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Injective Spaces. Part II[1]

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The articles [26], [11], [33], [34], [35], [8], [10], [9], [7], [28], [1], [21], [22], [24], [18], [25], [23], [14], [13], [27], [37], [15], [32], [2], [3], [4], [36], [12], [19], [29], [5], [30], [20], [31], [6], [17], and [16] provide the notation and terminology for this paper.

1. INJECTIVE SPACES

The following propositions are true:

(1) For every point p of the Sierpiński space such that $p = 0$ holds $\{p\}$ is closed.

(2) For every point p of the Sierpiński space such that $p = 1$ holds $\{p\}$ is non closed.

Let us observe that the Sierpiński space is non T_1.
Let us note that every top-lattice which is complete and Scott is also discernible.
Let us mention that there exists a T_0-space which is injective and strict.
Let us note that there exists a top-lattice which is complete, Scott, and strict.
The following propositions are true:

(3) Let I be a non empty set and T be a Scott topological augmentation of $\prod(I \longmapsto 2^1_{\subseteq})$. Then the carrier of $T =$ the carrier of $\prod(I \longmapsto$ the Sierpiński space$)$.

(4) Let L_1, L_2 be complete lattices, T_1 be a Scott topological augmentation of L_1, T_2 be a Scott topological augmentation of L_2, h be a map from L_1 into L_2, and H be a map from T_1 into T_2. If $h = H$ and h is isomorphic, then H is a homeomorphism.

(5) Let L_1, L_2 be complete lattices, T_1 be a Scott topological augmentation of L_1, and T_2 be a Scott topological augmentation of L_2. If L_1 and L_2 are isomorphic, then T_1 and T_2 are homeomorphic.

(6) Let S, T be non empty topological spaces. If S is injective and S and T are homeomorphic, then T is injective.

(7) Let L_1, L_2 be complete lattices, T_1 be a Scott topological augmentation of L_1, and T_2 be a Scott topological augmentation of L_2. If L_1 and L_2 are isomorphic and T_1 is injective, then T_2 is injective.

Let X, Y be non empty topological spaces. Let us observe that X is a topological retract of Y if and only if:

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(Def. 1) There exists a continuous map c from X into Y and there exists a continuous map r from Y into X such that $r \cdot c = \mathrm{id}_X$.

Next we state several propositions:

(8) Let S, T, X, Y be non empty topological spaces. Suppose that

(i) the topological structure of $S =$ the topological structure of T,

(ii) the topological structure of $X =$ the topological structure of Y, and

(iii) S is a topological retract of X.

Then T is a topological retract of Y.

(9) Let T, S_1, S_2 be non empty topological structures. Suppose S_1 and S_2 are homeomorphic and S_1 is a topological retract of T. Then S_2 is a topological retract of T.

(10) Let S, T be non empty topological spaces. Suppose T is injective and S is a topological retract of T. Then S is injective.

(11) Let J be an injective non empty topological space and Y be a non empty topological space. If J is a subspace of Y, then J is a topological retract of Y.

(12) For every complete continuous lattice L holds every Scott topological augmentation of L is injective.

Let L be a complete continuous lattice. Note that every topological augmentation of L which is Scott is also injective.

Let T be an injective non empty topological space. Note that the topological structure of T is injective.

2. Specialization Order

Let T be a topological structure. The functor ΩT yields a strict FR-structure and is defined by the conditions (Def. 2).

(Def. 2)(i) The topological structure of $\Omega T =$ the topological structure of T, and

(ii) for all elements x, y of ΩT holds $x \leq y$ iff there exists a subset Y of T such that $Y = \{y\}$ and $x \in \overline{Y}$.

Let T be an empty topological structure. Observe that ΩT is empty.
Let T be a non empty topological structure. One can check that ΩT is non empty.
Let T be a topological space. One can check that ΩT is topological space-like.
Let T be a topological structure. Note that ΩT is reflexive.
Let T be a topological structure. Observe that ΩT is transitive.
Let T be a T_0-space. One can verify that ΩT is antisymmetric.
We now state four propositions:

(13) Let S, T be topological spaces. Suppose the topological structure of $S =$ the topological structure of T. Then $\Omega S = \Omega T$.

(14) Let M be a non empty set and T be a non empty topological space. Then the relational structure of $\Omega \prod(M \longmapsto T) =$ the relational structure of $\prod(M \longmapsto \Omega T)$.

(15) For every Scott complete top-lattice S holds $\Omega S =$ the FR-structure of S.

(16) Let L be a complete lattice and S be a Scott topological augmentation of L. Then the relational structure of $\Omega S =$ the relational structure of L.

Let S be a Scott complete top-lattice. Observe that ΩS is complete.
We now state four propositions:

(17) Let T be a non empty topological space and S be a non empty subspace of T. Then ΩS is a full relational substructure of ΩT.

(18) Let S, T be topological spaces, h be a map from S into T, and g be a map from ΩS into ΩT. If $h = g$ and h is a homeomorphism, then g is isomorphic.

(19) For all topological spaces S, T such that S and T are homeomorphic holds ΩS and ΩT are isomorphic.

(20) For every injective T_0-space T holds ΩT is a complete continuous lattice.

Let T be an injective T_0-space. Note that ΩT is complete and continuous.
The following proposition is true

(21) For all non empty topological spaces X, Y holds every continuous map from ΩX into ΩY is monotone.

Let X, Y be non empty topological spaces. Note that every map from ΩX into ΩY which is continuous is also monotone.
The following proposition is true

(22) For every non empty topological space T and for every element x of ΩT holds $\overline{\{x\}} = {\downarrow}x$.

Let T be a non empty topological space and let x be an element of ΩT. Note that $\overline{\{x\}}$ is non empty, lower, and directed and ${\downarrow}x$ is closed.
One can prove the following proposition

(23) For every topological space X holds every open subset of ΩX is upper.

Let T be a topological space. Observe that every subset of ΩT which is open is also upper.
Let I be a non empty set, let S, T be non empty relational structures, let N be a net in T, and let i be an element of I. Let us assume that the carrier of $T \subseteq$ the carrier of S^I. The functor commute(N,i,S) yields a strict net in S and is defined by the conditions (Def. 3).

(Def. 3)(i) The relational structure of commute$(N,i,S) =$ the relational structure of N, and

(ii) the mapping of commute$(N,i,S) = ($commute(the mapping of $N))(i)$.

Next we state two propositions:

(24) Let X, Y be non empty topological spaces, N be a net in $[X \to \Omega Y]$, i be an element of N, and x be a point of X. Then (the mapping of commute$(N,x,\Omega Y))(i) =$ (the mapping of $N)(i)(x)$.

(25) Let X, Y be non empty topological spaces, N be an eventually-directed net in $[X \to \Omega Y]$, and x be a point of X. Then commute$(N,x,\Omega Y)$ is eventually-directed.

Let X, Y be non empty topological spaces, let N be an eventually-directed net in $[X \to \Omega Y]$, and let x be a point of X. Note that commute$(N,x,\Omega Y)$ is eventually-directed.
Let X, Y be non empty topological spaces. Note that every net in $[X \to \Omega Y]$ is function yielding.
We now state the proposition

(26) Let X be a non empty topological space, Y be a T_0-space, and N be a net in $[X \to \Omega Y]$. Suppose that for every point x of X holds sup commute$(N,x,\Omega Y)$ exists. Then sup rng(the mapping of N) exists in $(\Omega Y)^{\text{the carrier of } X}$.

3. Monotone Convergence Topological Spaces

Let T be a non empty topological space. We say that T is monotone convergence if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let D be a non empty directed subset of ΩT. Then $\sup D$ exists in ΩT and for every open subset V of T such that $\sup D \in V$ holds D meets V.

Next we state the proposition

(27) Let S, T be non empty topological spaces. Suppose the topological structure of $S =$ the topological structure of T and S is monotone convergence. Then T is monotone convergence.

One can check that every T_0-space which is trivial is also monotone convergence.
Let us observe that there exists a topological space which is strict, trivial, and non empty.
The following two propositions are true:

(28) Let S be a monotone convergence T_0-space and T be a T_0-space. If S and T are homeomorphic, then T is monotone convergence.

(29) Every Scott complete top-lattice is monotone convergence.

Let L be a complete lattice. One can verify that every Scott topological augmentation of L is monotone convergence.

Let L be a complete lattice and let S be a Scott topological augmentation of L. Note that the topological structure of S is monotone convergence.

We now state the proposition

(30) For every monotone convergence T_0-space X holds ΩX is up-complete.

Let X be a monotone convergence T_0-space. Note that ΩX is up-complete.
The following three propositions are true:

(31) Let X be a monotone convergence non empty topological space and N be an eventually-directed prenet over ΩX. Then $\sup N$ exists.

(32) Let X be a monotone convergence non empty topological space and N be an eventually-directed net in ΩX. Then $\sup N \in \operatorname{Lim} N$.

(33) For every monotone convergence non empty topological space X holds every eventually-directed net in ΩX is convergent.

Let X be a monotone convergence non empty topological space. One can check that every eventually-directed net in ΩX is convergent.

We now state two propositions:

(34) Let X be a non empty topological space. Suppose that for every eventually-directed net N in ΩX holds $\sup N$ exists and $\sup N \in \operatorname{Lim} N$. Then X is monotone convergence.

(35) Let X be a monotone convergence non empty topological space and Y be a T_0-space. Then every continuous map from ΩX into ΩY is directed-sups-preserving.

Let X be a monotone convergence non empty topological space and let Y be a T_0-space. Note that every map from ΩX into ΩY which is continuous is also directed-sups-preserving.

One can prove the following four propositions:

(36) Let T be a monotone convergence T_0-space and R be a T_0-space. If R is a topological retract of T, then R is monotone convergence.

(37) Let T be an injective T_0-space and S be a Scott topological augmentation of ΩT. Then the topological structure of $S =$ the topological structure of T.

(38) Every injective T_0-space is compact, locally-compact, and sober.

(39) Every injective T_0-space is monotone convergence.

Let us note that every T_0-space which is injective is also monotone convergence.
We now state four propositions:

(40) Let X be a non empty topological space, Y be a monotone convergence T_0-space, N be a net in $[X \to \Omega Y]$, and f, g be maps from X into ΩY. Suppose that

(i) $f = \bigsqcup_{((\Omega Y)^{\text{the carrier of } X})} \text{rng}\,(\text{the mapping of } N)$,

(ii) $\sup \text{rng}\,(\text{the mapping of } N)$ exists in $(\Omega Y)^{\text{the carrier of } X}$, and

(iii) $g \in \text{rng}\,(\text{the mapping of } N)$.

Then $g \leq f$.

(41) Let X be a non empty topological space, Y be a monotone convergence T_0-space, N be a net in $[X \to \Omega Y]$, x be a point of X, and f be a map from X into ΩY. Suppose for every point a of X holds $\text{commute}(N, a, \Omega Y)$ is eventually-directed and $f = \bigsqcup_{((\Omega Y)^{\text{the carrier of } X})} \text{rng}\,(\text{the mapping of } N)$. Then $f(x) = \sup \text{commute}(N, x, \Omega Y)$.

(42) Let X be a non empty topological space, Y be a monotone convergence T_0-space, and N be a net in $[X \to \Omega Y]$. Suppose that for every point x of X holds $\text{commute}(N, x, \Omega Y)$ is eventually-directed. Then $\bigsqcup_{((\Omega Y)^{\text{the carrier of } X})} \text{rng}\,(\text{the mapping of } N)$ is a continuous map from X into Y.

(43) Let X be a non empty topological space and Y be a monotone convergence T_0-space. Then $[X \to \Omega Y]$ is a directed-sups-inheriting relational substructure of $(\Omega Y)^{\text{the carrier of } X}$.

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