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Irreducible and Prime Elements[1]

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Summary. In the paper open and order generating subsets are defined. Irreducible and prime elements are also defined. The article includes definitions and facts presented in [16, pp. 68–72].

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The articles [22], [13], [26], [24], [15], [27], [1], [28], [9], [25], [21], [2], [4], [11], [12], [10], [3], [23], [20], [5], [18], [6], [14], [30], [19], [7], [17], [29], and [8] provide the notation and terminology for this paper.

1. Preliminaries

In this paper L is a lattice and l is an element of L.

The scheme *NonUniqExD1* deals with a non empty relational structure $\mathcal{A}$, a subset $\mathcal{B}$ of $\mathcal{A}$, a non empty subset $\mathcal{C}$ of $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a function f from $\mathcal{B}$ into $\mathcal{C}$ such that for every element e of $\mathcal{A}$ if $e \in \mathcal{B}$, then there exists an element u of $\mathcal{A}$ such that $u \in \mathcal{C}$ and $u = f(e)$ and $\mathcal{P}[e,u]$

provided the parameters meet the following requirement:

- For every element e of $\mathcal{A}$ such that $e \in \mathcal{B}$ there exists an element u of $\mathcal{A}$ such that $u \in \mathcal{C}$ and $\mathcal{P}[e,u]$.

Let L be a lattice, let A be a non empty subset of L, let f be a function from A into A, and let n be an element of $\mathbb{N}$. Then f^n is a function from A into A.

Let L be a lattice, let C, D be non empty subsets of L, let f be a function from C into D, and let c be an element of C. Then $f(c)$ is an element of L.

Let L be a non empty poset. One can check that every chain of L is filtered and directed.

One can verify that there exists a lattice which is strict, continuous, distributive, and lower-bounded.

Next we state three propositions:

(1) Let S, T be semilattices and f be a map from S into T. Then f is meet-preserving if and only if for all elements x, y of S holds $f(x \sqcap y) = f(x) \sqcap f(y)$.

(2) Let S, T be sup-semilattices and f be a map from S into T. Then f is join-preserving if and only if for all elements x, y of S holds $f(x \sqcup y) = f(x) \sqcup f(y)$.

(3) Let S, T be lattices and f be a map from S into T. Suppose T is distributive and f is meet-preserving, join-preserving, and one-to-one. Then S is distributive.

Let S, T be complete lattices. Note that there exists a map from S into T which is sups-preserving.

Next we state the proposition

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(4) Let S, T be complete lattices and f be a sups-preserving map from S into T. Suppose T is meet-continuous and f is meet-preserving and one-to-one. Then S is meet-continuous.

2. Open sets

Let L be a non empty reflexive relational structure and let X be a subset of L. We say that X is open if and only if:

(Def. 1) For every element x of L such that $x \in X$ there exists an element y of L such that $y \in X$ and $y \ll x$.

We now state two propositions:

(5) Let L be an up-complete lattice and X be an upper subset of L. Then X is open if and only if for every element x of L such that $x \in X$ holds $\twoheaddownarrow x$ meets X.

(6) Let L be an up-complete lattice and X be an upper subset of L. Then X is open if and only if $X = \bigcup\{\twoheaduparrow x; x$ ranges over elements of L: $x \in X\}$.

Let L be an up-complete lower-bounded lattice. One can verify that there exists a filter of L which is open.

We now state three propositions:

(7) For every lower-bounded continuous lattice L and for every element x of L holds $\twoheaduparrow x$ is open.

(8) Let L be a lower-bounded continuous lattice and x, y be elements of L. If $x \ll y$, then there exists an open filter F of L such that $y \in F$ and $F \subseteq \twoheaduparrow x$.

(9) Let L be a complete lattice, X be an open upper subset of L, and x be an element of L. If $x \in X^{\rm c}$, then there exists an element m of L such that $x \leq m$ and m is maximal in $X^{\rm c}$.

3. Irreducible elements

Let G be a non empty relational structure and let g be an element of G. We say that g is meet-irreducible if and only if:

(Def. 2) For all elements x, y of G such that $g = x \sqcap y$ holds $x = g$ or $y = g$.

We introduce g is irreducible as a synonym of g is meet-irreducible.

Let G be a non empty relational structure and let g be an element of G. We say that g is join-irreducible if and only if:

(Def. 3) For all elements x, y of G such that $g = x \sqcup y$ holds $x = g$ or $y = g$.

Let L be a non empty relational structure. The functor $\mathrm{IRR}(L)$ yields a subset of L and is defined by:

(Def. 4) For every element x of L holds $x \in \mathrm{IRR}(L)$ iff x is irreducible.

We now state the proposition

(10) For every upper-bounded antisymmetric non empty relational structure L with g.l.b.'s holds $\top_L$ is irreducible.

Let L be an upper-bounded antisymmetric non empty relational structure with g.l.b.'s. Note that there exists an element of L which is irreducible.

The following four propositions are true:

(11) Let L be a semilattice and x be an element of L. Then x is irreducible if and only if for every finite non empty subset A of L such that $x = \inf A$ holds $x \in A$.

(12) For every lattice L and for every element l of L such that $\uparrow l \setminus \{l\}$ is a filter of L holds l is irreducible.

(13) Let L be a lattice, p be an element of L, and F be a filter of L. If p is maximal in F^{c}, then p is irreducible.

(14) Let L be a lower-bounded continuous lattice and x, y be elements of L. Suppose $y \not\leq x$. Then there exists an element p of L such that p is irreducible and $x \leq p$ and $y \not\leq p$.

4. Order generating sets

Let L be a non empty relational structure and let X be a subset of L. We say that X is order-generating if and only if:

(Def. 5) For every element x of L holds $\inf \uparrow x \cap X$ exists in L and $x = \inf(\uparrow x \cap X)$.

We now state several propositions:

(15) Let L be an up-complete lower-bounded lattice and X be a subset of L. Then X is order-generating if and only if for every element l of L there exists a subset Y of X such that $l = \bigsqcap_L Y$.

(16) Let L be an up-complete lower-bounded lattice and X be a subset of L. Then X is order-generating if and only if for every subset Y of L such that $X \subseteq Y$ and for every subset Z of Y holds $\bigsqcap_L Z \in Y$ holds the carrier of $L = Y$.

(17) Let L be an up-complete lower-bounded lattice and X be a subset of L. Then X is order-generating if and only if for all elements l_1, l_2 of L such that $l_2 \not\leq l_1$ there exists an element p of L such that $p \in X$ and $l_1 \leq p$ and $l_2 \not\leq p$.

(18) Let L be a lower-bounded continuous lattice and X be a subset of L. If $X = \mathrm{IRR}(L) \setminus \{\top_L\}$, then X is order-generating.

(19) Let L be a lower-bounded continuous lattice and X, Y be subsets of L. If X is order-generating and $X \subseteq Y$, then Y is order-generating.

5. Prime elements

Let L be a non empty relational structure and let l be an element of L. We say that l is prime if and only if:

(Def. 6) For all elements x, y of L such that $x \sqcap y \leq l$ holds $x \leq l$ or $y \leq l$.

Let L be a non empty relational structure. The functor $\mathrm{PRIME}(L)$ yielding a subset of L is defined as follows:

(Def. 7) For every element x of L holds $x \in \mathrm{PRIME}(L)$ iff x is prime.

Let L be a non empty relational structure and let l be an element of L. We say that l is co-prime if and only if:

(Def. 8) $l^{\smile}$ is prime.

The following propositions are true:

(20) For every upper-bounded antisymmetric non empty relational structure L holds $\top_L$ is prime.

(21) For every lower-bounded antisymmetric non empty relational structure L holds $\bot_L$ is co-prime.

Let L be an upper-bounded antisymmetric non empty relational structure. Note that there exists an element of L which is prime.

Next we state a number of propositions:

(22) Let L be a semilattice and l be an element of L. Then l is prime if and only if for every finite non empty subset A of L such that $l \geq \inf A$ there exists an element a of L such that $a \in A$ and $l \geq a$.

(23) Let L be a sup-semilattice and x be an element of L. Then x is co-prime if and only if for every finite non empty subset A of L such that $x \leq \sup A$ there exists an element a of L such that $a \in A$ and $x \leq a$.

(24) For every lattice L and for every element l of L such that l is prime holds l is irreducible.

(25) Let given l. Then l is prime if and only if for every set x and for every map f from L into $2_{\subseteq}^{\{x\}}$ such that for every element p of L holds $f(p) = \emptyset$ iff $p \leq l$ holds f is meet-preserving and join-preserving.

(26) Let L be an upper-bounded lattice and l be an element of L. If $l \neq \top_L$, then l is prime iff $(\downarrow l)^{\mathrm{c}}$ is a filter of L.

(27) For every distributive lattice L and for every element l of L holds l is prime iff l is irreducible.

(28) For every distributive lattice L holds $\mathrm{PRIME}(L) = \mathrm{IRR}(L)$.

(29) Let L be a Boolean lattice and l be an element of L. Suppose $l \neq \top_L$. Then l is prime if and only if for every element x of L such that $x > l$ holds $x = \top_L$.

(30) Let L be a continuous distributive lower-bounded lattice and l be an element of L. Suppose $l \neq \top_L$. Then l is prime if and only if there exists an open filter F of L such that l is maximal in F^{c}.

(31) Let L be a relational structure and X be a subset of L. Then $\chi_{X,\text{the carrier of } L}$ is a map from L into $2_{\subseteq}^{\{\emptyset\}}$.

(32) Let L be a non empty relational structure and p, x be elements of L. Then $\chi_{(\downarrow p)^{\mathrm{c}},\text{the carrier of } L}(x) = \emptyset$ if and only if $x \leq p$.

(33) Let L be an upper-bounded lattice, f be a map from L into $2_{\subseteq}^{\{\emptyset\}}$, and p be a prime element of L. Suppose $\chi_{(\downarrow p)^{\mathrm{c}},\text{the carrier of } L} = f$. Then f is meet-preserving and join-preserving.

(34) For every complete lattice L such that $\mathrm{PRIME}(L)$ is order-generating holds L is distributive and meet-continuous.

(35) For every lower-bounded continuous lattice L holds L is distributive iff $\mathrm{PRIME}(L)$ is order-generating.

(36) For every lower-bounded continuous lattice L holds L is distributive iff L is Heyting.

(37) Let L be a continuous complete lattice. Suppose that for every element l of L there exists a subset X of L such that $l = \sup X$ and for every element x of L such that $x \in X$ holds x is co-prime. Let l be an element of L. Then $l = \bigsqcup_L(\twoheaddownarrow l \cap \mathrm{PRIME}(L^{\mathrm{op}}))$.

(38) Let L be a complete lattice. Then L is completely-distributive if and only if the following conditions are satisfied:

(i) L is continuous, and

(ii) for every element l of L there exists a subset X of L such that $l = \sup X$ and for every element x of L such that $x \in X$ holds x is co-prime.

(39) Let L be a complete lattice. Then L is completely-distributive if and only if L is distributive and continuous and L^{op} is continuous.

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