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Components and Basis of Topological Spaces[1]

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Summary. This article contains many facts about components and basis of topological spaces.

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The articles [20], [10], [24], [16], [13], [26], [22], [25], [8], [9], [7], [6], [12], [23], [18], [11], [1], [2], [17], [19], [21], [3], [4], [5], [14], and [15] provide the notation and terminology for this paper.

1. PRELIMINARIES

The scheme *SeqLambda1C* deals with a natural number $\mathcal{A}$, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, a unary functor $\mathcal{G}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

> There exists a finite sequence p of elements of $\mathcal{B}$ such that $\operatorname{len} p = \mathcal{A}$ and for every natural number i such that $i \in \operatorname{Seg} \mathcal{A}$ holds if $\mathcal{P}[i]$, then $p(i) = \mathcal{F}(i)$ and if not $\mathcal{P}[i]$, then $p(i) = \mathcal{G}(i)$

provided the following condition is satisfied:

- For every natural number i such that $i \in \operatorname{Seg} \mathcal{A}$ holds if $\mathcal{P}[i]$, then $\mathcal{F}(i) \in \mathcal{B}$ and if not $\mathcal{P}[i]$, then $\mathcal{G}(i) \in \mathcal{B}$.

Let X be a set and let p be a finite sequence of elements of 2^X. Then $\operatorname{rng} p$ is a family of subsets of X.

Let us observe that *Boolean* is finite.

The following two propositions are true:

(2)[1] For every natural number i and for every finite set D holds D^i is finite.

(3) For every finite set T holds every family of subsets of T is finite.

Let T be a finite set. One can check that every family of subsets of T is finite.

Let T be a finite 1-sorted structure. One can verify that every family of subsets of T is finite.

We now state the proposition

(4) For every non trivial set X and for every element x of X there exists a set y such that $y \in X$ and $x \neq y$.

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[1] The proposition (1) has been removed.

2. Components

Let X be a set, let p be a finite sequence of elements of 2^X, and let q be a finite sequence of elements of *Boolean*. The functor MergeSequence(p,q) yields a finite sequence of elements of 2^X and is defined by:

(Def. 1) len MergeSequence$(p,q) = \operatorname{len} p$ and for every natural number i such that $i \in \operatorname{dom} p$ holds $(\text{MergeSequence}(p,q))(i) = (q(i) = \mathit{true} \to p(i), X \setminus p(i))$.

Next we state a number of propositions:

(5) Let X be a set, p be a finite sequence of elements of 2^X, and q be a finite sequence of elements of *Boolean*. Then dom MergeSequence$(p,q) = \operatorname{dom} p$.

(6) Let X be a set, p be a finite sequence of elements of 2^X, q be a finite sequence of elements of *Boolean*, and i be a natural number. If $q(i) = \mathit{true}$, then $(\text{MergeSequence}(p,q))(i) = p(i)$.

(7) Let X be a set, p be a finite sequence of elements of 2^X, q be a finite sequence of elements of *Boolean*, and i be a natural number. If $i \in \operatorname{dom} p$ and $q(i) = \mathit{false}$, then $(\text{MergeSequence}(p,q))(i) = X \setminus p(i)$.

(8) For every set X and for every finite sequence q of elements of *Boolean* holds len MergeSequence$(\varepsilon_{2^X}, q) = 0$.

(9) For every set X and for every finite sequence q of elements of *Boolean* holds MergeSequence$(\varepsilon_{2^X}, q) = \varepsilon_{2^X}$.

(10) For every set X and for every element x of 2^X and for every finite sequence q of elements of *Boolean* holds len MergeSequence$(\langle x \rangle, q) = 1$.

(11) Let X be a set, x be an element of 2^X, and q be a finite sequence of elements of *Boolean*. Then

(i) if $q(1) = \mathit{true}$, then $(\text{MergeSequence}(\langle x \rangle, q))(1) = x$, and

(ii) if $q(1) = \mathit{false}$, then $(\text{MergeSequence}(\langle x \rangle, q))(1) = X \setminus x$.

(12) For every set X and for all elements x, y of 2^X and for every finite sequence q of elements of *Boolean* holds len MergeSequence$(\langle x, y \rangle, q) = 2$.

(13) Let X be a set, x, y be elements of 2^X, and q be a finite sequence of elements of *Boolean*. Then

(i) if $q(1) = \mathit{true}$, then $(\text{MergeSequence}(\langle x, y \rangle, q))(1) = x$,

(ii) if $q(1) = \mathit{false}$, then $(\text{MergeSequence}(\langle x, y \rangle, q))(1) = X \setminus x$,

(iii) if $q(2) = \mathit{true}$, then $(\text{MergeSequence}(\langle x, y \rangle, q))(2) = y$, and

(iv) if $q(2) = \mathit{false}$, then $(\text{MergeSequence}(\langle x, y \rangle, q))(2) = X \setminus y$.

(14) Let X be a set, x, y, z be elements of 2^X, and q be a finite sequence of elements of *Boolean*. Then len MergeSequence$(\langle x, y, z \rangle, q) = 3$.

(15) Let X be a set, x, y, z be elements of 2^X, and q be a finite sequence of elements of *Boolean*. Then

(i) if $q(1) = \mathit{true}$, then $(\text{MergeSequence}(\langle x, y, z \rangle, q))(1) = x$,

(ii) if $q(1) = \mathit{false}$, then $(\text{MergeSequence}(\langle x, y, z \rangle, q))(1) = X \setminus x$,

(iii) if $q(2) = \mathit{true}$, then $(\text{MergeSequence}(\langle x, y, z \rangle, q))(2) = y$,

(iv) if $q(2) = \mathit{false}$, then $(\text{MergeSequence}(\langle x, y, z \rangle, q))(2) = X \setminus y$,

(v) if $q(3) = \mathit{true}$, then $(\text{MergeSequence}(\langle x, y, z \rangle, q))(3) = z$, and

(vi) if $q(3) = \mathit{false}$, then $(\text{MergeSequence}(\langle x, y, z \rangle, q))(3) = X \setminus z$.

(16) Let X be a set and p be a finite sequence of elements of 2^X. Then $\{\mathrm{Intersect}(\mathrm{rng}\,\mathrm{MergeSequence}(p,q)); q$ ranges over finite sequences of elements of *Boolean*: $\mathrm{len}\,q = \mathrm{len}\,p\}$ is a family of subsets of X.

One can check that every finite sequence of elements of *Boolean* is boolean-valued.

Let X be a set and let Y be a finite family of subsets of X. The functor Components$Y$ yielding a family of subsets of X is defined by the condition (Def. 2).

(Def. 2) There exists a finite sequence p of elements of 2^X such that $\mathrm{len}\,p = \mathrm{card}\,Y$ and $\mathrm{rng}\,p = Y$ and $\mathrm{Components}\,Y = \{\mathrm{Intersect}(\mathrm{rng}\,\mathrm{MergeSequence}(p,q)); q$ ranges over finite sequences of elements of *Boolean*: $\mathrm{len}\,q = \mathrm{len}\,p\}$.

Let X be a set and let Y be a finite family of subsets of X. Observe that Components$Y$ is finite.

Next we state four propositions:

(17) For every set X and for every empty family Y of subsets of X holds $\mathrm{Components}\,Y = \{X\}$.

(18) For every set X and for all finite families Y, Z of subsets of X such that $Z \subseteq Y$ holds Components$Y$ is finer than Components$Z$.

(19) For every set X and for every finite family Y of subsets of X holds $\bigcup \mathrm{Components}\,Y = X$.

(20) Let X be a set, Y be a finite family of subsets of X, and A, B be sets. If $A \in \mathrm{Components}\,Y$ and $B \in \mathrm{Components}\,Y$ and $A \neq B$, then A misses B.

Let X be a set and let Y be a finite family of subsets of X. We say that Y is in general position if and only if:

(Def. 3) $\emptyset \notin \mathrm{Components}\,Y$.

We now state three propositions:

(21) Let X be a set and Y, Z be finite families of subsets of X. If Z is in general position and $Y \subseteq Z$, then Y is in general position.

(22) For every non empty set X holds every empty family of subsets of X is in general position.

(23) Let X be a non empty set and Y be a finite family of subsets of X. If Y is in general position, then Components$Y$ is a partition of X.

3. About Basis of Topological Spaces

The following propositions are true:

(24) For every non empty relational structure L holds Ω_L is infs-closed and sups-closed.

(25) For every non empty relational structure L holds Ω_L has bottom and top.

Let L be a non empty relational structure. Observe that Ω_L is infs-closed and sups-closed and has bottom and top.

We now state several propositions:

(26) For every continuous sup-semilattice L holds Ω_L is a CLbasis of L.

(27) For every up-complete non empty poset L such that L is finite holds the carrier of $L =$ the carrier of CompactSublatt(L).

(28) For every lower-bounded sup-semilattice L and for every subset B of L such that B is infinite holds $\overline{\overline{B}} = \overline{\overline{\mathrm{finsups}(B)}}$.

(29) For every T_0 non empty topological space T holds $\overline{\overline{\text{the carrier of } T}} \subseteq \overline{\overline{\text{the topology of } T}}$.

(30) Let T be a topological structure and X be a subset of T. Suppose X is open. Let B be a finite family of subsets of T. Suppose B is a basis of T. Let Y be a set. If $Y \in \mathrm{Components}B$, then X misses Y or $Y \subseteq X$.

(31) For every T_0 topological space T such that T is infinite holds every basis of T is infinite.

(32) Let T be a non empty topological space. Suppose T is finite. Let B be a basis of T and x be an element of T. Then $\bigcap\{A; A$ ranges over elements of the topology of T: $x \in A\} \in B$.

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