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Scott-Continuous Functions[1]

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The articles [18], [9], [23], [21], [1], [25], [24], [6], [8], [7], [17], [13], [22], [2], [3], [11], [4], [12], [26], [19], [5], [16], [15], [20], and [14] provide the notation and terminology for this paper.

1. Preliminaries

Let S be a non empty set and let a, b be elements of S. The functor $a,b,\ldots$ yielding a function from $\mathbb{N}$ into S is defined by the condition (Def. 1).

(Def. 1) Let i be a natural number. Then

- (i) if there exists a natural number k such that $i = 2 \cdot k$, then $(a,b,\ldots)(i) = a$, and
- (ii) if it is not true that there exists a natural number k such that $i = 2 \cdot k$, then $(a,b,\ldots)(i) = b$.

One can prove the following three propositions:

(1) Let S, T be non empty reflexive relational structures, f be a map from S into T, and P be a lower subset of T. If f is monotone, then $f^{-1}(P)$ is lower.

(2) Let S, T be non empty reflexive relational structures, f be a map from S into T, and P be an upper subset of T. If f is monotone, then $f^{-1}(P)$ is upper.

(3) Let S, T be reflexive antisymmetric non empty relational structures and f be a map from S into T. If f is directed-sups-preserving, then f is monotone.

Let S, T be reflexive antisymmetric non empty relational structures. Note that every map from S into T which is directed-sups-preserving is also monotone.

Next we state the proposition

(4) Let S, T be up-complete Scott top-lattices and f be a map from S into T. If f is continuous, then f is monotone.

Let S, T be up-complete Scott top-lattices. One can verify that every map from S into T which is continuous is also monotone.

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2. Poset of Continuous Maps

Let S be a set and let T be a reflexive relational structure. One can check that $S \longmapsto T$ is reflexive-yielding.

Let S be a non empty set and let T be a complete lattice. Observe that T^S is complete.

Let S, T be up-complete Scott top-lattices. The functor $\mathrm{SCMaps}(S,T)$ yielding a strict full relational substructure of $\mathrm{MonMaps}(S,T)$ is defined as follows:

(Def. 2) For every map f from S into T holds $f \in$ the carrier of $\mathrm{SCMaps}(S,T)$ iff f is continuous.

Let S, T be up-complete Scott top-lattices. Observe that $\mathrm{SCMaps}(S,T)$ is non empty.

3. Some Special Nets

Let S be a non empty relational structure and let a, b be elements of S. The functor $\mathrm{NetStr}(a,b)$ yields a strict non empty net structure over S and is defined by the conditions (Def. 3).

(Def. 3)(i) The carrier of $\mathrm{NetStr}(a,b) = \mathbb{N}$,

(ii) the mapping of $\mathrm{NetStr}(a,b) = a,b,\ldots$, and

(iii) for all elements i, j of $\mathrm{NetStr}(a,b)$ and for all natural numbers i', j' such that $i = i'$ and $j = j'$ holds $i \leq j$ iff $i' \leq j'$.

Let S be a non empty relational structure and let a, b be elements of S. Observe that $\mathrm{NetStr}(a,b)$ is reflexive, transitive, directed, and antisymmetric.

Next we state four propositions:

(5) Let S be a non empty relational structure, a, b be elements of S, and i be an element of $\mathrm{NetStr}(a,b)$. Then $(\mathrm{NetStr}(a,b))(i) = a$ or $(\mathrm{NetStr}(a,b))(i) = b$.

(6) Let S be a non empty relational structure, a, b be elements of S, i, j be elements of $\mathrm{NetStr}(a,b)$, and i', j' be natural numbers such that $i' = i$ and $j' = i' + 1$ and $j' = j$. Then

(i) if $(\mathrm{NetStr}(a,b))(i) = a$, then $(\mathrm{NetStr}(a,b))(j) = b$, and

(ii) if $(\mathrm{NetStr}(a,b))(i) = b$, then $(\mathrm{NetStr}(a,b))(j) = a$.

(7) For every poset S with g.l.b.'s and for all elements a, b of S holds $\liminf \mathrm{NetStr}(a,b) = a \sqcap b$.

(8) Let S, T be posets with g.l.b.'s, a, b be elements of S, and f be a map from S into T. Then $\liminf(f \cdot \mathrm{NetStr}(a,b)) = f(a) \sqcap f(b)$.

Let S be a non empty relational structure and let D be a non empty subset of S. The functor $\mathrm{NetStr}(D)$ yielding a strict net structure over S is defined by:

(Def. 4) $\mathrm{NetStr}(D) = \langle D, (\text{the internal relation of } S) |^2 D, \mathrm{id}_{\text{the carrier of } S} \restriction D \rangle$.

One can prove the following proposition

(9) Let S be a non empty reflexive relational structure and D be a non empty subset of S. Then $\mathrm{NetStr}(D) = \mathrm{NetStr}(D, \mathrm{id}_{\text{the carrier of } S} \restriction D)$.

Let S be a non empty reflexive relational structure and let D be a directed non empty subset of S. Observe that $\mathrm{NetStr}(D)$ is non empty, directed, and reflexive.

Let S be a non empty reflexive transitive relational structure and let D be a directed non empty subset of S. One can verify that $\mathrm{NetStr}(D)$ is transitive.

Let S be a non empty reflexive relational structure and let D be a directed non empty subset of S. Observe that $\mathrm{NetStr}(D)$ is monotone.

The following proposition is true

(10) For every up-complete lattice S and for every directed non empty subset D of S holds $\liminf \mathrm{NetStr}(D) = \sup D$.

4. Monotone Maps

The following propositions are true:

(11) Let S, T be lattices and f be a map from S into T. If for every net N in S holds $f(\liminf N) \leq \liminf(f \cdot N)$, then f is monotone.

(12) Let S, T be continuous lower-bounded lattices and f be a map from S into T. Suppose f is directed-sups-preserving. Let x be an element of S. Then $f(x) = \bigsqcup_T \{f(w); w$ ranges over elements of S: $w \ll x\}$.

(13) Let S be a lattice, T be an up-complete lower-bounded lattice, and f be a map from S into T. Suppose that for every element x of S holds $f(x) = \bigsqcup_T \{f(w); w$ ranges over elements of S: $w \ll x\}$. Then f is monotone.

(14) Let S be an up-complete lower-bounded lattice, T be a continuous lower-bounded lattice, and f be a map from S into T. Suppose that for every element x of S holds $f(x) = \bigsqcup_T \{f(w); w$ ranges over elements of S: $w \ll x\}$. Let x be an element of S and y be an element of T. Then $y \ll f(x)$ if and only if there exists an element w of S such that $w \ll x$ and $y \ll f(w)$.

(15) Let S, T be non empty relational structures, D be a subset of S, and f be a map from S into T. Suppose that

(i) $\sup D$ exists in S and $\sup f^\circ D$ exists in T, or

(ii) S is complete and antisymmetric and T is complete and antisymmetric.

If f is monotone, then $\sup(f^\circ D) \leq f(\sup D)$.

(16) Let S, T be non empty reflexive antisymmetric relational structures, D be a directed non empty subset of S, and f be a map from S into T. Suppose $\sup D$ exists in S and $\sup f^\circ D$ exists in T or S is up-complete and T is up-complete. If f is monotone, then $\sup(f^\circ D) \leq f(\sup D)$.

(17) Let S, T be non empty relational structures, D be a subset of S, and f be a map from S into T. Suppose that

(i) $\inf D$ exists in S and $\inf f^\circ D$ exists in T, or

(ii) S is complete and antisymmetric and T is complete and antisymmetric.

If f is monotone, then $f(\inf D) \leq \inf(f^\circ D)$.

(18) Let S, T be up-complete lattices, f be a map from S into T, and N be a monotone non empty net structure over S. If f is monotone, then $f \cdot N$ is monotone.

Let S, T be up-complete lattices, let f be a monotone map from S into T, and let N be a monotone non empty net structure over S. Note that $f \cdot N$ is monotone.

One can prove the following two propositions:

(19) Let S, T be up-complete lattices and f be a map from S into T. Suppose that for every net N in S holds $f(\liminf N) \leq \liminf(f \cdot N)$. Let D be a directed non empty subset of S. Then $\sup(f^\circ D) = f(\sup D)$.

(20) Let S, T be complete lattices, f be a map from S into T, N be a net in S, j be an element of N, and j' be an element of $f \cdot N$. Suppose $j' = j$. Suppose f is monotone. Then $f(\bigsqcap_S \{N(k); k$ ranges over elements of N: $k \geq j\}) \leq \bigsqcap_T \{(f \cdot N)(l); l$ ranges over elements of $f \cdot N : l \geq j'\}$.

5. Necessary and Sufficient Conditions of Scott-continuity

One can prove the following propositions:

(21) Let S, T be complete Scott top-lattices and f be a map from S into T. Then f is continuous if and only if for every net N in S holds $f(\liminf N) \leq \liminf(f \cdot N)$.

(22) Let S, T be complete Scott top-lattices and f be a map from S into T. Then f is continuous if and only if f is directed-sups-preserving.

Let S, T be complete Scott top-lattices. Observe that every map from S into T which is continuous is also directed-sups-preserving and every map from S into T which is directed-sups-preserving is also continuous.

The following propositions are true:

(23) Let S, T be continuous complete Scott top-lattices and f be a map from S into T. Then f is continuous if and only if for every element x of S and for every element y of T holds $y \ll f(x)$ iff there exists an element w of S such that $w \ll x$ and $y \ll f(w)$.

(24) Let S, T be continuous complete Scott top-lattices and f be a map from S into T. Then f is continuous if and only if for every element x of S holds $f(x) = \bigsqcup_T \{f(w); w$ ranges over elements of S: $w \ll x\}$.

(25) Let S be a lattice, T be a complete lattice, and f be a map from S into T. Suppose that for every element x of S holds $f(x) = \bigsqcup_T \{f(w); w$ ranges over elements of S: $w \leq x \wedge w$is compact$\}$. Then f is monotone.

(26) Let S, T be complete Scott top-lattices and f be a map from S into T. Suppose that for every element x of S holds $f(x) = \bigsqcup_T \{f(w); w$ ranges over elements of S: $w \leq x \wedge w$is compact$\}$. Let x be an element of S. Then $f(x) = \bigsqcup_T \{f(w); w$ ranges over elements of S: $w \ll x\}$.

(27) Let S, T be complete Scott top-lattices and f be a map from S into T. Suppose S is algebraic and T is algebraic. Then f is continuous if and only if for every element x of S and for every element k of T such that $k \in$ the carrier of CompactSublatt(T) holds $k \leq f(x)$ iff there exists an element j of S such that $j \in$ the carrier of CompactSublatt(S) and $j \leq x$ and $k \leq f(j)$.

(28) Let S, T be complete Scott top-lattices and f be a map from S into T. Suppose S is algebraic and T is algebraic. Then f is continuous if and only if for every element x of S holds $f(x) = \bigsqcup_T \{f(w); w$ ranges over elements of S: $w \leq x \wedge w$is compact$\}$.

(29) Let S, T be up-complete Scott non empty reflexive transitive antisymmetric topological space-like FR-structures and f be a map from S into T. If f is directed-sups-preserving, then f is continuous.

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