JOURNAL OF FORMALIZED MATHEMATICS
Volume 8, Released 1996, Published 2003
Inst. of Computer Science, Univ. of Białystok

Definitions and Properties of the Join and Meet of Subsets[1]

Artur Korniłowicz
Institute of Mathematics
Warsaw University
Białystok

Summary. This paper is the continuation of formalization of [4]. The definitions of meet and join of subsets of relational structures are introduced. The properties of these notions are proved.

MML Identifier: `YELLOW_4`.

WWW: `http://mizar.org/JFM/Vol8/yellow_4.html`

The articles [8], [10], [7], [1], [2], [9], [5], [3], and [6] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(1) Let L be a relational structure, X be a set, and a be an element of L. If $a \in X$ and sup X exists in L, then $a \leq \bigsqcup_L X$.

(2) Let L be a relational structure, X be a set, and a be an element of L. If $a \in X$ and inf X exists in L, then $\bigsqcap_L X \leq a$.

Let L be a relational structure and let A, B be subsets of L. We say that A is finer than B if and only if:

(Def. 1) For every element a of L such that $a \in A$ there exists an element b of L such that $b \in B$ and $a \leq b$.

We say that B is coarser than A if and only if:

(Def. 2) For every element b of L such that $b \in B$ there exists an element a of L such that $a \in A$ and $a \leq b$.

Let L be a non empty reflexive relational structure and let A, B be subsets of L. Let us note that the predicate A is finer than B is reflexive. Let us note that the predicate B is coarser than A is reflexive.

The following propositions are true:

(3) For every relational structure L and for every subset B of L holds $\emptyset_L$ is finer than B.

[1]This work was partially supported by Office of Naval Research Grant N00014-95-1-1336.

(4) Let L be a transitive relational structure and A, B, C be subsets of L. If A is finer than B and B is finer than C, then A is finer than C.

(5) For every relational structure L and for all subsets A, B of L such that B is finer than A and A is lower holds $B \subseteq A$.

(6) For every relational structure L and for every subset A of L holds $\emptyset_L$ is coarser than A.

(7) Let L be a transitive relational structure and A, B, C be subsets of L. If C is coarser than B and B is coarser than A, then C is coarser than A.

(8) Let L be a relational structure and A, B be subsets of L. If A is coarser than B and B is upper, then $A \subseteq B$.

2. The Join of Subsets

Let L be a non empty relational structure and let D_1, D_2 be subsets of L. The functor $D_1 \sqcup D_2$ yields a subset of L and is defined as follows:

(Def. 3) $D_1 \sqcup D_2 = \{x \sqcup y; x$ ranges over elements of L, y ranges over elements of L: $x \in D_1 \wedge y \in D_2\}$.

Let L be an antisymmetric relational structure with l.u.b.'s and let D_1, D_2 be subsets of L. Let us note that the functor $D_1 \sqcup D_2$ is commutative.

The following propositions are true:

(9) For every non empty relational structure L and for every subset X of L holds $X \sqcup \emptyset_L = \emptyset$.

(10) Let L be a non empty relational structure, X, Y be subsets of L, and x, y be elements of L. If $x \in X$ and $y \in Y$, then $x \sqcup y \in X \sqcup Y$.

(11) Let L be an antisymmetric relational structure with l.u.b.'s, A be a subset of L, and B be a non empty subset of L. Then A is finer than $A \sqcup B$.

(12) For every antisymmetric relational structure L with l.u.b.'s and for all subsets A, B of L holds $A \sqcup B$ is coarser than A.

(13) For every antisymmetric reflexive relational structure L with l.u.b.'s and for every subset A of L holds $A \subseteq A \sqcup A$.

(14) Let L be an antisymmetric transitive relational structure with l.u.b.'s and D_1, D_2, D_3 be subsets of L. Then $(D_1 \sqcup D_2) \sqcup D_3 = D_1 \sqcup (D_2 \sqcup D_3)$.

Let L be a non empty relational structure and let D_1, D_2 be non empty subsets of L. Observe that $D_1 \sqcup D_2$ is non empty.

Let L be a transitive antisymmetric relational structure with l.u.b.'s and let D_1, D_2 be directed subsets of L. Observe that $D_1 \sqcup D_2$ is directed.

Let L be a transitive antisymmetric relational structure with l.u.b.'s and let D_1, D_2 be filtered subsets of L. Observe that $D_1 \sqcup D_2$ is filtered.

Let L be a poset with l.u.b.'s and let D_1, D_2 be upper subsets of L. Observe that $D_1 \sqcup D_2$ is upper.

The following propositions are true:

(15) Let L be a non empty relational structure, Y be a subset of L, and x be an element of L. Then $\{x\} \sqcup Y = \{x \sqcup y; y$ ranges over elements of L: $y \in Y\}$.

(16) For every non empty relational structure L and for all subsets A, B, C of L holds $A \sqcup (B \cup C) = (A \sqcup B) \cup (A \sqcup C)$.

(17) Let L be an antisymmetric reflexive relational structure with l.u.b.'s and A, B, C be subsets of L. Then $A \cup (B \sqcup C) \subseteq (A \cup B) \sqcup (A \cup C)$.

(18) Let L be an antisymmetric relational structure with l.u.b.'s, A be an upper subset of L, and B, C be subsets of L. Then $(A \cup B) \sqcup (A \cup C) \subseteq A \cup (B \sqcup C)$.

(19) For every non empty relational structure L and for all elements x, y of L holds $\{x\} \sqcup \{y\} = \{x \sqcup y\}$.

(20) For every non empty relational structure L and for all elements x, y, z of L holds $\{x\} \sqcup \{y,z\} = \{x \sqcup y, x \sqcup z\}$.

(21) For every non empty relational structure L and for all subsets X_1, X_2, Y_1, Y_2 of L such that $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$ holds $X_1 \sqcup X_2 \subseteq Y_1 \sqcup Y_2$.

(22) Let L be a reflexive antisymmetric relational structure with l.u.b.'s, D be a subset of L, and x be an element of L. If $x \leq D$, then $\{x\} \sqcup D = D$.

(23) Let L be an antisymmetric relational structure with l.u.b.'s, D be a subset of L, and x be an element of L. Then $x \leq \{x\} \sqcup D$.

(24) Let L be a poset with l.u.b.'s, X be a subset of L, and x be an element of L. If $\inf \{x\} \sqcup X$ exists in L and $\inf X$ exists in L, then $x \sqcup \inf X \leq \inf(\{x\} \sqcup X)$.

(25) Let L be a complete transitive antisymmetric non empty relational structure, A be a subset of L, and B be a non empty subset of L. Then $A \leq \sup(A \sqcup B)$.

(26) Let L be a transitive antisymmetric relational structure with l.u.b.'s, a, b be elements of L, and A, B be subsets of L. If $a \leq A$ and $b \leq B$, then $a \sqcup b \leq A \sqcup B$.

(27) Let L be a transitive antisymmetric relational structure with l.u.b.'s, a, b be elements of L, and A, B be subsets of L. If $a \geq A$ and $b \geq B$, then $a \sqcup b \geq A \sqcup B$.

(28) For every complete non empty poset L and for all non empty subsets A, B of L holds $\sup(A \sqcup B) = \sup A \sqcup \sup B$.

(29) Let L be an antisymmetric relational structure with l.u.b.'s, X be a subset of L, and Y be a non empty subset of L. Then $X \subseteq {\downarrow}(X \sqcup Y)$.

(30) Let L be a poset with l.u.b.'s, x, y be elements of $\langle \mathrm{Ids}(L), \subseteq \rangle$, and X, Y be subsets of L. If $x = X$ and $y = Y$, then $x \sqcup y = {\downarrow}(X \sqcup Y)$.

(31) Let L be a non empty relational structure and D be a subset of $[:L, L:]$. Then $\bigcup\{X; X$ ranges over subsets of L: $\bigvee_{x:\text{element of } L} (X = \{x\} \sqcup \pi_2(D) \wedge x \in \pi_1(D))\} = \pi_1(D) \sqcup \pi_2(D)$.

(32) Let L be a transitive antisymmetric relational structure with l.u.b.'s and D_1, D_2 be subsets of L. Then ${\downarrow}({\downarrow}D_1 \sqcup {\downarrow}D_2) \subseteq {\downarrow}(D_1 \sqcup D_2)$.

(33) For every poset L with l.u.b.'s and for all subsets D_1, D_2 of L holds ${\downarrow}({\downarrow}D_1 \sqcup {\downarrow}D_2) = {\downarrow}(D_1 \sqcup D_2)$.

(34) Let L be a transitive antisymmetric relational structure with l.u.b.'s and D_1, D_2 be subsets of L. Then ${\uparrow}({\uparrow}D_1 \sqcup {\uparrow}D_2) \subseteq {\uparrow}(D_1 \sqcup D_2)$.

(35) For every poset L with l.u.b.'s and for all subsets D_1, D_2 of L holds ${\uparrow}({\uparrow}D_1 \sqcup {\uparrow}D_2) = {\uparrow}(D_1 \sqcup D_2)$.

3. The Meet of Subsets

Let L be a non empty relational structure and let D_1, D_2 be subsets of L. The functor $D_1 \sqcap D_2$ yielding a subset of L is defined by:

(Def. 4) $D_1 \sqcap D_2 = \{x \sqcap y; x$ ranges over elements of L, y ranges over elements of L: $x \in D_1 \wedge y \in D_2\}$.

Let L be an antisymmetric relational structure with g.l.b.'s and let D_1, D_2 be subsets of L. Let us notice that the functor $D_1 \sqcap D_2$ is commutative.

One can prove the following propositions:

(36) For every non empty relational structure L and for every subset X of L holds $X \sqcap \emptyset_L = \emptyset$.

(37) Let L be a non empty relational structure, X, Y be subsets of L, and x, y be elements of L. If $x \in X$ and $y \in Y$, then $x \sqcap y \in X \sqcap Y$.

(38) Let L be an antisymmetric relational structure with g.l.b.'s, A be a subset of L, and B be a non empty subset of L. Then A is coarser than $A \sqcap B$.

(39) For every antisymmetric relational structure L with g.l.b.'s and for all subsets A, B of L holds $A \sqcap B$ is finer than A.

(40) For every antisymmetric reflexive relational structure L with g.l.b.'s and for every subset A of L holds $A \subseteq A \sqcap A$.

(41) Let L be an antisymmetric transitive relational structure with g.l.b.'s and D_1, D_2, D_3 be subsets of L. Then $(D_1 \sqcap D_2) \sqcap D_3 = D_1 \sqcap (D_2 \sqcap D_3)$.

Let L be a non empty relational structure and let D_1, D_2 be non empty subsets of L. One can check that $D_1 \sqcap D_2$ is non empty.

Let L be a transitive antisymmetric relational structure with g.l.b.'s and let D_1, D_2 be directed subsets of L. One can check that $D_1 \sqcap D_2$ is directed.

Let L be a transitive antisymmetric relational structure with g.l.b.'s and let D_1, D_2 be filtered subsets of L. Note that $D_1 \sqcap D_2$ is filtered.

Let L be a semilattice and let D_1, D_2 be lower subsets of L. One can check that $D_1 \sqcap D_2$ is lower.

One can prove the following propositions:

(42) Let L be a non empty relational structure, Y be a subset of L, and x be an element of L. Then $\{x\} \sqcap Y = \{x \sqcap y; y$ ranges over elements of L: $y \in Y\}$.

(43) For every non empty relational structure L and for all subsets A, B, C of L holds $A \sqcap (B \cup C) = (A \sqcap B) \cup (A \sqcap C)$.

(44) Let L be an antisymmetric reflexive relational structure with g.l.b.'s and A, B, C be subsets of L. Then $A \cup (B \sqcap C) \subseteq (A \cup B) \sqcap (A \cup C)$.

(45) Let L be an antisymmetric relational structure with g.l.b.'s, A be a lower subset of L, and B, C be subsets of L. Then $(A \cup B) \sqcap (A \cup C) \subseteq A \cup (B \sqcap C)$.

(46) For every non empty relational structure L and for all elements x, y of L holds $\{x\} \sqcap \{y\} = \{x \sqcap y\}$.

(47) For every non empty relational structure L and for all elements x, y, z of L holds $\{x\} \sqcap \{y, z\} = \{x \sqcap y, x \sqcap z\}$.

(48) For every non empty relational structure L and for all subsets X_1, X_2, Y_1, Y_2 of L such that $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$ holds $X_1 \sqcap X_2 \subseteq Y_1 \sqcap Y_2$.

(49) For every antisymmetric reflexive relational structure L with g.l.b.'s and for all subsets A, B of L holds $A \cap B \subseteq A \sqcap B$.

(50) Let L be an antisymmetric reflexive relational structure with g.l.b.'s and A, B be lower subsets of L. Then $A \sqcap B = A \cap B$.

(51) Let L be a reflexive antisymmetric relational structure with g.l.b.'s, D be a subset of L, and x be an element of L. If $x \geq D$, then $\{x\} \sqcap D = D$.

(52) Let L be an antisymmetric relational structure with g.l.b.'s, D be a subset of L, and x be an element of L. Then $\{x\} \sqcap D \leq x$.

(53) Let L be a semilattice, X be a subset of L, and x be an element of L. If sup $\{x\} \sqcap X$ exists in L and sup X exists in L, then $\sup(\{x\} \sqcap X) \leq x \sqcap \sup X$.

(54) Let L be a complete transitive antisymmetric non empty relational structure, A be a subset of L, and B be a non empty subset of L. Then $A \geq \inf(A \sqcap B)$.

(55) Let L be a transitive antisymmetric relational structure with g.l.b.'s, a, b be elements of L, and A, B be subsets of L. If $a \leq A$ and $b \leq B$, then $a \sqcap b \leq A \sqcap B$.

(56) Let L be a transitive antisymmetric relational structure with g.l.b.'s, a, b be elements of L, and A, B be subsets of L. If $a \geq A$ and $b \geq B$, then $a \sqcap b \geq A \sqcap B$.

(57) For every complete non empty poset L and for all non empty subsets A, B of L holds $\inf(A \sqcap B) = \inf A \sqcap \inf B$.

(58) Let L be a semilattice, x, y be elements of $\langle \mathrm{Ids}(L), \subseteq \rangle$, and x_1, y_1 be subsets of L. If $x = x_1$ and $y = y_1$, then $x \sqcap y = x_1 \sqcap y_1$.

(59) Let L be an antisymmetric relational structure with g.l.b.'s, X be a subset of L, and Y be a non empty subset of L. Then $X \subseteq \uparrow(X \sqcap Y)$.

(60) Let L be a non empty relational structure and D be a subset of $[:L, L:]$. Then $\bigcup\{X; X$ ranges over subsets of L: $\bigvee_{x:\text{element of } L} (X = \{x\} \sqcap \pi_2(D) \ \wedge \ x \in \pi_1(D))\} = \pi_1(D) \sqcap \pi_2(D)$.

(61) Let L be a transitive antisymmetric relational structure with g.l.b.'s and D_1, D_2 be subsets of L. Then $\downarrow(\downarrow D_1 \sqcap \downarrow D_2) \subseteq \downarrow(D_1 \sqcap D_2)$.

(62) For every semilattice L and for all subsets D_1, D_2 of L holds $\downarrow(\downarrow D_1 \sqcap \downarrow D_2) = \downarrow(D_1 \sqcap D_2)$.

(63) Let L be a transitive antisymmetric relational structure with g.l.b.'s and D_1, D_2 be subsets of L. Then $\uparrow(\uparrow D_1 \sqcap \uparrow D_2) \subseteq \uparrow(D_1 \sqcap D_2)$.

(64) For every semilattice L and for all subsets D_1, D_2 of L holds $\uparrow(\uparrow D_1 \sqcap \uparrow D_2) = \uparrow(D_1 \sqcap D_2)$.

REFERENCES

[1] Grzegorz Bancerek. Complete lattices. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/lattice3.html.

[2] Grzegorz Bancerek. Bounds in posets and relational substructures. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/yellow_0.html.

[3] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/waybel_0.html.

[4] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[5] Adam Grabowski and Robert Milewski. Boolean posets, posets under inclusion and products of relational structures. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/yellow_1.html.

[6] Artur Korniłowicz. Cartesian products of relations and relational structures. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/yellow_3.html.

[7] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/pre_topc.html.

[8] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[9] Wojciech A. Trybulec. Partially ordered sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/orders_1.html.

[10] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

Received September 25, 1996

Published January 2, 2004