X = \emptyset$ iff $x \in X$.

(69) $\{x\} \setminus X = \emptyset$ or $\{x\} \setminus X = \{x\}$.

(70) $\{x,y\} \setminus X = \{x\}$ iff $x \notin X$ but $y \in X$ or $x = y$.

(72)[6] $\{x,y\} \setminus X = \{x,y\}$ iff $x \notin X$ and $y \notin X$.

(73) $\{x,y\} \setminus X = \emptyset$ iff $x \in X$ and $y \in X$.

(74) $\{x,y\} \setminus X = \emptyset$ or $\{x,y\} \setminus X = \{x\}$ or $\{x,y\} \setminus X = \{y\}$ or $\{x,y\} \setminus X = \{x,y\}$.

(75) $X \setminus \{x,y\} = \emptyset$ iff $X = \emptyset$ or $X = \{x\}$ or $X = \{y\}$ or $X = \{x,y\}$.

(79)[7] If $A \subseteq B$, then $2^A \subseteq 2^B$.

(80) $\{A\} \subseteq 2^A$.

(81) $2^A \cup 2^B \subseteq 2^{A \cup B}$.

---

[5] The propositions (61) and (62) have been removed.
[6] The proposition (71) has been removed.
[7] The propositions (76)–(78) have been removed.

(82) If $2^{A} \cup 2^{B} = 2^{A \cup B}$, then A and B are $\subseteq$-comparable.

(83) $2^{A \cap B} = 2^{A} \cap 2^{B}$.

(84) $2^{A \setminus B} \subseteq \{\emptyset\} \cup (2^{A} \setminus 2^{B})$.

(86)[8] $2^{A \setminus B} \cup 2^{B \setminus A} \subseteq 2^{A \dot{-} B}$.

(92)[9] If $X \in A$, then $X \subseteq \bigcup A$.

(93) $\bigcup \{X, Y\} = X \cup Y$.

(94) If for every X such that $X \in A$ holds $X \subseteq Z$, then $\bigcup A \subseteq Z$.

(95) If $A \subseteq B$, then $\bigcup A \subseteq \bigcup B$.

(96) $\bigcup (A \cup B) = \bigcup A \cup \bigcup B$.

(97) $\bigcup (A \cap B) \subseteq \bigcup A \cap \bigcup B$.

(98) If for every X such that $X \in A$ holds X misses B, then $\bigcup A$ misses B.

(99) $\bigcup (2^{A}) = A$.

(100) $A \subseteq 2^{\bigcup A}$.

(101) If for all X, Y such that $X \neq Y$ and $X \in A \cup B$ and $Y \in A \cup B$ holds X misses Y, then $\bigcup (A \cap B) = \bigcup A \cap \bigcup B$.

(102) If $z \in [: X, Y :]$, then there exist x, y such that $\langle x, y \rangle = z$.

(103) If $A \subseteq [: X, Y :]$ and $z \in A$, then there exist x, y such that $x \in X$ and $y \in Y$ and $z = \langle x, y \rangle$.

(104) If $z \in [: X_1, Y_1 :] \cap [: X_2, Y_2 :]$, then there exist x, y such that $z = \langle x, y \rangle$ and $x \in X_1 \cap X_2$ and $y \in Y_1 \cap Y_2$.

(105) $[: X, Y :] \subseteq 2^{2^{X \cup Y}}$.

(106) $\langle x, y \rangle \in [: X, Y :]$ iff $x \in X$ and $y \in Y$.

(107) If $\langle x, y \rangle \in [: X, Y :]$, then $\langle y, x \rangle \in [: Y, X :]$.

(108) If for all x, y holds $\langle x, y \rangle \in [: X_1, Y_1 :]$ iff $\langle x, y \rangle \in [: X_2, Y_2 :]$, then $[: X_1, Y_1 :] = [: X_2, Y_2 :]$.

(109) If $A \subseteq [: X, Y :]$ and for all x, y such that $\langle x, y \rangle \in A$ holds $\langle x, y \rangle \in B$, then $A \subseteq B$.

(110) If $A \subseteq [: X_1, Y_1 :]$ and $B \subseteq [: X_2, Y_2 :]$ and for all x, y holds $\langle x, y \rangle \in A$ iff $\langle x, y \rangle \in B$, then $A = B$.

(111) If for every z such that $z \in A$ there exist x, y such that $z = \langle x, y \rangle$ and for all x, y such that $\langle x, y \rangle \in A$ holds $\langle x, y \rangle \in B$, then $A \subseteq B$.

(112) Suppose that

(i) for every z such that $z \in A$ there exist x, y such that $z = \langle x, y \rangle$,

(ii) for every z such that $z \in B$ there exist x, y such that $z = \langle x, y \rangle$, and

(iii) for all x, y holds $\langle x, y \rangle \in A$ iff $\langle x, y \rangle \in B$.

Then $A = B$.

(113) $[: X, Y :] = \emptyset$ iff $X = \emptyset$ or $Y = \emptyset$.

(114) If $X \neq \emptyset$ and $Y \neq \emptyset$ and $[: X, Y :] = [: Y, X :]$, then $X = Y$.

(115) If $[: X, X :] = [: Y, Y :]$, then $X = Y$.

---

[8] The proposition (85) has been removed.

[9] The propositions (87)–(91) have been removed.

(116) If $X \subseteq [:X, X:]$, then $X = \emptyset$.

(117) If $Z \neq \emptyset$ and if $[:X, Z:] \subseteq [:Y, Z:]$ or $[:Z, X:] \subseteq [:Z, Y:]$, then $X \subseteq Y$.

(118) If $X \subseteq Y$, then $[:X, Z:] \subseteq [:Y, Z:]$ and $[:Z, X:] \subseteq [:Z, Y:]$.

(119) If $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$, then $[:X_1, X_2:] \subseteq [:Y_1, Y_2:]$.

(120) $[:X \cup Y, Z:] = [:X, Z:] \cup [:Y, Z:]$ and $[:Z, X \cup Y:] = [:Z, X:] \cup [:Z, Y:]$.

(121) $[:X_1 \cup X_2, Y_1 \cup Y_2:] = [:X_1, Y_1:] \cup [:X_1, Y_2:] \cup [:X_2, Y_1:] \cup [:X_2, Y_2:]$.

(122) $[:X \cap Y, Z:] = [:X, Z:] \cap [:Y, Z:]$ and $[:Z, X \cap Y:] = [:Z, X:] \cap [:Z, Y:]$.

(123) $[:X_1 \cap X_2, Y_1 \cap Y_2:] = [:X_1, Y_1:] \cap [:X_2, Y_2:]$.

(124) If $A \subseteq X$ and $B \subseteq Y$, then $[:A, Y:] \cap [:X, B:] = [:A, B:]$.

(125) $[:X \setminus Y, Z:] = [:X, Z:] \setminus [:Y, Z:]$ and $[:Z, X \setminus Y:] = [:Z, X:] \setminus [:Z, Y:]$.

(126) $[:X_1, X_2:] \setminus [:Y_1, Y_2:] = [:X_1 \setminus Y_1, X_2:] \cup [:X_1, X_2 \setminus Y_2:]$.

(127) If X_1 misses X_2 or Y_1 misses Y_2, then $[:X_1, Y_1:]$ misses $[:X_2, Y_2:]$.

(128) $\langle x, y \rangle \in [:\{z\}, Y:]$ iff $x = z$ and $y \in Y$.

(129) $\langle x, y \rangle \in [:X, \{z\}:]$ iff $x \in X$ and $y = z$.

(130) If $X \neq \emptyset$, then $[:\{x\}, X:] \neq \emptyset$ and $[:X, \{x\}:] \neq \emptyset$.

(131) If $x \neq y$, then $[:\{x\}, X:]$ misses $[:\{y\}, Y:]$ and $[:X, \{x\}:]$ misses $[:Y, \{y\}:]$.

(132) $[:\{x,y\}, X:] = [:\{x\}, X:] \cup [:\{y\}, X:]$ and $[:X, \{x,y\}:] = [:X, \{x\}:] \cup [:X, \{y\}:]$.

(134)[10] If $X_1 \neq \emptyset$ and $Y_1 \neq \emptyset$ and $[:X_1, Y_1:] = [:X_2, Y_2:]$, then $X_1 = X_2$ and $Y_1 = Y_2$.

(135) If $X \subseteq [:X, Y:]$ or $X \subseteq [:Y, X:]$, then $X = \emptyset$.

(136) There exists M such that

(i) $N \in M$,

(ii) for all X, Y such that $X \in M$ and $Y \subseteq X$ holds $Y \in M$,

(iii) for every X such that $X \in M$ holds $2^X \in M$, and

(iv) for every X such that $X \subseteq M$ holds $X \approx M$ or $X \in M$.

REFERENCES

[1] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

Received February 1, 1989

Published January 2, 2004

[10] The proposition (133) has been removed.