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The Reflection Theorem

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Summary. The goal is show that the reflection theorem holds. The theorem is as usual in the Morse-Kelley theory of classes (MK). That theory works with universal class which consists of all sets and every class is a subclass of it. In this paper (and in another Mizar articles) we work in Tarski-Grothendieck (TG) theory (see [15]) which ensures the existence of sets that have properties like universal class (i.e. this theory is stronger than MK). The sets are introduced in [13] and some concepts of MK are modeled. The concepts are: the class *On* of all ordinal numbers belonging to the universe, subclasses, transfinite sequences of non-empty elements of universe, etc. The reflection theorem states that if A_ξ is an increasing and continuous transfinite sequence of non-empty sets and class $A = \bigcup_{\xi \in On} A_\xi$, then for every formula H there is a strictly increasing continuous mapping $F : On \to On$ such that if $\varkappa$ is a critical number of F (i.e. $F(\varkappa) = \varkappa > 0$) and $f \in A_\varkappa^{\mathbf{VAR}}$, then $A, f \models H \equiv A_\varkappa, f \models H$. The proof is based on [11]. Besides, in the article it is shown that every universal class is a model of ZF set theory if ω (the first infinite ordinal number) belongs to it. Some propositions concerning ordinal numbers and sequences of them are also present.

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The articles [15], [14], [17], [16], [18], [9], [10], [3], [4], [5], [1], [12], [8], [13], [2], [6], and [7] provide the notation and terminology for this paper.

In this paper W is a universal class, H is a ZF-formula, and x, X are sets.

We now state several propositions:

(2)[1] $W \models$ the axiom of pairs.

(3) $W \models$ the axiom of unions.

(4) If $\omega \in W$, then $W \models$ the axiom of infinity.

(5) $W \models$ the axiom of power sets.

(6) For every H such that $\{x_0, x_1, x_2\}$ misses Free H holds $W \models$ the axiom of substitution for H.

(7) If $\omega \in W$, then W is a model of ZF.

For simplicity, we adopt the following rules: F is a function, A, B, C are ordinal numbers, a, b are ordinals of W, p_1 is a transfinite sequence of ordinals of W, and H is a ZF-formula.

Let us consider A, B. Let us observe that $A \subseteq B$ if and only if:

(Def. 1) For every C such that $C \in A$ holds $C \in B$.

[1] The proposition (1) has been removed.

In this article we present several logical schemes. The scheme *ALFA* deals with a non empty set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists F such that $\operatorname{dom} F = \mathcal{A}$ and for every element d of $\mathcal{A}$ there exists A such that $A = F(d)$ and $\mathcal{P}[d,A]$ and for every B such that $\mathcal{P}[d,B]$ holds $A \subseteq B$

provided the parameters satisfy the following condition:

- For every element d of $\mathcal{A}$ there exists A such that $\mathcal{P}[d,A]$.

The scheme *ALFA'Universe* deals with a universal class $\mathcal{A}$, a non empty set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

There exists F such that

(i) $\operatorname{dom} F = \mathcal{B}$, and

(ii) for every element d of $\mathcal{B}$ there exists an ordinal a of $\mathcal{A}$ such that $a = F(d)$ and $\mathcal{P}[d,a]$ and for every ordinal b of $\mathcal{A}$ such that $\mathcal{P}[d,b]$ holds $a \subseteq b$

provided the following condition is satisfied:

- For every element d of $\mathcal{B}$ there exists an ordinal a of $\mathcal{A}$ such that $\mathcal{P}[d,a]$.

Next we state the proposition

(8) x is an ordinal of W iff $x \in \operatorname{On} W$.

In the sequel p_2 is a sequence of ordinal numbers.

Now we present three schemes. The scheme *OrdSeqOfUnivEx* deals with a universal class $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a transfinite sequence p_1 of ordinals of $\mathcal{A}$ such that for every ordinal a of $\mathcal{A}$ holds $\mathcal{P}[a, p_1(a)]$

provided the following conditions are met:

- For all ordinals a, b_1, b_2 of $\mathcal{A}$ such that $\mathcal{P}[a,b_1]$ and $\mathcal{P}[a,b_2]$ holds $b_1 = b_2$, and
- For every ordinal a of $\mathcal{A}$ there exists an ordinal b of $\mathcal{A}$ such that $\mathcal{P}[a,b]$.

The scheme *UOS Exist* deals with a universal class $\mathcal{A}$, an ordinal $\mathcal{B}$ of $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding an ordinal of $\mathcal{A}$, and a binary functor $\mathcal{G}$ yielding an ordinal of $\mathcal{A}$, and states that:

There exists a transfinite sequence p_1 of ordinals of $\mathcal{A}$ such that

(i) $p_1(\mathbf{0}_{\mathcal{A}}) = \mathcal{B}$,

(ii) for every ordinal a of $\mathcal{A}$ holds $p_1(\operatorname{succ} a) = \mathcal{F}(a, p_1(a))$, and

(iii) for every ordinal a of $\mathcal{A}$ such that $a \neq \mathbf{0}_{\mathcal{A}}$ and a is a limit ordinal number holds $p_1(a) = \mathcal{G}(a, p_1 \restriction a)$

for all values of the parameters.

The scheme *Universe Ind* deals with a universal class $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every ordinal a of $\mathcal{A}$ holds $\mathcal{P}[a]$

provided the following requirements are met:

- $\mathcal{P}[\mathbf{0}_{\mathcal{A}}]$,
- For every ordinal a of $\mathcal{A}$ such that $\mathcal{P}[a]$ holds $\mathcal{P}[\operatorname{succ} a]$, and
- Let a be an ordinal of $\mathcal{A}$. Suppose $a \neq \mathbf{0}_{\mathcal{A}}$ and a is a limit ordinal number and for every ordinal b of $\mathcal{A}$ such that $b \in a$ holds $\mathcal{P}[b]$. Then $\mathcal{P}[a]$.

Let f be a function, let W be a universal class, and let a be an ordinal of W. The functor $\bigcup_a f$ yielding a set is defined as follows:

(Def. 2) $\bigcup_a f = \bigcup(W \restriction (f \restriction \mathbf{R}_a))$.

Next we state several propositions:

(10)[2] For every transfinite sequence L and for every A holds $L \restriction \mathbf{R}_A$ is a transfinite sequence.

(11) For every sequence L of ordinal numbers and for every A holds $L \restriction \mathbf{R}_A$ is a sequence of ordinal numbers.

(12) $\bigcup p_2$ is an ordinal number.

(13) $\bigcup(X \restriction p_2)$ is an ordinal number.

[2] The proposition (9) has been removed.

(14) $\mathrm{On}(\mathbf{R}_A) = A$.

(15) $p_2 \restriction \mathbf{R}_A = p_2 \restriction A$.

Let p_1 be a sequence of ordinal numbers, let W be a universal class, and let a be an ordinal of W. Then $\bigcup_a p_1$ is an ordinal of W.

The following proposition is true

(17)[3] For every transfinite sequence p_1 of ordinals of W holds $\bigcup_a p_1 = \bigcup (p_1 \restriction a)$ and $\bigcup_a (p_1 \restriction a) = \bigcup (p_1 \restriction a)$.

Let W be a universal class and let a, b be ordinals of W. Then $a \cup b$ is an ordinal of W.

Let us consider W. Observe that there exists an element of W which is non empty.

Let us consider W. A subclass of W is a non empty subset of W.

Let us consider W and let I_1 be a transfinite sequence of elements of W. We say that I_1 is non empty set yielding if and only if:

(Def. 5)[4] $\mathrm{dom}\, I_1 = \mathrm{On}\, W$.

Let us consider W. Observe that there exists a transfinite sequence of elements of W which is non empty set yielding and non-empty.

Let us consider W. A transfinite sequence of non empty sets from W is a non-empty non empty set yielding transfinite sequence of elements of W.

Let us consider W and let L be a transfinite sequence of non empty sets from W. Then $\bigcup L$ is a subclass of W. Let us consider a. Then $L(a)$ is a non empty element of W.

In the sequel L is a transfinite sequence of non empty sets from W and f is a function from VAR into $L(a)$.

Next we state several propositions:

(23)[5] $a \in \mathrm{dom}\, L$.

(24) $L(a) \subseteq \bigcup L$.

(25) $\mathbb{N} \approx \mathrm{VAR}$.

(27)[6] $\sup X \subseteq \mathrm{succ} \bigcup \mathrm{On}\, X$.

(28) If $X \in W$, then $\sup X \in W$.

(29) Suppose that

(i) $\omega \in W$,

(ii) for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$, and

(iii) for every a such that $a \neq \emptyset$ and a is a limit ordinal number holds $L(a) = \bigcup (L \restriction a)$.

Let given H. Then there exists p_1 such that

(iv) p_1 is increasing and continuous, and

(v) for every a such that $p_1(a) = a$ and $\emptyset \neq a$ and for every f holds $\bigcup L, (\bigcup L)[f] \models H$ iff $L(a), f \models H$.

[3] The proposition (16) has been removed.
[4] The definitions (Def. 3) and (Def. 4) have been removed.
[5] The propositions (18)–(22) have been removed.
[6] The proposition (26) has been removed.

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