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Models and Satisfiability

Grzegorz Bancerek
Warsaw University
Białystok

Summary. The article includes schemes of defining by structural induction, and definitions and theorems related to: the set of variables which have free occurrences in a ZF-formula, the set of all valuations of variables in a model, the set of all valuations which satisfy a ZF-formula in a model, the satisfiability of a ZF-formula in a model by a valuation, the validity of a ZF-formula in a model, the axioms of ZF-language, the model of the ZF set theory.

MML Identifier: `ZF_MODEL`.

WWW: `http://mizar.org/JFM/Vol1/zf_model.html`

The articles [7], [6], [5], [8], [9], [3], [1], [4], and [2] provide the notation and terminology for this paper.

For simplicity, we use the following convention: H, H' are ZF-formulae, x, y, z are variables, a, b, c are sets, and A, X are sets.

In this article we present several logical schemes. The scheme *ZFsch ex* deals with a binary functor $\mathcal{F}$ yielding a set, a binary functor $\mathcal{G}$ yielding a set, a unary functor $\mathcal{H}$ yielding a set, a binary functor $\mathcal{I}$ yielding a set, a binary functor $\mathcal{J}$ yielding a set, and a ZF-formula $\mathcal{A}$, and states that:

There exist a, A such that
(i) for all x, y holds $\langle x{=}y, \mathcal{F}(x,y)\rangle \in A$ and $\langle x\varepsilon y, \mathcal{G}(x,y)\rangle \in A$,
(ii) $\langle \mathcal{A}, a\rangle \in A$, and
(iii) for all H, a such that $\langle H, a\rangle \in A$ holds if H is an equality, then $a = \mathcal{F}(\mathrm{Var}_1(H), \mathrm{Var}_2(H))$ and if H is a membership, then $a = \mathcal{G}(\mathrm{Var}_1(H), \mathrm{Var}_2(H))$ and if H is negative, then there exists b such that $a = \mathcal{H}(b)$ and $\langle \mathrm{Arg}(H), b\rangle \in A$ and if H is conjunctive, then there exist b, c such that $a = \mathcal{I}(b,c)$ and $\langle \mathrm{LeftArg}(H), b\rangle \in A$ and $\langle \mathrm{RightArg}(H), c\rangle \in A$ and if H is universal, then there exists b such that $a = \mathcal{J}(\mathrm{Bound}(H), b)$ and $\langle \mathrm{Scope}(H), b\rangle \in A$

for all values of the parameters.

The scheme *ZFsch uniq* deals with a binary functor $\mathcal{F}$ yielding a set, a binary functor $\mathcal{G}$ yielding a set, a unary functor $\mathcal{H}$ yielding a set, a binary functor $\mathcal{I}$ yielding a set, a binary functor $\mathcal{J}$ yielding a set, a ZF-formula $\mathcal{A}$, a set $\mathcal{B}$, and a set $\mathcal{C}$, and states that:

$\mathcal{B} = \mathcal{C}$

provided the parameters have the following properties:

- There exists A such that
(i) for all x, y holds $\langle x{=}y, \mathcal{F}(x,y)\rangle \in A$ and $\langle x\varepsilon y, \mathcal{G}(x,y)\rangle \in A$,
(ii) $\langle \mathcal{A}, \mathcal{B}\rangle \in A$, and
(iii) for all H, a such that $\langle H, a\rangle \in A$ holds if H is an equality, then $a = \mathcal{F}(\mathrm{Var}_1(H), \mathrm{Var}_2(H))$ and if H is a membership, then $a = \mathcal{G}(\mathrm{Var}_1(H), \mathrm{Var}_2(H))$ and if H is negative, then there exists b such that $a = \mathcal{H}(b)$ and $\langle \mathrm{Arg}(H), b\rangle \in A$ and if H is conjunctive, then

there exist b, c such that $a = \mathcal{I}(b,c)$ and $\langle \mathrm{LeftArg}(H), b\rangle \in A$ and $\langle \mathrm{RightArg}(H), c\rangle \in A$ and if H is universal, then there exists b such that $a = \mathcal{J}(\mathrm{Bound}(H), b)$ and $\langle \mathrm{Scope}(H), b\rangle \in A$,

and

- There exists A such that
  (i) for all x, y holds $\langle x{=}y,\ \mathcal{F}(x,y)\rangle \in A$ and $\langle x\varepsilon y,\ \mathcal{G}(x,y)\rangle \in A$,
  (ii) $\langle \mathcal{A}, \mathcal{C}\rangle \in A$, and
  (iii) for all H, a such that $\langle H, a\rangle \in A$ holds if H is an equality, then $a = \mathcal{F}(\mathrm{Var}_1(H), \mathrm{Var}_2(H))$ and if H is a membership, then $a = \mathcal{G}(\mathrm{Var}_1(H), \mathrm{Var}_2(H))$ and if H is negative, then there exists b such that $a = \mathcal{H}(b)$ and $\langle \mathrm{Arg}(H), b\rangle \in A$ and if H is conjunctive, then there exist b, c such that $a = \mathcal{I}(b,c)$ and $\langle \mathrm{LeftArg}(H), b\rangle \in A$ and $\langle \mathrm{RightArg}(H), c\rangle \in A$ and if H is universal, then there exists b such that $a = \mathcal{J}(\mathrm{Bound}(H), b)$ and $\langle \mathrm{Scope}(H), b\rangle \in A$.

The scheme *ZFsch result* deals with a binary functor $\mathcal{F}$ yielding a set, a binary functor $\mathcal{G}$ yielding a set, a unary functor $\mathcal{H}$ yielding a set, a binary functor $\mathcal{I}$ yielding a set, a binary functor $\mathcal{J}$ yielding a set, a ZF-formula $\mathcal{A}$, and a unary functor $\mathcal{K}$ yielding a set, and states that:

(i) If $\mathcal{A}$ is an equality, then $\mathcal{K}(\mathcal{A}) = \mathcal{F}(\mathrm{Var}_1(\mathcal{A}), \mathrm{Var}_2(\mathcal{A}))$,
(ii) if $\mathcal{A}$ is a membership, then $\mathcal{K}(\mathcal{A}) = \mathcal{G}(\mathrm{Var}_1(\mathcal{A}), \mathrm{Var}_2(\mathcal{A}))$,
(iii) if $\mathcal{A}$ is negative, then $\mathcal{K}(\mathcal{A}) = \mathcal{H}(\mathcal{K}(\mathrm{Arg}(\mathcal{A})))$,
(iv) if $\mathcal{A}$ is conjunctive, then for all a, b such that $a = \mathcal{K}(\mathrm{LeftArg}(\mathcal{A}))$ and $b = \mathcal{K}(\mathrm{RightArg}(\mathcal{A}))$ holds $\mathcal{K}(\mathcal{A}) = \mathcal{I}(a,b)$, and
(v) if $\mathcal{A}$ is universal, then $\mathcal{K}(\mathcal{A}) = \mathcal{J}(\mathrm{Bound}(\mathcal{A}), \mathcal{K}(\mathrm{Scope}(\mathcal{A})))$

provided the following requirement is met:

- Let given H', a. Then $a = \mathcal{K}(H')$ if and only if there exists A such that for all x, y holds $\langle x{=}y,\ \mathcal{F}(x,y)\rangle \in A$ and $\langle x\varepsilon y,\ \mathcal{G}(x,y)\rangle \in A$ and $\langle H', a\rangle \in A$ and for all H, a such that $\langle H, a\rangle \in A$ holds if H is an equality, then $a = \mathcal{F}(\mathrm{Var}_1(H), \mathrm{Var}_2(H))$ and if H is a membership, then $a = \mathcal{G}(\mathrm{Var}_1(H), \mathrm{Var}_2(H))$ and if H is negative, then there exists b such that $a = \mathcal{H}(b)$ and $\langle \mathrm{Arg}(H), b\rangle \in A$ and if H is conjunctive, then there exist b, c such that $a = \mathcal{I}(b,c)$ and $\langle \mathrm{LeftArg}(H), b\rangle \in A$ and $\langle \mathrm{RightArg}(H), c\rangle \in A$ and if H is universal, then there exists b such that $a = \mathcal{J}(\mathrm{Bound}(H), b)$ and $\langle \mathrm{Scope}(H), b\rangle \in A$.

The scheme *ZFsch property* deals with a binary functor $\mathcal{F}$ yielding a set, a binary functor $\mathcal{G}$ yielding a set, a unary functor $\mathcal{H}$ yielding a set, a binary functor $\mathcal{I}$ yielding a set, a binary functor $\mathcal{J}$ yielding a set, a unary functor $\mathcal{K}$ yielding a set, a ZF-formula $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{K}(\mathcal{A})]$

provided the following conditions are met:

- Let given H', a. Then $a = \mathcal{K}(H')$ if and only if there exists A such that for all x, y holds $\langle x{=}y,\ \mathcal{F}(x,y)\rangle \in A$ and $\langle x\varepsilon y,\ \mathcal{G}(x,y)\rangle \in A$ and $\langle H', a\rangle \in A$ and for all H, a such that $\langle H, a\rangle \in A$ holds if H is an equality, then $a = \mathcal{F}(\mathrm{Var}_1(H), \mathrm{Var}_2(H))$ and if H is a membership, then $a = \mathcal{G}(\mathrm{Var}_1(H), \mathrm{Var}_2(H))$ and if H is negative, then there exists b such that $a = \mathcal{H}(b)$ and $\langle \mathrm{Arg}(H), b\rangle \in A$ and if H is conjunctive, then there exist b, c such that $a = \mathcal{I}(b,c)$ and $\langle \mathrm{LeftArg}(H), b\rangle \in A$ and $\langle \mathrm{RightArg}(H), c\rangle \in A$ and if H is universal, then there exists b such that $a = \mathcal{J}(\mathrm{Bound}(H), b)$ and $\langle \mathrm{Scope}(H), b\rangle \in A$,
- For all x, y holds $\mathcal{P}[\mathcal{F}(x,y)]$ and $\mathcal{P}[\mathcal{G}(x,y)]$,
- For every a such that $\mathcal{P}[a]$ holds $\mathcal{P}[\mathcal{H}(a)]$,
- For all a, b such that $\mathcal{P}[a]$ and $\mathcal{P}[b]$ holds $\mathcal{P}[\mathcal{I}(a,b)]$, and
- For all a, x such that $\mathcal{P}[a]$ holds $\mathcal{P}[\mathcal{J}(x,a)]$.

Let us consider H. The functor $\mathrm{Free}\,H$ yields a set and is defined by the condition (Def. 1).

(Def. 1) There exists A such that

(i) for all x, y holds $\langle x{=}y,\ \{x,y\}\rangle \in A$ and $\langle x\varepsilon y,\ \{x,y\}\rangle \in A$,

(ii) $\langle H, \mathrm{Free}\,H\rangle \in A$, and

(iii) for all H', a such that $\langle H', a\rangle \in A$ holds if H' is an equality, then $a = \{\mathrm{Var}_1(H'), \mathrm{Var}_2(H')\}$ and if H' is a membership, then $a = \{\mathrm{Var}_1(H'), \mathrm{Var}_2(H')\}$ and if H' is negative, then there exists b such that $a = b$ and $\langle \mathrm{Arg}(H'), b\rangle \in A$ and if H' is conjunctive, then there exist b, c such that $a = \bigcup\{b, c\}$ and $\langle \mathrm{LeftArg}(H'), b\rangle \in A$ and $\langle \mathrm{RightArg}(H'), c\rangle \in A$ and if H' is universal, then there exists b such that $a = \bigcup\{b\} \setminus \{\mathrm{Bound}(H')\}$ and $\langle \mathrm{Scope}(H'), b\rangle \in A$.

Let us consider H. Then $\mathrm{Free}\,H$ is a subset of VAR.
We now state the proposition

(1) Let given H. Then

(i) if H is an equality, then $\mathrm{Free}\,H = \{\mathrm{Var}_1(H), \mathrm{Var}_2(H)\}$,

(ii) if H is a membership, then $\mathrm{Free}\,H = \{\mathrm{Var}_1(H), \mathrm{Var}_2(H)\}$,

(iii) if H is negative, then $\mathrm{Free}\,H = \mathrm{Free}\,\mathrm{Arg}(H)$,

(iv) if H is conjunctive, then $\mathrm{Free}\,H = \mathrm{Free}\,\mathrm{LeftArg}(H) \cup \mathrm{Free}\,\mathrm{RightArg}(H)$, and

(v) if H is universal, then $\mathrm{Free}\,H = \mathrm{Free}\,\mathrm{Scope}(H) \setminus \{\mathrm{Bound}(H)\}$.

Let D be a non empty set. The functor $\mathrm{VAL}\,D$ yielding a set is defined as follows:

(Def. 2) $a \in \mathrm{VAL}\,D$ iff a is a function from VAR into D.

Let D be a non empty set. Note that $\mathrm{VAL}\,D$ is non empty.
We adopt the following convention: E denotes a non empty set, f, g denote functions from VAR into E, and v_1, v_2, v_3, v_4, v_5 denote elements of $\mathrm{VAL}\,E$.
Let us consider H, E. The functor $\mathrm{St}_E(H)$ yielding a set is defined by the condition (Def. 3).

(Def. 3) There exists A such that

(i) for all x, y holds $\langle x{=}y, \{v_1 : \bigwedge_f\ (f = v_1 \Rightarrow f(x) = f(y))\}\rangle \in A$ and $\langle x \varepsilon y, \{v_2 : \bigwedge_f\ (f = v_2 \Rightarrow f(x) \in f(y))\}\rangle \in A$,

(ii) $\langle H, \mathrm{St}_E(H)\rangle \in A$, and

(iii) for all H', a such that $\langle H', a\rangle \in A$ holds if H' is an equality, then $a = \{v_3 : \bigwedge_f\ (f = v_3 \Rightarrow f(\mathrm{Var}_1(H')) = f(\mathrm{Var}_2(H')))\}$ and if H' is a membership, then $a = \{v_4 : \bigwedge_f\ (f = v_4 \Rightarrow f(\mathrm{Var}_1(H')) \in f(\mathrm{Var}_2(H')))\}$ and if H' is negative, then there exists b such that $a = \mathrm{VAL}\,E \setminus \bigcup\{b\}$ and $\langle \mathrm{Arg}(H'), b\rangle \in A$ and if H' is conjunctive, then there exist b, c such that $a = \bigcup\{b\} \cap \bigcup\{c\}$ and $\langle \mathrm{LeftArg}(H'), b\rangle \in A$ and $\langle \mathrm{RightArg}(H'), c\rangle \in A$ and if H' is universal, then there exists b such that $a = \{v_5 : \bigwedge_{X,f}\ (X = b \wedge f = v_5 \Rightarrow f \in X \wedge \bigwedge_g\ (\bigwedge_y\ (g(y) \neq f(y) \Rightarrow \mathrm{Bound}(H') = y) \Rightarrow g \in X))\}$ and $\langle \mathrm{Scope}(H'), b\rangle \in A$.

Let us consider H, E. Then $\mathrm{St}_E(H)$ is a subset of $\mathrm{VAL}\,E$.
The following propositions are true:

(2) For all x, y, f holds $f(x) = f(y)$ iff $f \in \mathrm{St}_E(x{=}y)$.

(3) For all x, y, f holds $f(x) \in f(y)$ iff $f \in \mathrm{St}_E(x \varepsilon y)$.

(4) For all H, f holds $f \notin \mathrm{St}_E(H)$ iff $f \in \mathrm{St}_E(\neg H)$.

(5) For all H, H', f holds $f \in \mathrm{St}_E(H)$ and $f \in \mathrm{St}_E(H')$ iff $f \in \mathrm{St}_E(H \wedge H')$.

(6) Let given x, H, f. Then $f \in \mathrm{St}_E(H)$ and for every g such that for every y such that $g(y) \neq f(y)$ holds $x = y$ holds $g \in \mathrm{St}_E(H)$ if and only if $f \in \mathrm{St}_E(\forall_x H)$.

(7) If H is an equality, then for every f holds $f(\mathrm{Var}_1(H)) = f(\mathrm{Var}_2(H))$ iff $f \in \mathrm{St}_E(H)$.

(8) If H is a membership, then for every f holds $f(\mathrm{Var}_1(H)) \in f(\mathrm{Var}_2(H))$ iff $f \in \mathrm{St}_E(H)$.

(9) If H is negative, then for every f holds $f \notin \mathrm{St}_E(\mathrm{Arg}(H))$ iff $f \in \mathrm{St}_E(H)$.

(10) If H is conjunctive, then for every f holds $f \in \mathrm{St}_E(\mathrm{LeftArg}(H))$ and $f \in \mathrm{St}_E(\mathrm{RightArg}(H))$ iff $f \in \mathrm{St}_E(H)$.

(11) Suppose H is universal. Let given f. Then $f \in \mathrm{St}_E(\mathrm{Scope}(H))$ and for every g such that for every y such that $g(y) \neq f(y)$ holds $\mathrm{Bound}(H) = y$ holds $g \in \mathrm{St}_E(\mathrm{Scope}(H))$ if and only if $f \in \mathrm{St}_E(H)$.

Let D be a non empty set, let f be a function from VAR into D, and let us consider H. The predicate $D, f \models H$ is defined as follows:

(Def. 4) $f \in \mathrm{St}_D(H)$.

The following propositions are true:

(12) For all E, f, x, y holds $E, f \models x{=}y$ iff $f(x) = f(y)$.

(13) For all E, f, x, y holds $E, f \models x\varepsilon y$ iff $f(x) \in f(y)$.

(14) For all E, f, H holds $E, f \models H$ iff $E, f \not\models \neg H$.

(15) For all E, f, H, H' holds $E, f \models H \wedge H'$ iff $E, f \models H$ and $E, f \models H'$.

(16) For all E, f, H, x holds $E, f \models \forall_x H$ iff for every g such that for every y such that $g(y) \neq f(y)$ holds $x = y$ holds $E, g \models H$.

(17) For all E, f, H, H' holds $E, f \models H \vee H'$ iff $E, f \models H$ or $E, f \models H'$.

(18) For all E, f, H, H' holds $E, f \models H \Rightarrow H'$ iff if $E, f \models H$, then $E, f \models H'$.

(19) For all E, f, H, H' holds $E, f \models H \Leftrightarrow H'$ iff $E, f \models H$ iff $E, f \models H'$.

(20) For all E, f, H, x holds $E, f \models \exists_x H$ iff there exists g such that for every y such that $g(y) \neq f(y)$ holds $x = y$ and $E, g \models H$.

(21) For all E, f, x and for every element e of E there exists g such that $g(x) = e$ and for every z such that $z \neq x$ holds $g(z) = f(z)$.

(22) $E, f \models \forall_{x,y} H$ iff for every g such that for every z such that $g(z) \neq f(z)$ holds $x = z$ or $y = z$ holds $E, g \models H$.

(23) $E, f \models \exists_{x,y} H$ iff there exists g such that for every z such that $g(z) \neq f(z)$ holds $x = z$ or $y = z$ and $E, g \models H$.

Let us consider E, H. The predicate $E \models H$ is defined as follows:

(Def. 5) For every f holds $E, f \models H$.

Next we state the proposition

(25)[1] $E \models \forall_x H$ iff $E \models H$.

The ZF-formula the axiom of extensionality is defined as follows:

(Def. 6) The axiom of extensionality $= \forall_{\mathrm{x}_0,\mathrm{x}_1}(\forall_{\mathrm{x}_2}(\mathrm{x}_2\varepsilon(\mathrm{x}_0) \Leftrightarrow \mathrm{x}_2\varepsilon(\mathrm{x}_1)) \Rightarrow \mathrm{x}_0{=}(\mathrm{x}_1))$.

The ZF-formula the axiom of pairs is defined as follows:

(Def. 7) The axiom of pairs $= \forall_{\mathrm{x}_0,\mathrm{x}_1}\exists_{\mathrm{x}_2}\forall_{\mathrm{x}_3}(\mathrm{x}_3\varepsilon(\mathrm{x}_2) \Leftrightarrow \mathrm{x}_3{=}(\mathrm{x}_0) \vee \mathrm{x}_3{=}(\mathrm{x}_1))$.

The ZF-formula the axiom of unions is defined by:

(Def. 8) The axiom of unions $= \forall_{\mathrm{x}_0}\exists_{\mathrm{x}_1}\forall_{\mathrm{x}_2}(\mathrm{x}_2\varepsilon(\mathrm{x}_1) \Leftrightarrow \exists_{\mathrm{x}_3}(\mathrm{x}_2\varepsilon(\mathrm{x}_3) \wedge \mathrm{x}_3\varepsilon(\mathrm{x}_0)))$.

The ZF-formula the axiom of infinity is defined by:

[1] The proposition (24) has been removed.

(Def. 9) The axiom of infinity $= \exists_{x_0,x_1}(x_1 \varepsilon (x_0) \wedge \forall_{x_2}(x_2 \varepsilon (x_0) \Rightarrow \exists_{x_3}(x_3 \varepsilon (x_0) \wedge \neg x_3 {=} (x_2) \wedge \forall_{x_4}(x_4 \varepsilon (x_2) \Rightarrow x_4 \varepsilon (x_3))))))$.

The ZF-formula the axiom of power sets is defined as follows:

(Def. 10) The axiom of power sets $= \forall_{x_0} \exists_{x_1} \forall_{x_2}(x_2 \varepsilon (x_1) \Leftrightarrow \forall_{x_3}(x_3 \varepsilon (x_2) \Rightarrow x_3 \varepsilon (x_0)))$.

Let H be a ZF-formula. The axiom of substitution for H yielding a ZF-formula is defined as follows:

(Def. 11) The axiom of substitution for $H = \forall_{x_3} \exists_{x_0} \forall_{x_4}(H \Leftrightarrow x_4 {=} (x_0)) \Rightarrow \forall_{x_1} \exists_{x_2} \forall_{x_4}(x_4 \varepsilon (x_2) \Leftrightarrow \exists_{x_3}(x_3 \varepsilon (x_1) \wedge H))$.

Let us consider E. We say that E is model of ZF if and only if the conditions (Def. 12) are satisfied.

(Def. 12)(i) E is transitive,

(ii) $E \models$ the axiom of pairs,

(iii) $E \models$ the axiom of unions,

(iv) $E \models$ the axiom of infinity,

(v) $E \models$ the axiom of power sets, and

(vi) for every H such that $\{x_0, x_1, x_2\}$ misses Free$H$ holds $E \models$ the axiom of substitution for H.

We introduce E is a model of ZF as a synonym of E is model of ZF.

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