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Mostowski's Fundamental Operations — Part I

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Summary. In the chapter II.4 of his book [11] A. Mostowski introduces what he calls fundamental operations:
$A_1(a,b) = \{\{\langle 0,x\rangle,\langle 1,y\rangle\} : x \in y \wedge x \in a \wedge y \in a\}$,
$A_2(a,b) = \{a,b\}$,
$A_3(a,b) = \bigcup a$,
$A_4(a,b) = \{\{\langle x,y\rangle\} : x \in a \wedge y \in b\}$,
$A_5(a,b) = \{x \cup y : x \in a \wedge y \in b\}$,
$A_6(a,b) = \{x \setminus y : x \in a \wedge y \in b\}$,
$A_7(a,b) = \{x \circ y : x \in a \wedge y \in b\}$.
He proves that if a non-void class is closed under these operations then it is predicatively closed. Then he formulates sufficient criteria for a class to be a model of ZF set theory (theorem 4.12).

The article includes the translation of this part of Mostowski's book. The fundamental operations are defined (to be precise not these operations, but the notions of closure of a class with respect to them). Some properties of classes closed under these operations are proved. At last it is proved that if a non-void class X is closed with respect to the operations $A_1 - A_7$ then $D_H(a) \in X$ for every a in X and every H being formula of ZF language ($D_H(a)$ consists of all finite sequences with terms belonging to a which satisfy H in a).

MML Identifier: `ZF_FUND1`.

WWW: `http://mizar.org/JFM/Vol2/zf_fund1.html`

The articles [13], [9], [15], [4], [5], [6], [1], [10], [16], [12], [2], [3], [7], [8], and [14] provide the notation and terminology for this paper.

For simplicity, we use the following convention: V is a universal class, a, b, x, y are elements of V, X is a subclass of V, o, p, q, r, s, u, A, B are sets, n is an element of ω, f_1 is a finite subset of ω, E is a non empty set, f is a function from VAR into E, v_1, v_2 are elements of VAR, and H, H' are ZF-formulae.

Let us consider A, B. The functor $A\,B$ yields a set and is defined as follows:

(Def. 1) $p \in A\,B$ iff there exist q, r, s such that $p = \langle q, s\rangle$ and $\langle q, r\rangle \in A$ and $\langle r, s\rangle \in B$.

Let us consider V, x, y. Then $x\,y$ is an element of V.

The function decode from ω into VAR is defined by:

(Def. 2) For every element p of ω holds $\mathrm{decode}(p) = \mathrm{x}_{\mathrm{card}\,p}$.

Let us consider v_1. The functor ${}^{v_1}x$ yielding a natural number is defined by:

(Def. 3) $\mathrm{x}_{({}^{v_1}x)} = v_1$.

Let A be a finite subset of VAR. The functor $\mathrm{code}(A)$ yields a finite subset of ω and is defined by:

(Def. 4) $\mathrm{code}(A) = (\mathrm{decode}^{-1})^{\circ}A$.

Let us consider H. Then $\mathrm{Free}\,H$ is a finite subset of VAR.

Let us consider n. Then $\{n\}$ is a finite subset of ω.

Let us consider v_1. Then $\{v_1\}$ is a finite subset of VAR. Let us consider v_2. Then $\{v_1, v_2\}$ is a finite subset of VAR.

Let us consider H, E. The functor $\mathrm{D}_E(H)$ yields a set and is defined by:

(Def. 5) $p \in \mathrm{D}_E(H)$ iff there exists f such that $p = (f \cdot \mathrm{decode}) \restriction \mathrm{code}(\mathrm{Free}\,H)$ and $f \in \mathrm{St}_E(H)$.

Let us consider V, X. We say that X is closed w.r.t. A1 if and only if:

(Def. 6) For every a such that $a \in X$ holds $\{\{\langle \mathbf{0}_V, x\rangle, \langle \mathbf{1}_V, y\rangle\} : x \in y \wedge x \in a \wedge y \in a\} \in X$.

We say that X is closed w.r.t. A2 if and only if:

(Def. 7) For all a, b such that $a \in X$ and $b \in X$ holds $\{a, b\} \in X$.

We say that X is closed w.r.t. A3 if and only if:

(Def. 8) For every a such that $a \in X$ holds $\bigcup a \in X$.

We say that X is closed w.r.t. A4 if and only if:

(Def. 9) For all a, b such that $a \in X$ and $b \in X$ holds $\{\{\langle x, y\rangle\} : x \in a \wedge y \in b\} \in X$.

We say that X is closed w.r.t. A5 if and only if:

(Def. 10) For all a, b such that $a \in X$ and $b \in X$ holds $\{x \cup y : x \in a \wedge y \in b\} \in X$.

We say that X is closed w.r.t. A6 if and only if:

(Def. 11) For all a, b such that $a \in X$ and $b \in X$ holds $\{x \setminus y : x \in a \wedge y \in b\} \in X$.

We say that X is closed w.r.t. A7 if and only if:

(Def. 12) For all a, b such that $a \in X$ and $b \in X$ holds $\{x\, y : x \in a \wedge y \in b\} \in X$.

Let us consider V, X. We say that X is closed w.r.t. A1-A7 if and only if the condition (Def. 13) is satisfied.

(Def. 13) X is closed w.r.t. A1, closed w.r.t. A2, closed w.r.t. A3, closed w.r.t. A4, closed w.r.t. A5, closed w.r.t. A6, and closed w.r.t. A7.

Next we state a number of propositions:

(1) $X \subseteq V$ and if $o \in X$, then o is an element of V and if $o \in A$ and $A \in X$, then o is an element of V.

(2) If X is closed w.r.t. A1-A7, then $o \in X$ iff $\{o\} \in X$ and if $A \in X$, then $\bigcup A \in X$.

(3) If X is closed w.r.t. A1-A7, then $\emptyset \in X$.

(4) If X is closed w.r.t. A1-A7 and $A \in X$ and $B \in X$, then $A \cup B \in X$ and $A \setminus B \in X$ and $A\, B \in X$.

(5) If X is closed w.r.t. A1-A7 and $A \in X$ and $B \in X$, then $A \cap B \in X$.

(6) If X is closed w.r.t. A1-A7 and $o \in X$ and $p \in X$, then $\{o, p\} \in X$ and $\langle o, p\rangle \in X$.

(7) If X is closed w.r.t. A1-A7, then $\omega \subseteq X$.

(8) If X is closed w.r.t. A1-A7, then $\omega^{f_1} \subseteq X$.

(9) If X is closed w.r.t. A1-A7 and $a \in X$, then $a^{f_1} \in X$.

(10) If X is closed w.r.t. A1-A7 and $a \in \omega^{f_1}$ and $b \in X$, then $\{a\, x : x \in b\} \in X$.

(11) If X is closed w.r.t. A1-A7 and $n \in f_1$ and $a \in X$ and $b \in X$ and $b \subseteq a^{f_1}$, then $\{x : x \in a^{f_1 \setminus \{n\}} \wedge \bigvee_u \{\langle n, u\rangle\} \cup x \in b\} \in X$.

(12) If X is closed w.r.t. A1-A7 and $n \notin f_1$ and $a \in X$ and $b \in X$ and $b \subseteq a^{f_1}$, then $\{\{\langle n, x\rangle\} \cup y : x \in a \wedge y \in b\} \in X$.

(13) If X is closed w.r.t. A1-A7 and B is finite and for every o such that $o \in B$ holds $o \in X$, then $B \in X$.

(14) If X is closed w.r.t. A1-A7 and $A \subseteq X$ and $y \in A^{f_1}$, then $y \in X$.

(15) If X is closed w.r.t. A1-A7 and $n \notin f_1$ and $a \in X$ and $a \subseteq X$ and $y \in a^{f_1}$, then $\{\{\langle n, x\rangle\} \cup y : x \in a\} \in X$.

(16) Suppose X is closed w.r.t. A1-A7 and $n \notin f_1$ and $a \in X$ and $a \subseteq X$ and $y \in a^{f_1}$ and $b \subseteq a^{f_1 \cup \{n\}}$ and $b \in X$. Then $\{x : x \in a \wedge \{\langle n, x\rangle\} \cup y \in b\} \in X$.

(17) If X is closed w.r.t. A1-A7 and $a \in X$, then $\{\{\langle \mathbf{0}_V, x\rangle, \langle \mathbf{1}_V, x\rangle\} : x \in a\} \in X$.

(18) If X is closed w.r.t. A1-A7 and $E \in X$, then for all v_1, v_2 holds $\mathrm{D}_E(v_1{=}v_2) \in X$ and $\mathrm{D}_E(v_1 \varepsilon v_2) \in X$.

(19) If X is closed w.r.t. A1-A7 and $E \in X$, then for every H such that $\mathrm{D}_E(H) \in X$ holds $\mathrm{D}_E(\neg H) \in X$.

(20) If X is closed w.r.t. A1-A7 and $E \in X$, then for all H, H' such that $\mathrm{D}_E(H) \in X$ and $\mathrm{D}_E(H') \in X$ holds $\mathrm{D}_E(H \wedge H') \in X$.

(21) If X is closed w.r.t. A1-A7 and $E \in X$, then for all H, v_1 such that $\mathrm{D}_E(H) \in X$ holds $\mathrm{D}_E(\forall_{v_1} H) \in X$.

(22) If X is closed w.r.t. A1-A7 and $E \in X$, then $\mathrm{D}_E(H) \in X$.

(23) If X is closed w.r.t. A1-A7, then $n \in X$ and $\mathbf{0}_V \in X$ and $\mathbf{1}_V \in X$.

(24) $\{\langle o, p\rangle, \langle p, p\rangle\} \{\langle p, q\rangle\} = \{\langle o, q\rangle, \langle p, q\rangle\}$.

(25) If $p \neq r$, then $\{\langle o, p\rangle, \langle q, r\rangle\} \{\langle p, s\rangle, \langle r, t\rangle\} = \{\langle o, s\rangle, \langle q, t\rangle\}$.

(27)[1] $\mathrm{code}(\{v_1\}) = \{{}^{v_1}x\}$ and $\mathrm{code}(\{v_1, v_2\}) = \{{}^{v_1}x, {}^{v_2}x\}$.

(28) For every function f holds $\mathrm{dom} f = \{o, q\}$ iff $f = \{\langle o, f(o)\rangle, \langle q, f(q)\rangle\}$.

(29) $\mathrm{dom}\,\mathrm{decode} = \omega$ and $\mathrm{rng}\,\mathrm{decode} = \mathrm{VAR}$ and decode is one-to-one and decode^{-1} is one-to-one and $\mathrm{dom}(\mathrm{decode}^{-1}) = \mathrm{VAR}$ and $\mathrm{rng}(\mathrm{decode}^{-1}) = \omega$.

(30) For every finite subset A of VAR holds $A \approx \mathrm{code}(A)$.

(31) For every element A of ω holds $A = {}^{\mathrm{x}_{\mathrm{card}\,A}}x$.

(32) $\mathrm{dom}((f \cdot \mathrm{decode}) \restriction f_1) = f_1$ and $\mathrm{rng}((f \cdot \mathrm{decode}) \restriction f_1) \subseteq E$ and $(f \cdot \mathrm{decode}) \restriction f_1 \in E^{f_1}$ and $\mathrm{dom}(f \cdot \mathrm{decode}) = \omega$ and $\mathrm{rng}(f \cdot \mathrm{decode}) \subseteq E$.

(33) $\mathrm{decode}({}^{v_1}x) = v_1$ and $\mathrm{decode}^{-1}(v_1) = {}^{v_1}x$ and $(f \cdot \mathrm{decode})({}^{v_1}x) = f(v_1)$.

(34) For every finite subset A of VAR holds $p \in \mathrm{code}(A)$ iff there exists v_1 such that $v_1 \in A$ and $p = {}^{v_1}x$.

(35) For all finite subsets A, B of VAR holds $\mathrm{code}(A \cup B) = \mathrm{code}(A) \cup \mathrm{code}(B)$ and $\mathrm{code}(A \setminus B) = \mathrm{code}(A) \setminus \mathrm{code}(B)$.

(36) If $v_1 \in \mathrm{Free}\,H$, then $((f \cdot \mathrm{decode}) \restriction \mathrm{code}(\mathrm{Free}\,H))({}^{v_1}x) = f(v_1)$.

(37) For all functions f, g from VAR into E such that $(f \cdot \mathrm{decode}) \restriction \mathrm{code}(\mathrm{Free}\,H) = (g \cdot \mathrm{decode}) \restriction \mathrm{code}(\mathrm{Free}\,H)$ and $f \in \mathrm{St}_E(H)$ holds $g \in \mathrm{St}_E(H)$.

(38) If $p \in E^{f_1}$, then there exists f such that $p = (f \cdot \mathrm{decode}) \restriction f_1$.

[1] The proposition (26) has been removed.

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