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Cartesian Products of Relations and Relational Structures[1]

Artur Korniłowicz
Institute of Mathematics
Warsaw University
Białystok

Summary. In this paper the definitions of cartesian products of relations and relational structures are introduced. Facts about these notions are proved. This work is the continuation of formalization of [8].

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The articles [9], [7], [12], [13], [15], [14], [5], [10], [6], [1], [11], [2], [3], and [4] provide the notation and terminology for this paper.

1. Preliminaries

In this article we present several logical schemes. The scheme *FraenkelA2* deals with a non empty set $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a set, and two binary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(s,t); s$ ranges over elements of $\mathcal{A}, t$ ranges over elements of $\mathcal{A} : \mathcal{P}[s,t]\}$ is a subset of $\mathcal{A}$

provided the following condition is satisfied:

- For every element s of $\mathcal{A}$ and for every element t of $\mathcal{A}$ holds $\mathcal{F}(s,t) \in \mathcal{A}$.

The scheme *ExtensionalityR* deals with binary relations $\mathcal{A}$, $\mathcal{B}$ and a binary predicate $\mathcal{P}$, and states that:

$\mathcal{A} = \mathcal{B}$

provided the parameters meet the following conditions:

- For all sets a, b holds $\langle a, b\rangle \in \mathcal{A}$ iff $\mathcal{P}[a,b]$, and
- For all sets a, b holds $\langle a, b\rangle \in \mathcal{B}$ iff $\mathcal{P}[a,b]$.

Let X be an empty set. Note that $\pi_1(X)$ is empty and $\pi_2(X)$ is empty.

Let X, Y be non empty sets and let D be a non empty subset of $[:X, Y:]$. Observe that $\pi_1(D)$ is non empty and $\pi_2(D)$ is non empty.

Let L be a relational structure and let X be an empty subset of L. One can verify that $\downarrow X$ is empty and $\uparrow X$ is empty.

Next we state several propositions:

(1) For all sets X, Y and for every subset D of $[:X, Y:]$ holds $D \subseteq [:\pi_1(D), \pi_2(D):]$.

(2) Let L be a transitive antisymmetric relational structure with g.l.b.'s and a, b, c, d be elements of L. If $a \leq c$ and $b \leq d$, then $a \sqcap b \leq c \sqcap d$.

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(3) Let L be a transitive antisymmetric relational structure with l.u.b.'s and a, b, c, d be elements of L. If $a \leq c$ and $b \leq d$, then $a \sqcup b \leq c \sqcup d$.

(4) Let L be a complete reflexive antisymmetric non empty relational structure, D be a subset of L, and x be an element of L. If $x \in D$, then $\sup D \sqcap x = x$.

(5) Let L be a complete reflexive antisymmetric non empty relational structure, D be a subset of L, and x be an element of L. If $x \in D$, then $\inf D \sqcup x = x$.

(6) For every relational structure L and for all subsets X, Y of L such that $X \subseteq Y$ holds $\downarrow X \subseteq \downarrow Y$.

(7) For every relational structure L and for all subsets X, Y of L such that $X \subseteq Y$ holds $\uparrow X \subseteq \uparrow Y$.

(8) Let S, T be posets with g.l.b.'s, f be a map from S into T, and x, y be elements of S. Then f preserves inf of $\{x, y\}$ if and only if $f(x \sqcap y) = f(x) \sqcap f(y)$.

(9) Let S, T be posets with l.u.b.'s, f be a map from S into T, and x, y be elements of S. Then f preserves sup of $\{x, y\}$ if and only if $f(x \sqcup y) = f(x) \sqcup f(y)$.

Now we present four schemes. The scheme *Inf Union* deals with a complete antisymmetric non empty relational structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

$\bigsqcap_{\mathcal{A}}\{\bigsqcap_{\mathcal{A}} X; X$ ranges over subsets of $\mathcal{A} : \mathcal{P}[X]\} \geq \bigsqcap_{\mathcal{A}} \bigcup\{X; X$ ranges over subsets of $\mathcal{A} : \mathcal{P}[X]\}$

for all values of the parameters.

The scheme *Inf of Infs* deals with a complete lattice $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

$\bigsqcap_{\mathcal{A}}\{\bigsqcap_{\mathcal{A}} X; X$ ranges over subsets of $\mathcal{A} : \mathcal{P}[X]\} = \bigsqcap_{\mathcal{A}} \bigcup\{X; X$ ranges over subsets of $\mathcal{A} : \mathcal{P}[X]\}$

for all values of the parameters.

The scheme *Sup Union* deals with a complete antisymmetric non empty relational structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

$\bigsqcup_{\mathcal{A}}\{\bigsqcup_{\mathcal{A}} X; X$ ranges over subsets of $\mathcal{A} : \mathcal{P}[X]\} \leq \bigsqcup_{\mathcal{A}} \bigcup\{X; X$ ranges over subsets of $\mathcal{A} : \mathcal{P}[X]\}$

for all values of the parameters.

The scheme *Sup of Sups* deals with a complete lattice $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

$\bigsqcup_{\mathcal{A}}\{\bigsqcup_{\mathcal{A}} X; X$ ranges over subsets of $\mathcal{A} : \mathcal{P}[X]\} = \bigsqcup_{\mathcal{A}} \bigcup\{X; X$ ranges over subsets of $\mathcal{A} : \mathcal{P}[X]\}$

for all values of the parameters.

2. Properties of Cartesian Products of Relational Structures

Let P, R be binary relations. The functor $P \times R$ yields a binary relation and is defined as follows:

(Def. 1) For all sets x, y holds $\langle x, y \rangle \in P \times R$ iff there exist sets p, q, s, t such that $x = \langle p, q \rangle$ and $y = \langle s, t \rangle$ and $\langle p, s \rangle \in P$ and $\langle q, t \rangle \in R$.

The following proposition is true

(10) Let P, R be binary relations and x be a set. Then $x \in P \times R$ if and only if the following conditions are satisfied:

(i) $\langle (x_{\mathbf{1}})_{\mathbf{1}}, (x_{\mathbf{2}})_{\mathbf{1}} \rangle \in P$,

(ii) $\langle (x_{\mathbf{1}})_{\mathbf{2}}, (x_{\mathbf{2}})_{\mathbf{2}} \rangle \in R$,

(iii) there exist sets a, b such that $x = \langle a, b \rangle$,

(iv) there exist sets c, d such that $x_{\mathbf{1}} = \langle c, d \rangle$, and

(v) there exist sets e, f such that $x_{\mathbf{2}} = \langle e, f \rangle$.

Let A, B, X, Y be sets, let P be a relation between A and B, and let R be a relation between X and Y. Then $P \times R$ is a relation between $[:A, X:]$ and $[:B, Y:]$.

Let X, Y be relational structures. The functor $[:X, Y:]$ yields a strict relational structure and is defined by the conditions (Def. 2).

(Def. 2)(i) The carrier of $[: X, Y :] = [:$ the carrier of X, the carrier of $Y :]$, and

(ii) the internal relation of $[: X, Y :] =$ (the internal relation of X) $\times$ (the internal relation of Y).

Let L_1, L_2 be relational structures and let D be a subset of $[: L_1, L_2 :]$. Then $\pi_1(D)$ is a subset of L_1. Then $\pi_2(D)$ is a subset of L_2.

Let S_1, S_2 be relational structures, let D_1 be a subset of S_1, and let D_2 be a subset of S_2. Then $[: D_1, D_2 :]$ is a subset of $[: S_1, S_2 :]$.

Let S_1, S_2 be non empty relational structures, let x be an element of S_1, and let y be an element of S_2. Then $\langle x, y \rangle$ is an element of $[: S_1, S_2 :]$.

Let L_1, L_2 be non empty relational structures and let x be an element of $[: L_1, L_2 :]$. Then $x_{\mathbf{1}}$ is an element of L_1. Then $x_{\mathbf{2}}$ is an element of L_2.

One can prove the following propositions:

(11) Let S_1, S_2 be non empty relational structures, a, c be elements of S_1, and b, d be elements of S_2. Then $a \leq c$ and $b \leq d$ if and only if $\langle a, b \rangle \leq \langle c, d \rangle$.

(12) Let S_1, S_2 be non empty relational structures and x, y be elements of $[: S_1, S_2 :]$. Then $x \leq y$ if and only if the following conditions are satisfied:

(i) $x_{\mathbf{1}} \leq y_{\mathbf{1}}$, and

(ii) $x_{\mathbf{2}} \leq y_{\mathbf{2}}$.

(13) Let A, B be relational structures, C be a non empty relational structure, and f, g be maps from $[: A, B :]$ into C. Suppose that for every element x of A and for every element y of B holds $f(\langle x, y \rangle) = g(\langle x, y \rangle)$. Then $f = g$.

Let X, Y be non empty relational structures. One can verify that $[: X, Y :]$ is non empty.

Let X, Y be reflexive relational structures. Observe that $[: X, Y :]$ is reflexive.

Let X, Y be antisymmetric relational structures. One can verify that $[: X, Y :]$ is antisymmetric.

Let X, Y be transitive relational structures. Observe that $[: X, Y :]$ is transitive.

Let X, Y be relational structures with l.u.b.'s. Observe that $[: X, Y :]$ has l.u.b.'s.

Let X, Y be relational structures with g.l.b.'s. Observe that $[: X, Y :]$ has g.l.b.'s.

We now state several propositions:

(14) For all relational structures X, Y such that $[: X, Y :]$ is non empty holds X is non empty and Y is non empty.

(15) For all non empty relational structures X, Y such that $[: X, Y :]$ is reflexive holds X is reflexive and Y is reflexive.

(16) Let X, Y be non empty reflexive relational structures. If $[: X, Y :]$ is antisymmetric, then X is antisymmetric and Y is antisymmetric.

(17) Let X, Y be non empty reflexive relational structures. If $[: X, Y :]$ is transitive, then X is transitive and Y is transitive.

(18) For all non empty reflexive relational structures X, Y such that $[: X, Y :]$ has l.u.b.'s holds X has l.u.b.'s and Y has l.u.b.'s.

(19) For all non empty reflexive relational structures X, Y such that $[: X, Y :]$ has g.l.b.'s holds X has g.l.b.'s and Y has g.l.b.'s.

Let S_1, S_2 be relational structures, let D_1 be a directed subset of S_1, and let D_2 be a directed subset of S_2. Then $[: D_1, D_2 :]$ is a directed subset of $[: S_1, S_2 :]$.

The following propositions are true:

(20) Let S_1, S_2 be non empty relational structures, D_1 be a non empty subset of S_1, and D_2 be a non empty subset of S_2. If $[: D_1, D_2 :]$ is directed, then D_1 is directed and D_2 is directed.

(21) For all non empty relational structures S_1, S_2 and for every non empty subset D of $[: S_1, S_2 :]$ holds $\pi_1(D)$ is non empty and $\pi_2(D)$ is non empty.

(22) Let S_1, S_2 be non empty reflexive relational structures and D be a non empty directed subset of $[: S_1, S_2 :]$. Then $\pi_1(D)$ is directed and $\pi_2(D)$ is directed.

Let S_1, S_2 be relational structures, let D_1 be a filtered subset of S_1, and let D_2 be a filtered subset of S_2. Then $[: D_1, D_2 :]$ is a filtered subset of $[: S_1, S_2 :]$.

The following two propositions are true:

(23) Let S_1, S_2 be non empty relational structures, D_1 be a non empty subset of S_1, and D_2 be a non empty subset of S_2. If $[: D_1, D_2 :]$ is filtered, then D_1 is filtered and D_2 is filtered.

(24) Let S_1, S_2 be non empty reflexive relational structures and D be a non empty filtered subset of $[: S_1, S_2 :]$. Then $\pi_1(D)$ is filtered and $\pi_2(D)$ is filtered.

Let S_1, S_2 be relational structures, let D_1 be an upper subset of S_1, and let D_2 be an upper subset of S_2. Then $[: D_1, D_2 :]$ is an upper subset of $[: S_1, S_2 :]$.

Next we state two propositions:

(25) Let S_1, S_2 be non empty reflexive relational structures, D_1 be a non empty subset of S_1, and D_2 be a non empty subset of S_2. If $[: D_1, D_2 :]$ is upper, then D_1 is upper and D_2 is upper.

(26) Let S_1, S_2 be non empty reflexive relational structures and D be a non empty upper subset of $[: S_1, S_2 :]$. Then $\pi_1(D)$ is upper and $\pi_2(D)$ is upper.

Let S_1, S_2 be relational structures, let D_1 be a lower subset of S_1, and let D_2 be a lower subset of S_2. Then $[: D_1, D_2 :]$ is a lower subset of $[: S_1, S_2 :]$.

One can prove the following two propositions:

(27) Let S_1, S_2 be non empty reflexive relational structures, D_1 be a non empty subset of S_1, and D_2 be a non empty subset of S_2. If $[: D_1, D_2 :]$ is lower, then D_1 is lower and D_2 is lower.

(28) Let S_1, S_2 be non empty reflexive relational structures and D be a non empty lower subset of $[: S_1, S_2 :]$. Then $\pi_1(D)$ is lower and $\pi_2(D)$ is lower.

Let R be a relational structure. We say that R is void if and only if:

(Def. 3) The internal relation of R is empty.

Let us mention that every relational structure which is empty is also void.

Let us mention that there exists a poset which is non void, non empty, and strict.

Let us mention that every relational structure which is non void is also non empty.

One can verify that every relational structure which is non empty and reflexive is also non void.

Let R be a non void relational structure. Note that the internal relation of R is non empty.

Next we state a number of propositions:

(29) Let S_1, S_2 be non empty relational structures, D_1 be a non empty subset of S_1, D_2 be a non empty subset of S_2, x be an element of S_1, and y be an element of S_2. If $\langle x, y \rangle \geq [: D_1, D_2 :]$, then $x \geq D_1$ and $y \geq D_2$.

(30) Let S_1, S_2 be non empty relational structures, D_1 be a subset of S_1, D_2 be a subset of S_2, x be an element of S_1, and y be an element of S_2. If $x \geq D_1$ and $y \geq D_2$, then $\langle x, y \rangle \geq [: D_1, D_2 :]$.

(31) Let S_1, S_2 be non empty relational structures, D be a subset of $[: S_1, S_2 :]$, x be an element of S_1, and y be an element of S_2. Then $\langle x, y \rangle \geq D$ if and only if $x \geq \pi_1(D)$ and $y \geq \pi_2(D)$.

(32) Let S_1, S_2 be non empty relational structures, D_1 be a non empty subset of S_1, D_2 be a non empty subset of S_2, x be an element of S_1, and y be an element of S_2. If $\langle x, y \rangle \leq [: D_1, D_2 :]$, then $x \leq D_1$ and $y \leq D_2$.

(33) Let S_1, S_2 be non empty relational structures, D_1 be a subset of S_1, D_2 be a subset of S_2, x be an element of S_1, and y be an element of S_2. If $x \leq D_1$ and $y \leq D_2$, then $\langle x, y \rangle \leq [: D_1, D_2 :]$.

(34) Let S_1, S_2 be non empty relational structures, D be a subset of $[: S_1, S_2 :]$, x be an element of S_1, and y be an element of S_2. Then $\langle x, y\rangle \leq D$ if and only if $x \leq \pi_1(D)$ and $y \leq \pi_2(D)$.

(35) Let S_1, S_2 be antisymmetric non empty relational structures, D_1 be a subset of S_1, D_2 be a subset of S_2, x be an element of S_1, and y be an element of S_2. Suppose sup D_1 exists in S_1 and sup D_2 exists in S_2 and for every element b of $[: S_1, S_2 :]$ such that $b \geq [: D_1, D_2 :]$ holds $\langle x, y\rangle \leq b$. Then for every element c of S_1 such that $c \geq D_1$ holds $x \leq c$ and for every element d of S_2 such that $d \geq D_2$ holds $y \leq d$.

(36) Let S_1, S_2 be antisymmetric non empty relational structures, D_1 be a subset of S_1, D_2 be a subset of S_2, x be an element of S_1, and y be an element of S_2. Suppose inf D_1 exists in S_1 and inf D_2 exists in S_2 and for every element b of $[: S_1, S_2 :]$ such that $b \leq [: D_1, D_2 :]$ holds $\langle x, y\rangle \geq b$. Then for every element c of S_1 such that $c \leq D_1$ holds $x \geq c$ and for every element d of S_2 such that $d \leq D_2$ holds $y \geq d$.

(37) Let S_1, S_2 be antisymmetric non empty relational structures, D_1 be a non empty subset of S_1, D_2 be a non empty subset of S_2, x be an element of S_1, and y be an element of S_2. Suppose for every element c of S_1 such that $c \geq D_1$ holds $x \leq c$ and for every element d of S_2 such that $d \geq D_2$ holds $y \leq d$. Let b be an element of $[: S_1, S_2 :]$. If $b \geq [: D_1, D_2 :]$, then $\langle x, y\rangle \leq b$.

(38) Let S_1, S_2 be antisymmetric non empty relational structures, D_1 be a non empty subset of S_1, D_2 be a non empty subset of S_2, x be an element of S_1, and y be an element of S_2. Suppose for every element c of S_1 such that $c \geq D_1$ holds $x \geq c$ and for every element d of S_2 such that $d \geq D_2$ holds $y \geq d$. Let b be an element of $[: S_1, S_2 :]$. If $b \geq [: D_1, D_2 :]$, then $\langle x, y\rangle \geq b$.

(39) Let S_1, S_2 be antisymmetric non empty relational structures, D_1 be a non empty subset of S_1, and D_2 be a non empty subset of S_2. Then sup D_1 exists in S_1 and sup D_2 exists in S_2 if and only if sup $[: D_1, D_2 :]$ exists in $[: S_1, S_2 :]$.

(40) Let S_1, S_2 be antisymmetric non empty relational structures, D_1 be a non empty subset of S_1, and D_2 be a non empty subset of S_2. Then inf D_1 exists in S_1 and inf D_2 exists in S_2 if and only if inf $[: D_1, D_2 :]$ exists in $[: S_1, S_2 :]$.

(41) Let S_1, S_2 be antisymmetric non empty relational structures and D be a subset of $[: S_1, S_2 :]$. Then sup $\pi_1(D)$ exists in S_1 and sup $\pi_2(D)$ exists in S_2 if and only if sup D exists in $[: S_1, S_2 :]$.

(42) Let S_1, S_2 be antisymmetric non empty relational structures and D be a subset of $[: S_1, S_2 :]$. Then inf $\pi_1(D)$ exists in S_1 and inf $\pi_2(D)$ exists in S_2 if and only if inf D exists in $[: S_1, S_2 :]$.

(43) Let S_1, S_2 be antisymmetric non empty relational structures, D_1 be a non empty subset of S_1, and D_2 be a non empty subset of S_2. If sup D_1 exists in S_1 and sup D_2 exists in S_2, then $\sup [: D_1, D_2 :] = \langle \sup D_1, \sup D_2\rangle$.

(44) Let S_1, S_2 be antisymmetric non empty relational structures, D_1 be a non empty subset of S_1, and D_2 be a non empty subset of S_2. If inf D_1 exists in S_1 and inf D_2 exists in S_2, then $\inf [: D_1, D_2 :] = \langle \inf D_1, \inf D_2\rangle$.

Let X, Y be complete antisymmetric non empty relational structures. Observe that $[: X, Y :]$ is complete.

We now state several propositions:

(45) Let X, Y be non empty lower-bounded antisymmetric relational structures. If $[: X, Y :]$ is complete, then X is complete and Y is complete.

(46) Let L_1, L_2 be antisymmetric non empty relational structures and D be a non empty subset of $[: L_1, L_2 :]$. If $[: L_1, L_2 :]$ is complete or sup D exists in $[: L_1, L_2 :]$, then $\sup D = \langle \sup \pi_1(D), \sup \pi_2(D)\rangle$.

(47) Let L_1, L_2 be antisymmetric non empty relational structures and D be a non empty subset of $[: L_1, L_2 :]$. If $[: L_1, L_2 :]$ is complete or inf D exists in $[: L_1, L_2 :]$, then $\inf D = \langle \inf \pi_1(D), \inf \pi_2(D)\rangle$.

(48) For all non empty reflexive relational structures S_1, S_2 and for every non empty directed subset D of $[: S_1, S_2 :]$ holds $[: \pi_1(D), \pi_2(D) :] \subseteq \downarrow D$.

(49) For all non empty reflexive relational structures S_1, S_2 and for every non empty filtered subset D of $[: S_1, S_2 :]$ holds $[: \pi_1(D), \pi_2(D) :] \subseteq \uparrow D$.

The scheme *Kappa2DS* deals with non empty relational structures $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a binary functor $\mathcal{F}$ yielding a set, and states that:

There exists a map f from $[: \mathcal{A}, \mathcal{B} :]$ into $\mathcal{C}$ such that for every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $f(\langle x, y \rangle) = \mathcal{F}(x,y)$

provided the following requirement is met:

- For every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $\mathcal{F}(x,y)$ is an element of $\mathcal{C}$.

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