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# Algebraic and Arithmetic Lattices. Part II[1]

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**Summary.** The article is a translation of [10, pp. 89–92].

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The articles [16], [18], [19], [6], [7], [9], [17], [15], [2], [14], [1], [3], [11], [21], [4], [8], [5], [20], [12], and [13] provide the notation and terminology for this paper.

## 1. Preliminaries

One can prove the following propositions:

(1) Let $R$ be a relational structure and $S$ be a full relational substructure of $R$. Then every full relational substructure of $S$ is a full relational substructure of $R$.

(2) Let $X$ be a 1-sorted structure, $Y$, $Z$ be non empty 1-sorted structures, $f$ be a map from $X$ into $Y$, and $g$ be a map from $Y$ into $Z$. If $f$ is onto and $g$ is onto, then $g \cdot f$ is onto.

(3) For every 1-sorted structure $X$ and for every subset $Y$ of $X$ holds $(\mathrm{id}_X)^\circ Y = Y$.

(4) For every set $X$ and for every element $a$ of $2^X_\subseteq$ holds $\uparrow a = \{Y; Y$ ranges over subsets of $X$: $a \subseteq Y\}$.

(5) Let $L$ be an upper-bounded non empty antisymmetric relational structure and $a$ be an element of $L$. If $\top_L \leq a$, then $a = \top_L$.

(6) Let $S$, $T$ be non empty posets, $g$ be a map from $S$ into $T$, and $d$ be a map from $T$ into $S$. If $g$ is onto and $\langle g, d\rangle$ is Galois, then $T$ and $\mathrm{Im}\, d$ are isomorphic.

(7) Let $L_1$, $L_2$, $L_3$ be non empty posets, $g_1$ be a map from $L_1$ into $L_2$, $g_2$ be a map from $L_2$ into $L_3$, $d_1$ be a map from $L_2$ into $L_1$, and $d_2$ be a map from $L_3$ into $L_2$. If $\langle g_1, d_1\rangle$ is Galois and $\langle g_2, d_2\rangle$ is Galois, then $\langle g_2 \cdot g_1, d_1 \cdot d_2\rangle$ is Galois.

(8) Let $L_1$, $L_2$ be non empty posets, $f$ be a map from $L_1$ into $L_2$, and $f_1$ be a map from $L_2$ into $L_1$. Suppose $f_1 = (f$ **qua** function$)^{-1}$ and $f$ is isomorphic. Then $\langle f, f_1\rangle$ is Galois and $\langle f_1, f\rangle$ is Galois.

(9) For every set $X$ holds $2^X_\subseteq$ is arithmetic.

Let $X$ be a set. Observe that $2^X_\subseteq$ is arithmetic.
Next we state four propositions:

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(10) Let $L_1$, $L_2$ be up-complete non empty posets and $f$ be a map from $L_1$ into $L_2$. If $f$ is isomorphic, then for every element $x$ of $L_1$ holds $f^{\circ}{\downarrow}x = {\downarrow}f(x)$.

(11) For all non empty posets $L_1$, $L_2$ such that $L_1$ and $L_2$ are isomorphic and $L_1$ is continuous holds $L_2$ is continuous.

(12) Let $L_1$, $L_2$ be lattices. Suppose $L_1$ and $L_2$ are isomorphic and $L_1$ is lower-bounded and arithmetic. Then $L_2$ is arithmetic.

(13) Let $L_1$, $L_2$, $L_3$ be non empty posets, $f$ be a map from $L_1$ into $L_2$, and $g$ be a map from $L_2$ into $L_3$. Suppose $f$ is directed-sups-preserving and $g$ is directed-sups-preserving. Then $g \cdot f$ is directed-sups-preserving.

## 2. Maps Preserving Sup's and Inf's

Next we state several propositions:

(14) Let $L_1$, $L_2$ be non empty relational structures, $f$ be a map from $L_1$ into $L_2$, and $X$ be a subset of $\operatorname{Im} f$. Then $(f_{\circ})^{\circ}X = X$.

(15) Let $X$ be a set and $c$ be a map from $2^X_{\subseteq}$ into $2^X_{\subseteq}$. Suppose $c$ is idempotent and directed-sups-preserving. Then $c_{\circ}$ is directed-sups-preserving.

(16) Let $L$ be a continuous complete lattice and $p$ be a kernel map from $L$ into $L$. If $p$ is directed-sups-preserving, then $\operatorname{Im} p$ is a continuous lattice.

(17) Let $L$ be a continuous complete lattice and $p$ be a projection map from $L$ into $L$. If $p$ is directed-sups-preserving, then $\operatorname{Im} p$ is a continuous lattice.

(18) Let $L$ be a lower-bounded lattice. Then $L$ is continuous if and only if there exists an arithmetic lower-bounded lattice $A$ such that there exists a map from $A$ into $L$ which is onto, infs-preserving, and directed-sups-preserving.

(19) Let $L$ be a lower-bounded lattice. Then $L$ is continuous if and only if there exists an algebraic lower-bounded lattice $A$ such that there exists a map from $A$ into $L$ which is onto, infs-preserving, and directed-sups-preserving.

(20) Let $L$ be a lower-bounded lattice. Then

(i) if $L$ is continuous, then there exists a non empty set $X$ and there exists a projection map $p$ from $2^X_{\subseteq}$ into $2^X_{\subseteq}$ such that $p$ is directed-sups-preserving and $L$ and $\operatorname{Im} p$ are isomorphic, and

(ii) if there exists a set $X$ and there exists a projection map $p$ from $2^X_{\subseteq}$ into $2^X_{\subseteq}$ such that $p$ is directed-sups-preserving and $L$ and $\operatorname{Im} p$ are isomorphic, then $L$ is continuous.

## 3. Atoms Elements

The following proposition is true

(21) For every non empty relational structure $L$ and for every element $x$ of $L$ holds $x \in \mathrm{PRIME}(L^{\mathrm{op}})$ iff $x$ is co-prime.

Let $L$ be a non empty relational structure and let $a$ be an element of $L$. We say that $a$ is atom if and only if:

(Def. 1) $\perp_L < a$ and for every element $b$ of $L$ such that $\perp_L < b$ and $b \leq a$ holds $b = a$.

Let $L$ be a non empty relational structure. The functor $\mathrm{ATOM}(L)$ yielding a subset of $L$ is defined as follows:

(Def. 2) For every element $x$ of $L$ holds $x \in \mathrm{ATOM}(L)$ iff $x$ is atom.

We now state the proposition

(23)[1] For every Boolean lattice $L$ and for every element $a$ of $L$ holds $a$ is atom iff $a$ is co-prime and $a \neq \perp_L$.

Let $L$ be a Boolean lattice. Observe that every element of $L$ which is atom is also co-prime.
The following propositions are true:

(24) For every Boolean lattice $L$ holds $\mathrm{ATOM}(L) = \mathrm{PRIME}(L^{\mathrm{op}}) \setminus \{\perp_L\}$.

(25) For every Boolean lattice $L$ and for all elements $x$, $a$ of $L$ such that $a$ is atom holds $a \leq x$ iff $a \not\leq \neg x$.

(26) Let $L$ be a complete Boolean lattice, $X$ be a subset of $L$, and $x$ be an element of $L$. Then $x \sqcap \sup X = \bigsqcup_L \{x \sqcap y; y$ ranges over elements of $L$: $y \in X\}$.

(27) Let $L$ be a lower-bounded antisymmetric non empty relational structure with g.l.b.'s and $x$, $y$ be elements of $L$. If $x$ is atom and $y$ is atom and $x \neq y$, then $x \sqcap y = \perp_L$.

(28) Let $L$ be a complete Boolean lattice, $x$ be an element of $L$, and $A$ be a subset of $L$. If $A \subseteq \mathrm{ATOM}(L)$, then $x \in A$ iff $x$ is atom and $x \leq \sup A$.

(29) Let $L$ be a complete Boolean lattice and $X$, $Y$ be subsets of $L$. If $X \subseteq \mathrm{ATOM}(L)$ and $Y \subseteq \mathrm{ATOM}(L)$, then $X \subseteq Y$ iff $\sup X \leq \sup Y$.

## 4. More on the Boolean Lattice

The following propositions are true:

(30) For every Boolean lattice $L$ holds $L$ is arithmetic iff there exists a set $X$ such that $L$ and $2^X_\subseteq$ are isomorphic.

(31) For every Boolean lattice $L$ holds $L$ is arithmetic iff $L$ is algebraic.

(32) For every Boolean lattice $L$ holds $L$ is arithmetic iff $L$ is continuous.

(33) For every Boolean lattice $L$ holds $L$ is arithmetic iff $L$ is continuous and $L^{\mathrm{op}}$ is continuous.

(34) For every Boolean lattice $L$ holds $L$ is arithmetic iff $L$ is completely-distributive.

(35) Let $L$ be a Boolean lattice. Then $L$ is arithmetic if and only if the following conditions are satisfied:

(i) $L$ is complete, and

(ii) for every element $x$ of $L$ there exists a subset $X$ of $L$ such that $X \subseteq \mathrm{ATOM}(L)$ and $x = \sup X$.

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[1] The proposition (22) has been removed.

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